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## ITERATED AUTOMORPHISM ORBITS OF BOUNDED CONVEX DOMAINS IN $\mathbb{C}^n$


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# ITERATED AUTOMORPHISM ORBITS OF BOUNDED CONVEX DOMAINS IN $\mathbb{C}^n$

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**The classification of bounded domains in $\mathbb{C}^n$, with $n > 1$, is related to the geometric properties of the boundary. A conjecture of Greene and Krantz relates the geometry of the boundary with the group of biholomorphic self mappings of the domain. The Greene–Krantz conjecture, if true, can tell us much about the classification of smoothly bounded domains in $\mathbb{C}^n$. Much work has been done to attempt to solve this conjecture, though it has yet to be proved or disproved. However, there are numerous partial results which support the conjecture. In this paper, we prove a special case of the conjecture:**

**Theorem: Suppose $\Omega \subset \mathbb{C}^n$ is a bounded convex domain with $C^\infty$ boundary. Suppose there exists $\varphi \in \mathrm{Aut}(\Omega)$ and $p \in \Omega$ such that for the sequence of iterates $\{\varphi^j\} \subset \mathrm{Aut}(\Omega)$ we have $\varphi^j(p) \to x \in \partial\Omega$ nontangentially. Then $x$ is of finite type.**

## 1. Introduction

When studying domains in $\mathbb{C}^n$, we care about equivalence under biholomorphism. That is, two domains in $\mathbb{C}^n$ are equivalent if there is a biholomorphism between them. This equivalence is especially useful when our domains are endowed with the Kobayashi or Carathéodory metrics, for under these metrics, any biholomorphism preserves the distance between any two points. So no matter how much their Euclidean distances may differ, they are still the same distance apart in the Kobayashi metric. The Kobayashi metric will be an essential tool in whats follows. Certain properties of the automorphism group (biholomorphic self mappings) of the domain and the type (order of contact with a variety) of the boundary can be used to classify some bounded domains. It is a conjecture of Greene and Krantz that a smoothly bounded domain with a noncompact automorphism group is of finite type at any boundary orbit accumulation point. If this conjecture were true, it would classify all smoothly bounded domains in $\mathbb{C}^2$ with a noncompact automorphism group, for

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they would be, up to biholomorphism, the ball or a complex ellipsoid. The purpose of this paper is to prove the following:

**Theorem 1.1.** *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded convex domain with $C^\infty$ boundary. Suppose there exists $\varphi \in \mathrm{Aut}(\Omega)$ and $p \in \Omega$ such that for the sequence of iterates $\{\varphi^j\} \subset \mathrm{Aut}(\Omega)$, we have $\varphi^j(p) \to x \in \partial\Omega$ nontangentially. Then $x$ is of finite type.*

Our main theorem gives us a classification of domains with the properties of the hypothesis. Berteloot and Cœuré [1991] showed that if a smoothly bounded domain $\Omega \subset \mathbb{C}^2$ admits an automorphism accumulation point which is of finite type, then $\Omega$ is biholomorphic to a domain of the form $E_m = \{(z, w) \in \mathbb{C}^2 : |z|^{2m} + |w|^2 < 1\}$ for some $m > 0$.

**Corollary 1.2.** *Suppose $\Omega \subset \mathbb{C}^2$ is a bounded convex domain with $C^\infty$ boundary. Suppose there exist $\varphi \in \mathrm{Aut}(\Omega)$ and $p \in \Omega$ such that for the sequence of iterates $\{\varphi^j(p)\} \subset \mathrm{Aut}(\Omega)$, we have $\varphi^j(p) \to x \in \partial\Omega$ nontangentially. Then $\Omega$ is biholomorphic to an egg domain, $E_m$.*

For arbitrary dimension, $n$, Zimmer [2017] also showed that if a finite type boundary point of a smoothly bounded convex domain is also a nontangential automorphism orbit accumulation point, then the entire boundary is of finite type. Furthermore, Bedford and Pinchuk [1994] showed that smoothly bounded convex domains with finite type boundary and a noncompact automorphism group are biholomorphic to a certain class of polynomial domains. Therefore we have the following, which we state in the language of [Bedford and Pinchuk 1994].

**Corollary 1.3.** *Suppose $\Omega \subset \mathbb{C}^{n+1}$ is a bounded convex domain with $C^\infty$ boundary. Suppose there exist $\varphi \in \mathrm{Aut}(\Omega)$ and $p \in \Omega$ such that for the sequence of iterates $\{\varphi^j(p)\} \subset \mathrm{Aut}(\Omega)$, we have $\varphi^j(p) \to x \in \partial\Omega$ nontangentially. Then $\Omega$ is biholomorphic to a domain of the form*

$$\Big\{(w, z_1, \dots, z_n) \in \mathbb{C} \times \mathbb{C}^n : |w|^2 + \sum\nolimits_{\mathrm{wt}\, J = \mathrm{wt}\, K = \frac{1}{2}} a_{JK} z^J \bar{z}^K < 1\Big\},$$

*where $J = (j_1, \dots, j_n)$ and $K = (k_1, \dots k_n)$ are multi-indices, $a_{JK} = \bar{a}_{KJ}$, and* $\mathrm{wt}\, J = j_1\delta_1 + \cdots + j_n\delta_n$ *for some fixed $\delta_\ell = (2m_\ell)^{-1}$, with $m_\ell$ a positive integer.*

In Section 2 we discuss some notation and definitions to be used throughout. Sections 3, 4, and 5 cover the main ideas of the hypothesis of our result. Finally, in Section 6 we give a proof of the main result. This paper is part of the author's doctoral thesis. The author would like to thank Professor Bun Wong for all his help and guidance as well as the referees for their thoughtful revisions.

## 2. Preliminaries

For two open subsets $W, V \subset \mathbb{C}^n$, a function $f : W \to V$ is said to be a biholomorphism if $f$ is holomorphic and admits a holomorphic inverse $f^{-1} : V \to W$. We will denote by $\mathrm{Hol}(U, V)$ the collection of holomorphic maps from $U$ to $V$. The unit disk in $\mathbb{C}$ is given by

$$\Delta = \{z \in \mathbb{C} : |z| < 1\},$$

the upper half plane in $\mathbb{C}$ by

$$\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\},$$

and the unit polydisk in $\mathbb{C}^n$ by

$$\Delta^n = \Delta \times \cdots \times \Delta = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| < 1 \text{ for all } j = 1, \ldots, n\}.$$

Finally the unit ball in $\mathbb{C}^n$ is denoted

$$B^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 < 1\}.$$

**Definition 2.1.** Let $\Omega \subset \mathbb{C}^n$ be an open set with $C^k$ boundary. A function $\rho : \mathbb{C}^n \to \mathbb{R}$ is said to be a defining function for $\Omega$ if $\rho$ is $C^k$ and

(1) $\rho(x) < 0$ for all $x \in \Omega$,

(2) $\rho(x) > 0$ for all $x \notin \Omega$, and

(3) $\nabla\rho(x) \neq 0$ for all $x \in \partial\Omega$.

**Definition 2.2.** Let $\Omega \subset \mathbb{R}^n$ have a $C^1$ defining function $\rho$. Let $p \in \partial\Omega$. Then $w = (w_1, \ldots, w_n)$ is a tangent vector to $\partial\Omega$ at $p$ if

$$\sum_{k=1}^{n} \frac{\partial\rho}{\partial x_k}\bigg|_p w_k = 0.$$

In this case we write $w \in T_p(\partial\Omega)$.

**Definition 2.3.** Let $\Omega \subset \mathbb{C}^n$ be a domain with $C^2$ boundary, $p \in \partial\Omega$, and $\rho$ be a defining function for $\Omega$. We say that $p$ is a point of Levi pseudoconvexity if

$$\sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}\bigg|_p w_j \bar{w}_k \geq 0$$

for all $w \in \mathbb{C}^n$ such that

$$\sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j}\bigg|_p w_j = 0.$$

If instead, we have a strict inequality for all nonzero $w$ satisfying the second equation, we say that $x$ is a point of strict (Levi) pseudoconvexity. In general, when we say a boundary point is pseudoconvex we mean that it is weakly pseudoconvex.

The vectors satisfying the second equation in the above definition are called complex tangent vectors. We shall denote the complex tangent space by $T_p^{(1,0)}(\partial\Omega)$.

Note that $T_p^{(1,0)}(\partial\Omega) \subset T_p(\partial\Omega)$. We call

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}\bigg|_p w_j \bar{w}_k$$

the Levi form of $\rho$ at $p$. So $p \in \partial\Omega$ is a point of weak (respectively, strong) pseudoconvexity if its Levi form is positive semidefinite (respectively, positive definite).

**Definition 2.4.** Given $\Omega \subset \mathbb{C}^n$, $p \in \Omega$, and $v \in \mathbb{C}^n$, the Kobayashi pseudometric is given by

$$K_\Omega(p, v) = \inf\{|\zeta| : f \in \mathrm{Hol}(\Delta, \Omega),\, f(0) = p,\, f'(\zeta) = v\}.$$

The Poincaré metric coincides with this metric on $\Delta$ and $\mathcal{H}$. We will use Royden's integral formula [1971] for the Kobayashi pseudodistance.

**Definition 2.5.** The Kobayashi pseudodistance is given by

$$d_\Omega(z, w) = \inf_\gamma \int_0^1 K_\Omega(\gamma(t), \gamma'(t))\, dt,$$

where $z, w \in \Omega$ and $\gamma : [0, 1] \to \Omega$ is any piecewise $C^1$ curve such that $\gamma(0) = z$, $\gamma(1) = w$.

**Proposition 2.6.** *Let $U$, $V$ be domains in $\mathbb{C}^n$ and $f : U \to V$ be a holomorphic map. Then*

$$K_V(f(p), f'(v)) \le K_U(p, v)$$

*and*

$$d_V(f(z), f(w)) \le d_U(z, w).$$

For convex domains that do not contain any complex lines, Barth [1980] showed that the Kobayashi pseudodistance is an actual distance in the sense that $d_\Omega(z, w) > 0$ if $z \neq w$.

**Definition 2.7.** We say that a subset $\Omega \subset \mathbb{C}^n$ is $\mathbb{C}$-proper if $\Omega$ does not contain any nontrivial complex affine lines.

**Definition 2.8.** Let $\Omega \subset \mathbb{C}^n$ be a $\mathbb{C}$-proper open set. For $z \in \Omega$ and $v \in \mathbb{C}^n$, let $L(z, v) \subset \mathbb{C}^n$ be the complex line passing through $z$ in the direction of $v$. We set

$$\delta_\Omega(z, v) = d_{\mathrm{Euc}}(z, \partial\Omega \cap L(z, v))$$

and

$$\delta_\Omega(z) = d_{\mathrm{Euc}}(z, \partial\Omega).$$

That is, $\delta_\Omega(z, v)$ is the Euclidean distance from $z$ to $\partial\Omega$ in the complex direction of $v$ and $\delta_\Omega(z)$ is the overall Euclidean distance from $z$ to $\partial\Omega$.

Proposition 2.9 is a consequence of the definition of the Kobayashi metric.

**Proposition 2.9.** *Let $\Omega \subset \mathbb{C}^n$ be a domain, $z \in \Omega$, and $v \in \mathbb{C}^n$. Then*

$$K_\Omega(z, v) \leq \frac{\|v\|}{\delta_\Omega(z, v)}.$$

## 3. Variety type

**Definition 3.1.** Let $U \subset \mathbb{C}^n$. A subset $V \subset U$ is called a holomorphic variety if it is composed of the roots of a finite number of holomorphic functions. That is

$$V = \{z \in U : f_1(z) = f_2(z) = \cdots = f_k(z) = 0\}$$

where $f_i$ are holomorphic functions on $U$.

When a variety, $V$, is (complex) one-dimensional, then it can be parametrized. See [Gunning 1990] for a precise statement of the local parametrization theorem. We state only what is necessary for our purposes.

**Proposition 3.2.** *If $V \subset \mathbb{C}^n$ is a one-dimensional holomorphic variety and $p \in V$, then there is a neighborhood $U$ of $p$ and a nonconstant holomorphic function, $f : \Delta \to \mathbb{C}^n$ with $f(0) = p$ and $f(\Delta) \subset U \cap V$.*

We often refer to a one-dimensional holomorphic variety as a holomorphic disk or curve. When appropriate, we will refer to the image, $f(\Delta)$, as the holomorphic disk.

Given a smooth function $f : \mathbb{C} \to \mathbb{C}$ with $f(0) = 0$, we let $\nu(f)$ denote the order of vanishing of $f$ at 0. If $g : \mathbb{C} \to \mathbb{C}^n$ is a smooth function with $g(0) = 0$ we let $\nu(g) = \min_i \nu(g_i)$, where $g = (g_1, \ldots, g_d)$.

**Definition 3.3.** Let $\Omega$ be a smooth domain in $\mathbb{C}^n$ with $q = 0 \in \partial\Omega$. Let $\rho(z)$ be a defining function for $\Omega$ in a neighborhood of $q$. We say that $\partial\Omega$ is of finite type $C$ in the sense of D'Angelo if

$$\sup_f \left\{ \frac{\nu(\rho \circ f)}{\nu(f)} \right\} = C < \infty,$$

where $f$ ranges through nonconstant holomorphic parametrizations of one-dimensional holomorphic subvarieties of $\mathbb{C}^n$ with $f(0) = q$. We say that $\partial\Omega$ is of finite line type $L$ if

$$\sup_\ell \{\nu(\rho \circ \ell)\} = L < \infty,$$

where $\ell$ ranges through complex lines in $\mathbb{C}^n$ with $\ell(0) = q$.

Note that $\nu(\rho \circ \ell) \geq 2$ if and only if the image of $\ell$ is tangent to $\partial\Omega$ at $q$. So if we have a domain $\Omega \subset \mathbb{C}^n$ and a point $q \in \partial\Omega$ such that there is a holomorphic disk $V$ passing through $q$, the D'Angelo (or variety) type of $q$ is essentially a measurement of "how close" $V$ is to actually lying in $\partial\Omega$. Now if $V \subset \partial\Omega$ then $q$ would be a point of infinite type. When working with geometrically convex domains, one need only consider the line type rather than the more general variety type. This is due to McNeal, who gave the following proposition.

**Proposition 3.4** [McNeal 1992]. *Let $\Omega \subset \mathbb{C}^n$ be a convex domain with $q \in \partial\Omega$. Then $q$ is a point of finite variety type if and only if it is of finite line type.*

## 4. Automorphism orbits

For a domain $\Omega \subset \mathbb{C}^n$, the group of automorphisms will be denoted by $\mathrm{Aut}(\Omega)$. That is, $\mathrm{Aut}(\Omega)$ is the collection of biholomorphic self mappings of $\Omega$.

**Definition 4.1.** Let $\Omega \subset \mathbb{C}^n$ be a domain. We say $p \in \overline{\Omega}$ is an orbit accumulation point of $\mathrm{Aut}(\Omega)$ if there is a sequence $\{\varphi_k\} \subset \mathrm{Aut}(\Omega)$ and a point $q \in \Omega$ such that $\varphi_k(q) \to p$. If $p \in \partial\Omega$ then we say $p$ is a boundary orbit accumulation point for $\{\varphi_k\}$.

H. Cartan showed that for a bounded domain, $\Omega$, $\mathrm{Aut}(\Omega)$ is a Lie group which acts properly on $\Omega$; see [Narasimhan 1971]. We can determine the compactness of $\mathrm{Aut}(\Omega)$ by examining the orbits.

**Proposition 4.2.** *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded domain.* $\mathrm{Aut}(\Omega)$ *admits a boundary orbit accumulation point if and only if* $\mathrm{Aut}(\Omega)$ *is noncompact.*

The well-known ball characterization theorem of Bun Wong classifies all bounded strongly pseudoconvex domains with a noncompact automorphism group.

**Theorem 4.3** [Wong 1977]. *If $\Omega \subset \mathbb{C}^n$ is a strongly pseudoconvex bounded domain with a noncompact automorphism group, then $\Omega$ is biholomorphic to the unit ball $B^n$.*

We now have all the necessary machinery to state the Greene–Krantz conjecture.

**Conjecture 4.4** (Greene and Krantz). Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with smooth $C^\infty$ boundary. If $p \in \partial\Omega$ is a boundary orbit accumulation point for $\mathrm{Aut}(\Omega)$, then $\partial\Omega$ is of finite type at $p$.

In [Krantz 2016], the conjecture above is shown for convex domains in $\mathbb{C}^2$. The proof involves subelliptic estimates for the $\bar\partial$ problem.

One partial result to this conjecture shows that if there is a boundary orbit accumulation point, $x$, for a smoothly bounded convex domain, then there is no nontrivial holomorphic disk contained in the boundary and passing through $x$. However, this does not guarantee that $x$ is a point of finite type.

**Theorem 4.5** (Lee, Thomas and Wong [Lee et al. 2014]). *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded convex domain. Suppose there is a sequence $\{\varphi_j\} \subset$* $\mathrm{Aut}(\Omega)$ *such that $\varphi_j(z)$ converges nontangentially to some boundary point for all $z \in \Omega$. If $p \in \partial\Omega$*

*is an orbit accumulation point, then there does not exist any nontrivial complex analytic variety passing through* $p$ *and lying in* $\partial\Omega$.

In $\mathbb{C}^2$, Hamann and Wong showed that we can remove the nontangential condition in the above theorem.

**Theorem 4.6** [Hamann and Wong 2017]. *Let* $D$ *be a bounded convex domain in* $\mathbb{C}^2$ *with* $C^2$ *boundary. If* $p \in \partial D$ *is an orbit accumulation point, then* $\partial D$ *contains no nontrivial analytic variety passing through* $p$.

## 5. Nontangential convergence

The direction of travel of an automorphism orbit can yield certain conclusions. Nontangential convergence provides us with useful properties.

**Definition 5.1.** For a domain $\Omega \subset \mathbb{C}^n$ with $C^1$ boundary, a sequence $\{q_j\} \subset \Omega$, and a point $q \in \partial\Omega$, we say that $q_j \to q$ *nontangentially* if, for all $j$ large enough,

$$q_j \in \Gamma_\alpha(q) = \{z \in \Omega : \|z - q\| \le \alpha\delta_\Omega(z)\}$$

for some $\alpha > 1$. We say that $q_j \to q$ normally if the $q_j$'s approach $q$ along the real normal line to $\partial\Omega$ at $q$.

Let $n_q$ denote the inward-pointing normal vector to a $C^1$ domain $\Omega \in \mathbb{C}^n$ at boundary point $q \in \partial\Omega$.

**Lemma 5.2.** *Let* $\Omega \subset \mathbb{C}^n$ *be a convex domain with* $C^1$ *boundary. Let* $z \in \Omega$ *and* $q' = q + tn_q$ *for some* $t > 0$. *Then,*

$$\Gamma_\alpha(q) \subset \left\{z \in \Omega : 0 \le \angle zqq' \le \arccos\left(\frac{1}{\alpha}\right)\right\}.$$

*Proof.* Put $H = \{z \in \mathbb{C}^n : \mathrm{Im}(z_1) > 0\}$. We may assume $q = 0,\ n_q = (i, 0, \dots, 0)$, and $\Omega \subset H$. Then $\delta_\Omega(z) \le \delta_H(z) = \mathrm{Im}(z_1)$ which implies that $\|z - q\| \le \alpha \mathrm{Im}(z_1) = \alpha\|(\mathrm{Im}(z_1), 0, \dots, 0)\|$. Then, since

$$\cos(\angle zqq') = \frac{\|(\mathrm{Im}(z_1), 0, \dots, 0)\|}{\|z - q\|},$$

we have $\angle zqq' \le \arccos(1/\alpha)$. □

When $\partial\Omega$ admits a nontangential orbit accumulation point, Lee et al. [2014] showed that there is a sequence of points $\{p_j\} \subset \Omega$, within some fixed Kobayashi distance from $p \in \Omega$, such that the action of the sequence of automorphisms $\{\varphi_j\} \subset \mathrm{Aut}(\Omega)$ on the respective $p_j$'s approaches the accumulation point $q \in \partial\Omega$ along the real normal line to the boundary at $q$. To be precise:

**Lemma 5.3.** *Let* $\Omega \subset \mathbb{C}^n$ *be a convex domain with* $C^1$ *boundary. Suppose* $\{\varphi_j\} \subset \mathrm{Aut}(\Omega)$ *and* $\varphi_j(p) \to q \in \partial\Omega$ *nontangentially for some* $p \in \Omega$. *Then for sufficiently* *large $j$ there exists $\{p_j\} \subset \Omega$ such that $\varphi_j(p_j) \to q$ normally and $d_\Omega(p, p_j) \leq r$ for some $r > 0$.*

*Proof.* Let $\ell_q = \{q + tn_q : t \in \mathbb{R}\}$ and define $\pi : \mathbb{C}^n \to \ell_q$ as the projection mapping onto $\ell_q$. Put $q_j = \varphi_j(p)$, $\tilde{q}_j = \pi(q_j)$, and $p_j = \varphi^{-1}(\tilde{q}_j)$. Then $\tilde{q}_j \to q$ normally and $\|\tilde{q}_j - q_j\| \leq \|q_j - q\|$. Fix $j$ sufficiently large so that by Lemma 5.2

$$\frac{1}{\alpha} \leq \cos(\angle zqq') = \frac{\|\tilde{q}_j - q\|}{\|q_j - q\|}.$$

Let $\gamma(t) = (1-t)q_j + t\tilde{q}_j$. Then

$$d_\Omega(p, p_j) = d_\Omega(q_j, \tilde{q}_j) \leq \int_0^1 K_\Omega(\gamma(t), \gamma'(t))\, dt \leq \int_0^1 \frac{\|\gamma'(t)\|}{\delta_\Omega(\gamma(t), \gamma'(t))} dt$$

$$\leq \int_0^1 \frac{\|\gamma'(t)\|}{\delta_\Omega(\gamma(t))} dt \leq \int_0^1 \frac{\|\gamma'(t)\|\alpha}{\|\gamma(t) - q\|} dt \leq \frac{\|\tilde{q}_j - q_j\|\alpha}{\|\tilde{q}_j - q\|} \leq \frac{\|q_j - q\|\alpha}{\|\tilde{q}_j - q\|} \leq \alpha^2.$$

Finally, we let $r = \alpha^2$. □

Essentially, this gives us that the Kobayashi distance from each $\varphi_j(p)$ to the real normal line of the boundary at $q \in \partial\Omega$ remains bounded by a fixed constant.

## 6. Finite type

We will now be able to showcase a condition that guarantees finite type for some boundary point of a smoothly bounded convex domain.

**Definition 6.1.** For a domain $\Omega \subset \mathbb{C}^n$ denote by $B_\Omega(o, M)$ the closed ball centered at $o \in \Omega$ with Kobayashi radius $M$. That is,

$$B_\Omega(o, M) = \{z \in \Omega : d_\Omega(o, z) \leq M\}.$$

**Theorem 6.2** [Zimmer 2017]. *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded convex open set with $C^\infty$ boundary. If there exist $o \in \Omega$, $x \in \partial\Omega$, $M \geq 0$, and $T \in \mathbb{R}$ so that*

$$\{x + e^{-t} n_x : t > T\} \subset \operatorname{Aut}(\Omega) B_\Omega(o, M),$$

*then $x$ is of finite type in the sense of D'Angelo.*

We now state the main result.

**Theorem 6.3.** *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded convex domain with $C^\infty$ boundary. Suppose there exist $\varphi \in \operatorname{Aut}(\Omega)$ and $p \in \Omega$ such that for the sequence of iterates $\{\varphi^j\} \subset \operatorname{Aut}(\Omega)$ we have $\varphi^j(p) \to x \in \partial\Omega$ nontangentially. Then $x$ is of finite type.*

*Proof.* Since the set of automorphisms of $\Omega$ forms a group under composition of functions, we have $\varphi^j \in \operatorname{Aut}(\Omega)$ for all $j \in \mathbb{N}$. Put $M = d_\Omega(p, \varphi(p))$. We may assume $M > 0$, since otherwise, $\varphi$ would fix $p$. Then for every consecutive pair of iterates we have

$$\begin{aligned} d_\Omega(\varphi^j(p), \varphi^{j+1}(p)) &= d_\Omega(\varphi^j(p), \varphi^j(\varphi(p)) \\ &= d_\Omega(p, \varphi(p)) = M. \end{aligned}$$

By Lemma 5.3, there exists $\{p_j\} \subset \Omega$ such that $d_\Omega(p, p_j) \le r$ for some $r > 0$, and $\varphi^j(p_j) \to x$ normally. So we have

$$\begin{aligned} &d_\Omega(\varphi^j(p_j), \varphi^{j+1}(p_{j+1})) \\ &\qquad \le d_\Omega(\varphi^j(p_j), \varphi^j(p)) + d_\Omega(\varphi^j(p), \varphi^{j+1}(p)) + d_\Omega(\varphi^{j+1}(p), \varphi^{j+1}(p_{j+1})) \\ &\qquad \le r + d_\Omega(\varphi^j(p)), \varphi^{j+1}(p)) + r = 2r + M, \end{aligned}$$

for all $j \in \mathbb{N}$. By convexity, we may assume $x = 0$ and $n_x = (i, 0, \dots, 0)$. Also, since $\varphi^j(p_j) \to 0$ as $j \to \infty$, there are an infinitely many $j$ such that

$$|\varphi^j(p_j)| > |\varphi^{j+1}(p_{j+1})|.$$

Then in the argument that follows, we will see that there exists a fixed radius, $K$, such that for any $j$ with $|\varphi^j(p_j)| > |\varphi^{j+1}(p_{j+1})|$, the ball $B_\Omega(\varphi_j(p_j), K)$ contains both $\varphi^{j+1}(p_{j+1})$ and the real line segment connecting $\varphi^j(p_j)$ and $\varphi^{j+1}(p_{j+1})$. Now consider some $z, y, w \in \Omega$ that lie on the real normal line to $\partial\Omega$ at $x$ such that $|w| < |y| < |z|$. We claim that, for sufficiently small $|z|$, if $w \in B_\Omega(z, R)$ for some $R > 0$, then either $y \in B_\Omega(z, R)$ or $y \in B_\Omega(z, 1)$. If $z$ is sufficiently small, then there is a (complex) one-dimensional affine disk, $D$, centered at $z$, such that $D \subset \Omega \cap \{\zeta \in \Omega : \operatorname{Im}(\zeta_1) > 0\}$, $0 \in \partial D$, and $\partial D$ is tangent to $\partial\Omega$ at 0. Note that $D$ is essentially a copy of the unit disk under a translation and dilation and so $D$ is biholomorphic to $\Delta$. Now any geodesic under the Poincaré (equivalently, Kobayashi) metric passing through $z$ in $D$ is a straight line. Thus,

$$d_D(z, w) = d_D(z, y) + d_D(y, w).$$

Let $\pi : \mathbb{C}^n \to \mathbb{C}$ be the projection onto the first coordinate so $\pi(\Omega) \subset \mathcal{H}$. Then

$$d_\Omega(z, w) \ge d_{\pi(\Omega)}(\pi(z), \pi(w)) \ge d_{\mathcal{H}}(\pi(z), \pi(w)).$$

For simplicity, we may assume that $D$ has radius 1, so $z = (i, 0, \dots, 0)$. Then the mapping

$$\zeta \mapsto \left( i\frac{\zeta - i}{\zeta + i} + i, 0, \dots, 0 \right)$$

is a biholomorphism from $\mathcal{H}$ to $D$ where $z \mapsto z$ and any purely imaginary $ib \in \mathcal{H}$ is mapped to some $(ic, 0, \dots, 0) \in D$ where $ic$ is also purely imaginary. Now

$$d_{\mathcal{H}}(\pi(z), \pi(w)) = d_D\Big(z, w\frac{2}{|w|+1}\Big).$$

Note that since $|w| < 1$, then $\left|w\frac{2}{|w|+1}\right| > |w|$. So if $|y| > \left|w\frac{2}{|w|+1}\right|$, then

$$\begin{aligned} d_D\left(z, w\frac{2}{|w|+1}\right) &= d_D(z, y) + d_D\left(y, w\frac{2}{|w|+1}\right) \\ &\geq d_D(z, y) \\ &\geq d_\Omega(z, y) \end{aligned}$$

where the last inequality is given by the inclusion map from $D$ into $\Omega$. Otherwise, if $|y| \leq \left|w\frac{2}{|w|+1}\right|$, then

$$d_\Omega(y, w) \leq \int_0^1 \frac{|y-w|}{\delta_\Omega(w)}\, dt \leq \frac{\left|w\frac{2}{|w|+1}\right| - |w|}{|w|} = \frac{1-|w|}{1+|w|} < 1.$$

Therefore, if $w \in B_\Omega(z, R)$ for $z, w \in \{x + e^{-t} n_x : t > T\}$ with $T$ sufficiently large, then for all $y \in \{x + e^{-t} n_x : t > T\}$ with $|w| < |y| < |z|$, either $y \in B_\Omega(z, R)$ or $y \in B_\Omega(w, 1)$. Then since $d_\Omega(\varphi^j(p_j), \varphi^{j+1}(p_{j+1})) \leq 2r + M$, there is a $T > 0$ and some $K > 0$ such that

$$\{x + e^{-t} n_x : t > T\} \subset \bigcup_{j\in\mathbb{N}} B_\Omega(\varphi^j p_j, K),$$

and so

$$\{x + e^{-t} n_x : t > T\} \subset \mathrm{Aut}(\Omega) B_\Omega(p, K).$$

Thus, by Theorem 6.2, $x \in \partial\Omega$ is of finite type. □

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