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SHARP LOGARITHMIC SOBOLEV INEQUALITIES ALONG AN EXTENDED RICCI FLOW AND APPLICATIONS

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We prove a sharp Logarithmic Sobolev inequality along an extended Ricci flow. As applications, we derive an integral bound for the conjugate heat kernel and also obtain Lipschitz continuity of the pointed Nash entropy. Finally, based on these results, we prove an ε-regularity theorem for this extended Ricci flow.

1. Introduction

In this paper we study an extended Ricci flow as follows:

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}(g(t)) + 2d\phi(t) \otimes d\phi(t), \\ \frac{\partial}{\partial t} \phi(t) = \Delta_{g(t)} \phi(t), \\ g(0) = g_0, \ \phi(0) = \phi_0, \end{cases} \tag{1-1}$$

where $t \in [-T, 0]$, $g(t)$ are metrics, and $\phi(t) : (M, g) \to \mathbb{R}$ are smooth functions. This flow was introduced in [List 2008], where the author proved short time existence and long time existence if ϕ is a smooth function from M to $\mathbb{R}$. Later, R. Müller [2012] considered ϕ as a smooth map from (M, g) to (N, h) and proved some fundamental results for flow equation (1-1). The flow equations (1-1) come from static Einstein vacuum equations arising in the general relativity, and also arise as dimensional reductions of Ricci flow in higher dimensions. For more work on this flow, see [Fang and Zheng 2016b; 2016a; Guo et al. 2015a; 2015b; 2013; Li 2018; Liu and Wang 2017; Yang and Shen 2012]. Before stating our main results, we want to introduce some notation. Suppose $(M^n, g(t), \phi(t))|_{t\in[-T,0]}$ is an extended Ricci flow, fix $x, y \in M$; we use $d_t(x, y)$ to denote the distance between x and y at time t. We use $B_r(x, t)$ to denote the geodesic ball with radius r centered at x. We use Rm to denote the Riemannian curvature operator of the metric g, Ric the Ricci curvature, and R the scalar curvature. For the extended Ricci flow, we denote $\mathrm{Sic}(g(t)) = \mathrm{Ric}(g(t)) - d\phi(t) \otimes d\phi(t)$ and $S(g(t)) = R(g(t)) - |\nabla\phi(t)|^2_{g(t)}$.

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Given $(x_0, 0) \in M \times [-T, 0]$, we denote

$$H_{x_0}(y, s) = H(x_0, 0\,; y, s) = (4\pi|s|)^{-\frac{n}{2}} \exp(-f_{x_0}(y, s))$$

as the conjugate heat kernel based at $(x_0, 0)$, and

$$d\nu_{x_0}(y, s) = H_{x_0}(y, s)\mathrm{dvol}_{g(s)}(y)$$

as the associated probability measure. See Definition 2.2.

Definition 1.1. Let (M, g) be a smooth Riemannian manifold and ϕ and f be smooth functions. Given $\tau > 0$, we define the associated W entropy as

$$W(g, \phi, f, \tau) = \int_M [\tau(S + |\nabla f|^2) + f - n](4\pi\tau)^{-\frac{n}{2}} e^{-f}\, \mathrm{dvol}_g,$$

where $S = R - |\nabla\phi|^2$. Moreover, the μ entropy can also be defined:

$$\mu(g, \phi, \tau) = \inf\left\{ W(g, \phi, f, \tau) \;\Big|\; \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f}\, \mathrm{dvol}_g = 1 \right\}.$$

Next we state the Poincaré inequality and log-Sobolev inequality along the extended Ricci flow (1-1); previous work on Ricci flow was given in Hein and Naber [2014].

Theorem 1.2. *Let $(M^n, g(t), \phi(t))|_{t\in[-T,0]}$ be an extended Ricci flow* (1-1). *Fix a point $x_0 \in M$ in the final time slice and let $s \in [-T, 0]$.*

(1) *For all $u \in C_0^\infty(M)$ with $\int_M u\, d\nu_{x_0}(s) = 0$,*

$$\int_M u^2\, d\nu_{x_0}(s) \le 2|s| \int_M |\nabla u|^2_{g(s)}\, d\nu_{x_0}(s). \tag{1-2}$$

Equality holds if and only if $u \equiv 0$.

(2) *For all $u \in C_0^\infty(M)$ with $\int_M u^2\, d\nu_{x_0}(s) = 1$,*

$$\int_M u^2 \log u^2\, d\nu_{x_0}(s) \le 4|s| \int_M |\nabla u|^2_{g(s)}\, d\nu_{x_0}(s). \tag{1-3}$$

Equality holds if and only if $u \equiv 1$.

Remark. Following [Perelman 2002], Ni [2004] defined entropy for the linear heat equation on a complete Riemannian manifold. Under the Ricci nonnegativity assumption, he proved the monotonicity, and as an application, he characterized the Euclidean space using the sharp log-Sobolev inequality.

Next we state one important application of Theorem 1.2.

Theorem 1.3. *Let* $(M^n, g(t), \phi(t))|_{t\in[-T,0]}$ *be an extended Ricci flow and* $d\nu = d\nu_{x_0}(s)$ *be a heat kernel measure. Then the Gaussian concentration inequality*

$$\nu(A)\nu(B) \le \exp\biggl(-\frac{d_{g(s)}(A, B)^2}{8|s|}\biggr)$$

holds for $A, B \subseteq M$.

Corollary 1.4. *For any* $C > 0$ *there exists a* $C' = C'(n, C) > 0$ *such that the following holds*: *Let* $(M^n, g(t), \phi(t))$ *be an extended Ricci flow such that*

$$\sup_{t\in[s,0]} \|S(g(t))\|_\infty \le \frac{C}{|s|} \quad \text{and} \quad \inf_{\tau\in(0,2|s|)} \nu(g(s), \phi(s), \tau) \ge -C,$$

for some $s \in [-T, 0)$. *Let* $x_1, x_2 \in M$ *and* $r^2 = |s|$, *then*

$$\text{(1-4)} \qquad \frac{1}{\mathrm{Vol}(B_r(x_1, 0))} \int_{B_r(x_1,0)} H(x_2, 0; y, s)\, \mathrm{dvol}_{g(s)}(y) \le C' \exp\biggl(-\frac{-d_{g(s)}(B_r(x_1, 0), B_r(x_2, 0))^2}{C'|s|}\biggr).$$

Moreover, *we have the following distance distortion estimate*:

$$\text{(1-5)} \qquad d_{g(s)}(B_r(x_1, s), B_r(x_2, s)) \le C' d_{g(0)}(x_1, x_2).$$

Definition 1.5. The pointed W entropy at scale $|s|$ based at x_0 is defined by

$$W_{x_0}(s) = W(g(s), \phi(s), f_{x_0}(s), |s|).$$

The pointed Nash entropy at $(x_0, s) \in M \times [-T, 0]$ is defined as

$$N_{x_0}(s) = \frac{1}{|s|} \int_s^0 W_{x_0}(r)\, dr = \int_M f_{x_0}(s)\, d\nu_{x_0}(s) - \frac{n}{2}.$$

Now we can state the Lipschitz continuity of the pointed Nash entropy.

Theorem 1.6. *For each* $C > 0$, *there exists a* $C' = C'(n, C) > 0$ *such that the following holds*: *Let* $(M, g(t), \phi(t))$ *be an extended Ricci flow* (1-1) *such that*

$$\text{(1-6)} \qquad S(g(s)) \ge -\frac{C}{|s|} \quad \text{and} \quad \inf_{\tau\in(0,2|s|)} \mu(g(s), \phi(s), \tau) \ge -C$$

for some $s \in [-T, 0]$, *then the map*

$$x \in (M, g(0)) \to f_x(s) H_x(s) \in L^1(M, \mathrm{dvol}_{g(s)})$$

is globally $C'|s|^{-\frac{1}{2}}$ *Lipschitz. In particular*, *this means that*

$$|N_{x_1}(s) - N_{x_2}(s)| \le C'|s|^{-\frac{1}{2}} d_{g(0)}(x_1, x_2).$$

Definition 1.7. For an extended Ricci flow (1-1):

(1) Given $(x, t) \in M \times [-T, 0]$ and $r > 0$, we define the parabolic ball

$$P_r(x, t) = B_r(x, t) \times [t - r^2, t].$$

(2) Given $(x, t) \in M \times [-T, 0]$ and $r > 0$, we define the regularity scale

$$r_{|\mathrm{Rm}|}(x, t) = \sup\Big\{r > 0 : \sup_{P_r(x,t)} |\mathrm{Rm}| \le r^{-2}\Big\}.$$

Now we can state our main ε-regularity theorem.

Theorem 1.8. *For each $C > 0$, there exists $\varepsilon = \varepsilon(n, C) > 0$ such that the following holds*: *Let $(M, g(t), \phi(t))$ be an extended Ricci flow* (1-1) *such that*

$$S(g(s)) \ge -\frac{C}{|s|}, \qquad \inf_{\tau \in (0, 2|s|)} \mu(g(s), \phi(s), \tau) \ge -C, \qquad |\phi| \le C, \tag{1-7}$$

for some $s \in [-T, 0]$. If the pointed entropy satisfies

$$W_{x_0}(s) \ge -\varepsilon$$

for some point x_0 in the zero time slice, *then we have*

$$r_{|\mathrm{Rm}|}(x_0, 0)^2 \ge \varepsilon |s|.$$

Remark. Xu [2017] considered the short time asymptotics of Nash entropy on a complete Riemannian manifold with Ricci lower bound and gave interesting applications.

The paper is organized as follows. In Section 2 we review some background and preliminaries for the conjugate heat kernel, and we also prove a Bochner formula for any space-time function along the extended Ricci flow. In Section 3 we define the W entropy and obtain its monotonicity. As an application, we derive the κ noncollapsing property for the extended Ricci flow. We also clarify the relation between the pointed W entropy and the pointed Nash entropy. In Section 4 we derive various estimates. First, we derive a gradient estimate for the positive solution to the extended Ricci flow. Together with the monotonicity of W entropy, we prove the upper bound for the heat kernel. Second, we generalize Perelman's Harnack inequality to the extended Ricci flow, and based on this we prove the lower bound for the conjugate heat kernel. In Section 5, based on the results from previous sections, we prove the Poincaré inequality and the log-Sobolev inequality along the extended Ricci flow. As one application, we prove the Gaussian concentration inequality and then obtain an integral bound for the conjugate heat kernel. In Section 6, using the Poincaré inequality, we prove the Lipschitz continuity of the pointed Nash entropy. In Section 7, we derive the ε- regularity theorem; the key ingredients are the point picking argument and the Lipschitz continuity of the pointed Nash entropy.

2. Background and preliminary

Letting $(M^n, g(t), \phi(t))$ be an extended Ricci flow (1-1), we give the following definitions.

Definition 2.1. The heat operator and its conjugate are defined by

$$\square = \frac{\partial}{\partial t} - \Delta \quad \text{and} \quad \square^* = -\frac{\partial}{\partial t} - \Delta + S. \tag{2-8}$$

Definition 2.2. For $x, y \in M$ and $s < t$ in $[-T, 0]$, we let $H(x, t; y, s)$ denote the conjugate heat kernel based at (x, t), i.e, the unique minimal positive solution with $\lim_{s\to t} H(x, t; y, s) = \delta_x(y)$ of the conjugate heat equation

$$\square^*_{y,s} H(x, t; y, s) = \Big(-\frac{\partial}{\partial s} - \Delta_{y,g(s)} + S(y, s)\Big) H(x, t; y, s) = 0. \tag{2-9}$$

Lemma 2.3. *The conjugate heat equation satisfies the following properties*:

(1) $\int_M H(x, t; y, s)\, \mathrm{dvol}_{g(s)}(y) = 1$.

(2) $\int_M H(x, t; y, s)\, \mathrm{dvol}_{g(t)}(x) \le \exp(\rho(t - s))$, *where* $\rho = \|S(g(-T))^-\|_\infty$.

Proof. (1) Taking the derivative with respect to s, we get

$$\begin{aligned}
\frac{d}{ds}\int_M & H(x, t; y, s)\, \mathrm{dvol}_{g(s)}(y) \\
&= \int_M \Big(\frac{\partial}{\partial s} H(x, t; y, s) - H(x, t; y, s) S(y, s)\Big)\, \mathrm{dvol}_{g(s)}(y) \\
&= \int_M \Big(\frac{\partial}{\partial s} + \Delta_{g(s),y} - S\Big) H(x, t; y, s)\, \mathrm{dvol}_{g(s)}(y) = 0.
\end{aligned}$$

Due to $\lim_{s\to t} H(x, t; y, s) = \delta_x(y)$, we have $\int_M H(x, t; y, s)\, \mathrm{dvol}_{g(s)}(y) = 1$.

(2) Taking the derivative with respect to t,

$$\begin{aligned}
\frac{d}{dt}\int_M & H(x, t; y, s)\, \mathrm{dvol}_{g(t)}(x) \\
&= \int_M \Big(\frac{\partial}{\partial t} H(x, t; y, s) - H(x, t; y, s) S(x, t)\Big)\, \mathrm{dvol}_{g(t)}(x) \\
&= \int_M (\Delta_{g(t),x} H(x, t; y, s) - H(x, t; y, s) S(x, t))\, \mathrm{dvol}_{g(t)}(x) \\
&\le \rho \int_M H(x, t; y, s)\, \mathrm{dvol}_{g(t)}(x).
\end{aligned}$$

In the last inequality we need to use the evolution equation of S along the extended Ricci flow $\frac{\partial}{\partial t} S = \Delta S + 2|\operatorname{Sic}|^2 + 2(\Delta\phi)^2$. Applying the maximum principle, we know the minimum of S is increasing along the flow. Due to

$$\lim_{t\to s}\int_M H(x,t;y,s)\,\mathrm{dvol}_{g(t)}(x)=1,$$

we have

$$\int_M H(x,t;y,s)\,\mathrm{dvol}_{g(t)}(x)\le \exp(\rho(t-s)).\qquad\square$$

Lemma 2.4. *We have the following Bochner formula for all space-time functions u:*

$$\tfrac{1}{2}\square|\nabla u|^2=-|\nabla^2 u|^2+\langle\nabla\square u,\nabla u\rangle-\langle\nabla u,\nabla\phi\rangle^2. \tag{2-10}$$

Proof. Using the extended Ricci flow equation and the Bochner formula for function,

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}|\nabla u|^2&=\frac{\partial}{\partial t}\Big(\frac{1}{2}g^{ij}\nabla_i u\nabla_j u\Big)\\&=-\frac{1}{2}g^{ik}g^{jl}\frac{\partial}{\partial t}g_{kl}\nabla_i u\nabla_j u+g^{ij}\nabla_i\frac{\partial u}{\partial t}\cdot\nabla u\\&=\mathrm{Sic}(\nabla u,\nabla u)+\nabla\frac{\partial u}{\partial t}\cdot\nabla u,\end{aligned}$$

and

$$\frac{1}{2}\Delta|\nabla u|^2=|\nabla^2u|^2+\langle\nabla\Delta u,\nabla u\rangle+\mathrm{Ric}(\nabla u,\nabla u),$$

so

$$\begin{aligned}\frac{1}{2}\square|\nabla u|^2&=-|\nabla^2u|^2+\langle\nabla\square u,\nabla u\rangle+\mathrm{Sic}(\nabla u,\nabla u)-\mathrm{Ric}(\nabla u,\nabla u)\\&=-|\nabla^2u|^2+\langle\nabla\square u,\nabla u\rangle-\langle\nabla u,\nabla\phi\rangle^2.\end{aligned}\qquad\square$$

3. Monotonicity of entropy

Theorem 3.1. *Along the extended Ricci flow*

$$\begin{cases}\dfrac{\partial}{\partial t}g(t)=-2\,\mathrm{Sic}(g(t))=-2\mathrm{Ric}(g(t))+2d\phi(t)\otimes d\phi(t),\\ \dfrac{\partial}{\partial t}\phi(t)=\Delta_{g(t)}\phi(t),\\ \dfrac{\partial f}{\partial t}=-\Delta f+|\nabla f|^2-S(g(t))+\frac{n}{2\tau},\\ \dfrac{d\tau}{dt}=-1,\quad t\in[-T,0],\end{cases}$$

we have

$$\begin{aligned}&\frac{d}{dt}W(g(t),\phi(t),f(t),\tau(t))\\&\quad=2\tau\int_M\Big(|\,\mathrm{Sic}+\nabla^2f-\frac{1}{2\tau}g|^2+(\Delta\phi-\langle\nabla f,\nabla\phi\rangle)^2\Big)(4\pi\tau)^{-\frac{n}{2}}e^{-f}\,\mathrm{dvol}_{g(t)}.\end{aligned}$$

Define $\mu(g,\phi,\tau)=\inf\{W(g,\phi,f,\tau) \mid \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f}\,\mathrm{dvol}_g = 1\}$; by the above theorem, we have the following monotonicity fact.

Corollary 3.2. *For any fixed $t_0 \in \mathbb{R}$, the quantity $\mu(g(t),\phi(t),t_0-t)$ is non-decreasing in t. It is constant if and only if the flow is isometric to the gradient Ricci harmonic soliton with potential $f(t)$.*

Corollary 3.3. *Given $t_0 \in \mathbb{R}$, put $\mu_0 = \mu(g(-T),\phi(-T),t_0+T)$ and $\tau = t_0 - t$. Then,*

$$\int_M u^2 \log u^2 \,\mathrm{dvol}_{g(t)} \le \tau \int_M (4|\nabla u|^2 + S u^2)\,\mathrm{dvol}_{g(t)} - \frac{n}{2}\log(4\pi\tau) - n - \mu_0$$

for any $u \in C_0^\infty(M)$ with $\int_M u^2 \,\mathrm{dvol}_{g(t)} = 1$.

Proof. Recall $W(g,\phi,f,\tau) = \int_M [\tau(S+|\nabla f|^2)+f-n](4\pi\tau)^{-\frac{n}{2}} e^{-f}\,\mathrm{dvol}_g$, let $u^2 = (4\pi\tau)^{-\frac{n}{2}} e^{-f}$, then

$$W(g,\phi,f,\tau) = \int_M [\tau(Su^2 + 4|\nabla u|^2) - u^2 \log u^2]\,\mathrm{dvol}_g - \frac{n}{2}\log(4\pi\tau) - n.$$

By Theorem 3.1, we have

$$W(g(-T),\phi(-T),f(-T),\tau(-T)) \le W(g(t),\phi(t),f(t),\tau(t))$$

for $t \in [-T,t_0]$. Hence

$$\mu_0 \le \int_M [\tau(Su^2 + 4|\nabla u|^2) - u^2 \log u^2]\,\mathrm{dvol}_{g(t)} - \frac{n}{2}\log(4\pi\tau) - n. \qquad \square$$

Next we prove the κ noncollapsed property for the extended Ricci flow (1-1).

Theorem 3.4. *Fix $t_0 \in [-T,0]$, $x \in M$ and $r > 0$, and assume*

$$\inf_{\rho\in(0,r)} \mu(g(-T),\phi(-T),t_0+T+\rho^2) \ge -C \quad \textit{and} \quad \sup_{B_r(x,t)} S(g(t_0)) \le Cr^{-2}.$$

Define $\kappa = \exp(-(2^{n+4}+2C))$, then $\mathrm{Vol}(B_r(x,t_0)) \ge \kappa r^n$.

Proof. Given $\rho \in (0,r)$, define the function ψ as follows,

$$\begin{cases} \psi = 1 & \text{on } B_{\frac{\rho}{2}}(x,t_0), \\ \psi = 0 & \text{outside } B_\rho(x,t_0), \\ \psi \text{ is linear} & \text{on } B_\rho(x,t_0) \setminus B_{\frac{\rho}{2}}(x,t_0). \end{cases}$$

Denoting $\mu_0 = \mu(g(-T),\phi(-T),t_0+T+\rho^2)$ and $\tau(t) = t_0 - t + \rho^2$, applying the previous corollary to $u = \psi/\|\psi\|_2$, we have

$$\begin{aligned}\rho^2 \int_M (4|\nabla u|^2 + Su^2)\,\mathrm{dvol}_{g(t_0)} - \frac{n}{2}\log(4\pi\rho^2) - n - \mu_0 \\ \le \rho^2 \int_M \left(\frac{4|\nabla\psi|^2}{\|\psi\|_2^2} + S\frac{\psi^2}{\|\psi\|_2^2}\right)\mathrm{dvol}_{g(t_0)} - \frac{n}{2}\log(4\pi\rho^2) - n + C \\ \le \frac{16}{\mathrm{Vol}(B_{\frac{\rho}{2}}(x,t_0))}\big(\mathrm{Vol}(B_\rho(x,t_0)) - \mathrm{Vol}(B_{\frac{\rho}{2}}(x,t_0))\big) + 2C - \frac{n}{2}\log(4\pi\rho^2) - n.\end{aligned}$$

In the above calculation we use $|\nabla\psi| \le \frac{2}{\rho}$ on $B_\rho(x,t_0)\setminus B_{\frac{\rho}{2}}(x,t_0)$ and $\|\psi\|_2^2 \ge \mathrm{Vol}(B_{\frac{\rho}{2}}(x,t_0))$.

On the other hand,

$$\begin{aligned}\int_M u^2\log u^2\,\mathrm{dvol}_{g(t_0)} &= \int_{B_{t_0}(x,\rho)} \frac{\psi^2}{\|\psi\|_2^2}\log\frac{\psi^2}{\|\psi\|_2^2}\,\mathrm{dvol}_{g(t_0)} \\ &\ge \log\int_{B_\rho(x,t_0)}\left(\frac{\psi^2}{\|\psi\|_2^2}\right)^2\mathrm{dvol}_{g(t_0)} \ge \log\frac{1}{\mathrm{Vol}(B_\rho(x,t_0))},\end{aligned}$$

where in the first inequality we use the following Cauchy–Schwarz inequality:

$$\left(\int_{B_\rho(x,t_0)} \psi^2\,\mathrm{dvol}_{g(t_0)}\right)^2 \le \left(\int_{B_\rho(x,t_0)} \psi^4\,\mathrm{dvol}_{g(t_0)}\right)\mathrm{Vol}(B_\rho(x,t_0)).$$

So

$$\log\frac{1}{\mathrm{Vol}(B_\rho(x,t_0))} \le 16\left(\frac{\mathrm{Vol}(B_\rho(x,t_0))}{\mathrm{Vol}(B_{\frac{\rho}{2}}(x,t_0))} - 1\right) + 2C - \frac{n}{2}\log(4\pi\rho^2) - n.$$

It is easy to get the implication

$$\mathrm{Vol}(B_{\frac{\rho}{2}}(x,t_0)) \ge \kappa\left(\frac{\rho}{2}\right)^n \Rightarrow \mathrm{Vol}(B_\rho(x,t_0)) \ge \kappa\rho^n$$

for $\kappa = \exp(-(2^{n+4} + 2C))$. Then the claim follows by induction on ρ. □

Remark. Müller [2010] considered more general geometric flows which include Ricci flow, extended Ricci flow and Harmonic Ricci flow as special cases. In the same paper, he introduced more general reduced volume in analogy to Perelman's [2002] and proved its monotonicity. In [Müller 2012], he systematically studied the harmonic Ricci flow and proved some important fundamental estimates, in particular, he proved the κ noncollapsing result along harmonic Ricci flow. The W entropy for general geometric flow was discussed by Guo, Philipowski and Thalmaier [Guo et al. 2013], and they also proved the monotonicity of W entropy.

Proposition 3.5. *The following hold for $x \in M$ and $s \in [-T, 0]$.*

(1) $\lim_{s\to 0} W_x(s) = 0$.

(2) $\mu(g(-T), \phi(-T), T) \le W_x(s) \le 0$.

(3) $W_x(s) = -\int_s^0 2|r| \big(\int_M |\operatorname{Sic} + \nabla^2 f_x - \frac{g}{2|r|}|^2 + (\Delta\phi - \langle \nabla\phi, \nabla f_x \rangle)^2 \big)\, d\nu_x(r)\, dr$.

Proof. Recall $W_x(s) = W(g(s), \phi(s), f_x(x), |s|)$; (1) follows from the asymptotic expansion of the heat kernel at x. For (2) and (3),

$$\frac{d}{ds} W_x(s) = 2|s| \int_M \Big(|\operatorname{Sic} + \nabla^2 f_x - \frac{g}{2|s|}|^2 + (\Delta\phi - \langle \nabla\phi, \nabla f_x \rangle)^2 \Big)\, d\nu_x(s),$$

so

$$\begin{aligned} \mu(g(-T), \phi(-T), T) &\le W(g(-T), \phi(-T), f_x(-T), T) \\ &\le W(g(s), \phi(s), f_x(s), |s|) = W_x(s). \end{aligned}$$

After integrating, we get

$$W_x(s) = -\int_s^0 2|r| \int_M \bigg(\Big| \operatorname{Sic} + \nabla^2 f_x - \frac{g}{2|r|} \Big|^2 + (\Delta\phi - \langle \nabla\phi, \nabla f_x \rangle)^2 \bigg)\, d\nu_x(r)\, dr. \quad \square$$

Proposition 3.6. *The following hold for $x \in M$ and $s \in [-T, 0]$.*

(1) $W_x(s) \le N_x(s) \le 0$.

(2) $\dfrac{d}{ds} N_x(s) = \dfrac{1}{|s|}(N_x(s) - W_x(s)) \ge 0$.

(3) $N_x(s) = -\displaystyle\int_M \log H_x(s)\, d\nu_x(s) - \frac{n}{2}(\log(4\pi|s|) + 1) = \int_M f_x(s)\, d\nu_x(s) - \frac{n}{2}$.

(4) $N_x(s) = -\displaystyle\int_s^0 2|r| \Big(1 - \frac{r}{s}\Big) \int_M \Big(\Big| \operatorname{Sic} + \nabla^2 f_x - \frac{g}{2|r|} \Big|^2 + (\Delta\phi - \langle \nabla\phi, \nabla f_x \rangle)^2 \Big)\, d\nu_x(r) dr$.

Proof. (1) By the definition of $N_x(s)$ and the monotonicity of $W_x(s)$,

$$N_x(s) - W_x(s) = \frac{1}{|s|} \int_s^0 (W_x(r) - W_x(s))\, dr \ge 0.$$

(2) By direct calculation,

$$\begin{aligned} \frac{d}{ds} N_x(s) &= \frac{d}{ds} \Big(\frac{1}{|s|} \int_s^0 W_x(r)\, dr \Big) \\ &= \frac{1}{s^2} \int_s^0 W_x(r)\, dr + \frac{1}{s} W_x(s) = \frac{1}{|s|}(N_x(s) - W_x(s)). \end{aligned}$$

(3) Suppose $u(y, l) = H(x, 0; y, l)$, then u solves the conjugate heat equation $\frac{\partial u}{\partial l} + \Delta_{g(l)} u(y, l) - S(y, l) u(y, l) = 0$. Let $\tau(l) = -l$; by direct calculation,

$$\frac{d}{dl} \Big(-l \int_M u \log u \operatorname{dvol} \Big) = W(g(l), \phi(l), f_x(l), |l|) + n + \frac{n}{2} \log(4\pi|l|).$$

Integrating from s to 0,

$$s\int_M H_x(s)\log H_x(s)\,\mathrm{dvol}=\int_s^0 W_x(l)\,dl+\int_s^0\Bigl(n+\frac{n}{2}\log(4\pi|l|)\Bigr)\,dl,$$

hence

$$N_x(s)=-\int_M \log H_x(s)\,d\nu_x(s)-\frac{n}{2}(\log(4\pi|s|)+1)=\int_M f_x(s)\,d\nu_x(s)-\frac{n}{2},$$

where in the last equality we use $H_x(s)=(4\pi|s|)^{-\frac{n}{2}}e^{-f_x}$.
(4) Due to $\frac{d}{dl}N_x(l)=\frac{1}{|l|}(N_x(l)-W_x(l))$, we have $\frac{d}{dl}(lN_x(l))=W_x(l)$. Integrating from s to 0, we get

$$\begin{aligned}
&-sN_x(s)\\
&\quad=\int_s^0 W_x(r)\,dr\\
&\quad=-\int_s^0\int_r^0 2|\tau|\int_M\biggl(\Bigl|\mathrm{Sic}+\nabla^2 f_x-\frac{g}{2|\tau|}\Bigr|^2+(\Delta\phi-\langle\nabla f_x,\nabla\phi\rangle)^2\biggr)\,d\nu_x(\tau)d\tau dr\\
&\quad=-\int_s^0 2|r|(r-s)\int_M\biggl(\Bigl|\mathrm{Sic}+\nabla^2 f_x-\frac{g}{2|r|}\Bigr|^2+(\Delta\phi-\langle\nabla f_x,\nabla\phi\rangle)^2\biggr)\,d\nu_x(r)dr
\end{aligned}$$

so

$$\begin{aligned}
&N_x(s)=\\
&\quad-\int_s^0 2|r|\Bigl(1-\frac{r}{s}\Bigr)\int_M\biggl(\Bigl|\mathrm{Sic}+\nabla^2 f_x-\frac{g}{2|r|}\Bigr|^2+(\Delta\phi-\langle\nabla f_x,\nabla\phi\rangle)^2\biggr)\,d\nu_x(r)dr. \quad\square
\end{aligned}$$

4. Heat kernel estimate

At first we prove a gradient estimate for the heat equation along the extended Ricci flow (1-1).

Lemma 4.1. *Suppose u is a positive solution to the forward heat equation with a family of metrics evolving under the extended Ricci flow on $[0,T]$, then*

$$\frac{|\nabla u(x,t)|}{u(x,t)}\le\sqrt{\frac{1}{t}}\sqrt{\log\frac{A}{u(x,t)}}$$

for $A=\sup_{M\times[0,T]}u$ and $(x,t)\in M\times[0,T]$.

Proof. By direct calculation,

$$\begin{aligned}
\frac{\partial}{\partial t}\Bigl(u\log\frac{A}{u}\Bigr)&=\frac{\partial u}{\partial t}\log\frac{A}{u}-\frac{\partial u}{\partial t},\\
\Delta\Bigl(u\log\frac{A}{u}\Bigr)&=\Delta u\log\frac{A}{u}-\Delta u-\frac{|\nabla u|^2}{u},
\end{aligned}$$

which, combined with the heat equation, gives

$$\square\Big(u\log\frac{A}{u}\Big)=\frac{|\nabla u|^2}{u}.$$

Using the flow equation (1-1), we get

$$\frac{\partial}{\partial t}\frac{|\nabla u|^2}{u}=\frac{2\,\mathrm{Sic}(\nabla u,\nabla u)+2\nabla\frac{\partial u}{\partial t}\cdot\nabla u}{u}-\frac{\frac{\partial u}{\partial t}\cdot|\nabla u|^2}{u^2},$$
$$\Delta\frac{|\nabla u|^2}{u}=\frac{\Delta|\nabla u|^2}{u}-\frac{4\nabla^2u(\nabla u,\nabla u)}{u^2}+|\nabla u|^2\cdot\frac{2|\nabla u|^2-u\cdot\Delta u}{u^3}.$$

Combined with the Bochner formula, this gives

$$\square\frac{|\nabla u|^2}{u}=\frac{-2\langle\nabla u,\nabla\phi\rangle^2}{u}-\frac{2}{u}|\nabla^2u-\frac{du\otimes du}{u}|^2.$$

Consider the quantity $t\frac{|\nabla u|^2}{u}-u\log\frac{A}{u}$,

$$\square\Big(t\frac{|\nabla u|^2}{u}-u\log\frac{A}{u}\Big)=t\Big(\frac{-2\langle\nabla\phi,\nabla u\rangle^2}{u}-\frac{2}{u}|\nabla^2u-\frac{du\otimes du}{u}|^2\Big)\le 0.$$

By the maximum principle,

$$t\frac{|\nabla u|^2}{u}-u\log\frac{A}{u}\le 0,$$

so $|\nabla u|^2/u^2\le\frac{1}{t}\log\frac{A}{u}$. □

Now based on Corollary 3.3, we can use Davies's method to derive the L^∞ estimate for the heat kernel.

Theorem 4.2. *Define* $\rho=\|S(g(-T))^-\|_\infty$ *and*

$$\mu=\inf_{\tau\in(0,2T)}\mu(g(-T),\phi(-T),\tau).$$

Suppose $u:M\times[t_1,t_2]\to\mathbb{R}^+$ *with* $[t_1,t_2]\subseteq[-T,0]$ *is a positive solution to* $\frac{\partial u}{\partial s}=\Delta_{g(s)}u$, *then we have*

$$\|u(s)\|_\infty\le(4\pi(s-t_1))^{-\frac{n}{2}}\exp(\rho(s-t_1)-\mu)\|u(t_1)\|_1.$$

Proof. Given the flow and heat equation

$$\begin{cases}\dfrac{\partial g}{\partial t}=-2\,\mathrm{Sic}(g(t))=-2\mathrm{Ric}(g(t))+2d\phi(t)\otimes d\phi(t),\\ \dfrac{\partial}{\partial t}\phi(t)=\Delta_{g(t)}\phi(t),\\ \dfrac{\partial}{\partial t}u(t)=\Delta_{g(t)}u(t),\end{cases}$$

and letting $p(t)=(s-t_1)/(s-t)$, $t\in[t_1,s]$, with $p(t_1)=1$ and $p(s)=\infty$,

$$
\begin{aligned}
&\frac{d}{dt}\|u(t)\|_{p(t)} \\
&\quad = \frac{d}{dt}\left(\int_M u(t)^{p(t)}\,\mathrm{dvol}_{g(t)}\right)^{\frac{1}{p(t)}} \\
&\quad = -\frac{p'(t)}{p(t)^2}\|u(t)\|_{p(t)}\log\int_M u(t)^{p(t)}\,\mathrm{dvol}_{g(t)} + \frac{1}{p(t)}\left(\int_M u(t)^{p(t)}\,\mathrm{dvol}_{g(t)}\right)^{\frac{1}{p(t)}-1} \\
&\qquad \times\left[\int_M u(t)^{p(t)}p'(t)\log u(t)\,\mathrm{dvol}_{g(t)} + \int_M u(t)^{p(t)-1}(p(t)\Delta u - Su)\,\mathrm{dvol}_{g(t)}\right].
\end{aligned}
$$

Integrating by parts and multiplying by $p(t)^2\|u(t)\|_{p(t)}^{p(t)}$ gives

$$
\begin{aligned}
&p(t)^2\|u(t)\|_{p(t)}^{p(t)}\cdot\frac{\partial}{\partial t}\|u(t)\|_{p(t)} \\
&\quad = -p'(t)\|u(t)\|_{p(t)}^{p(t)+1}\log\int_M u(t)^{p(t)}\,\mathrm{dvol}_{g(t)} \\
&\qquad + p(t)\|u(t)\|_{p(t)}p'(t)\int_M u(t)^{p(t)}\log u(t)\,\mathrm{dvol}_{g(t)} \\
&\qquad - p(t)^2(p(t)-1)\|u(t)\|_{p(t)}\int_M u(t)^{p(t)-2}|\nabla u|^2\,\mathrm{dvol}_{g(t)} \\
&\qquad - p(t)\|u(t)\|_{p(t)}\int_M Su(t)^{p(t)}\,\mathrm{dvol}_{g(t)}.
\end{aligned}
$$

Dividing by $\|u(t)\|_{p(t)}$ on both sides,

$$
\begin{aligned}
&p(t)^2\|u(t)\|_{p(t)}^{p(t)}\cdot\frac{\partial}{\partial t}\log\|u(t)\|_{p(t)} \\
&\quad = -p'(t)\|u(t)\|_{p(t)}^{p(t)}\log\int_M u(t)^{p(t)}\,\mathrm{dvol}_{g(t)} \\
&\qquad + p(t)p'(t)\int_M u(t)^{p(t)}\log u(t)\,\mathrm{dvol}_{g(t)} \\
&\qquad - 4(p(t)-1)\int_M |\nabla u^{p(t)/2}|^2\,\mathrm{dvol}_{g(t)} - p(t)\int_M S(u^{p(t)/2})^2\,\mathrm{dvol}_{g(t)}.
\end{aligned}
$$

Define $v = u^{p(t)/2}/\|u^{p(t)/2}\|_2$, then $\|v\|_2 = 1$ and $v^2\log v^2 = p(t)v^2\log u - 2v^2\log\|u^p(t)/2\|_2$. So

$$
\begin{aligned}
&p(t)^2\frac{\partial}{\partial t}\log\|u(t)\|_{p(t)} \\
&\quad = p'(t)\int_M v^2\log v^2\,\mathrm{dvol}_{g(t)} - 4(p(t)-1)\int_M \left(|\nabla v|^2 + \tfrac{1}{4}Sv^2\right)\mathrm{dvol}_{g(t)} \\
&\qquad - \int_M Sv^2\,\mathrm{dvol}_{g(t)}
\end{aligned}
$$

$$= p'(t)\left[\int_M v^2 \log v^2 \,\mathrm{dvol}_{g(t)} - \frac{p(t)-1}{p'(t)} \int_M (4|\nabla v|^2 + Sv^2)\,\mathrm{dvol}_{g(t)}\right] - \int_M Sv^2 \,\mathrm{dvol}_{g(t)}$$

$$\le p'(t)\left[-\frac{n}{2}\log(4\pi)\frac{(t-t_1)(s-t)}{s-t_1} - n - \mu\right] + \rho.$$

Observe $p'(t)/p(t)^2 = 1/(s-t_1)$, hence

$$\frac{\partial}{\partial t}\log\|u(t)\|_{p(t)} \le \frac{1}{s-t_1}\left[-\frac{n}{2}\log(4\pi)\frac{(t-t_1)(s-t)}{s-t_1} - n - \mu\right] + \rho.$$

Integrating from t_1 to s with respect to t, we get

$$\log\frac{\|u(s)\|_\infty}{\|u(t_1)\|_1} \le -\frac{n}{2}\log(4\pi(s-t_1)) - \mu + (s-t_1)\rho,$$

so

$$\|u(s)\|_\infty \le (4\pi(s-t_1))^{-\frac{n}{2}}\exp(\rho(s-t_1)-\mu)\|u(t_1)\|_1. \qquad \square$$

Corollary 4.3. *Given any $C > 0$, there exists a $C' = C'(n, C) > 0$ such that if $S(g(-s)) \ge -\frac{C}{|s|}$ and $\inf_{\tau\in(0,2|s|)} \mu(g(s), \phi(s), \tau) \ge C$, if we denote $H(x, 0; y, s) = (4\pi|s|)^{-\frac{n}{2}}\exp(-f_x(y, s))$, then*

$$|\nabla_x f_x|^2 \le \frac{C'}{|s|}(C' + f_x)$$

at $(x, 0)$.

Proof. Fix y, s and let $u(x, t) = H(x, t; y, s)$, then u satisfies $\frac{\partial u}{\partial t} = \Delta_{g(t)} u$. Applying Theorem 4.2 we get

$$A = \sup_{[\frac{s}{2},0]\times M} u \le C'|s|^{-\frac{n}{2}},$$

then by Lemma 4.1 with $[t_1, t_2] = \left[\frac{s}{2}, 0\right]$,

$$|\nabla_x f_x|^2 = \frac{|\nabla u|^2}{u^2} \le \frac{1}{|s|/2}\log\frac{A}{u} = \frac{2}{|s|}(\log A - \log u)$$

$$\le \frac{2}{|s|}\left(\log C' - \frac{n}{2}\log|s| + \frac{n}{2}\log(4\pi|s|) + f_x\right) \le \frac{C'}{|s|}(C' + f_x). \qquad \square$$

Based on Perelman's Harnack inequality [Perelman 2002], Zhang [2012] obtained the lower bound for the heat kernel along Ricci flow, which can be used to derive the κ noninflating property for Ricci flow. Next we generalize Perelman's Harnack inequality to the extended Ricci flow.

Theorem 4.4. *Let $u = u(y, s) = H(x, t; y, s)$, $s < t$, and f be defined by $u = (4\pi(t-s))^{-\frac{n}{2}} e^{-f}$. Denote $\tau = t - s$ and let $P = P(u)$ be defined as*

$$\begin{aligned} P &= [\tau(2\Delta f - |\nabla f|^2 + S) + f - n]u \\ &= \tau\Big(-2\Delta u + \frac{|\nabla u|^2}{u} + Su\Big) - u\log u - \frac{n}{2}\log(4\pi\tau) - nu; \end{aligned}$$

then $P \le 0$. Moreover, for any smooth curve $c = c(s)$ on M,

$$-\frac{d}{ds} f(c(s), s) \le \frac{1}{2}(S(c(s), s) + |c'(s)|^2) - \frac{1}{2(t-s)} f(c(s), s).$$

Proof. By Lemma 6 in [Guo et al. 2015a], we know $P \le 0$, so

$$(t-s)(2\Delta f - |\nabla f|^2 + S) + f - n \le 0.$$

Since u solves the conjugate heat equation, we have

$$\frac{\partial f}{\partial s} = -\Delta f + |\nabla f|^2 - S + \frac{n}{2(t-s)}.$$

Combining the above two equations, we get

$$\frac{\partial f}{\partial s} + \frac{1}{2}S - \frac{1}{2}|\nabla f|^2 - \frac{f}{2(t-s)} \ge 0.$$

On the other hand,

$$-\frac{d}{ds} f(c(s), s) = -\frac{\partial f}{\partial s} - \langle \nabla f, c'(s)\rangle \le -\frac{\partial f}{\partial s} + \frac{1}{2}|\nabla f|^2 + \frac{1}{2}|c'(s)|^2.$$

The desired inequality follows from adding the last two inequalities. □

Remark. In [Cao et al. 2015], the authors considered the Harnack estimate for the conjugate heat kernel of general geometric flow, and our Theorem 4.4 is a special case of [Cao et al. 2015, Theorem 1.2]. This kind of estimate is used to derive smooth convergence of the conjugate heat kernel for our particular flow; for general geometric flow in [Cao et al. 2015], we don't even know the convergence of the flow.

Next we introduce the reduced length and prove a bound of the heat kernel which will be used in the following sections. Let $x, y \in M$, $0 \le s < t < T$ and consider a smooth curve $\gamma : [s, t] \to M$ connecting (y, s) and (x, t), i.e., $\gamma(s) = y$ and $\gamma(t) = x$. Its L length is defined as

$$L(\gamma) = \int_s^t \sqrt{t-\sigma}\,(|\gamma'(\sigma)|^2_{g(\sigma)} + S(\gamma(\sigma), \sigma))\, d\sigma.$$

The reduced distance between (x, t) and (y, s) is defined as

$$l_{(x,t)}(y, s) = \frac{1}{2\sqrt{t-s}} \inf\{L(\gamma) : \gamma : [s, t] \to M \text{ between } (y, s) \text{ and } (x, t)\}.$$

Choose $\gamma(\sigma) : [s, t] \to M$ to be the L geodesic between (y, s) and (x, t); from Theorem 4.4 we know

$$-\frac{d}{d\sigma}((t-\sigma)^{\frac{1}{2}} f(\gamma(\sigma), \sigma)) \le \frac{1}{2}(t-\sigma)^{\frac{1}{2}}(S(\gamma(\sigma), \sigma) + |\gamma'(\sigma)|^2).$$

Integrating from s to t, we have

$$(t-s)^{\frac{1}{2}} f(\gamma(s), s) \le \frac{1}{2}\int_s^t \sqrt{t-\sigma}(S(\gamma(\sigma), \sigma) + |\gamma'(\sigma)|^2)\, d\sigma,$$

hence

$$f(y, s) \le l_{(x,t)}(y, s),$$

i.e.,

$$H(x, t; y, s) \ge (4\pi(t-s))^{-\frac{n}{2}} e^{-l_{(x,t)}(y,s)}. \tag{4-11}$$

Now we are in a position to prove the lower bound of the heat kernel.

Theorem 4.5. *Define* $\rho = \|S(g(-T))^-\|_\infty$, $\mu = \inf_{\tau\in(0,2T)} \mu(g(-T), \phi(-T), \tau)$. *Denote* $\tau = t - s$ *for* $s < t$ *in* $[-T, 0]$, *then we have*

$$H(x, t; y, s) \ge (8\pi\tau)^{-\frac{n}{2}} \exp\left(-\frac{4d(x, y, t)^2}{\tau} - \frac{1}{\sqrt{\tau}}\int_s^t \sqrt{t-\sigma}\, S(y, \sigma)\, d\sigma - \rho\tau + \mu\right).$$

Proof. Let $u(y, s) = H(y, t; y, s)$, $s < t$, then u solves the conjugate heat equation $-\frac{\partial u}{\partial s} - \Delta_{g(s)} u + Su = 0$. Define a function f by $u(y, s) = (4\pi\tau)^{-\frac{n}{2}} e^{-f(y,s)}$; we need to use Theorem 4.4. Picking the curve $c(s)$ to be the fixed point, we have

$$-\frac{\partial f}{\partial s} \le \frac{1}{2} S(y, s) - \frac{1}{2\tau} f(y, s).$$

For any $s_2 < s_1 < t$, we integrate the above inequality to get

$$f(y, s_2)\sqrt{t-s_2} \le f(y, s_1)\sqrt{t-s_1} + \frac{1}{2}\int_{s_2}^{s_1} \sqrt{t-\sigma}\, S(y, \sigma)\, d\sigma.$$

When s_1 approaches t, $f(y, s_1)$ stays bounded because $H(y, t; y, s)(t-s)^{\frac{n}{2}}$ is bounded between two positive constants. Hence for $s \le t$, we have

$$f(y, s) \le \frac{1}{2\sqrt{\tau}}\int_s^t \sqrt{t-\sigma}\, S(y, \sigma)\, d\sigma,$$

so,

$$H(y, t; y, s) \ge \frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-\frac{1}{2\sqrt{\tau}}\int_s^t \sqrt{t-\sigma}\, S(y,\sigma)\, d\sigma}.$$

Next we will use the gradient estimate from Lemma 4.1 to get the lower bound for $H(x, t; y, s)$. Define $v(x, l) = H(x, l; y, s)$, then v satisfies $\frac{\partial v}{\partial l} = \Delta_{g(l)} v(l)$. On the interval $\left[\frac{t+s}{2}, t\right]$, applying Theorem 4.2 , we get

$$\|v(l)\|_\infty \le \left(4\pi \cdot \frac{t-s}{2}\right)^{-\frac{n}{2}} e^{\rho(t-s)-\mu} := A.$$

By Lemma 4.1, we have

$$\left|\nabla\sqrt{\log\frac{A}{v(x,t)}}\right| \le \frac{1}{\sqrt{\frac{1}{2}t-s}},$$

hence

$$\sqrt{\log\frac{A}{v(x,t)}} \le \sqrt{\log\frac{A}{v(y,t)}} + \frac{d(x,y,t)}{\sqrt{\frac{1}{2}t-s}}.$$

Using Cauchy–Schwarz,

$$\log\frac{A}{v(x,t)} \le \log\Bigl(\frac{A}{v(y,t)}\Bigr)^2 + \frac{4d(x,y,t)^2}{t-s}.$$

So

$$\begin{aligned} v(x,t) &\ge A^{-1}v(y,t)^2 e^{-\frac{4d(x,y,t)^2}{t-s}} \\ &\ge (2\pi(t-s))^{\frac{n}{2}} e^{-\rho(t-s)+\mu}(4\pi\tau)^{-n} e^{-\frac{1}{\sqrt{t-s}}\int_s^t \sqrt{t-\sigma}S(y,\sigma)\,d\sigma} e^{-\frac{4d(x,y,t)^2}{t-s}} \\ &= (8\pi\tau)^{-\frac{n}{2}} \exp\Bigl(-\frac{4d(x,y,t)^2}{t-s} - \frac{1}{\sqrt{t-s}}\int_s^t \sqrt{t-\sigma}S(y,\sigma)\,d\sigma - \rho\tau + \mu\Bigr). \qquad \square \end{aligned}$$

5. Log-Sobolev inequality and Gaussian concentration

Consider a smooth metric probability space $(M, g, d\nu)$, where $d\nu = e^{-h}dv_g$. If the so-called Bakry–Émery condition,

$$\mathrm{Ric} + \nabla^2 h \ge \tfrac{1}{2}g,$$

is satisfied, a celebrated theorem [Bakry and Émery 1985] asserts that $(M, g, d\nu)$ satisfies a logarithmic Sobolev inequality with a definite constant. More precisely, this means that for every smooth function v with compact support and $\int_M v^2\,d\nu = 1$,

$$\int_M v^2 \log v^2\,d\nu \le 4\int_M |\nabla v|^2\,d\nu.$$

Since the work of [Bakry and Émery 1985] there has been plenty of work on the characterization of the Ricci curvature bound using the log-Sobolev inequality; see [Bakry and Ledoux 2006; Cheng and Thalmaier 2018; Naber 2013]. The above log-Sobolev inequality has many important applications, for example, see [Carrillo and Ni 2009; Munteanu and Wang 2012; Wu and Zhang 2017].

With the same spirit, in this section we will prove Theorem 1.2, i.e., the Poincaré inequality (1-2) and the log-Sobolev inequality (1-3). As applications, we obtain Theorem 1.3 and Corollary 1.4.

In the following argument, for simplicity, we use $d\nu$ to denote $d\nu_{x_0}(y,s)$. We can rewrite the Poincaré inequality (1-2) and the log-Sobolev inequality (1-3) in the following way. For any $u \in C_0^\infty(M)$ with $u \geq 0$ in the second case,

$$\int_M u^2\,d\nu - \left(\int_M u\,d\nu\right)^2 \leq 2|s| \int_M |\nabla u|^2\,d\nu, \tag{5-12}$$

$$\int_M u \log u\,d\nu - \left(\int_M u\,d\nu\right)\log\left(\int_M u\,d\nu\right) \leq |s| \int_M \frac{|\nabla u|^2}{u}\,d\nu. \tag{5-13}$$

Theorem 1.2 can be derived using a similar idea to that in [Hein and Naber 2014], where the gradient estimate can be obtained by applying the "heat kernel homotopy" principle [Bakry and Ledoux 2006]. Given $s \leq t$ in $[-T, 0]$, we define $P_{st}u$ as

$$P_{st}u(x) = \int_M u(y) H(x,t;y,s)\,\mathrm{dvol}_{g(s)}(y).$$

Note that when s is fixed, $P_{st}u$ satisfies the heat equation.

Lemma 5.1. *If $U(t)$ are smooth functions on $M \times [-T, 0]$, then*

$$\frac{d}{dt} P_{t0} U(t) = P_{t0} \Box_t U(t).$$

Proof. When x, t are fixed, $H(x,t;y,s)$ satisfies the conjugate heat equation.

$$\begin{aligned}
\frac{d}{dt} P_{t0} U(t) &= \int_M \frac{\partial}{\partial t} U(y,t) H(x,0;y,t)\,\mathrm{dvol}_{g(t)}(y) \\
&\qquad + \int_M U(y,t)\left(\frac{\partial}{\partial t} - S(y,t)\right) H(x,0;y,t)\,\mathrm{dvol}_{g(t)}(y) \\
&= \int_M \frac{\partial}{\partial t} U(y,t) H(x,0;y,t)\,\mathrm{dvol}_{g(t)}(y) \\
&\qquad - \int_M U(y,t) \Delta_y H(x,0;y,t)\,\mathrm{dvol}_{g(t)}(y) \\
&= \int_M \left(\frac{\partial}{\partial t} U(y,t) - \Delta_y U(y,t)\right) H(x,0;y,t)\,\mathrm{dvol}_{g(t)}(y) \\
&= \int_M \Box_t U(y,t) H(x,0;y,t)\,\mathrm{dvol}_{g(t)}(y) = P_{t0} \Box_t U(t).
\end{aligned}$$

□

Lemma 5.2. *Let $u \in C_0^\infty(M)$ and $u(t) = P_{st}u$ so that $\Box_t u(t) = 0$. Suppose h and ψ are two smooth functions from $\mathbb{R}$ to $\mathbb{R}$.*

(1) *If $U(t) = h(u(t))$, then $\Box_t U(t) = -h''(u)|\nabla u(t)|^2_{g(t)}$.*

(2) *If* $U(t)=\psi(u(t))|\nabla u(t)|^2_{g(t)}$, *then*

$$\square_t U(t)=-2\psi(u)|\nabla^2 u|^2-2\psi(u)\langle\nabla\phi,\nabla u\rangle^2-\psi''(u)|\nabla u|^4-4\psi'(u)\nabla^2 u(\nabla u,\nabla u).$$

Proof. (1) $\square_t U(t)=(\frac{\partial}{\partial t}-\Delta)h(u(t))=h'\frac{\partial u}{\partial t}-(h''|\nabla u|^2+h'\Delta u)=-h''(u)|\nabla u|^2.$
(2) Using the Bochner formula (2-10), we get

$$\square_t|\nabla u|^2=-2|\nabla^2 u|^2-2\langle\nabla u,\nabla\phi\rangle^2.$$

So

$$\begin{aligned}\square_t(\psi(u)|\nabla u|^2)&=\square_t\psi(u)|\nabla u|^2+\psi(u)\square_t|\nabla u|^2-2\langle\nabla\psi(u),\nabla|\nabla u|^2\rangle\\&=-2\psi(u)|\nabla^2 u|^2-2\psi(u)\langle\nabla\phi,\nabla u\rangle^2\\&\qquad-\psi''(u)|\nabla u|^4-4\psi'(u)\nabla^2 u(\nabla u,\nabla u).\end{aligned}$$ □

Now we are going to prove the Poincaré inequality and the log-Sobolev inequality. Note that

$$\int_M h(u)\,d\nu-h\Big(\int_M u\,d\nu\Big)=-\int_s^0\frac{d}{dt}P_{t0}(h(P_{st}u))\,dt=\int_s^0 P_{t0}(h''(P_{st}u)|\nabla P_{st}u|^2_{g(t)})\,dt. \tag{5-14}$$

***Proof of the Poincaré inequality* (5-12).** Pick $h=x^2$; by (5-14) we have

$$\int_M u^2\,d\nu-\Big(\int_M u\,d\nu\Big)^2=2\int_s^0 P_{t0}(|\nabla P_{st}u|^2_{g(t)})\,dt.$$

Using Lemma 5.1 and Lemma 5.2,

$$\begin{aligned}\frac{\partial}{\partial r}P_{rt}(|\nabla P_{sr}u|^2_{g(r)})&=P_{rt}\square_r(|\nabla P_{sr}u|^2_{g(r)})\\&=P_{rt}\big(-2|\nabla^2 P_{sr}u|^2_{g(r)}-2\langle\nabla\phi,\nabla P_{sr}u\rangle^2\big).\end{aligned}$$

Integrating from s to t with respect to r,

$$|\nabla P_{st}u|^2_{g(t)}=P_{st}(|\nabla u|^2_{g(s)})-2\int_s^t P_{rt}(|\nabla^2 P_{sr}u|^2_{g(r)})\,dr-2\int_s^t P_{rt}(\langle\nabla\phi,\nabla P_{sr}u\rangle^2)\,dr,$$

so

$$\begin{aligned}\int_M u^2\,d\nu-\Big(\int_M u\,d\nu\Big)^2&=2\int_s^0 P_{t0}(|\nabla P_{st}u|^2_{g(t)})\,dt\le 2\int_s^0 P_{t0}P_{st}(|\nabla u|^2_{g(s)})\,dt\\&=2\int_s^0 P_{s0}|\nabla u|^2_{g(s)}\,dt=2|s|\int_M|\nabla u|^2_{g(s)}\,d\nu_{x_0}(s).\end{aligned}$$

It is easy to see that equality holds if and only if $\nabla^2 P_{sr}u\equiv 0$, i.e., u is constant. □

***Proof of the log-Sobolev inequality* (5-13).** Pick $h = x\log x$; by (5-14), we obtain

$$\int_M u\log u\,d\nu - \left(\int_M u\,d\nu\right)\log\left(\int_M u\,d\nu\right) = \int_s^0 P_{t0}\left(\frac{|\nabla P_{st}u|^2_{g(t)}}{P_{st}u}\right)dt.$$

Using Lemmas 5.1 and 5.2,

$$\begin{aligned}\frac{\partial}{\partial r}P_{rt}(P_{sr}u|\nabla\log P_{sr}u|^2_{g(r)}) &= P_{rt}\Box_r(P_{sr}u|\nabla\log P_{sr}u|^2_{g(r)})\\ &= P_{rt}\Box_r\left(\frac{|\nabla P_{sr}u|^2}{P_{sr}u}\right) = -2P_{rt}\left[P_{sr}u\left(|\nabla^2\log P_{sr}u|^2 + \frac{\langle\nabla\phi,\nabla P_{sr}u\rangle^2}{(P_{sr}u)^2}\right)\right],\end{aligned}$$

and integrating from s to t with respect to r,

$$\frac{|\nabla P_{st}u|^2_{g(t)}}{P_{st}u} = P_{st}\left(\frac{|\nabla u|^2_{g(s)}}{u}\right) - 2\int_s^t P_{rt}\left[P_{sr}u\left(|\nabla^2\log P_{sr}u|^2 + \frac{\langle\nabla\phi,\nabla P_{sr}u\rangle^2}{(P_{sr}u)^2}\right)\right]dr.$$

So

$$\begin{aligned}\int_M u\log u\,d\nu - \left(\int_M u\,d\nu\right)\log\left(\int_M u\,d\nu\right) &= \int_s^0 P_{t0}\left(\frac{|\nabla P_{st}u|^2_{g(t)}}{P_{st}u}\right)dt\\ &\le \int_s^0 P_{t0}P_{st}\left(\frac{|\nabla u|^2_{g(s)}}{u}\right)dt = \int_s^0 P_{s0}\left(\frac{|\nabla u|^2_{g(s)}}{u}\right)dt = |s|\int_M \frac{|\nabla u|^2_{g(s)}}{u}\,d\nu_{x_0}(s).\end{aligned}$$

One sees that equality holds if and only if $\nabla^2\log P_{sr}u \equiv 0$, i.e., u is constant. □

Next we will use the log-Sobolev inequality to derive Theorem 1.3, where the proof follows from standard theory in the metric measure space.

Proof of Theorem 1.3. Choose any $F \in C^\infty(M)$ with

$$\int_M F\,d\nu = 0, \quad |\nabla F| \le 1. \tag{5-15}$$

Define $U(\lambda) = \frac{1}{\lambda}\log\int_M e^{\lambda F}\,d\nu$, then

$$\lim_{\lambda\to 0}U(\lambda) = \lim_{\lambda\to 0}\frac{\int_M e^{\lambda F}F\,d\nu}{\int_M e^{\lambda F}\,d\nu} = \int_M F\,d\nu = 0.$$

Applying the log-Sobolev inequality to $u^2 = \frac{e^{\lambda F}}{\int_M e^{\lambda F}\,d\nu}$,

$$\int_M\left(\frac{e^{\lambda F}}{\int_M e^{\lambda F}\,d\nu}\log\frac{e^{\lambda F}}{\int_M e^{\lambda F}\,d\nu}\right)d\nu \le 4|s|\int_M\frac{\frac{\lambda^2}{4}e^{\lambda F}|\nabla F|^2}{\left(\int_M e^{\lambda F}\,d\nu\right)^2}d\nu,$$

so

$$\int_M e^{\lambda F}\,d\nu\int_M e^{\lambda F}\left(\lambda F - \log\int_M e^{\lambda F}\right)d\nu \le |s|\lambda^2\int_M e^{\lambda F}|\nabla F|^2\,d\nu \le |s|\lambda^2\int_M e^{\lambda F}\,d\nu,$$

i.e.,

$$\int_M e^{\lambda F}\left(\lambda F - \log \int_M e^{\lambda F}\right) d\nu \le |s|\lambda^2.$$

Hence,

$$\begin{aligned}\frac{d}{d\lambda}U = \frac{d}{d\lambda}\left(\frac{1}{\lambda}\log\int_M e^{\lambda F}\,d\nu\right) &= \frac{-1}{\lambda^2}\log\int_M e^{\lambda F}\,d\nu + \frac{1}{\lambda}\frac{\int_M e^{\lambda F} F\,d\nu}{\int_M e^{\lambda F}\,d\nu}\\ &= \frac{1}{\lambda^2}\frac{1}{\int_M e^{\lambda F}\,d\nu}\left(-\int_M e^{\lambda F}\,d\nu \log\int_M e^{\lambda F}\,d\nu + \lambda\int_M e^{\lambda F} F\,d\nu\right) \le |s|.\end{aligned}$$

In the last inequality we use $\int_M e^{\lambda F}\,d\nu \ge 1$ because $\log\int_M e^{\lambda F}\,d\nu \ge \int_M \lambda F\,d\nu = 0$.

Combining $\frac{d}{d\lambda}U \le |s|$ and $\lim_{\lambda\to 0} U(\lambda) = 0$, we obtain

$$\int_M e^{\lambda F}\,d\nu \le e^{|s|\lambda^2}$$

because any F satisfies (5-15).

Define $G(y) = d_{g(s)}(y, B)$ and $F = G - \int_M G\,d\nu$, then

$$\int_A e^{\lambda F(y_1)}\,d\nu(y_1) \le \int_M e^{\lambda F(y_1)}\,d\nu \le e^{|s|\lambda^2},$$

and

$$\int_B e^{-\lambda F(y_2)}\,d\nu(y_2) \le \int_M e^{-\lambda F(y_2)}\,d\nu \le e^{|s|\lambda^2}.$$

So

$$e^{\lambda d_{g(s)}(A,B)}\nu(A)\nu(B) \le \int_B\int_A e^{\lambda(F(y_1)-F(y_2))}\,d\nu(y_1)d\nu(y_2) \le e^{2|s|\lambda^2},$$

i.e.,

$$\nu(A)\nu(B) \le e^{2|s|\lambda^2 - \lambda d_{g(s)}(A,B)}.$$

Because

$$2|s|\lambda^2 - \lambda d_{g(s)}(A, B) = 2|s|\left(\lambda - \frac{d_{g(s)}(A, B)}{4|s|}\right)^2 - \frac{d_{g(s)}(A, B)^2}{8|s|} \ge -\frac{d_{g(s)}(A, B)^2}{8|s|},$$

we get

$$\nu(A)\nu(B) \le \exp\left(-\frac{d_{g(s)}(A, B)^2}{8|s|}\right). \qquad \square$$

Remark. Given $x_1,\ x_2 \in M$, take $A = B_r(x_1, s)$ and $B = B_r(x_2, s)$, where $r^2 = |s|$. Applying the above theorem to $d\nu = d\nu_{x_2}$,

$$\int_{B_r(x_1,s)} H(x_2, 0; y, s)\,\mathrm{dvol}_{g(s)}(y) \le \frac{1}{\nu_{x_2}(B_r(x_2, s))} \exp\biggl(-\frac{d_{g(s)}(B_r(x_1, s), B_r(x_2, s))^2}{8|s|}\biggr), \tag{5-16}$$

due to

$$d_{g(s)}(x_1, x_2) \le d_{g(s)}(B_r(x_1, s), (B_r(x_2, s))) + 2r,$$

hence

$$\tfrac{1}{2} d_{g(s)}(x_1, x_2)^2 \le d_{g(s)}(B_r(x_1, s), B_r(x_2, s))^2 + 4|s|.$$

So

$$\int_{B_r(x_1,s)} H(x_2, 0; y, s)\,\mathrm{dvol}_{g(s)}(y) \le \frac{C}{\nu_{x_2}(B_r(x_2, s))} \exp\biggl(-\frac{d_{g(s)}(x_1, x_2)^2}{C|s|}\biggr).$$

Together with Perelman's κ noncollapsing property, this can be used to derive certain upper bounds of the heat kernel [Wu $\ge$ 2019].

Proof of Corollary 1.4. Apply Theorem 1.3 with $x_0 = x_2,\ A = B_r(x_1, s)$ and $B = B_r(x_2, s)$. Using Theorem 4.5, we obtain

$$\inf_{B_r(x_2,s)} H(x_2, 0; y, s) \ge \frac{1}{C'} |s|^{-\frac{n}{2}}. \tag{5-17}$$

Due to the evolution equation of volume along (1-1),

$$\frac{d}{dt} \mathrm{Vol}_{g(t)}(B_r(x_2, 0)) = -\int_{B_r(x_2,0)} S(y, t)\,\mathrm{dvol}_{g(t)}(y) \le \frac{C}{|s|} \mathrm{Vol}_{g(t)}(B_r(x_2, 0)),$$

and integrating from s to 0 with respect to t, by Theorem 3.4 we have

$$\mathrm{Vol}_{g(s)}(B_r(x_2, 0)) \ge \frac{1}{C'} \mathrm{Vol}_{g(0)}(B_r(x_2, 0)) \ge \frac{1}{C'} r^n. \tag{5-18}$$

Combining (5-17), (5-18) and $\mathrm{Vol}_{g(s)}(B_r(x_1, 0)) \ge \frac{1}{C'} r^n$, we get

$$\frac{1}{\mathrm{Vol}_{g(s)}(B_r(x_1, 0))} \int_{B_r(x_1,0)} H(x_2, 0, y, s)\,\mathrm{dvol}_{g(s)}(y) \le C'|s|^{-\frac{n}{2}} \exp\biggl(-\frac{d_{g(s)}(B_r(x_1, 0), B_r(x_2, 0))}{C'|s|}\biggr).$$

From Theorem 4.5 again, we have

$$\inf_{B_r(x_1,0)} H(x_2, 0; y, s) \ge \frac{1}{C'} |s|^{-\frac{n}{2}} \exp\biggl(-\frac{d_{g(0)}(x_1, x_2)^2}{C'|s|}\biggr),$$

and combining this with (1-4), (1-5) follows.

6. Lipschitz continuity of pointed Nash entropy

Recall the pointed Nash entropy at $(x_0, s) \in M \times [-T, 0]$ is defined as

$$N_{x_0}(s) = \frac{1}{|s|}\int_s^0 W_{x_0}(r)\,dr = \int_M f_{x_0}(s)\,d\nu_{x_0}(s) - \frac{n}{2}.$$

Based on the Poincaré inequality (1-2) in Theorem 1.2, we can prove the Lipschitz continuity of the pointed Nash entropy.

Proof of Theorem 1.6. Define $F(x) = f_x(s)H_x(s)$, then

$$\begin{aligned}
\|F(x_1) - F(x_2)\| &= \|f_{x_1}(s)H_{x_1}(s) - f_{x_2}(s)H_{x_2}(s)\| \\
&= \int_M |f_{x_1}(s)H_{x_1}(s) - f_{x_2}(s)H_{x_2}(s)|\,\mathrm{dvol}_{g(s)}(y) \\
&\le \int_M \int_0^{d_{g(0)}(x_1,x_2)} |\nabla_{\gamma(t)}(f_{\gamma(t)}H_{\gamma(t)})|\,dt\,\mathrm{dvol}_{g(s)}(y) \\
&= \int_0^{d_{g(0)}(x_1,x_2)} \int_M |\nabla_{\gamma(t)}(f_{\gamma(t)}H_{\gamma(t)})|\,\mathrm{dvol}_{g(s)}(y)dt \\
&\le \sup_{x\in M}\int_M |\nabla_x(f_xH_x)|\,\mathrm{dvol}_{g(s)} \cdot d_{g(0)}(x_1, x_2),
\end{aligned}$$

where $\gamma(t)$ is a unit speed geodesic connecting x_1 and x_2 with respect to g(0). All we need to do is to estimate the integral,

$$\begin{aligned}
\int_M |\nabla_x(f_xH_x)|\,\mathrm{dvol}_{g(s)}(y) &= \int_M |\nabla_x f_x H_x - f_x H_x \nabla_x f_x|\,\mathrm{dvol}_{g(s)}(y) \\
&= \int_M |\nabla_x f_x - f_x\nabla_x f_x|\,d\nu_x(s) \le \|\nabla_x f_x\|_2(1 + \|f_x\|_2).
\end{aligned}$$

From the gradient estimate in Corollary 4.3, we know

$$|\nabla_x f_x|^2 \le \frac{C'}{|s|}(C' + f_x).$$

Hence

$$\int_M |\nabla_x(f_xH_x)|\,\mathrm{dvol}_{g(s)}(y) \le C'|s|^{-\frac{1}{2}}(1 + \|f_x\|_2^2) \le C'|s|^{-\frac{1}{2}},$$

where in the last inequality we use (3) from Theorem 6.1. □

Theorem 6.1. *Under the assumption* (1-6), *the following holds for* $f_x(s)$.

(1) $\int_M f_x\,d\nu \in [\frac{n}{2} - C, \frac{n}{2}]$.

(2) $\int_M |\nabla f_x|^2\,d\nu \le (\frac{n}{2} + C)\frac{1}{|s|}$.

(3) $\int_M |f_x|^2\,d\nu \le (n + 2 + C)^2$. *here we use* $d\nu$ *to denote* $d\nu_x(y, s)$.

Proof. (1) Applying Propositions 3.5 and 3.6, we have

$$-C \le \mu(g(s), \phi(s), |s|) \le W_x(s) \le N_x(s),$$

so $\int_M f_x \, d\nu \in \left[\frac{n}{2} - C, \frac{n}{2}\right]$.

(2) Recall $W_x(s) = \int_M (|s|(S + |\nabla f_x|^2) + f_x - n) \, d\nu$ and $N_x(s) = \int_M f_x(s) \, d\nu - \frac{n}{2}$, so

$$W_x(s) - N_x(s) = \int_M |s|(S + |\nabla f_x|^2) \, d\nu - \frac{n}{2} \le 0,$$

hence $\int_M (S + |\nabla f_x|^2) \, d\nu \le \frac{n}{2|s|}$.

(3) Applying the Poincaré inequality (1-2), we have

$$\begin{aligned} \int_M f_x^2 \, d\nu &\le 2|s| \int_M |\nabla f_x|^2 \, d\nu + \left(\int_M f_x \, d\nu \right)^2 \\ &\le 2|s| \left(\frac{n}{2} + c \right) \frac{1}{|s|} + \max\left\{ \left(\frac{n}{2} - C \right)^2, \left(\frac{n}{2} \right)^2 \right\} \le (n + C + 2)^2. \quad \square \end{aligned}$$

7. Proof of ε-regularity theorem

In this section we prove the ε-regularity theorem. In order to do that, we quote the derivative estimate to be used.

Lemma 7.1 [List 2008]. *Let $(M^n, g(t), \phi(t))$ be an extended Ricci flow* (1-1) *on $M \times [0, T)$ with initial data (g_0, ϕ_0), and assume* $\sup |\phi_0| \le C$, *then for all $t > 0$,*

$$\inf_{x \in M} \phi_0(x) \le \phi(x, t) \le \sup_{x \in M} \phi_0(x),$$
$$\sup_{x \in M} |\nabla \phi|^2(x, t) \le C^2 t^{-1}.$$

Proposition 7.2 [List 2008]. *Let $(M^n, g(t), \phi(t))$ be an extended Ricci flow* (1-1). *Fix $x_0 \in M$ and $r > 0$, if*

$$\sup_{B_r(x_0, s)} r^2 |\mathrm{Rm}| \le \widetilde{C}.$$

Denote $\Phi = (\mathrm{Rm}, \nabla^2 \phi)$, then the derivatives of Φ satisfy the inequality for all $m \ge 0$, and for all $t \in (0, s]$ the estimate

$$\sup_{B_{r/2}(x_0, t)} |\nabla^m \Phi|^2 \le C(n, m) \widetilde{C} (r^{-2} + t^{-1})^{m+2}$$

holds, where $C = C(n, m)$ is a constant only depending on n and m.

Next we prove a more restricted version of Theorem 1.8, whose proof is based on the point picking argument as in [Anderson 1990]. Once we have this, Theorem 1.8 can be derived using the Lipschitz continuity of the pointed Nash entropy in

Theorem 1.6. In the following argument, for simplicity, we define

$$t(y) = -\min\{T, r_{|\mathrm{Rm}|}(y,0)^2\}.$$

Theorem 7.3. *There exists an* $\varepsilon = \varepsilon(n, C) > 0$ *such that if*

$$N_y(t(y)) \geq -\varepsilon, \quad \textit{for all } y \in B_\delta(x,0),$$

where $0 < \delta \leq \sqrt{T}$, *then*

$$r_{|\mathrm{Rm}|}(y,0) \geq \varepsilon \cdot d_{g(0)}(y, \partial B_\delta(x,0)), \quad \textit{for all } y \in B_\delta(x,0).$$

Proof. Without loss of generality, we assume $\delta = 1 \leq T$. Suppose the contrary, then we have a sequence of the extended Ricci flow $(M_i, g_i(t), \phi_i(t))$ satisfying (1-7) and $x_i \in M_i$ such that

$$N_y(t(y)) \geq -\frac{1}{i}, \quad \text{for all } y \in B_1(x_i, 0), \tag{7-19}$$

but any point y_i minimizing the quantity

$$w(y) = \frac{r_{|\mathrm{Rm}|}(y,0)}{d_{g_i(0)}(y, \partial B_1(x_i,0))}$$

satisfies $0 < w(y_i) \leq \frac{1}{i}$.

Choose any such y_i and denote $r_i = r_{|\mathrm{Rm}|}(y_i, 0)$. Consider the rescaled extended Ricci flow $(M_i, \widetilde{g}_i(t), \widetilde{\phi}_i(t))$, where

$$\widetilde{g}_i(t) = \frac{1}{r_i^2} g_i(r_i^2 t), \quad \widetilde{\phi}_i(t) = \phi_i(r_i^2 t), \quad t \in \left[-\frac{1}{r_i^2}, 0\right].$$

Clearly $r_{|\mathrm{Rm}|}(y_i, \widetilde{g}_i(0)) = 1$ and

$$d_i = \frac{1}{2} d_{\widetilde{g}_i(0)}(y_i, \partial B_{\frac{1}{r_i}}(x_i, \widetilde{g}_i(0))) \geq \frac{i}{2}.$$

Because y_i minimizes w,

$$r_{|\mathrm{Rm}|}(y, \widetilde{g}_i(0)) \geq \frac{1}{2}, \quad \text{for all } y \in B_{d_i}(y_i, \widetilde{g}_i(0)). \tag{7-20}$$

This curvature bound, together with the assumption above and the κ noncollapsing property, implies that

$$\mathrm{Vol}_{\widetilde{g}_i(0)}(B_1(y,0)) \geq \kappa(n, C).$$

So we have a uniform curvature bound on $P_{1/4}(y, \widetilde{g}_i(0))$ for any $y \in B_{d_i}(y_i, \widetilde{g}_i(0))$. Then we have the smooth convergence

$$(M_i, \widetilde{g}_i(t), (y_i, 0)) \to (M_\infty, g_\infty(t), (y_\infty, 0)).$$

The limit is completely defined on $[-\frac{1}{16}, 0]$ and is of bounded curvature.

Now we have the heat kernel bound

$$(4\pi|t|)^{-\frac{n}{2}}\exp(-l_{(\widetilde{y}_i,0)}(y,t)) \le H(\widetilde{y}_i,0;y,t) \le C'(n,C)|t|^{-\frac{n}{2}}.$$

The lower bound is due to (4-11) and the upper bound is due to Theorem 4.2.

As before, we write $H(\widetilde{y}_i,0;y,t) = (4\pi|t|)^{-\frac{n}{2}}\exp(-f_i(y,t))$. By Lemma 4.1, the gradient of $H(\widetilde{y}_i,0;y,t)$ is uniformly bounded on any compact domain; (7-20) implies higher order derivatives of $H(\widetilde{y}_i,0;y,t)$ are also bounded, so the $f_i(y,t)$ converge to $f_\infty(y,t)$ smoothly on any compact subset. Because $|\widetilde{\phi}_i| \le C$, by Lemma 7.1 and Proposition 7.2 we know the various order derivatives of $\widetilde{\phi}_i$ are uniformly bounded. Equation (7-19) together with (4) in Proposition 3.6 gives

$$\int_{-\frac{1}{16}}^{0} 2|t|(1-16|t|)\int_{M_i}\left(\left|\mathrm{Sic}(\widetilde{g}_i)+\nabla^2 f_i-\frac{\widetilde{g}_i}{2|t|}\right|^2+(\Delta\widetilde{\phi}_i-\langle\nabla f_i,\nabla\widetilde{\phi}_i\rangle)^2\right)dv_{\widetilde{y}_i(t)}\,dt\le\frac{1}{i}.$$

Letting $i\to\infty$, we see f_∞ satisfies

$$\begin{cases}\mathrm{Sic}(g_\infty)+\nabla^2 f_\infty-\frac{g_\infty}{2|t|}=0,\\ \Delta\phi_\infty=\langle\nabla\phi_\infty,\nabla f_\infty\rangle.\end{cases} \tag{7-21}$$

Because $|\widetilde{\phi}_i|\le C$, after blowing up, $\widetilde{\phi}_i\to\phi_\infty=const$, so (7-21) can be simplified to

$$\mathrm{Ric}(g_\infty)+\nabla^2 f_\infty-\frac{g_\infty}{2|t|}=0,$$

which is nonflat and of bounded curvature on $\left[-\frac{1}{16},0\right]$. This is impossible, because the curvature at time t is $\frac{1}{16|t|}$ times the curvature at time $-\frac{1}{16}$, and hence tends to infinity as $t\to 0$. □

Proof of Theorem 1.8. Define $\varepsilon=\min\{\frac{1}{2}\varepsilon_{7.3},\frac{\varepsilon_{7.3}^2}{2C'}\}$ and $\delta=\frac{|s|^{\frac{1}{2}}\varepsilon_{7.3}}{2C'}$, where C' is the constant from Theorem 1.6 and $\varepsilon_{7.3}$ is the constant from Theorem 7.3. Assume $W_{x_0}(s)\ge-\varepsilon$, so $N_{x_0}(s)\ge-\varepsilon$. By Theorem 1.6, we have for any $x\in B_\delta(x_0,0)$,

$$N_x(s)\ge N_{x_0}(s)-C'|s|^{-\frac{1}{2}}d(x_0,x)\ge-\varepsilon_{7.3}.$$

Then we can apply Theorem 7.3 to get $r_{|\mathrm{Rm}|}(x_0,0)\ge\varepsilon_{7.3}\delta\ge\varepsilon|s|^{\frac{1}{2}}$. □

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