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CALABI–YAU 4-FOLDS OF BORCEA–VOISIN TYPE FROM F-THEORY

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We apply Borcea–Voisin's construction and give new examples of Calabi–Yau 4-folds Y, which admit an elliptic fibration onto a smooth 3-fold V, whose singular fibers of type I_5 lie above a del Pezzo surface $dP \subset V$. These are relevant models for F-theory according to Beasley et al. (2009a, 2009b). Moreover, we give the explicit equations of some of these Calabi–Yau 4-folds and their fibrations.

1. Introduction

New models of grand unified theory (GUT) have recently been developed using F-theory, a branch of string theory which provides a geometric realization of strongly coupled type IIB string theory backgrounds; see, e.g., [Beasley et al. 2009a; 2009b]. In particular, one can compactify F-theory on an elliptically fibered manifold, i.e., a fiber bundle whose general fiber is a torus.

We are interested in some of the mathematical questions posed by F-theory; above all, that of the construction of some of these models. For us, F-theory will be of the form $\mathbb{R}^{3,1} \times Y$, where Y is a Calabi–Yau 4-fold admitting an elliptic fibration with a section on a complex 3-fold V, namely:

$$\begin{array}{ccc} E & \hookrightarrow & Y \\ & & \big\downarrow \mathcal{E} \\ & & V \end{array}$$

In general, the elliptic fibers E of $\mathcal{E}$ degenerate over a locus contained in a complex codimension one sublocus $\Delta(\mathcal{E})$ of V, the discriminant of $\mathcal{E}$. According to theoretical speculation in physics, $\Delta(\mathcal{E})$ should contain del Pezzo surfaces above which the general fiber is a singular fiber of type I_5 (Figure 1): see, for instance, [Beasley et al. 2009a; Bini and Penegini 2017].

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The aim of this work is to investigate explicit examples of elliptically fibered Calabi–Yau 4-folds Y with this property by using a generalized Borcea–Voisin construction. The original Borcea–Voisin construction was described independently in [Borcea 1997] and [Voisin 1993], where the authors produced Calabi–Yau 3-folds starting from a K3 surface and an elliptic curve. Afterwards, generalization to higher dimensions was considered; see e.g., [Cynk and Hulek 2007; Dillies 2012]. There are two ways to construct 4-folds of Borcea–Voisin type, by using involutions, either starting from a pair of K3 surfaces, or considering a Calabi–Yau 3-fold and an elliptic curve. In this paper we will consider the former method. A first attempt to construct explicit examples of such Calabi–Yau 4-folds Y was made in [Bini and Penegini 2017], also using a generalized Borcea–Voisin's construction but applied to a product of a Calabi–Yau 3-fold and an elliptic curve. In that case the Calabi–Yau 3-fold was a complete intersection $(3,3)$ in $\mathbb{P}^5$ containing a del Pezzo surface of degree 6; this construction was inspired by [Kapustka 2015].

In order to construct a Calabi–Yau 4-fold Y with the elliptic fibration $\mathcal{E}$ as required one needs both a map to a smooth 3-fold V whose generic fibers are genus 1 curves and a distinguished del Pezzo surface dP in V. A natural way to produce these data is to consider two K3 surfaces S_1 and S_2 such that S_1 is the double cover of dP and S_2 admits an elliptic fibration $\pi : S_2 \to \mathbb{P}^1$. In this way we will obtain $\mathcal{E} : Y \to V \simeq dP \times \mathbb{P}^1$. To get Y from S_1 and S_2 we need a nonsymplectic involution on each surface. Since S_1 is a double cover of dP, it clearly admits the cover involution, denoted by ι_1, while the involution ι_2 on S_2 is induced by the elliptic involution on each smooth fiber of π. Thus, $(S_1 \times S_2)/(\iota_1 \times \iota_2)$ is a singular Calabi–Yau 4-fold which admits a crepant resolution Y obtained blowing up the singular locus. It follows at once that there is a map $Y \to (S_1/\iota_1) \times \mathbb{P}^1 \simeq dP \times \mathbb{P}^1$ whose generic fiber is a smooth genus 1 curve and the singular fibers lie either on $dP \times \Delta(\pi)$ or on $C \times \mathbb{P}^1$ (where $C \subset dP$ is the branch curve of $S_1 \to dP$ and $\Delta(\pi)$ is the discriminant of π). The discriminant $\Delta(\pi)$ consists of a finite number of points and generically the fibers of $\mathcal{E}$ over $dP \times \Delta(\pi)$ are of the same type as the fiber of π over $\Delta(\pi)$. Therefore the requirements on the singular fibers of $\mathcal{E}$ needed in F-theory reduce to a requirement on the elliptic fibration $\pi : S_2 \to \mathbb{P}^1$.

Moreover, we show that the choice of S_1 as double cover of a del Pezzo surface and of S_2 as elliptic fibration with specific reducible fibers can be easily modified to obtain Calabi–Yau 4-folds with elliptic fibrations with a different basis (isomorphic to $S_1/\iota_1 \times \mathbb{P}^1$) and reducible fibers (over $S_1/\iota_1 \times \Delta(\pi)$).

Our first result, proven in Propositions 3.1 and 4.2 (see also Section 4C) is:

Theorem 1.1. *Let dP be a del Pezzo surface of degree $9-n$ and $S_1 \to dP$ be a double cover with S_1 a K3 surface. Let $S_2 \to \mathbb{P}^1$ be an elliptic fibration on a K3 surface with singular fibers $mI_5 + (24-5m)I_1$. The blow up Y of $(S_1 \times S_2)/(\iota_1 \times \iota_2)$ along its singular locus is a crepant resolution. It is a Calabi–Yau 4-fold which* *admits an elliptic fibration $\mathcal{E} : Y \to dP \times \mathbb{P}^1$ whose discriminant contains m copies of dP above which the fibers are of type I_5. The Hodge numbers of Y depend only on n and m and are*

$$h^{1,1}(Y) = 5+n+2m, \qquad h^{2,1}(Y) = 2(15-n-m),$$
$$h^{2,2}(Y) = 4(138-9n-19m+2nm), \quad h^{3,1}(Y) = 137-11n-22m+2nm.$$

We also give more specific results on Y. Indeed, recalling that a del Pezzo surface is either $\mathbb{P}^1 \times \mathbb{P}^1$ or a blow up of $\mathbb{P}^2$ in n points $\beta : dP \to \mathbb{P}^2$, for $0 \le n \le 8$, we give a Weierstrass equation for the elliptic fibration $Y \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ or $Y \to \beta(dP) \times \mathbb{P}^1$, respectively, induced by $\mathcal{E}$; see (12) and (13). Moreover, for $n = 5, 6$ we provide the explicit Weierstrass equation of the fibration $\mathcal{E} : Y \to dP \times \mathbb{P}^1$; see (17) and (15).

For $m = 4$, there are two different choices for $\pi : S_2 \to \mathbb{P}^1$. One of them is characterized by the presence of a 5-torsion section for $\pi : S_2 \to \mathbb{P}^1$ and in this case the K3 surface S_2 is a $2:1$ cover of the rational surface with a level 5 structure; see [Balestrieri et al. 2018]. We observe that if $\pi : S_2 \to \mathbb{P}^1$ admits a 5-torsion section, the same is true for $\mathcal{E}$.

The particular construction of Y enables us to find two other distinguished fibrations (besides $\mathcal{E}$): one whose fibers are K3 surfaces and the other whose fibers are Calabi–Yau 3-folds of Borcea–Voisin type. So Y admits fibrations in Calabi–Yau manifolds of any possible dimension. Moreover, by the explicit description of these fibrations, we observe that $\mathcal{E}$ and the fibration in Calabi–Yau 3-folds are not isotrivial. So Y can be interpreted as a non (iso)trivial family of elliptic curves and of Calabi–Yau 3-folds.

The concrete geometric description and the explicit equation of Y are interesting in view of a possible application to F-theory and can be also used to specialize Y to some more specific Calabi–Yau 3-folds with extra symmetries. These specializations are intensively used in dimension 3 to construct Calabi–Yau 3-folds with prescribed Hodge numbers (see, e.g., [Constantin et al. 2017; Braun 2011]) and can be considered in higher dimensions.

The geometric description of the fibrations on Y and their projective realization is based on a detailed study of the linear systems of divisors on Y. In particular we consider divisors D_Y induced by divisors on S_1 and S_2. We relate the dimension of the spaces of sections of D_Y with the one of the associated divisors on S_1 and S_2. Thanks to this study we are also able to describe Y as a double cover of $\mathbb{P}^2 \times \mathbb{F}_4$ (where $\mathbb{F}_4$ is the Hirzebruch surface S_2/ι_2) and as an embedded variety in $\mathbb{P}^{59-n}$. The main results in this context are summarized in Propositions 6.1 and 6.2.

The paper is organized as follows. In Section 2, we recall the definition of Calabi–Yau manifold, K3 surface and del Pezzo surface. Moreover, we describe nonsymplectic involutions on K3 surfaces. Finally in Section 2E we introduce the Borcea–Voisin construction. Section 3 is devoted to presenting models Y for

F-theory described in the introduction. The Hodge numbers of Y are calculated in Section 4. Section 5 is devoted to the study of the linear systems on Y. The results are applied in Section 6 where several fibrations and projective models of Y are described. Finally, in Section 7 we provide the explicit equations for some of these models and fibrations.

Notation and conventions. We work over the field of complex numbers $\mathbb{C}$.

2. Preliminaries

Definition 2.1. A *Calabi–Yau* manifold X is a compact Kähler manifold with trivial canonical bundle such that $h^{i,0}(X) = 0$ if $0 < i < \dim X$.

A *K3* surface S is a Calabi–Yau manifold of dimension 2. The Hodge numbers of S are uniquely determined by these properties and are $h^{0,0}(S) = h^{2,0}(S) = 1$, $h^{1,0}(S) = 0$, and $h^{1,1}(S) = 20$.

2A. An involution ι on a K3 surface S can be either symplectic, i.e., it preserves the symplectic structure of the surface, or not, in which case we speak of nonsymplectic involution. In addition, an involution on a K3 surface is symplectic if and only if its fixed locus consists of isolated points; an involution on a K3 surface is nonsymplectic if and only if there are no isolated fixed points on S. These remarkable results depend on the possibility to linearize ι near the fixed locus. Moreover, the fixed locus of an involution on S is smooth. In particular, the fixed locus of a nonsymplectic involution on a K3 surface is either empty or consists of the disjoint union of smooth curves.

From now on we consider only nonsymplectic involutions ι on K3 surfaces S. As a consequence of the Hodge index theorem and of the adjunction formula, if the fixed locus contains at least one curve C of genus $g(C) := g \geq 2$, then all the other curves in the fixed locus are rational. On the other hand, if there is one curve of genus 1 in the fixed locus, then the other fixed curves are either rational curves or exactly one genus 1 curve.

So one obtains that the fixed locus of ι on S can be one of the following:

- Empty.
- The disjoint union of two smooth genus 1 curves E_1 and E_2.
- The disjoint union of k curves, such that $k-1$ are surely rational, with the remaining curve having genus $g \geq 0$.

If we exclude the first two cases $\big(\mathrm{Fix}_\iota(S) = \varnothing,\ \mathrm{Fix}_\iota(S) = E_1 \coprod E_2\big)$, the fixed locus can be topologically described by the two integers (g, k).

There is another point of view in the description of the involution ι on S. Indeed, ι^* acts on the second cohomology group of S and its action is related to the moduli space of K3 surfaces admitting a prescribed involution; this is due to the construction of the moduli space of the lattice polarized K3 surfaces. So we are interested in the description of the lattice $H^2(S,\mathbb{Z})^{\iota^*}$. This coincides with the invariant part of the Néron–Severi group $NS(S)^{\iota_*}$ since the automorphism is nonsymplectic, and thus acts on $H^{2,0}(S)$ as $-\mathrm{id}_{H^{2,0}(S)}$; see [Nikulin 1979, Section 4, 2°]. The lattice $H^2(S,\mathbb{Z})^{\iota^*}$ of rank $r:=\mathrm{rk}(H^2(S,\mathbb{Z})^{\iota^*})$ is known to be 2-elementary, i.e., its discriminant group is $(\mathbb{Z}/2\mathbb{Z})^a$. Hence one can attach to this lattice the two integers (r,a). A very deep and important result on the nonsymplectic involutions on K3 surfaces is that each admissible pair of integers (g,k) is uniquely associated to a pair of integers (r,a); see [Nikulin 1979, Theorem 4.2.2].

We observe that for several admissible choices of (r,a) this pair uniquely determines the lattice $H^2(S,\mathbb{Z})^{\iota^*}$, but there are some exceptions.

The relations between (g,k) and (r,a) are explicitly given by

$$g=\frac{22-r-a}{2},\quad k=\frac{r-a}{2}+1,\qquad r=10+k-g,\quad a=12-k-g. \tag{1}$$

2B. A surface dP is called a *del Pezzo surface* of degree d if the anticanonical bundle $-K_{dP}$ is ample and $K^2_{dP}=d$. Moreover,we say that dP is a *weak del Pezzo* surface if $-K_{dP}$ is big and nef.

The anticanonical map embeds dP in $\mathbb{P}^d$ as a surface of degree d. The del Pezzo surfaces are either $\mathbb{P}^1\times\mathbb{P}^1$ (which has degree 8) or a blow up of $\mathbb{P}^2$ in $9-d$ points in general position

$$\beta: dP\cong Bl_{9-d}(\mathbb{P}^2)\to\mathbb{P}^2; \tag{2}$$

see, e.g., [Dolgachev 2012].

2C. A double cover of a del Pezzo surface dP ramified along a smooth curve $C\in|-2K_{dP}|$ is a K3 surface S, endowed with the covering involution ι. Since dP is not a symplectic manifold, ι is nonsymplectic. For all the del Pezzo surfaces except $\mathbb{P}^1\times\mathbb{P}^1$, we can see S as the minimal resolution of a double cover of $\mathbb{P}^2$ branched along $\beta(C)$, which is a sextic with $9-d$ nodes. Let us denote by $\rho': S\to\mathbb{P}^2$ the composition of the double cover with the minimal resolution. The ramification divisor of ρ' is a genus $1+d$ smooth curve, which is the fixed locus of ι.

If the del Pezzo surface is $\mathbb{P}^1\times\mathbb{P}^1$, then S is a double cover of $\mathbb{P}^1\times\mathbb{P}^1$ branched along a smooth curve of bidegree $(4,4)$ and we denote by $\rho': S\to\mathbb{P}^1\times\mathbb{P}^1$ the double cover.

Definition 2.2. An *elliptic fibration* $\mathcal{E}: Y\to V$ is a surjective map with connected fibers between smooth manifolds such that: the general fiber of $\mathcal{E}$ is a smooth genus 1 curve; there is a rational map $O: V\dashrightarrow Y$ such that $\mathcal{E}\circ O=\mathrm{id}_V$. A *flat elliptic fibration* is an elliptic fibration with a flat map $\mathcal{E}$. In particular a flat elliptic fibration has equidimensional fibers.

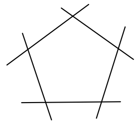

Figure 1. Fiber of type I_5.

2D. If Y is a surface then any elliptic fibration is flat. Moreover, on Y there is an involution ι which restricts to the elliptic involution on each smooth fiber. If Y is a K3 surface, then ι is a nonsymplectic involution.

2E. ***The generalized Borcea–Voisin construction.*** Let X_i, $i = 1, 2$, be a Calabi–Yau manifold endowed with an involution ι_i whose fixed locus has codimension 1. The quotient

$$(X_1 \times X_2)/(\iota_1 \times \iota_2)$$

admits a crepant resolution which is a Calabi–Yau manifold as well (see [Cynk and Hulek 2007]). We call *Borcea–Voisin* of X_1 and X_2 the Calabi–Yau $\mathrm{BV}(X_1, X_2)$ which is the blow up of $(X_1 \times X_2)/(\iota_1 \times \iota_2)$ in its singular locus.

2F. Let $b : \widetilde{X_1 \times X_2} \to X_1 \times X_2$ be the blow up of $X_1 \times X_2$ in the fixed locus of $\iota_1 \times \iota_2$. Let $\tilde{\iota}$ be the induced involution on $\widetilde{X_1 \times X_2}$ and $q : \widetilde{X_1 \times X_2} \to \widetilde{X_1 \times X_2}/\tilde{\iota} =: Y$ its quotient. The commutative diagram

$$\begin{array}{ccc}
\widetilde{X_1 \times X_2} & \xrightarrow{\quad b \quad} & X_1 \times X_2 \\
{\scriptstyle q}\big\downarrow & & \big\downarrow \\
BV(X_1, X_2) \cong Y & \longrightarrow & (X_1 \times X_2)/(\iota_1 \times \iota_2)
\end{array}$$

exhibits the Borcea–Voisin manifold as a smooth quotient.

3. The construction

3A. In the following we apply the just-described Borcea–Voisin construction in order to get a Calabi–Yau 4-fold Y together with a fibration $\mathcal{E} : Y \to V$ onto a smooth 3-fold V, with the following properties: the general fiber of $\mathcal{E}$ is a smooth elliptic curve E, the discriminant locus of $\mathcal{E}$ contains a del Pezzo surface dP, and for a generic point $p \in dP$ the singular fiber $\mathcal{E}^{-1}(p)$ is of type I_5 (see Figure 1).

3B. Let S_1 and S_2 be two K3 surfaces with the following properties:

(1) S_1 admits either a $2:1$ covering $\rho' : S_1 \to \mathbb{P}^2$, branched along a curve C, which is a (possibly singular and possibly reducible) sextic curve in $\mathbb{P}^2$, or a $2:1$ covering $\rho' : S_1 \to \mathbb{P}^1 \times \mathbb{P}^1$, branched along a curve C, which is a (possibly singular and possibly reducible) curve of bidegree $(4, 4)$ on $\mathbb{P}^1 \times \mathbb{P}^1$.

(2) S_2 admits an elliptic fibration $\pi : S_2 \to \mathbb{P}^1$, with discriminant locus $\Delta(\pi)$.

The surface S_1 has the covering involution ι_1, which is a nonsymplectic involution. Moreover, if the branch curve $C \subset \mathbb{P}^2$ (resp. $C \subset \mathbb{P}^1 \times \mathbb{P}^1$) is singular, then the double cover of $\mathbb{P}^2$ (resp. $\mathbb{P}^1 \times \mathbb{P}^1$) branched along C is singular. In this case the K3 surface S_1 is the minimal resolution of this last singular surface. The fixed locus of ι_1 consists of the strict transform $\tilde{C}$ of the branch curve, and possibly of some other smooth rational curves, W_i (which arise from the resolution of the triple points of C). Moreover, notice that if we choose $C \subset \mathbb{P}^2$ to be a sextic with $n < 9$ nodes in general position then ρ' factors through

$$\rho : S_1 \xrightarrow{2:1} dP := \mathrm{Bl}_n \mathbb{P}^2,$$

where dP is a del Pezzo surface of degree $d = 9 - n$. If C is a smooth curve, then $\widetilde{C} = C$ and we put $\rho = \rho'$ so we still have $\rho : S_1 \xrightarrow{2:1} dP$.

The second K3 surface S_2 admits a nonsymplectic involution too, as in Section 2D. This is the elliptic involution ι_2, which acts on the smooth fibers of π as the elliptic involution of each elliptic curve. In particular it fixes the 2-torsion group on each fiber. Therefore, it fixes the zero section O, which is a rational curve, and the trisection T (not necessarily irreducible) passing through the 2-torsion points of the fibers.

3C. Applying the Borcea–Voisin construction (Section 2E) to (S_1, ι_1) and (S_2, ι_2), we obtain a smooth Calabi–Yau 4-fold Y. In particular, the singular locus of the quotient $X := (S_1 \times S_2)/(\iota_1 \times \iota_2)$ is the image of the fixed locus of the product involution $\iota_1 \times \iota_2$. As the involution acts componentwise, we have

$$\mathrm{Fix}_{S_1 \times S_2}(\iota_1 \times \iota_2) = \mathrm{Fix}_{S_1} \iota_1 \times \mathrm{Fix}_{S_2} \iota_2,$$

therefore the fix locus consists of the *disjoint union* of

(1) the surface $\tilde{C} \times O$, where $O \simeq \mathbb{P}^1$ is the section of π,

(2) the surface $\tilde{C} \times T$, where T is the trisection of π,

and, possibly,

(3) the surfaces $\tilde{C} \times E_i$ (where $E_i \simeq \mathbb{P}^1$ are the fixed components in the reducible fibers of π),

(4) the surfaces $W_i \times O$, $W_i \times T$ and $W_i \times E_j$ (where $W_i \simeq \mathbb{P}^1$ are the rational curves fixed by ι_1 on S_1).

As in Section 2E we have the following commutative diagram.

$$\begin{array}{ccc} \widetilde{S_1 \times S_2} & \xrightarrow{b} & S_1 \times S_2 \\ {\scriptstyle q}\downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array} \tag{3}$$

3D. By construction, the smooth 4-fold Y comes with several fibrations. Let us analyze one of them and postpone the description of the others until Section 6.

Suppose that the del Pezzo surface is obtained by blowing up $\mathbb{P}^2$ in $9-d$ points in general position (the easier case $dP \simeq \mathbb{P}^1 \times \mathbb{P}^1$ can be studied in similar way). We have the fibration $Y \to \mathbb{P}^2 \times \mathbb{P}^1$ induced by the covering $\rho' : S_1 \to \mathbb{P}^2$ and the fibration $\pi : S_2 \to \mathbb{P}^1$. Recall from Section 3B that we can specialize the fibration if we require that ρ' is branched along a sextic with n nodes in general position. This further assumption yields

$$\begin{array}{c} Y \\ \big\downarrow{\scriptstyle \varphi} \\ dP \times \mathbb{P}^1 \end{array}$$

where dP is the del Pezzo surface obtained blowing up the nodes of the branch locus. The general fiber of φ is an elliptic curve. Indeed, let $(p,q) \in dP \times \mathbb{P}^1$ with $p \notin C$ and $q \notin \Delta(\pi)$. Then $(\varphi)^{-1}(p,q)$ is isomorphic to the smooth elliptic curve $\pi^{-1}(q)$. Hence the singular fibers lie on points $(p,q) \in dP \times \mathbb{P}^1$ of one of the following three types: $p \in C$, $q \notin \Delta(\pi)$; $p \notin C$, $q \in \Delta(\pi)$; $p \in C$, $q \in \Delta(\pi)$. We discuss these three cases separately.

Case 1: $(p,q) \in dP \times \mathbb{P}^1$ with $p \notin C$ and $q \in \Delta(\pi)$. Clearly $\pi^{-1}(q)$ is a singular curve, and since $p \notin C$, we get a singular fiber for φ,

$$\varphi^{-1}(p,q) \simeq \pi^{-1}(q). \tag{4}$$

Case 2: $(p,q) \in dP \times \mathbb{P}^1$ with $p \in C$ and $q \notin \Delta(\pi)$. Consider first $(\rho \times \pi)^{-1}(p,q)$ in $S_1 \times S_2$. This is a single copy of $\pi^{-1}(q)$, which is a smooth elliptic curve, over the point $p \in C \subseteq S_1$. In addition, this curve meets the fixed locus of $\iota_1 \times \iota_2$ in four distinct points: one of them corresponds to the intersection with $C \times O$ and the other three correspond to the intersections with $C \times T$. Notice that $\iota_1 \times \iota_2$ acts on $p \times \pi^{-1}(q)$ as the elliptic involution ι_2, hence the quotient curve is a rational curve. This discussion yields that $\varphi^{-1}(p,q)$ is a singular fiber of type I_0^*, where the central rational component is isomorphic to the quotient of $\pi^{-1}(q)/\iota_2$ and the other four rational curves are obtained by blowing up the intersection points described above.

Case 3: $(p,q) \in dP \times \mathbb{P}^1$ with $p \in C$ and $q \in \Delta(\pi)$. This time, $(\rho \times \pi)^{-1}(p,q)$ is the singular fiber $\pi^{-1}(q)$. Moreover, the quotient of this curve by ι_2 is determined by its singular fiber type. If ι_2 does not fix a component of $\pi^{-1}(q)$, then $(\rho \times \pi)^{-1}(p,q)$ meets the fixed locus of $\iota_1 \times \iota_2$ in a certain number of isolated points, depending on the fiber $\pi^{-1}(q)$ (which correspond to the intersection of the fiber with O and T). On the other hand, if ι_2 does fix a component of $\pi^{-1}(q)$, then there are curves in $(\rho \times \pi)^{-1}(p,q)$. In the latter case, $\phi^{-1}(p,q)$ contains a divisor.

In each of the previous cases, the fiber over (p, q) is not smooth and thus we obtain that the discriminant locus of φ is

$$\Delta(\varphi) = (C \times \mathbb{P}^1) \cup (dP \times \Delta(\pi)).$$

This discussion yields that the surface $dP \times \{q\} \subset \Delta(\varphi)$ for all $q \in \Delta(\pi)$ and for the generic point $p \in dP$ the fiber of φ over (p, q) is of the same type as the fiber of π over q. This implies the following proposition.

Proposition 3.1. *There exists a Calabi–Yau* 4*-fold with an elliptic fibration over* $dP \times \mathbb{P}^1$ *such that the discriminant locus contains a copy of dP. If*, *moreover*, *we assume that the generic fiber above it is reduced*, *i.e.*, *is of type* I_n, *II*, *III*, *IV*, *then it is possible to construct this elliptic fibration to be flat.*

Proof. We show firstly that if the singular fibers of the elliptic fibration π are of type I_n, *II*, *III* or *IV*, then our method produces an equidimensional fibration on Y. By our analysis in Section 3D, Case 3, it suffices to show that the elliptic involution ι_2 on S_2 does not fix any irreducible component of such fibers. As we already observed, the fixed locus of a nonsymplectic involution on a K3 surface consists of the disjoint union of smooth curves, which readily rules out the irreducible singular fibers (i.e., those of type I_1 and *II*). Consider now the case of the I_n singular fibers: call Γ_i, $i \in \mathbb{Z}/n\mathbb{Z}$, its irreducible components, in such a way that the component meeting the section O is Γ_0 and Γ_i intersects Γ_{i+1}. Consider then Γ_0; since it meets the section O (which is a component of $\operatorname{Fix}_{S_2} \iota_2$) we deduce that it is invariant but not fixed for ι_2, hence this involution must switch the two points where Γ_0 meets Γ_1 and Γ_{n-1} and has another fixed point. As a result, ι_2 switches Γ_1 and Γ_{n-1} and consequently switches Γ_i with Γ_{n-i} for $1 \leq i \leq \left[\frac{n-1}{2}\right]$. In the end, either we have a fixed point at the intersection of $\Gamma_{(n-1)/2}$ and $\Gamma_{(n+1)/2}$ if n is odd, or we have two fixed points on $\Gamma_{n/2}$ if n is even (in this case, this curve is ι_2-invariant). Consider now a fiber of type *III*, and let Γ_0 be the component meeting the section O and Γ_1 the other component. On Γ_0 the point $\Gamma_0 \cap O$ is fixed, and so Γ_0 is an invariant curve which is not fixed by ι_2. As there is only one singular point in the fiber, this point must be fixed as well, and so there cannot be other fixed points on Γ_0. As a consequence, if the trisection T does not meet Γ_1, then it must meet Γ_0 and this would imply that Γ_0 is fixed. So T must meet Γ_1, which prevents Γ_1 from being a fixed curve. Consider finally a fiber of type *IV*, and let Γ_0 be the component meeting the section O and Γ_1, Γ_2 be the other components. The unique singular point is necessarily a fixed point. As before, on Γ_0 we have two fixed points, the intersection with the section O and the singular point of the fiber, and this component is globally invariant and can not meet the trisection T in a point different from the singular point. If T does not meet Γ_0 at all (and thus does not pass through the singular point), then it must meet one between Γ_1 and Γ_2 with multiplicity 2, which would imply that component is fixed (since it has at least three fixed points). As it meets the trisection (which is fixed), this is absurd, and so T must pass through the singular point. As a consequence, neither Γ_1 nor Γ_2 are fixed since they meet T at least in the singular point.

This is enough to claim that the elliptic fibration on Y is equidimensional. Finally, as we are dealing with morphisms between smooth varieties, by [Nowak 1997, criterion for flatness] we deduce that our fibration is also flat. □

3E. We shall now discuss a special case of the elliptic fibration φ. Apparently, a good phenomenological model for F-theory (see the introduction and references therein) is the one where the discriminant locus contains a del Pezzo surface over which there are I_5 singular fibers. Indeed, F-theory on an elliptically fibered Calabi–Yau 4-fold Y with base B is equivalent to Type IIB string theory on B with a dilaton-axion $\tau = C_0 + ie^{-\phi}$ varying over this base. At each point in B the complex number τ can be identified with the complex structure modulus of the elliptic fiber over this point. For Y to be a Calabi–Yau 4-fold this fiber has to degenerate over divisors D_i in B. These degeneration loci encode the location of space-time filling seven-branes of Type IIB compactified on B. In the case of an SU(5) gauge group theory, D_i should be del Pezzo surfaces and a singular fiber splits into an I_5 Kodaira singular fiber; see, e.g., [Braun et al. 2013]. The choice of an SU(5) gauge group theory lies on the fact that it is the smallest simple Lie group which contains the standard model, and upon which the first grand unified theory was based. Besides SU(5), another group which seems to be interesting for the grand unified theories is E_6: the corresponding fibrations will have singular fibers of type IV^* on the del Pezzo in the discriminant. Let us discuss the situation of SU(5).

Remark 3.2. By Proposition 3.1 it is possible to construct elliptic fibrations with fibers I_5. Nevertheless, it is not possible to obtain elliptic fibrations such that *all* the singular fibers are of type I_5. Indeed, there are two different obstructions:

(1) The fibers obtained in Case 2 of Section 3D are of type I_0^* and this does not depend on the choice of the properties of the elliptic fibration $S_2 \to \mathbb{P}^1$.

(2) The singular fibers as in Case 1 of Section 3D depend only on the singular fibers of $S_2 \to \mathbb{P}^1$ and these cannot be only of type I_5, indeed $24 = \chi(S_2)$ is not divisible by 5.

However, it is known that there exist elliptic K3 surfaces with m fibers of type I_5 and all other singular fibers of type I_1 for $m = 1, 2, 3, 4$; see [Shimada 2000]. In this case the number of fibers of type I_1 is $24 - 5m$.

4. The Hodge numbers of Y

The aim of this section is the computation of the Hodge numbers of the constructed 4-folds.

4A. By (3) the cohomology of Y is given by the part of the cohomology of $\widetilde{S_1 \times S_2}$ which is invariant under $(\iota_1 \times \iota_2)^*$. The cohomology of $\widetilde{S_1 \times S_2}$ is essentially obtained as the sum of two different contributions: the pullback by b^* of the cohomology of $S_1 \times S_2$ and the part of the cohomology introduced by the blow up of the fixed locus $\mathrm{Fix}_{\iota_1 \times \iota_2}(S_1 \times S_2)$. The fixed locus $\mathrm{Fix}_{\iota_1 \times \iota_2}(S_1 \times S_2) = \mathrm{Fix}_{\iota_1}(S_1) \times \mathrm{Fix}_{\iota_2}(S_2)$ consists of surfaces, which are products of curves. So $b : \widetilde{S_1 \times S_2} \to S_1 \times S_2$ introduces exceptional divisors which are $\mathbb{P}^1$-bundles over surfaces which are products of curves. The Hodge diamonds of these exceptional 3-folds depends only on the genus of the curves in $\mathrm{Fix}_{\iota_1}(S_1)$ and $\mathrm{Fix}_{\iota_2}(S_2)$.

Since, up to an appropriate shift of the indices, the Hodge diamond of $\widetilde{S_1 \times S_2}$ is just the sum of the Hodge diamond of $S_1 \times S_2$ and of all the Hodge diamonds of the exceptional divisors, the Hodge diamond of $\widetilde{S_1 \times S_2}$ depends only on the properties of the fixed locus of ι_1 on S_1 and of ι_2 on S_2. Denoted by (g_i, k_i), $i = 1, 2$, the pair of integers which describes the fixed locus of ι_i on S_i, we obtain that the Hodge diamond of $\widetilde{S_1 \times S_2}$ depends only on the four integers (g_1, k_1, g_2, k_2).

Now we consider the quotient 4-fold Y. Its cohomology is the invariant cohomology of $\widetilde{S_1 \times S_2}$ for the action of $(\iota_1 \times \iota_2)^*$. Since the automorphism induced by $\iota_1 \times \iota_2$ on $\widetilde{S_1 \times S_2}$ acts trivially on the exceptional divisors, one has only to compute the invariant part of the cohomology of $S_1 \times S_2$ for the action of $(\iota_1 \times \iota_2)^*$. But this depends of course only on the properties of the action of ι_i^* on the cohomology of S_i. We observe that ι_i^* acts trivially on $H^0(S_i, \mathbb{Z})$, and that $H^1(S_i, \mathbb{Z})$ is empty. Denote by (r_i, a_i), $i = 1, 2$, the invariants of the lattice $H^2(S_i, \mathbb{Z})^{\iota_i^*}$; these determine uniquely $H^*(S_1 \times S_2, \mathbb{Z})^{(\iota_1 \times \iota_2)^*}$.

Thus the Hodge diamond of Y depends only on (g_i, k_i) and (r_i, a_i), $i = 1, 2$. By (1), it is immediate that the Hodge diamond of Y depends only either on (g_1, k_1, g_2, k_2) or equivalently on (r_1, a_1, r_2, a_2).

This result is already known, due to J. Dillies who computed the Hodge numbers of the Borcea–Voisin of the product of two K3 surfaces by means of the invariants (r_1, a_1, r_2, a_2) in [Dillies 2012]:

Proposition 4.1 [Dillies 2012, Section 7.2.1]. *Let ι_i be a nonsymplectic involution on S_i, $i = 1, 2$, such that its fixed locus is nonempty and does not consist of two curves of genus* 1. *Let Y be the Borcea–Voisin* 4*-fold of S_1 and S_2. Then*

$$
\begin{aligned}
h^{1,1}(Y) &= 1 + \frac{r_1 r_2}{4} - \frac{r_1 a_2}{4} - \frac{a_1 r_2}{4} + \frac{a_1 a_2}{4} + \frac{3r_1}{2} - \frac{a_1}{2} + \frac{3r_2}{2} - \frac{a_2}{2},\\
h^{2,1}(Y) &= 22 - \frac{r_1 r_2}{2} + \frac{a_1 a_2}{2} + 5r_1 - 6a_1 + 5r_2 - 6a_2,\\
h^{2,2}(Y) &= 648 + 3r_1 r_2 + a_1 a_2 - 30r_1 - 30r_2 - 12a_1 - 12a_2,\\
h^{3,1}(Y) &= 161 + \frac{r_1 r_2}{4} + \frac{a_1 a_2}{4} + \frac{r_1 a_2}{4} + \frac{a_1 r_2}{4} - \frac{13r_1}{2} - \frac{13r_2}{2} - \frac{11a_1}{2} - \frac{11a_2}{2}.
\end{aligned}
$$

4B. Now we apply these computations to our particular case: S_2 is an elliptic K3 surface with m fibers of type I_5 and S_1 is either the double cover of $\mathbb{P}^2$ branched along a sextic with n nodes or the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along a smooth bidegree (4, 4) curve. In this latter case we pose $n = 1$. We obtain the following proposition.

Proposition 4.2. *Let $m \geq 0$ be an integer, and suppose that $\pi : S_2 \to \mathbb{P}^1$ in an elliptic fibration with singular fibers of type $mI_5 + (24-5m)I_1$. Then*

$$h^{1,1}(Y) = 5 + n + 2m,$$
$$h^{2,1}(Y) = 2(15 - n - m),$$
$$h^{2,2}(Y) = 4(138 - 9n - 19m + 2nm),$$
$$h^{3,1}(Y) = 137 - 11n - 22m + 2nm.$$

Proof. In order to deduce the Hodge numbers of Y by Proposition 4.1, we have to compute the invariants (g_i, k_i) of the action of ι_i on S_i in our context. If the surface S_1 is a 2 : 1 cover of $\mathbb{P}^2$ branched on a sextic with n nodes and ι_1 is the cover involution, then the fixed locus of ι_1 is isomorphic to the branch curve and hence has genus $10-n$. So $(g_1, k_1) = (10-n, 1)$ and thus $r_1 = 1+n$ and $a_1 = 1+n$. If the surface S_1 is a 2 : 1 cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched on a smooth bidegree (4, 4) curve, and ι_1 is the cover involution, then the fixed locus of ι_1 is isomorphic to the branch curve and hence has genus $9 = 10 - n$. Also in this case $(g_1, k_1) = (10-n, 1)$ and thus $r_1 = 1+n$ and $a_1 = 1+n$.

The involution ι_2 on S_2 is the elliptic involution, and hence fixes the section of the fibration, which is a rational curve, and the trisection passing through the 2-torsion points of the fibers. Moreover, ι_2 does not fix components of the reducible fibers. So $k_2 = 2$ and it remains to compute the genus of the trisection. The Weierstrass equation of the elliptic fibration S_2 is $y^2 = x^3 + A(t)x + B(t)$ and the equation of the trisection T is $x^3 + A(t)x + B(t) = 0$, which exhibits T as 3 : 1 cover of $\mathbb{P}^1_t$ branched on the zero points of the discriminant $\Delta(t) = 4A(t)^3 + 27B(t)^2$. Under our assumptions, the discriminant has m roots of multiplicity 5 and $24 - 5m$ simple roots, so that T is a 3 : 1 cover branched in $24 - 5m + m = 24 - 4m$ points with multiplicity 2. Therefore, by the Riemann–Hurwitz formula, one obtains $2g(T) - 2 = -6 + 24 - 4m$, i.e., $g(T) = 10 - 2m$. Hence $k_2 = 2$, $g_2 = 10 - 2m$ and so $r_2 = 2 + 2m$ and $a_2 = 2m$. □

4C. *Proof of Theorem 1.1*. Theorem 1.1 states the existence of an elliptic fibration $\mathcal{E}$ on a certain Calabi–Yau 4-fold Y and contains the Hodge numbers of Y. The construction of the Calabi–Yau 4-fold Y is contained in Section 3C, the existence of the elliptic fibration $\mathcal{E}$ is proved in Proposition 3.1 and the Hodge numbers of Y are given in Proposition 4.2. This concludes the proof of Theorem 1.1.

4D. By construction, Y is obtained as a (desingularization of a) quotient of $S_1 \times S_2$ by $\iota_1 \times \iota_2$, so each complex deformation of the pairs (S_i, ι_i), where S_i is a K3 surface admitting a prescribed nonsymplectic involution ι_i, induces a complex deformation of Y. Since Y is a Calabi–Yau 4-fold, the dimension of the space of its complex deformations is $h^{3,1}(Y)$. By [Nikulin 1979], the dimension of the space of complex deformations of (S_i, ι_i) is $20 - r_i$, $i = 1, 2$. So $h^{3,1}(Y) \geq (20 - r_1) + (20 - r_2)$ and the equality holds if and only if all the deformations of Y are induced by deformations of (S_i, ι_i) (compare with the definition of Borcea–Voisin maximal family in [Cattaneo and Garbagnati 2016], where similar concepts are discussed on Calabi–Yau 3-folds of Borcea–Voisin type). By Proposition 4.2, one has $h^{3,1}(Y) = 137 - 11n - 22m + 2nm$, $r_1 = 1 + n$, $r_2 = 2 + 2m$ and thus $(20 - r_1) + (20 - r_2) = 37 - n - 2m$. Hence, $h^{3,1}(Y)$ is strictly bigger than $37 - n - 2m$ and therefore a part of the deformations of Y are not induced by deformations of the pairs (S_i, ι_i).

Let us now fix the del Pezzo surface dP, and then the K3 surface S_1, with its involution ι_1. The moduli space of K3 surfaces with m fibers of type I_5 as in Proposition 4.2 has dimension $18 - 4m$ (because this is the space of the $(U \oplus A_4^{\oplus m})$-polarized K3 surfaces). So, given a K3 surface S_2 as in Proposition 4.2, the moduli of Y are $137 - 11n - 22m + 2nm$ and the moduli of the K3 surfaces S_2 admitting the prescribed elliptic fibration are $18 - 4m$. In particular, any complex deformation of S_2 which preserves the elliptic fibration induces a complex deformation of Y, but there are a lot of deformations of Y which are not induced by those of S_2.

5. Linear systems on Y

5A. Here we state some general results on linear systems on the product of varieties with trivial canonical bundle, which will be applied to $S_1 \times S_2$.

Let X_1 and X_2 be two smooth varieties with trivial canonical bundle, and $\mathcal{L}_{X_1}$ and $\mathcal{L}_{X_2}$ be two line bundles on X_1 and X_2, respectively. Observe that we have a natural injective homomorphism

$$H^0(X_1, \mathcal{L}_{X_1}) \otimes H^0(X_2, \mathcal{L}_{X_2}) \to H^0(X_1 \times X_2, \pi_1^*\mathcal{L}_{X_1} \otimes \pi_2^*\mathcal{L}_{X_2})$$
$$s \otimes t \mapsto \pi_1^* s \cdot \pi_2^* t,$$

where the π_i's are the two projections. We now want to determine some conditions which guarantee that this map is an isomorphism.

Using the Hirzebruch–Riemann–Roch theorem,

$$\chi(X_1 \times X_2, \pi_1^*\mathcal{L}_{X_1} \otimes \pi_2^*\mathcal{L}_{X_2}) = \chi(X_1, \mathcal{L}_{X_1}) \cdot \chi(X_2, \mathcal{L}_{X_2}).$$

If $\mathcal{L}_{X_1}$ and $\mathcal{L}_{X_2}$ are nef and big line bundles such that $\pi_1^*\mathcal{L}_{X_1} \otimes \pi_2^*\mathcal{L}_{X_2}$ is still nef and big, then the above formula and Kawamata–Viehweg vanishing theorem lead to

$$h^0(X_1 \times X_2, \pi_1^*\mathcal{L}_{X_1} \otimes \pi_2^*\mathcal{L}_{X_2}) = h^0(X_1, \mathcal{L}_{X_1}) \cdot h^0(X_2, \mathcal{L}_{X_2}).$$

However, we are interested also in divisors which are not big and nef, therefore we need the following result.

Proposition 5.1. *Let X_1, X_2 be two smooth varieties of dimension n_1 and n_2, respectively. Assume that they have trivial canonical bundle $\omega_{X_i} = \mathcal{O}_{X_i}$ and that $h^{0,n_i-1}(X_i) = 0$. Let $D_i \subseteq X_i$ be a smooth irreducible codimension 1 subvariety. Then the canonical map*

$$H^0(X_1, \mathcal{O}_{X_1}(D_1)) \otimes H^0(X_2, \mathcal{O}_{X_2}(D_2)) \xrightarrow{\psi} H^0(X_1 \times X_2, \pi_1^*\mathcal{O}_{X_1}(D_1) \otimes \pi_2^*\mathcal{O}_{X_2}(D_2))$$

is an isomorphism.

Proof. By Künneth's formula

$$\begin{aligned} h^{0,n-1}(X_1 \times X_2) = &\ h^{0,n_1-1}(X_1) \cdot h^{0,n_2}(X_2) + h^{0,n_1}(X_1) \cdot h^{0,n_2-1}(X_2) \\ = &\ h^{0,n_1-1}(X_1) + h^{0,n_2-1}(X_2) = 0, \end{aligned}$$

where $n = n_1 + n_2 = \dim(X_1 \times X_2)$.

As already remarked, the ψ map is injective, so it suffices to show that the source and target spaces have the same dimension.

We begin with the computation of $h^0(X_i, \mathcal{O}_{X_i}(D_i))$. From the exact sequence

$$0 \to \mathcal{O}_{X_i}(-D_i) \to \mathcal{O}_{X_i} \to \mathcal{O}_{D_i} \to 0,$$

we deduce the exact piece

$$H^{n_i-1}(X_i, \mathcal{O}_{X_i}) \to H^{n_i-1}(D_i, \mathcal{O}_{D_i}) \to H^{n_i}(X_i, \mathcal{O}_{X_i}(-D_i)) \to H^{n_i}(X_i, \mathcal{O}_{X_i}) \to 0.$$

Since $H^{n_i-1}(X_i, \mathcal{O}_{X_i}) = 0$ by hypothesis, we get by Serre duality that

$$h^0(X_i, \mathcal{O}_{X_i}(D_i)) = h^{n_i}(X_i, \mathcal{O}_{X_i}(-D_i)) = h^{n_i-1}(D_i, \mathcal{O}_{D_i}) + 1.$$

Now we pass to the computation of $h^0(X_1 \times X_2, \pi_1^*\mathcal{O}_{X_1}(D_1) \otimes \pi_2^*\mathcal{O}_{X_2}(D_2))$. Let $D = D_1 \times X_2 \cup X_1 \times D_2$, and observe that

$$\pi_1^*\mathcal{O}_{X_1}(D_1) \otimes \pi_2^*\mathcal{O}_{X_2}(D_2) = \mathcal{O}_{X_1 \times X_2}(D).$$

By the previous part of the proof, we have

$$h^0(X_1 \times X_2, \pi_1^*\mathcal{O}_{X_1}(D_1) \otimes \pi_2^*\mathcal{O}_{X_2}(D_2)) = h^{n-1}(D, \mathcal{O}_D) + 1,$$

so we need to compute $h^{n-1}(D, \mathcal{O}_D)$ in this situation. Consider the diagram of inclusions

$$\begin{array}{ccc} X_1 \times D_2 & \xhookrightarrow{i_1} & D \\ \uparrow & \nearrow^{i} & \uparrow i_2 \\ D_1 \times D_2 & \hookrightarrow & D_1 \times X_2 \end{array}$$

and the short exact sequence

$$0 \to \mathcal{O}_D \to i_{1*}\mathcal{O}_{X_1\times D_2} \oplus i_{2*}\mathcal{O}_{D_1\times X_2} \to i_*\mathcal{O}_{D_1\times D_2} \to 0,$$

where

$$\mathcal{O}_D \to i_{1*}\mathcal{O}_{X_1\times D_2} \oplus i_{2*}\mathcal{O}_{D_1\times X_2}, \qquad s \mapsto (s_{|X_1\times D_2}, s_{|D_1\times X_2})$$

and

$$i_{1*}\mathcal{O}_{X_1\times D_2} \oplus i_{2*}\mathcal{O}_{D_1\times X_2} \to i_*\mathcal{O}_{D_1\times D_2}, \qquad (s_1, s_2) \mapsto s_{1|D_1\times D_2} - s_{2|D_1\times D_2}.$$

This sequence induces the exact piece

$$\begin{aligned} &H^{n-2}(D_1\times D_2, \mathcal{O}_{D_1\times D_2}) \to H^{n-1}(D, \mathcal{O}_D) \\ &\qquad \to H^{n-1}(X_1\times D_2, \mathcal{O}_{X_1\times D_2}) \oplus H^{n-1}(D_1\times X_2, \mathcal{O}_{D_1\times X_2}) \to 0, \end{aligned}$$

from which we have that

$$\begin{aligned} h^{n-1}(D, \mathcal{O}_D) \le h^{n-1}(X_1\times D_2, \mathcal{O}_{X_1\times D_2}) + h^{n-1}(D_1\times X_2, \mathcal{O}_{D_1\times X_2}) \\ + h^{n-2}(D_1\times D_2, \mathcal{O}_{D_1\times D_2}). \end{aligned}$$

These last numbers are easy to compute using Künneth's formula:

$$\begin{aligned} h^{n-1}(X_1\times D_2, \mathcal{O}_{X_1\times D_2}) &= \sum_{i=0}^{n-1} h^{0,i}(X_1)\cdot h^{0,n-1-i}(D_2) \\ &= h^{0,n_1}(X_1)\cdot h^{0,n_2-1}(D_2) = h^{0,n_2-1}(D_2); \\ h^{n-1}(D_1\times X_2, \mathcal{O}_{D_1\times X_2}) &= h^{0,n_1-1}(D_1); \\ h^{n-2}(D_1\times D_2, \mathcal{O}_{D_1\times D_2}) &= h^{0,n-2}(D_1\times D_2) \\ &= \sum_{i=0}^{n-2} h^{0,i}(D_1)\cdot h^{0,n-2-i}(D_2) \\ &= h^{0,n_1-1}(D_1)\cdot h^{0,n_2-1}(D_2), \end{aligned}$$

where we used the trivial observation that $h^{0,k}(D_i) = 0$ if $k \ge n_i$.

Finally, we have the following chain of inequalities:

$$\begin{aligned} &(h^{n_1-1}(D_1, \mathcal{O}_{D_1})+1)(h^{n_2-1}(D_2, \mathcal{O}_{D_2})+1) \\ &\qquad = h^0(X_1, \mathcal{O}_{X_1}(D_1))\cdot h^0(X_2, \mathcal{O}_{X_2}(D_2)) \\ &\qquad \le h^0(X_1\times X_2, \mathcal{O}_{X_1\times X_2}(D)) = h^{n-1}(D, \mathcal{O}_D)+1 \\ &\qquad \le h^{0,n_1-1}(D_1) + h^{0,n_2-1}(D_2) + h^{0,n_1-1}(D_1)\cdot h^{0,n_2-1}(D_2) + 1 \\ &\qquad = (h^{n_1-1}(D_1, \mathcal{O}_{D_1})+1)(h^{n_2-1}(D_2, \mathcal{O}_{D_2})+1), \end{aligned}$$

from which the proposition follows. □

Remark 5.2. Observe that this proposition can be deduced also from more general arguments; see, for instance, [Kashiwara and Schapira 1990, Exercise II.18] where a broader generalization of the Künneth formula is shown.

5B. In particular, this result applies when X_1 and X_2 are $K3$ surfaces or, more generally, when they are Calabi–Yau or hyperkähler manifolds.

By induction, it is easy to generalize this result to a finite number of factors. Notice that we require D_i to be smooth in order to use Künneth's formula. Indeed, there is a more general version of Proposition 5.1 for line bundles. Namely, if $\mathcal{L}_i$ are globally generated/base point free line bundles over X_i then their linear systems $|\mathcal{L}_i|$ have, by Bertini's theorem, a smooth irreducible member, and we can apply Proposition 5.1.

Let us denote $D_1+D_2 := \pi_1^*\mathcal{O}(D_1)+\pi_2^*\mathcal{O}(D_2)$. The linear system $|D_i|$ naturally defines the map $\varphi_{|D_i|}: X_i \to \mathbb{P}^{n_i}$. Denoting by $\sigma_{n_1,n_2}: \mathbb{P}^{n_1}\times\mathbb{P}^{n_2} \to \mathbb{P}^{n_1n_2+n_1+n_2}$ the Segre embedding, Proposition 5.1 implies that $\varphi_{|D_1+D_2|}$ coincides with $\sigma_{n_1,n_2}\circ(\varphi_{|D_1|}\times\varphi_{|D_2|})$.

Corollary 5.3. *Let S_i, $i=1,2$, be two $K3$ surfaces and D_i be an irreducible smooth curve of genus g_i on S_i. Then $h^0(S_1\times S_2, D_1+D_2) = (g_1+1)(g_2+1)$.*

5C. Use the same notation as in Section 3 diagram (3). On $S_1\times S_2$, let D be an invariant divisor (resp. an invariant line bundle $\mathcal{D}$) with respect to the $\iota_1\times\iota_2$ action. Moreover, denote by D_Y the divisor on Y such that $q^*D_Y = b^*D$ (resp. $\mathcal{D}_Y$ is the line bundle such that $q^*\mathcal{D}_Y = b^*\mathcal{D}$).

Since q is a double cover branched along a codimension 1 subvariety B, it is uniquely defined by a line bundle $\mathcal{L}$ on Y such that $\mathcal{L}^{\otimes 2} = \mathcal{O}_Y(B)$ and we have

$$H^0(\widetilde{S_1\times S_2}, q^*\mathcal{M}) = H^0(Y,\mathcal{M}) \oplus H^0(Y, \mathcal{M}\otimes\mathcal{L}^{\otimes -1})$$

for any line bundle $\mathcal{M}$ on Y.

The isomorphism $H^0(\widetilde{S_1\times S_2}, b^*\mathcal{D}) \simeq H^0(S_1\times S_2,\mathcal{D})$ yields

$$H^0(S_1\times S_2,\mathcal{D}) \simeq H^0(\widetilde{S_1\times S_2}, q^*\mathcal{D}_Y) \simeq H^0(Y,\mathcal{D}_Y)\oplus H^0(Y,\mathcal{D}_Y\otimes\mathcal{L}^{\otimes -1}).$$

As a consequence, one sees that the space $H^0(Y,\mathcal{D}_Y)$ corresponds to the invariant subspace of $H^0(S_1\times S_2,\mathcal{D})$ for the ι^* action, while $H^0(Y,\mathcal{D}_Y\otimes\mathcal{L}^{-1})$ corresponds to the anti-invariant one. This yields at once the commutative diagram

$$\begin{array}{ccccc} \widetilde{S_1\times S_2} & \xrightarrow{\ b\ } & S_1\times S_2 & \xrightarrow{\varphi_{|\mathcal{D}|}} & \mathbb{P}(H^0(S_1\times S_2,\mathcal{D})^\vee) \\ {\scriptstyle q}\big\downarrow & & \big\downarrow & & \big\downarrow \\ Y & \longrightarrow & X & \longrightarrow & \mathbb{P}(H^0(Y,\mathcal{D}_Y)^\vee), \\ & & \underbrace{\qquad\qquad}_{\varphi_{|\mathcal{D}_Y|}} & & \end{array} \tag{5}$$

where the vertical arrow on the right is the projection on $\mathbb{P}(H^0(Y,\mathcal{D}_Y)^\vee)$ with center $\mathbb{P}(H^0(Y, \mathcal{D}_Y \otimes \mathcal{L}^{-1})^\vee)$ (observe that both these two spaces are pointwise fixed for the induced action of ι on $\mathbb{P}(H^0(S_1 \times S_2, \mathcal{D})^\vee)$).

In what follows we denote by D_Y and L the divisors such that $\mathcal{D}_Y = \mathcal{O}(D_Y)$ and $\mathcal{L} = \mathcal{O}(L)$, so L is half of the branch divisor.

5D. Let D_i be a smooth irreducible curve on S_i such that the divisor D_i is invariant for ι_i. Then ι_i^* acts on $H^0(S_i, D_i)^\vee$. Let us denote by $H^0(S_i, D_i)_{\pm 1}$ the eigenspace relative to the eigenvalue ± 1 for the action of ι_i on $H^0(S_i, D_i)$. Let h_i be the dimension of $\mathbb{P}(H^0(S_i, D_i)^\vee_{+1})$.

Corollary 5.4. *Let S_i, D_i, D_Y, L and h_i be as above. Then $\varphi_{|D_Y|} : Y \to \mathbb{P}^N$ where $N := (h_1 + 1)(h_2 + 1) + (g(D_1) - h_1)(g(D_2) - h_2) - 1$ and $\varphi_{|D_Y - L|} : Y \to \mathbb{P}^M$ where $M := (h_1 + 1)(g(D_2) - h_2) + (g(D_1) - h_1)(h_2 + 1) - 1$.*

Proof. By Corollary 5.3 the map $\varphi_{|D_1 + D_2|}$ is a map from $S_1 \times S_2$ to the Segre embedding of $\mathbb{P}(H^0(S_1, D_1)^\vee)$ and $\mathbb{P}(H^0(S_2, D_2)^\vee)$. The action of the automorphism $\iota_1 \times \iota_2$ on $H^0(S_1 \times S_2, D_1 + D_2)$ is induced by the action of ι_i on $H^0(S_i, D_i)$ and in particular

$$H^0(S_1 \times S_2, D_1 + D_2)_{+1} = H^0(S_1, D_1)_{+1} \otimes H^0(S_2, D_2)_{+1} \oplus H^0(S_1, D_1)_{-1} \otimes H^0(S_2, D_2)_{-1},$$

whose dimension is $(h_1+1)(h_2+1)+(g(D_1)-h_1)(g(D_2)-h_2)$. By Section 5C, the divisors D_Y and $D_Y - L$ define on Y two maps whose target space is the projection of $\mathbb{P}(H^0(S_1 \times S_2, D)^\vee)$ to the eigenspaces for the action of $\iota_1 \times \iota_2$ and the image is the projection of $\varphi_{|D|}(S_1 \times S_2)$. So the target space of $\varphi_{|D_Y|}$ is

$$\mathbb{P}(H^0(S_1 \times S_2, D_1 + D_2)^\vee_{+1}),$$

whose dimension is $(h_1+1)(h_2+1)+(g(D_1)-h_1)(g(D_2)-h_2)-1$. One concludes similarly for $\varphi_{|D_Y - L|}$. □

Lemma 5.5. *Let D_i be an effective divisor on S_i invariant for ι_i, and h_i be the dimension of $\mathbb{P}(H^0(S_i, D_i)^\vee_{+1})$ for $i = 1, 2$. Denote by δ_{D_i} the divisor on Y such that $q^*(\delta_{D_i}) = b^*(\pi_i^*(D_i))$. Then*

$$H^0(S_1 \times S_2, \pi_i^*(D_i)) \simeq H^0(S_i, D_i) \quad \textit{and} \quad \dim(\mathbb{P}(H^0(Y, \delta_{D_i}))) = h_i,$$

for $i = 1, 2$.

6. Projective models and fibrations

The aim of this section is to apply the general results of the previous sections to our specific situation. So, let (S_1, ι_1) and (S_2, ι_2) be as in Section 3B (i.e., S_1 is a double cover of dP, ι_1 is the cover involution, S_2 is an elliptic fibration and ι_2 is the elliptic involution). To simplify the notation, from now on we assume that the del Pezzo surface dP is a blow up of $\mathbb{P}^2$, so that S_1 is a double cover of $\mathbb{P}^2$. The results of this section can be easily generalized to the case $dP \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

We now consider some interesting divisors on S_1 and S_2.

6A. Let $h \in \mathrm{Pic}(S_1)$ be the pullback of the hyperplane section of $\mathbb{P}^2$ by the generically $2:1$ map $\rho' : S_1 \to \mathbb{P}^2$. The divisor h is a nef and big divisor on S_1 and the map $\varphi_{|h|}$ is generically $2:1$ to the image (which is $\mathbb{P}^2$). The action of ι_1 is the identity on $H^0(S_1, h)^\vee$, since ι_1 is the cover involution.

We recall that the branch locus of ρ' is a sextic with n simple nodes in general position, for $0 \le n \le 8$. As explained in Section 3, in order to construct a smooth double cover we first blow up $\mathbb{P}^2$ at the n nodes of the sextic obtaining a del Pezzo surface dP. Thus on S_1 there are n rational curves, lying over these exceptional curves. We denote these curves by R_i, $i = 1, \dots, n$. We will denote by H the divisor $3h - \sum_{i=1}^n R_i$ if $n \ge 1$ or the divisor $3h$ if $n = 0$. Observe that H is the strict transform of the nodal sextic in $\mathbb{P}^2$.

For a generic choice of S_1 the Picard group of S_1 is generated by h and R_i. The divisor H is an ample divisor, because it has a positive intersection with all the effective -2 classes. Moreover, $H^2 = 18-2n > 2$, if $n \le 7$. By [Saint-Donat 1974], this divisor cannot be hyperelliptic and so the map $\varphi_{|H|}$ is $1:1$ onto its image in $\mathbb{P}^{10-n}$.

The divisor $\frac{1}{2}\rho_*(H)$ is the anticanonical divisor of the del Pezzo surface dP, which embeds dP in

$$\mathbb{P}^{9-n} = \mathbb{P}\big(H^0\big(dP, \tfrac{1}{2}\rho_*(H)\big)^\vee\big).$$

Since ι_1 is the cover involution of ρ, the action of ι_1^* on $H^0(S_1, H)^\vee$ has a $(10-n)$-dimensional eigenspace for the eigenvalue $+1$ and a 1-dimensional eigenspace for the eigenvalue -1. Observe that with this description, the projection

$$\mathbb{P}(H^0(S_1, H)^\vee) \to \mathbb{P}(H^0(S_1, H)^\vee_{+1})$$

from the point $\mathbb{P}(H^0(S_1, H)^\vee_{-1})$ coincides with the double cover ρ.

Notably, if $n = 6$, the del Pezzo surface dP is a cubic surface in $\mathbb{P}^3_{(x_0:x_1:x_2:x_3)}$, whose equation is $f_3(x_0 : x_1 : x_2 : x_3) = 0$. In this case the divisor H embeds the K3 surface S_1 in $\mathbb{P}^4$ as the complete intersection of a quadric with equation $x_4^2 = g_2(x_0 : x_1 : x_2 : x_3)$ and the cubic $f_3(x_0 : x_1 : x_2 : x_3) = 0$ and ι_1 acts multiplying x_4 by -1.

6B. Let S_2 be a K3 surface with an elliptic fibration. Generically $\mathrm{Pic}(S_2)$ is spanned by the divisors F and O, the class of the fiber and the class of the section, respectively. If S_2 has some other properties, for example some reducible fibers, then there are other divisors on S_2 linearly independent from F and O. In any case, it is still true that $\langle F, O\rangle$ is primitively embedded in $\mathrm{Pic}(S_2)$. We consider two divisors on S_2: F and $4F+2O$.

The divisor F is by definition the class of the fiber of the elliptic fibration on S_2, so that $\pi=\varphi_{|F|}: S_2\to\mathbb{P}^1$ is the elliptic fibration on S_2. In particular F is a nef divisor, but it is not big, and it is invariant for ι_2 (since ι_2 preserves the fibration). Moreover, ι_2 preserves each fiber of the fibration, therefore ι_2^* acts as the identity on $H^0(S_2, F)^\vee$.

It is easy to see that the divisor $4F+2O$ is a nef and big divisor. The map $\varphi_{|4F+2O|}$ contracts the zero section and possibly the nontrivial components of the reducible fibers of the fibration. We see that

$$\varphi_{|4F+2O|}: S_2 \xrightarrow{2:1} \varphi_{|4F+2O|}(S_2)$$

is a double cover, where $\varphi_{|4F+2O|}(S_2)$ is the cone over a rational normal curve of degree 4 in $\mathbb{P}^5$. Blowing up the vertex of $\varphi_{|4F+2O|}(S_2)$ we obtain a surface isomorphic to the Hirzebruch surface $\mathbb{F}_4$. The involution ι_2 is the associated cover involution; this means that ι_2^* acts as the identity on $H^0(S_2, 4F+2O)^\vee$.

6C. We observe that the divisors h, H, F and $4F+2O$ are invariant for the action of ι_i for some i. So, by Corollary 5.4, we get the following:

Proposition 6.1. *Let Y and the divisors on Y be as above, then*:

(1) *The map*

$$\varphi_{|(h+F)_Y|}:\quad \begin{array}{ccc} Y & \longrightarrow & \mathbb{P}^5 \\ & \searrow \quad \nearrow_{\sigma_{2,1}} & \\ & \mathbb{P}^2\times\mathbb{P}^1 & \end{array}$$

is an elliptic fibration on the image of $\mathbb{P}^2\times\mathbb{P}^1$ *by the Segre embedding.*

(2) *The map*

$$\varphi_{|(H+F)_Y|}:\quad \begin{array}{ccc} Y & \longrightarrow & \mathbb{P}^{19-2n} \\ & \searrow \quad \nearrow_{\sigma_{9-n,1}} & \\ & \mathbb{P}^{9-n}\times\mathbb{P}^1 & \end{array}$$

is the same elliptic fibration as in (1) *with a different projective model of the basis*, *i.e.*, *the image of $dP\times\mathbb{P}^1$ via $\sigma_{9-n,1}$.*

(3) *The map*

$$\varphi_{|(h+(4F+2O))_Y|}:\quad \begin{array}{ccc} Y & \longrightarrow & \mathbb{P}^{17} \\ & \searrow \quad \nearrow_{\sigma_{2,5}} & \\ & \mathbb{P}^2\times\mathbb{P}^5 & \end{array}$$

is a generically $2:1$ *map onto its image contained in* $\sigma_{2,5}(\mathbb{P}^2\times\mathbb{P}^5)$.

(4) *The map*

$$\varphi_{|(H+(4F+2O))_Y|}: \quad Y \longrightarrow \mathbb{P}^{59-6n}, \qquad Y \searrow \mathbb{P}^{9-n}\times\mathbb{P}^5 \overset{\sigma_{9-n,5}}{\nearrow} \mathbb{P}^{59-6n}$$

is birational onto its image contained in $\sigma_{9-n,5}(\mathbb{P}^{9-n}\times\mathbb{P}^5)$.

Proof. The points (1) and (2) are proved in Section 6D. The points (3) and (4) are proved in Section 6E. □

Proposition 6.2. *Using the same notation as for Lemma 5.5 we have*:

(1) $\varphi_{|\delta_h|}: Y\to\mathbb{P}^2$ *is an isotrivial fibration in* $K3$ *surfaces whose generic fiber is isomorphic to* S_2.

(2) $\varphi_{|\delta_H|}: Y\to\mathbb{P}^{9-n}$ *is the same fibration as in* (1) *with a different projective model of the basis.*

(3) $\varphi_{|\delta_F|}: Y\to\mathbb{P}^1$ *is a fibration in Calabi–Yau* 3*-folds whose generic fiber is the Borcea–Voisin of the* $K3$ *surface* S_1 *and the elliptic fiber of the fibration* π.

(4) $\varphi_{|\delta_{4F+2O}|}: Y\to\mathbb{P}^5$ *is an isotrivial fibration in* $K3$ *surfaces whose generic fiber is isomorphic to* S_1.

Proof. The proof is explained in Section 6D, where all the previous maps are described in detail. □

6D. *Fibrations on Y*. As the natural map $\rho'\times\pi: S_1\times S_2\to\mathbb{P}^2\times\mathbb{P}^1$ satisfies $(\rho\times\pi)\circ\iota=\rho\times\pi$, we have an induced map $X\to\mathbb{P}^2\times\mathbb{P}^1$. The composition of this map with the resolution $Y\to X$ and with the two projections then gives

(1) an elliptic fibration $\mathcal{E}: Y\to\mathbb{P}^2\times\mathbb{P}^1$,

(2) a K3-fibration $\mathcal{G}: Y\to\mathbb{P}^2$,

(3) a fibration in elliptically fibered 3-folds $\mathcal{H}: Y\to\mathbb{P}^1$.

We describe these fibrations:

(1) The map $\mathcal{E}: Y\to\mathbb{P}^2\times\mathbb{P}^1$ is induced by the divisor $(h+F)_Y$ since $\varphi_{|h|}: S_1\to\mathbb{P}^2$ and $\varphi_{|F|}: S_2\to\mathbb{P}^1$. We already described the properties and the singular fibers for this fibration in Section 3D.

The composition of $\varphi_{|H|}(S_1)$ and the projection to the invariant subspace of $\mathbb{P}^{10-n}$ exhibits S_1 as the double cover of the del Pezzo surface dP anticanonically embedded in $\mathbb{P}^{9-n}$. The del Pezzo surface dP is the blow up of $\mathbb{P}^2$ in n points and the double cover $S_1\to dP$ corresponds (after the blow up) to the double cover $\varphi_{|h|}: S_1\to\mathbb{P}^2$ since $H=3h-\sum_{i=1}^n R_i$. Thus, the map $\varphi_{|(H+F)_Y|}$ is the same fibration as $\varphi_{|(h+F)_Y|}$, with a different model for the basis (which is now $dP\times\mathbb{P}^1$).

(2) The map $\mathcal{G} : Y \to \mathbb{P}^2$ is induced by δ_h. The fibers of these fibrations are isomorphic to S_2 since we have the commutative diagram

$$\begin{array}{ccccc} S_1\times S_2 & \longrightarrow & S_1 & & \\ \big\downarrow{\scriptstyle /\iota_1\times\iota_2} & & \big\downarrow{\scriptstyle\rho} & \searrow^{\varphi_{|h|}} & \\ X & \longrightarrow & dP & \longrightarrow & \mathbb{P}^2 \end{array}$$

The singular fibers of $\mathcal{G}$ lie over the branch curve $C \subset \mathbb{P}^2$ of the double cover $S_1 \to \mathbb{P}^2$. Let $P \in C$. It is easy to see that $(\rho' \times \pi)^{-1}(pr_{\mathbb{P}^2}^{-1}(P))$ is given by $P \times S_2$, and so in the quotient X we see a surface isomorphic to S_2/ι_2, which is a surface obtained from $\mathbb{F}_4$ by means of blow ups. Moreover, under the blow up $Y \to X$ we add a certain number of ruled surfaces: these are all disjoint from each other, and meet the blow up of $\mathbb{F}_4$ on the base curve of the rulings, i.e., on the section O, on the trisection T and possibly on the rational fixed components E_i (which are necessarily contained in reducible not-reduced fibers).

For the same reason as above, $\varphi_{|\delta_H|}$ is the fibration $\mathcal{G}$ with a different description of the basis.

(3) The fibration $\mathcal{H}$ is induced by δ_F. For every $t \in \mathbb{P}^1$, we denote by F_t the elliptic fiber of $S_2 \to \mathbb{P}^1$ over t. The inclusion $S_1 \times F_t \subset S_1 \times S_2$ induces

$$\begin{array}{ccccccc} & & S_1\times F_t & \hookrightarrow & S_1\times S_2 & & \\ & & \big\downarrow{\scriptstyle /\iota_1\times(\iota_2)_{|F_t}} & & \big\downarrow{\scriptstyle /\iota_1\times\iota_2} & & \\ BV(S_1,F_t) & \longrightarrow & (S_1\times F_t)/(\iota_1\times(\iota_2)_{|F_t}) & \hookrightarrow & X & \longleftarrow & Y \end{array}$$

So the fibers of φ_{δ_F} are Borcea–Voisin Calabi–Yau 3-folds which are elliptically fibered by definition. The singular fibers lie on $\Delta(\pi)$.

(4) Moreover, there is another K3-fibration. Indeed, the map $\varphi_{|\delta_{4F+2O}|}$ gives an isotrivial fibration in K3 surfaces isomorphic to S_1 and with basis the cone over the rational normal curve in $\mathbb{P}^4$, by the diagram

$$\begin{array}{ccccc} S_1\times S_2 & \longrightarrow & S_2 & & \\ \big\downarrow{\scriptstyle /\iota_1\times\iota_2} & & \big\downarrow{\scriptstyle /\iota_2} & & \\ X & \longrightarrow & (S_2/\iota_2) & \longrightarrow & \mathbb{P}^5 \end{array}$$

6E. ***Projective models.*** By the diagram

$$\begin{array}{ccccc} & & S_1\times S_2 & \xrightarrow[4:1]{\varphi_{|h|}\times\varphi_{|4F+2O|}} & \mathbb{P}^2\times\mathbb{P}^5 \xhookrightarrow[\sigma_{2,5}]{} \mathbb{P}^{17} \\ & & \big\downarrow{\scriptstyle 2:1} & \nearrow_{2:1} & \\ Y & \longrightarrow & X & & \end{array}$$

we can describe the map induced by the linear system $|(h+4F+2O)_Y|$ on Y as a double cover of the image (under the Segre embedding of the ambient spaces) of $\varphi_{|h|}(S_1)\times\varphi_{|4F+2O|}(S_2)$, which is the product of $\mathbb{P}^2$ with the cone over the rational normal curve of degree 4. This map is generically $2:1$, and its branch locus is given by the union of the product of the sextic curve in $\mathbb{P}^2$ with the vertex of the cone (the fiber over such points is a curve) and the product of the sextic with the trisection; the generic fiber is a single point, but there may be points where the fiber is a curve. The last case occurs only if the fibration $\pi: S_2\to\mathbb{P}^1$ has reducible nonreduced fibers.

To describe the map induced by $|(H+4F+2O)_Y|$ we use the diagram

$$\begin{array}{ccccccc} & & S_1\times S_2 & \xrightarrow[2:1]{\varphi_{|H|}\times\varphi_{|4F+2O|}} & \mathbb{P}^{10-n}\times\mathbb{P}^5 & \xhookrightarrow[\sigma_{10-n,5}]{} & \mathbb{P}^{65-6n}\\ & & {\scriptstyle 2:1}\downarrow & & \downarrow & & \\ Y & \longrightarrow & X & \xrightarrow[1:1]{\varphi_{|(H+4F+2O)_Y|}} & \mathbb{P}^{9-n}\times\mathbb{P}^5 & \xhookrightarrow[\sigma_{9-n,5}]{} & \mathbb{P}^{59-6n}\end{array}$$

where

$$\mathbb{P}^{10-n}\times\mathbb{P}^5\to\mathbb{P}^{9-n}\times\mathbb{P}^5$$

is induced by the projection of $\mathbb{P}^{10-n}=\mathbb{P}(H^0(S_1,H)^\vee)$ to $\mathbb{P}(H^0(S_1,H)^\vee_{+1})$. Recall that H is an ample divisor on S_1 (indeed, it is very ample), so the image of $\varphi_{|H|}\times\varphi_{|4F+2O|}$ is the product of S_1 and the cone over the rational normal curve of degree 4. Observe that generically this map is $2:1$, and so it descends to a $1:1$ map on X and on Y. So $\varphi_{|(H+4F+2O)_Y|}$ maps Y onto the product of dP with the cone over the rational normal curve of degree 4.

7. Explicit equations of *Y*

The aim of this section is to give some explicit equations for the projective models described above, in terms of the corresponding equations for S_i.

With a slight abuse, in this section we will substitute $\mathbb{F}_4$ to its singular model as the cone on the rational normal curve of degree 4. In this way we will obtain better models for Y.

7A. If S_1 is the double cover of $\mathbb{P}^2_{(x_0:x_1:x_2)}$ we assume its equation to be

$$w^2=f_6(x_0:x_1:x_2) \tag{6}$$

so that the curve C is $V(f_6(x_0:x_1:x_2))$. We assume that C is irreducible, even if some of the following results can be easily generalized. The cover involution ι_1 acts as $(w;(x_0:x_1:x_2))\mapsto(-w;(x_0:x_1:x_2))$.

If S_1 is the double cover of $\mathbb{P}^1_{(x_0:x_1)}\times\mathbb{P}^1_{(x_2:x_3)}$ we assume its equation to be

$$w^2=f_{4,4}((x_0:x_1),(x_2:x_3)) \tag{7}$$

so that the curve C is $V(f_{4,4}((x_0:x_1),(x_2:x_3)))$.

In the following we give the details of our computations under the assumption that S_1 is a cover of $\mathbb{P}^2$, and we only state the main results in the case where S_1 is a cover of $\mathbb{P}^1 \times \mathbb{P}^1$.

7B. Before giving the description of S_2, we make a little digression on the Weierstrass equation of an elliptic fibration. In particular, let $Y \to V$ be an elliptic fibration and

$$y^2 = x^3 + Ax + B \tag{8}$$

be an equation for its Weierstrass model. The condition that Y is a Calabi–Yau variety is equivalent to

$$A \in H^0(V, -4K_V), \qquad B \in H^0(V, -6K_V).$$

The discriminant Δ is then an element of $H^0(V, -12K_V)$.

In particular if V is $\mathbb{P}^m$ (resp. $\mathbb{P}^n \times \mathbb{P}^m$), the functions A, B and Δ are homogeneous polynomials of degree $4m+4$, $6m+6$ and $12m+12$ (resp. of bidegree $(4n+4, 4m+4)$, $(6n+6, 6m+6)$ and $(12n+12, 12m+12)$).

We observe that, if V is $\mathbb{P}^m$ (resp. $\mathbb{P}^n \times \mathbb{P}^m$), requiring that all the singular fibers of the elliptic fibration (8) are of type I_5 implies that $m \equiv 4 \mod 5$ (resp. $n \equiv 4 \mod 5$ and $m \equiv 4 \mod 5$). When V is a 3-fold, this gives a stronger version of Remark 3.2.

7C. Let S_2 be the elliptic K3 surface whose Weierstrass equation is

$$y^2 = x^3 + A(t:s)x + B(t:s), \tag{9}$$

where (according to the previous section) $A(t:s)$ and $B(t:s)$ are homogeneous polynomials of degree 8 and 12, respectively. For generic choices of $A(t:s)$ and $B(t:s)$, the elliptic fibration (9) has 24 nodal curves as unique singular fibers. For specific choices one can obtain other singular and reducible fibers. The cover involution ι_2 acts as

$$(y, x; (t:s)) \mapsto (-y, x; (t:s)).$$

Equivalently S_2 is the double cover of the Hirzebruch surface $\mathbb{F}_4$ given by

$$u^2 = z(x^3 + A(t:s)xz^2 + B(t:s)z^3), \tag{10}$$

where the coordinates (t, s, x, z) are the homogeneous toric coordinates of $\mathbb{F}_4$; see, e.g., [Cattaneo and Garbagnati 2016, §2.3]. The action of ι_2 on these coordinates is $(u, t, s, x, z) \mapsto (-u, t, s, x, z)$. Observe that the curve on $\mathbb{F}_4$ defined by

$$z(x^3 + A(t:s)xz^2 + B(t:s)z^3) = 0$$

is linearly equivalent to $-2K_{\mathbb{F}_4}$.

7C1. The choice of particular polynomials in (9) is associated to the choice of particular fibers of the fibration. Indeed, this elliptic fibration has an I_5-fiber in $(\bar{t}:\bar{s})$ if and only if the following three conditions hold:

(1) $A(\bar{t}:\bar{s})\neq 0$.

(2) $B(\bar{t}:\bar{s})\neq 0$.

(3) Δ vanishes of order 5 in $(\bar{t}:\bar{s})$, where $\Delta := 4A^3+27B^2$.

Up to standard transformations one can assume that the fiber of type I_5 is over $t=0$ and

$$A(t:s) := t^8 + \sum_{i=1}^{7} a_i t^i s^{8-i} - 3s^8,$$

$$\begin{aligned} B(t:s) := b_{12}t^{12} + \sum_{i=5}^{11} b_i t^i s^{12-i} + \left(-a_4 + \frac{a_1^4}{1728} + \frac{a_3a_1}{6} + \frac{a_2^2}{12} + \frac{a_2a_1^2}{72}\right)t^4s^8 \\ + \left(-a_3 + \frac{a_2a_1}{6} + \frac{a_1^3}{216}\right)t^3s^9 + \left(-a_2 + \frac{a_1^2}{12}\right)t^2s^{10} - a_1t^1s^{11} + 2s^{12}. \end{aligned}$$

We observe that the polynomials $A(t:s)$ and $B(t:s)$ depend on 14 parameters and, indeed, 14 is exactly the dimension of the family of K3 surfaces whose generic member has an elliptic fibration with one fiber of type I_5.

We already noticed that an elliptic fibration on a K3 surface has at most four fibers of type I_5 and indeed there are two distinct families of K3 surfaces with this property: the Mordell–Weil group of the generic member of one of these families is trivial, the one of the other is $\mathbb{Z}/5\mathbb{Z}$, [Shimada 2000, Case 2345, Table 1].

The K3 surfaces of the latter family are known to be double covers of the extremal rational elliptic surface [1, 1, 5, 5] whose Mordell–Weil group is $\mathbb{Z}/5\mathbb{Z}$; see [Schütt and Shioda 2010, Section 9.1] for the definition of the extremal rational elliptic surface. By this property, it is easy to find the Weierstrass equation of the K3 surface (as described in [Balestrieri et al. 2018, Section 4.2.2]). Indeed, the equation of the rigid rational fibration over $\mathbb{P}^1_{(\mu)}$ is

$$y^2 = x^3 + A(\mu)x + B(\mu), \tag{11}$$

where

$$\begin{aligned} A(\mu) &:= -\tfrac{1}{48}\mu^4 - \tfrac{1}{4}\mu^3\lambda - \tfrac{7}{24}\mu^2\lambda^2 + \tfrac{1}{4}\mu\lambda^3 - \tfrac{1}{48}\lambda^4, \\ B(\mu) &:= \tfrac{1}{864}\mu^6 + \tfrac{1}{48}\mu^5\lambda + \tfrac{25}{288}\mu^4\lambda^2 + \tfrac{25}{288}\mu^2\lambda^4 - \tfrac{1}{48}\mu\lambda^5 + \tfrac{1}{864}\lambda^6. \end{aligned}$$

In order to obtain the two-dimensional family of K3 surfaces we are looking for, it suffices to apply a base change of order two $f:\mathbb{P}^1_{(t:s)}\to\mathbb{P}^1_{(\mu:\lambda)}$ to the rational elliptic surface. In particular, if f branches over $(p_1:1)$ and $(p_2:1)$, the base change $\mu = p_1t^2 + s^2$, $\lambda = t^2 + s^2/p_2$ produces the required K3 surface if the fibers over $(p_1 : 1)$ and $(p_2 : 1)$ of the rational elliptic surface are smooth.

7D. ***The elliptic fibration $\mathcal{E}$.*** Let us now consider (6) for S_1 and (9) for S_2. The action of $\iota_1 \times \iota_2$ on $S_1 \times S_2$ leaves the functions $Y := yw^3$, $X := xw^2$, x_0, x_1, x_2, t, s invariant. Hence an equation for a birational model of Y expressed in these coordinates is

$$Y^2 = X^3 + A(t:s) f_6^2(x_0 : x_1 : x_2) X + B(t:s) f_6^3(x_0 : x_1 : x_2). \tag{12}$$

The previous equation is a Weierstrass form for the elliptic fibration

$$\mathcal{E} : Y \to \mathbb{P}^2_{(x_0:x_1:x_2)} \times \mathbb{P}^1_{(t:s)}.$$

Observe that the coefficient $A(t:s) f_6^2(x_0 : x_1 : x_2)$ and $B(t:s) f_6^3(x_0 : x_1 : x_2)$ are bihomogeneous on $\mathbb{P}^2 \times \mathbb{P}^1$ of bidegree (12, 8) and (18, 12) respectively, so by Section 7B we have another proof that the total space of the elliptic fibration $\mathcal{E}$ is indeed a Calabi–Yau variety.

One can check the properties of this fibration described in Section 3D directly by the computation of the discriminant of the Weierstrass equation (12), indeed

$$\Delta(\mathcal{E}) = f_6^6(x_0 : x_1 : x_2)(4A^3(t:s) + 27B^2(t:s)) = f_6^6(x_0 : x_1 : x_2)\Delta(\pi).$$

In this birational model, the basis of the fibration is $\mathbb{P}^2 \times \mathbb{P}^1$ and the del Pezzo surface contained in the discriminant is the blow up of $\mathbb{P}^2$ in the singular points of $f_6(x_0 : x_1 : x_2)$. The singular fibers due to the factor $\Delta(\pi)$ in $\Delta(\mathcal{E})$ are not generically modified by the blow up of $\mathbb{P}^2$ in n points, so that over the generic point of $\mathbb{P}^2$ (and thus of the del Pezzo surface), the singular fibers of $\mathcal{E}$ correspond to singular fibers of π.

If the equation of S_1 is (7), the Weierstrass equation of $\mathcal{E}$ is

$$Y^2 = X^3 + A(t:s) f_{4,4}^2((x_0:x_1), (x_2:x_3))X + B(t:s) f_{4,4}^3((x_0:x_1), (x_2:x_3)). \tag{13}$$

In some special cases it is also possible to write more explicitly a Weierstrass form of this elliptic fibration with basis the product of the del Pezzo surface and $\mathbb{P}^1_{(t:s)}$, as we see in Sections 7D1 and 7D2.

Remark 7.1. A generalization of this construction produces 4-folds with Kodaira dimension equal to $-\infty$ (resp. > 0) with an elliptic fibration. Indeed, it suffices to consider S_2 which is no longer a K3 surface, but a surface with Kodaira dimension $-\infty$ (resp. > 0) admitting an elliptic fibration with basis $\mathbb{P}^1$. So the equation of S_2 is $y^2 = x^3 + A(t:s)x + B(t:s)$ with $\deg(A(t:s)) = 4m$ and $\deg(B(t:s)) = 6m$ for $m = 1$ (resp. $m > 2$). The surface S_2 admits the elliptic involution ι_2 and $(S_1 \times S_2)/\iota_1 \times \iota_2$ admits a Weierstrass equation analogous to (12) or to (13).

7D1. *The case $n=6$.* Let us assume that $C \subset \mathbb{P}^2$ has $n=6$ nodes in general position. In this case the del Pezzo surface dP has degree 3 and is canonically embedded as a cubic in $\mathbb{P}^3_{(y_0:y_1:y_2:y_3)}$. So it admits an equation of the form $g_3(y_0:y_1:y_2:y_3)=0$. The image of C under this embedding is the complete intersection of $g_3=0$ and a quadric $g_2(y_0:y_1:y_2:y_3)=0$ in $\mathbb{P}^3$.

The K3 surface S_1 is embedded by $\varphi_{|H|}$ in $\mathbb{P}^4(y_0:y_1:y_2:y_3:y_4)$ as the complete intersection of a cubic and a quadric, and since it is the double cover of dP, its equation is

$$\begin{cases} y_4^2 = g_2(y_0:y_1:y_2:y_3), \\ 0 = g_3(y_0:y_1:y_2:y_3). \end{cases} \tag{14}$$

The involution ι_1 acts on $\mathbb{P}^4$, changing only the sign of y_4.

With the same argument as before, this leads to the following equation for a birational model of Y:

$$\begin{cases} Y^2 = X^3 + A(t:s)g_2^2(y_0:y_1:y_2:y_3)X + B(t:s)g_2^3(y_0:y_1:y_2:y_3), \\ g_3(y_0:y_1:y_2:y_3)=0. \end{cases} \tag{15}$$

The first equation is the Weierstrass form of an elliptic fibration with basis $\mathbb{P}^3 \times \mathbb{P}^1$ and the second equation corresponds to restricting this equation to the del Pezzo surface embedded in the first factor (i.e., in $\mathbb{P}^3$).

Corollary 7.2. *The equation*

$$\begin{cases} Y^2 = X^3 + \left(\sum_{i=0}^{8} a_i t^i s^{8-i}\right) g_2^2(y_0:y_1:y_2:y_3)X + \left(\sum_{i=0}^{12} b_i t^i s^{12-i}\right) g_2^3(y_0:y_1:y_2:y_3), \\ g_3(y_0:y_1:y_2:y_3)=0, \end{cases}$$

where g_i is a homogenous polynomial of degree i in $\mathbb{C}[y_0:y_1:y_2:y_3]$,

$$a_0=-3, \quad b_0=2, \quad b_1=-a_1, \quad b_2=-a_2+\frac{a_1^2}{12},$$
$$b_3=-a_3+\frac{a_2a_1}{6}+\frac{a_1^3}{216} \qquad b_4=-a_4+\frac{a_1^4}{1728}+\frac{a_3a_1}{6}+\frac{a_2^2}{12}+\frac{a_2a_1^2}{72},$$

describes a birational model of a Calabi–Yau 4-fold with an elliptic fibration such that the fibers over the del Pezzo surface $(g_3(y_0:y_1:y_2:y_3)=0)\times(t=0)\subset\mathbb{P}^3\times\mathbb{P}^1_t$ are generically of type I_5.

The other singular fibers are described by the zeros of the discriminant

$$g_2^6(y_0:y_1:y_2:y_3)\left(4\left(\sum_{i=0}^{8} a_i t^i s^{8-i}\right)^3 + 27\left(\sum_{i=0}^{12} b_i t^i s^{12-i}\right)^2\right).$$

Remark 7.3. With the same process one obtains the equation of an elliptic fibration over $dP \times \mathbb{P}^1$ such that there are $m \leq 4$ del Pezzo surfaces in $dP \times \mathbb{P}^1$ over each of which the general fiber is of type I_5. To do this it suffices to specialize the coefficients a_i and b_i according to the conditions described in Section 7C1. In the case $m = 4$ there are two different specializations; one of them is associated to the presence of a 5-torsion section and its equation is the given in Section 7C1.

7D2. *The case $n = 5$.* The treatment of the case $n = 5$ is similar to that for $n = 6$. So let us assume that $C \subset \mathbb{P}^2$ has $n = 5$ nodes in general position. In this case the del Pezzo surface dP has degree 4 and is canonically embedded in $\mathbb{P}^4_{(y_0:y_1:y_2:y_3:y_4)}$ as the complete intersection of two quadrics $q_2 = 0$ and $q_2' = 0$. The image of C under this embedding is the complete intersection of the del Pezzo with a quadric $q_2'' = 0$.

The K3 surface S_1 is embedded by $\varphi_{|H|}$ in $\mathbb{P}^5(y_0 : y_1 : y_2 : y_3 : y_4 : y_5)$ as the complete intersection of three quadrics, and since it is the double cover of dP, its equation is

$$\begin{cases} y_5^2 = q_2''(y_0 : y_1 : y_2 : y_3 : y_4), \\ 0 = q_2'(y_0 : y_1 : y_2 : y_3 : y_4), \\ 0 = q_2(y_0 : y_1 : y_2 : y_3 : y_4). \end{cases} \tag{16}$$

The involution ι_1 acts on $\mathbb{P}^5$ changing only the sign of y_5.

Hence a birational model of Y is:

$$\begin{cases} Y^2 = X^3 + A(t:s)q_2''^2(y_0:y_1:y_2:y_3:y_4)X + B(t:s)q_2''^3(y_0:y_1:y_2:y_3:y_4), \\ q_2'(y_0:y_1:y_2:y_3:y_4) = 0, \\ q_2(y_0:y_1:y_2:y_3:y_4) = 0. \end{cases} \tag{17}$$

The first equation is the Weierstrass form of an elliptic fibration with basis $\mathbb{P}^4 \times \mathbb{P}^1$ and the other two equations restrict this equation to the del Pezzo surface embedded in the first factor (i.e., in $\mathbb{P}^4$).

Remark 7.4. It is possible to obtain explicit equations for the elliptic fibrations with fiber(s) of type I_5 as in Corollary 7.2.

7E. ***The double cover $Y \to \mathbb{P}^2 \times \mathbb{F}_4$.*** Let us consider the equations (6) for S_1 and (10) for S_2. The functions

$$W := uw, \quad x_0, \quad x_1, \quad x_2, \quad t, \quad s, \quad x, \quad z$$

are invariant for $\iota_1 \times \iota_2$ and they satisfy the equation

$$W^2 = f_6(x_0 : x_1 : x_2)z(x^3 + A(t:s)xz^2 + B(t:s)z^3). \tag{18}$$

This equation exhibits a birational model of Y as a double cover of the rational

4-fold $\mathbb{P}^2 \times \mathbb{F}_4$ branched over a divisor in $|-2K_{\mathbb{P}^2\times\mathbb{F}_4}|$. In particular this is the equation associated to the linear system $|(h+4F+2O)_Y|$.

The projections of (18) give different descriptions of projective models: the one associated to the linear system $|\delta_h|$ is obtained by the projection to $\mathbb{P}^2$; the one associated to $|\delta_{4F+2O}|$ is obtained by the projection to $\mathbb{F}_4 \subset \mathbb{P}^5$; the one associated to the linear system $|\delta_F|$ is obtained by the projection to $\mathbb{P}^1_{(t:s)}$.

Consider first the composition with the projection on $\mathbb{P}^2$ to obtain an equation for $\mathcal{G}$. Fix a point $(\bar{x}_0 : \bar{x}_1 : \bar{x}_2) \in \mathbb{P}^2$ and assume that $f_6(\bar{x}_0 : \bar{x}_1 : \bar{x}_2) \neq 0$. Then the corresponding fiber has equation

$$W^2 = f_6(\bar{x}_0 : \bar{x}_1 : \bar{x}_2)z(x^3 + A(t:s)xz^2 + B(t:s)z^3),$$

which is easily seen to be isomorphic to S_2 (substitute W with $\sqrt{f_6(\bar{x}_0 : \bar{x}_1 : \bar{x}_2)}W$ to find an equation equivalent to (10)).

Consider now the composition with the projection on $\mathbb{F}_4$. Fix a point $(\bar{t}, \bar{s}, \bar{x}, \bar{z}) \in \mathbb{F}_4$ which does not lie on the negative curve nor on the trisection. Then the corresponding fiber is

$$W^2 = f_6(x_0 : x_1 : x_2)\bar{z}(\bar{x}^3 + A(\bar{t}:\bar{s})\bar{x}\bar{z}^2 + B(\bar{t}:\bar{s})\bar{z}^3),$$

which is a K3 surface isomorphic to S_1.

Finally we give an equation for $\mathcal{H}$. Let us put $z = 1$ in (18) and perform the change of coordinates $w \mapsto w/f_6$, $x \mapsto x/f_6$. Multiplying the resulting equation by f_6^2, we obtain

$$w^2 = x^3 + A(t:s)f_6^2(x_0 : x_1 : x_2)x + B(t:s)f_6^3(x_0 : x_1 : x_2).$$

For every fixed $(\bar{t} : \bar{s}) \in \mathbb{P}^1$, this is the equation of a Calabi–Yau 3-fold of Borcea–Voisin type obtained from the K3 surface $w^2 = f_6(x_0 : x_1 : x_2)$ and the elliptic curve $y^2 = x^3 + A(\bar{t}:\bar{s})x + B(\bar{t}:\bar{s})$; see [Cattaneo and Garbagnati 2016, Section 4.4].

7E1. We now want to describe what happens if the sextic curve in $\mathbb{P}^2$ has $n = 6$ or $n = 5$ nodes.

Assume first that $\rho' : S_1 \to \mathbb{P}^2$ is branched along a sextic with 6 nodes. Then we can use (14) and (10) to describe S_1 and S_2, respectively, and using the same argument as before (i.e., put $W = y_4 u$) we obtain the equation

$$\begin{cases} W^2 = g_2(y_0 : y_1 : y_2 : y_3)z(x^3 + A(t:s)xz^2 + B(t:s)z^3), \\ 0 = g_3(y_0 : y_1 : y_2 : y_3), \end{cases}$$

which exhibits Y as the double cover of $dP \times \mathbb{F}^4$. Let us denote by $U \to \mathbb{P}^3 \times \mathbb{F}_4$ the double cover branched on $g_2(y_0 : y_1 : y_2 : y_3)z(x^3 + A(t:s)xz^2 + B(t:s)z^3)$. The branch divisor is $2H_{\mathbb{P}^3} - 2K_{\mathbb{F}_4}$ and so Y is a section of the anticanonical bundle of U.

With a further change of variables, where the only nonidentic transformations are $W' = g_2 W$ and $x' = g_2 x$, we then find the following equation for a birational model of Y (we drop the primes for simplicity of notation):

$$\begin{cases} W^2 = z(x^3 + A(t:s)g_2^2(y_0:y_1:y_2:y_3)xz^2 + B(t:s)g_2^3(y_0:y_1:y_2:y_3)z^3), \\ 0 = g_3(y_0:y_1:y_2:y_3). \end{cases}$$

Here the first equation gives an elliptic fibration over $\mathbb{P}^3 \times \mathbb{P}^1$ as a double cover, while the second restricts this fibration to $dP \times \mathbb{P}^1$.

Analogously, if $n = 5$, then S_1 and S_2 are described by (16) and (10), respectively, so that we have the following equation for Y:

$$\begin{cases} W^2 = q_2'' z(x^3 + Axz^2 + Bz^3), \\ 0 = q_2', \\ 0 = q_2, \end{cases}$$

with the same considerations as the case just treated.

7F. *An involution on Y.* By construction Y admits an involution ι induced by $\iota_1 \times \mathrm{id} \in \mathrm{Aut}(S_1 \times S_2)$ and acting as -1 on $H^{4,0}(Y)$. Since

$$\iota_1 \times \mathrm{id} = (\iota_1 \times \iota_2) \circ (\mathrm{id} \times \iota_2),$$

ι is equivalently induced by $\mathrm{id} \times \iota_2$. The involution ι has a clear geometric interpretation in several models described above. By Section 6E, Y is a $2:1$ cover of $\mathbb{P}^2 \times \mathbb{F}_4$ whose equation is given in (18). The involution ι is the cover involution, indeed it acts as -1 on the variable $W := uw$, and by (6) the map $\iota_1 \times \mathrm{id}$ acts as -1 on w.

By Section 6D, Y admits the elliptic fibration $\mathcal{E}$ whose equation is given in (12). The involution ι is the elliptic involution, indeed it acts as -1 on the variable $Y := yw^3$, and by (9) the map $\mathrm{id} \times \iota_2$ acts as -1 on y.

Hence Y/ι is birational to $\mathbb{P}^2 \times \mathbb{F}_4$ and admits a fibration in rational curves, whose fibers are the quotient of the fibers of the elliptic fibration $\mathcal{E}$.

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