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PARTIAL REGULARITY OF HARMONIC MAPS FROM A RIEMANNIAN MANIFOLD INTO A LORENTZIAN MANIFOLD

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We study the partial regularity theorem for stationary harmonic maps from a Riemannian manifold into a Lorentzian manifold. For a weakly stationary harmonic map (u, v) from a smooth bounded open domain $\Omega \subset \mathbb{R}^m$ to a Lorentzian manifold with Dirichlet boundary condition, we prove that it is smooth outside a closed set whose $(m-2)$-dimensional Hausdorff measure is zero. Moreover, if the target manifold N does not admit any harmonic spheres S^l, $l = 2, \dots, m-1$, we show (u, v) is smooth.

1. Introduction

Suppose (M, g) and (N, h_N) are two compact Riemannian manifolds of dimensions m and n respectively. For a map $u \in C^1(M, N)$, the energy functional of u is defined as

$$E(u) = \frac{1}{2}\int_M |\nabla u|^2 \, d\,\mathrm{vol}_g . \tag{1-1}$$

A critical point of the energy functional E is called a harmonic map. By Nash's embedding theorem, we can embed N isometrically into some Euclidean space $\mathbb{R}^K$ and the corresponding Euler–Lagrange equation is

$$\Delta_g u = A(u)(\nabla u, \nabla u),$$

where Δ_g is the Laplace–Beltrami operator on M with respect to g and A is the second fundamental form of $N \subset \mathbb{R}^K$.

Harmonic map is a very important notion in geometric analysis which has been widely studied in the past decades. Physically, harmonic maps come from the nonlinear sigma model, which plays an important role in quantum field and string theory. From the perspective of general relativity, it is natural to consider the targets of harmonic maps to be Lorentzian manifolds. Geometrically, the link between harmonic maps into S^4_1 and the conformal Gauss maps of Willmore surfaces in S^3

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also leads to such harmonic maps [Bryant 1984]. The work on minimal surfaces in anti-de Sitter spaces and its applications in theoretical physics also shows the importance of such maps [Alday and Maldacena 2009]. In this paper, we shall focus on the interior partial regularity of stationary harmonic maps from a compact Riemannian manifold of dimension m (≥ 3) into a Lorentzian manifold.

We now proceed to introduce the model. Let $N \times \mathbb{R}$ be a Lorentzian manifold equipped with a warped product metric

$$h = h_N - \beta (d\theta)^2,$$

where $(\mathbb{R}, d\theta^2)$ is the standard 1-dimensional Euclidean space and β is a positive smooth function on (N, h_N). Since N is compact, there exist positive constants λ_1 and λ_2 such that

$$0 < \lambda_1 \leq \beta(y) \leq \lambda_2 < \infty \quad \text{and} \quad |\nabla \beta(y)| \leq \lambda_2 \quad \text{for all } y \in N.$$

Set

$$W^{1,2}(M, N \times \mathbb{R}) := \{u \in W^{1,2}(M, \mathbb{R}^K),\ v \in W^{1,2}(M, \mathbb{R}) \mid u(x) \in N \text{ for a.e. } x \in M\}.$$

For $(u, v) \in W^{1,2}(M, N \times \mathbb{R})$, we consider the functional

$$E_h(u, v; M) = \frac{1}{2} \int_M \{|\nabla u|^2 - \beta(u)|\nabla v|^2\} \, d\,\mathrm{vol}_g, \tag{1-2}$$

which is called the Lorentzian energy of the map (u, v) on M. A critical point (u, v) of the functional (1-2) is called a harmonic map from (M, g) into the Lorentzian manifold $(N \times \mathbb{R}, h)$.

When the target manifold is a Lorentzian manifold, the existence of geodesics was studied in [Benci et al. 1991] and Greco [1993; 1997] constructed a smooth harmonic map via some developed variational methods. Recently, Han, Jost, Liu and Zhao [Han et al. 2019] investigated a parabolic-elliptic system for maps and got a global existence result by assuming either some geometric conditions on the target manifold or small energy of the initial maps. The result implies the existence of a harmonic map in a given homotopy class. The blowup behavior for Lorentzian harmonic maps was studied in [Han et al. 2017b], and for approximate Lorentzian harmonic maps and Lorentzian harmonic maps, flow from a Riemann surface was studied in [Han et al. 2019; 2017a]. For the global weak solution of Lorentzian harmonic map flow, one can refer to [Han et al. 2018]. The regularity theory was studied in [Isobe 1998; Zhu 2013] for dimension 2 and in [Isobe 1997] for higher dimensions on some kinds of minimal type solutions.

Via direct calculations, Zhu [2013] derived the Euler–Lagrange equations for (1-2),

$$\begin{cases} -\Delta u = A(u)(\nabla u, \nabla u) - B^{\top}(u)|\nabla v|^2 & \text{in } M, \\ -\operatorname{div}(\beta(u)\nabla v) = 0, & \text{in } M, \end{cases} \tag{1-3}$$

where A is the second fundamental form of N in $\mathbb{R}^K$, $B(u) := (B^1, B^2, \dots, B^K)$ with

$$B^j := -\frac{1}{2}\frac{\partial \beta(u)}{\partial y^j}$$

and $B^\top$ is the tangential part of B along the map u.

Definition 1.1. We call $(u, v) \in W^{1,2}(\Omega, N \times \mathbb{R})$ a weakly Lorentzian harmonic map with Dirichlet boundary data

$$(u, v)|_{\partial\Omega} = (\phi, \psi)$$

if it is a weak solution of (1-3) with boundary data (ϕ, ψ).

Similar to harmonic maps, we introduce the notion of stationary Lorentzian harmonic maps.

Definition 1.2. A weakly Lorentzian harmonic map $(u, v) \in W^{1,2}(\Omega, N \times \mathbb{R})$ is called a stationary Lorentzian harmonic map if it is also a critical point of E_h with respect to the domain variations; i.e., for any $Y \in C_0^\infty(\Omega, \mathbb{R}^m)$, it holds

$$\frac{d}{dt}\Big|_{t=0} \int_\Omega \frac{1}{2}(|\nabla u_t|^2 - \beta(u_t)|\nabla v_t|^2)\, d\,\mathrm{vol}_g = 0,$$

where $u_t(x) = u(x + tY(x))$ and $v_t(x) = v(x + tY(x))$.

Our first main result is the following small-energy regularity theorem.

Theorem 1.3. *For $m \geq 2$ and any $\alpha \in (0, 1)$, there exists an $\epsilon_0 > 0$ depending only on m, α and (N, h_N) such that if $(u, v) \in W^{1,2}(\Omega, N \times \mathbb{R})$ is a weakly Lorentzian harmonic map satisfying*

$$\sup_{x \in B_{r_0}(x_0),\, 0 < r \leq r_0} r^{2-m} \int_{B_r(x)} |\nabla u|^2 \, d\,\mathrm{vol}_g \leq \epsilon_0^2, \tag{1-4}$$

then $(u, v) \in C^\infty(B_{r_0/2}(x_0))$. Moreover, it satisfies the estimate

$$\begin{aligned} r_0\|\nabla u\|_{L^\infty(B_{r_0/2}(x_0))} + r_0\|\nabla v\|_{L^\infty(B_{r_0/2}(x_0))} + r_0^{1+\alpha}\|\nabla u\|_{C^\alpha(B_{r_0/2}(x_0))} \\ + r_0^{1+\alpha}\|\nabla v\|_{C^\alpha(B_{r_0/2}(x_0))} \\ \leq C\big(r_0^{1-m/2}\|(\nabla u, \nabla v)\|_{L^2(B_{r_0}(x_0))} + r_0^{2-m}\|(\nabla u, \nabla v)\|^2_{L^2(B_{r_0}(x_0))} \\ + r_0^{4-2m}\|\nabla v\|^4_{L^2(B_{r_0}(x_0))}\big), \end{aligned} \tag{1-5}$$

where $C = C(m, \lambda_1, \lambda_2, \alpha, N)$ is a positive constant and

$$\|(\nabla u, \nabla v)\|^2_{L^2(B_{r_0}(x_0))} := \|\nabla u\|^2_{L^2(B_{r_0}(x_0))} + \|\nabla v\|^2_{L^2(B_{r_0}(x_0))}.$$

In this paper, we can get the following interior partial regularity theorem. For a similar result for harmonic maps, one can refer to [Bethuel 1993; Evans 1991; Li and Tian 1998]. For results on gauge theory, one can refer to [Tian 2000].

Theorem 1.4. *For $m \geq 2$, let $(u, v) \in W^{1,2}(\Omega, N \times \mathbb{R})$ be a stationary Lorentzian harmonic map with Dirichlet boundary data $(u, v)|_{\partial\Omega} = (\phi, \psi)$, where $\psi \in C^1(\partial\Omega)$. Then there exists a closed subset $S(u) \subset \Omega$, with $H^{m-2}(S(u)) = 0$, such that $(u, v) \in C^\infty(\Omega \setminus S(u))$.*

Remark 1.5. The boundary assumption $\psi \in C^1(\partial\Omega)$ is used to derive the estimate $\|v\|_{W^{1,p}(\Omega)}$ for some $p > m$. See Lemma 2.1. In fact, by the classical theory of the Laplace operator and the following proof in this paper, one may find that it is enough to assume that $\psi \in W^{1-1/p,p}(\partial\Omega)$ for some $p > m$.

Furthermore, we have:

Theorem 1.6. *Under the same assumption as the above theorem, if N does not admit harmonic spheres S^l, $l = 2, \ldots, m-1$, then (u, v) is smooth.*

To prove the partial regularity results, we first need to establish the monotonicity formula for stationary Lorentzian harmonic maps. Thanks to the elliptic estimates of the v-equation of divergence forms, we can control the additional terms (corresponding to harmonic maps) in the monotonicity formula. Secondly, we need to study the energy concentration set of a blow-up sequence of stationary Lorentzian harmonic maps. Here, we follow Lin's scheme [1999] to get the first bubble which is a nonconstant harmonic sphere. The proof is based on the analysis of defect measure using geometric measure theory.

The rest of paper is organized as follows. In Section 2, we establish the monotonicity formula for stationary Lorentzian harmonic maps which is crucial in the proof of our main theorems. In Section 3, we prove the small-energy regularity theorem, Theorem 1.3, and then the partial regularity theorem, Theorem 1.4, follows immediately from a standard monotonicity formula argument. Theorem 1.6 will be proved in Section 4.

2. Monotonicity formula

In this section, we firstly derive the monotonicity formula for stationary Lorentzian harmonic maps. Secondly, for reader's convenience, we recall a regularity theorem in [Sharp 2014] which will be used in the proof.

Thanks to the divergence structure of v-equation, we have the following estimate.

Lemma 2.1. *Let $(u, v) \in W^{1,2}(\Omega, N \times \mathbb{R})$ be a weakly Lorentzian harmonic map with Dirichlet boundary data (ϕ, ψ), where $\psi \in C^1(\partial\Omega)$. Then $v \in W^{1,p}(\Omega)$ for any $1 < p < \infty$ and*

$$\|\nabla v\|_{L^p} \leq C(p, \lambda_1, \lambda_2, \Omega)\|\psi\|_{C^1(\partial\Omega)}. \tag{2-1}$$

Proof. Let v be the unique smooth solution of the equation

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v(x) = \psi & \text{on } \partial\Omega, \end{cases}$$

which satisfies

$$\|v\|_{C^1(\bar{\Omega})} \le C(\Omega)\|\psi\|_{C^1(\partial\Omega)}.$$

We call v an extension of ψ and for simplicity, we still denote it by $\psi \in C^1(\bar{\Omega})$. It is easy to see that $v - \psi \in W_0^{1,2}(\Omega)$ is a weak solution of

$$-\operatorname{div}(\beta(u)\nabla(v-\psi)) = \operatorname{div}(\beta(u)\nabla\psi).$$

By the standard theory of the second elliptic operator of divergence forms, see Theorem 1 in [Meyers 1963], we obtain that $v \in W^{1,p}$ for any $1 < p < \infty$ and satisfies

$$\|\nabla v\|_{L^p} \le C(p, \lambda_1, \lambda_2, \Omega)\|\nabla\psi\|_{L^p} \le C(p, \lambda_1, \lambda_2, \Omega)\|\psi\|_{C^1(\partial\Omega)}. \qquad \square$$

Next, we derive the stationary identity for stationary Lorentzian harmonic maps.

Lemma 2.2. *Let $(u, v) \in W^{1,2}(\Omega, N \times \mathbb{R})$ be a weakly Lorentzian harmonic map. Then (u, v) is stationary if and only if for any $Y \in C_0^\infty(\Omega, \mathbb{R}^m)$, there holds*

$$\int_\Omega \left(\left\langle \frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\gamma} \right\rangle - \beta(u) \left\langle \frac{\partial v}{\partial x^\alpha}, \frac{\partial v}{\partial x^\gamma} \right\rangle - \frac{1}{2}(|\nabla u|^2 - \beta(u)|\nabla v|^2)\delta_{\alpha\gamma} \right) \frac{\partial Y^\gamma}{\partial x^\alpha}\, dx = 0. \tag{2-2}$$

Proof. For any $Y \in C_0^\infty(\Omega, \mathbb{R}^m)$, let $t \in \mathbb{R}$ small enough and $y = F_t(x) := x + tY(x)$ and $x = F_t^{-1}(y)$. By Definition 1.2, (u, v) is stationary if and only if

$$\frac{d}{dt}\Big|_{t=0} \int_\Omega \frac{1}{2}(|\nabla u_t|^2 - \beta(u_t)|\nabla v_t|^2)\, dx = 0,$$

where $u_t(x) = u(F_t(x))$ and $v_t(x) = v(F_t(x))$.

On the one hand, by a standard calculation, see, e.g., [Lin and Wang 2008], we have

$$\frac{d}{dt}\Big|_{t=0} \frac{1}{2} \int_\Omega |\nabla u_t|^2\, dx = \int_\Omega \left(\left\langle \frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\gamma} \right\rangle - \frac{1}{2}|\nabla u|^2 \delta_{\alpha\gamma} \right) \frac{\partial Y^\gamma}{\partial x^\alpha}\, dx. \tag{2-3}$$

On the other hand, computing directly, we obtain

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} &\left(\frac{1}{2}\beta(u_t)|\nabla v_t|^2 \right) \\ &= \frac{1}{2}\frac{\partial\beta(u)}{\partial x^\alpha} Y^\alpha |\nabla v|^2 + \beta(u) \left\langle \frac{\partial v}{\partial x^\alpha}, \frac{\partial v}{\partial x^\gamma} \right\rangle \frac{\partial Y^\gamma}{\partial x^\alpha} + \beta(u) \left\langle \frac{\partial^2 v}{\partial x^\alpha \partial x^\gamma}, \frac{\partial v}{\partial x^\gamma} \right\rangle Y^\alpha \\ &= \frac{1}{2}\frac{\partial}{\partial x^\alpha}(\beta(u)|\nabla v|^2) Y^\alpha + \beta(u) \left\langle \frac{\partial v}{\partial x^\alpha}, \frac{\partial v}{\partial x^\gamma} \right\rangle \frac{\partial Y^\gamma}{\partial x^\alpha}. \end{aligned}$$

Thus,

$$\frac{d}{dt}\Big|_{t=0}\frac{1}{2}\int_\Omega \beta(u_t)|\nabla v_t|^2\,dx=\int_\Omega \beta(u)\Big(\Big\langle\frac{\partial v}{\partial x^\alpha},\frac{\partial v}{\partial x^\gamma}\Big\rangle-\frac{1}{2}|\nabla v|^2\delta_{\alpha\gamma}\Big)\frac{\partial Y^\gamma}{\partial x^\alpha}\,dx. \tag{2-4}$$

Combining (2-3) with (2-4), we will get the conclusion of the lemma. □

Now, we can derive the monotonicity formula for stationary Lorentzian harmonic maps.

Lemma 2.3. *Let* $(u,v)\in W^{1,2}(\Omega, N\times\mathbb{R})$ *be a stationary Lorentzian harmonic map. Then for any* $x_0\in\Omega$ *and* $0<r_1\le r_2<\operatorname{dist}(x_0,\partial\Omega)$, *there holds*

$$\begin{aligned} r_2^{2-m}\int_{B_{r_2}(x_0)}(|\nabla u|^2-\beta(u)|\nabla v|^2)\,dx-r_1^{2-m}\int_{B_{r_1}(x_0)}(|\nabla u|^2-\beta(u)|\nabla v|^2)\,dx \\ =2\int_{B_{r_2}(x_0)\setminus B_{r_1}(x_0)}|x-x_0|^{2-m}\Big(\Big|\frac{\partial u}{\partial r}\Big|^2-\beta(u)\Big|\frac{\partial v}{\partial r}\Big|^2\Big)\,dx, \end{aligned} \tag{2-5}$$

where $\partial_r=\partial/\partial r=\partial/\partial|x-x_0|$.

Proof. For simplicity, we assume $x_0=0\in\Omega$. For any $\epsilon>0$ and $0<r<\operatorname{dist}(0,\partial\Omega)$, let $\varphi_\epsilon(x)=\varphi_\epsilon(|x|)\in C_0^\infty(B_r)$ be such that

$$0\le\varphi_\epsilon(x)\le 1 \quad\text{and}\quad \varphi_\epsilon(x)|_{B_{(1-\epsilon)r}}=1.$$

Taking $Y(x)=x\varphi_\epsilon(x)$ in the formula (2-2) and noting that

$$\frac{\partial Y^\gamma}{\partial x^\alpha}=\varphi_\epsilon(x)\delta_{\alpha,\gamma}+\frac{x^\alpha x^\gamma}{|x|}\varphi'_\epsilon(x),$$

we have

$$\begin{aligned}\Big(1-\frac{m}{2}\Big)\int_{B_r}(|\nabla u|^2-\beta(u)|\nabla v|^2)\varphi_\epsilon(x)\,dx \\ =\int_{B_r}\Big(-\Big|\frac{\partial u}{\partial r}\Big|^2+\beta(u)\Big|\frac{\partial v}{\partial r}\Big|^2+\frac{1}{2}(|\nabla u|^2-\beta(u)|\nabla v|^2)\Big)|x|\varphi'_\epsilon(x)\,dx.\end{aligned}$$

Letting $\epsilon\to 0$, we get

$$\begin{aligned}(2-m)\int_{B_r}(|\nabla u|^2-\beta(u)|\nabla v|^2)\,dx+r\int_{\partial B_r}(|\nabla u|^2-\beta(u)|\nabla v|^2) \\ =2r\int_{\partial B_r}\Big(\Big|\frac{\partial u}{\partial r}\Big|^2-\beta(u)\Big|\frac{\partial v}{\partial r}\Big|^2\Big),\end{aligned}$$

which yields

$$\frac{d}{dr}\Big(r^{2-m}\int_{B_r}(|\nabla u|^2-\beta(u)|\nabla v|^2)\,dx\Big)=r^{2-m}\int_{\partial B_r}\Big(\Big|\frac{\partial u}{\partial r}\Big|^2-\beta(u)\Big|\frac{\partial v}{\partial r}\Big|^2\Big).$$

The conclusion of the lemma follows by integrating r from r_1 to r_2. □

As a direct corollary of above monotonicity formula, we have:

Corollary 2.4. *Let* $(u, v) \in W^{1,2}(\Omega, N \times \mathbb{R})$ *be a stationary Lorentzian harmonic map with Dirichlet boundary data* (ϕ, ψ)*. Then for any* $x_0 \in \Omega$ *and* $0 < r_1 \le r_2 < \operatorname{dist}(x_0, \partial\Omega)$*, there holds*

$$r_1^{2-m}\int_{B_{r_1}(x_0)} |\nabla u|^2\,dx \\ \le r_2^{2-m}\int_{B_{r_2}(x_0)} |\nabla u|^2 dx + C(m, p, \lambda_1, \lambda_2, \Omega, \|\psi\|_{C^1(\partial\Omega)})(r_2)^{2-2m/p}.$$

Proof. By Lemma 2.3, we have

$$\begin{aligned} r_1^{2-m}&\int_{B_{r_1}(z)} |\nabla u|^2\,dx \\ &\le r_2^{2-m}\int_{B_{r_2}(x_0)} (|\nabla u|^2 - \beta(u)|\nabla v|^2)\,dx + r_1^{2-m}\int_{B_{r_1}(x_0)} \beta(u)|\nabla v|^2\,dx \\ &\qquad + 2\int_{B_{r_2}(x_0)} |x-x_0|^{2-m}\beta(u)\Big|\frac{\partial v}{\partial|x-x_0|}\Big|^2\,dx \\ &\le r_2^{2-m}\int_{B_{r_2}(x_0)} |\nabla u|^2\,dx + C(m, \lambda_2)(r_2)^{2-2m/p}\|\nabla v\|^2_{L^p} \\ &\le r_2^{2-m}\int_{B_{r_2}(x_0)} |\nabla u|^2\,dx + C(m, p, \lambda_1, \lambda_2, \Omega, \|\psi\|_{C^1(\partial\Omega)})(r_2)^{2-2m/p}, \end{aligned}$$

where the second inequality follows from Young's inequality:

$$\begin{aligned} \int_{B_r} |x|^{2-m}|\nabla v|^2\,dx &\le \|\nabla v\|^2_{L^p}\,\||x|^{2-m}\|_{L^{p/(p-2)}(B_r)} \\ &\le C(m, p, \lambda_1, \lambda_2, \Omega, \|\psi\|_{C^1(\partial\Omega)})(r)^{2-2m/p}. \end{aligned} \tag{2-6}$$ □

In the end of this section, we want to recall a regularity theorem for a system of critical PDE in [Sharp 2014]. Systems of this form were introduced and studied by [Rivière and Struwe 2008]. For this, let us first recall the definition of Morrey spaces; see [Giaquinta 1983].

Definition 2.5. For $p \ge 1,\ 0 < \mu \le m$, and a domain $U \subset \mathbb{R}^m$, the Morrey space $M^{p,\mu}(U)$ is defined by

$$M^{p,\mu}(U) := \{f \in L^p_{\mathrm{loc}}(U) \mid \|f\|_{M^{p,\mu}(U)} < \infty\},$$

where

$$\|f\|^p_{M^{p,\mu}(U)} := \sup_{B_r\subset U} r^{\mu-m}\int_{B_r} |f|^p.$$

Theorem 2.6 [Sharp 2014, Theorem 1.2]. *For every* $m \ge 2$ *and* $p \in \left(\frac{m}{2}, m\right)$*, there exists* $\epsilon = \epsilon(m, d, p) > 0$ *and* $C = C(m, d, p) > 0$ *with the following property.*

Suppose that $u \in W^{1,2}(B_1, \mathbb{R}^d)$, $\nabla u \in M^{2,2}(B_1, \mathbb{R}^d)$, $\Omega \in M^{2,2}(B_1, \mathrm{so}(d) \otimes \wedge^1 \mathbb{R}^m)$ *and* $f \in L^p(B_1, \mathbb{R}^d)$ *satisfy*

$$\Delta u = \Omega \cdot \nabla u + f \quad \text{in } x B_1 \tag{2-7}$$

weakly. If $\|\Omega\|_{M^{2,2}(B_1)} \le \epsilon$, *then*

$$\|\nabla^2 u\|_{M^{2p/m,2}(B_{1/2})} + \|\nabla u\|_{M^{2p/(m-p),2}(B_{1/2})} \le C(\|u\|_{L^1(B_1)} + \|f\|_{L^p(B_1)}).$$

3. Proofs of Theorems 1.3 and 1.4

Proof of Theorem 1.3. Without loss of generality, we may assume $r_0 = 1$ and

$$\frac{1}{|B_1|} \int_{B_1} v \, dx = 0.$$

Take a cut-off function $\eta \in C_0^\infty(B_1)$ such that $0 \le \eta \le 1$, $\eta|_{B_{7/8}} \equiv 1$ and $|\nabla \eta| \le C$. By a direct computation, we get

$$\operatorname{div}(\beta(u)\nabla(\eta v)) = \operatorname{div}(\beta(u)\nabla\eta v) + \beta(u)\nabla\eta\nabla v \quad \text{in } B_1.$$

Then according to the standard theory of the second elliptic operator of divergence forms, see Theorem 1 in [Meyers 1963], we have $v \in W^{1,2m/(m-2)}(B_{7/8})$ and

$$\begin{aligned} \|\nabla v\|_{L^{2m/(m-2)}(B_{7/8})} &\le C(m, \lambda_1, \lambda_2)(\|\nabla \eta v\|_{L^{2m/(m-2)}(B_1)} + \|\beta(u)\nabla\eta\nabla v\|_{L^2(B_1)}) \\ &\le C(m, \lambda_1, \lambda_2)\|\nabla v\|_{L^2(B_1)}, \end{aligned}$$

where the last inequality follows from Sobolev's embedding $W^{1,2} \hookrightarrow L^{2m/(m-2)}$ and Poincaré's inequality

$$\|v\|_{L^2(B_1)} \le C(m)\|\nabla v\|_{L^2(B_1)}.$$

Using Theorem 1 in [Meyers 1963] and by a bootstrap argument, it is easy to see that $v \in W^{1,p}(B_{3/4})$ for any $1 < p < \infty$ and

$$\|\nabla v\|_{L^p(B_{3/4})} \le C(m, p, \lambda_1, \lambda_2)\|\nabla v\|_{L^2(B_1)}. \tag{3-1}$$

It is well known that the equation of u can be written in the form of (2-7) with

$$|\Omega| \le C(N)|\nabla u| \quad \text{and} \quad |f| \le C(\lambda_2, N)|\nabla v|^2.$$

By Theorem 2.6 and (3-1), taking $\epsilon_0 = \epsilon_0(m, p, N)$ sufficiently small, we know $u \in W^{1,p}(B_{5/8})$ for any $m < p < \infty$ and

$$\begin{aligned} \|\nabla u\|_{L^p(B_{5/8})} &\le C(m, p, \lambda_1, \lambda_2, N)(\|\nabla u\|_{L^2(B_1)} + \||\nabla v|^2\|_{L^{mp/(2+p)}(B_1)}) \\ &\le C(m, p, \lambda_1, \lambda_2, N)(\|\nabla u\|_{L^2(B_1)} + \|\nabla v\|^2_{L^2(B_1)}). \end{aligned}$$

Applying $W^{2,p}$ estimates of the Laplacian operator, we obtain

$$\begin{aligned}&\|\nabla u\|_{W^{1,p}(B_{9/16})}\\&\quad\le C(m,p,\lambda_2,N)(\|\nabla u\|^2_{L^{2p}(B_{5/8})}+\|\nabla v\|^2_{L^{2p}(B_{5/8})}+\|\nabla u\|_{L^2(B_{5/8})})\\&\quad\le C(m,p,\lambda_1,\lambda_2,N)(\|\nabla u\|_{L^2(B_{5/8})}+\|\nabla u\|^2_{L^2(B_1)}+\|\nabla v\|^2_{L^2(B_1)}+\|\nabla v\|^4_{L^2(B_1)})\end{aligned}$$

and

$$\begin{aligned}\|\nabla v\|_{W^{1,p}(B_{9/16})}&\le C(m,p,\lambda_1,\lambda_2,N)(\||\nabla u||\nabla v|\|_{L^p(B_{5/8})}+\|\nabla v\|_{L^2(B_{5/8})})\\&\le C(m,p,\lambda_1,\lambda_2,N)\|\nabla v\|_{L^2(B_1)}(1+\|\nabla u\|_{L^2(B_1)}+\|\nabla v\|^2_{L^2(B_1)}).\end{aligned}$$

By Sobolev's embedding theorem, we see that $(\nabla u,\nabla v)\in C^\alpha(B_{9/16})$ for any $\alpha=1-m/p\in(0,1)$ and the estimate (1-5) holds. Then the high regularity follows from the classical Schauder estimates of the Laplacian operator and a standard bootstrap argument. □

Now, we prove our main theorem, Theorem 1.4.

Proof of Theorem 1.4. Define

$$S(u):=\left\{x\in\Omega\;\middle|\;\liminf_{r\searrow 0} r^{2-n}\int_{B_r(x)}|\nabla u|^2\ge\frac{\epsilon_0^2}{2^m}\right\},\tag{3-2}$$

where $\epsilon_0>0$ is the constant in Theorem 1.3. It is well known that $H^{n-2}(S(u))=0$. Next, we will show $S(u)$ is a closed set and $(u,v)\in C^\infty(\Omega\setminus S(\phi))$.

For any $x_0\in\Omega\setminus S(u)$ and $\epsilon>0$, there exists $0<r_0<\epsilon$ such that

$$(2r_0)^{2-m}\int_{B_{2r_0}(x_0)}|\nabla u|^2\,dx<\frac{\epsilon_0^2}{2^m}.\tag{3-3}$$

Therefore,

$$\sup_{z\in B_{r_0}(x_0)} r_0^{2-m}\int_{B_{r_0}(z)}|\nabla u|^2\,dx\le r_0^{2-m}\int_{B_{2r_0}(x_0)}|\nabla u|^2\,dx<\frac{2^{m-2}\epsilon_0^2}{2^m}.\tag{3-4}$$

By Corollary 2.4, we have

$$\begin{aligned}&\sup_{z\in B_{r_0}(x_0),\,0<r\le r_0} r^{2-m}\int_{B_r(z)}|\nabla u|^2\,dx\\&\qquad\le\sup_{z\in B_{r_0}(x_0)} r_0^{2-m}\int_{B_{r_0}(z)}|\nabla u|^2\,dx+C(m,p,\lambda_1,\lambda_2,\|\psi\|_{C^1(\partial\Omega)})(r_0)^{2-2m/p}\\&\qquad\le\frac{2^{m-2}\epsilon_0^2}{2^m}+C_1(m,p,\lambda_1,\lambda_2,\|\psi\|_{C^1(\partial\Omega)})(r_0)^{2-2m/p}\end{aligned}\tag{3-5}$$

for some $m<p<\infty$, where $C_1(m,p,\lambda_1,\lambda_2,\|\psi\|_{C^1(\partial\Omega)})$ is a positive constant.

Taking

$$\epsilon \le \left(\frac{\epsilon_0^2}{4C_1(m, p, \lambda_1, \lambda_2, \|\psi\|_{C^1(\partial\Omega)})} \right)^{2m/p-2},$$

we get

$$\sup_{z \in B_{r_0}(x_0),\, 0<r\le r_0} r^{2-m} \int_{B_r(z)} |\nabla u|^2 \, dx \le \frac{\epsilon_0^2}{2}. \tag{3-6}$$

Then Theorem 1.3 tells us that $(u, v) \in C^\infty(B_{r_0/2}(x_0))$, which implies $B_{r_0/4}(x_0) \subset \Omega \setminus S(u)$. □

4. Proof of Theorem 1.6

In this section, we will study the blow-up behavior of a sequence of stationary Lorentzian harmonic maps $\{(u_n, v_n)\}$ with Dirichlet boundary data (ϕ, ψ) and with bounded energy

$$E(u_n, v_n) = \frac{1}{2} \int_\Omega (|\nabla u_n|^2 + |\nabla v_n|^2) \, dx \le \Lambda.$$

Due to the weak compactness, we may assume $u_n \rightharpoonup u$ weakly in $W^{1,2}(\Omega, N)$ and

$$\mu_n := |\nabla u_n|^2 dx \to \mu := |\nabla u|^2 dx + \nu$$

in the sense of Radon measures, where ν is a nonnegative Radon measure by Fatou's lemma and is usually called the defect measure.

Without loss of generality, we assume $B_1(0) \subseteq \Omega$. Similar to harmonic maps [Lin 1999], we define the energy concentration set Σ as

$$\Sigma = \left\{ x \in B_1(0) \;\middle|\; \liminf_{r \searrow 0} \liminf_{n\to\infty} r^{2-n} \int_{B_r(x)} |\nabla u_n|^2 \, dx \ge \frac{\epsilon_0^2}{2^m} \right\}, \tag{4-1}$$

where ϵ_0 is the constant in Theorem 1.3.

Denoting by $\operatorname{spt}(\nu)$ the support set of ν and defining

$$\operatorname{sing}(u) := \{x \in B_1(0) \mid u \text{ is not smooth at } x\},$$

we have:

Lemma 4.1. *Suppose $\{(u_n, v_n)\}$ is a sequence of stationary Lorentzian harmonic maps with Dirichlet boundary data $(u_n, v_n)|_\Omega = (\phi, \psi)$ and bounded energy $E(u_n, v_n) \le \Lambda$; then the energy concentration set Σ is closed in B_1 and*

$$H^{m-2}(\Sigma) \le C(m, \epsilon_0, \Lambda).$$

Moreover, there holds

$$\Sigma = \operatorname{spt}(\nu) \cup \operatorname{sing}(u). \tag{4-2}$$

Proof. For $x_0 \in B_1 \setminus \Sigma$, by the definition of Σ, we know that for any positive constant

$$\epsilon \le \left(\frac{\epsilon_0^2}{4C_1(m, p, \lambda_1, \lambda_2, \|\psi\|_{C^1(\partial\Omega)})} \right)^{2m/p-2},$$

where $C_1(m, p, \lambda_1, \lambda_2, \|\psi\|_{C^1(\partial\Omega)})$ is the constant in (3-5), there exists a positive constant $r_0 < \epsilon$ and a subsequence of $\{n\}$ (also denoted by $\{n\}$), such that, for any n,

$$(2r_0)^{2-m} \int_{B_{2r_0}(x)} |\nabla u_n|^2 \, dx < \frac{\epsilon_0^2}{2^m},$$

which implies (similar to deriving (3-6))

$$\sup_{z \in B_{r_0}(x),\, 0<r\le r_0} r^{2-m} \int_{B_r(z)} |\nabla u_n|^2 \, dx < \frac{\epsilon_0^2}{2}.$$

By Theorem 1.4, we know

$$\|\nabla u_n\|_{L^\infty(B_{r_0/2}(x_0))} + \|\nabla v_n\|_{L^\infty(B_{r_0/2}(x_0))} \le C(m, \lambda_1, \lambda_2, \Lambda, N) r_0^{-m/2}. \tag{4-3}$$

Then, it is easy to see that there exists a small positive constant $r_1 = r_1(m, r_0, \lambda_1, \lambda_2, \Lambda, \epsilon_0, N)$, such that, whenever $r \le r_1$,

$$\sup_{x \in B_{r_0/4}(x_0)} r^{2-m} \int_{B_r(x)} |\nabla u_n|^2 \, dx < \frac{\epsilon_0^2}{2^{m+1}}.$$

Thus, $B_{r_0/4}(x_0) \subset B_1 \setminus \Sigma$. So, Σ is a closed set.

It is standard to get $H^{m-2}(\Sigma) \le C$ by a covering lemma; see [Lin 1999].

For (4-2), on the one hand, let $x_0 \in B_1 \setminus \Sigma$. Then (4-3) holds and by standard elliptic estimates of the Laplace operator, we have

$$\|u_n\|_{C^{1+\alpha}(B_{r_0/4}(x_0))} + \|v_n\|_{C^{1+\alpha}(B_{r_0/4}(x_0))} \le C \tag{4-4}$$

for some $0 < \alpha < 1$. Thus, up to a subsequence of $\{u_n, v_n\}$, $u_n \to u$ strongly in $W^{1,2}$ and $u \in C^\infty(B_{r_0/8}(x_0))$, which implies that $x_0 \notin \operatorname{sing}(u)$ and $x_0 \notin \operatorname{spt} \nu$ since $\nu \equiv 0$ on $B_{r_0/8}(x_0)$.

On the other hand, if $x_0 \in \Sigma$, by the definition, for any $r > 0$ sufficiently small, we have

$$\liminf_{n\to\infty} \frac{\mu_n(B_r(x_0))}{r^{m-2}} \ge \frac{\epsilon_0^2}{2^{m+1}},$$

which implies

$$\frac{\mu(B_r(x_0))}{r^{m-2}} \ge \frac{\epsilon_0^2}{2^{m+1}}$$

for a.e. $r > 0$. Suppose $x_0 \notin \operatorname{sing}(\phi)$; then

$$r^{2-m} \int_{B_r(x_0)} |\nabla u|^2 \, dx \le \frac{\epsilon_0^2}{2^{m+2}}$$

whenever $r > 0$ is small enough. Then we have

$$\frac{\nu(B_r(x_0))}{r^{m-2}} \geq \frac{\epsilon_0^2}{2^{m+2}}$$

for all small positive $r > 0$ and $x_0 \in \operatorname{spt} \nu$. □

Lemma 4.2. *Under the same assumption as above lemma, the limit*

$$\theta_\nu(x) := \lim_{r\to 0} \frac{\nu(B_r(x))}{r^{m-2}} \tag{4-5}$$

exists for H^{m-2}-a.e. $x \in \Sigma$. Moreover,

$$\frac{\epsilon_0^2}{2^m} \leq \theta_\nu(x) \leq C(m, \lambda_1, \lambda_2, \Lambda, N, \|\psi\|_{C^1(\partial\Omega)})\delta_0^{2-m},$$

where $\delta_0 := \operatorname{dist}(B_1(0), \partial\Omega)$.

Proof. Let $x \in \Omega$ and $s_i \to 0$, $t_i \to 0$ be two arbitrary positive sequences. By Corollary 2.4, we have

$$\frac{\mu_n(B_{s_i}(x))}{s_i^{m-2}} \leq \frac{\mu_n(B_{t_j}(x))}{t_j^{m-2}} + C(m, p, \lambda_1, \lambda_2, \Lambda, N, \|\psi\|_{C^1(\partial\Omega)})(t_j)^{2-2m/p} \tag{4-6}$$

for $s_i \leq t_j$ and some $m < p < \infty$. Letting firstly $i \to \infty$ and secondly $j \to \infty$, we get

$$\limsup_{r\to 0} \frac{\mu(B_r(x))}{r^{m-2}} \leq \liminf_{r\to 0} \frac{\mu(B_r(x))}{r^{m-2}}.$$

Thus,

$$\lim_{r\to 0} \frac{\mu(B_r(x))}{r^{m-2}}$$

exists. Noting that for H^{m-2}-a.e. $x \in \Omega$,

$$\lim_{r\to 0} r^{2-m} \int_{B_r(x)} |\nabla u|^2 \, dx = 0, \tag{4-7}$$

we have

$$\lim_{r\to 0} \frac{\nu(B_r(x))}{r^{m-2}} = \lim_{r\to 0} \frac{\mu(B_r(x))}{r^{m-2}}.$$

It is easy to see from (4-6) (taking $p = 2m$) that

$$\begin{aligned} r^{2-m}\mu(B_r(x)) &\leq C(\Lambda)\delta_0^{2-m} + C(m, \lambda_1, \lambda_2, \Lambda, N, \|\psi\|_{C^1(\partial\Omega)})\delta_0 \\ &\leq C(m, \lambda_1, \lambda_2, \Lambda, N, \|\psi\|_{C^1(\partial\Omega)})\delta_0^{2-m}, \end{aligned}$$

which implies $\mu \lfloor \Sigma$ is absolutely continuous with respect to $H^{m-2} \lfloor \Sigma$. By the Radon–Nikodym theorem, we know that there exists a measurable function $\theta(x)$

such that

$$\mu \lfloor \Sigma = \theta(x) H^{m-2} \lfloor \Sigma.$$

Noting that for H^{m-2}-a.e. $x \in \Sigma$,

$$2^{2-m} \leq \liminf_{r \to 0} \frac{H^{m-2}(\Sigma \cap B_r(x))}{r^{m-2}} \leq \limsup_{r \to 0} \frac{H^{m-2}(\Sigma \cap B_r(x))}{r^{m-2}} \leq 1$$

and by (4-7), we have

$$\nu \lfloor \Sigma = \theta(x) H^{m-2} \lfloor \Sigma$$

and

$$\frac{\epsilon_0^2}{2^m} \leq \theta_\nu(x) = \theta(x) \leq C(m, \lambda_1, \lambda_2, \Lambda, N, \|\psi\|_{C^1(\partial\Omega)})\delta_0^{2-m}. \qquad \square$$

Since ν is absolutely continuous with respect to $H^{m-2} \lfloor \Sigma$ and $\nu = 0$ outside Σ, $\theta_\nu(x)$ is positive for ν-a.e. $x \in \Omega$. Hence by [Preiss 1987], we have:

Corollary 4.3. *The set of energy concentration points Σ is $(m-2)$-rectifiable.*

For any $y \in \Sigma$ and $\lambda > 0$, we define a scaled Radon measure $\mu_{y,\lambda}$ by

$$\mu_{y,\lambda}(A) = \lambda^{2-m} \mu(y + \lambda A).$$

A Radon measure μ_* is called the tangent measure of μ at y if

$$\mu_{y,\lambda} \to \mu_*$$

in the sense of Radon measures as $r \searrow 0$; see [Federer 1969; Simon 1983].

Lemma 4.4. *Suppose $H^{m-2}(\Sigma) > 0$. Then there exists a nonconstant harmonic sphere S^2 into N.*

Proof. Since Σ is $(m-2)$-rectifiable and $H^{m-2}(\Sigma) > 0$, we know there exists a point $x_0 \in \Sigma$ such that ν has a tangent measure ν_* at x_0 and

$$\nu_* = \theta_\nu(x_0) H^{m-2} \lfloor \Sigma_*,$$

where $\Sigma_* \subset \mathbb{R}^m$ is an $(m-2)$-dimensional linear subspace which is usually called the tangent space of Σ at x_0. Without loss of generality, we may assume $x_0 = 0$ and $\Sigma_* = \mathbb{R}^{m-2} \times \{(0, 0)\}$.

By a similar diagonal argument as that in [Lin 1999], there exists a sequence $r_n \to 0$ such that

$$\tilde{\mu}_n^1 := |\nabla \tilde{u}_n^1|^2 dx \to \nu_*$$

in the sense of Radon measures, where $\tilde{u}_n^1(x) := u_n(x_0 + r_n x)$.

Set $\tilde{v}_n^1(x) := v_n(x_0 + r_n x)$. It is easy to see that $(\tilde{u}_n^1, \tilde{v}_n^1)$ is also a stationary Lorentzian harmonic map. By Lemma 2.3, we have

$$r_2^{2-m}\int_{B_{r_2}(0)}(|\nabla\tilde{u}_n^1|^2 - \beta(\tilde{u}_n^1)|\nabla\tilde{v}_n^1|^2)\,dx - r_1^{2-m}\int_{B_{r_1}(0)}(|\nabla\tilde{u}_n^1|^2 - \beta(\tilde{u}_n^1)|\nabla\tilde{v}_n^1|^2)\,dx = 2\int_{r_1}^{r_2} r^{2-m}\int_{\partial B_r(0)}\left(\left|\frac{\partial\tilde{u}_n^1}{\partial|x|}\right|^2 - \beta(\tilde{u}_n^1)\left|\frac{\partial\tilde{v}_n^1}{\partial|x|}\right|^2\right)dH^{n-1}\,dr. \tag{4-8}$$

By Young's inequality, there holds

$$\int_{B_r}|x|^{2-m}|\nabla\tilde{v}_n^1|^2\,dx \le (r_n)^{2-2m/p}\|\nabla v_n\|^2_{L^p(B_{r_n r})}\||x|^{2-m}\|_{L^{p/(p-2)}(B_r)} \le C(m, p, \lambda_1, \lambda_2, \Lambda, N, \|\psi\|_{C^1(\partial\Omega)})(r_n r)^{2-2m/p}. \tag{4-9}$$

Letting $n \to \infty$ in (4-8) and noting that

$$r_2^{2-m}\nu_*(B_{r_2}(0)) = r_1^{2-m}\nu_*(B_{r_1}(0)),$$

we get

$$\lim_{n\to\infty}\int_{B_2(0)}\left|\frac{\partial\tilde{u}_n^1}{\partial|x|}\right|^2 dx = 0. \tag{4-10}$$

Similarly, since $\nu_{*y,r} = \nu_*$ for any $y \in \Sigma_*$ and $r > 0$, we also have

$$\lim_{n\to\infty}\int_{B_2(0)}\left|\frac{\partial\tilde{u}_n^1}{\partial|x-y|}\right|^2 dx = 0 \quad \text{for } y \in \Sigma_* \cap B_2. \tag{4-11}$$

This implies

$$\lim_{n\to\infty}\sum_{k=1}^{m-2}\int_{B_2(0)}\left|\frac{\partial\tilde{u}_n^1}{\partial x^k}\right|^2 dx = 0. \tag{4-12}$$

Let $x' = (x_1, \dots, x_{m-2}),\ x'' = (x_{m-1}, x_m)$, and define $f_n : B_1^{m-2} \to \mathbb{R}$ by

$$f_n(x') := \sum_{k=1}^{m-2}\int_{B_1^2(0)}\left|\frac{\partial\tilde{u}_n^1}{\partial x_k}\right|^2(x', x'')\,dx''.$$

Then, (4-12) tells us

$$\lim_{n\to\infty}\|f_n(x')\|_{L^1(B_1^{m-2}(0))} = 0.$$

Denote by $M(f_n)(x')$ the Hardy–Littlewood maximal function; i.e.,

$$M(f_n)(x) = \sup_{0<r<1/2} r^{2-m}\int_{B_r^{m-2}(x)} f_n(x')\,dx', \quad x \in B_{1/2}^{m-2}(0).$$

By a weak L^1-estimate, for any $\rho > 0$, we have

$$|\{x \in B^{m-2}_{1/2}(0) \mid M(f_n) > \rho\}| \le \frac{C(m)}{\rho} \|f_n\|_{L^1(B^{m-2}_{1/2}(0))},$$

which implies

$$\left|\left\{x \in B^{m-2}_{1/2}(0) \;\middle|\; \limsup_{n\to\infty} M(f_n) > 0\right\}\right| = 0.$$

Combining this with Theorem 1.4, we know there exists a sequence of points $\{x'_n \in B^{m-2}_{1/2}(0)\}$ such that $(\tilde{u}^1_n, \tilde{v}^1_n)$ is smooth near (x'_n, x'') for all $x'' \in B^2_1(0)$ and

$$\lim_{n\to\infty} M(f_n)(x'_n) = 0. \tag{4-13}$$

By the blow-up argument in [Lin 1999], we can find sequences $\{\sigma_n\}$ and $\{x''_n\} \subset B^2_{1/2}(0)$ such that $\sigma_n \to 0$, $x''_n \to (0,0)$ and

$$\max_{x'' \in B^2_{1/2}(0)} \sigma_n^{2-m} \int_{B^{m-2}_{\sigma_n}(x'_n) \times B^2_{\sigma_n}(x'')} |\nabla \tilde{u}^1_n|^2 \, dx = \frac{\epsilon_0^2}{C_2(m)}, \tag{4-14}$$

where the maximum is achieved at the point x''_n and $C_2(m) > 2^m$ is a positive constant to be determined later.

In fact, define

$$g_n(\sigma) := \max_{x'' \in B^2_{1/2}(0)} \sigma^{2-m} \int_{B^{m-2}_{\sigma}(x'_n) \times B^2_{\sigma}(x'')} |\nabla \tilde{u}^1_n|^2 \, dx.$$

On the one hand, noting that (u_n, v_n) is smooth near $x'_n \times B^2_1(0)$, we have

$$\lim_{\sigma\to 0} g_n(\sigma) = 0.$$

On the other hand, for any $\sigma > 0$, when n is big enough, it must hold that $g_n(\sigma) \ge \epsilon_0^2/2^m$, for otherwise, by Theorem 1.3, $\tilde{u}^1_n$ will converge strongly in $W^{1,2}$ to a constant map, which contradicts $\tilde{\mu}_n \to \nu_*$. Thus, there exists σ_n such that $g_n(\sigma_n) = \epsilon_0^2/C_2(m)$ and we may assume the maximum is achieved at x''_n. Next, we show $\sigma_n \to 0$ and $x''_n \to (0,0)$.

If $\sigma_n \ge \delta > 0$, by Corollary 2.4, we have

$$\begin{aligned}
\frac{\epsilon_0^2}{C_2(m)} &= \limsup_{n\to\infty} g_n(\sigma_n) \\
&\ge \limsup_{n\to\infty} \bigl(g_n(\delta) - C(m, p, \lambda_1, \lambda_2, \Omega, \|\psi\|_{C^1(\partial\Omega)})(r_n\delta)^{2-2m/p}\bigr) \ge \frac{\epsilon_0^2}{2^m},
\end{aligned}$$

which is a contradiction.

If $x''_n \to x''_0 \in B^2_{1/2}(0)$ and $x''_0 \ne (0,0)$, for any $\sigma < \frac{1}{2}|x''_0|$

$$\frac{\epsilon_0^2}{2^m} \le \limsup_{n\to\infty} g_n(\sigma) \le \sigma^{2-m} \nu_*(B^{m-2}_1(0) \times B^2_{2\sigma}(x''_0)) = 0.$$

This is also a contradiction.

Let $x_n = (x_n', x_n'')$ and

$$(\tilde{u}_n^2(x), \tilde{v}_n^2(x)) := (\tilde{u}_n^1(x_n + \sigma_n x), \tilde{v}_n^1(x_n + \sigma_n x)).$$

Then $(\tilde{u}_n^2(x), \tilde{v}_n^2(x))$ is a stationary Lorentzian harmonic map defined on $B_{R_n}^{m-2}(0) \times B_{R_n}^2(0)$, where $R_n = 1/(4\sigma_n)$ which tends to infinity as $n \to \infty$.

By (4-13), we have

$$\begin{aligned}
\text{(4-15)} \quad \lim_{n\to\infty} \sup_{0<R<R_n} R^{2-m} \int_{B_R^{m-2}(0)\times B_{R_n}^2(0)} \sum_{k=1}^{m-2} \left| \frac{\partial \tilde{u}_n^2}{\partial x_k} \right|^2 dx \\
= \lim_{n\to\infty} \sup_{0<R<R_n} (\sigma_n R)^{2-m} \int_{B_{\sigma_n R}^{m-2}(x_n')\times B_{\sigma_n R_n}^2(x_n'')} \sum_{k=1}^{m-2} \left| \frac{\partial \tilde{u}_n^1}{\partial x_k} \right|^2 dx \\
\le \lim_{n\to\infty} M(f_n)(x_n') = 0.
\end{aligned}$$

By (4-14), we get

$$\text{(4-16)} \quad \frac{\epsilon_0^2}{C_2(m)} = \int_{B_1^{m-2}(0)\times B_1^2(0)} |\nabla \tilde{u}_n^2|^2 \, dx = \max_{x''\in B_{R_n-1}^2(0)} \int_{B_1^{m-2}(0)\times B_1^2(x'')} |\nabla \tilde{u}_n^2|^2 \, dx.$$

By Corollary 2.4, for any $R > 0$, we obtain

$$\begin{aligned}
\text{(4-17)} \quad \int_{B_R^{m-2}(0)\times B_R^2(0)} |\nabla \tilde{u}_n^2|^2 \, dx = (\sigma_n)^{2-m} \int_{B_{\sigma_n R}^{m-2}(x_n')\times B_{\sigma_n R}^2(x_n'')} |\nabla \tilde{u}_n^1|^2 \, dx \\
\le C(m, \lambda_1, \lambda_2, \delta_0, \Lambda, \Omega, \|\psi\|_{C^1(\partial\Omega)}) R^{m-2},
\end{aligned}$$

when n is big enough.

Let $\zeta \in C_0^\infty(B_1^{m-2}(0))$ and $\eta \in C_0^\infty(B_1^2(0))$ be two cut-off functions such that $0 \le \zeta \le 1$, $\zeta|_{B_{1/2}^{m-2}(0)} \equiv 1$, $0 \le \zeta \le 1$, and $\eta|_{B_{1/2}^2(0)} \equiv 1$. Similar to [Lin 1999], for any $R > 0$, we define $F_n(a) : B_6^{m-2}(0) \times B_R^2(0) \to \mathbb{R}$ as

$$F_n(a) = \int_{B_1^{m-2}(0)\times B_1^2(0)} |\nabla \tilde{u}_n^2|^2 (a+x) \zeta(x') \eta(x'') \, dx.$$

Computing directly, one has

$$\begin{aligned}
\frac{\partial F_n(a)}{\partial a_k} &= \int_{B_1^{m-2}(0)\times B_1^2(0)} \frac{\partial}{\partial x_k} |\nabla \tilde{u}_n^2|^2 (a+x) \zeta(x') \eta(x'') \, dx \\
&= 2 \int_{B_1^{m-2}(0)\times B_1^2(0)} \left\langle \frac{\partial \tilde{u}_n^2}{\partial x_l}, \frac{\partial^2 \tilde{u}_n^2}{\partial x_l \partial x_k} \right\rangle (a+x) \zeta(x') \eta(x'') \, dx \\
&= -2 \int_{B_1^{m-2}(0)\times B_1^2(0)} \left\langle \Delta \tilde{u}_n^2, \frac{\partial \tilde{u}_n^2}{\partial x_k} \right\rangle (a+x) \zeta(x') \eta(x'') \, dx \\
&\quad - 2 \int_{B_1^{m-2}(0)\times B_1^2(0)} \left\langle \frac{\partial \tilde{u}_n^2}{\partial x_l}, \frac{\partial \tilde{u}_n^2}{\partial x_k} \right\rangle (a+x) \frac{\partial}{\partial x_l} (\zeta(x') \eta(x'')) \, dx.
\end{aligned}$$

On the one hand, by (1-3), we have

$$
\begin{aligned}
-2\int_{B_1^{m-2}(0)\times B_1^2(0)} &\left\langle \Delta\tilde{u}_n^2, \frac{\partial\tilde{u}_n^2}{\partial x_k}\right\rangle (a+x)\zeta(x')\eta(x'')\,dx \\
&= -2\int_{B_1^{m-2}(0)\times B_1^2(0)} \left\langle B^\top(\tilde{u}_n^2)|\nabla\tilde{v}_n^2|^2, \frac{\partial\tilde{u}_n^2}{\partial x_k}\right\rangle (a+x)\zeta(x')\eta(x'')\,dx \\
&\le C\left(\int_{B_{R+1}^{m-2}(0)\times B_{R+1}^2(0)} |\nabla\tilde{v}_n^2|^4\,dx\right)^{1/2}\left(\int_{B_{R+1}^{m-2}(0)\times B_{R+1}^2(0)} \left|\frac{\partial\tilde{u}_n^2}{\partial x_k}\right|^2 dx\right)^{1/2}.
\end{aligned}
$$

On the other hand, by Holder's inequality, we have

$$
\begin{aligned}
-2\int_{B_1^{m-2}(0)\times B_1^2(0)} &\left\langle \frac{\partial\tilde{u}_n^2}{\partial x_l}, \frac{\partial\tilde{u}_n^2}{\partial x_k}\right\rangle (a+x)\frac{\partial}{\partial x_l}(\zeta(x')\eta(x''))\,dx \\
&\le C\left(\int_{B_{R+1}^{m-2}(0)\times B_{R+1}^2(0)} |\nabla\tilde{u}_n^2|^2\,dx\right)^{1/2}\left(\int_{B_{R+1}^{m-2}(0)\times B_{R+1}^2(0)} \left|\frac{\partial\tilde{u}_n^2}{\partial x_k}\right|^2 dx\right)^{1/2}.
\end{aligned}
$$

Combining these together and letting $n\to\infty$, we obtain

$$
\frac{\partial F_n(a)}{\partial a_k}\to 0, \quad k=1,\dots,m-2,
$$

uniformly in $B_2^{m-2}(0)\times B_R^2(0)$ for any fixed $R>0$.

Thus, for any $a=(a',a'')=B_6^{m-2}(0)\times B_R^2(0)$,

$$
\begin{aligned}
\int_{B_{1/2}^{m-2}(a')\times B_{1/2}^2(a'')} |\nabla\tilde{u}_n^2|^2\,dx &\le F_n(a)\le F_n((0,a''))+C(m)\sum_{k=1}^{m-2}\left|\frac{\partial F_n(a)}{\partial a_k}\right| \\
&\le \int_{B_1^{m-2}(0)\times B_1^2(a'')} |\nabla\tilde{u}_n^2|^2\,dx + C(m)\sum_{k=1}^{m-2}\left|\frac{\partial F_n(a)}{\partial a_k}\right| \\
&\le \frac{\epsilon_0^2}{C_2(m)} + C(m)\sum_{k=1}^{m-2}\left|\frac{\partial F_n(a)}{\partial a_k}\right|.
\end{aligned}
$$

Therefore, when n is big enough, we have

$$
6^{2-m}\int_{B_6^{m-2}(0)\times B_6^2(0)} |\nabla\tilde{u}_n^2|^2(x',x''+b)\,dx \le \frac{C(m)\epsilon_0^2}{C_2(m)} \quad \text{for all } b\in B_R^2(0). \tag{4-18}
$$

Taking $C_2(m)\ge 2^m C(m)$, by Corollary 2.4, we have

$$
\begin{aligned}
\sup_{x_0\in B_3(0),\,0<r\le 3} & r^{2-m}\int_{B_r(x_0)} |\nabla\tilde{u}_n^2|^2(x',x''+b)\,dx \\
&\le \sup_{x_0\in B_3(0)} 3^{2-m}\int_{B_3(x_0)} |\nabla\tilde{u}_n^2|^2(x',x''+b)\,dx \\
&\qquad + C(m,p,\lambda_1,\lambda_2,\Omega,\|\psi\|_{C^1(\partial M)})(\sigma_n r_n)^{2-2m/p}
\end{aligned}
$$

$$\le 2^{m-2}6^{2-m}\int_{B_6^{m-2}(0)\times B_6^2(0)}|\nabla\tilde{u}_n^2|^2(x',x''+b)\,dx + C(m,p,\lambda_1,\lambda_2,\Omega,\|\psi\|_{C^1(\partial M)})(\sigma_n r_n)^{2-2m/p}$$

$$\le \frac{2^{m-2}C(m)\epsilon_0^2}{C_2(m)} + C(m,p,\lambda_1,\lambda_2,\Omega,\|\psi\|_{C^1(\partial M)})(\sigma_n r_n)^{2-2m/p} \le \frac{\epsilon_0^2}{2}$$

for some $m < p < \infty$, whenever n is large enough.

By Theorem 1.3, we know $(\tilde{u}_n^2, \tilde{v}_n^2)$ subconverges to a Lorentzian harmonic map $(\tilde{u}, \tilde{v})$ in $C^1_{\text{loc}}(B^{m-2}_{3/2}(0)\times\mathbb{R}^2)$. Moreover, by (4-15)-(4-17), for any $R > 0$, we have

$$\int_{B_R(0)}\sum_{k=1}^{m-2}\left|\frac{\partial\tilde{u}}{\partial x_k}\right|^2 dx = 0,$$

and

$$\int_{B_1(0)}|\nabla\tilde{u}|^2\,dx = \frac{\epsilon_0^2}{C_2(m)},$$

$$\int_{B_R(0)}|\nabla\tilde{u}|^2\,dx \le C(m,\lambda_1,\lambda_2,\delta_0,\Lambda,\Omega,\|\psi\|_{C^1(\partial\Omega)})R^{m-2}.$$

Furthermore, since

$$\int_{B_R(0)}|\nabla\tilde{v}|^2\,dx = \lim_{n\to\infty}\int_{B_R(0)}|\nabla\tilde{v}_n^2|^2\,dx \le \lim_{n\to\infty}(r_n\sigma_n)^{2-2m/p}R^{m(1-2/p)}\|\nabla v_n\|^2_{L^p} = 0,$$

we know $\tilde{v}$ is a constant and $\tilde{u}:\mathbb{R}^2\to N$ is a nonconstant harmonic map with finite energy. By the conformal theory of harmonic maps in dimension 2, $\tilde{u}$ can be extended to a nonconstant harmonic sphere. □

Proof of Theorem 1.6. The conclusion of Theorem 1.6 standardly follows from Lemma 4.4 and the Federer dimensions reduction argument, which is similar to [Schoen and Uhlenbeck 1982] for minimizing harmonic maps. We omit the details here. □

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