Pacific Journal of Mathematics

KMS CONDITIONS, STANDARD REAL SUBSPACES AND REFLECTION POSITIVITY ON THE CIRCLE GROUP

KARL-HERMANN NEEB AND GESTUR ÓLAFSSON

Volume 299 No. 1 March 2019

PACIFIC JOURNAL OF MATHEMATICS
Vol. 299, No. 1, 2019
dx.doi.org/10.2140/pjm.2019.299.117

KMS CONDITIONS, STANDARD REAL SUBSPACES AND REFLECTION POSITIVITY ON THE CIRCLE GROUP

KARL-HERMANN NEEB AND GESTUR ÓLAFSSON

We continue our investigations of the representation theoretic side of reflection positivity by studying positive definite functions ψ on the additive group $(\mathbb{R}, +)$ satisfying a suitably defined KMS condition. These functions take values in the space $\mathrm{Bil}(V)$ of bilinear forms on a real vector space V. As in quantum statistical mechanics, the KMS condition is defined in terms of an analytic continuation of ψ to the strip

$$\{z \in \mathbb{C} : 0 \leq \mathrm{Im} z \leq \beta\}$$

with a coupling condition $\psi(i\beta + t) = \overline{\psi(t)}$ on the boundary. Our first main result consists of a characterization of these functions in terms of modular objects (Δ, J) (J an antilinear involution and $\Delta > 0$ selfadjoint with $J\Delta J = \Delta^{-1}$) and an integral representation.

Our second main result is the existence of a $\mathrm{Bil}(V)$-valued positive definite function f on the group $\mathbb{R}_\tau = \mathbb{R} \rtimes \{\mathrm{id}_\mathbb{R}, \tau\}$ with $\tau(t) = -t$ satisfying $f(t, \tau) = \psi(it)$ for $0 \leq t \leq \beta$. We thus obtain a 2β-periodic unitary one-parameter group on the GNS space $\mathcal{H}_f$ for which the one-parameter group on the GNS space $\mathcal{H}_\psi$ is obtained by Osterwalder–Schrader quantization.

Finally, we show that the building blocks of these representations arise from bundle-valued Sobolev spaces corresponding to the kernels

$$(\lambda^2 - d^2/dt^2)^{-1}$$

on the circle $\mathbb{R}/\beta\mathbb{Z}$ of length β.

MSC2010: primary 43A35, 43A65, 47L30; secondary 47L90, 81T05.
Keywords: KMS condition, reflection positivity, standard real subspace.

1. Introduction

In this note we continue our investigations of the mathematical foundations of *reflection positivity*, a basic concept in constructive quantum field theory [Glimm and Jaffe 1981; Klein and Landau 1983; Jorgensen and Ólafsson 1998; 2000; De Angelis et al. 1986; Jaffe and Ritter 2007]. Originally, reflection positivity, also called Osterwalder–Schrader positivity, arises as a requirement on the euclidean side to establish a duality between euclidean and relativistic quantum field theories [Osterwalder and Schrader 1973]. It is closely related to "Wick rotation" or "analytic continuation" in the time variable from the real to the imaginary axis.

The underlying fundamental concept is that of a *reflection positive Hilbert space*, introduced in [Neeb and Ólafsson 2014]. This is a triple $(\mathcal{E}, \mathcal{E}_+, \theta)$, where $\mathcal{E}$ is a Hilbert space, $\theta : \mathcal{E} \to \mathcal{E}$ is a unitary involution and $\mathcal{E}_+$ is a closed subspace of $\mathcal{E}$ which is θ-positive in the sense that $\langle \theta v, v\rangle \geq 0$ for $v \in \mathcal{E}_+$.

In [Neeb and Ólafsson 2014], we introduced the concept of a reflection positive cyclic representation $(\pi, \mathcal{E}, v)$, where $(\mathcal{E}, \mathcal{E}_+, \theta)$ is a reflection positive Hilbert space and $v \in \mathcal{E}$ a θ-fixed vector (or, more generally, a distribution vector). In the present paper we shall see that, to treat reflection positive representations of the circle group $G = \mathbb{T}$ corresponding to unitary representations of the dual group $G^c \cong \mathbb{R}$ arising from KMS states, or from their modular objects (Δ, J),[1] we are forced to work in a more general framework, where the representations are generated by the image of an $\mathbb{R}$-linear map $j : V \to \mathcal{E}$ from a real vector space V into the representation space $\mathcal{E}$ and where $j(V)$ does not consist of θ-fixed vectors.

To explain the corresponding concept of a *reflection positive representation*, we start with a *symmetric Lie group*, i.e., a pair (G, τ), where $\tau \in \mathrm{Aut}(G)$ is an involution. Then we form the extended group $G_\tau := G \rtimes \{\mathbf{1}, \tau\}$. Let $(U, \mathcal{E})$ be a unitary representation of G_τ and let $j : V \to \mathcal{E}$ be a linear map from the real vector space V to $\mathcal{E}$. Then $(U, \mathcal{E}, j, V)$ is called *reflection positive with respect to a subset* $G_+ \subseteq G$ if the closed subspace $\mathcal{E}_+$ generated by $U_{G_+}^{-1} j(V)$ defines a reflection positive Hilbert space $(\mathcal{E}, \mathcal{E}_+, U_\tau)$. Generalizing the well-known Gelfand–Naimark–Segal (GNS) construction leads to an encoding of representations generated by $j(V)$ in terms of form-valued positive definite functions $\psi(g)(v, w) := \langle j(v), U_g j(w)\rangle$ [Neeb and Ólafsson 2015b].

This paper continues the investigations started in [Neeb and Ólafsson 2015b], where we studied reflection positive representations of the circle group and their connections to KMS states, which was largely motivated by the work of Klein and Landau [1981] (see also [Cuniberti et al. 2001]). A long-term goal of this project is

[1] Recall that KMS stands for Kubo–Martin–Schwinger; see [Bratteli and Robinson 1981, §5.3.1] for more on KMS states and their interpretation in quantum statistical mechanics as thermal equilibrium states.

to combine our representation theoretic approach to reflection positivity with KMS states of operator algebras and Borchers triples corresponding to modular inclusions [Buchholz et al. 2011; Borchers 1992; Longo 2008; Schlingemann 1999].

A crucial step in this direction is the concept of a positive definite function satisfying a KMS condition that can be formulated as follows: First, let V be a real vector space and $\mathrm{Bil}(V)$ be the space of real bilinear maps $V \times V \to \mathbb{C}$. A function $\psi : \mathbb{R} \to \mathrm{Bil}(V)$ is said to be *positive definite* if the kernel $\psi(t-s)(v, w)$ on $\mathbb{R} \times V$ is positive definite. For $\beta > 0$, we consider the open strip $\mathcal{S}_\beta := \{z \in \mathbb{C} : 0 < \mathrm{Im}\, z < \beta\}$. We say that a positive definite function $\psi : \mathbb{R} \to \mathrm{Bil}(V)$ satisfies the *KMS condition* for $\beta > 0$ if ψ extends to a function $\overline{\mathcal{S}_\beta} \to \mathrm{Bil}(V)$ which is pointwise continuous and pointwise holomorphic on the interior $\mathcal{S}_\beta$, and satisfies

$$\psi(i\beta + t) = \overline{\psi(t)} \quad \text{for } t \in \mathbb{R}.$$

The central idea in the classification of positive definite functions satisfying a KMS condition is to relate them to *standard real subspaces* of a (complex) Hilbert space; these are closed real subspaces $V \subseteq \mathcal{H}$ for which $V \cap iV = \{0\}$ and $V + iV$ is dense (cf. Definition 2.4). Any such subspace determines a pair (Δ, J) of *modular objects*, where Δ is a positive selfadjoint operator and J is an antilinear involution satisfying $J\Delta J = \Delta^{-1}$. The connection is established by $V = \mathrm{Fix}(J\Delta^{1/2}) = \{v \in \mathcal{D}(\Delta^{1/2}) : J\Delta^{1/2} v = v\}$. Our first main result is the following characterization of the KMS condition in terms of standard real subspaces. Here we write $\mathrm{Bil}^+(V) \subseteq \mathrm{Bil}(V)$ for the convex cone of all those bilinear forms f for which the sesquilinear extension to $V_\mathbb{C} \times V_\mathbb{C}$ is positive semidefinite.

Theorem 2.6 (characterization of the KMS condition). *Let V be a real vector space and $\psi : \mathbb{R} \to \mathrm{Bil}(V)$ be a pointwise continuous positive definite function. Then the following are equivalent*:

(i) *ψ satisfies the KMS condition for $\beta > 0$.*

(ii) *There exists a standard real subspace V_1 in a Hilbert space $\mathcal{H}$ and a linear map $j : V \to V_1$ such that*

(1) $$\psi(t)(v, w) = \langle j(v), \Delta^{-it/\beta} j(w)\rangle \quad \textit{for } t \in \mathbb{R},\ v, w \in V.$$

(iii) *There exists a $\mathrm{Bil}^+(V)$-valued regular Borel measure μ on $\mathbb{R}$ satisfying*

$$\psi(t) = \int_\mathbb{R} e^{it\lambda}\, d\mu(\lambda), \quad \textit{where } d\mu(-\lambda) = e^{-\beta\lambda} d\overline{\mu}(\lambda).$$

If these conditions are satisfied, then the function $\psi : \overline{\mathcal{S}_\beta} \to \mathrm{Bil}(V)$ is pointwise bounded.

The equivalence of (i) and (ii) in Theorem 2.6 describes the tight connection between the KMS condition and the modular objects associated to a standard real subspace. Part (iii) provides an integral representation that can be viewed as a classification result.

For a function ψ satisfying the β-KMS condition, analytic continuation leads to the operator-valued function

$$\varphi : [0, \beta] \to B(V_{\mathbb{C}}), \qquad \langle v, \varphi(t)w \rangle = \psi(it)(v, w).$$

This function satisfies $\varphi(\beta) = \overline{\varphi(0)}$, and hence extends uniquely to a (weak operator) continuous function $\varphi : \mathbb{R} \to B(V_{\mathbb{C}})$ satisfying

$$\varphi(t + \beta) = \overline{\varphi(t)} \quad \text{for } t \in \mathbb{R}. \tag{2}$$

Recall the group $\mathbb{R}_\tau := \mathbb{R} \rtimes \{1, \tau\}$ with $\tau(t) = -t$. In Theorem 4.5 we show that there exists a positive definite function

$$f : \mathbb{R}_\tau \to \mathrm{Bil}(V) \quad \text{satisfying } f(t, \tau) = \varphi(t).$$

The function f is 2β-periodic, hence factors through a function on $\mathbb{T}_{2\beta,\tau} := \mathbb{R}_\tau/\mathbb{Z}2\beta \cong \mathrm{O}_2(\mathbb{R})$. This leads to a natural "euclidean" counterpart of the unitary one-parameter group $U_t = \Delta^{-it/\beta}$ associated to the KMS positive definite function ψ. To understand the structure of the positive definite functions which arise in this way, and the corresponding unitary representations of $\mathbb{T}_{2\beta,\tau}$, we write $f = f_+ + f_-$ with $f_+(\beta + t, \tau^\varepsilon) = f_+(t, \tau^\varepsilon)$ (the bosonic part) and $f_-(\beta + t, \tau^\varepsilon) = -f_-(t, \tau^\varepsilon)$ (the fermionic part). Then $f_\pm$ are both positive definite and combine to a matrix valued positive definite function

$$f^\sharp := \begin{pmatrix} f_+ & 0 \\ 0 & f_- \end{pmatrix} : \mathbb{R}_\tau \to M_2(B(V_{\mathbb{C}})) \cong B(V_{\mathbb{C}}^2)$$

(Lemma 4.12). Neglecting an additive summand which is constant, we can now define a unitary representation of the subgroup $P := (\mathbb{Z}\beta)_\tau$ on $V_{\mathbb{C}}^2$ by

$$\rho(\beta, \mathbf{1}) := \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \quad \text{and} \quad \rho(0, \tau) := \begin{pmatrix} \mathbf{1} & 0 \\ 0 & iI \end{pmatrix},$$

where I is a complex structure on V. Then we have the relation

$$f^\sharp(hg) = \rho(h) f^\sharp(g) \quad \text{for } h \in P,\ g \in \mathbb{R}_\tau,$$

which determines in particular how $f^\sharp$ is obtained from the function φ above. For the special case where the real representation corresponding to ψ is isotypic, or the associated modular operator Δ is a multiple of the identity, the GNS representation $(U^{f^\sharp}, \mathcal{H}_{f^\sharp})$ can be realized on the Hilbert space completion of

$$\Gamma_\rho := \{s \in C^\infty(\mathbb{R}_\tau, V_{\mathbb{C}}^2) : s(hg) = \rho(h)s(g) \text{ for all } g \in \mathbb{R}_\tau,\ h \in P\}$$

with respect to the scalar product

$$\langle s_1, s_2\rangle := \frac{1}{2\beta}\int_0^{2\beta} \langle s_1(t,\mathbf{1}), ((\lambda^2-\Delta)^{-1}s_2)(t,\mathbf{1})\rangle\, dt, \qquad \text{where } \Delta = \frac{d^2}{dt^2}.$$

On this space, $\mathbb{R}_\tau$ acts by right translation. This provides a natural "euclidean realization" of our representation on the Riemannian manifold $\mathbb{T}_\beta \cong \mathbb{S}^1$ in the spirit of [De Angelis et al. 1986; Dimock 2004; Jaffe and Ritter 2007]. The "periodicity in imaginary time" that we also observe here has been studied in detail from a physics perspective by Fulling and Ruijsenaars [1987].

We conclude this paper with a short Section 5, in which we prove a version of Theorem 2.6 for $\beta = \infty$ which connects naturally to our previous work on dilations of semigroups of contractions in [Neeb and Ólafsson 2015a]. In two appendices we provide some background material. Appendix A recalls some facts on positive definite kernels and discusses in particular the connection between complex and real-valued kernels. Appendix B discusses standard real subspaces in terms of skew-symmetric contractions on real Hilbert spaces. This perspective was crucial for the present paper, and we expect it to be useful in other contexts as well.

In a subsequent paper [Neeb and Ólafsson 2019], we extend the results obtained here for the group $G = \mathrm{O}_2(\mathbb{R}) = \mathrm{SO}_2(\mathbb{R})_\tau$ to more general groups such as $\mathrm{O}_{n+1}(\mathbb{R})$ (where reflection positivity refers to the sphere $\mathbb{S}^n$) and $\mathrm{O}_{1,n}(\mathbb{R})$ (where reflection positivity refers to the n-dimensional hyperbolic space $\mathbb{H}^n$). Eventually, we would like to see how our representation theoretic analysis can be blended with the existing work on relativistic KMS conditions [Bros and Buchholz 1994; Gérard and Jäkel 2007] and in particular with [Barata et al. 2013; 2016]. The close connection between modular objects (Δ, J) and standard real subspaces was first explored by Rieffel and van Daele [1977]. They also define a notion of a KMS condition for a unitary one-parameter group $(U_t)_{t\in\mathbb{R}}$ on a complex Hilbert space $\mathcal{H}$ with a real subspace $V \subseteq \mathcal{H}$. In our terms, their condition means that the function $\psi : \mathbb{R} \to \mathrm{Bil}(V)$, $\psi(t) = \langle v, U_t w\rangle$ satisfies the KMS condition for $\beta = -1$ (which refers to a function on the strip $\{-1 < \operatorname{Im} z < 0\}$). From [Rieffel and van Daele 1977, Proposition 3.7], one can easily derive the implication (ii) $\Rightarrow$ (i) of Theorem 2.6 (cf. also [Longo 2008, Proposition 3.7]). In this case, [Rieffel and van Daele 1977, Theorem 3.8] even implies that $U_t = \Delta^{-it/\beta}$ is the unique unitary one-parameter group satisfying the KMS condition for β. From [Rieffel and van Daele 1977, Theorem 3.9], one can also derive the implication (i) $\Rightarrow$ (ii). Instead of Δ, Rieffel and van Daele work with the bounded operator $R = 2(\mathbf{1}+\Delta)^{-1}$ which is the sum of the orthogonal projections of the real Hilbert space $\mathcal{H}$ onto the closed subspaces V and iV. In our context, this operator appears as $\mathbf{1} + i\widehat{C}$ for the skew-hermitian operator $\widehat{C} = i\,\frac{\Delta-\mathbf{1}}{\Delta+\mathbf{1}}$ (Lemma 4.2).

In the context of free fields, the interplay between standard real subspaces and von Neumann algebras of operators on Fock space has already been studied by Araki [1963] and Eckmann and Osterwalder [1973]. The connection between the KMS condition and the modular theory of von Neumann algebras has already been observed and studied in [Haag et al. 1967]. We refer to [Yngvason 1994] for some particularly interesting concrete subspaces corresponding to fields on light rays and to [Ramacher 2000] for descriptions of standard real subspaces in terms of boundary values of holomorphic functions. Numerical aspects of the KMS condition and rather general holomorphic extension aspects have recently been studied in [De Micheli and Viano 2012].

Notation. We follow the "physics convention" that the scalar product $\langle \cdot, \cdot \rangle$ on a complex Hilbert space is linear in the second argument.

For a real vector space V, we write $\mathrm{Bil}(V)$ for the complex vector space of complex-valued bilinear forms $V \times V \to \mathbb{C}$. For $h \in \mathrm{Bil}(V)$, we write $\bar{h}$ for the pointwise complex conjugate and put $h^\top(v, w) := h(w, v)$ and $h^* := \bar{h}^\top$. We say that h is *hermitian* if $\bar{h} = h^\top$, which means that $\mathrm{Re}\, h$ is symmetric and $\mathrm{Im}\, h$ is skew-symmetric. We write $\mathrm{Herm}(V) \subseteq \mathrm{Bil}(V)$ for the real subspace of hermitian forms.

Every $h \in \mathrm{Bil}(V)$ extends canonically to a sesquilinear form on $V_{\mathbb{C}}$ (linear in the second argument),

$$h_{\mathbb{C}}(v + iw, v' + iw') := h(v, v') - ih(w, v') + ih(v, w') + h(w, w').$$

We may therefore identify $\mathrm{Bil}(V)$ with the space $\mathrm{Sesq}(V_{\mathbb{C}})$ of sesquilinear forms on the complex vector space $V_{\mathbb{C}}$. We write $\mathrm{Bil}^+(V) \subseteq \mathrm{Bil}(V)$ for the convex cone of all those bilinear forms f for which the sesquilinear extension to $V_{\mathbb{C}} \times V_{\mathbb{C}}$ is positive semidefinite, i.e., for which h defines a positive definite kernel on V.

2. Positive definite functions and KMS conditions

Throughout this section V is an arbitrary real vector space. We recall from Definition A.3 that a function $\psi : \mathbb{R} \to \mathrm{Bil}(V)$ is called *positive definite* if the kernel $K((t, v), (s, w)) := \psi(t - s)(v, w)$ on $\mathbb{R} \times V$ is positive definite. The main result of this section is Theorem 2.6. This result leads in particular to the analytic continuation of ψ to the strip $\mathcal{S}_\beta$. We also explain how the corresponding representation of $\mathbb{R}$ can be realized in a Hilbert space consisting of holomorphic functions on the strip $\mathcal{S}_{\beta/2}$ with continuous boundary values (Proposition 2.9).

We call a function $\psi : \overline{\mathcal{S}_\beta} \to \mathrm{Bil}(V)$ *pointwise continuous* if, for all $v, w \in V$, the function $\psi^{v,w}(z) := \psi(z)(v, w)$ is continuous. Moreover, we say that ψ is *pointwise holomorphic in* $\mathcal{S}_\beta$, if, for all $v, w \in V$, the function $\psi^{v,w}|_{\mathcal{S}_\beta}$ is holomorphic. By the Schwarz reflection principle, any pointwise continuous pointwise holomorphic function ψ is uniquely determined by its restriction to $\mathbb{R}$.

Definition 2.1. For $\beta > 0$, let $\mathcal{S}_\beta := \{z \in \mathbb{C} : 0 < \operatorname{Im} z < \beta\}$. For a real vector space V, we say that a positive definite function $\psi : \mathbb{R} \to \mathrm{Bil}(V)$ satisfies the *KMS condition* for $\beta > 0$ if ψ extends to a function $\psi : \overline{\mathcal{S}_\beta} \to \mathrm{Bil}(V)$ which is pointwise continuous, pointwise holomorphic on $\mathcal{S}_\beta$, and satisfies

$$\psi(i\beta + t) = \overline{\psi(t)} \quad \text{for } t \in \mathbb{R}. \tag{3}$$

Lemma 2.2. *Suppose that* $\psi : \mathbb{R} \to \mathrm{Bil}(V)$ *satisfies the KMS condition for* $\beta > 0$. *Then*

$$\psi(-\overline{z}) = \psi(z)^* \quad \textit{and} \quad \psi(i\beta + \overline{z}) = \overline{\psi(z)} \qquad \textit{for } z \in \overline{\mathcal{S}_\beta}. \tag{4}$$

The function $\varphi : [0, \beta] \to \mathrm{Bil}(V)$, $\varphi(t) := \psi(it)$ *has hermitian values and satisfies*

$$\varphi(\beta - t) = \overline{\varphi(t)} \quad \textit{for } 0 \le t \le \beta. \tag{5}$$

It extends to a unique pointwise continuous symmetric 2β*-periodic function* $\varphi : \mathbb{R} \to \mathrm{Herm}(V)$ *satisfying*

$$\varphi(\beta + t) = \overline{\varphi(t)} \quad \textit{for } t \in \mathbb{R}.$$

Proof. Note that $\psi(-t) = \psi(t)^*$ holds for every positive definite function $\psi : \mathbb{R} \to \mathrm{Bil}(V)$. By analytic continuation (and the Schwarz reflection principle), this leads to the first part of (4). Likewise, condition (3) leads to the second part of (4). This in turn implies (5), and the remainder is clear. □

Remark 2.3. Note that (4) implies in particular that $\psi(i\beta/2 + t)$ is real-valued for $t \in \mathbb{R}$ (cf. [Rieffel and van Daele 1977, Proposition 3.5]).

We now introduce standard real subspaces $V \subseteq \mathcal{H}$ and the associated modular objects (Δ, J).

Definition 2.4. A closed real subspace V of a complex Hilbert space $\mathcal{H}$ is said to be *standard* if

$$V \cap iV = \{0\} \quad \text{and} \quad \overline{V + iV} = \mathcal{H}.$$

For every standard real subspace $V \subseteq \mathcal{H}$, we define an unbounded antilinear operator

$$S : \mathcal{D}(S) = V + iV \to \mathcal{H}, \quad S(v + iw) := v - iw, \quad v, w \in V.$$

Then S is closed and has a polar decomposition $S = J\Delta^{1/2}$, where J is an antiunitary involution and Δ a positive selfadjoint operator (cf. [Neeb and Ólafsson 2015b, Lemma 4.2]; see also [Bratteli and Robinson 1979, Proposition 2.5.11; Longo 2008, Proposition 3.3]). We call (Δ, J) the *modular objects* of V.

Remark 2.5. (a) From $S^2 = \mathrm{id}$, it follows that the modular objects (Δ, J) of a standard real subspace satisfy the modular relation

$$J\Delta J = \Delta^{-1}. \tag{6}$$

If, conversely, (Δ, J) is a pair of a positive selfadjoint operator Δ and an antilinear involution J satisfying (6), then $S := J\Delta^{1/2}$ is an unbounded antilinear involution with $\mathcal{D}(S) = \mathcal{D}(\Delta^{1/2})$ whose fixed point space $\mathrm{Fix}(S)$ is a standard real subspace. Thus standard real subspaces are parametrized by pairs (Δ, J) satisfying (6) (cf. [Longo 2008, Proposition 3.2] and [Neeb and Ólafsson 2015b, Lemma 4.4]).

(b) As the unitary one-parameter group Δ^{it} commutes with J and Δ, it leaves the real subspace $V = \mathrm{Fix}(S)$ invariant.

We now come to the proof of Theorem 2.6.

Theorem 2.6 (Characterization of the KMS condition). *Let V be a real vector space and let $\psi : \mathbb{R} \to \mathrm{Bil}(V)$ be a pointwise continuous positive definite function. Then the following are equivalent*:

(i) *ψ satisfies the KMS condition for $\beta > 0$.*

(ii) *There exists a standard real subspace V_1 in a Hilbert space $\mathcal{H}$ and a linear map $j : V \to V_1$ such that*

$$\psi(t)(v, w) = \langle j(v), \Delta^{-it/\beta} j(w)\rangle \quad \textit{for } t \in \mathbb{R}, v, w \in V. \tag{7}$$

(iii) *There exists a $\mathrm{Bil}^+(V)$-valued regular Borel measure μ on $\mathbb{R}$ satisfying*

$$d\mu(-\lambda) = e^{-\beta\lambda} d\overline{\mu}(\lambda),$$

such that

$$\psi(t) = \int_{\mathbb{R}} e^{it\lambda}\, d\mu(\lambda) = \widehat{\mu}(t).$$

If these conditions are satisfied, then the function

$$\psi : \overline{\mathcal{S}_\beta} \to \mathrm{Bil}(V)$$

is pointwise bounded.

Proof. (i) $\Rightarrow$ (ii): From the GNS construction (Proposition A.4), we obtain a continuous unitary representation $(U, \mathcal{H})$ and a linear map $j : V \to \mathcal{H}$ such that

$$\psi(t)(v, w) = \langle j(v), U_t j(w)\rangle \quad \text{for } t \in \mathbb{R},\ v, w \in V.$$

We further assume that the range of the map

$$\zeta : \mathbb{R} \times V \to \mathcal{H}, \quad \zeta(t, v) := U_t j(v)$$

spans a dense subspace. Using Stone's theorem, we write $U_t = e^{-itH}$ for a selfadjoint operator H on $\mathcal{H}$ and consider the positive selfadjoint operator

$$\Delta := e^{\beta H} \quad \text{satisfying } U_t = \Delta^{-it/\beta} \ \text{for } t \in \mathbb{R}.$$

With the $B(\mathcal{H})$-valued spectral measure P on $\mathbb{R}$ with $H = \int_{\mathbb{R}} \lambda \, dP(\lambda)$, we thus obtain

$$\psi(t)(v, w) = \langle j(v), e^{-itH} j(w)\rangle = \int_{\mathbb{R}} e^{-it\lambda} \, dP^{j(v),j(w)}(\lambda),$$

where $P^{v,w} = \langle v, P(\,\cdot\,)w\rangle$. The KMS condition for ψ gives that, for each $v \in V$, the function $\psi(t)(v, v)$ extends holomorphically to $\overline{\mathcal{S}_\beta}$, which implies that the integral $\int_{\mathbb{R}} e^{\beta\lambda} \, dP^{j(v),j(v)}(\lambda)$ is finite, and hence that $j(V) \subseteq \mathcal{D}(\Delta^{1/2})$ [Neeb and Ólafsson 2015b, Lemma B.4]. The uniqueness of analytic continuation (Schwarz' principle) now implies

$$\begin{aligned}(8) \qquad \psi(x+iy)(v, w) &= \int_{\mathbb{R}} e^{-i(x+iy)\lambda} \, dP^{j(v),j(w)}(\lambda) \\ &= \langle \Delta^{y/2\beta} j(v), \Delta^{-ix/\beta} \Delta^{y/2\beta} j(w)\rangle\end{aligned}$$

for $v, w \in V$ and $0 \leq y \leq \beta$. Since $\mathcal{D}(\Delta^{1/2})$ is U-invariant, we obtain from the KMS condition,

$$\begin{aligned}\langle \Delta^{1/2}\zeta(t, v), \Delta^{1/2}\zeta(s, w)\rangle = \langle \Delta^{1/2} j(v), \Delta^{1/2} U_{s-t} j(w)\rangle &= \psi(i\beta + s - t)(v, w) \\ = \overline{\psi(s-t)(v, w)} &= \overline{\langle \zeta(t, v), \zeta(s, w)\rangle}.\end{aligned}$$

This implies the existence of a unique antilinear isometry $J : \mathcal{H} \to \mathcal{H}$ with

$$J\zeta(t, v) = \Delta^{1/2}\zeta(t, v) \quad \text{for all } t \in \mathbb{R},\ v \in V.$$

Then

$$U_s J\zeta(t, v) = \Delta^{1/2}\zeta(t+s, v) = J\zeta(t+s, v) = JU_s\zeta(t, v) \quad \text{for } t, s \in \mathbb{R},\ v \in V$$

shows that J commutes with every U_t. This implies that $J\Delta^{1/2}J^{-1} = \Delta^{-1/2}$, so

$$\zeta(t, v) = J^{-1}\Delta^{1/2}\zeta(t, v) = \Delta^{-1/2}J^{-1}\zeta(t, v),$$

which in turn implies

$$J\zeta(t, v) = \Delta^{1/2}\zeta(t, v) = J^{-1}\zeta(t, v) \quad \text{for } t \in \mathbb{R},\ v \in V.$$

Since the range of ζ is total, it follows that $J^{-1} = J$, so J is an anti-unitary involution. Therefore (Δ, J) is the modular object of the standard real subspace $V_1 := \mathrm{Fix}(S)$ for the unbounded antilinear involution $S := J\Delta^{1/2}$ (Remark 2.5).

For $v \in V$, we now have $j(v) \in \mathcal{D}(S) = \mathcal{D}(\Delta^{1/2})$ and $Sj(v) = J\Delta^{1/2}j(v) = J^2 j(v) = j(v)$, so that $j(V) \subseteq V_1$. This completes the proof of (ii).

(ii) $\Rightarrow$ (iii): For $v, w \in V$ we have

$$\psi(t)(v, w) = \langle j(v), \Delta^{-it/\beta} j(w)\rangle = \int_{\mathbb{R}} e^{it\lambda} \langle j(v), dP(\lambda) j(w)\rangle,$$

where P is the spectral measure of the selfadjoint operator $L := -\frac{1}{\beta}\log\Delta$ (the

Liouvillian). We therefore consider the $\mathrm{Bil}^+(V)$-valued measure defined by

$$\mu(\,\cdot\,)(v, w) := \langle j(v), P(\,\cdot\,)j(w)\rangle = P^{j(v),j(w)}.$$

It remains to show that $d\mu(-\lambda) = e^{-\beta\lambda}d\overline{\mu}(\lambda)$, which means that $r_*\mu = e_{-\beta}\overline{\mu}$ holds for $r(\lambda) = -\lambda$. To verify this relation, we first observe that $JLJ = -L$ implies that $JPJ = r_*P$. This leads to

$$\begin{aligned}\overline{\mu(\,\cdot\,)(v, w)} &= \langle P(\,\cdot\,)j(w), j(v)\rangle = \langle P(\,\cdot\,)Sj(w), Sj(v)\rangle \\ &= \langle P(\,\cdot\,)J\Delta^{1/2}j(w), J\Delta^{1/2}j(v)\rangle \\ &= \langle JP(\,\cdot\,)J\Delta^{1/2}j(v), \Delta^{1/2}j(w)\rangle = \langle (r_*P)(\,\cdot\,)\Delta^{1/2}j(v), \Delta^{1/2}j(w)\rangle \\ &= e_\beta \cdot \langle (r_*P)(\,\cdot\,)j(v), j(w)\rangle = e_\beta \cdot (r_*\mu)(\,\cdot\,)(v, w).\end{aligned}$$

This implies that $\overline{\mu} = e_\beta \cdot r_*\mu$.

(iii) $\Rightarrow$ (i): Condition (iii) implies that $\psi(0) = \mu(\mathbb{R})$ exists, so that μ is a pointwise finite measure. Further, the relation $r_*\mu = e_{-\beta}\overline{\mu}$ implies that the measure $e_{-\beta}\mu$ is also finite. Therefore the integral

$$\psi(z) := \int_{\mathbb{R}} e^{iz\lambda}\, d\mu(\lambda) \tag{9}$$

exists pointwise and extends ψ to $\overline{\mathcal{S}_\beta}$ in such a way that this extension is pointwise continuous on $\overline{\mathcal{S}_\beta}$ and pointwise holomorphic on the interior. The relation $r_*\mu = e_{-\beta}\overline{\mu}$ further leads to

$$\begin{aligned}\psi(i\beta + t) &= \int_{\mathbb{R}} e^{\lambda(-\beta+it)}\, d\mu(\lambda) = \int_{\mathbb{R}} e_{-\beta}(\lambda)e^{i\lambda t}\, d\mu(\lambda) \\ &= \int_{\mathbb{R}} e^{i\lambda t}\, d(r_*\overline{\mu})(\lambda) = \int_{\mathbb{R}} e^{-i\lambda t}\, d\overline{\mu}(\lambda) = \overline{\psi(t)}.\end{aligned}$$

Therefore ψ satisfies the KMS condition for β.

We finally assume that (i)–(iii) are satisfied and show that ψ is pointwise bounded on $\overline{\mathcal{S}_\beta}$. Since each $\psi(z)$ extends to a sesquilinear form $\psi(z)_{\mathbb{C}}$ on $V_{\mathbb{C}}$, in view of the polarization identity, it suffices to show the boundedness of the functions $z \mapsto \psi(z)_{\mathbb{C}}(v, v)$ for $v \in V_{\mathbb{C}}$. For the positive measure $\mu^{v,v}(E) := \mu(E)_{\mathbb{C}}(v, v)$, we obtain from (9) the estimate

$$|\psi(z)_{\mathbb{C}}(v, v)| \le \int_{\mathbb{R}} |e^{-i\lambda z}|\, d\mu^{v,v}(\lambda) = \int_{\mathbb{R}} e^{\lambda\, \mathrm{Im}\, z}\, d\mu^{v,v}(\lambda).$$

The convexity of the function on the right, the Laplace transform of the finite positive measure $\mu^{v,v}$, and $\psi(\beta i)(v, v) = \|\Delta^{1/2}j(v)\|^2 < \infty$ now imply the boundedness of $\psi(z)_{\mathbb{C}}(v, v)$. $\square$

Remark 2.7. A special case worth noting arises from a C^*-dynamical system $(\mathcal{A}, \mathbb{R}, \alpha)$ for $V := \mathcal{A}_h := \{A \in \mathcal{A} : A^* = A\}$ and an invariant state ω on $\mathcal{A}$. Such a state is a β-KMS state if and only if

$$\psi : \mathbb{R} \to \mathrm{Bil}(\mathcal{A}_h), \quad \psi(t)(A, B) := \omega(A\alpha_t(B))$$

satisfies the KMS condition for $\beta > 0$ (cf. [Neeb and Ólafsson 2015b, Proposition 5.2; Rieffel and van Daele 1977, Theorem 4.10; Bratteli and Robinson 1981]). If $(\pi_\omega, U^\omega, \mathcal{H}_\omega, \Omega)$ is the corresponding covariant GNS representation of $(\mathcal{A}, \mathbb{R})$,

$$\omega(A) = \langle \Omega, \pi_\omega(A)\Omega \rangle \quad \text{for } A \in \mathcal{A} \qquad \text{and} \qquad U_t^\omega \Omega = \Omega \quad \text{for } t \in \mathbb{R}.$$

Therefore

$$\begin{aligned} \psi(t)(A, B) &= \omega(A\alpha_t(B)) = \langle \Omega, \pi_\omega(A\alpha_t(B))\Omega \rangle \\ &= \langle \Omega, \pi_\omega(A) U_t^\omega \pi_\omega(B) U_{-t}^\omega \Omega \rangle = \langle \pi_\omega(A)\Omega, U_t^\omega \pi_\omega(B)\Omega \rangle \end{aligned}$$

for $A, B \in \mathcal{A}_h$. The corresponding standard real subspace of $\mathcal{H}_\omega$ is $V_1 := \overline{\pi_\omega(\mathcal{A}_h)\Omega}$.

Corollary 2.8. *If $\psi : \mathbb{R} \to \mathrm{Bil}(V)$ satisfies the β-KMS condition, then the kernel*

$$K : \overline{\mathcal{S}_{\beta/2}} \times \overline{\mathcal{S}_{\beta/2}} \to \mathrm{Bil}(V), \qquad K(z, w)(\xi, \eta) := \psi(z - \overline{w})(\xi, \eta) \tag{10}$$

is positive definite.

Proof. This follows immediately from the following relation that we derive from (8):

$$\begin{aligned} K(z, w)(\xi, \eta) &= \psi(z - \overline{w})(\xi, \eta) \\ &= \langle \Delta^{-\frac{i\overline{z}}{\beta}} j(\xi), \Delta^{-\frac{i\overline{w}}{\beta}} j(\eta) \rangle \quad \text{for } \xi, \eta \in V,\ z, w \in \overline{\mathcal{S}_{\beta/2}}. \end{aligned}$$ □

Now that we know from Corollary 2.8 that the kernel K in (10) is positive definite, we obtain a corresponding reproducing kernel Hilbert space consisting of functions on $\overline{\mathcal{S}_{\beta/2}} \times V$ which are linear in the second argument and holomorphic on $\mathcal{S}_{\beta/2}$ in the first. We may therefore think of these functions as having values in the algebraic dual space $V^* := \mathrm{Hom}(V, \mathbb{R})$ of V. We write $\mathcal{O}(\overline{\mathcal{S}_{\beta/2}}, V^*)$ for the space of those functions $f : \overline{\mathcal{S}_{\beta/2}} \to V^*$ with the property that, for every $\eta \in V$, the function $z \mapsto f(z)(\eta)$ is continuous on $\overline{\mathcal{S}_{\beta/2}}$ and holomorphic on the open strip $\mathcal{S}_{\beta/2}$.

Proposition 2.9 (Realization of $\mathcal{H}_\psi$ on $\mathcal{O}(\overline{\mathcal{S}_{\beta/2}}, V^*)$). *Assume that $\psi : \mathbb{R} \to \mathrm{Bil}(V)$ satisfies the KMS condition for $\beta > 0$ and let $\psi : \overline{\mathcal{S}_\beta} \to \mathrm{Bil}(V)$ denote the corresponding extension and $\mathcal{H}_\psi \subseteq \mathcal{O}(\overline{\mathcal{S}_{\beta/2}}, V^*)$ denote the Hilbert space with reproducing kernel*

$$K(z, w)(\xi, \eta) := \psi(z - \overline{w})(\xi, \eta) \quad \text{for } \xi, \eta \in V,$$

i.e.,

$$f(z)(\xi) = \langle K_{z,\xi}, f \rangle \quad \text{for } f \in \mathcal{H}_\psi, \ \text{where } K_{z,\xi}(w)(\eta) = \psi(w - \overline{z})(\eta, \xi).$$

Then

$$(U_t^\psi f)(z) := f(z+t), \qquad t \in \mathbb{R}, z \in \overline{\mathcal{S}_{\beta/2}},$$

defines a unitary one-parameter group on $\mathcal{H}_\psi$,

$$j : V \to \mathcal{H}_\psi, \quad j(\eta)(z) := \psi(z)(\,\cdot\,, \eta)$$

is a linear map with U^ψ*-cyclic range, and*

$$\psi(t)(\xi, \eta) = \langle j(\xi), U_t^\psi j(\eta)\rangle \quad \textit{for } t \in \mathbb{R}, \xi, \eta \in V.$$

The anti-unitary involution on $\mathcal{H}_\psi$ *corresponding to the standard real subspace* $V_1 \subseteq \mathcal{H}_\psi$ *from Theorem 2.6 is given by*

$$(J_1 f)(z) := \overline{f\left(\overline{z} + \frac{i\beta}{2}\right)}. \tag{11}$$

Proof. First we recall that the natural reproducing kernel Hilbert space $\mathcal{H}_\psi = \mathcal{H}_K$ is generated by the function $K_{(w,\eta)}$ satisfying

$$\begin{aligned} K_{(w,\eta)}(z)(\xi) = \langle K_{(z,\xi)}, K_{(w,\eta)}\rangle &= K(z, w)(\xi, \eta) \\ &= \psi(z - \overline{w})(\xi, \eta). \end{aligned}$$

As a function of z, the kernel K is continuous on $\overline{\mathcal{S}_{\beta/2}}$ and holomorphic on the interior. Therefore [Neeb 2000, Proposition I.1.9] implies that $\mathcal{H}_\psi$ is a subspace of $\mathcal{O}(\overline{\mathcal{S}_{\beta/2}}, V^*)$, where, for every $f \in \mathcal{H}_\psi$ and $\xi \in V$, we have

$$f(z)(\xi) = \langle K_{(z,\xi)}, f\rangle.$$

That the formula for U_t^ψ defines a unitary one-parameter group on $\mathcal{H}_\psi$ follows directly from the invariance of the kernel K under the action of $\mathbb{R}$ on $\overline{\mathcal{S}_\beta}$ by translation.

Next we observe that

$$\begin{aligned} \langle j(\xi), U_t^\psi j(\eta)\rangle &= \langle K_{(0,\xi)}, U_t^\psi K_{(0,\eta)}\rangle \\ &= \langle K_{(0,\xi)}, K_{(-t,\eta)}\rangle = \psi(t)(\xi, \eta). \end{aligned}$$

To see that $j(V)$ is U^ψ-cyclic, we have to show that the elements $U_t^\psi j(\eta) = K_{(-t,\eta)}$ form a total subset of $\mathcal{H}_\psi$. This means that any $f \in \mathcal{H}_\psi$ with

$$0 = \langle K_{(t,\eta)}, f\rangle = f(t)(\eta)$$

for every $t \in \mathbb{R}$ and $\eta \in V$ vanishes. As the function $t \mapsto f(t)(\eta)$ extends to a continuous function on $\overline{\mathcal{S}_{\beta/2}}$, holomorphic on the interior, it vanishes by the Schwarz reflection principle. Further, η was arbitrary, so $f = 0$ follows.

Now we turn to the involution J_1. As $K_{(w,\eta)}(z) = \psi(z - \overline{w})(\cdot\,,\eta)$, the operator J_1 on $\mathcal{O}(\overline{\mathcal{S}_{\beta/2}}, V^*)$, defined by the right hand side of (11) satisfies

$$(J_1 K_{(w,\eta)})(z) = \overline{K_{(w,\eta)}\left(\overline{z} + \frac{i\beta}{2}\right)} = \overline{\psi\left(\overline{z} + \frac{i\beta}{2} - \overline{w}\right)(\cdot\,,\eta)} \tag{12}$$
$$= \psi\left(i\beta + z - \frac{i\beta}{2} - w\right)(\cdot\,,\eta) = \psi\left(z + \frac{i\beta}{2} - w\right)(\cdot\,,\eta)$$
$$= K_{(\overline{w}+i\beta/2,\eta)}(z).$$

Here we have used that $\overline{\psi(z)} = \psi(i\beta + \overline{z})$ (Lemma 2.2). From

$$\langle K_{(\overline{w}+i\beta/2,\eta)}, K_{(\overline{z}+i\beta/2,\xi)}\rangle = \overline{K(\overline{z} + i\beta/2, \overline{w} + i\beta/2)(\xi,\eta)} = \overline{\psi(i\beta + \overline{z} - w)(\xi,\eta)}$$
$$= \psi(z - \overline{w})(\xi,\eta) = \langle K_{(z,\xi)}, K_{(w,\eta)}\rangle,$$

it now follows that the operator J_1 in (11) leaves the subspace $\mathcal{H}_\psi$ invariant and defines an antilinear isometry on this space. From the explicit formula it follows that J_1 is an involution. It is also clear that J_1 commutes with the unitary operators $(U_t f)(z) = f(z+t)$.

The relation $U_t K_{(w,\eta)} = K_{(w-t,\eta)}$ leads by analytic continuation to

$$J_1 K_{(0,\eta)} = K_{(i\beta/2,\eta)} = \Delta^{1/2} K_{(0,\eta)}.$$

In the proof of Theorem 2.6, we have seen that, for $\eta \in V$ and $t \in \mathbb{R}$, the anti-unitary involution J corresponding to the associated standard real subspace V_1 satisfies

$$Jj(\eta) = \Delta^{1/2} j(\eta).$$

As both J and J_1 commute with every U_t and the subset $\{U_t j(\eta) : t \in \mathbb{R}, \eta \in V\}$ is total in $\mathcal{H}_\psi$, we conclude that $J_1 = J$. □

3. Form-valued reflection positive functions

In this section we discuss reflection positivity on the level of form-valued positive definite functions. This is particularly well adapted to reflection positive Hilbert spaces $(\mathcal{E}, \mathcal{E}_+, \theta)$, for which $\mathcal{E}_+$ is generated by elements of the form $U_g^{-1} j(v)$, where g is contained in a certain subset $G_+ \subseteq G$ which is not necessarily a subsemigroup, and $j : V \to \mathcal{H}$ is a linear map for which $U_G j(V)$ spans a dense subspace of $\mathcal{E}$. In particular, we present a version of the GNS construction in this context (Proposition 3.9) and we briefly discuss it more specifically for the trivial group $G = \{\mathbf{1}\}$ (Section 3B) and the 2-element group (Section 3C). The latter case shows explicitly that the cone of reflection positive functions does not adapt naturally to the decomposition into even and odd functions. Put differently, if a reflection positive representation decomposes into two subrepresentations, the summands need not be reflection positive (see also [Neeb and Ólafsson 2014]).

3A. *Reflection positivity and form-valued functions.* Let (G, τ) be a *symmetric Lie group*, i.e., G is a Lie group and $\tau \in \mathrm{Aut}(G)$ with $\tau^2 = \mathrm{id}_G$. In the following we write $G_\tau := G \rtimes \{\mathbf{1}, \tau\}$ and $g^\sharp := \tau(g)^{-1}$ [Neeb and Ólafsson 2014]. In this section we introduce reflection positive functions on G_τ with values in $\mathrm{Bil}(V)$ for a real vector space V.

Definition 3.1. Let $\mathcal{E}$ be a Hilbert space and let $\theta \in \mathrm{U}(\mathcal{E})$ be an involution. A closed subspace $\mathcal{E}_+ \subseteq \mathcal{E}$ is called *θ-positive* if $\langle \theta v, v\rangle \geq 0$ for $v \in \mathcal{E}_+$. We then call the triple $(\mathcal{E}, \mathcal{E}_+, \theta)$ a *reflection positive Hilbert space*. For a reflection positive Hilbert space we put $\mathcal{N} := \{v \in \mathcal{E}_+ : \langle \theta v, v\rangle = 0\}$ and write $q : \mathcal{E}_+ \to \mathcal{E}_+/\mathcal{N}$, $v \mapsto \widehat{v} = q(v)$ for the quotient map and $\widehat{\mathcal{E}}$ for the Hilbert completion of $\mathcal{E}_+/\mathcal{N}$ with respect to the norm $\|\widehat{v}\|_{\widehat{\mathcal{E}}} := \|\widehat{v}\| := \sqrt{\langle \theta v, v\rangle}$.

Example 3.2. Suppose that $K\colon X \times X \to \mathbb{C}$ is a positive definite kernel on the set X and that $\tau\colon X \to X$ is an involution leaving K invariant. We further assume that $X_+ \subseteq X$ is a subset with the property that the kernel $K^\tau(x, y) := K(\tau x, y)$ is also positive definite on X_+.

Let $\mathcal{E} := \mathcal{H}_K \subseteq \mathbb{C}^X$ denote the corresponding reproducing kernel Hilbert space generated by elements $(K_x)_{x\in X}$ with $\langle K_x, K_y\rangle = K(x, y)$. Then the closed subspace $\mathcal{E}_+ \subseteq \mathcal{E}$ generated by $(K_x)_{x\in X_+}$ is θ-positive for $(\theta f)(x) := f(\tau x)$. We thus obtain a reflection positive Hilbert space $(\mathcal{E}, \mathcal{E}_+, \theta)$. We call such kernels K *reflection positive* with respect to (X, X_+, τ).

Definition 3.3. Let $G_+ \subseteq G$ be a subset. Let V be a real vector space and let $j : V \to \mathcal{H}$ be a linear map whose range is cyclic for the unitary representation $(U, \mathcal{E})$ of G_τ. Then we say that $(U, \mathcal{E}, j, V)$ is *reflection positive with respect to $G_+ \subseteq G$* if, for $\mathcal{E}_+ := \overline{\mathrm{span}\, U_{G_+}^{-1} j(V)}$, the triple $(\mathcal{E}, \mathcal{E}_+, U_\tau)$ is a reflection positive Hilbert space.

Definition 3.4. Let V be a real vector space. We call a function $\varphi : G_\tau \to \mathrm{Bil}(V)$ *reflection positive with respect to the subset G_+ of G* if

(RP1) φ is positive definite and

(RP2) the kernel $(s, t) \mapsto \varphi(st^\sharp\tau) = \varphi(s\tau t^{-1})$ is positive definite on G_+.

Remark 3.5. Let $\varphi : G_\tau \to \mathrm{Bil}(V)$ be a positive definite function, so that the kernel $K((x, v), (y, w)) := \varphi(xy^{-1})(v, w)$ on $G_\tau \times V$ is positive definite. The involution τ acts on $G_\tau \times V$ by $\tau.(g, v) := (g\tau, v)$ and the corresponding kernel $K^\tau((x, v), (y, w)) := K((x\tau, v), (y, w)) = \varphi(x\tau y^{-1})(v, w)$ is positive definite on $G_+ \times V$ if and only if φ is reflection positive in the sense of Example 3.2.

Positive definite functions on G extend canonically to G_τ if they are τ-invariant:

Lemma 3.6. *Let V be a real vector space and let (G, τ) be a symmetric Lie group. Then the following assertions hold*:

(i) *If* $\varphi : G \to \mathrm{Bil}(V)$ *is a positive definite function which is* τ*-invariant in the sense that* $\varphi \circ \tau = \varphi$, *then* $\widehat{\varphi}(g, \tau) := \varphi(g)$ *defines an extension to* G_τ *which is positive definite and* τ*-biinvariant.*

(ii) *Let* $(U, \mathcal{H})$ *be a unitary representation of* G_τ, *let* $\theta := U_\tau$, *let* $j : V \to \mathcal{H}$ *be a linear map, and let* $\varphi(g)(v, w) = \langle j(v), U_g j(w)\rangle$ *be the corresponding* Bil(V)*-valued positive definite function. Then the following are equivalent*:

(a) $\theta j(v) = j(v)$ *for every* $v \in V$.

(b) φ *is* τ*-biinvariant.*

(c) φ *is left* τ*-invariant.*

Proof. (i) From the GNS construction (Proposition A.4), we obtain a continuous unitary representation $(U, \mathcal{H})$ of G and a linear map $j : V \to \mathcal{H}$ such that

$$\varphi(g)(v, w) = \langle j(v), U_g j(w)\rangle \quad \text{for } g \in G,\ v, w \in V.$$

As $\varphi(g)(v, w) = \varphi(\tau(g))(v, w)$, the uniqueness in the GNS construction provide a unitary operator $\theta : \mathcal{H} \to \mathcal{H}$ with

$$\theta U_g j(v) = U_{\tau(g)} j(v) \quad \text{for } g \in G,\ v \in V.$$

Note that θ fixes each $j(v)$. Therefore $U_\tau := \theta$ defines an extension of G to a unitary representation of G_τ on $\mathcal{H}$. Hence $\psi(g)(v, w) = \langle j(v), U_g j(w)\rangle$ defines a positive definite Bil(V)-valued function on G_τ which satisfies

$$\begin{aligned}\psi(g, \tau)(v, w) &= \langle \theta j(v), U_g j(w)\rangle \\ &= \langle j(v), U_g j(w)\rangle = \varphi(g)(v, w) \quad \text{for } g \in G, v, w \in V.\end{aligned}$$

(ii) Clearly, (a) $\Rightarrow$ (b) $\Rightarrow$ (c). It remains to show that (c) implies (a). So we assume that $\varphi(\tau g) = \varphi(g)$ for $g \in G_\tau$. This means that, for every $v, w \in V$, we have

$$\begin{aligned}\langle j(v), U_g j(w)\rangle &= \varphi(g)(v, w) = \varphi(\tau g)(v, w) \\ &= \langle j(v), \theta U_g j(w)\rangle = \langle \theta j(v), U_g j(w)\rangle.\end{aligned}$$

Since $U_{G_\tau} j(V)$ is total in $\mathcal{H}$, this implies that $\theta j(v) = j(v)$ for every $v \in V$. □

Remark 3.7. (a) As G_τ consists of the two cosets G and $G\tau = G \times \{\tau\}$, every function φ on G_τ is given by a pair of functions on G:

$$\varphi_\pm : G \to \mathrm{Bil}(V), \quad \varphi_+(g) := \varphi(g, \mathbf{1}), \qquad \varphi_-(g) := \varphi(g, \tau).$$

Then (RP2) is a condition on φ_- alone, and (RP1) is a condition on the pair (φ_+, φ_-).

(b) If φ is reflection positive, then its complex conjugate $\overline{\varphi}$ is also reflection positive because the convex cone of positive definite kernels on a set is stable under complex conjugation. This implies in particular that $\mathrm{Re}\,\varphi = \frac{1}{2}(\varphi + \overline{\varphi})$ is reflection positive (cf. Theorem A.13).

The following lemma provides a tool which is sometimes convenient to verify positive definiteness of a function on the extended group G_τ in terms of a kernel on the original group G.

Lemma 3.8. *Every function $\varphi : G_\tau \to B(V)$ leads to a $M_2(B(V))$-valued kernel*

$$Q : G \times G \to M_2(B(V)) \cong B(V \oplus V), \quad Q(g, h) = \begin{pmatrix} \varphi(gh^{-1}) & \varphi(g\tau h^{-1}) \\ \varphi(g\tau h^{-1}) & \varphi(gh^{-1}) \end{pmatrix},$$

and the function φ on G_τ is positive definite if and only if Q is positive definite.

Proof. That Q is positive definite is equivalent to the existence of a Hilbert space $\mathcal{H}$ and a map

$$\ell : G \to B(\mathcal{H}, V \oplus V) \cong B(\mathcal{H}, V)^{\oplus 2} \quad \text{with} \;\; Q(x, y) = \ell(x)\ell(y)^* \;\; \text{for} \;\; x, y \in G$$

(cf. [Neeb 2000, Theorem I.1.4]). If ℓ is such a map, then it can be written as $\ell(x) = (\ell_1(x), \ell_2(x))$ with $\ell_j(x) \in B(\mathcal{H}, V)$. We thus obtain

$$Q(x, y) = \ell(x)\ell(y)^* = \begin{pmatrix} \ell_1(x)\ell_1(y)^* & \ell_1(x)\ell_2(y)^* \\ \ell_2(x)\ell_1(y)^* & \ell_2(x)\ell_2(y)^* \end{pmatrix}$$

and thus

$$\ell_1(x)\ell_1(y)^* = \ell_2(x)\ell_2(y)^* \quad \text{and} \quad \ell_1(x)\ell_2(y)^* = \ell_2(x)\ell_1(y)^*.$$

Therefore

$$j : G_\tau \to B(\mathcal{H}, V), \qquad j(x, \mathbf{1}) := \ell_1(x), \quad j(x, \tau) := \ell_2(x),$$

satisfies

$$j(x, \mathbf{1}) j(y, \mathbf{1})^* = \ell_1(x)\ell_1(y)^* = \varphi(xy^{-1}),$$
$$j(x, \tau) j(y, \tau)^* = \ell_2(x)\ell_2(y)^* = \varphi(xy^{-1})$$

and

$$j(x, \mathbf{1}) j(y, \tau)^* = \ell_1(x)\ell_2(y)^* = \varphi(x\tau y^{-1}),$$
$$j(x, \tau) j(y, \mathbf{1})^* = \ell_2(x)\ell_1(y)^* = \varphi(xy^{-1}).$$

We therefore have $\varphi(xy^{-1}) = j(x)j(y)^*$ for $x, y \in G_\tau$, and thus φ is positive definite.

If, conversely, φ is positive definite and $j : G_\tau \to B(\mathcal{H}, V)$ is such that $\varphi(x^{-1}y) = j(x)j(y)^*$ for $x, y \in G_\tau$, then $\ell(x) := (j(x, \mathbf{1}), j(x, \tau)) \in B(\mathcal{H}, V \oplus V)$ defines a map with $Q(x, y) = \ell(x)\ell(y)^*$ for $x, y \in G$. □

Proposition 3.9 (GNS construction for reflection positive functions). *Let V be a real vector space, let $(U, \mathcal{E})$ be a unitary representation of G_τ and put $\theta := U_\tau$. Then the following assertions hold*:

(i) *If* $(U, \mathcal{H}, j, V)$ *is reflection positive with respect to* G_+, *then*

$$\varphi(g)(v, w) := \langle j(v), U_g j(w)\rangle, \qquad g \in G_\tau,\ v, w \in V,$$

is a reflection positive $\mathrm{Bil}(V)$*-valued function.*

(ii) *If* $\varphi : G_\tau \to \mathrm{Bil}(V)$ *is a reflection positive function with respect to* G_+, *then the corresponding GNS representation* $(U^\varphi, \mathcal{H}_\varphi, j, V)$ *is a reflection positive representation, where* $\mathcal{H}_\varphi \subseteq \mathbb{C}^{G_\tau \times V}$ *is the Hilbert subspace with reproducing kernel* $K((x, v), (y, w)) := \varphi(xy^{-1})(v, w)$ *on which* G_τ *acts by*

$$(U^\varphi_g f)(x, v) := f(xg, v).$$

Proof. (i) For $s, t \in G_+$, we have

$$\begin{aligned}\varphi(s\tau t^{-1})(v, w) &= \langle j(v), U_{s\tau t^{-1}} j(w)\rangle = \langle U_{s^{-1}} j(v), U_\tau U_{t^{-1}} j(w)\rangle \\ &= \langle \theta U_{s^{-1}} j(v), U_{t^{-1}} j(w)\rangle,\end{aligned}$$

so that the kernel $(\varphi(s\tau t^{-1}))_{s,t\in G_+}$ is positive definite.

(ii) Recall the relation $\varphi(g)(v, w) = \langle j(v), U_g j(w)\rangle$ for $g \in G$, $v, w \in V$ from Proposition A.4. Moreover, $(\theta f)(x, v) = f(x\tau, v)$, and

$$\langle \theta U^\varphi_{s^{-1}} j(v), U^\varphi_{t^{-1}} j(w)\rangle = \langle j(v), U^\varphi_{s\tau t^{-1}} j(w)\rangle = \varphi(s\tau t^{-1})(v, w),$$

so the positive definiteness of the kernel $(\varphi(s\tau t^{-1}))_{s,t\in G_+}$ implies that we obtain, with $\mathcal{E} = \mathcal{H}_\varphi$ and $\mathcal{E}_+ := \overline{\mathrm{span}\,(U^\varphi_{G_+})^{-1} j(V)}$, a reflection positive Hilbert space $(\mathcal{E}, \mathcal{E}_+, \theta)$. □

3B. ***Reflection positivity for the trivial group.*** In this short section we discuss the case of the 2-element group $T = \{\mathbf{1}, \tau\}$ in some detail. It corresponds to G_τ where $G = \{\mathbf{1}\}$ is trivial, but it already demonstrates how the intricate structure of a reflection positive Hilbert space $(\mathcal{E}, \mathcal{E}_+, \theta)$ can be encoded in terms of positive definite functions on T.

A unitary representation $(U, \mathcal{E})$ of T is nothing but the specification of a unitary operator $\theta = U_\tau$ on $\mathcal{E}$. We write $\mathcal{E} = \mathcal{E}^1 \oplus \mathcal{E}^{-1}$ for the eigenspace decomposition of $\mathcal{E}$ under θ and $p^{\pm 1} : \mathcal{E} \to \mathcal{E}^{\pm 1}$ for the orthogonal projections.

Suppose, in addition, that V is a real or complex Hilbert space and that $j : V \to \mathcal{E}$ is a continuous linear map whose range generates $\mathcal{E}$ under the representation U, i.e., the projections $p^{\pm 1}(j(V)) \subseteq \mathcal{E}^{\pm 1}$ are dense subspaces. In view of the GNS construction, the data $(\mathcal{E}, U, j, V)$ is encoded in the operator-valued positive definite function

$$\varphi : T \to B(V), \quad \varphi(g) = j^* U_g j.$$

For a function $\varphi : T \to B(V)$, let $A := \varphi(\mathbf{1})$ and $B := \varphi(\tau)$. Then φ is positive definite if and only if $A = A^* \geq 0$, $B = B^*$, and the operator matrix

$$\begin{pmatrix} \varphi(\mathbf{1}) & \varphi(\tau) \\ \varphi(\tau) & \varphi(\mathbf{1}) \end{pmatrix} = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \in M_2(B(V)) \cong B(V \oplus V)$$

defines a positive operator (Lemma 3.8 and [Neeb 2000, Remark I.1.3]). This is equivalent to

$$|\langle Bv, w\rangle|^2 \leq \langle Av, v\rangle\langle Aw, w\rangle \quad \text{for } v, w \in V \tag{13}$$

(cf. Corollary A.9). Note that (13) holds if $A = \mathbf{1}$ and $\|B\| \leq 1$. If, more generally, A is invertible, then (13) is equivalent to $\|A^{-1/2}BA^{-1/2}\| \leq \mathbf{1}$. Here $A = j^*j$ basically encodes how V is mapped into $\mathcal{E}$ and B encodes the unitary involution θ.

The function φ is reflection positive with respect to $G_+ = \{\mathbf{1}\}$ if and only if $B = \varphi(\tau) \geq 0$, which means that $j(V)$ is θ-positive. In this sense reflection positive functions on T encode reflection positive Hilbert spaces $(\mathcal{E}, \mathcal{E}_+, \theta)$ by $\theta = U_\tau$ and $\mathcal{E}_+ := \overline{j(V)}$. A pair (A, B) of hermitian operators on V corresponds to a reflection positive function $\varphi : T \to B(V)$ if and only if $0 \leq B \leq A$. By the Cauchy–Schwarz inequality, this is equivalent to (13) if A and B are positive operators. This shows that

$$\varphi = \varphi_0 + \varphi_1 \quad \text{with } \varphi_0(\mathbf{1}) = A - B,\ \varphi_0(\tau) = 0 \text{ and } \varphi_1(\mathbf{1}) = \varphi_1(\tau) = B,$$

where both functions φ_0 and φ_1 are reflection positive. The function φ_0 corresponds to the case where $\mathcal{E}_+ \perp \theta\mathcal{E}_+$, so that $\widehat{\mathcal{E}} = \{0\}$, and the constant function φ_1 corresponds to the trivial representation of T, and hence to $\theta = \mathbf{1}$, which means that $q : \mathcal{E}_+ \to \widehat{\mathcal{E}}$ is isometric.

Replacing V by $\mathcal{E}_+$, we see that reflection positive functions $\varphi : T \to B(\mathcal{E}_+)$ with $\varphi(\mathbf{1}) = \mathbf{1}$ encode reflection positive Hilbert spaces $(\mathcal{E}, \mathcal{E}_+, \theta)$ for which $p^{\pm 1}(\mathcal{E}_+)$ is dense in $\mathcal{E}^{\pm 1}$. By (13), these configurations are parametrized by the hermitian contractions $B = \varphi(\tau)$ on $\mathcal{E}_+$. For $v, w \in \mathcal{E}_+$, we then have

$$\langle v, \theta w\rangle = \langle v, Bw\rangle.$$

Therefore the 1-eigenspace $\ker(B - \mathbf{1})$ corresponds to the maximal subspace in $\mathcal{E}_+$ on which the map $q : \mathcal{E}_+ \to \widehat{\mathcal{E}}$ is isometric. We also observe that $\ker B = \ker q$. In this sense the operator B describes how $\widehat{\mathcal{E}}$ is obtained from the Hilbert space $\mathcal{E}_+$.

Remark 3.10. Suppose that θ is a unitary involution on $\mathcal{E}$ with the eigenspaces $\mathcal{E}^{\pm 1}$. If $\mathcal{K} \subseteq \mathcal{E}$ is a θ-positive subspace, then clearly $\mathcal{K} \cap \mathcal{E}^{-1} = \{0\}$ and this implies that $\mathcal{K}$ is the graph $\Gamma(Z)$ of the operator

$$Z : \mathcal{D}(Z) := \{v_+ \in \mathcal{E}^1 : (\exists v_- \in \mathcal{E}^{-1})\ (v_+, v_-) \in \mathcal{K}\} \to \mathcal{E}^{-1}, \quad v_+ \mapsto v_-.$$

That $\Gamma(Z)$ is a θ-positive subspace is equivalent to $\|Z\| \leq 1$. Therefore the closedness of $\mathcal{K}$ shows that $\mathcal{D}(Z)$ is a closed subspace of $\mathcal{E}^1$ (cf. [Jorgensen 2002, Lemma 5.1]). If $p^1(\mathcal{K}) = \mathcal{D}(Z)$ is dense in $\widehat{\mathcal{E}}$, the closedness of $\mathcal{D}(Z)$ implies that $Z \in B(\mathcal{E}^1, \mathcal{E}^{-1})$. The density of $p^{-1}(\mathcal{K}) = Z(\mathcal{E}^1)$ is equivalent to Z having dense range.

From this perspective, we can also generate the configuration $(\mathcal{E}, \mathcal{E}_+, \theta)$ in terms of $\mathcal{E}^1$. Then $j(v) = (v, Zv) \in \mathcal{E}^1 \oplus \mathcal{E}^{-1}$ defines a linear map $j : \mathcal{E}^1 \to \mathcal{E}$ whose range is $\mathcal{K}$. The corresponding $B(\mathcal{E}^1)$-valued positive definite function on T is given by

$$\psi(\mathbf{1}) = j^* j = \mathbf{1} + Z^* Z \quad \text{and} \quad \psi(\tau) = j^* \theta j = \mathbf{1} - Z^* Z.$$

The polar decomposition of $j : \mathcal{E}^1 \to \mathcal{K}$ takes the form

$$j = U\sqrt{j^* j} = U\sqrt{\mathbf{1} + Z^* Z},$$

where $U : \mathcal{E}^1 \to \mathcal{K}$ is unitary. Therefore the corresponding $B(\mathcal{K})$-valued positive definite function on T is given by

$$\varphi(\mathbf{1}) = \mathbf{1} \quad \text{and} \quad \varphi(\tau) = U \frac{\mathbf{1} - Z^* Z}{\mathbf{1} + Z^* Z} U^{-1}$$

because $j^* \varphi(\tau) j = j^* \theta j = \mathbf{1} - Z^* Z$ implies

$$\begin{aligned}\varphi(\tau) &= (j^*)^{-1}(\mathbf{1} - Z^* Z) j^{-1} = U(\mathbf{1} + Z^* Z)^{-1/2}(\mathbf{1} - Z^* Z)(\mathbf{1} + Z^* Z)^{-1/2} U^{-1} \\ &= U \frac{\mathbf{1} - Z^* Z}{\mathbf{1} + Z^* Z} U^{-1}.\end{aligned}$$

Relating this to the preceding discussion, we see that $U \ker Z \subseteq \mathcal{E}_+$ is the maximal subspace on which q is isometric and

$$U\{v \in \mathcal{E}^1 : \|Zv\| = \|v\|\} = U \ker(\mathbf{1} - Z^* Z) = \ker q.$$

In particular, q is injective if and only if Z is a strict contraction.

3C. *Reflection positivity for the 2-element group.* In this subsection, we take a closer look at the 2-element group $G = \{\mathbf{1}, \sigma\}$ because it nicely illustrates that if a reflection positive representation decomposes into two subrepresentations, then the summands need not be reflection positive (see also [Neeb and Ólafsson 2014]). On the level of positive definite functions, this is reflected in the fact that the cone of reflection positive functions does not adapt to the decomposition into even and odd functions.

We consider the 2-element group $G := \{\mathbf{1}, \sigma\}$, which leads to the Klein-4-group

$$G_\tau := G \rtimes \{\mathbf{1}, \tau\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

We consider reflection positivity with respect to the subset $G_+ := \{\mathbf{1}\}$.

Any unitary representation $(U, \mathcal{E})$ of G_τ decomposes into four eigenspaces

$$\mathcal{E} = \mathcal{E}^{1,1} \oplus \mathcal{E}^{-1,1} \oplus \mathcal{E}^{1,-1} \oplus \mathcal{E}^{-1,-1}, \qquad \mathcal{E}^{\varepsilon_1,\varepsilon_2} = \{v \in \mathcal{E} : U_\sigma v = \varepsilon_1 v, U_\tau v = \varepsilon_2 v\},$$

and for $\theta := U_\tau$, the subspace $\mathcal{E}^\theta = \mathcal{E}^{1,1} \oplus \mathcal{E}^{-1,1}$ is U_σ-invariant. For $v = (a, b, c, d)$, we then have

$$U_\tau v = (a, b, -c, -d) \quad \text{and} \quad U_\sigma v = (a, -b, c, -d).$$

Assume $\mathcal{E}_+ = \mathbb{C}v$ for a single vector v. Then reflection positivity corresponds to

$$\langle v, \theta v\rangle = |a|^2 + |b|^2 - |c|^2 - |d|^2 \geq 0.$$

With respect to U_σ, we have

$$v = v_1 + v_{-1} = (a, 0, c, 0) + (0, b, 0, d)$$

and

$$\langle U_\sigma v, \theta U_\sigma v\rangle = \langle v, \theta v\rangle \geq 0 \quad \text{and} \quad \langle U_\sigma v, \theta v\rangle = |a|^2 - |b|^2 - |c|^2 + |d|^2.$$

Therefore the subspace $\mathbb{C}v + \mathbb{C}U_\sigma v$ is θ-positive if and only if

$$\pm(|a|^2 - |b|^2 - |c|^2 + |d|^2) \leq |a|^2 + |b|^2 - |c|^2 - |d|^2,$$

which is equivalent to

$$|d| \leq |b| \quad \text{and} \quad |c| \leq |a|.$$

Clearly, these two conditions are strictly stronger than the θ-positivity of $\mathbb{C}v$.

For the corresponding positive definite function $f(g) = \langle v, U_g v\rangle$ we have

$$f(\mathbf{1}) = |a|^2 + |b|^2 + |c|^2 + |d|^2, \qquad f(\tau) = |a|^2 + |b|^2 - |c|^2 - |d|^2,$$
$$f(\sigma) = |a|^2 - |b|^2 + |c|^2 - |d|^2, \qquad f(\sigma\tau) = |a|^2 - |b|^2 - |c|^2 + |d|^2.$$

Decomposing $f = f_1 + f_{-1}$ with respect to the left action of σ, we obtain

$$f_1(\mathbf{1}) = f_1(\sigma) = |a|^2 + |c|^2, \qquad f_1(\tau) = f_1(\sigma\tau) = |a|^2 - |c|^2$$

and

$$f_{-1}(\mathbf{1}) = -f_1(\sigma) = |b|^2 + |d|^2, \qquad f_{-1}(\tau) = -f_{-1}(\sigma\tau) = |b|^2 - |d|^2.$$

Both functions $f_{\pm 1}(g) = \langle v_{\pm 1}, U_g v_{\pm 1}\rangle$ are positive definite, but they are reflection positive if and only if $|c| \leq |a|$ and $|d| \leq |b|$.

Note that, even for $U_\sigma = \mathbf{1}$ and $U_\sigma = -\mathbf{1}$, there exist nontrivial reflection positive representations with $\langle v, \theta v\rangle > 0$.

4. Reflection positive functions and KMS conditions

In this section we build the bridge from positive definite functions $\psi : \mathbb{R} \to \mathrm{Bil}(V)$ satisfying the KMS condition for $\beta > 0$ to reflection positive functions on the group $\mathbb{T}_{2\beta,\tau} \cong \mathrm{O}_2(\mathbb{R})$. We have already seen in Lemma 2.2 that analytic continuation leads to a 2β-periodic function $\varphi : \mathbb{R} \to \mathrm{Bil}(V)$ satisfying $\varphi(t + \beta) = \overline{\varphi(t)}$ for $t \in \mathbb{R}$

and $\varphi(t) = \psi(it)$ for $0 \le t \le \beta$. In this section we show the existence of a positive definite function $f : \mathbb{R}_\tau \to \mathrm{Bil}(V)$ with $f(t, \tau) = \varphi(t)$ for $t \in \mathbb{R}$. By construction, f is then reflection positive with respect to the interval $[0, \beta/2] = G_+ \subseteq G = \mathbb{R}$ in the sense of Definition 3.4.

Since we can build on Theorem 2.6, our first goal is to express, for a standard real subspace $V \subseteq \mathcal{H}$, the $\mathrm{Bil}(V)$-valued function

$$\begin{aligned} &\varphi : [0, \beta] \to \mathrm{Bil}(V), \\ &\varphi(t)(v, w) := \psi(it)(v, w) = \langle \Delta^{t/2\beta} v, \Delta^{t/2\beta} w\rangle \quad \text{for } v, w \in V, 0 \le t \le \beta \end{aligned} \tag{14}$$

from (8) in the proof of Theorem 2.6 as a $B(V_{\mathbb{C}})$-valued function. To this end, we shall need the description of V in terms of a skew-symmetric strict contraction C on V (Lemma B.9), and this leads to a quite explicit description of φ that we then use to prove our main theorem.

4A. ***From form-valued to operator-valued functions.*** In the following it will be more convenient to work with operator-valued functions instead of form-valued ones. The translation is achieved by the following lemma. For its formulation, we recall the polar decomposition of bounded skew-symmetric operators on real Hilbert spaces.

Remark 4.1. (polar decomposition of skew-symmetric operators) Let $D^\top = -D$ be an injective skew-symmetric operator on the real Hilbert space V and let $D = I|D|$ be its polar decomposition. Then $\mathrm{im}(D)$ is dense because D is injective, and therefore I defines an isometry $V \to V$. From

$$I|D| = D = -D^\top = -|D|I^{-1} = -I^{-1}(I|D|I^{-1})$$

it follows that $I^2 = -\mathbf{1}$, i.e., that I is a complex structure and that $|D|$ commutes with I.

Lemma 4.2. *Let $V \subseteq \mathcal{H}$ be a standard real subspace with modular objects (Δ, J), let $\widehat{C} := i\frac{\Delta - \mathbf{1}}{\Delta + \mathbf{1}}$, and let $C := \widehat{C}|_V \in B(V)$ be the skew-symmetric strict contraction from Lemma B.9. We assume that* $\ker C = \{0\}$, *so the polar decomposition $C = I|C|$ defines a complex structure I on V. Consider the skew-symmetric operator*

$$D := \log\left(\frac{\mathbf{1} - |C|}{\mathbf{1} + |C|}\right) I.$$

Then the function $\varphi(t)(v, w) = \langle \Delta^{t/2} v, \Delta^{t/2} w\rangle$ from (14) has the form

$$\varphi(t)(v, w) = \langle v, \widetilde{\varphi}(t) w\rangle_{V_{\mathbb{C}}} \quad \text{for } t \in [0, 1],\ v, w \in V_{\mathbb{C}}, \tag{15}$$

where the function $\widetilde{\varphi} : [0, 1] \to B(V_{\mathbb{C}})$ is given by

$$\widetilde{\varphi}(t) = (\mathbf{1} + iC)^{1-t}(\mathbf{1} - iC)^t = \frac{e^{-t|D|} + e^{-(1-t)|D|}}{\mathbf{1} + e^{-|D|}} + iI\frac{e^{-t|D|} - e^{-(1-t)|D|}}{\mathbf{1} + e^{-|D|}}.$$

Note that $\widetilde{\varphi}(0) = \mathbf{1} + iC \neq \mathbf{1}$ if $C \neq 0$.

Proof. Since C is a skew-symmetric contraction on V, the operators $\mathbf{1} \pm iC$ on $V_{\mathbb{C}}$ are symmetric, so that we obtain a function

$$\widetilde{\varphi} : [0, 1] \to B(V_{\mathbb{C}}), \qquad \widetilde{\varphi}(t) := (\mathbf{1} + iC)^{1-t}(\mathbf{1} - iC)^t, \quad 0 \leq t \leq 1.$$

Therefore both sides of (15) are defined, and we have to show that

$$\langle v, \widetilde{\varphi}(t) w\rangle_{V_{\mathbb{C}}} = \langle \Delta^{t/2} v, \Delta^{t/2} w\rangle \quad \text{for } v, w \in V_{\mathbb{C}}. \tag{16}$$

For the skew-hermitian contraction $\widehat{C}$ on $\mathcal{H}$, we likewise obtain bounded operators

$$\widehat{\varphi}(t) := (\mathbf{1} + i\widehat{C})^{1-t}(\mathbf{1} - i\widehat{C})^t, \quad 0 \leq t \leq 1,$$

and the continuity of the inclusion $V_{\mathbb{C}} \hookrightarrow \mathcal{H}$ implies that

$$\widehat{\varphi}(t)|_{V_{\mathbb{C}}} = \widetilde{\varphi}(t) : V_{\mathbb{C}} \to V_{\mathbb{C}}.$$

From the relation

$$\Delta = \frac{\mathbf{1} - i\widehat{C}}{\mathbf{1} + i\widehat{C}},$$

we further obtain the identity

$$\widehat{\varphi}(t) = (\mathbf{1} + i\widehat{C})\Delta^t$$

of selfadjoint operators on $\mathcal{H}$. Let $V'_{\mathbb{C}}$ denote the domain of the (possibly) unbounded selfadjoint operator $\frac{\mathbf{1}-iC}{\mathbf{1}+iC}$ on $V_{\mathbb{C}}$. Then, for $0 \leq t \leq 1$, $V'_{\mathbb{C}}$ is a dense subspace which is contained in the domain of $\left(\frac{\mathbf{1}-iC}{\mathbf{1}+iC}\right)^t$. For $w \in V'_{\mathbb{C}}$, we have

$$\widetilde{\varphi}(t) w = (\mathbf{1} + iC)\left(\frac{\mathbf{1} - iC}{\mathbf{1} + iC}\right)^t w \quad \text{for } 0 \leq t \leq 1.$$

For $v \in V_{\mathbb{C}}$ and $\widetilde{w} := \left(\frac{\mathbf{1}-iC}{\mathbf{1}+iC}\right)^t w$ we now obtain with (39) from Lemma B.9 the relation

$$\begin{aligned}\langle v, \widetilde{\varphi}(t) w\rangle_{V_{\mathbb{C}}} &= \langle v, (\mathbf{1} + iC)\widetilde{w}\rangle_{V_{\mathbb{C}}} = \langle v, \widetilde{w}\rangle_{\mathcal{H}} = \left\langle v, \left(\frac{\mathbf{1} - iC}{\mathbf{1} + iC}\right)^t w\right\rangle_{\mathcal{H}} \\ &= \left\langle v, \left(\frac{\mathbf{1} - i\widehat{C}}{\mathbf{1} + i\widehat{C}}\right)^t w\right\rangle_{\mathcal{H}} = \langle v, \Delta^t w\rangle_{\mathcal{H}} = \langle \Delta^{t/2} v, \Delta^{t/2} w\rangle_{\mathcal{H}}.\end{aligned}$$

Since both sides of (16) define continuous hermitian forms on $V_{\mathbb{C}}$ and the preceding calculation shows that equality holds on a dense subspace, we obtain (16) for all $v, w \in V_{\mathbb{C}}$.

Next we observe that the polar decomposition of D is given by

$$D = -I|D| \quad \text{and} \quad |D| = \log\left(\frac{\mathbf{1} + |C|}{\mathbf{1} - |C|}\right).$$

The operator $|D|$ satisfies

$$e^{\mp|D|} = \frac{\mathbf{1} \mp |C|}{\mathbf{1} \pm |C|} \quad \text{and} \quad 1 + e^{\mp|D|} = \frac{2}{1 \pm |C|}. \tag{17}$$

Since iI is an involution with the two eigenvalues ± 1, comparing the action on both eigenspaces shows that, for $0 \leq t \leq 1$, we have

$$\left(\frac{\mathbf{1} - iC}{\mathbf{1} + iC}\right)^t = \left(\frac{\mathbf{1} - iI|C|}{\mathbf{1} + iI|C|}\right)^t = e^{-t|D|iI}.$$

The assertion of the lemma now follows from

$$\begin{aligned}
\widetilde{\varphi}(t) &= (\mathbf{1} + iC)\left(\frac{\mathbf{1} - iC}{\mathbf{1} + iC}\right)^t = (\mathbf{1} + iI|C|)e^{-t|D|iI} \\
&= (\mathbf{1} + iI|C|)\left(e^{t|D|}\frac{\mathbf{1} - iI}{2} + e^{-t|D|}\frac{\mathbf{1} + iI}{2}\right) \\
&= e^{-t|D|}(\mathbf{1} + |C|)\frac{\mathbf{1} + iI}{2} + e^{t|D|}(\mathbf{1} - |C|)\frac{\mathbf{1} - iI}{2} \\
&= (\mathbf{1} + |C|)\left(e^{-t|D|}\frac{\mathbf{1} + iI}{2} + e^{-(1-t)|D|}\frac{\mathbf{1} - iI}{2}\right) \\
&= (\mathbf{1} + e^{-|D|})^{-1}\big(e^{-t|D|}(\mathbf{1} + iI) + e^{-(1-t)|D|}(\mathbf{1} - iI)\big) \\
&= \frac{e^{-t|D|} + e^{-(1-t)|D|}}{\mathbf{1} + e^{-|D|}} + iI\frac{e^{-t|D|} - e^{-(1-t)|D|}}{\mathbf{1} + e^{-|D|}}.
\end{aligned}$$

□

Remark 4.3. (a) Since C is a strict contraction on V, $\mathbf{1} + iC$ is injective on $V_{\mathbb{C}}$, so

$$\widetilde{H} := \log\left(\frac{\mathbf{1} + iC}{\mathbf{1} - iC}\right) = \log\left(\frac{\mathbf{1} + iI|C|}{\mathbf{1} - iI|C|}\right) = iI|D| = -iD$$

also defines a selfadjoint operator on the complex Hilbert space $V_{\mathbb{C}}$.

Next we observe that $\widetilde{H}$ is a restriction of

$$L := \log\left(\frac{\mathbf{1} + i\widehat{C}}{\mathbf{1} - i\widehat{C}}\right) = -\log\Delta,$$

the infinitesimal generator of the one-parameter group $U_t = \Delta^{-it}$. For the orthogonal one-parameter group $U_t^V := U_t|_V \in \mathrm{O}(V)$, it follows that its infinitesimal generator is a skew-adjoint extension of the skew-adjoint operator D on V, and hence coincides with D. We therefore have

$$e^{tD} = \Delta^{-it}|_V \quad \text{for } t \in \mathbb{R}. \tag{18}$$

This provides an alternative characterization of the operator D in Lemma 4.2.

(b) Let $(V, (\cdot\,,\cdot))$ be a real Hilbert space and $(U_t)_{t\in\mathbb{R}}$ be an orthogonal strongly continuous one-parameter group with skew-symmetric infinitesimal generator D, i.e., $U_t = e^{tD}$ for $t \in \mathbb{R}$. Let us assume that $\ker D = \{0\}$, i.e., the subspace V^U of U-fixed points in V is trivial. Then the polar decomposition $D = I|D|$ can be used to define a skew-symmetric contraction

$$C := (-I)\frac{\mathbf{1}-e^{-|D|}}{\mathbf{1}+e^{-|D|}} \quad \text{with } |C| = \frac{\mathbf{1}-e^{-|D|}}{\mathbf{1}+e^{-|D|}}.$$

Then the hermitian form

$$h(v, w) := (v, w) + i(v, Cw)$$

defines a positive definite kernel on V (Lemma A.10). Let $\mathcal{H}$ denote the corresponding reproducing kernel space and let $j : V \to \mathcal{H}$ be the natural map. By construction, $|C|$ has no fixed points, so that $\mathbf{1} + C^2$ is injective, and therefore Lemma A.10(iii) implies that the complex linear extension $j_{\mathbb{C}} : V_{\mathbb{C}} \to \mathcal{H}$ is injective. As the real part of h is the original scalar product on V, the inclusion $V \hookrightarrow \mathcal{H}$ is isometric, so that $V \cong j(V)$ is a standard real subspace of $\mathcal{H}$. Since h is U-invariant, it defines a unitary one-parameter group $\widehat{U}$ on $\mathcal{H}$. Finally (18) implies that $\widehat{U}_t = \Delta^{-it}$ for $t \in \mathbb{R}$ and the modular operator Δ corresponding to $j(V)$. This shows that every orthogonal one-parameter group on a real Hilbert space is of the form (18) for a naturally defined embedding $V \hookrightarrow \mathcal{H}$ as a standard real subspace.

Before we turn to the associated reflection positive functions, we need the following technical lemma on Fourier expansions of certain operator-valued functions. In [Cuniberti et al. 2001], this is called the Matsubara formalism. (In view of [Dereziński and Gérard 2013, Definition 18.49], we have

$$u_B^+(t) = G_{E,\beta}(t)\cdot\frac{2B(\mathbf{1}-e^{-\beta B})}{\mathbf{1}+e^{-\beta B}},$$

where $G_{E,\beta}$ is the euclidean thermal Green's function associated to the positive operator $\varepsilon = B$.)

Lemma 4.4. *Let $B \geq 0$ be a selfadjoint operator on the complex Hilbert space $\mathcal{H}$ and let $\beta > 0$. We consider the operator-valued functions $u_B^\pm : \mathbb{R} \to B(\mathcal{H})$ satisfying*

$$u_B^\pm(t+\beta) = \pm u_B^\pm(t) \quad \textit{and} \quad u_B^\pm(t) = \frac{e^{-tB} \pm e^{-(\beta-t)B}}{\mathbf{1}+e^{-\beta B}} \quad \textit{for } 0 \leq t \leq \beta.$$

Then $u_B^\pm$ are weakly continuous symmetric 2β-periodic with the Fourier expansions

$$u_B^+(t) = \sum_{n\in\mathbb{Z}} c_{2n}^B e^{2n\pi i t/\beta} \quad \textit{and} \quad u_B^-(t) = \sum_{n\in\mathbb{Z}} c_{2n+1}^B e^{(2n+1)\pi i t/\beta}$$

with

$$c_n^B = c_{-n}^B = \frac{(\mathbf{1}-(-1)^n e^{-\beta B})}{\mathbf{1}+e^{-\beta B}}\cdot\frac{2\beta B}{(\beta B)^2+(n\pi)^2} \quad \textit{for } n \in \mathbb{Z}.$$

Proof. (a) Every 2β-periodic continuous function $\xi : \mathbb{R} \to \mathbb{C}$ has a Fourier expansion

$$\xi(t) = \sum_{n\in\mathbb{Z}} c_n e^{\pi int/\beta} \quad \text{with} \quad c_n = \frac{1}{2\beta}\int_0^{2\beta} \xi(t) e^{-\pi int/\beta}\, dt.$$

For the β-periodic function with $u^+(t) = u^+_\lambda(t) := (e^{-t\lambda} + e^{-(\beta-t)\lambda})/(1+e^{-\beta\lambda})$ for $0 \leq t \leq \beta$ we have $u^+(t+\beta) = u^+(t)$, so that only even terms appear:

$$u^+(t) = \sum_{n\in\mathbb{Z}} c_{2n} e^{\pi i 2nt/\beta}, \quad c_{2n} = \frac{1-e^{-\beta\lambda}}{1+e^{-\beta\lambda}} \frac{2\beta\lambda}{(\beta\lambda)^2 + (2\pi n)^2}.$$

To obtain this formula, we first calculate

$$\begin{aligned} a_{\lambda,n} &:= \frac{1}{\beta}\int_0^\beta e^{-t\lambda} e^{-\pi int/\beta}\, dt = \int_0^1 e^{-(\beta\lambda+\pi in)t}\, dt \\ &= \frac{1-e^{-(\beta\lambda+\pi in)}}{\beta\lambda + \pi i 2n} = \frac{1-(-1)^n e^{-\beta\lambda}}{\beta\lambda+\pi i 2n}. \end{aligned}$$

Therefore

$$\begin{aligned} (1+e^{-\lambda\beta})c_{2n} &= a_{\lambda,2n} + e^{-\beta\lambda} a_{-\lambda,2n} = \frac{1-e^{-\beta\lambda}}{\beta\lambda + 2n\pi i} + e^{-\beta\lambda}\frac{1-e^{\beta\lambda}}{-\beta\lambda+2n\pi i} \\ &= \frac{1-e^{-\beta\lambda}}{\beta\lambda+2n\pi i} + \frac{1-e^{-\beta\lambda}}{\beta\lambda - 2n\pi i} = \frac{(1-e^{-\beta\lambda})2\beta\lambda}{(\beta\lambda)^2 + (2n\pi)^2} \end{aligned}$$

For the 2β-periodic function with $u^-(t) = u^-_\lambda(t) := (e^{-t\lambda} - e^{-(\beta-t)\lambda})/(1+e^{-\beta\lambda})$ for $0 \leq t \leq \beta$ and $u^-(t+\beta) = -u^-(t)$ only odd terms appear:

$$u^-(t) = \sum_{n\in\mathbb{Z}} c_{2n+1} e^{\pi i(2n+1)t/\beta}, \quad c_{2n+1} = \frac{2\beta\lambda}{(\beta\lambda)^2 + ((2n+1)\pi)^2}.$$

This follows from

$$\begin{aligned} c_{2n+1} &= \frac{a_{\lambda,2n+1} - e^{-\beta\lambda} a_{-\lambda,2n+1}}{1+e^{-\beta\lambda}} \\ &= \frac{1}{\beta\lambda + (2n+1)\pi i} - \frac{e^{-\beta\lambda}(1+e^{\beta\lambda})}{1+e^{-\beta\lambda}} \frac{1}{-\beta\lambda + (2n+1)\pi i} \\ &= \frac{1}{\beta\lambda+(2n+1)\pi i} + \frac{1}{\beta\lambda - (2n+1)\pi i} = \frac{2\beta\lambda}{(\beta\lambda)^2 + ((2n+1)\pi)^2}. \end{aligned}$$

Note that

$$c_n = c_{-n} = \frac{1-(-1)^n e^{-\beta\lambda}}{1+e^{-\beta\lambda}} \frac{2\beta\lambda}{(\beta\lambda)^2 + (n\pi)^2} \quad \text{for } n \in \mathbb{Z}.$$

(b) If P denotes the spectral measure of B, we have for $v \in \mathcal{H}$ the relation

$$\langle v, Bv\rangle = \int_0^\infty x\, dP^{v,v}(x) \quad \text{with } P^{v,v} = \langle v, P(\cdot)v\rangle.$$

This leads for $0 \le t \le 2\beta$ to

$$\langle v, u_B^\pm(t)v\rangle = \int_0^\infty u_\lambda^\pm(t)\, dP^{v,v}(\lambda).$$

For the operator-valued Fourier coefficients, we thus obtain

$$\begin{aligned}\langle v, c_n^B v\rangle &= \int_{\mathbb{R}} c_n(\lambda)\, dP^{v,v}(\lambda) = \int_{\mathbb{R}} \frac{1-(-1)^n e^{-\beta\lambda}}{1+e^{-\beta\lambda}} \frac{2\beta\lambda}{(\beta\lambda)^2+(n\pi)^2}\, dP^{v,v}(\lambda)\\ &= \Big\langle v, \frac{(\mathbf{1}-(-1)^n e^{-\beta B})}{\mathbf{1}+e^{-\beta B}} \frac{2\beta B}{(\beta B)^2+(n\pi)^2} v\Big\rangle.\end{aligned}$$

This proves the assertion. □

4B. *Existence of reflection positive extensions.* We now come to one of our main results on reflection positive extensions. It shows that, for every positive definite function $\psi : \mathbb{R} \to \mathrm{Bil}(V)$ satisfying the β-KMS condition, there exists a reflection positive function $f : G_\tau \to B(V_{\mathbb{C}})$ satisfying

$$\psi(it)(v, w) = \langle v, f(it, \tau)w\rangle$$

for $v, w \in V$, $0 \le t \le \beta$. Then the corresponding GNS representation $(U^f, \mathcal{H}_f)$ of the group $(\mathbb{T}_{2\beta})_\tau \cong \mathrm{O}_2(\mathbb{R})$ is a "euclidean realization" of the unitary one-parameter group $(\Delta^{-it/\beta})_{t\in\mathbb{R}}$ corresponding to ψ in the sense that it is obtained by Osterwalder–Schrader quantization from U^f (cf. [Neeb and Ólafsson 2014]). The following theorem generalizes the results of [Neeb and Ólafsson 2015b] dealing with the scalar-valued case.

Theorem 4.5 (Reflection positive extensions). *Let $V \subseteq \mathcal{H}$ be a standard real subspace and let $C = I|C|$ be the corresponding skew-symmetric strict contraction on V. We assume that* $\ker C = \{0\}$, *so that I defines a complex structure on V. We define a weakly continuous function $\widetilde{\varphi} : \mathbb{R} \to B(V_{\mathbb{C}})$ by*

$$\widetilde{\varphi}(t) = (\mathbf{1}+iC)^{1-t/\beta}(\mathbf{1}-iC)^{t/\beta} \quad \text{for } 0 \le t \le \beta \quad \text{and} \quad \widetilde{\varphi}(t+\beta) = \overline{\widetilde{\varphi}(t)} \quad \text{for } t \in \mathbb{R}.$$

Write

$$\widetilde{\varphi}(t) = u^+(t) + iIu^-(t) \quad \text{with } u^\pm(t) \in B(V),\ u^\pm(t+\beta) = \pm u^\pm(t).$$

Then

$$f : \mathbb{R}_\tau \to B(V_{\mathbb{C}}), \qquad f(t, \tau^\varepsilon) := u^+(t) + (iI)^\varepsilon u^-(t), \qquad t \in \mathbb{R}, \varepsilon \in \{0, 1\},$$

is a weak-operator continuous positive definite function satisfying $f(t, \tau) = \widetilde{\varphi}(t)$.

It is reflection positive with respect to the subset $[0, \beta/2] \subseteq \mathbb{R}$ *in the sense that the kernel*

$$f((t,\tau)(-s,\mathbf{1})) = f(t+s,\tau), \qquad 0 \le s,t \le \beta/2,$$

is positive definite.

Proof. We may, without loss of generality, assume that $\beta = 1$. Recall the operator D from Lemma 4.2. With this lemma, we write

$$\widetilde{\varphi}(t) = (\mathbf{1}+e^{-|D|})^{-1}\big(e^{-t|D|}+e^{-(1-t)|D|}+iI(e^{-t|D|}-e^{-(1-t)|D|})\big) \quad \text{for } 0 \le t \le 1.$$

Using Lemma 4.4 with $\beta = 1$ and $B = |D|$, we get

$$\widetilde{\varphi}(t) = u^+_{|D|}(t) + iIu^-_{|D|}(t) \quad \text{for } t \in \mathbb{R}.$$

(a) We define $f_1 : \mathbb{R}_\tau \to B(V_\mathbb{C})$ by $f_1(t,\tau^\varepsilon) := u^+_{|D|}(t)$ for $t \in \mathbb{R}$, $\varepsilon \in \{0,1\}$. To see that f_1 is positive definite, it suffices to verify this for its restriction to $\mathbb{R}$ (Lemma 3.6), which follows from the positivity of the Fourier coefficients in the expansion

$$u^+_{|D|}(t) = \sum_{n\in\mathbb{Z}} c^{|D|}_{2n} e^{2n\pi it} \quad \text{with} \quad c^{|D|}_{2n} = \frac{\mathbf{1}-e^{-|D|}}{\mathbf{1}+e^{-|D|}} \frac{2|D|}{|D|^2+(2n\pi)^2\mathbf{1}} \ge 0$$

(Lemma 4.4). Note that f_1 is 1-periodic.

(b) Likewise, the function $f_2 : \mathbb{R}_\tau \to B(V_\mathbb{C})$ defined by $f_2(t,\tau^\varepsilon) := u^-_{|D|}(t)$ for $t \in \mathbb{R}$, $\varepsilon \in \{0,1\}$, is positive definite because the Fourier coefficients

$$c^{|D|}_{2n+1} = \frac{2|D|}{|D|^2+((2n+1)\pi)^2\mathbf{1}} \ge 0 \quad \text{for } n \in \mathbb{Z}$$

are positive. Note that $f_2(t+1,\tau^\varepsilon) = -f_2(t,\tau^\varepsilon)$ for $t \in \mathbb{R}$, $\varepsilon \in \{0,1\}$.

(c) We now consider the function

$$\tilde{f}_2(g) := h(g)f_2(g) \quad \text{with } h(t,\tau^\varepsilon) = (iI)^\varepsilon \text{ for } t \in \mathbb{R},\ \varepsilon \in \{0,1\}.$$

Since $h(g)$ commutes with $f_2(g')$ for $g, g' \in \mathbb{R}_\tau$, the function $\tilde{f}_2$ is positive definite if h is positive definite (Lemma A.6). As h is constant on the two $\mathbb{R}$-cosets and its restriction to the 2-element subgroup $\{\mathbf{1},\tau\}$ is a unitary representation, h is positive definite. We conclude that the $B(V_\mathbb{C})$-valued function $f := f_1 + \tilde{f}_2$ on $\mathbb{R}_\tau$ is positive definite. □

Corollary 4.6. *Let V be a real vector space and let $\psi : \mathbb{R} \to \mathrm{Bil}(V)$ be a continuous positive definite function satisfying the β-KMS condition. Then there exists a pointwise continuous function $f : \mathbb{R}_\tau \to \mathrm{Bil}(V)$ which is reflection positive with respect to the subset $[0,\beta/2] \subseteq \mathbb{R}$ and which satisfies*

$$f(t,\tau) = \psi(it) \quad \textit{for } 0 \le t \le \beta \qquad \textit{and} \qquad f(t+\beta,\tau) = \overline{f(t,\tau)} \quad \textit{for } t \in \mathbb{R}.$$

Remark 4.7. The function $\tilde{f}_2$ in the proof of Theorem 4.5 is not reflection positive because $\tilde{f}_2(\beta, \tau)$ is a negative operator. This also shows that the natural decomposition $f = f_1 + \tilde{f}_2$ into even and odd parts is not compatible with reflection positivity.

4C. ***Integral representation of reflection positive functions.*** We now describe an integral representation of the reflection positive function $f : \mathbb{R}_\tau \to \mathrm{Bil}(V)$ which corresponds to the decomposition of the corresponding unitary representation of $\mathbb{R}_\tau$. With

$$\widetilde{\varphi}(t) = (\mathbf{1} + iC)^{1-t/\beta}(\mathbf{1} - iC)^{t/\beta} \quad \text{for } 0 \le t \le \beta,$$

where $C \in B(V)$ is a skew-symmetric strict contraction, we first decompose V into $V_0 := \ker C$ and $V_1 := V_0^\perp = \overline{CV}$. Then the polar decomposition $C = I|C|$ yields a complex structure I on V_1. Accordingly, we write $\widetilde{\varphi} = \widetilde{\varphi}_0 + \widetilde{\varphi}_1$, where $\widetilde{\varphi}_0 = \mathbf{1}$ is constant. This component leads to the constant function $f_0(t, \tau) = \mathbf{1}$. We now assume that $V = V_1$, i.e., that C is injective. Then I is a complex structure on V.

Proposition 4.8. *If* $\ker C = \{0\}$ *and* P *denotes the spectral measure of the symmetric operator* $|D| = \frac{1}{\beta} \log \frac{\mathbf{1}+|C|}{\mathbf{1}-|C|}$ *on* V, *then we have the integral representation*

$$f(t, \tau^\varepsilon) = \int_{(0,\infty)} u_\lambda^+(t) + u_\lambda^-(t)(iI)^\varepsilon \, dP(\lambda), \tag{19}$$

where $u_\lambda^\pm : \mathbb{R} \to \mathbb{R}$ *are defined by* $u_\lambda^\pm(t + \beta) = \pm u_\lambda^\pm(t)$ *and*

$$u_\lambda^\pm(t) := \frac{e^{-t\lambda} \pm e^{-(\beta-t)\lambda}}{1 + e^{-\beta\lambda}} \quad \textit{for } 0 \le t \le \beta.$$

Proof. First we observe that $|D|$ is a positive symmetric operator with trivial kernel which commutes with I. We therefore have $|D| = \int_{(0,\infty)} \lambda \, dP(\lambda)$. With the notation from Lemma 4.4, we then have

$$f(t, \tau^\varepsilon) = u_{|D|}^+(t) + u_{|D|}^-(t)(iI)^\varepsilon \quad \text{for } t \in \mathbb{R},\ \varepsilon \in \{0, 1\}.$$

From the integral representations $u_{|D|}^\pm(t) = \int_{(0,\infty)} u_\lambda^\pm(t) \, dP(\lambda)$, we obtain (19). □

Remark 4.9. (a) For $0 \le t \le \beta$, we have in particular

$$f(t, \tau^\varepsilon) = \int_{(0,\infty)} \frac{e^{-t\lambda} + e^{-(\beta-t)\lambda}}{1 + e^{-\beta\lambda}} \mathbf{1} + \frac{e^{-t\lambda} - e^{-(\beta-t)\lambda}}{1 + e^{-\beta\lambda}} (iI)^\varepsilon \, dP(\lambda).$$

(b) The most basic type is obtained for $D = \lambda\mathbf{1}$, $\lambda > 0$, which, for $0 \le t \le \beta$, leads to

$$f(t, \tau^\varepsilon) = \frac{(e^{-t\lambda} + e^{-(\beta-t)\lambda})\mathbf{1} + (e^{-t\lambda} - e^{-(\beta-t)\lambda})(iI)^\varepsilon}{1 + e^{-\beta\lambda}} = u_\lambda^+(t)\mathbf{1} + u_\lambda^-(t)(iI)^\varepsilon.$$

The simplest nontrivial example arises for $V = \mathbb{R}^2$ with $I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(c) Every Borel spectral measure P on $(0, \infty)$ which commutes with I defines a positive operator $|D| = \int_0^\infty \lambda\, dP(\lambda)$ and we may put $D := -I|D|$. Then $\ker |D| = 0$, so

$$|C| := \frac{e^{\beta|D|} - \mathbf{1}}{e^{\beta|D|} + \mathbf{1}} = \tanh\Big(\frac{\beta|D|}{2}\Big)$$

is a symmetric contraction with trivial kernel commuting with I, and therefore $C := I|C|$ is a skew-symmetric contraction with polar decomposition $I|C|$ and

$$|D| = \frac{1}{\beta} \log\Big(\frac{\mathbf{1} + |C|}{\mathbf{1} - |C|}\Big).$$

4D. *Characterizing reflection positive extensions.* In Theorem 4.5 we obtained positive definite extensions to all of $\mathbb{R}_\tau$ for certain functions on the coset $\mathbb{R} \rtimes \{\tau\}$. In this section we describe an intrinsic characterization of those weakly continuous reflection positive functions $f : \mathbb{R}_\tau \to B(V_\mathbb{C})$ arising from this construction. First we observe that we can recover ψ from f:

Lemma 4.10. *If $f : \mathbb{R}_\tau \to \mathrm{Bil}(V)$ is reflection positive and pointwise continuous, then there exists a unique β-KMS positive definite function $\psi : \mathbb{R} \to \mathrm{Bil}(V)$ with*

$$f(t, \tau) = \psi(it) \quad \text{for } 0 \le t \le \beta.$$

Proof. First we observe that the function $\varphi(t) := f(t, \tau)$ has values in $\mathrm{Herm}(V_\mathbb{C})$ and satisfies

$$\varphi(t + \beta) = \overline{\varphi(t)} \quad \text{for } t \in \mathbb{R}. \tag{20}$$

Reflection positivity implies that the kernel $\varphi\big(\frac{t+s}{2}\big)$ for $0 \le t, s \le \beta$ is positive definite. By [Neeb and Ólafsson 2015b, Theorem B.3], there exists a $\mathrm{Bil}^+(V)$-valued measure μ such that

$$\varphi(t) = \int_\mathbb{R} e^{-\lambda t}\, d\mu(\lambda) \quad \text{for } 0 < t < \beta. \tag{21}$$

The continuity of φ on $[0, \beta]$ actually implies that the integral representation also holds on the closed interval $[0, \beta]$ by the monotone convergence theorem. In particular, the measure μ is finite. Therefore its Fourier transform $\psi(t) := \int_\mathbb{R} e^{it\lambda}\, d\mu(\lambda)$ is a pointwise continuous $\mathrm{Bil}(V)$-valued positive definite function on $\mathbb{R}$. Further, (20) implies

$$e^{\beta\lambda}\, d\mu(-\lambda) = d\overline{\mu}(\lambda), \tag{22}$$

and $\varphi(t) = \psi(it)$ holds for the β-KMS function $\psi : \mathbb{R} \to \mathrm{Bil}(V)$ by Theorem 4.5. □

Before we describe a realization of the GNS representation $(U^f, \mathcal{H}_f)$ in spaces of sections of a vector bundle, let us recall the general background for this.

Remark 4.11. For a $B(V)$-valued positive definite function $f : G \to B(V)$, the reproducing kernel Hilbert space with kernel $K(g,h) = \varphi(gh^{-1}) = K_g K_h^*$ is generated by the functions

$$K_{h,w} := K_h^* w \quad \text{with} \quad K_{h,w}(g) = K_g K_h^* w = K(g,h)w = \varphi(gh^{-1})w.$$

The group G acts on this space by right translations

$$(U_g s)(h) := s(hg).$$

If $P \subseteq G$ is a subgroup and (ρ, V) is a unitary representation for which

$$f(hg) = \rho(h) f(g) \quad \text{for all } g \in G,\ h \in P,$$

then

$$\mathcal{H}_f \subseteq \mathcal{F}(G,V)_\rho := \{s : G \to V : s(hg) = \rho(h)s(g) \text{ for all } g \in G, h \in P\}.$$

Therefore $\mathcal{H}_f$ can be identified with a space of sections of the associated bundle

$$\mathbb{V} := (V \times_P G) = (V \times G)/P,$$

where P acts on the trivial vector bundle $V \times G$ over G by $h.(v,g) = (\rho(h)v, hg)$.

To derive a suitable characterization of the functions f arising in Theorem 4.5, we identify 2β-periodic function s on $\mathbb{R}$ with pairs of function (s_0, s_1) via $s = s_0 + s_1$, where s_0 is β-periodic and $s_1(\beta + t) = -s_1(t)$. Accordingly, any 2β-periodic function $s : \mathbb{R} \to V_\mathbb{C}$ defines a function

$$\tilde{s} : \mathbb{R} \to V_\mathbb{C}^2, \quad \tilde{s} = (s_1, s_2) \quad \text{with } \tilde{s}(\beta + t) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tilde{s}(t).$$

In this sense, $\tilde{s}$ is a section of the vector bundle over $\mathbb{T}_\beta$ with fiber $V_\mathbb{C}^2$ defined by the representation of $\beta\mathbb{Z}$, specified by

$$\rho(\beta) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}.$$

Splitting the $B(V)$-valued positive definite function

$$f : \mathbb{R}_\tau \to B(V), \qquad f(t, \tau^\varepsilon) = u_{|D|}^+(t) + u_{|D|}^-(t)(iI)^\varepsilon \quad \text{for } t \in \mathbb{R}, \varepsilon \in \{0,1\}$$

into even and odd parts with respect to the β-translation, we obtain:

Lemma 4.12. *For the subgroup $P := (\mathbb{Z}\beta)_\tau \cong \mathbb{Z}\beta \rtimes \{\mathbf{1}, \tau\}$ of $G := \mathbb{R}_\tau$, we consider the unitary representation $\rho : P \to \mathrm{U}(V_\mathbb{C}^2)$ defined by*

$$\rho(\beta, \mathbf{1}) := \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \quad \text{and} \quad \rho(0, \tau) := \begin{pmatrix} \mathbf{1} & 0 \\ 0 & iI \end{pmatrix},$$

where I is a complex structure on the real Hilbert space V commuting with the *positive operator* $|D|$. *Then*

$$f^\sharp : \mathbb{R}_\tau \to B(V^2) \cong M_2(B(V)), \qquad f^\sharp(t, \tau^\varepsilon) := \begin{pmatrix} u^+_{|D|}(t) & 0 \\ 0 & u^-_{|D|}(t)(iI)^\varepsilon \end{pmatrix}$$

is a positive definite function satisfying

$$f^\sharp(hg) = \rho(h) f^\sharp(g) \quad \text{for } h \in P, g \in G. \tag{23}$$

The corresponding GNS representation $(U^{f^\sharp}, \mathcal{H}_{f^\sharp})$ *is equivalent to the GNS representation* $(U^f, \mathcal{H}_f)$.

Proof. The first assertion follows from

$$f^\sharp((0,\tau)(t,\tau^\varepsilon)) = f^\sharp(-t, \tau^{\varepsilon+1}) = \begin{pmatrix} u^+_{|D|}(-t) & 0 \\ 0 & u^-_{|D|}(-t)(iI)^{\varepsilon+1} \end{pmatrix}$$
$$= \begin{pmatrix} u^+_{|D|}(t) & 0 \\ 0 & u^-_{|D|}(t)(iI)^{\varepsilon+1} \end{pmatrix}$$

and

$$f^\sharp(\beta + t, \tau^\varepsilon) = \begin{pmatrix} u^+_{|D|}(t) & 0 \\ 0 & -u^-_{|D|}(t)(iI)^\varepsilon \end{pmatrix}.$$

As the GNS representation $(U^f, \mathcal{H}_f)$ decomposes under the involution U^f_β into ± 1-eigenspaces, this representation is equivalent to the GNS representation $(U^{f^\sharp}, \mathcal{H}_{f^\sharp})$ corresponding to $f^\sharp$. □

Remark 4.13. (a) The preceding lemma implies that, if the complex structure I on V is fixed, then the relation (23) determines $f^\sharp$ completely in terms of the function

$$[0,\beta] \to M_2(B(V)), \quad t \mapsto f^\sharp(t,\tau) = \begin{pmatrix} \operatorname{Re}\varphi(t) & 0 \\ 0 & i\operatorname{Im}\varphi(t) \end{pmatrix},$$

so that φ determines f in a natural way.

(b) This lemma also shows that we may identify the Hilbert space $\mathcal{H}_f \cong \mathcal{H}_{f^\sharp}$ as a space of section of a Hilbert bundle $V^2 \times_\rho G$ over the circle $\mathbb{T}_\beta \cong \mathbb{R}_\tau / H$ with fiber V^2.

(c) Every function $s : \mathbb{R}_\tau \to V^2$ satisfying $s(hg) = \rho(h)s(g)$ for $h \in (\beta\mathbb{Z})_\tau$ is determined by its restriction $\tilde{s}$ to the subgroup $\mathbb{R}$, which satisfies

$$\tilde{s}(\beta + t) = \rho(\beta, \mathbf{1})\tilde{s}(t) \quad \text{for } t \in \mathbb{R}.$$

The action of τ is in this picture given by

$$(\tau.\tilde{s})(t) := s(t,\tau) = s((0,\tau)(-t,\mathbf{1})) = \rho(\tau)\tilde{s}(-t). \tag{24}$$

Remark 4.14. (a) In view of (22), there exists a $\mathrm{Bil}^+(V)$-valued measure ν on $[0, \infty)$ for which we can write

$$d\mu(\lambda) = d\nu(\lambda) + e^{\beta\lambda} d\overline{\nu}(-\lambda). \tag{25}$$

This leads, for $0 \le t \le \beta$ and $\nu = \nu_1 + i\nu_2$, to

$$\varphi(t) = \int_0^\infty e^{-t\lambda} + e^{-(\beta-t)\lambda}\, d\nu_1(\lambda) + i \int_0^\infty e^{-t\lambda} - e^{-(\beta-t)\lambda}\, d\nu_2(\lambda). \tag{26}$$

In particular, the most elementary nontrivial examples correspond to the Dirac measures of the form $\nu = \delta_\lambda \cdot (\gamma + i\omega)$, where δ_λ is the Dirac measure in $\lambda > 0$:

$$\varphi(t) = (e^{-t\lambda} + e^{-(\beta-t)\lambda})\gamma + i(e^{-t\lambda} - e^{-(\beta-t)\lambda})\omega = e^{-t\lambda} h + e^{-(\beta-t)\lambda}\bar{h},$$

where $h := \gamma + i\omega \in \mathrm{Bil}^+(V)$. Writing $\omega(v, w) = \gamma(v, Cw)$ (Corollary A.9) and replacing V by the real Hilbert space defined by the positive semidefinite form γ on V, we obtain the $B(V_{\mathbb{C}})$-valued function

$$\widetilde{\varphi}(t) = (e^{-t\lambda} + e^{-(\beta-t)\lambda}) + iC(e^{-t\lambda} - e^{-(\beta-t)\lambda}) = e^{-t\lambda}(\mathbf{1} + iC) + e^{-(\beta-t)\lambda}(\mathbf{1} - iC)$$

for $0 \le t \le \beta$, which leads to

$$f(t, \tau^\varepsilon) = (1 + e^{-\beta\lambda})(u_\lambda^+(t)\mathbf{1} + u_\lambda^-(t)|C|(iI)^\varepsilon) \quad \text{for } t \in \mathbb{R}, \varepsilon \in \{0, 1\}.$$

(b) This can also be formulated in terms of forms. With $\gamma(v, w) = \langle v, w\rangle_V$ and

$$h(v, w) = \gamma(v, w) + i\omega(v, w) = \langle v, (\mathbf{1} + iC)w\rangle_{V_{\mathbb{C}}} = \langle v, (\mathbf{1} + iI|C|)w\rangle_{V_{\mathbb{C}}},$$

we get

$$f(t, \tau^\varepsilon)(v, w) = \langle v, (u_\lambda^+(t)\mathbf{1} + u_\lambda^-(t)|C|(iI)^\varepsilon)w\rangle.$$

4E. ***Realization by resolvents of the Laplacian.*** We have seen in the preceding subsection how to obtain a realization of the Hilbert space $\mathcal{H}_f$ as a space $\mathcal{H}_{f^\sharp}$ of sections of a Hilbert bundle $\mathbb{V}$ with fiber $V_{\mathbb{C}}^2$ over the circle $\mathbb{T}_\beta = \mathbb{R}/\beta\mathbb{Z}$. In this section we provide an analytic description of the scalar product on this space if $|D| = \lambda\mathbf{1}$ for some $\lambda > 0$. We shall see that it has a natural description in terms of the resolvent $(\lambda^2 - \Delta)^{-1}$ of the Laplacian of $\mathbb{T}_\beta$ acting on sections of the bundle $\mathbb{V}$.

On the circle group $\mathbb{T}_{2\beta}$, we consider the normalized Haar measure given by

$$\int_{\mathbb{T}_{2\beta}} h(t)\, d\mu_{\mathbb{T}_{2\beta}} = \frac{1}{2\beta}\int_0^{2\beta} h(t)\, dt,$$

where we identify functions h on $\mathbb{T}_{2\beta}$ with 2β-periodic functions on $\mathbb{R}$.

As in Lemma 4.12, we write

$$f^\sharp(t, \tau^\varepsilon) = \begin{pmatrix} u_\lambda^+(t)\mathbf{1} & 0 \\ 0 & u_\lambda^-(t)(iI)^\varepsilon \end{pmatrix} \in B(V_{\mathbb{C}}^2) \cong M_2(B(V_{\mathbb{C}})),$$

For $\chi_n(t) = e^{\pi i n t/\beta}$ we then have

$$u_\lambda^+ = \sum_{n\in\mathbb{Z}} c_{2n}^\lambda \chi_{2n} \quad \text{and} \quad u_\lambda^- = \sum_{n\in\mathbb{Z}} c_{2n+1}^\lambda \chi_{2n+1},$$

where

$$c_n^\lambda = c_{-n}^\lambda = \frac{1-(-1)^n e^{-\beta\lambda}}{1+e^{-\beta\lambda}} \cdot \frac{2\beta\lambda}{(\beta\lambda)^2 + (n\pi)^2} = \frac{1-(-1)^n e^{-\beta\lambda}}{1+e^{-\beta\lambda}} \cdot \frac{2\lambda}{\beta} \cdot \frac{1}{\lambda^2 + (n\pi/\beta)^2}$$

for $n \in \mathbb{Z}$ (the rightmost factors are called bosonic Matsubara coefficients if n is even and fermionic if n is odd [Dereziński and Gérard 2013, §18]). With

$$c_+^\lambda := \frac{1-e^{-\beta\lambda}}{1+e^{-\beta\lambda}} \frac{2\lambda}{\beta} = \tanh\left(\frac{\beta\lambda}{2}\right)\frac{2\lambda}{\beta} \quad \text{and} \quad c_-^\lambda := \frac{2\lambda}{\beta}, \tag{27}$$

we thus obtain

$$c_{2n}^\lambda = \frac{c_+^\lambda}{\lambda^2 + (2n\pi/\beta)^2}, \qquad c_{2n+1}^\lambda = \frac{c_-^\lambda}{\lambda^2 + ((2n+1)\pi/\beta)^2}. \tag{28}$$

The following proposition shows that the positive operator $(\lambda^2 - \Delta)^{-1}$ on the Hilbert space of L^2-section of $\mathbb{V}$ defines a unitary representation of $\mathbb{R}_\tau$ which is unitarily equivalent to the representation on $\mathcal{H}_f$ (cf. Lemma 4.12).

Proposition 4.15. *For $\lambda > 0$, let $\mathcal{H}_\lambda$ be the Hilbert space obtained by completing the space*

$$\Gamma_\rho := \{s \in C^\infty(\mathbb{R}_\tau, V_\mathbb{C}^2) : s(hg) = \rho(h)s(g) \text{ for all } g \in \mathbb{R}_\tau,\ h \in (\mathbb{Z}\beta)_\tau)\}$$

with respect to

$$\langle s_1, s_2\rangle := \frac{1}{2\beta}\int_0^{2\beta} \langle s_1(t,\mathbf{1}), ((\lambda^2-\Delta)^{-1}s_2)(t,\mathbf{1})\rangle\, dt.$$

On $\mathcal{H}_\lambda$ we have a natural unitary representation U^λ of $\mathbb{R}_\tau$ by right translation which is unitarily equivalent to the GNS representation $(U^{f^\sharp}, \mathcal{H}_{f^\sharp})$. Here the corresponding j-map is given by

$$j : V \to \mathcal{H}_\lambda, \quad j\begin{pmatrix} v_1 \\ v_2\end{pmatrix} = \sqrt{c_+^\lambda}\sum_{n\in\mathbb{Z}} \chi_{2n}\begin{pmatrix} v_1 \\ 0\end{pmatrix} + \sqrt{c_-^\lambda}\sum_{n\in\mathbb{Z}} \chi_{2n+1}\begin{pmatrix} 0 \\ v_2\end{pmatrix}. \tag{29}$$

Proof. We identify Γ_ρ with the space

$$\{s \in C^\infty(\mathbb{R}, V_\mathbb{C}^2) : s(\beta + t) = \rho(\beta)s(t) \text{ for all } t \in \mathbb{R}\}$$

(Remark 4.13). Then $s = \binom{s_+}{s_-}$, where s_+ is β-periodic and s_- is β-antiperiodic. Accordingly, we have an orthogonal decomposition $\mathcal{H}_\lambda = \mathcal{H}_\lambda^+ \oplus \mathcal{H}_\lambda^-$, where $\mathcal{H}_\lambda^\pm =$

$\{s \in \mathcal{H}_\lambda : s(\beta + t) = \pm s(t) \text{ for all } t \in \mathbb{R}\}$. Then U^λ is given by

$$(U_t^\lambda s)(x) = s(t+x) \quad \text{for } t, x \in \mathbb{R} \quad \text{and} \quad (U_\tau^\lambda s)(x) = \begin{pmatrix} s_+(-x) \\ (iI)s_-(-x) \end{pmatrix}.$$

From the Fourier expansion $s = \sum_{n\in\mathbb{Z}} \chi_n s_n$ and the orthonormality of the χ_n, we then derive

$$\langle s_1, s_2\rangle_{\mathcal{H}_\lambda} = \sum_{n\in\mathbb{Z}} \frac{\langle s_{1,n}, s_{2,n}\rangle}{\lambda^2 + (n\pi/\beta)^2}. \tag{30}$$

For the map $j : V \to \mathcal{H}_\lambda$ in (29), the image is $U^\lambda_\mathbb{R}$-generating for $\mathcal{H}_\lambda$ because the projection onto each Fourier component generates the first or the second component of $V^2_\mathbb{C}$, according to parity. Therefore the unitary representation $(U^\lambda, \mathcal{H}_\lambda)$ is equivalent to the GNS representation of the positive definite function $\tilde{f} : \mathbb{R}_\tau \to B(V^2_\mathbb{C})$, given by

$$\langle v, \tilde{f}(g)w\rangle = \langle j(v), U^\tau_g j(w)\rangle_{\mathcal{H}_\lambda}.$$

From

$$U^\lambda_{(t,\tau^\varepsilon)} j(v) = \sqrt{c^\lambda_+}\sum_{n\in\mathbb{Z}} \chi_{2n}\chi_{2n}(t)\begin{pmatrix} v_1 \\ 0\end{pmatrix} + \sqrt{c^\lambda_-}\sum_{n\in\mathbb{Z}} \chi_{2n+1}\chi_{2n+1}(t)\begin{pmatrix} 0 \\ (iI)^\varepsilon v_2\end{pmatrix},$$

we derive with (28),

$$\begin{aligned}\langle v, \tilde{f}(t,\tau^\varepsilon)w\rangle &= c^\lambda_+ \sum_{n\in\mathbb{Z}} \frac{\chi_{2n}(t)}{\lambda^2 + (2n\pi/\beta)^2}\langle v_1, w_1\rangle \\ &\qquad + c^\lambda_- \sum_{n\in\mathbb{Z}} \frac{\chi_{2n+1}(t)}{\lambda^2 + ((2n+1)\pi/\beta)^2}\langle v_1, (iI)^\varepsilon w_2\rangle \\ &= \sum_{n\in\mathbb{Z}} \chi_{2n}(t)c^\lambda_{2n}\langle v_1, w_1\rangle + \sum_{n\in\mathbb{Z}} \chi_{2n+1}(t)c^\lambda_{2n+1}\langle v_2, (iI)^\varepsilon w_2\rangle \\ &= \langle v_1, u^+_\lambda(t)w_1\rangle + \langle v_2, u^-_\lambda(t)(iI)^\varepsilon w_2\rangle = \langle v, f^\sharp(t,\tau^\varepsilon)w\rangle.\end{aligned}$$

This shows that $\tilde{f} = f^\sharp$, which completes the proof. □

Remark 4.16. From $u^+_\lambda = \sum_{n\in\mathbb{Z}} c^\lambda_{2n}\chi_{2n}$, it follows that

$$(\lambda^2 - \Delta)u^+_\lambda = \sum_{n\in\mathbb{Z}} c^\lambda_{2n}\left(\lambda^2 + \frac{(2\pi n)^2}{\beta^2}\right)\chi_{2n} = c^\lambda_+ \sum_{n\in\mathbb{Z}} \chi_{2n} = c^\lambda_+\delta_0,$$

where the latter relation means that

$$s_+(0) = \frac{1}{2\beta}\sum_{n\in\mathbb{Z}} \int_0^{2\beta} s_+(t)\chi_{2n}(t)\,dt$$

for every smooth β-periodic function s_+ on $\mathbb{R}$. This relation can also be written as

$$(\lambda^2 - \Delta)^{-1}\delta_0 = \frac{1}{c_+^\lambda} u_\lambda^+.$$

From $u_\lambda^- = \sum_{n\in\mathbb{Z}} c_{2n+1}^\lambda \chi_{2n+1}$, it follows that

$$(\lambda^2 - \Delta)u_\lambda^- = \sum_{n\in\mathbb{Z}} c_{2n+1}^\lambda \Big(\lambda^2 + \frac{(2n+1)^2\pi^2}{\beta^2}\Big)\chi_{2n+1} = c_-^\lambda \chi_1 \sum_{n\in\mathbb{Z}} \chi_{2n}.$$

As every smooth β-antiperiodic function s_- is of the form $s_- = \chi_{-1}s_+$, where s_+ is β-periodic, we obtain, in the sense of distributions,

$$\langle (\lambda^2 - \Delta)u_\lambda^-, s_-\rangle = c_-^\lambda s_+(0) = c_-^\lambda s_-(0) = \langle c_-^\lambda \delta_0, s_-\rangle,$$

and therefore

$$(\lambda^2 - \Delta)^{-1}\delta_0 = \frac{1}{c_-^\lambda} u_\lambda^-$$

on β-antiperiodic functions. Combining all this, we get

$$((\lambda^2 - \Delta) f^\sharp)(t, \tau^\varepsilon) = \begin{pmatrix} (\lambda^2 - \Delta)u_\lambda^+ \mathbf{1} & 0 \\ 0 & (\lambda^2 - \Delta)u_\lambda^-(iI)^\varepsilon \end{pmatrix} = \delta_0 \begin{pmatrix} c_+^\lambda \mathbf{1} & 0 \\ 0 & c_-^\lambda (iI)^\varepsilon \end{pmatrix}$$

as an operator-valued distribution on the space of smooth sections of $\mathbb{V}$ (cf. also the discussion of thermal euclidean Green's functions in [Dereziński and Gérard 2013, Definition 18.49]).

5. The case $\beta = \infty$

In the context of C^*-dynamical systems, it is well known that the positive energy condition for the unitary one-parameter group implementing the automorphisms of a C^*-algebra $\mathcal{A}$ in a representation can be viewed as a KMS condition for $\beta = \infty$ (cf. [Bratteli and Robinson 1981]). For reflection positive representations of $G = \mathbb{R}$, this case corresponds to $G_+ = \mathbb{R}_+$, which has been treated in [Neeb and Ólafsson 2014; 2015a] (cf. also the discussion of euclidean Green's functions in [Dereziński and Gérard 2013, Definition 18.48]). The following theorem makes this analogy also transparent in the context of our Theorem 2.6.

If $\psi : \mathbb{R} \to \mathrm{Bil}(V)$ is a positive definite function satisfying the KMS condition for $\beta > 0$, then its extension to $\overline{\mathcal{S}_\beta}$ is pointwise bounded (Theorem 2.6). This observation explains the assumptions in the following theorem.

Theorem 5.1 (KMS condition for $\beta = \infty$). *Let V be a real vector space and let $\psi : \mathbb{R} \to \mathrm{Bil}(V)$ be a pointwise continuous positive definite function. Then the following are equivalent*:

(i) *ψ extends to a pointwise bounded function on the closed upper half plane which is pointwise holomorphic on $\mathbb{C}_+$.*

(ii) *There exists a* $\mathrm{Bil}^+(V)$*-valued regular Borel measure μ on* $[0,\infty)$ *satisfying*

$$\psi(t) = \int_0^\infty e^{it\lambda}\, d\mu(\lambda).$$

(iii) *The GNS representation $(U^\psi, \mathcal{H}_\psi)$ has spectrum contained in* $[0,\infty)$.

If this is the case, then the function

$$f(t,\tau^\varepsilon) := \psi(i|t|) \quad \textit{for } t \in \mathbb{R}, \varepsilon \in \{0,1\},$$

on $\mathbb{R}_\tau$ is reflection positive with respect to $\mathbb{R}_+ = [0,\infty)$.

Proof. (i) $\Rightarrow$ (ii): First we use [Neeb and Ólafsson 2015b, Proposition B.1] to write φ as the Fourier transform of a $\mathrm{Bil}^+(V)$-valued regular Borel measure μ on $\mathbb{R}$: $\psi(t) = \int_{\mathbb{R}} e^{it\lambda}\, d\mu(\lambda)$. Evaluating in $v \in V_{\mathbb{C}}$, we obtain for the positive measure $\mu^{v,v} := \mu(\cdot)(v,v)$ the relation

$$\psi(t)(v,v) = \int_{\mathbb{R}} e^{it\lambda}\, d\mu^{v,v}(\lambda).$$

This function extends to a bounded holomorphic function ψ on $\mathbb{C}_+$. In particular, the Laplace transform $\mathcal{L}(\mu^{v,v})(t) = \psi(it)(v,v)$ is bounded, which implies that $\mathrm{supp}(\mu^{v,v}) \subseteq [0,\infty)$ (cf. [Neeb 2000, Remark V.4.12]). This implies that μ is supported on $[0,\infty)$.

(ii) $\Rightarrow$ (iii): Write $U_t := U_t^\psi = e^{itH}$ with the selfadjoint generator H. We show that $H \geq 0$. Let E be the spectral measure of H, so that $H = \int_{\mathbb{R}} \lambda\, dE(\lambda)$ and $U_t = \int_{\mathbb{R}} e^{it\lambda}\, dE(\lambda)$. It suffices to show that, for every $f \in L^1(\mathbb{R})$ for which the Fourier transform $\hat{f}(\lambda) = \int_{\mathbb{R}} e^{i\lambda t} f(t)\, dt$ vanishes on $\mathbb{R}_+$, the operator

$$\begin{aligned} U_f = \int_{\mathbb{R}} f(t) e^{itH}\, dt &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(t) e^{it\lambda}\, dE(\lambda)\, dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(t) e^{it\lambda}\, dt\, dE(\lambda) = \int_{\mathbb{R}} \hat{f}(\lambda)\, dE(\lambda) = \hat{f}(H) \end{aligned}$$

vanishes. For $v, w \in V$, we obtain with (ii) that

$$\begin{aligned} \langle j(v), U_f j(w)\rangle &= \int_{\mathbb{R}} f(t) \langle j(v), U_t j(w)\rangle\, dt = \int_{\mathbb{R}} f(t) \int_0^\infty e^{it\lambda}\, d\mu^{v,w}(\lambda)\, dt \\ &= \int_0^\infty \int_{\mathbb{R}} f(t) e^{it\lambda}\, dt\, d\mu^{v,w}(\lambda) = \int_0^\infty \hat{f}(\lambda)\, d\mu^{v,w}(\lambda) = 0 \end{aligned}$$

if $\hat{f}$ vanishes on $\mathbb{R}_+$. This proves that $j(V) \subseteq \ker(U_f)$ and since U_f is an intertwining operator and the subspace $j(V) \subseteq \mathcal{H}_\psi$ is generating, it follows that $U_f = 0$. This implies that $H \geq 0$.

(iii) ⇒ (i): Write $U_t := U_t^{\psi} = e^{itH}$ and assume that $H \geq 0$. The spectral calculus for selfadjoint operators now implies that $\widehat{U}_z := e^{izH}$, $\operatorname{Im} z \geq 0$ defines a strongly continuous representation on the upper half plane $\mathbb{C}_+$ which is holomorphic on the interior and whose range consists of contractions ([Neeb 2000, Chapter VI]). Then

$$\widehat{\psi}(z)(v, w) = \langle j(v), \widehat{U}_z j(w)\rangle = \langle j(v), e^{izH} j(w)\rangle, \qquad v, w \in V, \operatorname{Im} z \geq 0,$$

provides the bounded analytic extension of ψ to $\mathbb{C}_+$.

Now we assume that (i)–(iii) are satisfied. Writing $\psi(t)(v, w) = \langle j(v), U_t j(w)\rangle$ for a linear map $j : V \to \mathcal{H}$ and a unitary one-parameter group $U_t = e^{itH}$ on $\mathcal{H}$, we have $H \geq 0$ by (iii) and

$$f(t, \tau^{\varepsilon}) = \langle j(v), e^{-|t|H} j(w)\rangle,$$

so that the positive definiteness of f follows from the positive definiteness of the function $t \mapsto e^{-|t|H}$ on $\mathbb{R}$ [Neeb and Ólafsson 2014, Proposition 4.1]. □

Appendix A. Some background on positive definite kernels

In this appendix we collect precise statements of some basic facts on positive definite kernels and functions to keep the paper more self-contained.

Form-valued positive definite kernels.

Definition A.1. Let X be a set and V be a real vector space. We write $\mathrm{Bil}(V) = \mathrm{Bil}(V, \mathbb{C})$ for the space of complex-valued bilinear forms on V. We call a map $K : X \times X \to \mathrm{Bil}(V)$ *a positive definite kernel* if the associated scalar-valued kernel

$$K^{\flat} : (X \times V) \times (X \times V) \to \mathbb{C}, \qquad K^{\flat}((x, v), (y, w)) := K(x, y)(v, w)$$

is positive definite.[2] The corresponding reproducing kernel Hilbert space $\mathcal{H}_{K^{\flat}} \subseteq \mathbb{C}^{X \times V}$ is generated by the elements $K^{\flat}_{x,v}$, $x \in X$, $v \in V$, with the inner product

$$\langle K^{\flat}_{(x,v)}, K^{\flat}_{(y,w)}\rangle = K(x, y)(v, w) =: K^{\flat}((x, v), (y, w)) =: K^{\flat}_{y,w}(x, v),$$

so that, for all $f \in \mathcal{H}_{K^{\flat}}$, we have

$$f(x, v) = \langle K^{\flat}_{x,v}, f\rangle.$$

We identify $\mathcal{H}_{K^{\flat}}$ with a subspace of $(V^*)^X$ by identifying $f \in \mathcal{H}_{K^{\flat}}$ with the function $f^* : X \to V^*$, $f^*(x) := f(x, \cdot)$. We call

$$\mathcal{H}_K := \{f^* : f \in \mathcal{H}_{K^{\flat}}\} \subseteq (V^*)^X$$

[2] This definition is adapted to our convention that scalar products are linear in the second argument. Accordingly, a kernel $K : X \times X \to \mathrm{Bil}(V)$ is positive definite in the sense of Definition A.1 if and only if the kernel $(x, y) \mapsto K(x, y)^{\top}$ is positive definite in the sense of [Neeb and Ólafsson 2015b].

the (*vector-valued*) *reproducing kernel space associated to* K. The elements

$$K_{x,v} := (K^\flat_{x,v})^* \quad \text{with} \ \ K_{x,v}(y) = K(y,x)(\,\cdot\,,v) \ \ \text{for} \ \ x,y \in X,\ v,w \in V,$$

then form a dense subspace of $\mathcal{H}_K$ with

$$\langle K_{x,v}, K_{y,w}\rangle = K(x,y)(v,w). \tag{31}$$

Example A.2. If V is a complex Hilbert space, X is a set and $K : X \times X \to B(V)$ is an operator-valued kernel, then K is called positive definite if the corresponding kernel

$$\widetilde{K} : (X \times V) \times (X \times V) \to \mathbb{C}, \quad \widetilde{K}((x,v),(y,w)) := \langle v, K(x,y)w\rangle$$

is positive definite [Neeb 2000, Definition I.1.1], and this means that the kernel

$$K' : X \times X \to \mathrm{Sesq}(V) \subseteq \mathrm{Bil}(V), \quad K'(x,y)(v,w) := \langle v, K(x,y)w\rangle$$

is positive definite.

If $X = G$ is a group and the kernel K is invariant under right translations, then it is of the form $K(g,h) = \varphi(gh^{-1})$ for a function $\varphi : G \to \mathrm{Bil}(V)$.

Definition A.3. Let G be a group and let V be a real vector space. A function $\varphi : G \to \mathrm{Bil}(V)$ is said to be *positive definite* if the $\mathrm{Bil}(V)$-valued kernel $K(g,h) := \varphi(gh^{-1})$ is positive definite.

The following proposition ([Neeb and Ólafsson 2015b, Proposition A.4]) generalizes the GNS construction to form-valued positive definite functions on groups.

Proposition A.4 (GNS-construction). *Let V be a real vector space.*

(a) *Let $\varphi : G \to \mathrm{Bil}(V)$ be a positive definite function. Then $(U^\varphi_g f)(h) := f(hg)$ defines a unitary representation of G on the reproducing kernel Hilbert space $\mathcal{H}_\varphi \subseteq (V^*)^G$ with kernel $K(g,h) = \varphi(gh^{-1})$ and the range of the map*

$$j : V \to \mathcal{H}_\varphi, \quad j(v)(g)(w) := \varphi(g)(w,v), \qquad j(v) = K^\flat_{\mathbf{1},v},$$

is a cyclic subspace, i.e., $U^\varphi_G j(V)$ spans a dense subspace of $\mathcal{H}$. We then have

$$\varphi(g)(v,w) = \langle j(v), U^\varphi_g j(w)\rangle \quad \textit{for} \ \ g \in G,\ v,w, \in V. \tag{32}$$

(b) *If, conversely, $(U,\mathcal{H})$ is a unitary representation of G and $j : V \to \mathcal{H}$ a linear map whose range is cyclic, then*

$$\varphi : G \to \mathrm{Bil}(V), \quad \varphi(g)(v,w) := \langle j(v), U_g j(w)\rangle$$

is a $\mathrm{Bil}(V)$-valued positive definite function and $(U,\mathcal{H})$ is unitarily equivalent to $(U^\varphi,\mathcal{H}_\varphi)$.

Remark A.5. If $\varphi : G \to \mathrm{Bil}(V)$ is a positive definite function, then (32) shows that, if $\widetilde{V} := \overline{j(V)}$, which is the real Hilbert space defined by completing V with respect to the positive semidefinite form $\varphi(\mathbf{1})$, then

$$\widetilde{\varphi}(g)(v, w) = \langle v, U_g w\rangle$$

defines a positive definite function

$$\widetilde{\varphi} : G \to \mathrm{Bil}(\widetilde{V}) \quad \text{with } \widetilde{\varphi}(g)(j(v), j(w)) = \varphi(g)(v, w) \text{ for } v, w \in V.$$

Therefore it often suffices to consider $\mathrm{Bil}(V)$-valued positive definite functions for a real Hilbert space V for which $\varphi(\mathbf{1})$ is a positive definite hermitian form on V whose real part is the scalar product on V. In terms of (32), this means that $j : V \to \mathcal{H}$ is an isometric embedding of the real Hilbert space V.

Products of operator-valued kernels.

Lemma A.6. *If $K_j : X \times X \to B(V)$, $j = 1, 2$, are two positive definite kernels with the property that*

$$K_1(x, y)K_2(x', y') = K_2(x', y')K_1(x, y) \quad \textit{for } x, x', y, y' \in X,$$

then the product kernel $K := K_1 \cdot K_2$ is also positive definite.

Proof. Let $x_1, \ldots, x_k$. We have to show that the operator

$$C := (K_1(x_j, x_k)K_2(x_j, x_k))_{1\leq j,k\leq n} \in M_n(B(V)) \cong B(V^n)$$

is positive (cf. [Neeb 2000, Remark I.1.3]).

Let $\mathcal{A}_j \subseteq B(V)$ denote the von Neumann algebra generated by the values of K_j. Then $\mathcal{A}_1$ and $\mathcal{A}_2$ commute. Further, the matrices

$$A^{(\ell)} := (K_\ell(x_j, x_k))_{1\leq j,k\leq n} \in M_n(\mathcal{A}_\ell), \qquad \ell = 1, 2,$$

are positive, so [Lance 1995, Lemma 4.3] implies that the matrix

$$D := (K_1(x_j, x_k) \otimes K_2(x_j, x_k)) \in M_n(\mathcal{A}_1 \otimes \mathcal{A}_2)$$

is positive. Since C is the image of D under the canonical representation of $M_n(\mathcal{A}_1 \otimes \mathcal{A}_2)$ on V^n, it follows that C is positive. □

From real to complex-valued kernels. In this section we take a brief look at the interplay between real and complex-valued positive definite kernels. Here Corollary A.9 is of central importance because it shows how the positive definiteness of a complex-valued form $h = \gamma + i\omega$ on a real vector space V leads to a skew-symmetric contraction on the real Hilbert space V_γ.

Lemma A.7. *Let $K : X \times X \to \mathbb{C}$ be a positive definite kernel. Then the corresponding Hilbert space $\mathcal{H}_K \subseteq \mathbb{C}^X$ is invariant under complex conjugation such that $\sigma(f) := \overline{f}$ defines an antilinear isometry on $\mathcal{H}_K$ if and only if K is real-valued.*

Proof. The invariance requirement implies the relation

$$\langle f, K_x\rangle = \overline{f(x)} = \langle K_x, \sigma(f)\rangle = \langle f, \sigma(K_x)\rangle \quad \text{for } f \in \mathcal{H}_K,$$

and therefore $\sigma(K_x) = K_x$, i.e., K is real-valued. If, conversely, K is real-valued, then $\mathcal{H}_K = \mathcal{H}_K^{\mathbb{R}} \oplus i\mathcal{H}_K^{\mathbb{R}}$ is an orthogonal sum of real Hilbert spaces, so that complex conjugation acts on $\mathcal{H}_K$ as an isometry. □

Proposition A.8. *Let $A, B : X \times X \to \mathbb{R}$ be real kernels on the set X. Then the kernel*

$$K = A + iB : X \times X \to \mathbb{C}$$

is positive definite if and only if

(a) *A is positive definite*, *and*

(b) *there exists a skew-symmetric contractive operator C on the real reproducing kernel Hilbert space $\mathcal{H}_A^{\mathbb{R}} \subseteq \mathbb{R}^X$ with*

$$B(x, y) = \langle A_x, CA_y\rangle = (CA_y)(x) \quad \textit{for } x, y \in X.$$

Proof. **Necessity:** If K is positive definite, then so is $\overline{K} = A - iB$, and this implies that $A = \frac{1}{2}(K + \overline{K})$ is positive definite. As $A - iB = 2A - K$ is positive definite, [Neeb 2000, Theorem I.2.8][3] implies the existence of a bounded operator $D \geq 0$ on the complex reproducing kernel Hilbert space $\mathcal{H}_A \subseteq \mathbb{C}^X$ with

$$K_y(x) = K(x, y) = \langle A_x, DA_y\rangle = (DA_y)(x) \quad \text{for } x, y \in X.$$

From Lemma A.7 we know that $\mathcal{H}_A = \mathcal{H}_A^{\mathbb{R}} \oplus i\mathcal{H}_A^{\mathbb{R}}$. From the relation $A_y + iB_y = DA_y$ for every $y \in X$ and the fact that B is real-valued it thus follows that $D = \mathbf{1} + iC$ for a bounded operator C on $\mathcal{H}_A^{\mathbb{R}}$ satisfying $CA_y = B_y$ for every $y \in X$. Now $D = D^* \geq 0$ implies that $C = -C^\top$ is a contraction and

$$B(x, y) = (CA_y)(x) = \langle A_x, CA_y\rangle \quad \text{for } x, y \in X.$$

Sufficiency: Suppose, conversely, that A is positive definite and that C is a skew-symmetric contraction on the real Hilbert space $\mathcal{H}_A^{\mathbb{R}}$. Then the hermitian operator $\mathbf{1} + iC$ on $\mathcal{H}_A^{\mathbb{C}}$ is nonnegative, and therefore its symbol

$$K(x, y) := ((\mathbf{1} + iC)A_y)(x) = A(x, y) + i(CA_y)(x)$$

is a positive definite kernel on X. □

[3] For two positive definite kernels K and Q on a set X, the relation $\mathcal{H}_K \subseteq \mathcal{H}_Q$ is equivalent to $\lambda Q - K$ being positive definite for some $\lambda > 0$, and this in turn is equivalent to the existence of a bounded positive operator B on $\mathcal{H}_Q$ with $\|B\| \leq \lambda$ satisfying $K(x, y) = \langle Q_x, BQ_y\rangle = (BQ_y)(x)$ for $x, y \in X$ [Neeb 2000, Theorem I.2.8].

Corollary A.9. *Let V be a real vector space, let $\gamma : V \times V \to \mathbb{R}$ be a symmetric and $\omega : V \times V \to \mathbb{R}$ be a skew-symmetric bilinear form and consider the corresponding hermitian form $h := \gamma + i\omega$. Then the following are equivalent*:

(i) *h is a positive definite kernel on V.*

(ii) *γ is positive semidefinite and there exists a skew-symmetric bounded operator C on the real Hilbert space V_γ obtained by completing $V/\{v \in V : \gamma(v, v) = 0\}$ such that $\omega(v, w) = \langle [v], C[w]\rangle_{V_\gamma}$, where $[v]$ denotes the image of v in V_γ.*

(iii) *γ is positive semidefinite and*

$$\omega(v, w)^2 \le \gamma(v, v)\gamma(w, w) \quad \textit{for } v, w \in V. \tag{33}$$

Proof. (i) $\Leftrightarrow$ (ii): In view of Proposition A.8, the kernel h is positive definite if and only if the kernel γ is positive definite, i.e., γ is a positive semidefinite form, and the kernel ω can be written as

$$\omega(v, w) = \langle [v], C[w]\rangle_{V_\gamma} \quad \text{for } v, w \in V, \tag{34}$$

where C is a skew-symmetric contraction on the real Hilbert space V_γ.

(ii) $\Rightarrow$ (iii): (34) and $\|C\| \le 1$ imply that

$$\omega(v, w)^2 \le \|C\|^2 \|[v]\|^2 \|[w]\|^2 = \gamma(w, w)\gamma(v, v).$$

(iii) $\Rightarrow$ (ii): Suppose, conversely, that γ is positive semidefinite and that (33) is satisfied. Then ω defines a continuous bilinear form on the real Hilbert space V_γ with norm ≤ 1. Hence there exists a skew-symmetric contraction $C \in B(V_\gamma)$ satisfying (34). This proves the corollary. ☐

Lemma A.10. *Let $h = \gamma + i\omega$ be a positive definite kernel as in Corollary A.9, let $\mathcal{H}_h \subseteq \mathrm{Hom}(V, \mathbb{C})$ be the corresponding reproducing kernel Hilbert space and let $j : V \to \mathcal{H}_h$, $j(v) = h(\cdot, v)$ be the canonical map. The following assertions hold*:

(i) *j is injective if and only if γ is positive definite, i.e., defines an inner product on V.*

(ii) *The complex linear extension $j_\mathbb{C} : V_\mathbb{C} \to \mathcal{H}_h$, $v + iw \mapsto j(v) + i \cdot j(w)$ is injective if and only if*

$$\omega(v, w)^2 < \gamma(v, v)\gamma(w, w) \quad \textit{for } 0 \neq v, w \in V.$$

(iii) *Suppose that γ is positive definite, that (V, γ) is complete and that $\omega(v, w) = \langle [v], C[w]\rangle$ for an operator C on $\mathcal{H}_\gamma^\mathbb{R} \cong (V, \gamma)$. Then $j_\mathbb{C}$ is injective if and only if $\|Cv\| < \|v\|$ for every nonzero $v \in \mathcal{H}_\gamma^\mathbb{R}$.*

Proof. (i) In view of $\langle j(v), j(w)\rangle = \langle h(\cdot, v), h(\cdot, w)\rangle = h(v, w)$, we have $\|j(v)\|^2 = h(v, v) = \gamma(v, v)$, so that j is injective if and only if γ is positive definite.

(ii) First we calculate

$$\begin{aligned}\|j_{\mathbb{C}}(v+iw)\|^2 = \|j(v)+i\cdot j(w)\|^2 &= \gamma(v,v)+\gamma(w,w)+2\,\mathrm{Re}\langle j(v), i\cdot j(w)\rangle \\ &= \gamma(v,v)+\gamma(w,w)+2\,\mathrm{Re}\, ih(w,v) \\ &= \gamma(v,v)+\gamma(w,w)+2\omega(v,w).\end{aligned}$$

Writing $\omega(v,w)=\langle\gamma_w, C\gamma_v\rangle$ as in (34), it follows that $j_{\mathbb{C}}(v+iw)=0$ is equivalent to

$$2\langle\gamma_v, C\gamma_w\rangle = \langle\gamma_v,\gamma_v\rangle+\langle\gamma_w,\gamma_w\rangle. \tag{35}$$

Next we observe that $j(v)=-i\cdot j(w)$ implies $\gamma(v,v)=\|j(v)\|^2=\|j(w)\|^2=\gamma(w,w)$, which leads to

$$\langle\gamma_v, C\gamma_w\rangle = \|\gamma_v\|^2 = \|\gamma_w\|^2 = \|\gamma_v\|\cdot\|\gamma_w\|.$$

As C is a contraction, this is equivalent to $C\gamma_v=\gamma_w$ by the Cauchy–Schwarz inequality.

If, conversely, there exists a nonzero $v\in V$ with $C\gamma_v=\gamma_w$ and $\gamma(v,v)=\gamma(w,w)$, then $j_{\mathbb{C}}(v+iw)=0$ by (35). This proves (ii).

(iii) If (V,γ) is complete, $j(V)\cong(V,\gamma)$ is closed in $\mathcal{H}_h$. Therefore $Cj(V)\subseteq j(V)$, and (iii) follows from the preceding discussion. □

Remark A.11. If $V\subseteq\mathcal{H}$ is a standard real subspace (Definition 2.4), then the kernel $h(v,w):=\langle v,w\rangle$ on V has the property that the corresponding reproducing kernel Hilbert space is $\mathcal{H}$ and the inclusion is the corresponding map $j: V\to\mathcal{H}$. In particular, its complex linear extension is injective.

If, conversely, $h=\gamma+i\omega$ is a positive definite bilinear kernel on a real vector space V, then $j(V)$ is a standard real subspace of the corresponding complex Hilbert space $\mathcal{H}_h$ if and only if (V,γ) is complete (which is equivalent to the closedness of $j(V)$) and the complex linear extension $j_{\mathbb{C}}: V_{\mathbb{C}}\to\mathcal{H}_h$ is injective, which is equivalent to $j(V)\cap i\cdot j(V)=\{0\}$ (cf. Lemma A.10(iii)).

Example A.12. Consider the context of Proposition A.8, where $K=A+iB$ is a positive definite kernel and $C\in B(\mathcal{H}_A^{\mathbb{R}})$ is such that $B_y=CA_y$ for $y\in X$. Then

$$V:=(\mathbf{1}+iC)\mathcal{H}_A^{\mathbb{R}}\subseteq\mathcal{H}_A$$

is a real subspace. For the isometric antilinear involution defined on $\mathcal{H}_A$ by $\sigma(f)=\overline{f}$, we then have for every $f\in\mathcal{H}_A^{\mathbb{R}}$ the relation

$$\langle\sigma(\mathbf{1}+iC)f, (\mathbf{1}+iC)f\rangle = \|f\|^2-\|Cf\|^2\geq 0.$$

Therefore $(\mathcal{H}_A, V, \sigma)$ is a reflection positive real Hilbert space (Proposition B.3).

Real parts of positive definite functions. Let $\varphi : G \to \mathbb{C}$ be a positive definite function on the group G. Then $\overline{\varphi}$ is also positive definite, so that $\operatorname{Re}\varphi = \frac{1}{2}(\varphi + \overline{\varphi})$ is positive definite as well. From Lemma A.7(a) we know that a positive definite function φ on G is real-valued if and only if the corresponding reproducing kernel Hilbert space $\mathcal{H}_\varphi$ is invariant under conjugation with $\|\overline{f}\| = \|f\|$ for $f \in \mathcal{H}_\varphi$. Based on these observations, one would like to understand the set of all positive definite functions with a given real part. A natural description of this set in the spirit of the present paper is provided by the following theorem.

Theorem A.13 (Complex extensions of real positive definite functions). *Let $\varphi : G \to \mathbb{R}$ be a positive definite function and let $(U^\varphi, \mathcal{H}_\varphi^\mathbb{R})$ denote the corresponding orthogonal representation on the real reproducing kernel space $\mathcal{H}_\varphi^\mathbb{R} \subseteq \mathbb{R}^G$ by right translations*: $(U^\varphi(g)f)(h) := f(hg)$. *Then the following assertions hold*:

(a) *For each skew-symmetric contraction C on $\mathcal{H}_\varphi$ commuting with $U^\varphi(G)$, the function $\varphi_C := \varphi + iC\varphi \in \mathcal{H}_\varphi \subseteq \mathbb{C}^G$ is positive definite. Here we consider φ as an element of the real Hilbert space $\mathcal{H}_\varphi^\mathbb{R} \subseteq \mathbb{R}^G$.*

(b) *Each positive definite function $\widehat{\varphi}$ with* $\operatorname{Re}\widehat{\varphi} = \varphi$ *is of the form φ_C for a unique skew-symmetric contraction C on $\mathcal{H}_\varphi$ commuting with $U^\varphi(G)$.*

Proof. (a) Clearly $\mathcal{H}_\varphi = \mathcal{H}_\varphi^\mathbb{R} \oplus i\mathcal{H}_\varphi^\mathbb{R}$ is the Hilbert space complexification of $\mathcal{H}_\varphi^\mathbb{R}$ (Lemma A.7). On $\mathcal{H}_\varphi$ the operator $B := \mathbf{1} + iC$ is positive because it is hermitian and $\|C\| \leq 1$. Let $K(x, y) := \varphi(xy^{-1})$ be the kernel corresponding to φ which satisfies $K_y = U^\varphi(y)^{-1}\varphi$. Then the associated kernel

$$\begin{aligned} K^B(x, y) &:= \langle BK_y, K_x\rangle = \langle BU^\varphi(y)^{-1}\varphi, U^\varphi(x)^{-1}\varphi\rangle \\ &= \langle U^\varphi(y)^{-1}B\varphi, U^\varphi(x)^{-1}\varphi\rangle \\ &= \langle U^\varphi(xy^{-1})(\mathbf{1} + iC)\varphi, \varphi\rangle = ((\mathbf{1} + iC)\varphi)(xy^{-1}) \end{aligned}$$

is positive definite (cf. [Neeb 2000, Lemma I.2.4]), and this means that $\varphi + iC\varphi$ is a positive definite function.

(b) If $\widehat{\varphi} = \varphi + i\psi$ is positive definite with φ, ψ real-valued, then write $K = A + iB$ for the corresponding kernels:

$$K(x, y) = \widehat{\varphi}(xy^{-1}), \qquad A(x, y) = \varphi(xy^{-1}) \qquad \text{and} \qquad B(x, y) = \psi(xy^{-1}).$$

Then Proposition A.8 implies that φ is positive definite and that there exists a skew-symmetric contraction $C \in B(\mathcal{H}_\varphi^\mathbb{R})$ with

$$\psi(xy^{-1}) = (CA_y)(x) = \langle CU^\varphi(y)^{-1}\varphi, U^\varphi(x)^{-1}\varphi\rangle.$$

Since this kernel on $G \times G$ is invariant under right translations and $U^\varphi(G)\varphi$ is total in $\mathcal{H}_\varphi^{\mathbb{R}}$, it follows that C commutes with $U^\varphi(G)$. This in turn leads to

$$\psi(xy^{-1}) = \langle C\varphi, U^\varphi(yx^{-1})\varphi\rangle = (C\varphi)(xy^{-1})$$

and hence to $\psi = C\varphi$. □

Appendix B. Standard real subspaces via contractions

In this section we show how standard real subspaces can be parametrized in a very convenient way by skew-symmetric contractions in real Hilbert spaces. The survey article [Longo 2008] is an excellent source for the theory of standard real subspaces.

Skew symmetric contractions.

Lemma B.1. *Let C_V be a skew-symmetric contraction on the real Hilbert space E and $V := (\mathbf{1} + iC_V)E \subseteq E_{\mathbb{C}}$. For $0 \neq v \in E$, the following are equivalent*:

(i) $C_V^2 v = -v$.

(ii) $\|C_V v\| = \|v\|$.

(iii) *There exists* $0 \neq w \in V$ *with* $\langle C_V v, w\rangle = \|v\|\|w\|$.

(iv) $(\mathbf{1} + iC_V)v \in V \cap iV$.

Proof. (i) ⇔ (ii): First we observe that $\|v\|^2 - \|C_V v\|^2 = \langle(\mathbf{1} + C_V^2)v, v\rangle$. In view of the positivity of $\mathbf{1} + C_V^2$, the relation $\langle(\mathbf{1} + C_V^2)v, v\rangle = 0$ is equivalent to $(\mathbf{1} + C_V^2)v = 0$.

(ii) ⇔ (iii) follows from $\max\{\langle C_V v, w\rangle : w \in E, \|w\| \leq 1\} = \|C_V v\| \leq \|v\|$.

(iv) ⇔ (i): For $w \in E$, the condition $(\mathbf{1} + iC_V)v = i(\mathbf{1} + iC_V)w$ is equivalent to $C_V w = -v$ and $w = C_V v$. Such an element w exists if and only if $C_V^2 v = -v$. □

Lemma B.2. *For a skew-symmetric contraction C_V on the real Hilbert space E and $V := (\mathbf{1} + iC_V)E \subseteq E_{\mathbb{C}}$, the following are equivalent*:

(i) $C_V^2 + \mathbf{1}$ *is injective.*

(ii) $\|C_V v\| < \|v\|$ *for every nonzero* $v \in E$.

(iii) $\langle C_V v, w\rangle < \|v\|\|w\|$ *for nonzero elements* $v, w \in E$.

(iv) $V \cap iV = \{0\}$.

(v) *The operators* $\mathbf{1} \pm iC_V$ *on* $E_{\mathbb{C}}$ *are injective.*

(vi) $V + iV$ *is dense in* $E_{\mathbb{C}}$.

(vii) V *is a standard real subspace.*

Proof. The equivalence of (i)–(iv) follows immediately from Lemma B.1.

Further, (iv) can also be formulated as: $(\mathbf{1}+iC_V)(v+iw)=0$ for $v, w \in E$ implies $v+iw=0$, which in turn means that $\mathbf{1}+iC_V$ is injective. This in turn is equivalent to $\mathbf{1}-iC_V$ being injective. Therefore (iv) is equivalent to (v).

As $V+iV=(\mathbf{1}+iC_V)E_{\mathbb{C}}=\mathrm{im}(\mathbf{1}+iC_V)$, this complex subspace is dense if and only if the hermitian operator $\mathbf{1}+iC_V$ has dense range, and this is equivalent to $\mathbf{1}+iC_V$ being injective. Therefore (v) and (vi) are also equivalent.

Next we observe that V is closed because

$$\|(\mathbf{1}+iC_V)v\|^2=\|v\|^2+\|C_Vv\|^2\geq\|v\|^2 \quad \text{for } v\in E$$

shows that the range V of the operator $\mathbf{1}+iC_V : E\to E_{\mathbb{C}}$ is closed. Since (iv) and (vi) are equivalent, they are therefore equivalent to V being a standard real subspace. □

Proposition B.3. *Let E be a real Hilbert space, C_V be a skew-symmetric contraction on E, let $E_{\mathbb{C}}$ be the complexification of E and let $\sigma : E_{\mathbb{C}}\to E_{\mathbb{C}}$, $a+ib\mapsto a-ib$ be complex conjugation on $E_{\mathbb{C}}$. Then the real subspace*

$$V:=(\mathbf{1}+iC_V)E\subseteq E_{\mathbb{C}}$$

has the following properties:

(i) *Let $E_0=\ker(C_V^2+\mathbf{1})$ and $E_1=E_0^{\perp}$, so that $E=E_0\oplus E_1$. Then $C_0:=C_V|_{E_0}$ is a complex structure on E_0 and $V_0:=(\mathbf{1}+iC_V)E_0\subseteq E_{\mathbb{C}}$ is the $(-i)$-eigenspace of C_V. It coincides with $V\cap iV$. In particular it is a complex subspace of $E_{\mathbb{C}}$. The subspace $V_1:=(\mathbf{1}+iC_V)E_1$ is a standard real subspace of $E_{1,\mathbb{C}}$.*

(ii) *If $V=V_1$, then the corresponding modular objects are given by*

$$(\Delta, J)=\left(\left(\frac{\mathbf{1}-iC_V}{\mathbf{1}+iC_V}\right)^2, \sigma\right).$$

Proof. (i) For $a, b\in E$, the relation $C_V(a+ib)=-i(a+ib)$ is equivalent to $C_Va=b$ and $C_Vb=-a$, i.e., to $a+ib\in V_0$. Therefore V_0 is the $(-i)$-eigenspace of C_V in $E_{\mathbb{C}}$. From Lemma B.1(iv) we further obtain $V\cap iV=V_0$. For $V_1:=(\mathbf{1}+iC_V)E_1$, we thus have $V_1\cap iV_1=\{0\}$, so that Lemma B.2(vii) implies that V_1 is a standard real subspace of $E_{1,\mathbb{C}}$.

(ii) If $V=V_1$, then

$$\Delta:=\left(\frac{\mathbf{1}-iC_V}{\mathbf{1}+iC_V}\right)^2 \tag{36}$$

is a positive selfadjoint operator on $E_{\mathbb{C}}$ with domain $(\mathbf{1}+iC_V)^2E_{\mathbb{C}}$. Further $\Delta^{1/2}=(\mathbf{1}-iC_V)(\mathbf{1}+iC_V)^{-1}$ has domain $V_{\mathbb{C}}$

Since $\sigma\Delta\sigma = \Delta^{-1}$ by (36), $S := \sigma\Delta^{1/2}$ is an unbounded antilinear involution with

$$\mathrm{Fix}(S) = \{\xi \in \mathcal{D}(\Delta^{1/2}) = V_{\mathbb{C}} : S\xi = \xi\}.$$

For $\xi = (\mathbf{1} + iC_V)v,\ v \in E_{\mathbb{C}}$, we have

$$S\xi = \sigma\Delta^{1/2}\xi = \sigma(\mathbf{1} - iC_V)v = (\mathbf{1} + iC_V)\sigma(v),$$

so $S\xi = \xi$ is equivalent to $v \in V$. We conclude that $\mathrm{Fix}(S) = V$. This proves (ii). □

Remark B.4. Let C be a skew-symmetric contraction on the real Hilbert space E. Then the selfadjoint operator $C^2 + \mathbf{1}$ is invertible if and only if $-\mathbf{1} \notin \mathrm{Spec}(C^2)$, which is equivalent to $\mathbf{1} \notin \mathrm{Spec}(iC)$, where iC is considered as a selfadjoint operator on the complex Hilbert space $E_{\mathbb{C}}$. This, in turn, is equivalent to the invertibility of $\mathbf{1} + iC$ and hence to the boundedness of $(\mathbf{1} - iC)(\mathbf{1} + iC)^{-1}$.

Real reflection positivity and standard subspaces. In this section we relate standard real subspaces to reflection positive real Hilbert spaces of the form $(E_{\mathbb{C}}, V, \sigma)$, where σ is the complex conjugation on the complexification $E_{\mathbb{C}}$ of a real Hilbert space. This sheds an interesting light on the close connection between standard real subspaces and reflection positivity.

Lemma B.5. *Let E be a real Hilbert space and $E_{\mathbb{C}}$ be its complexification. On $E_{\mathbb{C}}$ we consider the antilinear isometry defined by $\sigma(a + ib) := a - ib$. A real subspace $V \subseteq E_{\mathbb{C}}$ has the property that the form $(v, w) \mapsto \langle \sigma v, w\rangle$ is real-valued and positive semidefinite on V if and only if there exists a skew-symmetric contraction $C_V : \mathcal{D}(C_V) \to E$ with $V = (\mathbf{1} + iC_V)(\mathcal{D}(C_V))$. The subspace V is closed if and only if $\mathcal{D}(C_V)$ is closed.*

Proof. First, let $C_V : \mathcal{D}(C_V) \to E$ be a skew-symmetric contraction and put $V := (\mathbf{1} + iC_V)\mathcal{D}(C_V)$. For $v, w \in \mathcal{D}(C_V)$, we then have

$$\begin{aligned}\langle \sigma((\mathbf{1}+iC_V)v), (\mathbf{1}+iC_V)w\rangle &= \langle (\mathbf{1}-iC_V)v), (\mathbf{1}+iC_V)w\rangle \\ &= \langle v, w\rangle + \langle -iC_V v, w\rangle + \langle v, iC_V w\rangle - \langle C_V v, C_V w\rangle \\ &= \langle v, w\rangle - \langle C_V v, C_V w\rangle = \langle (\mathbf{1}+C_V^2)v, w\rangle \in \mathbb{R}.\end{aligned}$$

Moreover $\mathbf{1} + C_V^2 \geq 0$ implies that the form is positive semidefinite.

Conversely, let $V \subseteq E_{\mathbb{C}}$ be a real subspace which is σ-positive in the sense that the form $f(v, w) := \langle \sigma v, w\rangle$ is real-valued and positive semidefinite. This assumption implies that $V \cap iE = \{0\}$. Hence there exists a real linear operator $C_V : \mathcal{D}(C_V) \to E$ for which $V = (\mathbf{1} + iC_V)\mathcal{D}(C_V)$. Since

$$\begin{aligned}\langle \sigma(v + iC_V v), w + iC_V w\rangle &= \langle v - iC_V v, w + iC_V w\rangle \\ &= \langle v, w\rangle - \langle C_V v, C_V w\rangle + i(\langle C_V v, w\rangle + \langle C_V w, v\rangle)\end{aligned}$$

is supposed to be real-valued,

$$\langle C_V v, w\rangle + \langle v, C_V w\rangle = 0 \quad \text{for } v, w \in E.$$

This means that C_V is skew-symmetric on $\mathcal{D}(C_V)$. Further, the positivity assumption implies that $\|C_V v\| \leq \|v\|$ for $v \in E$.

The subspace V is closed if and only if the graph of C_V is closed, which is equivalent to the closedness of $\mathcal{D}(C_V)$ because C_V is a contraction. □

Proposition B.6. *Let E be a real Hilbert space, let C_V be a skew-symmetric contraction on E, let $E_{\mathbb{C}}$ be the complexification of E and let $\sigma : E_{\mathbb{C}} \to E_{\mathbb{C}}$, $a + ib \mapsto a - ib$ be complex conjugation on $E_{\mathbb{C}}$. Then the real subspace*

$$V := (\mathbf{1} + iC_V)E \subseteq E_{\mathbb{C}}$$

has the following properties:

(i) *V is closed and σ-positive, so that $(E_{\mathbb{C}}, V, \sigma)$ is a reflection positive real Hilbert space.*

(ii) *$V^\perp = i\sigma(V)$, i.e., the bilinear form $\gamma_\sigma(\xi, \eta) := \langle \sigma\xi, \eta\rangle$ on V is real-valued.*

(iii) *The null space of the positive semidefinite form γ_σ on V coincides with the $(-i)$-eigenspace V_0 of C_V on $E_{\mathbb{C}}$. If $V_0 = \{0\}$, then the unbounded positive operator*

$$F := \sqrt{\frac{\mathbf{1} - iC_V}{\mathbf{1} + iC_V}} : V \to E_{\mathbb{C}}$$

satisfies $\|F\xi\|^2 = \langle \sigma\xi, \xi\rangle$ for $\xi \in V$, so that we can identify the real Hilbert space completion $\widehat{V}$ of V with respect to γ_σ with $\overline{F(V)}$. We further have $\sigma F \sigma = F^{-1}$.

Proof. (i) The subspace V is closed because

$$\|(\mathbf{1} + iC_V)v\|^2 = \|v\|^2 + \|C_V v\|^2 \geq \|v\|^2 \quad \text{for } v \in E$$

shows that the range of the operator $\mathbf{1} + iC_V : E \to V$ is closed.

For the complex conjugation σ on $E_{\mathbb{C}}$, we have for $v, w \in E$ the relation

$$\begin{aligned}\gamma_\sigma((\mathbf{1}+iC_V)v, (\mathbf{1}+iC_V)w) &= \langle \sigma(\mathbf{1}+iC_V)v, (\mathbf{1}+iC_V)w\rangle \\ &= \langle (\mathbf{1}-iC_V)v, (\mathbf{1}+iC_V)w\rangle \\ &= \langle (\mathbf{1}+iC_V)(\mathbf{1}-iC_V)v, w\rangle = \langle (\mathbf{1}+C_V^2)v, w\rangle \in \mathbb{R}\end{aligned}$$

and thus

$$\gamma_\sigma((\mathbf{1} + iC_V)v, (\mathbf{1} + iC_V)v) = \|v\|^2 - \|C_V v\|^2 \geq 0.$$

(ii) An element $a+ib \in E_{\mathbb{C}}$ $(a, b \in E)$ is orthogonal to V with respect to the real scalar product if and only if

$$0 = \mathrm{Re}\langle a+ib, v+iC_V v\rangle = \langle a, v\rangle + \langle b, C_V v\rangle = \langle a - C_V b, v\rangle$$

for every $v \in E$; this is equivalent to $C_V b = a$, i.e., to $a+ib = i(b - iC_V b) \in i\sigma(V)$.
(iii) An element $\xi := (\mathbf{1}+iC_V)v \in V$ satisfies $\langle \sigma\xi, \xi\rangle = 0$ if and only if $C_V^2 v = -v$, which is equivalent to

$$(\mathbf{1} - iC_V)\xi = (\mathbf{1} - iC_V)(\mathbf{1}+iC_V)v = (\mathbf{1}+C_V^2)v = 0,$$

i.e., to $C_V\xi = -i\xi$. This implies that $V_0 \subseteq V$ is the nullspace of γ_σ.

Now we assume that $V_0 = \{0\}$ and $V = V_1$. As $\mathbf{1} \pm iC_V$ are nonnegative hermitian operators on $E_{\mathbb{C}}$, they have a nonnegative square root and $(\mathbf{1}+iC_V)^{-1/2}$ is an unbounded operator whose domain is

$$\sqrt{\mathbf{1}+iC_V}\,E_{\mathbb{C}} \supseteq \sqrt{\mathbf{1}+iC_V}\sqrt{\mathbf{1}+iC_V}\,E_{\mathbb{C}} = (\mathbf{1}+iC_V)E_{\mathbb{C}}.$$

This leads to an unbounded symmetric operator

$$F := \sqrt{\frac{\mathbf{1}-iC_V}{\mathbf{1}+iC_V}} : V \to E_{\mathbb{C}}.$$

For $\xi = (\mathbf{1}+iC_V)v,\ v \in E$, we have

$$F\xi = \sqrt{(\mathbf{1}-iC_V)(\mathbf{1}+iC_V)}v = \sqrt{\mathbf{1}+C_V^2}\,v,$$

so $\|F\xi\|^2 = \langle(\mathbf{1}+C_V^2)v, v\rangle = \langle\sigma\xi, \xi\rangle$. Therefore $F : V \to \widehat{V} := \overline{F(V)} \subseteq E_{\mathbb{C}}$ is the canonical map of the reflection positive real Hilbert space $(E_{\mathbb{C}}, V, \sigma)$. It satisfies

$$\sigma F\sigma = \sqrt{\frac{\mathbf{1}+iC_V}{\mathbf{1}-iC_V}} = F^{-1}. \qquad \square$$

Remark B.7. Since $U_t = \Delta^{-it}$ acts on the reflection positive Hilbert space $(E_{\mathbb{C}}, V, \sigma)$ by automorphisms, it induces on the corresponding real Hilbert space $\widehat{V}$ an orthogonal representation. The natural map $\sqrt{\mathbf{1}+C_V^2} : E \to \widehat{V}$ in Proposition B.6 intertwines the orthogonal representations $U_t|_E$ and $U_t|_{\widehat{V}}$.

The following proposition asserts that all standard real subspaces are of the form described in Proposition B.3.

Proposition B.8. *Let $V \subseteq \mathcal{H}$ be a standard real subspace with modular objects (Δ, J). Then $E := \mathrm{Fix}(J)$ is a real Hilbert space with $\mathcal{H} \cong E_{\mathbb{C}}$ and there exists a skew-symmetric strict contraction $C_V : E \to E$ with $V = (\mathbf{1}+iC_V)E$. Then $\mathcal{D}(\Delta) \cap V$ is dense in V.*

Proof. First we observe that V is J-positive:

$$\langle J\xi, \xi\rangle = \langle JS\xi, \xi\rangle = \langle \Delta^{1/2}\xi, \xi\rangle \geq 0.$$

This implies the existence of a contraction $C_V : \mathcal{D}(C_V) \to E$ with

$$V = \Gamma(C_V) := (\mathbf{1} + iC_V)\mathcal{D}(C_V)$$

(Section 3B). That C_V is strict follows from Lemma B.2. From the real orthogonal decomposition $\mathcal{H} = V \oplus iJ(V)$ [Neeb and Ólafsson 2015b, Lemma 4.2(iv)], we now obtain

$$V^{\perp} = iJ(V) = i(\mathbf{1} - iC_V)\mathcal{D}(C_V) = i\Gamma(-C_V) = (C_V + i\mathbf{1})\mathcal{D}(C_V),$$

where $\perp$ refers to the real-valued scalar product $\mathrm{Re}\langle\cdot,\cdot\rangle$ on $\mathcal{H} \cong E \oplus iE$.

If $a \in E \cap \mathcal{D}(C_V)^{\perp}$, then $a \in V^{\perp} = iJ(V) = i\Gamma(-C_V)$ leads to $a = C_V 0 = 0$. Therefore $\mathcal{D}(C_V)$ is dense in E. As V is closed and $\mathbf{1} + iC_V : \mathcal{D}(C_V) \to V$ is a topological isomorphism, it follows that $\mathcal{D}(C_V)$ is closed, and thus $\mathcal{D}(C_V) = E$.

As $\gamma_J(\xi, \eta) := \langle J\xi, \eta\rangle$ is real-valued on V (recall $JV = (iV)^{\perp}$), we obtain for $v, w \in V$ the relation

$$\begin{aligned} 0 &= \mathrm{Im}\langle J(\mathbf{1} + iC_V)v, (\mathbf{1} + iC_V)w\rangle = \mathrm{Im}\langle(\mathbf{1} - iC_V)v, (\mathbf{1} + iC_V)w\rangle \\ &= \mathrm{Im}\langle(\mathbf{1} - iC_V^{\top})(\mathbf{1} - iC_V)v, w\rangle = -\langle(C_V^{\top} + C_V)v, w\rangle, \end{aligned}$$

so that $C_V^{\top} = -C_V$ (Lemma B.5).

It remains to show that $\mathcal{D}(\Delta) \cap V$ is dense in V. Since C_V is a strict contraction, the kernel of $\mathbf{1} + C_V^2$ is trivial, i.e., -1 is not an eigenvalue of C_V^2. Let $E_n \subseteq E$ be the spectral subspace of C_V^2 for the subset $[-1 + 1/n, 1]$. This subspace is C_V-invariant and the union of these subspace is dense in E because -1 is not an eigenvalue. As $(\mathbf{1} + iC_V)E_n \subseteq \mathcal{D}(\Delta)$, it follows that $\mathcal{D}(\Delta) \cap V$ is dense in V. □

Contractions and modular objects. The following lemma describes the complex-valued scalar product on a standard real subspace in terms of the corresponding modular objects (Δ, J).

Lemma B.9. *Let $V \subseteq \mathcal{H}$ be a standard real subspace, (Δ, J) be the corresponding modular objects and*

$$\langle v, w\rangle_{\mathcal{H}} = \gamma(v, w) + i\omega(v, w)$$

be the corresponding hermitian positive definite form on V; in particular $\langle v, w\rangle_V = \gamma(v, w)$. Then

$$\begin{aligned} \gamma(v, w) &= \frac{1}{2}\big(\langle v, w\rangle + \langle\Delta^{1/2}v, \Delta^{1/2}w\rangle\big), \\ \omega(v, w) &= \frac{1}{2i}\big(\langle v, w\rangle - \langle\Delta^{1/2}v, \Delta^{1/2}w\rangle\big). \end{aligned} \tag{37}$$

In particular, we have a strict contraction C on V satisfying

$$\omega(v, w) = \gamma(v, Cw) \quad \textit{and} \quad C = \widehat{C}|_V, \tag{38}$$

where

$$\widehat{C} = i\frac{\Delta - \mathbf{1}}{\Delta + \mathbf{1}} = i\frac{\Delta^{1/2} - \Delta^{-1/2}}{\Delta^{1/2} + \Delta^{-1/2}} = i\tanh\Big(\frac{\log\Delta}{2}\Big).$$

Moreover,

$$\langle v, w\rangle_{\mathcal{H}} = \langle v, (\mathbf{1} + iC)w\rangle_{V_{\mathbb{C}}} \quad \textit{for } v, w \in V_{\mathbb{C}}, \tag{39}$$

so that the map

$$\Phi := \sqrt{\mathbf{1} + iC} : V_{\mathbb{C}} \to V_{\mathbb{C}}$$

extends to a unitary isomorphism $\mathcal{H} \hookrightarrow V_{\mathbb{C}}$.

Proof. As $V \subseteq \mathcal{D}(\Delta^{1/2})$ and $v = Sv = J\Delta^{1/2}v$ or $v \in V$ (Remark 2.5), we obtain

$$\langle \Delta^{1/2}v, \Delta^{1/2}w\rangle = \langle Jv, Jw\rangle = \langle w, v\rangle = \overline{\langle v, w\rangle} \quad \text{for } v, w \in V.$$

This implies (37). Next we note that

$$B := \frac{\Delta - \mathbf{1}}{\Delta + \mathbf{1}}$$

is a bounded operator on $\mathcal{H}$ which can also be written as

$$B = \frac{\Delta^{1/2} - \Delta^{-1/2}}{\Delta^{1/2} + \Delta^{-1/2}}.$$

In this form we see that $JBJ = -B$. We also note that B commutes with Δ, and hence preserves $\mathcal{D}(\Delta^{1/2})$. This leads to

$$SB = J\Delta^{1/2}B = -BS,$$

and therefore to $BV = B\,\mathrm{Fix}(S) \subseteq i\,\mathrm{Fix}(S) = iV$. In particular, $\widehat{C} := iB$ restricts to a bounded skew-symmetric operator $C : V \to V$. If v, w are contained in the dense subspace $V \cap \mathcal{D}(\Delta)$ of V (Proposition B.8), we obtain

$$\begin{aligned}
\gamma(v, Cw) &= \tfrac{1}{2}\big(\langle v, Cw\rangle + \langle \Delta^{1/2}v, \Delta^{1/2}Cw\rangle\big) \\
&= \tfrac{1}{2}\langle (\mathbf{1} + \Delta)v, Cw\rangle \\
&= \tfrac{1}{2}\langle v, (\mathbf{1} + \Delta)\widehat{C}w\rangle \\
&= \frac{1}{2i}\langle v, (\mathbf{1} - \Delta)w\rangle = \omega(v, w).
\end{aligned}$$

Since ω and $\gamma(\,\cdot\,, C\cdot\,)$ are continuous on V, they coincide on all of V. By Lemma B.2, the operator C is a strict contraction. By (38), we have for $v, w \in V$ the relation (39), and since both sides are sesquilinear, it also holds for $v, w \in V_{\mathbb{C}}$. This implies the existence of an isometric extension $\Phi : \mathcal{H} \to V_{\mathbb{C}}$ of the operator $\sqrt{\mathbf{1}+iC}$ on $V_{\mathbb{C}}$. To see that Φ is unitary, we observe that

$$\operatorname{im}(\Phi)^{\perp} = ((\mathbf{1}+iC)^{1/2} V_{\mathbb{C}})^{\perp} = \ker(\mathbf{1}+iC)^{1/2} = \ker(\mathbf{1}+iC),$$

and this space is trivial by Lemma B.2. □

References

[Araki 1963] H. Araki, "A lattice of von Neumann algebras associated with the quantum theory of a free Bose field", *J. Math. Phys.* **4** (1963), 1343–1362. MR Zbl

[Barata et al. 2013] J. C. A. Barata, C. D. Jäkel, and J. Mund, "The $P(\varphi)_2$ model on the de Sitter space", preprint, 2013. arXiv

[Barata et al. 2016] J. C. A. Barata, C. D. Jäkel, and J. Mund, "Interacting quantum fields on de Sitter space", 2016. To appear in *Mem. Amer. Math. Soc.* arXiv

[Borchers 1992] H.-J. Borchers, "The CPT-theorem in two-dimensional theories of local observables", *Comm. Math. Phys.* **143**:2 (1992), 315–332. MR Zbl

[Bratteli and Robinson 1979] O. Bratteli and D. W. Robinson, *Operator algebras and quantum statistical mechanics, I*, Springer, 1979. MR Zbl

[Bratteli and Robinson 1981] O. Bratteli and D. W. Robinson, *Operator algebras and quantum statistical mechanics, II*, Springer, 1981. MR Zbl

[Bros and Buchholz 1994] J. Bros and D. Buchholz, "Towards a relativistic KMS-condition", *Nuclear Phys. B* **429**:2 (1994), 291–318. MR Zbl

[Buchholz et al. 2011] D. Buchholz, G. Lechner, and S. J. Summers, "Warped convolutions, Rieffel deformations and the construction of quantum field theories", *Comm. Math. Phys.* **304**:1 (2011), 95–123. MR Zbl

[Cuniberti et al. 2001] G. Cuniberti, E. De Micheli, and G. A. Viano, "Reconstructing the thermal Green functions at real times from those at imaginary times", *Comm. Math. Phys.* **216**:1 (2001), 59–83. MR Zbl

[De Angelis et al. 1986] G. F. De Angelis, D. de Falco, and G. Di Genova, "Random fields on Riemannian manifolds: a constructive approach", *Comm. Math. Phys.* **103**:2 (1986), 297–303. MR Zbl

[De Micheli and Viano 2012] E. De Micheli and G. A. Viano, "Holomorphic extensions associated with series expansions", *Forum Math.* **24**:6 (2012), 1269–1316. MR Zbl

[Dereziński and Gérard 2013] J. Dereziński and C. Gérard, *Mathematics of quantization and quantum fields*, Cambridge Univ. Press, 2013. MR Zbl

[Dimock 2004] J. Dimock, "Markov quantum fields on a manifold", *Rev. Math. Phys.* **16**:2 (2004), 243–255. MR Zbl

[Eckmann and Osterwalder 1973] J.-P. Eckmann and K. Osterwalder, "An application of Tomita's theory of modular Hilbert algebras: duality for free Bose fields", *J. Funct. Anal.* **13**:1 (1973), 1–12. MR Zbl

[Fulling and Ruijsenaars 1987] S. A. Fulling and S. N. M. Ruijsenaars, "Temperature, periodicity and horizons", *Phys. Rep.* **152**:3 (1987), 135–176. MR

[Gérard and Jäkel 2007] C. Gérard and C. D. Jäkel, "On the relativistic KMS condition for the $P(\varphi)_2$ model", pp. 125–140 in *Rigorous quantum field theory*, edited by A. Boutet de Monvel et al., Progr. Math. **251**, Birkhäuser, Basel, 2007. MR Zbl

[Glimm and Jaffe 1981] J. Glimm and A. Jaffe, *Quantum physics: a functional integral point of view*, Springer, 1981. MR Zbl

[Haag et al. 1967] R. Haag, N. M. Hugenholtz, and M. Winnink, "On the equilibrium states in quantum statistical mechanics", *Comm. Math. Phys.* **5** (1967), 215–236. MR Zbl

[Jaffe and Ritter 2007] A. Jaffe and G. Ritter, "Quantum field theory on curved backgrounds, I: The Euclidean functional integral", *Comm. Math. Phys.* **270**:2 (2007), 545–572. MR Zbl

[Jorgensen 2002] P. E. T. Jorgensen, "Diagonalizing operators with reflection symmetry", *J. Funct. Anal.* **190**:1 (2002), 93–132. MR Zbl

[Jorgensen and Ólafsson 1998] P. E. T. Jorgensen and G. Ólafsson, "Unitary representations of Lie groups with reflection symmetry", *J. Funct. Anal.* **158**:1 (1998), 26–88. MR Zbl

[Jorgensen and Ólafsson 2000] P. E. T. Jorgensen and G. Ólafsson, "Unitary representations and Osterwalder–Schrader duality", pp. 333–401 in *The mathematical legacy of Harish-Chandra* (Baltimore, 1998), edited by R. S. Doran and V. S. Varadarajan, Proc. Sympos. Pure Math. **68**, Amer. Math. Soc., Providence, RI, 2000. MR Zbl

[Klein and Landau 1981] A. Klein and L. J. Landau, "Periodic Gaussian Osterwalder–Schrader positive processes and the two-sided Markov property on the circle", *Pacific J. Math.* **94**:2 (1981), 341–367. MR Zbl

[Klein and Landau 1983] A. Klein and L. J. Landau, "From the Euclidean group to the Poincaré group via Osterwalder–Schrader positivity", *Comm. Math. Phys.* **87**:4 (1983), 469–484. MR Zbl

[Lance 1995] E. C. Lance, *Hilbert C^*-modules*, London Math. Soc. Lecture Note Series **210**, Cambridge Univ. Press, 1995. MR Zbl

[Longo 2008] R. Longo, "Real Hilbert subspaces, modular theory, $\mathrm{SL}(2, \mathbb{R})$ and CFT", pp. 33–91 in *Von Neumann algebras in Sibiu* (Sibiu, Romania, 2007), edited by K. Dykema and F. Rădulescu, Theta Ser. Adv. Math. **10**, Theta, Bucharest, 2008. MR Zbl

[Neeb 2000] K.-H. Neeb, *Holomorphy and convexity in Lie theory*, de Gruyter Expositions in Math. **28**, de Gruyter, Berlin, 2000. MR Zbl

[Neeb and Ólafsson 2014] K.-H. Neeb and G. Ólafsson, "Reflection positivity and conformal symmetry", *J. Funct. Anal.* **266**:4 (2014), 2174–2224. MR Zbl

[Neeb and Ólafsson 2015a] K.-H. Neeb and G. Ólafsson, "Reflection positive one-parameter groups and dilations", *Complex Anal. Oper. Theory* **9**:3 (2015), 653–721. MR Zbl

[Neeb and Ólafsson 2015b] K.-H. Neeb and G. Ólafsson, "Reflection positivity for the circle group", *J. Phys. Conf. Ser.* **597**:1 (2015), art. id. 012004.

[Neeb and Ólafsson 2019] K.-H. Neeb and G. Ólafsson, "Reflection positivity on spheres", (2019).

[Osterwalder and Schrader 1973] K. Osterwalder and R. Schrader, "Axioms for Euclidean Green's functions", *Comm. Math. Phys.* **31**:2 (1973), 83–112. MR Zbl

[Ramacher 2000] P. Ramacher, "Modular localization of elementary systems in the theory of Wigner", *J. Math. Phys.* **41**:9 (2000), 6079–6089. MR Zbl

[Rieffel and van Daele 1977] M. A. Rieffel and A. van Daele, "A bounded operator approach to Tomita–Takesaki theory", *Pacific J. Math.* **69**:1 (1977), 187–221. MR Zbl

[Schlingemann 1999] D. Schlingemann, "From Euclidean field theory to quantum field theory", *Rev. Math. Phys.* **11**:9 (1999), 1151–1178. MR Zbl

[Yngvason 1994] J. Yngvason, "A note on essential duality", *Lett. Math. Phys.* **31**:2 (1994), 127–141. MR Zbl

Received October 31, 2016.

Karl-Hermann Neeb
Department Mathematik
Friedrich-Alexander Universität Erlangen-Nürnberg
Erlangen
Germany
neeb@math.fau.de

Gestur Ólafsson
Department of Mathematics
Louisiana State University
Baton Rouge, LA
United States
olafsson@math.lsu.edu

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2019 is US $490/year for the electronic version, and $665/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 299 No. 1 March 2019