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IMPROVED BUCKLEY'S THEOREM ON LOCALLY COMPACT ABELIAN GROUPS

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We present sharp quantitative weighted norm inequalities for the Hardy–Littlewood maximal function in the context of locally compact abelian groups, obtaining an improved version of the so-called Buckley's theorem. On the way, we prove a precise reverse Hölder inequality for Muckenhoupt A_∞ weights and provide a valid version of the "open property" for Muckenhoupt A_p weights.

1. Introduction and main results

The study of weighted norm inequalities for maximal type operators is one of the central topics in harmonic analysis that began with the celebrated theorem of Muckenhoupt [1972]. It states that the class of weights (nonnegative locally integrable functions) characterizing the boundedness of the Hardy–Littlewood maximal function M on the weighted space $L^p(\mathbb{R}^n, w dx)$ is the so-called Muckenhoupt A_p class (see below for the precise definitions). It is important to remark that Muckenhoupt's result is qualitative, that is, it does not provide any precise information on how the operator norm of M depends on the underlying weight in $w \in A_p$. The first quantitative result on the boundedness for the maximal function in $\mathbb{R}^n$ dates back to the 90s, is due to Buckley [1993], and gives the best possible power dependence on the A_p constant $[w]_{A_p}$. More precisely, Buckley proved

$$\|M\|_{L^p(\mathbb{R}^n, w dx) \to L^p(\mathbb{R}^n, w dx)} \le C[w]_{A_p}^{1/(p-1)}, \qquad 1 < p < \infty. \tag{1-1}$$

Recently a simpler and elegant proof was presented by Lerner [2008], who used a very clever argument composing weighted versions of the maximal function. Since then, finer improvements have been found. In particular, there is in [Hytönen et al. 2012] a sharp mixed bound valid in the context of spaces of homogeneous type.

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Our purpose here is to obtain sharp quantitative norm estimates in the context of locally compact abelian groups (LCA groups). The modern approach to this problem is to use a sharp version of the reverse Hölder inequality (RHI) with a precise quantitative expression for the exponent to derive a proper open property for the A_p classes. Then an interpolation type argument allows us to prove the desired bound.

In the rest of the introduction we first describe in detail the context where we will work in and then properly state the results that we will prove.

1A. ***Muckenhoupt weights and maximal functions on LCA groups.*** In the euclidean setting, the standard way to introduce A_p weights is by considering averages over cubes, balls or more general families of convex sets. In any case, the family is built using some specific metric. In our context of LCA groups we lack such a concept. However there are many LCA groups where we do have the possibility of considering a family of *base sets* satisfying the other fundamental property of the basis of cubes or balls: any point has a family of decreasing base sets shrinking to it and, in addition, the whole space can be covered by the increasing union of such a family.

In order to properly define the A_p classes, let us fix an LCA group G with a measure μ that is inner regular and such that $\mu(K) < \infty$ for every compact set $K \subset G$. Notice that μ does need to be the Haar measure because we do not assume μ to be translation invariant. The reader can find a comprehensive treatment of harmonic analysis on LCA groups in [Hewitt and Ross 1970; 1963; Rudin 1962]. The general assumption on the group will be that it admits a sequence of neighborhoods of 0 with certain properties that we described in the next definition (cf. [Edwards and Gaudry 1977, Section 2.1]).

Definition 1.1. A collection $\{U_i\}_{i\in\mathbb{Z}}$ is a covering family for G if

(1) $\{U_i\}_{i\in\mathbb{Z}}$ is an increasing base of relatively compact neighborhoods of 0, $\bigcup_{i\in\mathbb{Z}} U_i = G$ and $\bigcap_i U_i = \{0\}$.

(2) There exists a positive constant $D \geq 1$ and an increasing function $\theta : \mathbb{Z} \to \mathbb{Z}$ such that for any $i \in \mathbb{Z}$ and any $x \in G$,
   - $i \leq \theta(i)$,
   - $U_i - U_i \subset U_{\theta(i)}$,
   - $\mu(x + U_{\theta(i)}) \leq D\mu(x + U_i)$.

We will refer to the third condition as the *doubling* property of the measure μ with respect to θ and we will call D the doubling constant. In the case of $\mathbb{R}^n$ equipped with the natural metric and measure, we can consider the family of dyadic cubes of sidelength 2^i or the euclidean balls $B(x, 2^i)$ for $i \in \mathbb{Z}$. The doubling constant of the Lebesgue measure in this context is 2^n and the function θ can be taken to be $\theta(i) = i + 1$. Therefore, the intuition here is that the index i in the above definition can be seen as a sort of *radius* or *size* of the given set U_i.

For each $x \in G$, the set $x + U_i$ will be called a *base set* and the collection of all base sets will be denoted by

$$\mathcal{B} := \{x + U_i : x \in G,\ i \in \mathbb{Z}\}. \tag{1-2}$$

The notion of base sets allows us to define a direct analogue of the Hardy–Littlewood maximal function:

$$Mf(x) = \sup_{x \in U \in \mathcal{B}} \fint_U |f|\,d\mu := \sup_{x \in U \in \mathcal{B}} \frac{1}{\mu(U)} \int_U |f|\,d\mu, \tag{1-3}$$

where the supremum is taken over the sets $U \in \mathcal{B}$ with positive measure.

As we already mentioned, our purpose here is to prove sharp weighted norm inequalities for this operator in $L^p(G, w\,d\mu)$, where w is a weight on G. Firstly, recall that the celebrated Muckenhoupt's theorem asserts that the class of weights characterizing the boundedness of M on $L^p(\mathbb{R}^n, w\,dx)$, $p > 1$, is the Muckenhoupt A_p class defined in $\mathbb{R}^n$ by

$$[w]_{A_p(\mathbb{R}^n, dx)} := \sup_Q \biggl(\fint_Q w\,d\mu\biggr)\biggl(\fint_Q w^{1-p'}\,d\mu\biggr)^{p-1} < \infty. \tag{1-4}$$

Here p' denotes the conjugate exponent of p defined by the condition $\frac{1}{p} + \frac{1}{p'} = 1$. In the case of LCA groups the analogue of (1-4) is obtained by replacing the cubes by base sets. More precisely, a weight w is an $A_p = A_p(G, d\mu)$ weight if

$$[w]_{A_p} := \sup_{U \in \mathcal{B}} \biggl(\fint_U w\,d\mu\biggr)\biggl(\fint_U w^{1-p'}\,d\mu\biggr)^{p-1} < \infty. \tag{1-5}$$

The limiting case of (1-5), when $p = 1$, defines the class A_1; that is, the set of weights w such that

$$[w]_{A_1} := \sup_{U \in \mathcal{B}} \biggl(\fint_U w\,d\mu\biggr) \operatorname*{ess\,sup}_U (w^{-1}) < +\infty,$$

which is equivalent to w having the property

$$Mw(x) \le [w]_{A_1} w(x) \qquad \mu\text{-almost everywhere } x \in G.$$

As in the usual setting of $\mathbb{R}^n$ we will also often refer to $\sigma := w^{1-p'}$ as the *dual weight* for w. It is easy to verify that $w \in A_p$ if and only if $\sigma \in A_{p'}$.

The family of A_p classes is increasing and this motivates the definition of the larger class A_∞ as the union $A_\infty = \bigcup_{p \ge 1} A_p$. There are many characterizations of the class A_∞ (see [Duoandikoetxea et al. 2016] or the more classical reference [Grafakos 2004]). Some of them are given in terms of the finiteness of some A_∞

constant suitably defined. The classical definition consists in taking the limit on the A_p constant as p goes to infinity, namely:

$$(w)_{A_\infty} := \sup_{U\in\mathcal{B}}\Big(\fint w\,d\mu\Big)\exp\fint_U \log(w^{-1})\,d\mu. \tag{1-6}$$

However, the modern tendency is to consider the so-called Fujii–Wilson constant implicitly introduced by Fujii [1977/78], and later rediscovered by Wilson [1987; 2008], and here we choose to follow this approach by defining the A_∞ constant as

$$[w]_{A_\infty} := \sup_{U\in\mathcal{B}} \frac{1}{w(U)}\int_U M(w\chi_U)\,d\mu, \tag{1-7}$$

where $w(U)=\int_U w\,d\mu$.

1B. *Our contribution.* As we have already seen, there is a proper — and natural — way to define the A_p and A_∞ classes on LCA groups having covering families. In contrast with the case $p<\infty$, it is not immediate that the weight w belongs to A_∞ when any of constants defined on (1-6) and (1-7) is finite. In fact, a weight w is in A_∞ (that is, in some A_p) if and only if it satisfies the reverse Hölder inequality, which says

$$\Big(\fint_U w^r\,d\mu\Big)^{1/r} \le C\fint_{\widehat{U}} w\,d\mu$$

for some $r>1$ and where $\widehat{U}$ is an open set defined in terms of U (in the euclidean case $\widehat{U}=U$ and in the case of spaces of homogeneous type, it is a dilation of U). This is a very well known result in the qualitative case. Concerning the quantitative aspect, a sharp result in terms of $[w]_{A_\infty}$ in the context of spaces of homogeneous type was proved recently in [Hytönen et al. 2012].

Our first result is the following version of the RHI. Note that, as in [Hytönen et al. 2012], we are able to precisely describe the exponent r in terms of the constant $[w]_{A_\infty}$.

Theorem 1.2 (sharp weak reverse Hölder inequality). *Let $w\in A_\infty$. Define the exponent $r(w)$ as*

$$r(w)=1+\frac{1}{4D^{10}[w]_{A_\infty}-1},$$

where D is the doubling constant. Then, for a fixed $U=x_0+U_{i_0}\in\mathcal{B}$, the following inequality holds:

$$\Big(\fint_U w^{r(w)}\,d\mu\Big)^{1/r(w)} \le 2D^2\fint_{\widehat{U}} w\,d\mu, \tag{1-8}$$

where $\widehat{U}$ is the union of the base sets $\{x+U_i : x\in U,\ i\le i_0\}$.

Once we have proven such RHI, we are able to provide a quantitative open property for A_p classes. It is very well known that the A_p classes are open in the sense that if $w \in A_p$ for some $p > 1$, then w also belongs to some $A_{p-\varepsilon}$ for some $\varepsilon > 0$. But the best possible ε in this property is not completely characterized. Another related interesting and still open question (even in the euclidean setting) is to determine, given a weight $w \in A_\infty$, the smallest $p > 1$ such that $w \in A_p$. There are some estimates in [Hagelstein and Parissis 2016] but there is no proof of their sharpness.

Here we will deduce from Theorem 1.2 an open property for A_p classes in LCA groups with some control on the constants. More precisely, given $w \in A_p$ for $1 < p < \infty$ we will obtain that $w \in A_{p-\varepsilon}$ for $\varepsilon = (p-1)/(C[\sigma]_{A_\infty})$ with $C = 4D^{10}$. Further, $[w]_{A_{p-\varepsilon}} \le 2^{p-1} D^{4p-2}[w]_{A_p}$ (see Lemma 3.1).

In a recent article, Sauer [2015] proved a weighted bound for the maximal function for LCA groups following Lerner's approach. Additionally, he asked whether it is possible to obtain the sharp result from Buckley in this general setting. In our main theorem we answer this question in the affirmative and moreover, we provide a better mixed bound. By a mixed bound we understand a bound that depends on $[w]_{A_p}$ and $[w]_{A_\infty}$ of the form $\varphi([w]_{A_p}[w]_{A_\infty})$ where φ is some nonnegative function, typically a power function. Since $[w]_{A_\infty} \le [w]_{A_p}$ always, usually mixed type bounds are sharper than estimates involving only the A_p constant.

A result in this direction was obtained in [Hytönen et al. 2012] where the authors proved an improvement of Buckley's result (1-1) in terms of mixed bounds for spaces of homogeneous type, namely

$$\|M\|_{L^p_w \to L^p_w} \le C([w]_{A_p}[\sigma]_{A_\infty})^{1/p} \le C[w]_{A_p}^{1/(p-1)}.$$

Our main result provides an extension of the above estimate to the context of LCA groups and we will obtain it as a consequence of the RHI and the open property. We remark here that the lack of geometry in this setting constitutes a major obstacle to overcome.

Theorem 1.3. *Let M be the Hardy–Littlewood maximal function defined in (1-3) and let $1 < p < \infty$. Then there is a structural constant $C > 0$ such that*

$$\|Mf\|_{L^p_w(G)} \le C([w]_{A_p}[\sigma]_{A_\infty})^{1/p}\|f\|_{L^p_w(G)}. \tag{1-9}$$

In particular,

$$\|M\|_{L^p(w)} \le C[w]_{A_p}^{1/(p-1)}. \tag{1-10}$$

1C. ***Outline.*** The paper is organized as follows. In Section 2 we give some preliminary results. We prove the engulfing property in this context that will be used several times throughout the paper. We also define the local maximal function, prove a crucial covering lemma (Lemma 2.7) and show a localization property of the local maximal function. In Section 3 we give the proofs of the results described in Section 1B.

2. Preliminaries

In this section we provide some properties of covering families that we will use. Furthermore, we will introduce a local maximal function which will be crucial to proving the RHI.

As we already mentioned in the introduction, the family of dyadic cubes of sidelength 2^i or the euclidean balls $B(x, 2^i)$ for $i \in \mathbb{Z}$ are covering families for $G = \mathbb{R}$. Other examples are presented below.

Example 2.1. (1) When $G = \mathbb{T} = \left\{e^{2\pi i t} : t \in \left[-\frac{1}{2}, \frac{1}{2}\right)\right\}$ with the Haar measure, consider $U_k \subseteq G$ defined as $U_0 = \mathbb{T}$ and for $k \in \mathbb{N}$, $U_k = \{0\}$ and

$$U_{-k} = \left\{e^{2\pi i t} : |t| < \frac{1}{2^{k+1}}\right\}.$$

Then, $\{U_k\}_{k\in\mathbb{Z}}$ is a covering family for $\mathbb{T}$ with $\theta(k) = k + 1$ and $D = 2$.

(2) For $G = \mathbb{Z}$, take $U_i = \{k \in \mathbb{Z} : |k| \le 2^{i-1}\}$ for $i \ge 1$ and $U_i = \{0\}$ otherwise. Then $\{U_i\}_{i\in\mathbb{Z}}$ is a covering family for $\mathbb{Z}$ with $\theta(i) = i + 1$ and $D = 2$.

(3) Let G be an LCA group with Haar measure μ and let H be a compact and open subgroup of G with $\mu(H) = 1$. Consider an expansive automorphism $A : G \to G$ with respect to H, which means that $H \subsetneq AH$ and $\bigcap_{i<0} A^i H = \{0\}$. If, additionally, $G = \bigcup_{i\in\mathbb{Z}} A^i H$, then $\{A^i H\}_{i\in\mathbb{Z}}$ is a covering family for G. Indeed, since $H \subsetneq AH$ and H is a group, $A^i H - A^i H = A^i H \subseteq A^{i+1} H$ so $\theta(i) = i + 1$. To see that the doubling property is satisfied, note that μ_A defined as $\mu_A(B) := \mu(AB)$ for $B \subseteq G$ a Borel set, is a Haar measure on G. Thus, there is a positive number α such that $\mu_A = \alpha\mu$. The constant α is the so-called *modulus of* A and is denoted by $\alpha = |A|$. Then, $\mu(A^{i+1} H) = \mu_A(A^i H) = |A|\mu(A^i H)$ for $i \in \mathbb{Z}$. Observe that G/H is discrete and AH/H is finite, so AH is the union of finitely many cosets of the quotient G/H, say $\{H + s_j\}_{j=1}^r$. Therefore, $|A| = |A|\mu(H) = \mu(AH) = r$, and $r \ge 2$ since $H \subsetneq AH$. Thus we can take $D = |A| \ge 2$. A structure of this type is considered in [Benedetto and Benedetto 2004] for constructing wavelets on LCA groups with open and compact subgroups.

For a concrete example of this situation, consider the p-adic group $G = \mathbb{Q}_p$ where $p \ge 2$ is a prime number consisting of all formal Laurent series in p with coefficients $\{0, 1, \dots, p-1\}$, that is,

$$\mathbb{Q}_p = \left\{\sum_{n\ge n_0} a_n p^n : n_0 \in \mathbb{Z},\ a_n \in \{0, 1, \dots, p-1\}\right\}.$$

As a compact and open subgroup we can consider $H = \mathbb{Z}_p$ which is

$$\mathbb{Z}_p = \left\{\sum_{n\ge 0} a_n p^n : a_n \in \{0, 1, \dots, p-1\}\right\}.$$

Take $A : \mathbb{Q}_p \to \mathbb{Q}_p$ to be the automorphism defined as $A(x) = p^{-1}x$. Then, A is expansive with respect to $\mathbb{Z}_p$ and it can be easily checked that $\mathbb{Q}_p = \bigcup_{i\in\mathbb{Z}} A^i\mathbb{Z}_p$. Then, $\{A^i\mathbb{Z}_p\}_{i\in\mathbb{Z}}$ is a covering family for $\mathbb{Q}_p$ and in this case, $D = |A| = p$.

Let $\{U_i\}_{i\in\mathbb{Z}}$ be a fixed covering family for G. From now on, we assume the sets U_i to be symmetric. This is not a restriction at all because one can always consider the new family of base sets formed by the difference sets $U_i - U_i$ which increases the doubling constant from D to D^2. We denote $2U_i := U_i - U_i = U_i + U_i$.

Any covering family has the so-called *engulfing* property:

Lemma 2.2. *Let U, V be two base sets such that $U = x + U_i$ and $V = y + U_j$ with $i \leq j$ and $x, y \in G$. If $U \cap V \neq \varnothing$, then $x + U_i \subset y + U_{\theta^2(j)}$.*

Proof. There are two points $u_i \in U_i$ and $u_j \in U_j$ such that $x + u_i = y + u_j$. Then $x = y + u_i - u_j \in y + U_j - U_j \subset y + U_{\theta(j)}$ and therefore

$$x + U_i \subset y + U_{\theta(j)} + U_{\theta(j)} \subset y + U_{\theta^2(j)}$$

(recall that we assume that the base sets are symmetric). □

Remark 2.3. For a given $V \in \mathcal{B}$, where $\mathcal{B}$ is the base of G defined as in (1-2), we will denote by $j(V) \in \mathbb{Z}$ the maximum integer such that $V = x + U_{j(V)}$ for some $x \in G$. To see that such a number exists, let us define $N(V) = \{j \in \mathbb{Z} : \exists\, x \in G,\ V = x + U_j\}$ and show that $\sup N(V) < \infty$. If $\sup N(V) = \infty$, we could find a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in G and a sequence of integer indices $\{i_n\}_{n\in\mathbb{N}}$ such that $i_n \to \infty$ as $n \to \infty$ and

$$V = x_n + U_{i_n} \qquad \text{for all } n \in \mathbb{N}.$$

By compactness of $\overline{V}$ we can assume (relabeling) that the sequence converges to some $x \in G$, which we can assume to be the origin. Now we claim that, for any $j \in \mathbb{N}$, there is some $m \in \mathbb{N}$ such that $U_j \subset x_m + U_{i_m}$ and from this fact would follow that $\mu(V) = \infty$, but this implies that $\infty = \mu(V) \leq \mu(\overline{V}) < \infty$ which is a contradiction. To verify the claim, fix U_j and choose n_0 such that $x_n \in U_j$ and $i_n \geq j$ for all $n \geq n_0$. Then,

$$U_j \cap x_n + U_{i_n} \neq \varnothing$$

for all $n \geq n_0$. Furthermore, the above still holds if we replace x_n by any x_m with $m \geq n \geq n_0$ since $x_m \in U_j$ and $x_m \in x_m + U_{i_n}$. Therefore by the engulfing property (see e.g., Lemma 2.2) we obtain

$$U_j \subset x_m + U_{\theta^2(i_n)} \subset x_m + U_{i_m}$$

for any m such that $i_m \geq \theta^2(i_n)$.

In order to introduce the local maximal function, we first define a local base for a fixed base set U.

Definition 2.4. Let $U \in \mathcal{B}$ be a fixed base set and let $k := j(U)$. The local base $\mathcal{B}_U$ is defined as

$$\mathcal{B}_U := \{y + U_j : y \in U,\ j \le k\}. \tag{2-1}$$

We also defined the *enlarged* set $\widehat{U}$ by the formula

$$\widehat{U} := \bigcup_{V \in \mathcal{B}_U} V. \tag{2-2}$$

Lemma 2.5. *Let $U = x + U_k$ be a fixed base set in $\mathcal{B}$ and set $k = j(U)$. We then have the following geometric properties*:

(a) *If $V \in \mathcal{B}_U$ then $V \subset x + U_{\theta(k)}$.*

(b) *For any $z \in U$,*

$$\widehat{U} \subset z + U_{\theta^2(k)},$$

where $\widehat{U}$ is as in (2-2). As a consequence of this last property, we obtain

$$\mu(\widehat{U}) \le \mu(z + U_{\theta^2(k)}) \le D^2 \mu(z + U_k)$$

for any $z \in U$. In particular, $\mu(\widehat{U}) \le D^2 \mu(U)$, since $U = x + U_k$.

Proof. (a) Let $V = y + U_j$ with $j \le k$ and take any $z \in V$. Then $z = y + u_j$ with $u_j \in U_j \subset U_k$. Since $y \in U$ we can write $y = x + u_k,\ u_k \in U_k$. Then we have

$$z = x + u_j + u_k \in x + U_k + U_k \subset x + U_{\theta(k)}.$$

(b) Let $V \in \mathcal{B}_U,\ V = y + U_j$ with $y \in U,\ j \le k$. By Lemma 2.2, since $V \cap U \ne \varnothing$, $V \subset x + U_{\theta(k)}$. Take any $z \in U,\ z = x + u_k,\ u_k \in U_k$. Then,

$$V \subset x + U_{\theta(k)} = z - u_k + U_{\theta(k)} \subset z - U_k + U_{\theta(k)} \subset z - U_{\theta(k)} + U_{\theta(k)} \subset z + U_{\theta^2(k)}. \qquad \square$$

We now define the local maximal function as

$$M_U f(y) := \sup_{y \in V \in \mathcal{B}_U} \fint_V |f(z)|\, d\mu(z) \tag{2-3}$$

for any $y \in \widehat{U}$ and $M_U f(y) = 0$ otherwise.

Remark 2.6. (a) In [Hewitt and Ross 1970, Theorem 44.18], a version of the Lebesgue differentiation theorem is proven with respect to the Haar measure for LCA groups having D'-sequences (cf. [Hewitt and Ross 1970, Definition 44.10]). A careful reading of the proof of [Hewitt and Ross 1970, Theorem 44.18] reveals that the result is still true with the obvious changes for measures which are not translation invariant. Thus, since a covering family is in particular a D'-sequence, we have that the Lebesgue differentiation theorem holds in our context.

(*b*) As a consequence of the Lebesgue differentiation theorem, we have the elementary but important property of the local maximal function:

$$f(x) \le M_U f(x) \qquad \mu\text{-almost everywhere } x \in U.$$

Consider now, for a fixed $U \in \mathcal{B}$, the level set for the local maximal function acting on a weight w at scale $\lambda > 0$:

$$\Omega_\lambda := \{x \in \widehat{U} : M_U w(x) > \lambda\}. \tag{2-4}$$

A key instrument will be a Calderón–Zygmund (C–Z) decomposition of Ω_λ. We will obtain it by using an adapted version of a covering lemma from [Edwards and Gaudry 1977, Lemma 2.2.1]. Although the proof follows standard arguments, we include it here for completeness. When w is a nonnegative and locally integrable function on G and $V \subseteq G$ is relatively compact, we denote the average of w on V as w_V; that is, $w_V = ⨍_V w\, d\mu$.

Lemma 2.7. *Let $U \in \mathcal{B}$ be a fixed base set in G and let w be a nonnegative and integrable function supported on $\widehat{U}$. For $\lambda > w_{\widehat{U}}$, define Ω_λ as in (2-4). If Ω_λ is nonempty, there exists a finite or countable index set Q and a family $\{y_i + U_{\alpha_i}\}_{i \in Q}$ of pairwise disjoint base sets from $\mathcal{B}_U$ such that*:

(a) *The sequence $\{\alpha_i\}_{i \in Q}$ is decreasing.*

(b) $\displaystyle\bigcup_{i \in Q} y_i + U_{\alpha_i} \subset \Omega_\lambda \subset \bigcup_{i \in Q} y_i + U_{\theta^2(\alpha_i)}$.

(c) *For any $i \in Q$,*

$$\lambda < ⨍_{y_i + U_{\alpha_i}} w\, d\mu.$$

(d) *Given $r > \alpha_i$ for some $i \in Q$,*

$$⨍_{y_i + U_r} w\, d\mu \le D^2 \lambda. \tag{2-5}$$

Proof. Suppose that there is no finite sequence of points in Ω_λ such that the conclusion holds (in that case, there is nothing to prove). For $x \in \Omega_\lambda$, define

$$\alpha(x) = \max\left\{ j \in \mathbb{Z} : \exists\, V = y + U_j \in \mathcal{B}_U,\, x \in V,\, ⨍_V w\, d\mu > \lambda \right\}. \tag{2-6}$$

Since $V = y + U_j \in \mathcal{B}_U$ implies $j \le j(U)$, we have that α is well defined. Consider now, for each $x \in \Omega_\lambda$, a base set $V_x \in \mathcal{B}_U$, $V_x := y_x + U_{\alpha(x)}$ such that $x \in V_x$. In other words, one of the base sets in $\mathcal{B}$ containing the point x where the map α attains its value. Observe that, in particular, $\alpha(y_x) \ge \alpha(x)$.

We start by picking x_1 as an extremal point for α, that is $\alpha(x_1) \ge \alpha(x)$ for all $x \in \Omega_\lambda$. Put $\alpha_1 = \alpha(x_1)$ and $y_1 := y_{x_1}$ such that $V_{x_1} = y_1 + U_{\alpha_1}$. Note that, since $\alpha_1 \le$

$\alpha(y_1) \le \alpha(x_1) = \alpha_1$, we also have $\alpha(y_1) = \alpha_1$. Now suppose that we have chosen the first n points $y_1, \dots, y_n$ and their respective base sets $U_{\alpha_1}, \dots, U_{\alpha_n}$ such that

- the sets $V_j := y_j + U_{\alpha_j}$, $1 \le j \le n$, are pairwise disjoint,
- $\alpha_j := \alpha(y_j) \ge \alpha(x)$ for all $x \in A_{j-1}$, where

$$A_j := \Omega_\lambda \setminus \bigcup_{\ell \le j} y_\ell + U_{\theta^2(\alpha_\ell)}, \qquad 1 \le j \le n. \tag{2-7}$$

Since we are assuming that this procedure never ends, $A_j \ne \varnothing$ for all $1 \le j \le n$. Therefore we can choose $x_{n+1} \in A_n$ such that $\alpha_{n+1} := \alpha(x_{n+1}) \ge \alpha(x)$ for all $x \in A_n$. This means that there is a base set $V_{n+1} := y_{n+1} + U_{\alpha_{n+1}}$ and in particular $w_{V_{n+1}} > \lambda$ and $\alpha(y_{n+1}) = \alpha_{n+1}$. Note that this construction produces a decreasing sequence $\{\alpha_n\}_{n \in \mathbb{N}}$. Let's see that $V_{n+1} \cap V_j = \varnothing$ for all $1 \le j \le n$. Supposing that this is not the case, we could find $u \in U_{\alpha_{n+1}}$ and $v \in U_{\alpha_j}$ for some $j \le n$ such that

$$y_{n+1} + u = y_j + v.$$

Since $x_{n+1} \in V_{n+1}$, we have that for some $z \in U_{\alpha_{n+1}}$,

$$x_{n+1} = y_{n+1} + z = y_j + v - u + z \in y_j + U_{\alpha_j} - U_{\alpha_{n+1}} + U_{\alpha_{n+1}}.$$

Since $U_{\alpha_{n+1}} \subset U_{\alpha_j}$ and trivially $U_{\alpha_j} \subset U_{\theta(\alpha_j)}$, we get

$$x_{n+1} \in y_j + U_{\theta^2(\alpha_j)},$$

which is a contradiction by the choice of x_{n+1}.

We are left to prove that this procedure exhausts the set Ω_λ. If not, there is a point $x \in A_n$ with $\alpha(x) \le \alpha_n$ for all $n \ge 1$. Define the set S as

$$S := \{y_n : n \in \mathbb{N}\}.$$

Since

$$S \subset \{z \in \Omega_\lambda : \alpha(z) \ge \alpha(x)\} \subset \widehat{U}$$

and $\widehat{U}$ is contained in some base set (see item (b) in Lemma 2.5), we conclude that S is relatively compact.

By monotonicity of α, we have $U_{\alpha_n} \subset U_{\alpha_1}$. Therefore the set

$$F := \bigcup_n (y_n + U_{\alpha_n}) \subset S + U_{\alpha_1}$$

is also relatively compact and this implies $\mu(\overline{F}) < \infty$. Now consider $N \in \mathbb{Z}$ such that $\overline{S} \subset U_N$ and an integer $r > 0$ such that $\theta^r(\alpha(x)) \ge N$. Then for any $n \in \mathbb{N}$, $y_n \in S \subset U_N \subset U_{\theta^r(\alpha(x))}$ and thus $0 \in y_n + U_{\theta^r(\alpha(x))}$. Further, we get

$$U_N = 0 + U_N \subset y_n + U_{\theta^r(\alpha(x))} + U_N \subset y_n + 2U_{\theta^r(\alpha(x))} \subset y_n + U_{\theta^{r+1}(\alpha(x))}.$$

The doubling property shows

$$\mu(U_N) \le D^{r+1}\mu(y_n + U_{\alpha(x)})$$

and this implies

$$\mu(F) = \sum_n \mu(y_n + U_{\alpha_n}) \ge \sum_n \mu(y_n + U_{\alpha(x)}) \ge D^{-(r+1)} \sum_n \mu(U_N) = \infty.$$

This contradicts the condition $\mu(\bar{F}) < \infty$ and we conclude with the proof of items (a), (b) and (c) of the lemma.

We prove now item (d). To control the average on $y_i + U_r$, we consider two cases: first we consider $r \le k := j(U)$. Then $y_i + U_r \in \mathcal{B}_U$ and by maximality we have $⨍_{y_i+U_r} w\,d\mu \le \lambda$. Indeed, if not we would have $\alpha_i = \alpha(y_i) \ge r > \alpha_i$. Second, where $r > k$, we have $\theta^2(r) > \theta^2(k)$ and thus, by Lemma 2.5,

$$y_i + U_{\theta^2(r)} \supset y_i + U_{\theta^2(k)} \supset \widehat{U}.$$

Therefore, since $w = 0$ almost everywhere $\widehat{U}^c$,

$$⨍_{y_i+U_r} w\,d\mu \le \frac{\mu(\widehat{U})}{\mu(y_i + U_r)} ⨍_{\widehat{U}} w\,d\mu \le D^2\lambda.$$

The lemma is now completely proven. □

Now we present a localization argument for the local maximal function M_U. The idea is better understood when considering the usual dyadic maximal function M_Q^d localized on a cube Q in $\mathbb{R}^n$. Suppose that the level set $\Omega_\lambda = \{x \in Q : M_Q^d w(x) > \lambda\}$ for $\lambda > w_Q$ is decomposed into dyadic subcubes of Q such that $Q = \bigcup_i Q_i$ and the cubes Q_i are maximal with respect to the condition $w_{Q_i} > \lambda$. Then the conclusion is that for any $x \in Q_i$, the equality $M_Q^d w(x) = M_Q^d(w\chi_{Q_i})(x)$ holds. In this more general setting, the analogous result is contained in Lemma 2.8 which does not have a direct proof as in the dyadic case.

For simplicity in the exposition, we introduce the following notation. Given a base set of the form $V = y + U_j$, we denote by V^* the dilation of V by θ, i.e., $V^* = y + U_{\theta(j)}$. Further iterations of this operation are defined recursively, that is, $V^{**} = (V^*)^*$ and V^{n*} for n iterations of the dilation operation.

Lemma 2.8. *Let $U \in \mathcal{B}$ be a fixed base set and consider $w = w\chi_{\widehat{U}}$ a nonnegative and integrable function on $\widehat{U}$ where $\widehat{U}$ is as in (2-2). For $\lambda > w_{\widehat{U}}$, let Ω_λ be defined as above and let $\{V_i\}_{i\in Q} = \{y_i + U_{\alpha_i}\}_{i\in Q}$ be the C–Z decomposition of Ω_λ given by Lemma 2.7. Then, for $L = D^6$, any $i \in Q$ and any $x \in V_i^{**} \cap \Omega_{L\lambda}$,*

$$M_U w(x) \le M_U(w\chi_{V_i^{4*}})(x). \tag{2-8}$$

Proof. Let $x \in V_i^{**} \cap \Omega_{L\lambda}$. Then there exists $V \in \mathcal{B}_U$, $V = y + U_j$, with $y \in U$ and $j \le j(U)$ such that $x \in V$ and $w_V > L\lambda$. We claim $j \le \theta^2(\alpha_i)$. To see that this is in fact true, suppose towards a contradiction, that $j > \theta^2(\alpha_i)$. Then, $V \subset y_i + U_{\theta^2(j)}$. Indeed, if $z \in V$ then $z = y + w$ with $w \in U_j$. On the other hand, since $x \in V_i^{**} \cap V$, $x = y_i + u = y + v$ with $u \in U_{\theta^2(\alpha_i)}$ and $v \in U_j$. Then

$$z = y + v - v + w = x - v + w = y_i + u - v + w.$$

Since $U_{\theta^2(\alpha_i)} \subset U_j$, we get that $z \in y_i + U_j + U_{\theta(j)} \subset y_i + U_{\theta^2(j)}$. As a consequence,

$$\fint_V w\,d\mu \le \frac{\mu(y_i + U_{\theta^2(j)})}{\mu(V)} \fint_{y_i + U_{\theta^2(j)}} w\,d\mu.$$

We note that since $\theta^2(\alpha_i) < j$, $x \in V \cap V_i^{**} \subset V \cap (y_i + U_j)$ and then, by the engulfing property we have that $y_i + U_j \subset y + U_{\theta^2(j)}$. Thus, using the doubling property of the measure μ we obtain

$$\frac{\mu(y_i + U_{\theta^2(j)})}{\mu(y + U_j)} \le D^2 \frac{\mu(y_i + U_j)}{\mu(y + U_j)} \le D^2 \frac{\mu(y + U_{\theta^2(j)})}{\mu(y + U_j)} \le D^4.$$

Furthermore, since $\theta^2(j) \ge j > \theta^2(\alpha_i) \ge \alpha_i$, by item (4) in Lemma 2.7,

$$\fint_{y_i + U_{\theta^2(j)}} w\,d\mu \le D^2\lambda$$

and we can conclude that

$$L\lambda < \fint_V w\,d\mu \le D^6\lambda = L\lambda,$$

which gives a contradiction. Hence, the claim $j \le \theta^2(\alpha_i)$ holds.

Now, using Lemma 2.2 we have $V \subset V_i^{4*}$ and then

$$\fint_V w\,d\mu = \fint_V w\chi_{V_i^{4*}}\,d\mu \le M(w\chi_{V_i^{4*}})(x),$$

which proves inequality (2-8). □

3. Proof of the main results

We present here the proof of Theorem 1.2.

Proof of Theorem 1.2. **Step 1.** We start with the following estimate for the local maximal function. Let $U = x_0 + U_k$ be a fixed base set. We claim that, for $\varepsilon = 1/(4D^{10}[w]_{A_\infty} - 1)$,

(3-1) $$\fint_{\widehat{U}} (M_U w)^{1+\varepsilon}\,d\mu \le 2[w]_{A_\infty} \left(\fint_{\widehat{U}} w\,d\mu \right)^{1+\varepsilon}.$$

Recall that we may assume that the weight w is supported on $\widehat{U}$. Let Ω_λ be defined as in (2-4). We write the norm using the layer cake formula as

$$\begin{aligned}\int_{\widehat{U}}(M_U w)^{1+\varepsilon}\,d\mu &= \int_0^\infty \varepsilon\lambda^{\varepsilon-1}M_U w(\Omega_\lambda)\,d\lambda\\ &= \int_0^{w_{\widehat{U}}} \varepsilon\lambda^{\varepsilon-1}M_U w(\Omega_\lambda)\,d\lambda + \int_{w_{\widehat{U}}}^\infty \varepsilon\lambda^{\varepsilon-1}M_U w(\Omega_\lambda)\,d\lambda\\ &= I + II.\end{aligned}$$

The first term is easily controlled by using the A_∞ constant of w (see (1-7)):

$$\begin{aligned}I \le M_U w(\widehat{U})w_{\widehat{U}}^{\varepsilon} = w_{\widehat{U}}^{\varepsilon}\int_{\widehat{U}} M_U w\,d\mu &\\ &\le w_{\widehat{U}}^{\varepsilon}\int_{y+U_{\theta^2(k)}} M_U(w\chi_{y+U_{\theta^2(k)}})\,d\mu\\ &\le w_{\widehat{U}}^{\varepsilon}[w]_{A_\infty} w(y+U_{\theta^2(k)})\\ &= w_{\widehat{U}}^{\varepsilon}[w]_{A_\infty} w(\widehat{U}),\end{aligned}$$

where $y \in U$ and we used Lemma 2.5 and the definition of $[w]_{A_\infty}$.

Now, for each $\lambda > w_{\widehat{U}}$ we consider $\{V_i\}_{i\in Q}$ the C–Z decomposition of Ω_λ from Lemma 2.7 to control *II*. We have

$$M_U w(\Omega_\lambda) \le \sum_i M_U w(V_i^{**}).$$

For any $i \in Q$ we write $V_i^{**} = V_1 \cup V_2$ with $V_1 := V_i^{**} \cap \Omega_{L\lambda}$ and $V_2 := V_i^{**} \setminus \Omega_{L\lambda}$ where $L = D^6$. Thus, by Lemma 2.8 and the A_∞ property (1-7) we have

$$\begin{aligned}M_U w(V_i^{**}) &= \int_{V_1} M_U w\,d\mu + \int_{V_2} M_U w\,d\mu\\ &\le \int_{V_1} M_U(w\chi_{V_i^{4*}})(x)\,d\mu + L\lambda\mu(V_2)\\ &\le [w]_{A_\infty} w(V_i^{4*}) + L\lambda\mu(V_i^{4*}) = ([w]_{A_\infty} w_{V_i^{4*}} + L\lambda)\mu(V_i^{4*})\\ &\le ([w]_{A_\infty}\lambda D^2 + L\lambda)D^4\mu(V_i) \le 2[w]_{A_\infty}\lambda D^{10}\mu(V_i),\end{aligned}$$

where in the last inequality we have used (2-5) and the doubling property of μ. This gives

$$\begin{aligned}M_U w(\Omega_\lambda) &\le \sum_i M_U w(V_i^{**}) \le 2[w]_{A_\infty}\lambda D^{10}\sum_i \mu(V_i)\\ &\le 2[w]_{A_\infty}\lambda D^{10}\mu(\Omega_\lambda).\end{aligned}$$

Thus,

$$\begin{aligned} II &\leq 2[w]_{A_\infty} D^{10} \int_0^\infty \varepsilon \lambda^\varepsilon \mu(\Omega_\lambda)\, d\lambda \\ &= 2[w]_{A_\infty} D^{10} \frac{\varepsilon}{\varepsilon+1} \int_{\widehat{U}} M_U w^{1+\varepsilon}\, d\mu. \end{aligned}$$

Therefore, gathering all the estimations and averaging over $\widehat{U}$,

$$\Big(1 - 2[w]_{A_\infty} D^{10} \frac{\varepsilon}{\varepsilon+1}\Big) \fint_{\widehat{U}} M_U w^{1+\varepsilon}\, d\mu \leq w_{\widehat{U}}^{1+\varepsilon}.$$

Choosing $\varepsilon \leq 1/(4[w]_{A_\infty} D^{10} - 1)$ we get that $1 - 2[w]_{A_\infty} D^{10} \varepsilon/(\varepsilon+1) \geq \frac{1}{2}$ and we obtain the desired estimate (3-1).

Step 2. Now we proceed to prove the main estimate (1-8). By Remark 2.6, we have that $w(x) \leq M_U w(x)$ holds on U. Then we obtain

$$\int_U w^{1+\varepsilon}\, d\mu \leq \int_U (M_U w)^\varepsilon w\, d\mu \leq \int_{\widehat{U}} (M_U w)^\varepsilon w\, d\mu.$$

Once again we use the layer cake formula combined with the C–Z decomposition of Ω_λ and proceeding much as above, we obtain

$$\begin{aligned} \int_{\widehat{U}} (M_U w)^\varepsilon w\, d\mu &= \int_0^\infty \varepsilon \lambda^{\varepsilon-1} w(\Omega_\lambda)\, d\lambda \\ &= \int_0^{w_{\widehat{U}}} \varepsilon \lambda^{\varepsilon-1} w(\Omega_\lambda)\, d\lambda + \int_{w_{\widehat{U}}}^\infty \varepsilon \lambda^{\varepsilon-1} w(\Omega_\lambda)\, d\lambda \\ &\leq w(\widehat{U}) w_{\widehat{U}}^\varepsilon + \int_{w_{\widehat{U}}}^\infty \varepsilon \lambda^{\varepsilon-1} \sum_i w(V_i^{**})\, d\lambda \\ &\leq w(\widehat{U}) w_{\widehat{U}}^\varepsilon + D^2 \int_{w_{\widehat{U}}}^\infty \varepsilon \lambda^{\varepsilon} \sum_i \mu(V_i^{**})\, d\lambda \\ &\leq w(\widehat{U}) w_{\widehat{U}}^\varepsilon + D^4 \int_{w_{\widehat{U}}}^\infty \varepsilon \lambda^{\varepsilon} \sum_i \mu(V_i)\, d\lambda \\ &\leq w(\widehat{U}) w_{\widehat{U}}^\varepsilon + D^4 \int_0^\infty \varepsilon \lambda^{\varepsilon} \mu(\Omega_\lambda)\, d\lambda \\ &\leq w(\widehat{U}) w_{\widehat{U}}^\varepsilon + \frac{D^4 \varepsilon}{\varepsilon+1} \int_{\widehat{U}} (M_U w)^{1+\varepsilon}\, d\mu. \end{aligned}$$

Therefore, averaging over U, using $\mu(\widehat{U}) \leq D^2 \mu(U)$ and (3-1), we have

$$\fint_U w^{1+\varepsilon}\, d\mu \leq D^2 w_{\widehat{U}}^{\varepsilon+1} + \frac{2D^6 \varepsilon [w]_{A_\infty}}{\varepsilon+1} \Big(\fint_{\widehat{U}} w\, d\mu\Big)^{1+\varepsilon}.$$

By our previous choice of ε,

$$\frac{2D^6\varepsilon[w]_{A_\infty}}{\varepsilon+1} \le \frac{2D^{10}\varepsilon[w]_{A_\infty}}{\varepsilon+1} \le \frac{1}{2}$$

and we conclude that

$$\fint_U w^{1+\varepsilon}\,d\mu \le 2D^2\left(\fint_{\widehat{U}} w\,d\mu\right)^{1+\varepsilon}. \qquad \square$$

We present now some classical applications of the RHI to weighted norm inequalities for maximal functions. One crucial property of A_p classes is the well known open condition. In the next lemma we provide a quantitative version of it.

Lemma 3.1. *For* $1 < p < \infty$, *let* $w \in A_p$. *Then, for* $\varepsilon = (p-1)/(C[\sigma]_{A_\infty})$ *with* $C = 4D^{10}$ *and* $\sigma = w^{1-p'}$, *we have that* $w \in A_{p-\varepsilon}$. *Further,*

$$[w]_{A_{p-\varepsilon}} \le 2^{p-1}D^{4p-2}[w]_{A_p}.$$

Proof. Let $w \in A_p$. The $A_{p-\varepsilon}$ condition for w takes the form

$$\sup_{U\in\mathcal{B}}\left(\fint_U w\,d\mu\right)\left(\fint_U w^{1-(p-\varepsilon)'}\,d\mu\right)^{p-\varepsilon-1} < \infty.$$

Recall that the dual weight of w, $\sigma = w^{1-p'}$, is also in A_∞. Therefore it satisfies an RHI with exponent $r(\sigma)$ given by Theorem 1.2. Choose ε such that $1-(p-\varepsilon)' = (1-p')r(\sigma)$, namely $\varepsilon = (p-1)/(r(\sigma)')$ which is equivalent to the condition $r(\sigma) = (p-1)/(p-\varepsilon-1)$. Then we obtain

$$\begin{aligned}\left(\fint_U w^{1-(p-\varepsilon)'}\,d\mu\right)^{p-\varepsilon-1} &= \left(\fint_U \sigma^{(1-p')r(\sigma)}\,d\mu\right)^{(p-1)/(r(\sigma))}\\ &\le (2D^2\fint_{\widehat{U}}\sigma\,d\mu)^{p-1},\end{aligned}$$

for any $U \in \mathcal{B}$. Now, for $U = x+U_k \in \mathcal{B}$, recall that $U^{**} = x+U_{\theta^2(k)}$ and that $\widehat{U} \subset U^{**}$. Then,

$$\left(\fint_U w\,d\mu\right)\left(\fint_U w^{1-(p-\varepsilon)'}\,d\mu\right)^{p-\varepsilon-1} \le C\left(\fint_{U^{**}} w\,d\mu\right)\left(\fint_{U^{**}}\sigma\,d\mu\right)^{p-1}$$

with $C = 2^{p-1}D^{4p-2}$. We conclude that

$$[w]_{A_{p-\varepsilon}} \le 2^{p-1}D^{4p-2}[w]_{A_p}. \qquad \square$$

In what follows we will need the fact that the maximal function M maps $L^{q,\infty}_w(G)$ to itself with operator norm bounded by $C[w]_{A_q}^{1/q}$ for some $C > 0$. Without presenting any details on weak norms and Lorentz spaces, we include here a quantitative estimate on the size of level sets of the maximal function.

Lemma 3.2. *Let $1 \le q < \infty$ and let M be the maximal function defined in (1-3). Then, for any $f \in L^q_w(G)$,*

$$\sup_{\lambda>0} \lambda^q w(\{x \in G : Mf(x) > \lambda\}) \le D^{2q}[w]_{A_q} \|f\|^q_{L^q_w}. \tag{3-2}$$

Proof. For any locally integrable function f and any $\lambda > 0$, let Ω_λ be the level set $\Omega_\lambda := \{x \in G : Mf(x) > \lambda\}$. We also define some sort of *truncated* maximal operator as follows: for any $K \in \mathbb{Z}$, let M_K be the averaging operator given by

$$M_K f(x) = \sup_{V \in \mathcal{B}_K(x)} \fint_V |f(z)|\, d\mu, \tag{3-3}$$

where the supremum is taken over the subfamily $\mathcal{B}_K$ of $\mathcal{B}$ consisting of all base sets of the form $y + U_i$ with $y \in G$ and $i \le K$ containing the point x, i.e.,

$$\mathcal{B}_K(x) := \{V = y + U_i : x \in V, i \le K\}. \tag{3-4}$$

For each K we consider the corresponding level set $\Omega^K_\lambda := \{x \in G : M_K f(x) > \lambda\}$. We clearly have that the family $\{\Omega^K_\lambda\}$ is increasing in K and also $\Omega_\lambda = \bigcup_K \Omega^K_\lambda$. We therefore may compute the value of $w(\Omega_\lambda)$ as the limit of $w(\Omega^K_\lambda)$. In addition, we recall that the group G is σ-compact since $G = \bigcup_{r\in\mathbb{Z}} \overline{U}_r$. We will again use a limiting argument to compute $w(\Omega^K_\lambda)$ as the limit of $w(\Omega^K_\lambda \cap U_r)$ with $r \to +\infty$.

Now for $K \in \mathbb{Z}$ fixed, choose $r \in \mathbb{Z}$ such that $r \ge K$. A simple Vitali's covering lemma can be applied now to $\Omega^K_\lambda \cap U_r$. We want to select a countable subfamily of disjoint base sets whose dilates cover $\Omega^K_\lambda \cap U_r$. More precisely, suppose that the set $\Omega^K_\lambda \cap U_r$ is nonempty. For each $x \in \Omega^K_\lambda \cap U_r$, there exists a base set V_x of the form $V_x = y_x + U_{i_x}$ such that

$$\fint_{V_x} |f(z)|\, d\mu > \lambda. \tag{3-5}$$

Since $i_x \le K$ for all $x \in \Omega^K_\lambda \cap U_r$, there is some $i_1 = \max\{i_x\}$. We start the recursive selection procedure by picking one of these largest base sets as $V_1 = y_1 + U_{i_1}$. Now suppose that the first $V_1, V_2, \ldots, V_k$ sets have been selected. We pick V_{k+1} verifying that $V_{k+1} = y_{k+1} + U_{i_{k+1}}$ where $i_{k+1} = \max\{i_x : y_x + U_{i_x} \cap V_j = \varnothing,\ j = 1, \ldots, k\}$.

This process generates a sequence of disjoint base sets $\{V_k\}$. We note that the index sequence $\{i_k\}$ goes to $-\infty$ as k goes to infinity. If not, since it is decreasing, there would be some $i_0 = i_k$ for all $k \ge k_0$. Then we have that $V_k \cap U_r \ne \varnothing$ and $i_k \le K \le r$ and by the engulfing property, $V_k \subset U^{**}_r$ for all $k \ge k_0$. In particular, the set $S = \{y_k : k \ge k_0\} \subset U^{**}_r$ is relatively compact. Then, considering the set

$$F = \bigcup_{k \ge k_0} V_k \subset S + U_{i_0}$$

and proceeding as in Lemma 2.7 we get a contradiction.

We claim now that

$$\Omega_\lambda^K \cap U_r \subset \bigcup_{k\in\mathbb{N}} V_k^{**}.$$

To verify this, consider some $x \in \Omega_\lambda^K \cap U_r$ and the corresponding $V_x = y_x + U_{i_x}$. Suppose first that V_x intersects some V_k. Let k_0 be the smallest $k \in \mathbb{N}$ such that $V_x \cap V_k \neq \varnothing$. Then we have that $i_x \leq i_{k_0}$, since i_{k_0} was selected as the largest index among all the sets V_x disjoint from $V_1, \ldots, V_{k_0-1}$ (and by hypothesis V_x is one of them). Then the engulfing property yields

$$V_x = y_x + U_{i_x} \subset y_{k_0} + U_{\theta^2(i_{k_0})} = V_{k_0}^{**}.$$

We are left to consider the case when $V_x \cap V_k = \varnothing$ for all $k \in \mathbb{N}$. But in this case, we would have that $i_x \leq i_k$ for all k and this is a contradiction since we saw that $i_k \to -\infty$.

Summing up, we find a countable collection of base sets $\{V_k\}_k$ such that

$$\fint_{V_k} f\, d\mu > \lambda \quad \text{and} \quad \Omega_\lambda^K \cap U_r \subset \bigcup_k V_k^{**}.$$

Then we can compute

$$\begin{aligned}
\lambda^q w(\Omega_\lambda^K \cap U_r) &\leq \sum_k \lambda^q w(V_k^{**}) \leq \sum_k w(V_k^{**}) \left(\fint_{V_k} w^{-1/q} w^{1/q} |f| \right)^q \\
&\leq \sum_k \frac{w(V_k^{**})}{\mu(V_k)^q} \left(\int_{V_k} w^{1-q'}\, d\mu \right)^{q-1} \left(\int_{V_k} |f|^q w\, d\mu \right) \\
&\leq D^{2q} \sum_k \frac{w(V_k^{**})}{\mu(V_k^{**})^q} \left(\int_{V_k^{**}} w^{1-q'}\, d\mu \right)^{q-1} \left(\int_{V_k} |f|^q w\, d\mu \right) \\
&\leq D^{2q} [w]_{A_q} \sum_k \int_{V_k} |f|^q w\, d\mu \\
&\leq D^{2q} [w]_{A_q} \|f\|_{L_w^q}^q.
\end{aligned}$$

From this estimate we conclude that

$$\lambda^q w(\Omega_\lambda) \leq D^{2q} [w]_{A_q} \|f\|_{L_w^q}^q$$

for any $\lambda > 0$. □

Now we are able to present the proof of the sharp version of Buckley's theorem for the maximal function M on $L^p(G)$, $p > 1$.

Proof of Theorem 1.3. The idea is to use a sort of interpolation type argument, exploiting the sublinearity of the maximal operator M and the weak type estimate for M from Lemma 3.2. For any $f \in L^p_w(G)$ and any $t > 0$, define the truncation $f_t := f\chi_{\{|f|>t\}}$. Then, an easy computation of the averages defining M gives

$$\{x \in G : Mf(x) > 2t\} \subset \{x \in G : Mf_t(x) > t\}.$$

Now we compute the L^p_w norm as follows:

$$\begin{aligned}\|Mf\|^p_{L^p_w(G)} &= \int_0^\infty pt^{p-1} w(\{x \in G : Mf(x) > t\})\,dt \\ &= 2^p \int_0^\infty pt^{p-1} w(\{x \in G : Mf(x) > 2t\})\,dt \\ &\le 2^p \int_0^\infty pt^{p-1} w(\{x \in G : Mf_t(x) > t\})\,dt.\end{aligned}$$

We recall the open property for Muckenhoupt weights: any $w \in A_p$ also belongs to $A_{p-\varepsilon}$ for some explicit $\varepsilon > 0$ (see Lemma 3.1). Using the estimate of Lemma 3.2 for $q = p - \varepsilon$, we obtain

$$\begin{aligned}\|Mf\|^p_{L^p_w(G)} &\le 2^p p D^{2(p-\varepsilon)} [w]_{A_{p-\varepsilon}} \int_0^\infty t^{\varepsilon-1} \int_G f_t^{p-\varepsilon}(x) w(x)\,d\mu\,dt \\ &= \frac{2^p p D^{2(p-\varepsilon)} [w]_{A_{p-\varepsilon}}}{\varepsilon} \int_G |f(x)|^p w\,d\mu \\ &\le \frac{p 2^{2p-1} D^{6p-2} [w]_{A_p}}{\varepsilon} \|f\|^p_{L^p_w(G)},\end{aligned} \tag{3-6}$$

where in the last inequality we used Lemma 3.1. Noticing that in Lemma 3.1, $\varepsilon = (p-1)/(4D^{10}[\sigma]_{A_\infty})$, we finally conclude from (3-6) that

$$\|Mf\|_{L^p_w(G)} \le C([w]_{A_p}[\sigma]_{A_\infty})^{1/p} \|f\|_{L^p_w(G)}$$

and the proof of (1-9) is complete.

Finally, since $[\sigma]_{A_\infty} \le [\sigma]_{A_{p'}} = [w]^{p'-1}_{A_p}$, (1-10) follows from (1-9). □

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