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SYMMETRY AND NONEXISTENCE OF SOLUTIONS FOR A FULLY NONLINEAR NONLOCAL SYSTEM

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We study the system involving fully nonlinear nonlocal operators:

$$F_\alpha(u(x)) = C_{n,\alpha}\,\mathrm{PV}\int_{\mathbb{R}^n} \frac{G(u(x)-u(y))}{|x-y|^{n+\alpha}}\,dy = f(u(x),v(x)),$$
$$F_\beta(v(x)) = C_{n,\beta}\,\mathrm{PV}\int_{\mathbb{R}^n} \frac{G(v(x)-v(y))}{|x-y|^{n+\beta}}\,dy = g(u(x),v(x)).$$

We will prove the symmetry and monotonicity for positive solutions to the nonlinear system in whole space by using the method of moving planes. To achieve it, a narrow region principle and a decay at infinity are established. Further more, nonexistence of positive solutions to the nonlinear system on a half space is derived. In addition, the symmetry and monotonicity in whole space for positive solutions to a fully nonlinear nonlocal system

$$F_\alpha(u(x)) = -u^p(x) + v^q(x), \qquad F_\beta(v(x)) = -v^p(x) + u^q(x)$$

can be derived.

1. Introduction

We are interested in the general nonlinear system involving fully nonlinear nonlocal operators:

$$F_\alpha(u(x)) = f(u(x),v(x)), \qquad F_\beta(v(x)) = g(u(x),v(x))$$

with

$$F_\alpha(u(x)) = C_{n,\alpha}\,\mathrm{PV}\int_{\mathbb{R}^n} \frac{G(u(x)-u(y))}{|x-y|^{n+\alpha}}\,dy,$$

where the PV stands for the Cauchy principal value, G is a nonlinear operator and is at least local Lipschitz continuous with $G(0)=0$ and $0<\alpha,\beta<2$. The operator F_α was introduced by Caffarelli and Silvestre [2009].

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In order to make the integral significative, we require

$$u(x) \in C^{1,1}_{\text{loc}} \cap L_\alpha \quad \text{and} \quad v(x) \in C^{1,1}_{\text{loc}} \cap L_\beta$$

with

$$L_\alpha = \left\{ u : \mathbb{R}^n \to \mathbb{R} \;\Big|\; \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+\alpha}}\, dx < \infty \right\},$$

and L_β is defined similarly.

The special case is that when $G(\cdot)$ is an identity map, F_α becomes the usual fractional Laplacian $(-\Delta)^{\alpha/2}$. It is the nonlocal nature of fractional operators that makes them difficult to study. To circumvent this, Caffarelli and Silvestre [2007] introduced the *extension method*, which turns the nonlocal problem involving the fractional Laplacian into a local one in higher dimensions. A series of fruitful results show that this method has been applied successfully to treat equations involving the fractional Laplacian (see [Brändle et al. 2013; Chen and Zhu 2016; Gilbarg and Trudinger 1977]). Another way is using *the integral equations method*, such as *the method of moving planes in integral forms* (see [Cao and Chen 2013; Cao and Dai 2013; Li and Zhuo 2010; Lu and Zhu 2012; Zhuo et al. 2016]) and *regularity lifting* to investigate equations involving fractional Laplacian by showing that they are equivalent to corresponding integral equations (see [Chen et al. 2005; 2006; 2015]). For more articles concerning the method of moving planes for nonlocal equations and for integral equations, see [Gilbarg and Trudinger 1977; Hang et al. 2009; 2012; Hang 2007; Lei et al. 2012; Li 2017; Li and Ma 2017; Lu and Zhu 2011; 2012; Ma and Chen 2006; Ma and Zhao 2010; Wang and Niu 2017].

Chen, Li, and Li [Chen et al. 2017b] developed a systematic approach to carry out the method of moving planes for equations involving fractional Laplacian directly. Subsequently, by using this direct method, many authors investigated different equations involving fractional Laplace; see, for example, [Cheng et al. 2017a; 2017b; Cheng 2017; Li and Ma 2017; Liu and Ma 2016; Zhang et al. 2017].

However, for the fully nonlinear nonlocal equations, so far as we know, there is neither any corresponding *extension method* nor equivalent integral equations that one can begin to work. Chen, Li, and Li [Chen et al. 2017a], developed a new method that can deal with the fully nonlinear nonlocal equations directly. Then with the help of the direct method of moving planes, Wang and Yu [2017] studied a fully nonlinear nonlocal system where $u(x)$ and $v(x)$ belong to different nonhomogeneous terms. Wang and Niu [2017] studied a fully nonlinear nonlocal system with special nonhomogeneous terms which have $u(x)$ and $v(x)$ simultaneously while $u(x)$ and $v(x)$ have positive coefficients.

In this paper, we extend the direct method in [Chen et al. 2017a] to more general fully nonlinear nonlocal systems:

$$\text{(1-1)}\qquad \begin{aligned} F_\alpha(u(x)) &= f(u(x), v(x)), \quad x \in \mathbb{R}^n, \\ F_\beta(v(x)) &= g(u(x), v(x)), \quad x \in \mathbb{R}^n, \\ u(x) > 0, \quad v(x) &> 0, \quad x \in \mathbb{R}^n, \end{aligned}$$

and

$$\text{(1-2)}\qquad \begin{aligned} F_\alpha(u(x)) &= f(u(x), v(x)), \quad x \in \mathbb{R}^n_+, \\ F_\beta(v(x)) &= g(u(x), v(x)), \quad x \in \mathbb{R}^n_+, \\ u(x) \equiv 0, \quad v(x) &\equiv 0, \quad x \notin \mathbb{R}^n_+, \end{aligned}$$

where f, g are continuous functions. It is worth mentioning that (1-1) is more general than the system in [Wang and Yu 2017] and is different from the system in [Wang and Niu 2017]. Because (1-1) can be allowed, $u(x)$ or $v(x)$ have negative coefficients.

We first establish the *narrow region principle* and *decay at infinity* for the systems and they will play important roles in carrying out the method of moving planes.

To state them, let

$$T_\lambda = \{x \in \mathbb{R}^n \mid x_1 = \lambda, \lambda \in \mathbb{R}\}$$

be the moving plane, and denote by

$$\Sigma_\lambda = \{x \in \mathbb{R}^n \mid x_1 < \lambda\}$$

the left region of the plane T_λ, by

$$x^\lambda = (2\lambda - x_1, x_2, \dots, x_n)$$

the reflection of x about T_λ, and let

$$U_\lambda(x) = u_\lambda(x) - u(x), \qquad V_\lambda(x) = v_\lambda(x) - v(x)$$

with

$$u_\lambda(x) = u(x^\lambda), \qquad v_\lambda(x) = v(x^\lambda).$$

For simplicity of notation, in the following, we denote $U_\lambda(x)$ by $U(x)$ and $V_\lambda(x)$ by $V(x)$. Throughout this paper, we assume that

$$\text{(1-3)}\qquad G \in C^1(\mathbb{R}), \quad G(0) = 0, \quad \text{and} \quad G'(t) \ge c_0 > 0 \quad \forall\, t \in \mathbb{R}.$$

Theorem 1.1 (narrow region principle). *Let $\Omega \subset \Sigma_\lambda$ be a bounded narrow region contained in the strip*

$$\{x \mid \lambda - l < x_1 < \lambda\}$$

with small $l > 0$. *Suppose that* $U(x) \in L_\alpha \cap C^{1,1}_{\text{loc}}(\Omega)$, $V(x) \in L_\beta \cap C^{1,1}_{\text{loc}}(\Omega)$, $U(x)$ *and* $V(x)$ *are both lower semicontinuous on* $\overline{\Omega}$, *and satisfy*

$$\begin{cases} F_\alpha(u_\lambda(x)) - F_\alpha(u(x)) + c_{11}(x)U(x) + c_{12}(x)V(x) \geq 0, & x \in \Omega, \\ F_\beta(v_\lambda(x)) - F_\beta(v(x)) + c_{21}(x)U(x) + c_{22}(x)V(x) \geq 0, & x \in \Omega, \\ U(x),\, V(x) \geq 0, & x \in \Sigma_\lambda \setminus \Omega, \\ U(x^\lambda) = -U(x), \quad V(x^\lambda) = -V(x), & x \in \Sigma_\lambda, \end{cases} \tag{1-4}$$

if $c_{12}(x) < 0$, $c_{21}(x) < 0$, *and* $c_{ij}(x)$, $i, j = 1, 2$, *are bounded from below in* Ω, *then for sufficiently small* l, *we have*

$$U(x) \geq 0, \quad V(x) \geq 0 \quad \textit{in} \quad \Omega; \tag{1-5}$$

furthermore, *if* $U(x)$ or $V(x)$ *equals* 0 *at some point in* Ω, *then*

$$U(x) = V(x) \equiv 0, \quad x \in \mathbb{R}^n. \tag{1-6}$$

These conclusions hold for the unbounded region Ω *if we further assume that*

$$\varliminf_{|x|\to\infty} U(x) \geq 0, \qquad \varliminf_{|x|\to\infty} V(x) \geq 0. \tag{1-7}$$

Theorem 1.2 (decay at infinity). *Let* Ω *be an unbounded region in* Σ_λ. *Assume that* $U(x) \in C^{1,1}_{\text{loc}}(\Omega) \cap L_\alpha(\mathbb{R}^n)$, $V(x) \in C^{1,1}_{\text{loc}}(\Omega) \cap L_\beta(\mathbb{R}^n)$ *are both lower semicontinuous and satisfy*

$$\begin{cases} F_\alpha(u_\lambda(x)) - F_\alpha(u(x)) + c_{11}(x)U(x) + c_{12}(x)V(x) \geq 0, & x \in \Omega, \\ F_\beta(v_\lambda(x)) - F_\beta(v(x)) + c_{21}(x)U(x) + c_{22}(x)V(x) \geq 0, & x \in \Omega, \\ U(x),\, V(x) \geq 0, & x \in \Sigma_\lambda \setminus \Omega, \\ U(x^\lambda) = -U(x), \quad V(x^\lambda) = -V(x), & x \in \Sigma_\lambda, \end{cases} \tag{1-8}$$

with

$$c_{11}(x), c_{12}(x) \sim o\left(\frac{1}{|x|^\alpha}\right), \quad c_{21}(x), c_{22}(x) \sim o\left(\frac{1}{|x|^\beta}\right) \quad \textit{for } |x| \textit{ large} \tag{1-9}$$

and $c_{12}(x), c_{21}(x) < 0$, *then there exists a constant* $R_0 > 0$ *depending only on* $c_{ij}(x)$ *such that if*

$$U(\tilde{x}) = \min_\Omega U(x) < 0 \quad \textit{and} \quad V(\bar{x}) = \min_\Omega V(x) < 0,$$

then

$$|\tilde{x}| \leq R_0 \quad \textit{or} \quad |\bar{x}| \leq R_0. \tag{1-10}$$

Based on Theorems 1.1 and 1.2, we apply the *method of moving planes* to obtain symmetry and monotonicity of positive solutions to (1-1) in $\mathbb{R}^n$, as well as nonexistence of positive solutions to (1-2) on the half space $\mathbb{R}^n_+$.

Theorem 1.3. *Assume that* $u(x) \in L_\alpha \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, $v(x) \in L_\beta \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ *are positive solutions of system* (1-1). *Suppose that for some* $\gamma_1, \gamma_2 > 0$,

$$u(x) = o\left(\frac{1}{|x|^{\gamma_1}}\right), \quad v(x) = o\left(\frac{1}{|x|^{\gamma_2}}\right) \quad \text{as } |x| \to \infty \tag{1-11}$$

and f, g *are continuous functions satisfying*

$$\begin{aligned} &\text{(i)} \quad \text{for fixed } t: \quad -C_1 s^p \le f_1'(s,t) < 0, \quad 0 < g_1'(s,t) \le C_2 s^q; \\ &\text{(ii)} \quad \text{for fixed } s: \quad 0 < f_2'(s,t) \le C_3 t^p, \quad -C_4 t^q \le g_2'(s,t) < 0; \end{aligned} \tag{1-12}$$

with $\min\{p\gamma_1, p\gamma_2\} \ge \alpha$, $\min\{q\gamma_1, q\gamma_2\} \ge \beta$, *and* $C_i > 0$, $i = 1, 2, 3, 4$.

Then $u(x)$ *and* $v(x)$ *must be radially symmetric and monotone decreasing about some point in* $\mathbb{R}^n$.

Theorem 1.4. *Assume that* $u(x) \in L_\alpha \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n_+)$, $v(x) \in L_\beta \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n_+)$ *are nonnegative solutions to system* (1-2) *where* f, g *are nonnegative continuous functions and* u, v *are lower semicontinuous on* $\overline{\mathbb{R}^n_+}$. *Suppose*

$$\lim_{|x|\to\infty} u(x) = 0, \qquad \lim_{|x|\to\infty} v(x) = 0, \tag{1-13}$$

then $u(x) \equiv 0$, $v(x) \equiv 0$.

In Section 2, we prove Theorems 1.1 and 1.2 with a key inequality (2-2) below. Sections 3 and 4 are devoted to the proofs of Theorems 1.3 and 1.4, respectively, by using the previous results and the method of moving planes. In Section 5, we will consider the fully nonlinear nonlocal system

$$\begin{aligned} F_\alpha(u(x)) &= -u^p(x) + v^q(x), & x &\in \mathbb{R}^n, \\ F_\beta(v(x)) &= -v^p(x) + u^q(x), & x &\in \mathbb{R}^n, \\ u(x), v(x) &> 0, & x &\in \mathbb{R}^n, \end{aligned}$$

where $p, q > 0$. And it is a specific case of (1-1).

2. Proofs of Theorems 1.1 and 1.2

Let

$$F_\alpha(u(x)) = C_{n,\alpha}\,\mathrm{PV}\int_{\mathbb{R}^n} \frac{G(u(x)-u(y))}{|x-y|^{n+\alpha}}\,dy \\ = C_{n,\alpha} \lim_{\epsilon\to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{G(u(x)-u(y))}{|x-y|^{n+\alpha}}\,dy,$$

and use c and C for general various positive constants that are usually different in different contexts.

We first introduce a lemma which is often called the strong maximum principle to F_α.

Lemma 2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$. Assume that $u(x) \in C^{1,1}_{\mathrm{loc}} \cap L_\alpha(\mathbb{R}^n)$ is lower semicontinuous on $\overline{\Omega}$ and satisfies*

$$\begin{aligned} F_\alpha(u(x)) &\geq 0, \quad x \in \Omega, \\ u(x) &\geq 0, \quad x \in \Omega^c. \end{aligned} \tag{2-1}$$

If $u(x)$ attains 0 somewhere in Ω, then

$$u(x) \equiv 0, \quad x \in \mathbb{R}^n.$$

The proof of this lemma was completed in [Wang and Yu 2017], we omit the details here.

Proof of Theorem 1.1. If (1-5) does not hold, without loss of generality, we assume $U(x) < 0$ at some point in Ω. By the lower semicontinuity of $U(x)$ on $\overline{\Omega}$, we know that there exists some $\tilde{x} \in \Omega$ such that

$$U(\tilde{x}) = \min_{\Omega} U(x) < 0.$$

It follows from (1-4) that $\tilde{x}$ must be in the interior of Ω. Then we have

$$\begin{aligned} &F_\alpha(u_\lambda(\tilde{x})) - F_\alpha(u(\tilde{x})) \\ &\quad = C_{n,\alpha}\,\mathrm{PV}\int_{\mathbb{R}^n} \frac{G(u_\lambda(\tilde{x}) - u_\lambda(y)) - G(u(\tilde{x}) - u(y))}{|\tilde{x} - y|^{n+\alpha}}\,dy \\ &\quad = C_{n,\alpha}\,\mathrm{PV}\int_{\Sigma_\lambda} \frac{G(u_\lambda(\tilde{x}) - u_\lambda(y)) - G(u(\tilde{x}) - u(y))}{|\tilde{x} - y|^{n+\alpha}}\,dy \\ &\qquad + C_{n,\alpha}\,\mathrm{PV}\int_{\Sigma_\lambda} \frac{G(u_\lambda(\tilde{x}) - u(y)) - G(u(\tilde{x}) - u_\lambda(y))}{|\tilde{x} - y^\lambda|^{n+\alpha}}\,dy \\ &\quad \leq C_{n,\alpha}\,\mathrm{PV}\int_{\Sigma_\lambda} \frac{G(u_\lambda(\tilde{x}) - u_\lambda(y)) - G(u(\tilde{x}) - u(y))}{|\tilde{x} - y^\lambda|^{n+\alpha}}\,dy \\ &\qquad + C_{n,\alpha}\,\mathrm{PV}\int_{\Sigma_\lambda} \frac{G(u_\lambda(\tilde{x}) - u(y)) - G(u(\tilde{x}) - u_\lambda(y))}{|\tilde{x} - y^\lambda|^{n+\alpha}}\,dy \\ &\quad = C_{n,\alpha}\,\mathrm{PV}\int_{\Sigma_\lambda} \frac{2G'(\cdot)U(\tilde{x})}{|\tilde{x} - y^\lambda|^{n+\alpha}}\,dy \\ &\quad \leq 2C_{n,\alpha}U(\tilde{x})\int_{\Sigma_\lambda} \frac{1}{|\tilde{x} - y^\lambda|^{n+\alpha}}\,dy. \end{aligned} \tag{2-2}$$

Let $D = \{y \mid l < y_1 - \tilde{x}_1 < 1, |y' - \tilde{x}'| < 1\}$, $s = y_1 - \tilde{x}_1$, $\tau = |y' - \tilde{x}'|$, and ω_{n-2} be the area of an $(n-2)$-dimensional unit sphere. Here we write $x = (x_1, x')$. Then we have

$$\begin{aligned}
\text{(2-3)} \quad \int_{\Sigma_\lambda} \frac{1}{|\tilde{x} - y^\lambda|^{n+\alpha}}\, dy &\geq \int_D \frac{1}{|\tilde{x} - y|^{n+\alpha}}\, dy = \int_l^1 \int_0^1 \frac{\omega_{n-2}\tau^{n-2}}{(s^2+\tau^2)^{(n+\alpha)/2}}\, d\tau\, ds \\
&= \int_l^1 \int_0^{1/s} \frac{\omega_{n-2}(st)^{n-2}s}{s^{n+\alpha}(1+t^2)^{(n+\alpha)/2}}\, dt\, ds \\
&= \int_l^1 \frac{1}{s^{1+\alpha}} \int_0^{1/s} \frac{\omega_{n-2}t^{n-2}}{(1+t^2)^{(n+\alpha)/2}}\, dt\, ds \\
&\geq \int_l^1 \frac{1}{s^{1+\alpha}} \int_0^{1} \frac{\omega_{n-2}t^{n-2}}{(1+t^2)^{(n+\alpha)/2}}\, dt\, ds \\
&\geq C \int_l^1 \frac{1}{s^{1+\alpha}}\, ds \geq \frac{C}{l^\alpha}.
\end{aligned}$$

Thus from (2-2) and the fact that $c_{11}(x)$ is bounded from below in Ω,

$$\text{(2-4)} \qquad F_\alpha(u_\lambda(\tilde{x})) - F_\alpha(u(\tilde{x})) + c_{11}(\tilde{x})u(\tilde{x}) \leq \frac{C}{l^\alpha} U(\tilde{x}) < 0.$$

Together (2-4) with (1-4), we have

$$\text{(2-5)} \qquad U(\tilde{x}) \geq -cl^\alpha c_{12}(\tilde{x}) V(\tilde{x}).$$

From (2-5) and the lower semicontinuity of $V(x)$ on $\overline{\Omega}$, we know that there exists $\bar{x}$ in Ω such that

$$V(\bar{x}) = \min_{\Omega} V(x) < 0.$$

Similar to (2-4), it is easy to see that

$$F_\beta(v_\lambda(\bar{x})) - F_\beta(v(\bar{x})) + c_{22}(\bar{x})V(\bar{x}) \leq \frac{C}{l^\beta} V(\bar{x}) < 0.$$

Together with (2-5), for l sufficiently small, we have

$$\begin{aligned}
0 &\leq F_\beta(v_\lambda(\bar{x})) - F_\beta(v(\bar{x})) + c_{21}(\bar{x})U(\bar{x}) + c_{22}(\bar{x})V(\bar{x}) \\
&\leq \frac{C}{l^\beta} V(\bar{x}) + c_{21}(\bar{x})U(\tilde{x}) \\
&\leq \frac{C}{l^\beta} V(\bar{x}) - cc_{21}(\bar{x}) l^\alpha c_{12}(\tilde{x}) V(\tilde{x}) \\
&\leq \frac{C}{l^\beta} V(\bar{x}) - cc_{21}(\bar{x}) l^\alpha c_{12}(\tilde{x}) V(\bar{x}) \\
&\leq \frac{C}{l^\beta} V(\bar{x})(1 - c_{21}(\bar{x}) c_{12}(\tilde{x}) l^{\alpha+\beta}) < 0.
\end{aligned}$$

This contradiction shows that (1-5) must be true. If Ω is unbounded, then (1-5) is easily obtained by using (1-7).

To prove (1-6), without loss of generality, we suppose that there exists $\eta \in \Omega$ such that

$$U(\eta) = 0.$$

Combining the fact

$$\frac{1}{|x-y|} > \frac{1}{|x-y^\lambda|} \quad \forall\, x, y \in \Sigma_\lambda,$$

we have

$$\begin{aligned}
&F_\alpha(u_\lambda(\eta)) - F_\alpha(u(\eta)) \\
&\quad = C_{n,\alpha}\,\mathrm{PV}\!\int_{\mathbb{R}^n} \frac{G(u_\lambda(\eta)-u_\lambda(y)) - G(u(\eta)-u(y))}{|\eta-y|^{n+\alpha}}\,dy \\
&\quad = C_{n,\alpha}\,\mathrm{PV}\!\int_{\Sigma_\lambda} \frac{G(u_\lambda(\eta)-u_\lambda(y)) - G(u(\eta)-u(y))}{|\eta-y|^{n+\alpha}}\,dy \\
&\qquad\qquad + C_{n,\alpha}\,\mathrm{PV}\!\int_{\Sigma_\lambda} \frac{G(u_\lambda(\eta)-u(y)) - G(u(\eta)-u_\lambda(y))}{|\eta-y^\lambda|^{n+\alpha}}\,dy \\
&\quad = C_{n,\alpha}\,\mathrm{PV}\!\int_{\Sigma_\lambda} [G(u_\lambda(\eta)-u_\lambda(y)) - G(u(\eta)-u(y))]\left(\frac{1}{|\eta-y|^{n+\alpha}} - \frac{1}{|\eta-y^\lambda|^{n+\alpha}}\right) dy \\
&\qquad\qquad + C_{n,\alpha}\,\mathrm{PV}\!\int_{\Sigma_\lambda} \frac{1}{|\eta-y^\lambda|^{n+\alpha}} \big(G(u_\lambda(\eta)-u(y)) - G(u(\eta)-u_\lambda(y)) \\
&\qquad\qquad\qquad\qquad + G(u_\lambda(\eta)-u_\lambda(y)) - G(u(\eta)-u(y))\big)\,dy \\
&\quad = C_{n,\alpha}\,\mathrm{PV}\!\int_{\Sigma_\lambda} [G(u_\lambda(\eta)-u_\lambda(y)) - G(u(\eta)-u(y))]\left(\frac{1}{|\eta-y|^{n+\alpha}} - \frac{1}{|\eta-y^\lambda|^{n+\alpha}}\right) dy \\
&\qquad\qquad + C_{n,\alpha}\,\mathrm{PV}\!\int_{\Sigma_\lambda} \frac{1}{|\eta-y^\lambda|^{n+\alpha}} \big(G(u_\lambda(\eta)-u(y)) - G(u(\eta)-u(y)) \\
&\qquad\qquad\qquad\qquad + G(u_\lambda(\eta)-u_\lambda(y)) - G(u(\eta)-u_\lambda(y))\big)\,dy \\
&\quad = C_{n,\alpha} G'(\cdot)\int_{\Sigma_\lambda} (U(\eta)-U(y))\left(\frac{1}{|\eta-y|^{n+\alpha}} - \frac{1}{|\eta-y^\lambda|^{n+\alpha}}\right) dy \\
&\qquad\qquad + C_{n,\alpha}\int_{\Sigma_\lambda} \frac{G'(\cdot)U(\eta) + G'(\cdot)U(\eta)}{|\eta-y^\lambda|^{n+\alpha}}\,dy \\
&\quad \le -C_{n,\alpha} c_0 \int_{\Sigma_\lambda} U(y)\left(\frac{1}{|\eta-y|^{n+\alpha}} - \frac{1}{|\eta-y^\lambda|^{n+\alpha}}\right) dy.
\end{aligned}$$

That is,

$$\begin{aligned}
\text{(2-6)}\quad & F_\alpha(u_\lambda(\eta)) - F_\alpha(u(\eta)) + c_{11}(\eta)U(\eta) \\
&\qquad \le -C_{n,\alpha} c \int_{\Sigma_\lambda} U(y)\left(\frac{1}{|\eta-y|^{n+\alpha}} - \frac{1}{|\eta-y^\lambda|^{n+\alpha}}\right) dy.
\end{aligned}$$

If $U(x) \not\equiv 0$, $x \in \Sigma_\lambda$, then (2-6) implies

$$F_\alpha(u_\lambda(\eta)) - F_\alpha(u(\eta)) + c_{11}(\eta)U(\eta) < 0.$$

Together with (1-4), it is easy to see that $V(\eta) < 0$. This contradicts with (1-5). Hence $U(x)$ must be identically 0 in Σ_λ. Since

$$U(x^\lambda) = -U(x), \quad x \in \Sigma_\lambda,$$

it gives

$$U(x) \equiv 0, \quad x \in \mathbb{R}^n.$$

Together with the first equation in (1-4), we see

$$V(x) \leq 0, \quad x \in \Sigma_\lambda.$$

Noting we already have

$$V(x) \geq 0, \quad x \in \Sigma_\lambda,$$

it must hold

$$V(x) = 0, \quad x \in \Sigma_\lambda.$$

Recalling $V(x^\lambda) = -V(x)$, we deduce

$$V(x) \equiv 0, \quad x \in \mathbb{R}^n.$$

Similarly, one can show that if $V(x)$ attains 0 at some point in Ω, then both $U(x)$ and $V(x)$ are identically 0 in $\mathbb{R}^n$. This completes the proof. □

Proof of Theorem 1.2. Assume that there exists $\tilde{x} \in \Omega$ such that

$$U(\tilde{x}) = \min_\Omega U(x) < 0.$$

Using the key inequality (2-2), we have

$$F_\alpha(u_\lambda(\tilde{x})) - F_\alpha(u(\tilde{x})) \leq 2C_{n,\alpha}c_0 U(\tilde{x}) \int_{\Sigma_\lambda} \frac{1}{|\tilde{x} - y^\lambda|^{n+\alpha}}\, dy.$$

For each fixed $\lambda \in \mathbb{R}$, there exists $C > 0$ such that for $\tilde{x} \in \Sigma_\lambda$ and $|\tilde{x}|$ sufficiently large,

$$\int_{\Sigma_\lambda} \frac{1}{|\tilde{x} - y^\lambda|^{n+\alpha}}\, dy \geq \int_{B_{3|\tilde{x}|}(\tilde{x}) \setminus B_{2|\tilde{x}|}(\tilde{x})} \frac{1}{|\tilde{x} - y|}\, dy \sim \frac{C}{|\tilde{x}|^\alpha}. \tag{2-7}$$

Hence, from (2-7) and (1-9), we have

$$F_\alpha(u_\lambda(\tilde{x})) - F_\alpha(u(\tilde{x})) + c_{11}(\tilde{x})U(\tilde{x}) \leq \frac{C}{|\tilde{x}|^\alpha} U(\tilde{x}) < 0. \tag{2-8}$$

Together (2-8) with (1-8), it is easy to know

$$V(\tilde{x}) < 0, \tag{2-9}$$

and

$$U(\tilde{x}) \geq -cc_{12}(\tilde{x})|\tilde{x}|^{\alpha} V(\tilde{x}). \tag{2-10}$$

From (2-9) and the lower semicontinuity of $V(x)$ on $\overline{\Omega}$, there exists $\bar{x}$ such that

$$V(\bar{x}) = \min_{\Omega} V(x) < 0.$$

Similarly to (2-8), we can derive

$$F_\beta(v_\lambda(\bar{x})) - F_\beta(v(\bar{x})) + c_{22}(\bar{x})V(\bar{x}) \leq \frac{C}{|\bar{x}|^{\beta}} V(\bar{x}) < 0. \tag{2-11}$$

Combining (1-8), (1-10), and (2-11), for λ sufficiently negative, it follows that

$$\begin{aligned} 0 &\leq F_\beta(v_\lambda(\bar{x})) - F_\beta(v(\bar{x})) + c_{21}(\bar{x})U(\bar{x}) + c_{22}(\bar{x})V(\bar{x}) \\ &\leq \frac{C}{|\bar{x}|^{\beta}} V(\bar{x}) + c_{21}(\bar{x})U(\tilde{x}) \\ &\leq \frac{C}{|\bar{x}|^{\beta}} V(\bar{x}) - cc_{21}(\bar{x})c_{12}(\tilde{x})|\tilde{x}|^{\alpha} V(\tilde{x}) \\ &\leq \frac{C}{|\bar{x}|^{\beta}} V(\bar{x}) - cc_{21}(\bar{x})c_{12}(\tilde{x})|\tilde{x}|^{\alpha} V(\bar{x}) \\ &\leq \frac{C}{l^{\beta}} V(\bar{x})(1 - c_{12}(\tilde{x})|\tilde{x}|^{\alpha} c_{21}(\bar{x})|\bar{x}|^{\beta}) < 0. \end{aligned} \tag{2-12}$$

The last inequality follows from assumption (1-9). This contradiction shows that (1-10) must be true. □

3. Symmetry of solutions in the whole space $\mathbb{R}^n$

Proof of Theorem 1.3. Choose an arbitrary direction for the x_1-axis. Let

$$T_\lambda = \{x \in \mathbb{R}^n \mid x_1 = \lambda, \lambda \in \mathbb{R}\}, \qquad \Sigma_\lambda = \{x \in \mathbb{R}^n \mid x_1 < \lambda\},$$
$$x^\lambda = (2\lambda - x_1, x'), \quad u_\lambda(x) = u(x^\lambda),$$
$$U_\lambda(x) = u_\lambda(x) - u(x), \quad V_\lambda(x) = v_\lambda(x) - v(x).$$

Step 1: Starting moving the plane T_λ from $-\infty$ to the right along the x_1-axis. We need to show that for λ sufficiently negative,

$$U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda. \tag{3-1}$$

By the assumption (1-11), for fixed λ and $x \in \Sigma_\lambda$, we know that

$$u(x) \to 0 \text{ as } |x| \to +\infty.$$

Since $|x^\lambda| \to +\infty$, as $|x| \to +\infty$, we have

$$u_\lambda(x) = u(x^\lambda) \to 0.$$

Hence for $x \in \Sigma_\lambda$,

$$U_\lambda(x) \to 0 \text{ as } |x| \to +\infty. \tag{3-2}$$

Similarly, one can show that for $x \in \Sigma_\lambda$,

$$V_\lambda(x) \to 0 \text{ as } |x| \to +\infty.$$

If

$$\Sigma_\lambda^- = \{x \in \Sigma_\lambda \mid U_\lambda(x) < 0\} \neq \varnothing,$$

then by the lower semicontinuity of $U_\lambda(x)$, there must exist some $\tilde{x} \in \Sigma_\lambda$ such that

$$U_\lambda(\tilde{x}) = \min_{\Sigma_\lambda} U(x) < 0.$$

Let

$$\begin{aligned} I &= f(u_\lambda(\tilde{x}), v_\lambda(\tilde{x})) - f(u(\tilde{x}), v_\lambda(\tilde{x})), \\ J &= f(u(\tilde{x}), v_\lambda(\tilde{x})) - f(u(\tilde{x}), v(\tilde{x})). \end{aligned}$$

Then

$$\begin{aligned} I + J &= f(u_\lambda(\tilde{x}), v_\lambda(\tilde{x})) - f(u(\tilde{x}), v(\tilde{x})) \\ &= F_\alpha(u_\lambda(\tilde{x})) - F_\alpha(u(\tilde{x})) \\ &\le 2C_{n,\alpha} c_0 U_\lambda(\tilde{x}) \int_{\Sigma_\lambda} \frac{1}{|\tilde{x} - y^\lambda|^{n+\alpha}}\, dy \\ &< 0. \end{aligned} \tag{3-3}$$

By the mean value theorem and the assumption (1-12), we have

$$I = f_1'(\xi_\lambda(\tilde{x}), v_\lambda(\tilde{x}))U_\lambda(\tilde{x}) > 0 \quad \text{and} \quad J = f_2'(u(\tilde{x}), \eta_\lambda(\tilde{x}))V_\lambda(\tilde{x}), \tag{3-4}$$

where $\xi_\lambda(\tilde{x})$ is between $u_\lambda(\tilde{x})$ and $u(\tilde{x})$; $\eta_\lambda(\tilde{x})$ is between $v_\lambda(\tilde{x})$ and $v(\tilde{x})$. Together (3-3) with (3-4) and (1-12), it is easy to see that

$$V_\lambda(\tilde{x}) < 0.$$

This implies that there exists some $\bar{x} \in \Sigma_\lambda$ such that

$$V_\lambda(\bar{x}) = \min_{\Sigma_\lambda} V(x) < 0.$$

By the mean value theorem again, we have

$$\begin{aligned} F_\alpha(u_\lambda(\tilde{x})) - F_\alpha(u(\tilde{x})) &= I + J \\ &\ge f_1'(\xi_\lambda(\tilde{x}), v_\lambda(\tilde{x}))U_\lambda(\tilde{x}) + f_2'(u(\tilde{x}), \eta_\lambda(\tilde{x}))V_\lambda(\tilde{x}). \end{aligned}$$

By the decay assumptions (1-11) and (1-12), we deduce that

$$f_1'(\xi_\lambda(\tilde{x}), v_\lambda(\tilde{x})),\ f_2'(u(\tilde{x}), \eta_\lambda(\tilde{x})) \sim o\left(\frac{1}{|\tilde{x}|^\alpha}\right).$$

Hence

$$F_\alpha(u_\lambda(\tilde{x})) - F_\alpha(u(\tilde{x})) + c_{11}(\tilde{x})U_\lambda(\tilde{x}) + c_{12}(\tilde{x})V_\lambda(\tilde{x}) \geq 0,$$

where

$$c_{11}(\tilde{x}) = -f_1'(\xi_\lambda(\tilde{x}), v_\lambda(\tilde{x})) \quad \text{and} \quad c_{12}(\tilde{x}) = -f_2'(u(\tilde{x}), \eta_\lambda(\tilde{x})).$$

Similarly, we have

$$F_\beta(v_\lambda(\bar{x})) - F_\beta(v(\bar{x})) + c_{21}(\bar{x})U_\lambda(\bar{x}) + c_{22}(\bar{x})V_\lambda(\bar{x}) \geq 0,$$

where

$$c_{21}(\bar{x}) = -g_1'(\hat{\xi}_\lambda(\bar{x}), v_\lambda(\bar{x})) \quad \text{and} \quad c_{22}(\bar{x}) = -g_2'(u(\bar{x}), \hat{\eta}_\lambda(\bar{x}))$$

with

$$c_{21}(\bar{x}), c_{22}(\bar{x}) \sim o\left(\frac{1}{|\bar{x}|^\beta}\right).$$

Consequently, there exists $R_0 > 0$, such that if $\tilde{x}$ and $\bar{x}$ are negative minima of $U_\lambda(x)$ and $V_\lambda(x)$ in Σ_λ respectively, then by (1-2) we know that

$$|\tilde{x}| \leq R_0 \quad \text{or} \quad |\bar{x}| \leq R_0. \tag{3-5}$$

Without loss of generality, we may assume

$$|\tilde{x}| \leq R_0. \tag{3-6}$$

Combining (3-2) with the fact that $U_\lambda(x) = 0$, $x \in T_\lambda$, it is easy to see if $U_\lambda(x) < 0$ at some point in Σ_λ, then $U_\lambda(x)$ must have a negative minimum in Σ_λ. For λ sufficiently negative, it contradicts (3-6). Hence we have for λ sufficiently negative,

$$U_\lambda(x) \geq 0. \tag{3-7}$$

It follows that $U_\lambda(x) \geq 0$ in Σ_λ. Otherwise, there exists $\bar{x}$ in Σ_λ such that

$$V_\lambda(\bar{x}) = \min_{\Sigma_\lambda} V(x) < 0.$$

From (2-11), we have

$$F_\beta(v_\lambda(\bar{x})) - F_\beta(v(\bar{x})) + c_{22}(\bar{x})V_\lambda(\bar{x}) < 0. \tag{3-8}$$

Combining (1-8) with (3-7), however, we have

$$F_\beta(v_\lambda(\bar{x})) - F_\beta(v(\bar{x})) + c_{22}(\bar{x})V_\lambda(\bar{x}) \geq 0.$$

This is a contradiction with (3-8) and $V_\lambda(x)$ cannot attain its negative value in Σ_λ. It follows that (3-1) must be true. This completes the preparation for the moving planes.

Step 2: Keep moving the plane to the limiting position T_{λ_0} as long as (3-1) holds.

Let

$$\lambda_0 = \sup\{\lambda \mid U_\mu(x), V_\mu(x) \geq 0,\ x \in \Sigma_\mu, \mu \leq \lambda\}.$$

Obviously,

$$\lambda_0 < \infty. \tag{3-9}$$

Otherwise, if $\lambda_0 = \infty$, then for any $\lambda > 0$,

$$\begin{aligned} &u(0^\lambda) > u(0) > 0, \qquad v(0^\lambda) > v(0) > 0, \\ &u(0^\lambda) \sim o\Big(\frac{1}{|0^\lambda|^{\gamma_1}}\Big), \quad v(0^\lambda) \sim o\Big(\frac{1}{|0^\lambda|^{\gamma_2}}\Big), \quad \lambda \to \infty. \end{aligned}$$

This is a contradiction and (3-9) is true.

Now we point out that

$$U_{\lambda_0}(x) \equiv 0, \quad V_{\lambda_0}(x) \equiv 0, \quad x \in \Sigma_{\lambda_0}. \tag{3-10}$$

If (3-10) is not true, then from the proof of Theorem 1.1, we only have the case that $U_{\lambda_0}(x) \geq 0$ and $V_{\lambda_0}(x) \geq 0$ but $U_{\lambda_0}(x) \not\equiv 0$ and $V_{\lambda_0}(x) \not\equiv 0$.

In what follows, we will show that the plane T_λ can be moved further to the right. More rigorously, there exists some $\epsilon > 0$, such that for any $\lambda \in [\lambda_0, \lambda_0 + \epsilon)$,

$$U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda. \tag{3-11}$$

This contradicts the definition of λ_0 and hence (3-10) must be true.

Now we prove (3-11) by using Theorems 1.1 and 1.2. From Theorem 1.1, we have

$$U_{\lambda_0}(x) > 0, \quad V_{\lambda_0}(x) > 0, \quad x \in \Sigma_{\lambda_0}.$$

Let R_0 be the constant determined in Theorem 1.2. It follows that for any $\delta > 0$,

$$U_{\lambda_0}(x) \geq c_0 > 0, \quad V_{\lambda_0}(x) \geq c_0 > 0, \quad x \in \overline{\Sigma_{\lambda_0-\delta} \cap B_{R_0}(0)}.$$

Together with the continuity of $U_\lambda(x)$ and $V_\lambda(x)$ with respect to λ, there exists $\epsilon > 0$, such that for all $\lambda \in [\lambda_0, \lambda_0 + \epsilon)$, we have

$$U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \overline{\Sigma_{\lambda_0-\delta} \cap B_{R_0}(0)}. \tag{3-12}$$

Suppose that (3-11) is not true. By the proofs of Theorems 1.1 and 1.2, we know that if one of $U_\lambda(x)$ and $V_\lambda(x)$ becomes the negative minimum value at some point in Σ_λ, then there exist $\tilde{x}$ and $\bar{x}$ which are the negative minima of $U_\lambda(x)$ and

$V_\lambda(x)$ in Σ_λ respectively. Additionally, by Theorem 1.2, at least one of them lies in $(\Sigma_\lambda \setminus \Sigma_{\lambda_0-\delta}) \cap B_{R_0}(0)$. Here we consider two possibilities.

Case 1: One of the negative minima of $U_\lambda(x)$ and $V_\lambda(x)$ lies in $B_{R_0}(0)$, i.e., in the narrow region $\Sigma_{\lambda_0+\epsilon} \setminus \Sigma_{\lambda_0-\delta}$, and the other is outside of $B_{R_0}(0)$. Without loss of generality, we assume the negative minimum of $U_\lambda(x)$ lies in $B_{R_0}(0)$. From (2-5), we have

$$U_\lambda(\tilde{x}) \geq -cl^\alpha c_{12}(\tilde{x}) V_\lambda(\tilde{x}) \tag{3-13}$$

and

$$\begin{aligned}
0 &\leq F_\beta(v_\lambda(\bar{x})) - F_\beta(v(\bar{x})) + c_{21}(\bar{x}) U_\lambda(\bar{x}) + c_{22}(\bar{x}) V_\lambda(\bar{x}) \\
&\leq \frac{C}{|\bar{x}|^\beta} V_\lambda(\bar{x}) + c_{21}(\bar{x}) U_\lambda(\tilde{x}) \\
&\leq \frac{C}{|\bar{x}|^\beta} V_\lambda(\bar{x}) - cc_{21}(\bar{x}) c_{12}(\tilde{x}) l^\alpha V_\lambda(\tilde{x}) \\
&\leq \frac{C}{|\bar{x}|^\beta} V_\lambda(\bar{x}) - cc_{21}(\bar{x}) c_{12}(\tilde{x}) l^\alpha V_\lambda(\bar{x}) \\
&\leq \frac{C}{|\bar{x}|^\beta} V_\lambda(\bar{x}) (1 - c_{12}(\tilde{x}) l^\alpha c_{21}(\bar{x}) |\bar{x}|^\beta).
\end{aligned}$$

Hence,

$$1 \leq c_{12}(\tilde{x}) l^\alpha c_{21}(\bar{x}) |\bar{x}|^\beta. \tag{3-14}$$

By (1-9), we know that $c_{21}(\bar{x})|\bar{x}|^\beta$ is small for $|\bar{x}|$ sufficiently large. Since $l = \epsilon + \delta$ is very narrow and $c_{12}(\tilde{x})$ is bounded from below in $\Sigma_{\lambda_0+\epsilon} \setminus \Sigma_{\lambda_0-\delta}$, it is easy to see that $c_{12}(\tilde{x}) l^\alpha$ can be small. Consequently,

$$c_{12}(\tilde{x}) l^\alpha c_{21}(\bar{x}) |\bar{x}|^\beta < 1.$$

This is a contradiction with (3-14) and (3-11) is proved.

Case 2: Both of the negative minima of $U_\lambda(x)$ and $V_\lambda(x)$ lie in $B_{R_0}(0)$, i.e., they are all in the narrow region $\Sigma_{\lambda_0+\epsilon} \setminus \Sigma_{\lambda_0-\delta}$.

Recalling (2-4),

$$F_\alpha(u_\lambda(\tilde{x})) - F_\alpha(u(\tilde{x})) + c_{11}(\tilde{x}) U_\lambda(\tilde{x}) \leq \frac{C}{l^\alpha} U_\lambda(\tilde{x}) < 0, \tag{3-15}$$

where $l = \delta + \epsilon$. Together with (1-4), it implies

$$U_\lambda(\tilde{x}) \geq -cc_{12}(\tilde{x}) l^\alpha V_\lambda(\tilde{x}). \tag{3-16}$$

Similarly to (3-15), we have

$$F_\beta(v_\lambda(\bar{x})) - F_\beta(v(\bar{x})) + c_{22}(\bar{x}) V_\lambda(\bar{x}) \leq \frac{C}{l^\beta} V_\lambda(\bar{x}) < 0.$$

Noting (3-16), for l sufficiently small, it gives

$$\begin{aligned}
0 &\le F_\beta(v_\lambda(\bar{x})) - F_\beta(v(\bar{x})) + c_{21}(\bar{x})U_\lambda(\bar{x}) + c_{22}(\bar{x})V_\lambda(\bar{x}) \\
&\le \frac{C}{l^\beta} V_\lambda(\bar{x}) + c_{21}(\bar{x})U_\lambda(\tilde{x}) \\
&\le \frac{C}{l^\beta} V_\lambda(\bar{x}) - cc_{21}(\bar{x})c_{12}(\tilde{x})l^\alpha V_\lambda(\tilde{x}) \\
&\le \frac{C}{l^\beta} V_\lambda(\bar{x}) - cc_{21}(\bar{x})c_{12}(\tilde{x})l^\alpha V_\lambda(\bar{x}) \\
&\le \frac{C}{l^\beta} V_\lambda(\bar{x})(1 - c_{12}(\tilde{x})c_{21}(\bar{x})l^{\alpha+\beta}) < 0.
\end{aligned}$$

This contradiction shows that (3-11) has to be true.

Now we have proved that $U_{\lambda_0}(x) \equiv 0$, $V_{\lambda_0}(x) \equiv 0$, $x \in \Sigma_{\lambda_0}$. Since the x_1-direction can be chosen arbitrarily, we actually indicate that $u(x)$ and $v(x)$ are radically symmetric about some point x^0. Also the monotonicity follows easily from the argument. This completes the proof of Theorem 1.3. □

4. Nonexistence of positive solutions on a half space $\mathbb{R}^n_+$

In this section, we investigate the system (1-2).

Proof of Theorem 1.4. Based on (1-3), from the proof of Lemma 2.1 in [Wang and Yu 2017], one can see that either

$$u(x) > 0,\ v(x) > 0 \quad \text{or} \quad u(x) \equiv 0,\ v(x) \equiv 0 \quad \text{for } x \in \mathbb{R}^n_+,$$

where $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > 0\}$. In fact, assume $u(x) \not\equiv 0$, and there exists $x^0 \in \mathbb{R}^n_+$ such that $u(x^0) = 0$, then

$$\begin{aligned}
F_\alpha(u(x^0)) &= \int_{\mathbb{R}^n} \frac{G(u(x^0) - u(y))}{|x^0 - y|^{n+\alpha}}\,dy = \int_{\mathbb{R}^n} \frac{G(u(x^0) - u(y)) - G(0)}{|x^0 - y|^{n+\alpha}}\,dy \\
&= \int_{\mathbb{R}^n} \frac{G'(\cdot)(u(x^0) - u(y))}{|x^0 - y|^{n+\alpha}}\,dy \le c_0 \int_{\mathbb{R}^n} \frac{-u(y)}{|x^0 - y|^{n+\alpha}}\,dy < 0,
\end{aligned}$$

i.e., $0 \le f(u(x), v(x)) = F_\alpha(u(x)) < 0$, which is impossible. Hence if $u(x)$ or $v(x)$ attains 0 somewhere in $\mathbb{R}^n_+$, then $u(x) = v(x) \equiv 0$, $x \in \mathbb{R}^n$.

Now we always assume that $u(x) > 0$ and $v(x) > 0$ in $\mathbb{R}^n_+$. Let us carry on the method of moving planes to the solution u along the x_n-direction.

Denote

$$T_\lambda = \{x \in \mathbb{R}^n_+ \mid x_n = \lambda, \lambda > 0\}, \qquad \Sigma_\lambda = \{x \in \mathbb{R}^n_+ \mid 0 < x_n < \lambda\}.$$

Let

$$x^\lambda = (x_1, x_2, \ldots, 2\lambda - x_n)$$

be the reflection of x about the plane T_λ, and

$$U_\lambda(x) = u_\lambda(x) - u(x), \quad V_\lambda(x) = v_\lambda(x) - v(x).$$

Using the key inequality (2-2) obtained in the proof of Theorem 1.1, we only need to take $\Sigma = \Sigma_\lambda \cup \mathbb{R}^n_-$, where $\mathbb{R}^n_- = \{x \in \mathbb{R}^n \mid x_n \leq 0\}$.

Step 1: It is obvious that, for $\lambda \leq 0$, we have

$$U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \mathbb{R}^n_-. \tag{4-1}$$

For $\lambda > 0$ sufficiently small, Σ_λ is a narrow region, we have immediately

$$U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda. \tag{4-2}$$

Step 2: Since (4-2) provides a starting point, we move the plane T_λ upward as long as (4-2) holds. Define

$$\lambda_0 = \sup\{\lambda \geq 0 \mid U_\mu(x) \geq 0, V_\mu(x) \geq 0, x \in \Sigma_\mu, \mu \leq \lambda\}.$$

We show that

$$\lambda_0 = \infty. \tag{4-3}$$

Otherwise, if $\lambda_0 < \infty$, we show that the plane T_λ can be moved further up. To be more rigorous, there exists some $\epsilon > 0$, such that, for any $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$,

$$U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda.$$

This is a contradiction with the definition of λ_0. Hence, (4-3) holds.

By using Theorem 1.1, Theorem 1.2, and similar arguments as in Section 3, we can prove that

$$U_{\lambda_0} \equiv 0, \quad V_{\lambda_0} \equiv 0, \quad x \in \Sigma_{\lambda_0}, \quad \lambda_0 = \infty,$$

which implies

$$u(x_1, \dots, x_{n-1}, 2\lambda_0) = u(x_1, \dots, x_{n-1}, 0) = 0,$$
$$v(x_1, \dots, x_{n-1}, 2\lambda_0) = v(x_1, \dots, x_{n-1}, 0) = 0.$$

This is impossible, because we have assumed that $u(x), v(x) > 0$ in $\mathbb{R}^n_+$.

Therefore, (4-3) must be valid and the solutions $u(x), v(x)$ are increasing with respect to x_n. This contradicts (1-13) and completes the proof of Theorem 1.4. □

5. Application to fully nonlinear nonlocal system

In this section, we consider

$$
\begin{aligned}
F_\alpha(u(x)) &= -u^p(x) + v^q(x), && x \in \mathbb{R}^n,\\
F_\beta(v(x)) &= -v^p(x) + u^q(x), && x \in \mathbb{R}^n,\\
u(x), v(x) &> 0, && x \in \mathbb{R}^n.
\end{aligned}
\tag{5-1}
$$

Obviously, (5-1) is a specific case of (1-1) and we have the similar conclusion here.

Theorem 5.1. *Assume that $u(x) \in L_\alpha \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, $v(x) \in L_\beta \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ are positive solutions of system (5-1). Suppose that for some $\gamma_1, \gamma_2 > 0$, $u(x), v(x)$ satisfy the assumption (1-11) and*

$$\min\{(p-1)\gamma_1, (q-1)\gamma_1\} > \alpha, \quad \min\{(p-1)\gamma_2, (q-1)\gamma_2\} > \beta.$$

Then $u(x)$, $v(x)$ must be radially symmetric and monotone decreasing about some point in $\mathbb{R}^n$.

By using Theorem 1.3, we can prove Theorem 5.1 directly. Notice that, if we let $f(u(x), v(x)) = -u^p(x) + v^q(x)$ and $g(u(x), v(x)) = -v^p(x) + u^q(x)$, it is easy to see that f, g satisfy the assumption (1-12). For convenience, we omit the proof of Theorem 5.1 here.

References

[Brändle et al. 2013] C. Brändle, E. Colorado, A. de Pablo, and U. Sánchez, "A concave-convex elliptic problem involving the fractional Laplacian", *Proc. Roy. Soc. Edinburgh Sect. A* **143**:1 (2013), 39–71. MR Zbl

[Caffarelli and Silvestre 2007] L. Caffarelli and L. Silvestre, "An extension problem related to the fractional Laplacian", *Comm. Partial Differential Equations* **32**:7-9 (2007), 1245–1260. MR Zbl

[Caffarelli and Silvestre 2009] L. Caffarelli and L. Silvestre, "Regularity theory for fully nonlinear integro-differential equations", *Comm. Pure Appl. Math.* **62**:5 (2009), 597–638. MR Zbl

[Cao and Chen 2013] L. Cao and W. Chen, "Liouville type theorems for poly-harmonic Navier problems", *Discrete Contin. Dyn. Syst.* **33**:9 (2013), 3937–3955. MR Zbl

[Cao and Dai 2013] L. Cao and Z. Dai, "A Liouville-type theorem for an integral system on a half-space R^n_+", *J. Inequal. Appl.* **2013** (2013), art. id. 37. MR Zbl

[Chen and Zhu 2016] W. Chen and J. Zhu, "Indefinite fractional elliptic problem and Liouville theorems", *J. Differential Equations* **260**:5 (2016), 4758–4785. MR Zbl

[Chen et al. 2005] W. Chen, C. Li, and B. Ou, "Qualitative properties of solutions for an integral equation", *Discrete Contin. Dyn. Syst.* **12**:2 (2005), 347–354. MR Zbl

[Chen et al. 2006] W. Chen, C. Li, and B. Ou, "Classification of solutions for an integral equation", *Comm. Pure Appl. Math.* **59**:3 (2006), 330–343. MR Zbl

[Chen et al. 2015] W. Chen, Y. Fang, and R. Yang, "Liouville theorems involving the fractional Laplacian on a half space", *Adv. Math.* **274** (2015), 167–198. MR Zbl

[Chen et al. 2017a] W. Chen, C. Li, and G. Li, "Maximum principles for a fully nonlinear fractional order equation and symmetry of solutions", *Calc. Var. Partial Differential Equations* **56**:2 (2017), art. id. 29. MR Zbl

[Chen et al. 2017b] W. Chen, C. Li, and Y. Li, "A direct method of moving planes for the fractional Laplacian", *Adv. Math.* **308** (2017), 404–437. MR Zbl

[Cheng 2017] T. Cheng, "Monotonicity and symmetry of solutions to fractional Laplacian equation", *Discrete Contin. Dyn. Syst.* **37**:7 (2017), 3587–3599. MR Zbl

[Cheng et al. 2017a] C. Cheng, Z. Lü, and Y. Lü, "A direct method of moving planes for the system of the fractional Laplacian", *Pacific J. Math.* **290**:2 (2017), 301–320. MR Zbl

[Cheng et al. 2017b] T. Cheng, G. Huang, and C. Li, "The maximum principles for fractional Laplacian equations and their applications", *Commun. Contemp. Math.* **19**:6 (2017), art. id. 1750018. MR Zbl

[Gilbarg and Trudinger 1977] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Grundlehren der Math. Wissenschaften **224**, Springer, 1977. MR Zbl

[Han et al. 2012] X. Han, G. Lu, and J. Zhu, "Characterization of balls in terms of Bessel-potential integral equation", *J. Differential Equations* **252**:2 (2012), 1589–1602. MR Zbl

[Hang 2007] F. Hang, "On the integral systems related to Hardy–Littlewood–Sobolev inequality", *Math. Res. Lett.* **14**:3 (2007), 373–383. MR Zbl

[Hang et al. 2009] F. Hang, X. Wang, and X. Yan, "An integral equation in conformal geometry", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26**:1 (2009), 1–21. MR Zbl

[Lei et al. 2012] Y. Lei, C. Li, and C. Ma, "Asymptotic radial symmetry and growth estimates of positive solutions to weighted Hardy–Littlewood–Sobolev system of integral equations", *Calc. Var. Partial Differential Equations* **45**:1-2 (2012), 43–61. MR Zbl

[Li 2017] J. Li, "Monotonicity and symmetry of fractional Lane–Emden-type equation in the parabolic domain", *Complex Var. Elliptic Equ.* **62**:1 (2017), 135–147. MR Zbl

[Li and Ma 2017] Y. Li and P. Ma, "Symmetry of solutions for a fractional system", *Sci. China Math.* **60**:10 (2017), 1805–1824. MR Zbl

[Li and Zhuo 2010] D. Li and R. Zhuo, "An integral equation on half space", *Proc. Amer. Math. Soc.* **138**:8 (2010), 2779–2791. MR Zbl

[Liu and Ma 2016] B. Liu and L. Ma, "Radial symmetry results for fractional Laplacian systems", *Nonlinear Anal.* **146** (2016), 120–135. MR Zbl

[Lu and Zhu 2011] G. Lu and J. Zhu, "Symmetry and regularity of extremals of an integral equation related to the Hardy–Sobolev inequality", *Calc. Var. Partial Differential Equations* **42**:3-4 (2011), 563–577. MR Zbl

[Lu and Zhu 2012] G. Lu and J. Zhu, "An overdetermined problem in Riesz-potential and fractional Laplacian", *Nonlinear Anal.* **75**:6 (2012), 3036–3048. MR Zbl

[Ma and Chen 2006] L. Ma and D. Chen, "A Liouville type theorem for an integral system", *Commun. Pure Appl. Anal.* **5**:4 (2006), 855–859. MR Zbl

[Ma and Zhao 2010] L. Ma and L. Zhao, "Classification of positive solitary solutions of the nonlinear Choquard equation", *Arch. Ration. Mech. Anal.* **195**:2 (2010), 455–467. MR Zbl

[Wang and Niu 2017] P. Wang and P. Niu, "A direct method of moving planes for a fully nonlinear nonlocal system", *Commun. Pure Appl. Anal.* **16**:5 (2017), 1707–1718. MR Zbl

[Wang and Yu 2017] P. Wang and M. Yu, "Solutions of fully nonlinear nonlocal systems", *J. Math. Anal. Appl.* **450**:2 (2017), 982–995. MR Zbl

[Zhang et al. 2017] L. Zhang, M. Yu, and J. He, "A Liouville theorem for a class of fractional systems in $\mathbb{R}^n_+$", *J. Differential Equations* **263**:9 (2017), 6025–6065. MR Zbl

[Zhuo et al. 2016] R. Zhuo, W. Chen, X. Cui, and Z. Yuan, "Symmetry and non-existence of solutions for a nonlinear system involving the fractional Laplacian", *Discrete Contin. Dyn. Syst.* **36**:2 (2016), 1125–1141. MR Zbl

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