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JUNCHEOL PYO

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# RIGIDITY THEOREMS OF HYPERSURFACES WITH FREE BOUNDARY IN A WEDGE IN A SPACE FORM

JUNCHEOL PYO

*Dedicated to Professor Jaigyoung Choe in honor of his 65th birthday.*

**This paper presents some rigidity results about compact hypersurfaces with free boundary in a wedge in a space form. First, we prove that every compact immersed stable constant mean curvature hypersurface with free boundary in a wedge is part of an intrinsic sphere centered at a point of the edge of the wedge. Second, we show that the same rigidity result holds for a compact embedded constant higher-order mean curvature hypersurface with free boundary in a wedge. Finally, we extend this result to a compact immersed hypersurface with free boundary in a wedge that has the additional property that the ratio of two higher-order mean curvatures is constant.**

**The same conclusions hold for a compact hypersurface with free boundary that lies in a half-space in a space form.**

## 1. Introduction

The set of all points at a given positive intrinsic distance from a fixed point in a manifold will be called an *intrinsic sphere*. Intrinsic spheres in space forms have been characterized in a number of different ways. Among all hypersurfaces of a given volume bounding a domain in a space form, an intrinsic sphere has the least area; that is, it is the boundary of an isoperimetric domain in a space form. Every smooth boundary of an isoperimetric domain is a stable constant mean curvature (CMC) hypersurface. Barbosa and do Carmo [1984] proved that an intrinsic sphere in Euclidean space is the only closed stable immersed CMC hypersurface; Barbosa, do Carmo, and Eschenburg [Barbosa et al. 1988] extended this result to other space forms.

We call a hypersurface a *totally geodesic hypersurface* if all of its intrinsic geodesics are also geodesic curves in the ambient manifold. Totally geodesic hypersurfaces and intrinsic spheres are the only totally umbilic hypersurfaces. The

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mean curvature as well as the higher-order mean curvature are all constant on such a surface. Alexandrov [1962] proved that a closed embedded CMC hypersurface in Euclidean space must be an intrinsic sphere. This result has also been generalized to hyperbolic space and the open hemisphere. Ros [1988] generalized Alexandrov's results to a closed embedded hypersurface in Euclidean space of constant scalar curvature. Using the so-called Alexandrov reflection method, Korevaar [1988] gave another proof of Ros' result and extended it to the hyperbolic space and the open hemisphere. Ros [1987] generalized Alexandrov's result to hypersurfaces of constant higher-order mean curvature in Euclidean space; Montiel and Ros [1991] settled Alexandrov's result for other space forms.

Koh and Lee [2001] characterized intrinsic spheres in a space form in terms of the ratio of two higher-order mean curvatures. They proved that a closed hypersurface in a space form is an intrinsic sphere if it has constant ratio $H_r/H_l$, where $l<r$, and nonvanishing $H_l$, where $H_r$ is the $r$-th order mean curvature of the hypersurface.

It is natural to extend the above results for closed surfaces to compact surfaces with nonempty boundary in a domain. When the domain is a ball, Nitsche [1985] showed that an immersed disk-type CMC surface in a ball which intersects the boundary sphere orthogonally is part of a sphere, and Souam [1997] extended Nitsche's result to other space forms. Presently, only partial results are known for higher-dimensional stable CMC hypersurfaces in a ball [Ros and Vergasta 1995; Souam 1997].

Recently, when the domain is a wedge in Euclidean space, López [2014] showed that a compact connected CMC surface orthogonally meeting the boundary of the wedge in $\mathbb{R}^3$ is part of sphere if it is either stable or embedded. In this paper, we extend López's results both to other space forms and to a higher-dimensional case. We now establish some notation that will be used throughout the remainder of this paper.

Let $\overline{M}^{n+1}(k)$ be the $(n+1)$-dimensional simply connected space form of constant sectional curvature $k$. By changing the metric conformally we may assume that $k=0$ or $k=\pm 1$; that is, $\overline{M}^{n+1}(0)=\mathbb{R}^{n+1}$, $\overline{M}^{n+1}(-1)=\mathbb{H}^{n+1}$, and $\overline{M}^{n+1}(1)=\mathbb{S}^{n+1}$. When $k=1$, we consider the open hemisphere $\mathbb{S}^{n+1}_{+}$ rather than the whole sphere.

Let $\Pi_1$ and $\Pi_2$ be two totally geodesics in $\overline{M}^{n+1}(k)$ which intersect. By $\Pi_1$ and $\Pi_2$, $\overline{M}^{n+1}(k)$ is divided into four connected domains. Choosing any of the four domains and then taking closure of the domain, we have a wedge-shaped closed connected domain $W\subset\overline{M}^{n+1}(k)$. For simplicity, we refer to $W$ as a *wedge*. Let $M^n$ be an $n$-dimensional compact connected orientable Riemannian manifold with nonempty boundary $\partial M$. Let $\psi : M\to\overline{M}^{n+1}(k)$ be an isometric immersion, and we identify $M$ with $\psi(M)$. In this paper, we consider a hypersurface $M$ in a wedge $W$, which means that there exists an immersion $\psi : M\to W$ such that $\psi(\mathrm{int}(M))\subset\mathrm{int}(W)$ and $\psi(\partial M)\subset\partial W$, where $\partial W$ is the boundary of $W$ and $\mathrm{int}(A)$ denotes the interior of a set $A$. The $(n-1)$-dimensional totally geodesic

$E = \Pi_1 \cap \Pi_2$ is called the *edge* of $W$. Since we consider the open hemisphere $\mathbb{S}^{n+1}_+$ rather than the whole sphere, for $k = 1$, the edge is a connected totally geodesic. In the other cases ($k = 0, -1$), clearly the edge is a connected one. Throughout this paper, we assume that $\partial M \cap (\Pi_1 \setminus E) \neq \varnothing \neq \partial M \cap (\Pi_2 \setminus E)$ and all hypersurfaces are connected. We call $M$ a *hypersurface with free boundary* in $W$ when $M$ intersects $\partial W$ orthogonally along $\partial M$.

First, in Section 3, we prove:

> *A compact immersed stable CMC hypersurface with free boundary in a wedge in a space form is part of an intrinsic sphere centered at a point of the edge of the wedge.*

This is a generalization of Barbosa, do Carmo, and Eschenburg's results [Barbosa and do Carmo 1984; Barbosa et al. 1988] for hypersurfaces with free boundary in a wedge. A CMC hypersurface $M$ in a wedge $W$ is called a *capillary* hypersurface if $M$ meets the boundary of $W$ with a constant angle along $\partial M$. McCuan [1997] and Park [2005] showed that a capillary surface in a wedge that is topologically an annulus is part of a sphere. Recently, Choe and Koiso [2016] proved that a compact capillary hypersurface in a wedge that is disjoint from the edge is part of an intrinsic sphere if the boundary of the capillary surface is embedded for the surface case, or if the boundary is convex for the higher-dimensional case. More results and more physical explanation for capillary surfaces can be found in [Concus and Finn 1998; Concus et al. 2001; Finn 1986].

Ros [1987] obtained an interesting inequality for closed hypersurfaces of positive mean curvature. When the mean curvature is a positive constant, a linear isoperimetric inequality for closed hypersurfaces of nonvanishing mean curvature is satisfied. This inequality was extended to other space forms by Brendle [2013] and Qiu and Xia [2015] in different ways. In Section 4, we extend these results to compact hypersurfaces with nonempty boundary. Besides Reilly's formula, somewhat surprisingly, many geometric and rigidity results can be deduced from the so-called Minkowski formula; see, for example, [Koh 1998; Montiel and Ros 1991; Ros 1987]. Montiel and Ros [1991] extended the Minkowski formula in space forms. In Section 5, we extend the Minkowski formula for closed hypersurfaces to hypersurfaces with free boundary in a wedge. Choe and Park [2011] extended the Minkowski formula for hypersurfaces with free boundary in a cone in Euclidean space. Second, in Section 6, we extend the result of [Montiel and Ros 1991] to hypersurfaces with free boundary in a wedge using a Ros-type inequality and a Minkowski-type formula for compact hypersurfaces with free boundary:

> *A compact embedded constant higher-order mean curvature hypersurface with free boundary in a wedge in a space form is part of an intrinsic sphere centered at a point of the edge of the wedge.*

In the last section, we extend the results of [Koh and Lee 2001] to hypersurfaces with free boundary. In this case, the same rigidity holds when the hypersurface lies in the wedge near the boundary. More precisely, we prove:

> *If a compact immersed hypersurface has nonempty boundary such that near the boundary the hypersurface meets the boundary of a wedge orthogonally along the boundary, then it is part of an intrinsic sphere if $H_l$ does not vanish and the ratio $H_r/H_l$ is constant for $0 < l < r$.*

Note that there are no a priori restrictions on the topology of the hypersurface $M$; that is, it may have some genus or boundary components. Also note that the proof works in the case that the boundary $\partial M$ lies in a hyperplane, that is, we obtain similar results for $M$ in a half-space in a space form.

## 2. Preliminaries

For $\overline{M}^{n+1}(k)$, when $k=-1$, use the hyperboloid model and when $k=1$, take the usual embedding to $\mathbb{R}^{n+2}$. More precisely, let $\mathbb{L}^{n+2}$ be the $(n+2)$-dimensional Lorentz–Minkowski space with the Lorentzian metric

$$\langle x, y\rangle = x_1 y_1 + \cdots + x_{n+1} y_{n+1} - x_{n+2} y_{n+2}.$$

Then, $\overline{M}^{n+1}(-1) \subset \mathbb{L}^{n+2}$ is defined as

$$\{x \in \mathbb{L}^{n+2} \mid |x|^2 = -1, x_{n+2} \geq 1\}.$$

Let $\psi : M \to \overline{M}^{n+1}(k)$ be an immersion. If $k=-1$, we regard this immersion as $\psi : M \to \mathbb{L}^{n+2}$, and if $k=1$ we regard it as $\psi : M \to \mathbb{R}^{n+2}$.

Denote by $\bar{\nabla}$, $\bar{\Delta}$, and $\bar{\nabla}^2$ the gradient, the Laplacian, and the Hessian on $\overline{M}^{n+1}(k)$, respectively, and denote by $\nabla$, $\Delta$, $N$, $\sigma$, and $H$ the gradient, the Laplacian, the unit outward normal vector field whenever this makes sense, the second fundamental form, and the normalized mean curvature on $M \subset \overline{M}^{n+1}(k)$, respectively. Let $dV$, $dA$, and $ds$ be canonical measures of $\overline{M}^{n+1}(k)$, $M$, and $\partial M$, respectively.

We recall the formal definition of stability of CMC hypersurfaces; see [Barbosa et al. 1988; Ros and Vergasta 1995; Souam 1997] for further details. Let $W \subset \overline{M}^{n+1}(k)$ be a wedge. A CMC hypersurface with free boundary in $W$ arises from a critical point of the area functional for all volume-preserving variations in $W$. More precisely, let $\psi : M \to W \subset \overline{M}^{n+1}(k)$ be an immersion such that $\psi(\mathrm{int}(M)) \subset \mathrm{int}(W)$ and $\psi(\partial M) \subset \partial W$. A variation of $\psi$ is a smooth family of proper hypersurfaces in $W$ given by a 1-parameter family of immersions $\Psi_t : M \times (-\epsilon, \epsilon) \to W$ with $\Psi_0 = \psi$.

The area function is defined by

$$A(t) = \int_M dA_t,$$

where $dA_t$ is the volume form of $\Psi_t(M)$. The volume function enclosing the space between $\psi(M)$ and $\Psi_t(M)$ is defined by

$$V(t) = \int_{M\times[0,t]} \Psi^* \, dV,$$

where $dV$ is the volume form of $\overline{M}^{n+1}(k)$. The variation is said to be *volume-preserving* if $V(t) = V(0)$ for all $t$.

With the associated variational vector field $Y = (\partial\Psi/\partial t)|_{t=0}$, the first variation formulas of the area and the volume are

$$A'(0) = -n \int_M H f \, dA + \int_{\partial M} \langle Y, \nu\rangle \, ds, \tag{1}$$

$$V'(0) = \int_M f \, dA, \tag{2}$$

where $f = \langle Y, N\rangle$. The variation is called *normal* if $Y = fN$, and *admissible* if $\Psi_t(\mathrm{int}(M)) \subset \mathrm{int}(W)$ and $\Psi_t(\partial M) \subset \partial W$ for all $t$.

From (1) and (2), $\psi$ is a critical point of the area functional $A(t)$ for all volume-preserving and admissible variations if and only if $\psi(M) \subset W$ is a CMC hypersurface with free boundary.

By a standard computation, the second variation formula of any admissible volume-preserving normal variation is

$$A''(0) = -\int_M (f\,\Delta f + (|\sigma|^2 + kn) f^2) \, dA + \int_{\partial M} \left( f \frac{\partial f}{\partial \nu} - II(N, N) f^2 \right) ds,$$

where $II$ is the second fundamental form of $\partial W$ in $\overline{M}^{n+1}(k)$.

A stationary immersion $\psi : M \to W$ is called *stable* if $A''(0) \geq 0$ for any admissible volume-preserving normal variation of $\psi$. Let $\mathcal{F} = \left\{ f \in H^1(M) \,\middle|\, \int_M f \, dA = 0 \right\}$, where $H^1(M)$ denotes the first Sobolev space of $M$, and we define the *index form* $\mathcal{I}$ of $\psi$ as the symmetric bilinear form on $H^1(M)$ given by

$$\mathcal{I}(f, g) = \int_M (\langle\nabla f, \nabla g\rangle - (|\sigma|^2 + kn) f g) \, dA - \int_{\partial M} II(N, N) f g \, ds.$$

It follows that the stationary immersion $\psi$ is stable if and only if $\mathcal{I}(f, f) \geq 0$ for any $f \in \mathcal{F}$.

## 3. Stable CMC surfaces with free boundary in a wedge

**Theorem 1.** *Let $W$ be a wedge in $\overline{M}^{n+1}(k)$. If $M$ is a compact immersed stable CMC ($H \neq 0$) hypersurface with free boundary in $W$, then it is part of an intrinsic sphere centered at a point of the edge of $W$.*

*Proof.* Suppose the wedge $W$ is determined by $\Pi_1$ and $\Pi_2$ and the edge $E$ is given by $E = \Pi_1 \cap \Pi_2$. By an isometry in $\overline{M}^{n+1}(k)$, we assume that $E$ contains the origin of $\mathbb{R}^{n+1}$ for $k=0$ case, the north pole $(0,\dots,0,1)\in\mathbb{R}^{n+2}$ for $k=1$ case and the point $(0,\dots,0,1)\in\mathbb{L}^{n+2}$ for $k=-1$ case. Let $\eta_i,\ i=1,2$, be the unit normal vector of $\Pi_i,\ i=1,2$, outward-pointing with respect to $W$. Let $N$ be the unit normal vector field of $M$. Let $\nu$ be the outward unit conormal vector field along $\partial M$ which means that $\nu$ is tangential to $M$ and normal to $\partial M$. The free boundary condition implies that $\nu=\eta_i$ on $\partial M\cap\Pi_i,\ i=1,2$.

In $\mathbb{R}^{n+1}$, $\mathbb{H}^{n+1}$, and $\mathbb{S}^{n+1}_+$, the only totally umbilic hypersurfaces are the geodesic hypersurfaces and the intrinsic spheres; see Chapter 7 of [Spivak 1975]. Since $M$ is assumed to satisfy $\partial M\cap\Pi_1\neq\varnothing\neq\partial M\cap\Pi_2$, the only possibility to be a compact surface with free boundary in a wedge is that $M$ is part of an intrinsic sphere centered at a point on the edge $E\subset W$. So, we claim that $M$ is totally umbilic. Proving this claim completes the proof.

Case: $k=0$. Let $h=\langle\psi,N\rangle$ be the support function of $\psi$. From a direct computation in [Barbosa and do Carmo 1984, Lemma 3.5], we have

$$\Delta h = nH-|\sigma|^2h. \tag{3}$$

Since $\Pi_1$ and $\Pi_2$ are totally geodesics and $M$ intersects $\partial W$ orthogonally along $\partial M$, $\nu$ is a principal direction of $\psi$ along $\partial M$. More precisely, it follows that, for any tangent vector field $X$ of $\partial M$, we have

$$\langle\bar\nabla_\nu N, X\rangle = -\langle\sigma(X,\nu),N\rangle = -\langle\bar\nabla_X\nu,N\rangle=0,$$

where the last equality follows from the fact that $\nu$ is constant on $\partial M$. Hence, for a function $\lambda$,

$$\frac{\partial h}{\partial\nu}=\langle\nu,N\rangle+\langle\psi,\bar\nabla_\nu N\rangle=\langle\nu,N\rangle+\langle\psi,\lambda\nu\rangle=0. \tag{4}$$

Integrating (3) on $M$ and applying Stokes' theorem, by (4) we have

$$\int_M nH-|\sigma|^2h\,dA=0. \tag{5}$$

From a direct computation, we have

$$\Delta|\psi|^2=2n(1-H\langle\psi,N\rangle);$$

by integrating on $M$ and applying Stokes' theorem, we obtain

$$\int_M(1-H\langle\psi,N\rangle)\,dA=\frac{1}{n}\int_{\partial M}\langle\psi,\nu\rangle\,ds.$$

Since $E$ contains the origin of $\mathbb{R}^{n+1}$ and $\nu=\eta_i$ on $\Pi_i,\ i=1,2$, respectively, $\langle\psi,\nu\rangle=0$ on $\partial M$. So, for a hypersurface with free boundary, we also have the Minkowski-type formula

$$\int_M(1-H\langle\psi,N\rangle)\,dA=0.$$

Set $u = 1 - Hh,\ u \in \mathcal{F}$. Since $\partial u/\partial \nu = 0$ by (4) and $II \equiv 0$ on $\partial W$, the second variation formula becomes

$$A''(0) = -\int_M (u\,\Delta u + |\sigma|^2 u^2)\,dA.$$

From a direct computation using (3), $u\,\Delta u + |\sigma|^2 u^2 = u(|\sigma|^2 - nH^2)$. Thus

$$\begin{aligned}
0 \le \mathcal{I}(u,u) &= -\int_M (u\,\Delta u + |\sigma|^2 u^2)\,dA && \text{(by stability condition)}\\
&= -\int_M ((1-Hh)(|\sigma|^2 - nH^2))\,dA\\
&= -\int_M (|\sigma|^2 - nH^2)\,dA + \int_M nH^2\,dA - \int_M nH^3 h\,dA && \text{(by (5))}\\
&= -\int_M (|\sigma|^2 - nH^2)\,dA + nH^2 \int_M u\,dA\\
&= -\int_M (|\sigma|^2 - nH^2)\,dA \le 0 && \text{(by } nH^2 \le |\sigma|^2\text{)}.
\end{aligned}$$

It follows that $|\sigma|^2 = nH^2$ on $M$; that is, all points of $M$ are umbilic.

From now on we consider the case $k \neq 0$. We first recall the following identities [Barbosa et al. 1988, Lemma 3.3]:

$$\Delta\psi = -nHN - kn\psi, \tag{6}$$

$$\Delta N = -|\sigma|^2 N - knH\psi. \tag{7}$$

Case: $k = 1$. Let $\overline{\psi} = \int_M \psi\,dA$ and $\overline{N} = \int_M N\,dA$. We claim that $\overline{N}$ belongs to the vector space spanned by $\{\overline{\psi}, \eta_1, \eta_2\}$.

Integrating (6) and applying Stokes' theorem, we obtain

$$\begin{aligned}
-nH \int_M N\,dA &= n\int_M \psi\,dA + \int_M \Delta\psi\,dA\\
&= n\overline{\psi} + \mathrm{Vol}(\partial M \cap \Pi_1)\eta_1 + \mathrm{Vol}(\partial M \cap \Pi_2)\eta_2.
\end{aligned}$$

Therefore $\overline{N}$ is spanned by $\{\overline{\psi}, \eta_1, \eta_2\}$, completing the claim.

Now, choose $n-1$ vectors $\{v_1, \dots, v_{n-1}\}$ in $\mathbb{R}^{n+2}$ such that

$$\langle \overline{\psi}, v_i\rangle = \langle \eta_1, v_i\rangle = \langle \eta_2, v_i\rangle = 0, \quad i = 1, \dots, n-1.$$

Clearly, $\langle \overline{N}, v_i\rangle = 0$ for $i = 1, \dots, n-1$.

For each $i = 1, \dots, n-1$, define $f_i = \langle \psi, v_i\rangle$ and $g_i = \langle N, v_i\rangle$. Since $\langle \overline{\psi}, v_i\rangle = \langle \overline{N}, v_i\rangle = 0,\ i = 1, \dots, n-1$, we have $\int_M f_i\,dA = \int_M g_i\,dA = 0$.

From (6) and (7), for each $i = 1, \dots, n-1$, we deduce

$$\Delta f_i + nf_i = -nHg_i, \tag{8}$$

$$\Delta g_i + |\sigma|^2 g_i = -nHf_i. \tag{9}$$

Recall $\nu$ is the outward unit conormal vector field along $\partial M$ and $\eta_i = \nu$ on $\partial M \cap \Pi_i$ for $i = 1, 2$. Along the boundary $\partial M$,

$$\frac{\partial f_i}{\partial \nu} = \frac{\partial}{\partial \nu}\langle \psi, v_i\rangle = \langle \nu, v_i\rangle = 0, \quad i = 1, \dots, n-1. \tag{10}$$

Since $\nu$ is a principal direction of $\psi$ along $\partial M$, for a function $\lambda$,

$$\frac{\partial g_i}{\partial \nu} = \frac{\partial}{\partial \nu}\langle N, v_i\rangle = \langle \bar{\nabla}_\nu N, v_i\rangle = \langle \lambda\nu, v_i\rangle = 0, \quad i = 1, \dots, n-1. \tag{11}$$

By combining (8)–(11), for each $i = 1, \dots, n-1$, the index form is

$$\mathcal{I}(f_i, f_i) = n\int_M H f_i g_i \, dA - \int_M |\sigma|^2 f_i^2 \, dA, \tag{12}$$

$$\mathcal{I}(g_i, g_i) = n\int_M H f_i g_i \, dA - n\int_M g_i^2 \, dA, \tag{13}$$

and summing up,

$$\begin{aligned}
0 &\le \sum_{i=1}^{n-1} \mathcal{I}(f_i, f_i) + \mathcal{I}(g_i, g_i) && \text{(by stability condition)}\\
&= -\sum_{i=1}^{n-1}\left(\int_M (|\sigma|^2 f_i^2 - 2nHf_i g_i + n g_i^2)\, dA\right)\\
&\le -n\sum_{i=1}^{n-1}\left(\int_M (H^2 f_i^2 - 2Hf_i g_i + g_i^2)\, dA\right) && \text{(by } nH^2 \le |\sigma|^2\text{)} \qquad (14)\\
&= -n\sum_{i=1}^{n-1}\int_M (Hf_i - g_i)^2\, dA \le 0.
\end{aligned}$$

Since inequality (14) turns to equality, we have

$$\sum_{i=1}^{n-1}\int_M (|\sigma|^2 - nH^2) f_i^2 \, dA = 0, \tag{15}$$

and by $nH^2 \le |\sigma|^2$ again, we obtain

$$(|\sigma|^2 - nH^2)\left(\sum_{i=1}^{n-1} f_i^2\right) = 0 \quad \text{on } M. \tag{16}$$

For each $i = 1, \dots, n-1$, the zero set of $f_i$ in $M$ is the set of points that belong to $M$ and the hyperplane $\{x \in \mathbb{R}^{n+2} \mid \langle x, v_i\rangle = 0\}$, so, the zero set of $\sum_{i=1}^{n-1} f_i^2$ in $M$ is the set of points that belong to $M$ and the three-dimensional subspace which is orthogonal to $\{v_i \mid i = 1, \dots, n-1\}$.

For $n \ge 3$, the zero set of $\sum_{i=1}^{n-1} f_i^2$ in $M$ has measure zero in $M$. From (15), $|\sigma|^2 = nH^2$ on $M$; that is, $M$ is totally umbilic. For the surfaces case ($n = 2$), we use the reductio ad absurdum argument. Suppose $M$ is not totally umbilic, then the umbilic points are isolated by the holomorphic Hopf differential. By (15), $f_1 \equiv 0$ on $M$, and it follows that $M$ is a surface in a three-dimensional subspace that is orthogonal to $v_1$ and hence it is totally geodesic, a contradiction. Therefore, $M$ is totally umbilic. This completes the claim when $k = 1$.

Case: $k = -1$. Let $v = (0, \dots, 1) \in E$. Define $f = \langle \psi, v \rangle$ and $g = \langle N, v \rangle$. By (6) and (7), a direct computation yields

$$\tag{17} \tfrac{1}{2}\Delta f^2 = f\,\Delta f + |\nabla f|^2 = -nHfg + nf^2 + |\nabla f|^2,$$

$$\tag{18} \tfrac{1}{2}\Delta g^2 = g\,\Delta g + |\nabla g|^2 = -|\sigma|^2 g^2 + nHfg + |\nabla g|^2,$$

$$\tag{19} \begin{aligned} \Delta(fg) &= f\,\Delta g + g\,\Delta f + 2\langle \nabla f, \nabla g\rangle \\ &= -|\sigma|^2 fg + nHf^2 - nHg^2 + nfg + 2\langle \nabla f, \nabla g\rangle, \end{aligned}$$

and

$$\tfrac{1}{2}H^2\,\Delta f^2 - H\,\Delta(fg) + \tfrac{1}{2}\Delta g^2 = |H\nabla f - \nabla g|^2 - (|\sigma|^2 - nH^2)(g^2 - Hfg).$$

Recall $\eta_i = \nu$ on $\partial M \cap \Pi_i,\ i = 1, 2$. Along the boundary $\partial M$,

$$\tag{20} \frac{\partial f}{\partial \nu} = \frac{\partial}{\partial \nu}\langle \psi, v\rangle = \langle \nu, v\rangle = 0.$$

Similar to the case when $k = 0$, $\nu$ is a principal direction of $\psi$ along $\partial M$. Hence,

$$\tag{21} \frac{\partial g}{\partial \nu} = \frac{\partial}{\partial \nu}\langle N, v\rangle = \langle \bar{\nabla}_\nu N, v\rangle = 0.$$

From (20) and (21), we get $\int_M \frac{1}{2}H^2\,\Delta f^2 - H\,\Delta(fg) + \frac{1}{2}\Delta g^2\,dA = 0$, and hence

$$\tag{22} \int_M (|\sigma|^2 - nH^2)(g^2 - Hfg)\,dA = \int_M |H\nabla f - \nabla g|^2\,dA.$$

Define $u = Hg - f$. Since $u = Hg - f = -\frac{1}{n}\Delta f$ and by Stokes' theorem,

$$\tag{23} \int_M u\,dA = -\frac{1}{n}\int_{\partial M} \langle \nu, v\rangle\,ds = 0.$$

From (22) and (23),

$$\tag{24} \begin{aligned} \mathcal{I}(u, u) &= \int_M (|\sigma|^2 - nH^2)(Hfg - f^2)\,dA \\ &= \int_M (|\sigma|^2 - nH^2)(g^2 - f^2)\,dA - \int_M |H\nabla f - \nabla g|^2\,dA. \end{aligned}$$

To simplify computations, we choose an orthonormal frame $\{e_A \mid A = 0, \dots, n+1\}$ around a point $\psi(p),\ p \in M$, such that $e_0 = \psi,\ e_{n+1} = N$ and $e_1, \dots, e_n$ are tangential to $\psi(M)$.

With this frame,

$$v=-\langle\psi,v\rangle\psi+\langle N,v\rangle N+\sum_{i=1}^{n}\langle e_i,v\rangle e_i,$$

and

$$\nabla f=\sum_{i=1}^{n}\langle e_i,v\rangle e_i.$$

It follows that

$$\begin{aligned}-1=\langle v,v\rangle&=-\langle\psi,v\rangle^2+\langle N,v\rangle^2+\sum_{i=1}^{n}\langle e_i,v\rangle^2\\&=-f^2+g^2+|\nabla f|^2.\end{aligned}$$

Hence,

$$g^2-f^2=-(1+|\nabla f|^2). \tag{25}$$

By (24) and (25),

$$\begin{aligned}0&\le\mathcal{I}(u,u) &&\text{(by stability condition)}\\&=-\int_M(|\sigma|^2-nH^2)(1+|\nabla f|^2)\,dA-\int_M|H\nabla f-\nabla g|^2\,dA\\&\le 0 &&\text{(from } nH^2\le|\sigma|^2\text{)},\end{aligned}$$

that is, all points of $M$ are umbilic, and hence, the conclusion for the case $k=-1$ follows. □

Observe that the proof also holds when the boundary $\partial M$ lies in a hyperplane. This gives rise to the following theorem.

**Theorem 2.** *Let $\mathcal{H}$ be a half-space in $\overline{M}^{n+1}(k)$ determined by a hyperplane $P$. Let $M$ be a compact immersed stable CMC hypersurface with free boundary in $\mathcal{H}$. Then $M$ is an intrinsic hemisphere centered at a point of $P$.*

## 4. Ros-type inequality

In this section, we extend the Ros-type inequality for closed hypersurfaces to compact hypersurfaces with free boundary in a wedge.

**Theorem 3.** *Let $W\subset\overline{M}^{n+1}(k)$ be a wedge, and $E$ be the edge of $W$. Let $M$ be a compact embedded hypersurface with free boundary in $W$. Let $\Omega$ be the compact domain enclosed by $M$ and $\partial W$. Defining $r(x)=\mathrm{dist}_{\overline{M}^{n+1}(k)}(x,v)$ for a fixed point $v\in E$,*

$$V_k(x)=\begin{cases}1 & \text{if } k=0,\\ \cos r(x) & \text{if } k=1,\\ \cosh r(x) & \text{if } k=-1.\end{cases}$$

*If the mean curvature $H$ is positive on $M$, then*

$$\int_M \frac{V_k}{H}\,dA \geq (n+1)\int_\Omega V_k\,dV, \tag{26}$$

*and equality holds if and only if $M$ is part of an intrinsic sphere.*

*Proof.* Take $\Omega_\epsilon \subset \Omega$ to be a domain with a smooth boundary obtained from $\Omega$ by rounding off the singular part of $\partial\Omega$ in a small distance $\epsilon > 0$. Let $N$ be the outward unit normal vector field of $\partial\Omega$; it is the same one on $M$ as in the previous section.

From a direct computation, $\bar{\nabla}^2 V_k = -kV_k g$, where $g$ is the metric of $\overline{M}^{n+1}(k)$. For any smooth function $f$ on $\Omega_\epsilon$, the Reilly-type formula is given by

$$\begin{aligned}\int_{\Omega_\epsilon} V_k\Big((\bar{\Delta} f + k(n+1)f)^2 - |\bar{\nabla}^2 f + kfg|^2\Big)\,dV \\ = \int_{\partial\Omega_\epsilon} V_k(2u\,\Delta z + nHu^2 + \sigma(\nabla z, \nabla z) + 2nkuz)\,dA \\ + \int_{\partial\Omega_\epsilon} \bar{\nabla}_N V_k(|\nabla z|^2 - nkz^2)\,dA,\end{aligned} \tag{27}$$

where $z = f|_M$ and $u = \bar{\nabla}_N f$. Equation (27) is a particular case of the general Reilly-type formula in a Riemannian manifold; see [Qiu and Xia 2015, Theorem 1.1].

<u>Case: $k = 0$</u>. Let $f : \Omega_\epsilon \to \mathbb{R}$ be the solution to the mixed boundary value problem

$$\begin{cases} \bar{\Delta} f = 1 & \text{in } \Omega_\epsilon, \\ f = 0 & \text{on } \partial\Omega_\epsilon \setminus \partial W, \\ u = \partial f/\partial N = 0 & \text{on } \partial\Omega_\epsilon \cap \partial W. \end{cases}$$

Equation (27) becomes the classical Reilly formula

$$\int_{\Omega_\epsilon} ((\bar{\Delta} f)^2 - |\bar{\nabla}^2 f|^2)\,dV = \int_{\partial\Omega_\epsilon\setminus\partial W} nHu^2\,dA + \int_{\partial\Omega_\epsilon\cap\partial W} \sigma(\nabla z, \nabla z)\,dA. \tag{28}$$

Since $\partial W$ is composed of part of a totally geodesic, $\sigma \equiv 0$ on $\partial W$. From the Cauchy–Schwarz inequality, (28) becomes

$$\frac{\mathrm{Vol}(\Omega_\epsilon)}{n+1} \geq \int_{\partial\Omega_\epsilon\setminus\partial W} Hu^2\,dA. \tag{29}$$

On the other hand,

$$\begin{aligned}(\mathrm{Vol}(\Omega_\epsilon))^2 = \Big(\int_{\Omega_\epsilon} \bar{\Delta} f\,dV\Big)^2 = \Big(\int_{\partial\Omega_\epsilon\setminus\partial W} u\,dA\Big)^2 \\ \leq \int_{\partial\Omega_\epsilon\setminus\partial W} Hu^2\,dA \int_{\partial\Omega_\epsilon\setminus\partial W} \frac{1}{H}\,dA \leq \frac{\mathrm{Vol}(\Omega_\epsilon)}{n+1}\int_{\partial\Omega_\epsilon\setminus\partial W}\frac{1}{H}\,dA,\end{aligned} \tag{30}$$

where the first inequality comes from the Hölder inequality and the second inequality is a consequence of (29). Therefore, letting $\epsilon \to 0$, we obtain (26).

When equality occurs, the Cauchy–Schwarz inequality implies that the Hessian $\bar{\nabla}^2 f$ is proportional to the identity matrix. Because $\bar{\Delta} f = 1$ on $\Omega$, $\bar{\nabla}^2 f = \frac{1}{n} g$ in $\Omega$. With $f = 0$ on $M$, the conclusion follows from the Obata-type result that $M$ is part of an intrinsic sphere. This completes the proof when $k = 0$; see [Reilly 1980, Theorem B].

Case: $k \neq 0$. Let $f : \Omega_\epsilon \to \mathbb{R}$ be the solution to the mixed boundary value problem

$$\begin{cases} \bar{\Delta} f + k(n+1) f = 1 & \text{in } \Omega_\epsilon, \\ f = 0 & \text{on } \partial\Omega_\epsilon \setminus \partial W, \\ u = \partial f / \partial N = 0 & \text{on } \partial\Omega_\epsilon \cap \partial W. \end{cases} \tag{31}$$

From the Cauchy–Schwarz inequality,

$$\frac{n}{n+1} \int_{\Omega_\epsilon} V_k (\bar{\Delta} f + k(n+1) f)^2 \, dV \geq \int_{\Omega_\epsilon} V_k ((\bar{\Delta} f + k(n+1) f)^2 - |\bar{\nabla}^2 f + k f g|^2) \, dV. \tag{32}$$

We deal with $\partial\Omega_\epsilon$ in two parts, $\partial\Omega_\epsilon \setminus \partial W$ and $\partial\Omega_\epsilon \cap \partial W$.

On $\partial\Omega_\epsilon \setminus \partial W$, $z = f|_{\partial\Omega_\epsilon \setminus \partial W} = 0$, and

$$\int_{\partial\Omega_\epsilon \setminus \partial W} V_k (2u \, \Delta z + nHu^2 + \sigma(\nabla z, \nabla z) + 2nkuz) \, dA + \int_{\partial\Omega_\epsilon \setminus \partial W} \bar{\nabla}_N V_k (|\nabla z|^2 - nkz^2) \, dA = \int_{\partial\Omega_\epsilon \setminus \partial W} n V_k H u^2 \, dA. \tag{33}$$

On $\partial\Omega_\epsilon \cap \partial W$, $u = 0$. Since $\partial W$ is part of a totally geodesic, $\sigma(\nabla z, \nabla z) = 0$. Since $N = \eta_i$ on $\Pi_i$, $i = 1, 2$, and $\bar{\nabla} r(x) \subset \partial W$, we have $V_k(x) = \cos r(x)$ and $\bar{\nabla}_N V_k = -\sin r(x) g(\bar{\nabla} r(x), N) = 0$ or $V_k(x) = \cosh r(x)$ and $\bar{\nabla}_N V_k = -\sinh r(x) g(\bar{\nabla} r(x), N) = 0$ on $\partial W$. Then, we obtain

$$\int_{\partial\Omega_\epsilon \cap \partial W} V_k (2u \, \Delta z + nHu^2 + \sigma(\nabla z, \nabla z) + 2nkuz) \, dA + \int_{\partial\Omega_\epsilon \cap \partial W} \bar{\nabla}_N V_k (|\nabla z|^2 - nkz^2) \, dA = 0. \tag{34}$$

Then, from (31)–(34), we have

$$\frac{1}{n+1} \int_{\Omega_\epsilon} V_k \, dV \geq \int_{\partial\Omega_\epsilon \setminus \partial W} V_k H u^2 \, dA. \tag{35}$$

Because $\bar{\Delta} V_k = -(n+1) k V_k$ and $\bar{\nabla}_N V_k = 0$ on $\partial\Omega_\epsilon \cap \partial W$, the Green's formula implies

$$\int_{\Omega_\epsilon} V_k \, dV = \int_{\partial\Omega_\epsilon \setminus \partial W} V_k u \, dA. \tag{36}$$

On the other hand,

$$\left(\int_{\Omega_\epsilon} V_k\,dV\right)^2 = \left(\int_{\partial\Omega_\epsilon\setminus\partial W} V_k u\,dA\right)^2 \le \int_{\partial\Omega_\epsilon\setminus\partial W} V_k H u^2\,dA \int_{\partial\Omega_\epsilon\setminus\partial W} \frac{V_k}{H}\,dA \le \frac{1}{n+1}\int_{\Omega_\epsilon} V_k\,dV \int_{\partial\Omega_\epsilon\setminus\partial W} \frac{V_k}{H}\,dA, \tag{37}$$

where the first inequality follows from the Hölder inequality and the second inequality follows from (35).

Therefore, letting $\epsilon \to 0$ we obtain (26).

Combining (32) and (35)–(37) and the equality in (26),

$$|\bar\nabla^2 f + kfg|^2 = \frac{1}{n+1}(\bar\Delta f + k(n+1)f)^2.$$

Since $\bar\Delta f + k(n+1)f = 1$, we have

$$\bar\nabla^2\Big(f + \frac{1}{n+1}\Big) = -k\Big(f + \frac{1}{n+1}\Big)g \quad \text{in } \Omega.$$

With $f + 1/(n+1) = 1/(n+1)$ on $M$, the conclusion follows from the Obata-type result [Reilly 1980, Theorem B] that $M$ is part of an intrinsic sphere. □

The above result is counterpart of the Ros-type inequality for closed hypersurfaces in [Brendle 2013, Theorem 3.5]. Qiu and Xia [2015] also gave another proof of a Ros-type inequality for closed hypersurfaces in manifolds which include space forms.

If the boundary of the compact hypersurface lies in a hyperplane of $\overline{M}^{n+1}(k)$ we conclude an analogous result:

**Theorem 4.** *Let $\mathcal{H}$ be a half-space in $\overline{M}^{n+1}(k)$ determined by a hyperplane $P$. Let $M$ be a compact embedded hypersurface with free boundary in $\mathcal{H}$. Let $\Omega$ be the compact domain enclosed by $M$ and $P$. Defining $r(x) = \operatorname{dist}_{\overline{M}^{n+1}(k)}(x, v)$ for a fixed point $v \in P$,*

$$V_k(x) = \begin{cases} 1 & \text{if } k = 0, \\ \cos r(x) & \text{if } k = 1, \\ \cosh r(x) & \text{if } k = -1. \end{cases}$$

*If the mean curvature $H$ is positive on $M$, then*

$$\int_M \frac{V_k}{H}\,dA \ge (n+1)\int_\Omega V_k\,dV,$$

*and equality holds if and only if $M$ is an intrinsic hemisphere centered at a point of $P$.*

## 5. Minkowski-type formula

With the unit normal vector field $N$ of $M$, we denote by $\kappa_i$, $i = 1, \ldots, n$, the principal curvatures of $M$. For any $r = 1, \ldots, n$, the *mean curvature of order* $r$, $H_r$, is defined by the identity

$$P_n(t) := (1+\kappa_1 t)\cdots(1+\kappa_n t) = 1 + \binom{n}{1} H_1 t + \cdots + \binom{n}{n} H_n t^n \tag{38}$$

for any real number $t$. Note that $H_1$ is the normalized mean curvature of $M$, $H_2$ is the scalar curvature of $M$ up to a constant, and $H_n$ is the Gauss–Kronecker curvature of $M$. For convenience, we define $H_0 = 1$.

For higher-order mean curvatures, the following inequalities hold:

**Lemma 5.** *If there is a point of $M$ where all the principal curvatures are positive and $H_r > 0$, $r = 1, \ldots, n$, on $M$, then*:

(i) $H_l > 0$ *if* $l < r$.

(ii) $H_r / H_l \le H_{r-1}/H_{l-1}$ *for any* $l < r$.

(iii) $H_s^{(s-1)/s} \le H_{s-1}$ *and* $H_s^{1/s} \le H_1 = H$, *where equality holds only at umbilic points if* $s > 1$.

*Proof.* For (i) and (iii), see, for example, Lemma 1 of [Montiel and Ros 1991]. For (ii), see, for example, Section 12 of [Beckenbach and Bellman 1961]. □

Besides Reilly's formula, somewhat surprisingly, many geometric and rigidity results can be deduced from the so-called Minkowski formula; see, for example, [Montiel and Ros 1991; Ros 1987]. Montiel and Ros [1991] extended the Minkowski formula in space forms and gave another characterization of an intrinsic sphere. We now extend the Minkowski formula for closed hypersurfaces to hypersurfaces with free boundary in a wedge.

We include the proof of the Minkowski formula for closed hypersurfaces in space forms for the reader's convenience (see [Montiel and Ros 1991] for further details), and then generalize it to hypersurfaces with free boundary.

<u>Case: $k = 0$</u>. From a direct computation, we have

$$\Delta |\psi|^2 = 2n(1 - H\langle \psi, N\rangle). \tag{39}$$

For a real number $t$ close enough to 0, the parallel hypersurface is given by

$$\psi_t = \exp_\psi tN = \psi + tN$$

and this is also an immersion.

If $dA$ and $\kappa_1, \ldots, \kappa_n$ denote the volume form and the principal curvatures of $\psi(M)$, respectively, then the volume form of $\psi_t(M) = M_t$ is given by

$$dA_t = (1+\kappa_1 t)\cdots(1+\kappa_n t)\, dA = P_n(t)\, dA,$$

where $P_n$ is as in (38). From a direct computation, the mean curvature $H(t)$ of $M_t$ is

$$H(t) = \frac{1}{n}\sum_i \frac{\kappa_i}{1+\kappa_i t} = \frac{1}{n}\frac{P_n'(t)}{P_n(t)}. \tag{40}$$

Integrating (39) on $M_t$ gives,

$$\begin{aligned} 0 &= \int_M (1 - H(t)\langle \psi + tN, N\rangle)\, dA_t \\ &= \int_M \Big(P_n(t) - \frac{t}{n}P_n'(t) - \frac{1}{n}P_n'(t)\langle \psi, N\rangle\Big)\, dA, \end{aligned} \tag{41}$$

where the second equality follows from (38) and (40). Because (41) holds for any real variable $t$, all of its coefficients vanish. As a result, we obtain the Minkowski-type identity

$$\int_M H_{r-1} - H_r\langle \psi, N\rangle\, dA = 0, \quad r = 1, \ldots, n. \tag{42}$$

Case: $k \neq 0$. Because of the similarity between $\overline{M}^{n+1}(-1)$ and $\overline{M}^{n+1}(1)$, we focus on $k = -1$. From a direct computation, for any $v \in \mathbb{L}^{n+2}$, we have

$$\Delta\langle \psi, v\rangle = n(\langle \psi, v\rangle - H\langle N, v\rangle), \tag{43}$$

and then, integration on $M$ and applying the Stokes' theorem yield

$$\int_M (\langle \psi, v\rangle - H\langle N, v\rangle)\, dA = 0. \tag{44}$$

For a real number $t$ close enough to 0, the parallel hypersurface is given by

$$\psi_t = \exp_\psi(tN) = \cosh t\,\psi + \sinh t\, N$$

and this is also an immersion.

If $dA$ and $\kappa_1, \ldots, \kappa_n$ denote the volume form and the principal curvatures of $\psi(M)$, respectively, then the volume form of $M_t$ is given by

$$\begin{aligned} dA_t &= (\cosh t + \kappa_1 \sinh t)\cdots(\cosh t + \kappa_n \sinh t)\, dA \\ &= \cosh^n t\, P_n(\tanh t)\, dA, \end{aligned}$$

where $P_n$ is as in (38). From a direct computation, the mean curvature $H(t)$ of $\psi_t$ is

$$H(t) = \frac{n\cosh t \sinh t\, P_n(\tanh t) + P_n'(\tanh t)}{n\cosh^2 t\, P_n(\tanh t)}. \tag{45}$$

Integrating (44) on $M_t$ and using (38) and (45), we have

$$\int_M (nP_n(\tanh t) - \tanh t\, P_n'(\tanh t))\langle \psi, v\rangle - P_n'(\tanh t)\langle N, v\rangle\, dA = 0. \tag{46}$$

Equation (46) holds for any variable $\tanh t$. By comparing its coefficients, we obtain the Minkowski-type identity

$$\int_M H_{r-1}\langle\psi, v\rangle - H_r\langle N, v\rangle\, dA = 0, \quad r = 1, \dots, n.$$

Similarly for the case $k = 1$, we have the following identities:

**Minkowski-type identity** [Montiel and Ros 1991]. *Let $\psi : M \to \overline{M}^{n+1}(k)$ be a closed orientable immersed hypersurface. For any $r = 1, \dots, n$, the following hold*:

(a) *If $k = 0$, then $\int_M H_{r-1} - H_r\langle\psi, N\rangle\, dA = 0$.*

(b) *If $k = -1$, then $\int_M H_{r-1}\langle\psi, v\rangle - H_r\langle N, v\rangle\, dA = 0$ for any $v \in \mathbb{L}^{n+2}$.*

(c) *If $k = 1$, then $\int_M H_{r-1}\langle\psi, v\rangle + H_r\langle N, v\rangle\, dA = 0$ for any $v \in \mathbb{R}^{n+2}$.*

We extend the Minkowski-type identity to immersed hypersurfaces with free boundary in a wedge in a space form.

**Proposition 6.** *Let $W \subset \overline{M}^{n+1}(k)$ be a wedge and $E$ be the edge of $W$. Let $M$ be a compact immersed hypersurface in $\overline{M}^{n+1}(k)$ with $\partial M \subset \partial W$ such that near $\partial M$, $M$ lies inside of $W$ and perpendicular to $\partial W$. Then, for any $r = 1, \dots, n$ we obtain*:

(a) *If $k = 0$, then $\int_M H_{r-1} - H_r\langle\psi, N\rangle\, dA = 0$.*

(b) *If $k = -1$, then $\int_M H_{r-1}\langle\psi, v\rangle - H_r\langle N, v\rangle\, dA = 0$ for any $v \in E$.*

(c) *If $k = 1$, then $\int_M H_{r-1}\langle\psi, v\rangle + H_r\langle N, v\rangle\, dA = 0$ for any $v \in E$.*

*Proof.* By an isometry in $\mathbb{R}^{n+1}$, we assume that $E$ contains the origin of $\mathbb{R}^{n+1}$. For sufficiently small $t$, the parallel hypersurface $\psi_t(M) = M_t$ is an immersed hypersurface. Since $W$ is a wedge and $M$ is a hypersurface with free boundary, $\partial M_t$ lies on $\partial W$ and $M_t$ intersects $\partial W$ orthogonally along $\partial M_t$. Integrating (39) on $M_t$ and applying Stokes' theorem, we have

$$\int_M \Big(P_n(t) - \frac{t}{n}P_n'(t) - \frac{1}{n}P_n'(t)\langle\psi, N\rangle\Big)\, dA = \frac{1}{2n}\int_{\partial M_t} \frac{\partial|\psi + tN|^2}{\partial\nu_t}\, ds, \tag{47}$$

where $\nu_t$ is the outward unit conormal vector field to $\partial M_t$. Since $\partial M_t$ lies on $\partial W$ and $M_t$ intersects $\partial W$ orthogonally along $\partial M_t$, $\partial|\psi + tN|^2/\partial\nu_t = 0$ on $\partial M_t$. Then (47) is the same as (41). The conclusion follows the same argument as that of the closed case.

Because of the similarity between the two cases ($k = \pm 1$), we consider only the case $k = -1$.

Recall $\eta_i$, $i = 1, 2$, is the unit normal vector of $\Pi_i$, $i = 1, 2$. By an isometry in $\overline{M}^{n+1}(-1)$, we assume that $v = (0, \dots, 0, 1) \in E$ and $\langle v, \eta_i\rangle = 0$, $i = 1, 2$. For sufficiently small $t$, the parallel hypersurface $\psi_t(M) = M_t$ is an immersed hypersurface. Since $W$ is a wedge and $M$ is a hypersurface with free boundary, $\partial M_t$ lies on $\partial W$ and $M_t$ intersects $\partial W$ orthogonally along $\partial M_t$.

Integrating (43) on $M_t$ and applying Stokes' theorem give

$$\int_{M_t} (\langle \psi_t, v\rangle - H(t)\langle N_t, v\rangle)\, dA_t - \frac{1}{n}\int_{\partial M_t} \langle \nu_t, v\rangle\, ds = 0, \tag{48}$$

where $\nu_t$ is the outward unit conormal vector field to $\partial M_t$.

Since $M_t$ intersects $\partial W$ orthogonally along $\partial M_t$, $\nu_t = \eta_i$ on $\partial M_t \cap \Pi_i$, $i = 1, 2$, and then, $\langle \nu_t, v\rangle \equiv 0$ on $\partial M_t$. Then (48) is the same as (44). The conclusion follows the same argument as that of the closed case. □

Using the same argument, a similar result holds if the boundary of a hypersurface with free boundary lies in a hyperplane of $\overline{M}^{n+1}(k)$.

## 6. Constant-$H_r$ embedded hypersurfaces with free boundary

**Theorem 7.** *Let $W \subset \overline{M}^{n+1}(k)$ be a wedge. Let $M \subset W$ be a compact embedded constant-$H_r$ $(r = 1, \dots, n)$ hypersurface with free boundary. Then $M$ is part of an intrinsic sphere centered at a point of the edge of $W$.*

*Proof.* Denote by $\Omega$ the compact domain enclosed by $M$ and $\partial W$.

For the case $k = 0$, by an isometry in $\mathbb{R}^{n+1}$, we assume that $E$ contains the origin of $\mathbb{R}^{n+1}$. Because the unit normal vector to $\partial\Omega \cap \partial W$ is perpendicular to the position vector $\psi$,

$$\mathrm{Vol}(\Omega) = \frac{1}{n+1}\int_M \langle \psi, N\rangle\, dA, \tag{49}$$

where $\mathrm{Vol}(\Omega)$ is the volume of $\Omega$ and $N$ is the outward unit vector field of $M$.

From (a) of Proposition 6 and (49), we have

$$\int_M H_{r-1}\, dA = H_r \int_M \langle \psi, N\rangle\, dA = (n+1) H_r \,\mathrm{Vol}(\Omega).$$

Denote by $S(r)$ the intrinsic sphere of radius $r$ centered at the origin. For sufficiently large $r$, $M$ is contained inside of $S(r)$. Decreasing $r \searrow 0$, we can find $r_0 > 0$ such that $S(r) \cap M = \varnothing$ for $r > r_0$ but $S(r_0) \cap M \neq \varnothing$. That is, $S(r_0)$ is the first touching to $M$ at a point $q \in S(r_0) \cap M$. At the touching point $q$, all the principal curvatures of $M$ and $H_1$ are positive by comparison with $S(r_0)$. This argument also holds in $\overline{M}^{n+1}(-1)$ without any change. When $\mathbb{S}^{n+1}_+$, if $r$ close enough to $\frac{\pi}{2}$, $M$ is contained inside of $S(r)$; thus, the Euclidean argument also holds.

From (iii) of Lemma 5,

$$\int_M H_{r-1}\, dA \geq \int_M H_r^{(r-1)/r}\, dA,$$

and then,

$$(n+1)\operatorname{Vol}(\Omega) \geq \int_M H_r^{-1/r}\, dA \geq \int_M \frac{1}{H}\, dA. \tag{50}$$

Comparing (26) and (50), $M$ is part of an intrinsic sphere by Theorem 3.

Now, we consider $k \neq 0$ case. From a direct computation, we have $\bar{\Delta}\langle \psi, v\rangle = -k(n+1)\langle \psi, v\rangle$ for any $v \in E$. Integrating on $\Omega$ and using Stokes' theorem, we have

$$-k(n+1)\int_\Omega \langle \psi, v\rangle\, dV = \int_M \langle N, v\rangle\, dA + \int_{\partial\Omega\cap\partial W} \langle \nu, v\rangle\, dA,$$

where $N$ and $\nu$ are the outward unit normal vector fields of $M$ and $\partial\Omega \cap \partial W$, respectively.

By an isometry of $\overline{M}^{n+1}(k)$, we assume $v = (0, \dots, 0, 1) \in E$ and $\langle \eta_i, v\rangle = 0$, $i = 1, 2$. With $v$, $\langle \nu, v\rangle \equiv 0$ on $\partial\Omega \cap \partial W$, that is,

$$-k(n+1)\int_\Omega \langle \psi, v\rangle\, dV = \int_M \langle N, v\rangle\, dA. \tag{51}$$

Let $r(x) = \operatorname{dist}(x, v)$ be the distance function from $v$ to $x$ in $\overline{M}^{n+1}(k)$. If $k = -1$, then $\langle \psi, v\rangle = -\cosh r(\psi)$ and if $k = 1$, then $\langle \psi, v\rangle = \cos r(\psi)$; that is, $k\langle \psi, v\rangle = V_k(\psi)$ in $\overline{M}^{n+1}(k)$.

From (b) of Proposition 6, we have

$$\int_M H_{r-1} V_k(\psi) + H_r \langle N, v\rangle\, dA = 0.$$

Since $H_r$ is constant and (51), $(n+1)H_r \int_\Omega V_k\, dV = \int_M H_{r-1} V_k\, dA$. By the same argument for the $k=0$ case, there exists a point in $M$ such that all the principal curvatures are positive. From (iii) of Lemma 5,

$$(n+1)H_r \int_\Omega V_k\, dV = \int_M H_{r-1} V_k\, dA \geq \int_M H_r^{(r-1)/r} V_k\, dA,$$

and then,

$$(n+1)\int_\Omega V_k\, dV \geq \int_M H_r^{-1/r} V_k\, dA \geq \int_M \frac{V_k}{H}\, dA. \tag{52}$$

Comparing (26) and (52) and using the results of Theorem 3, we conclude that $M$ is part of an intrinsic sphere. □

As before, when $\partial M$ lies in a hyperplane, the following conclusion holds.

**Theorem 8.** *Let $\mathcal{H}$ be a half-space in $\overline{M}^{n+1}(k)$ determined by a hyperplane $P$. Let $M$ be a compact embedded constant-$H_r$ ($r = 1, \dots, n$) hypersurface with free boundary in $\mathcal{H}$. Then $M$ is an intrinsic hemisphere centered at a point of $P$.*

## 7. Constant-$H_r/H_l$ immersed hypersurfaces with free boundary

Using the Minkowski formula and the inequalities for higher-order mean curvatures (Lemma 5), Koh and Lee [2001] gave characterizations of an intrinsic sphere in space forms. In Proposition 6, the Minkowski formula is extended to hypersurfaces with free boundary in space forms; then, Koh and Lee's results are naturally extended for hypersurfaces with free boundary. For the reader's convenience, we give the proof in detail.

**Theorem 9.** *Let $W \subset \overline{M}^{n+1}(k)$ be a wedge. Let $M$ be a compact immersed hypersurface in $\overline{M}^{n+1}(k)$ with $\partial M \subset \partial W$ such that near $\partial M$, $M$ lies inside of $W$ and meets $\partial W$ perpendicularly along $\partial M$. If, for $r, l = 1, \dots, n$ and $r > l$, the ratio $H_r/H_l$ is constant and $H_l$ does not vanish on $M$, then it is part of an intrinsic sphere centered at a point of the edge of $W$.*

*Proof.* For the case $k = 0$, by an isometry in $\mathbb{R}^{n+1}$, we assume that $E$ contains the origin of $\mathbb{R}^{n+1}$. By the same argument as the proof of Theorem 7, there is an elliptic point $q$ in $M$; that is, all the principal curvatures are positive, and clearly, both $H_r$ and $H_l$ are positive at $q$. Because $\alpha = H_r/H_l$ is constant and $H_l$ does not vanish on $M$, the curvatures $H_r$, $H_l$ are positive on $M$ and $\alpha > 0$. By (i) of Lemma 5, $H_s > 0$ if $s < r$. By (ii) of Lemma 5,

$$0 < \alpha = \frac{H_r}{H_l} \le \frac{H_{r-1}}{H_{l-1}}. \tag{53}$$

Because $H_r = \alpha H_l$ and by (a) of Proposition 6,

$$\int_M H_{r-1} - \alpha H_l \langle \psi, N\rangle \, dA = 0. \tag{54}$$

Because $\alpha > 0$ is constant and by (a) of Proposition 6,

$$\int_M \alpha (H_{l-1} - H_l \langle \psi, N\rangle) \, dA = 0. \tag{55}$$

Combining (54) and (55) yields

$$\int_M (H_{r-1} - \alpha H_{l-1}) \, dA = 0.$$

From (53),

$$\frac{H_r}{H_l} = \frac{H_{r-1}}{H_{l-1}} = \alpha \quad \text{on } M.$$

Proceeding inductively, and defining $p = r - l$, we obtain

$$\frac{H_{p+1}}{H_1} = \frac{H_p}{H_0} = H_p \quad \text{on } M; \tag{56}$$

that is, $H_{p+1}/H_p = H_1$.

On the other hand, by (ii) of Lemma 5,

$$H_{p+1}/H_p \leq H_p/H_{p-1} \leq \cdots \leq H_1. \tag{57}$$

Combining (56) and (57) gives,

$$H_{p+1}/H_p = H_p/H_{p-1} = \cdots = H_1,$$

and therefore,

$$H_r = H_1^r, \quad r = 1, 2, \ldots, p+1.$$

By (iii) of Lemma 5, $M$ is part of an intrinsic sphere.

By an isometry in $\overline{M}^{n+1}(-1)$, we assume that $E$ contains $v = (0, \ldots, 0, 1) \in \mathbb{L}^{n+2}$. As before there exists a point $q$ such that all the principal curvatures are positive, and clearly, both $H_r$ and $H_l$ are positive at $q$. Because $\alpha = H_r/H_l$ is constant and $H_l$ does not vanish on $M$, the curvatures $H_r$, $H_l$ are positive on $M$ and $\alpha > 0$.

By (ii) of Lemma 5,

$$0 < \alpha = \frac{H_r}{H_l} \leq \frac{H_{r-1}}{H_{l-1}}. \tag{58}$$

Because $H_r = \alpha H_l$ and by Proposition 6,

$$\int_M H_{r-1}\langle\psi, v\rangle - \alpha H_l\langle N, v\rangle \, dA = 0. \tag{59}$$

Because $\alpha > 0$ is constant and by Proposition 6,

$$\int_M \alpha(H_{l-1}\langle\psi, v\rangle - H_l\langle N, v\rangle) \, dA = 0. \tag{60}$$

Combining (59) and (60) yields,

$$\int_M (H_{r-1} - \alpha H_{l-1})\langle\psi, v\rangle \, dA = 0.$$

Because $\langle\psi, v\rangle \leq -1$ on $M$ and by (58),

$$\frac{H_r}{H_l} = \frac{H_{r-1}}{H_{l-1}} = \alpha \quad \text{on } M.$$

Proceeding inductively, and defining $p = r - l$, we obtain

$$\frac{H_{p+1}}{H_1} = \frac{H_p}{H_0} = H_p \quad \text{on } M; \tag{61}$$

that is, $H_{p+1}/H_p = H_1$.

On the other hand, by (ii) of Lemma 5,

$$H_{p+1}/H_p \leq H_p/H_{p-1} \leq \cdots \leq H_1. \tag{62}$$

Combining (61) and (62) gives,

$$H_{p+1}/H_p = H_p/H_{p-1} = \cdots = H_1,$$

and therefore,

$$H_r = H_1^r, \quad r = 1, 2, \ldots, p+1.$$

By (iii) of Lemma 5, $M$ is part of an intrinsic sphere.

For the case $k = 1$, we assume that the edge $E$ contains $v = (0, 0, \ldots, 1) \in \mathbb{R}^{n+2}$. Because $\psi : M \to \mathbb{S}_+^{n+1}$, we have $\langle \psi, v \rangle > 0$. By the same argument for $k = -1$, the conclusion follows as for the $k = 1$ case. □

**Theorem 10.** *Let $P$ be a hyperplane in $\overline{M}^{n+1}(k)$. Let $M$ be a compact immersed hypersurface in $\overline{M}^{n+1}(k)$ with $\partial M \subset P$ such that near $\partial M$, $M$ lies on one side of $P$ and meets $P$ perpendicularly along $\partial M$. If, for $r, l = 1, \ldots, n$ and $r > l$, the ratio $H_r/H_l$ is constant and $H_l$ does not vanish on $M$, then it is an intrinsic hemisphere centered at a point of $P$.*

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JUNCHEOL PYO
jcpyo@pusan.ac.kr
DEPARTMENT OF MATHEMATICS
PUSAN NATIONAL UNIVERSITY
BUSAN
SOUTH KOREA

and

KOREA INSTITUTE FOR ADVANCED STUDY
SEOUL
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