Pacific Journal of Mathematics

Volume 300 No. 1 May 2019

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 300, No. 1, 2019
dx.doi.org/10.2140/pjm.2019.300.1

BRAID GROUP REPRESENTATIONS FROM BRAIDING GAPPED BOUNDARIES OF DIJKGRAAF–WITTEN THEORIES

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We study representations of the braid groups from braiding gapped boundaries of Dijkgraaf–Witten theories and their twisted generalizations, which are (twisted) quantum doubled topological orders in two spatial dimensions. We show that the braid representations associated to Lagrangian algebras are all monomial with respect to some specific bases. We give explicit formulas for the monomial matrices and the ground state degeneracy of the Kitaev models that are Hamiltonian realizations of Dijkgraaf–Witten theories. Our results imply that braiding gapped boundaries alone cannot provide universal gate sets for topological quantum computing with gapped boundaries.

1. Introduction

Interesting new directions in topological quantum computing include its extension from anyons to gapped boundaries and symmetry defects, with the hope that anyonic systems with nonuniversal computational power can be enhanced to achieve universality. Enrichment of topological physics in two spatial dimensions by gapped boundaries has been investigated intensively, but their computing power has not been analyzed in detail yet. One interesting case is gapped boundaries of Dijkgraaf–Witten theories both for their experimental relevance and as theoretical exemplars (see [Cong et al. 2016; 2017a; 2017b]).

In this paper, we study representations of the braid groups from braiding gapped boundaries of Dijkgraaf–Witten theories and their twisted generalizations, which are (twisted) quantum doubled topological orders in two spatial dimensions. We show that the resulting braid (pure braid) representations are all monomial with respect to some specific bases, and their entries are roots of unity; hence all such representation

Galindo was partially supported by Fondo de Investigaciones de la Facultad de Ciencias de la Universidad de los Andes, Convocatoria 2018-2019 para la Financiación de Programas de Investigación, programa "Simetría T (inversión temporal) en categorías de fusión y modulares", Escobar was partially supported by the Convocatoria para la Financiación de Proyectos de Investigación 2017-1 funds from Facultad de Ciencias de la Universidad de los Andes, and Wang by NSF grant DMS-1411212.
MSC2010: 18D10, 20C35, 20F36.
Keywords: topological quantum computation, modular categories, braid group.

images of the braid groups are finite groups. The finiteness of the image of the braid representation from braiding gapped boundaries of twisted Dijkgraaf–Witten theories follows directly from [Etingof et al. 2008], since they are braid representations obtained from group-theoretical braided fusion categories. Besides, we give explicit formulas for the monomial matrices and the ground state degeneracy of the Kitaev models that are Hamiltonian realizations of Dijkgraaf–Witten theories. The universal gate sets from [Cong et al. 2017c] include a nontopological measurement primitive. Our results imply that braiding gapped boundaries alone cannot provide universal gate sets for topological quantum computing with gapped boundaries.

For a topological order of the form $\mathcal{C} = \mathcal{Z}(\mathcal{S})$, were $\mathcal{S}$ is some unitary fusion category, gapped boundaries are modeled by Lagrangian algebras (see [Cong et al. 2016]). For these models the ground manifolds have the form

$$\mathrm{Hom}_{\mathcal{C}}(1, A_1 \otimes \cdots \otimes A_n),$$

where the A_i's are the Lagrangian algebras modeling the gapped boundaries; see [Cong et al. 2016, Section 3] for details. Recall that a Lagrangian algebra in any modular (tensor) category is a commutative étale algebra whose quantum dimension is maximal. A group theoretical modular category (GTMC) admitting a Lagrangian algebra is a category of the form $\mathcal{C} = \mathcal{Z}(\mathrm{Vec}_G^\omega)$ for some finite group G and some $\omega \in Z^3(G, \mathbb{C}^\times)$, where $\mathcal{Z}$ denotes the Drinfeld center and Vec_G^ω is the category of finite-dimensional G-graded vector spaces with associativity constraint twisted by $\omega \in H^3(G, \mathbb{C}^\times)$; see [Davydov et al. 2013; Davydov and Simmons 2017].

Kitaev [2003] proposed Hamiltonian realizations of Dijkgraaf–Witten theories, whose topological orders are GTMCs. Moreover, extensions of these Hamiltonian realizations to surfaces with boundaries can be constructed from Lagrangian algebras [Bravyi and Kitaev 1998; Bombin and Martin-Delgado 2008; Beigi et al. 2011; Kitaev and Kong 2012].

Lagrangian algebras in GTMCs are in one-to-one correspondence with indecomposable module categories of Vec_G^ω [Davydov et al. 2013], which are in bijection with pairs (H, γ), where H is a subgroup of G and $\gamma \in C^2(H, \mathbb{C}^\times)$ such that $\delta(\gamma) = \omega|_{H^{\times 3}}$, all up to conjugation [Natale 2017]. A more direct description of the relationship between Lagrangian algebras and pairs (H, γ) can be found in [Davydov 2010].

Recently, a quantum computing scheme to use gapped boundaries to achieve universality has been proposed [Cong et al. 2016; 2017a; 2017b; 2017c]. Braiding gapped boundaries can be either added to braiding anyons as in Kitaev's original proposal or as new computing primitives supplemented with other topological operations. Gapped boundaries lead to additional degeneracy of the topologically protected subspace, which potentially allows the implementation of more powerful gates. More precisely, the new gates come from representation matrices of the braid groups, $\mathcal{B}_n$, on objects of the GTMCs that are tensor products of Lagrangian algebras. However, a characterization of the computational power of these new braid representations, mathematically a study of the representation images, was left as an important open problem [Cong et al. 2016; 2017c].

The goal of this paper is to provide such a characterization. We find a canonical monomial structure for Lagrangian algebras in $\mathcal{Z}(\mathrm{Vec}_G^\omega)$, which allows us to compute things more easily. This paper is organized as follows. Section 2 develops the theory of monomial representations. Specifically, it shows how to calculate invariants for a representation of G using the monomial structure. In Section 3 we recall the notion of a monomial twisted Yetter–Drinfeld module. We use the theory developed in Section 2 to give an explicit description and a basis for $\mathrm{Hom}_{\mathcal{Z}(\mathrm{Vec}_G^\omega)}(\mathbb{C}, V^{\otimes n})$ if V is a monomial object. Next, we describe the representation of $\mathcal{B}_n$ with respect to this basis. Theorem 3.3 states the representation is monomial and Theorem 3.4 gives an explicit formula for the nonzero entries. In Section 4 we prove that every Lagrangian algebra in $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ has a canonical monomial structure. Then the results of Section 3 are applied to Lagrangian algebras in $\mathcal{Z}(\mathrm{Vec}_G^\omega)$. We finish the section developing some examples and applications.

2. Monomial representations

In this section, we recall some basic definitions and results on monomial representations of groups.

Definition 2.1. A *monomial space* is a triple $\boldsymbol{V} = (V, X, (V_x)_{x\in X})$, where

(i) V is a finite-dimensional complex vector space,

(ii) X is a finite set,

(iii) $(V_x)_{x\in X}$ is a family of one-dimensional subspaces of V, indexed by X, such that $V = \bigoplus_{x\in X} V_x$.

Let G be a group. By a *monomial representation* of G on $\boldsymbol{V}$ we mean a group homomorphism

$$\Gamma : G \to \mathrm{GL}(V)$$

such that for every $g \in G$, $\Gamma(g)$ permutes the V_x's; hence, Γ induces an action by permutation of G on X. We will denote $\Gamma(g)(v)$ by $g \triangleright v$.

If V is a representation of G, we denote by V^G the subspace of G-invariant vectors, i.e.,

$$V^G = \{v \in V : g \triangleright v = v, \text{ for all } g \in G\}.$$

For each $x \in X$, we denote by $\mathrm{Sta}_G(x)$ the stabilizer of x and by $\mathcal{O}_G(x)$ the

G-orbit of x. For G finite, and a representation V define

$$\mathrm{Av}_G : V \to V, \qquad v \mapsto \frac{1}{|G|}\sum_{g\in G} g \triangleright v.$$

It is easy to see that Av_G is a G-linear projection onto V^G. We define

$$\mathrm{Av}_G(V_{\mathcal{O}}) := \mathrm{Av}_G(V_x), \qquad x \in \mathcal{O}(x),$$

since for any $x' \in \mathcal{O}_G(x)$, $\mathrm{Av}_G(V_x) = \mathrm{Av}_G(V_{x'})$.

We say that an element $x \in X$ is *regular* under the monomial action of G if $\Gamma(g)$ is the identity map on V_x, for all $g \in \mathrm{Sta}_G(x)$.

Let us write X/G for the set of orbits of the action of G on X, and $\tilde{X}$ for the regular ones.

Proposition 2.2 [Karpilovsky 1985, Lemma 9.1]. *Let $\boldsymbol{V} = (V, X, (V_x)_{x\in X})$ be a monomial representation of G.*

(a) *$x \in X$ is a regular element if and only if* $\mathrm{Av}_G(V_x) \neq 0$.
(b) *If $x \in X$ is a regular element under the monomial action of G, then so are all elements in the G-orbit of x.*
(c) *The triple*
$$\boldsymbol{V}^G = (V^G, \tilde{X}, (\mathrm{Av}_G(V_{\mathcal{O}}))_{\mathcal{O}\in\tilde{X}})$$
is a monomial space.
(d) *The dimension of V^G is equal to the number of regular G-orbits under the monomial action of G on X.*

Let $\boldsymbol{V} = (V, X, (V_x)_{x\in X})$ and $\boldsymbol{V}' = (V', Y, (V'_y)_{y\in Y})$ be monomial spaces. A linear isomorphism $T : V \to V'$ is called an *isomorphism of monomial spaces* if $T(V_x) = V'_y$ for any $x \in X$.

Proposition 2.3. *Let $\boldsymbol{V} = (V, X, (V_x)_{x\in X})$ and $\boldsymbol{V}' = (V', Y, (V'_y)_{y\in Y})$ be monomial representations of a finite group G. If $T : V \to V'$ is a G-linear isomorphism of monomial spaces, then $T|_{V^G} : \boldsymbol{V}^G \to \boldsymbol{V}'^G$ is an isomorphism of monomial spaces.*

Proof. Clearly, $T|_{V^G} : V^G \to V'^G$ is a linear isomorphism. Let $x \in X$ be a regular element. Since T is an isomorphism of monomial spaces, there is some $y \in Y$ such that $T(V_x) = V'_y$. In that case,

$$\mathrm{Av}_G(V'_y) = \mathrm{Av}_G(T(V_x)) = T(\mathrm{Av}_G(V_x)).$$

This implies y is regular, because $\mathrm{Av}_G(V_x) \neq \{0\}$ and T is an isomorphism. It also gives $T|_{V^G}(\mathrm{Av}_G(V_{\mathcal{O}(x)})) = \mathrm{Av}_G(V'_{\mathcal{O}(y)})$, which means $T|_{V^G}$ is an isomorphism of monomial spaces. $\square$

3. Monomial representation of the braid group

In this section we recall the notion of monomial twisted Yetter–Drinfeld module introduced in [Galindo and Rowell 2014, Definition 4.12] and prove that the representation of the braid groups $\mathcal{B}_n$ over $\mathrm{Hom}_{\mathcal{Z}(\mathrm{Vec}_G^\omega)}(\mathbb{C}, V^{\otimes n})$ is monomial if V is monomial.

3A. ***Dijkgraaf–Witten theories.*** Let G be a discrete group. A (normalized) 3-cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$ is a map $\omega : G \times G \times G \to \mathbb{C}^\times$ such that

$$\omega(ab, c, d)\omega(a, b, cd) = \omega(a, b, c)\omega(a, bc, d)\omega(b, c, d), \quad \omega(a, 1, b) = 1,$$

for all $a, b, c, d \in G$.

Let us recall the description of the modular category $\mathcal{Z}(\mathrm{Vec}_G^\omega)$, the Drinfeld center of the category Vec_G^ω sometimes called the category of twisted Yetter–Drinfeld modules. The category $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ is braided equivalent to the representations of the twisted Drinfeld double defined by Dijkgraaf, Pasquier and Roche [Dijkgraaf et al. 1991, Section 3.2].

Given $\omega \in Z^3(G; \mathbb{C}^\times)$, we define

$$\omega(g, g'; h) := \frac{\omega({}^{gg'}h, g, g')\omega(g, g', h)}{\omega(g, {}^{g'}h, g')}, \quad \omega(g; f, h) := \frac{\omega({}^{g}f, g, h)}{\omega(g, f, h)\omega({}^{g}f, {}^{g}h, g)},$$

for $f, g, g', h \in G$.

The objects of $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ are G-graded vector spaces $V = \bigoplus_{g\in G} V_g$ with a linear map $\triangleright : \mathbb{C}^\omega G \otimes V \to V$ such that $1 \triangleright v = v$ for all $v \in V$, and

$$(gh) \triangleright v = \omega(g, h; k)(g \triangleright (h \triangleright v)), \qquad g, h, k \in G, \ v \in V_k,$$

satisfying the compatibility condition

$$g \triangleright V_h \subseteq V_{ghg^{-1}}, \quad g, h \in G.$$

Morphisms in $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ are G-linear G-homogeneous maps. The tensor product of $V = \oplus_{g\in G} V$ and $W = \oplus_{g\in G} w$ is $V \otimes W$ as vector space, with

$$(V \otimes W)_g = \bigoplus_{h\in G} V_h \otimes W_{h^{-1}g},$$

and for all $v \in V_g, \ w \in W_l$,

$$h \triangleright (v \otimes w) = \omega(h; g, l)(h \triangleright v) \otimes (h \triangleright w).$$

For $V, W, Z \in \mathcal{Z}(\mathrm{Vec}_G^\omega)$, the associativity constraint is defined by

$$\begin{aligned} a_{V,W,Z} &: (V \otimes W) \otimes Z \to V \otimes (W \otimes Z), \\ &(v_g \otimes w_h) \otimes z_k \mapsto \omega(g, h, k)^{-1} v_g \otimes (w_h \otimes z_k) \end{aligned}$$

for all $g, h, k \in G,\ v_g \in V_x,\ w_h \in W_h,\ z_k \in Z_k$. The category is tensor braided, with braiding $c_{V,W} : V \otimes W \to W \otimes V,\ V, W \in \mathcal{Z}(\mathrm{Vec}_G^\omega)$,

$$c_{V,W}(v \otimes w) = (g \triangleright w) \otimes v, \qquad g \in G,\ v \in V_g,\ w \in W.$$

3B. ***Braid group representation of twisted Yetter–Drinfeld modules.*** Since the braided category $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ is not strict, we must be careful about the way we associate terms when we consider tensor products with more than two objects. For a list of objects $A_1, A_2, \dots, A_n \in \mathcal{Z}(\mathrm{Vec}_G^\omega)$, we define

$$A_1 \otimes \cdots \otimes A_n := (\cdots (A_1 \otimes A_2) \otimes \cdots \otimes A_{n-1}) \otimes A_n,$$

and an isomorphism by

$$\sigma_i' = (a^{-1}_{A_1\otimes\cdots\otimes A_{i-1}, A_{i+1}, A_i} \otimes \mathrm{id}_{A_{i+2}\otimes\cdots\otimes A_n}) \circ (\mathrm{id}_{A_1\otimes\cdots A_{i-1}} \otimes c_{A_i, A_{i+1}} \otimes \mathrm{id}_{A_{i+2}\otimes\cdots\otimes A_n}) \circ (a_{A_1\otimes\cdots\otimes A_{i-1}, A_i, A_{i+1}} \otimes \mathrm{id}_{A_{i+2}\otimes\cdots\otimes A_n}), \tag{1}$$

where $a_{V,W,Z}$ denotes the associativity constraint.

If $A = A_1 = \cdots = A_n$, there exists a unique group homomorphism

$$\rho_n : \mathcal{B}_n \to \mathrm{Aut}_{\mathcal{Z}(\mathrm{Vec}_G^\omega)}(A^{\otimes n})$$

sending the generator $\sigma_i \in \mathcal{B}_n$ to σ_i'.

In general, the pure braid group $\mathcal{P}_n$ acts on $A_1 \otimes \cdots \otimes A_n$, in the sense that there exists a group homomorphism $\rho_n : \mathcal{P}_n \to \mathrm{Aut}_{\mathcal{Z}(\mathrm{Vec}_G^\omega)}(A_1 \otimes \cdots \otimes A_n)$.

3C. ***Crossed G-sets.*** Let G be a group. We will recall the definition of (left) *crossed G-set* introduced in [Freyd and Yetter 1989]. A crossed G-set is a left G-set X and a grading function $|-| : X \to G$ such that

$$|gx| = g|x|g^{-1},$$

for all $x \in X$, $g \in G$. If X and Y are crossed G-sets, a G-equivariant map $f : X \to Y$ is a morphism of crossed G-sets if $|f(x)| = |x|$ for all $x \in X$.

If X and Y are crossed G-sets, the cartesian product $X \times Y$ is a crossed G-set with the diagonal action and grading map $|(x, y)| = |x|\,|y|$.

The category of crossed G-sets is a braided category with braiding

$$\begin{aligned} c_{X,Y} : X \times Y &\to Y \times X, \\ (x, y) &\mapsto (|x| \triangleright y, x). \end{aligned}$$

Thus, given a crossed G-set X the braid group $\mathcal{B}_n$ acts on X^n, in the following way:

$$\sigma_i' := \mathrm{id}_{X^{i-1}} \times c_{X,X} \times \mathrm{id}_{X^{n-(i-1)}}\,.$$

3D. ***Monomial objects of $\mathcal{Z}(\mathrm{Vec}_G^\omega)$.*** Let G be a finite group and $\omega \in Z^3(G, \mathbb{C}^\times)$ be a 3-cocycle.

Definition 3.1 [Galindo and Rowell 2014]. A monomial Yetter–Drinfeld module is a monomial space $\boldsymbol{V} = (V, X, (V_x)_{x\in X})$ such that $V \in \mathcal{Z}(\mathrm{Vec}_G^\omega)$, the twisted G-action $\triangleright$ permutes the V_x's, and each V_x is G-homogeneous.

Remark 3.2. (a) If $\boldsymbol{V} = (V, X, (V_x)_{x\in X})$ is a monomial Yetter–Drinfeld module, the set X is a crossed G-set with the induced G-action and the grading map.

(b) If $\boldsymbol{V} = (V, X, (V_x)_{x\in X})$ is a monomial Yetter–Drinfeld module, the action of G on $(V_e, X_e, (V_x)_{x\in X_e})$ is monomial, where $X_e := \{x \in X : |x| = e\}$ and $V_e = \oplus_{x\in X_e} V_x$.

Theorem 3.3. *Let G be a finite group, $\omega \in Z^3(G, \mathbb{C}^\times)$. If $\boldsymbol{V} = (V, X, (V_x)_{x\in X})$ is a monomial Yetter–Drinfeld module in $\mathcal{Z}(\mathrm{Vec}_G^\omega)$, then*

(a) *the action of $\mathcal{B}_n$ on $\mathrm{Hom}_{\mathcal{Z}(\mathrm{Vec}_G^\omega)}(\mathbb{C}, V^{\otimes n})$ is monomial,*

(b) *the dimension of $\mathrm{Hom}_{\mathcal{Z}(\mathrm{Vec}_G^\omega)}(\mathbb{C}, V^{\otimes n})$ is equal to the number of regular G-orbits under the monomial action of G on*

$$(X^n)_e := \{(x_1, \dots, x_n) : |x_1| \cdots |x_n| = e\}.$$

Proof. The action of G on $(V_e^{\otimes n}, (X^n)_e, (V_x)_{x\in X_e})$ is monomial. Hence, by Proposition 2.2, the triple

$$\boldsymbol{V}_e^G := \Big((V_e^{\otimes n})^G, \widetilde{(X^n)_e}, (\mathrm{Av}_G((V_e^{\otimes n})_{\mathcal{O}}))_{\mathcal{O}\in\widetilde{(X^n)_e}}\Big)$$

is a monomial space. Since $\mathrm{Hom}_{\mathcal{Z}(\mathrm{Vec}_G^\omega)}(\mathbb{C}, V^{\otimes n}) = (V^{\otimes n})_e^G$, and each of the automorphisms σ' are morphisms in $\mathcal{Z}(\mathrm{Vec}_G^\omega)$,

$$\sigma'|_{V_e^{\otimes n}} : (V_e^{\otimes n}, (X^n)_e, (V_x)_{x\in X_e}) \to (V_e^{\otimes n}, (X^n)_e, (V_x)_{x\in X_e})$$

is a G-linear isomorphism of monomial spaces. It follows from Proposition 2.3 that $\sigma'|_{(V^{\otimes n})_e^G}$ is an isomorphism of monomial spaces. Thus, the linear representation

$$\begin{aligned}\rho_n : \mathcal{B}_n &\to \mathrm{GL}((V_e^{\otimes n})^G),\\ \sigma &\mapsto \sigma'\end{aligned}$$

is a monomial representation of $\mathcal{B}_n$. The second part follows immediately from Proposition 2.2. □

3E. *Monomial matrices of the braid representation.* In this subsection we obtain concrete formulas for the monomial braid representations associated to a monomial Yetter–Drinfeld module.

Let G be a finite group, $\omega \in Z^3(G, \mathbb{C}^\times)$, and $\boldsymbol{V} = (V, X, (V_x)_{x\in X})$ be a monomial Yetter–Drinfeld module. If we fix nonzero vectors $\mathcal{S} := \{v_x \in V_x : x \in X\}$, the twisted G-action defines a map

$$\lambda_X : G \times X \to \mathbb{C}^\times,$$

by $g \triangleright v_x = \lambda_X(g; x)v_{gx}$, where $g \in G$, $x \in X$.

For the monomial Yetter–Drinfeld module $\boldsymbol{V}^{\otimes n} = (V^{\otimes n}, X^n, (V_x)_{x \in X^n})$ and the basis $\mathcal{S}^{\otimes n} := \{v_{x_1} \otimes \cdots \otimes v_{x_n} : v_{x_i} \in S,\ 1 \le i \le n\}$, the action is determined by the map $\lambda_{X^n} : G \times X^n \to \mathbb{C}^\times$,

$$\begin{aligned}(2)\quad \lambda_{X^n}(g; x_1, \dots, x_n) := \prod_{i=1}^{n} &\lambda_X(g; x_i)\omega(g; |x_1|\,|x_2| \cdots |x_{n-1}|, |x_n|)^{-1} \\ &\times \omega(g; |x_1| \cdots |x_{n-2}|, |x_{n-1}|)^{-1} \cdots \omega(g; |x_1|, |x_2|)^{-1},\end{aligned}$$

that is,

$$g \triangleright (v_{x_1} \otimes \cdots \otimes v_{x_n}) = \lambda_{X^n}(g; x_1, \dots, x_n)(v_{gx_1} \otimes \cdots \otimes v_{gx_n}),$$

for all $g \in G$, $x_1, x_2, \dots, x_n \in X$. Hence an element $(x_1, \dots, x_n) \in (X^n)_e$ is regular if and only if

$$(3)\qquad \lambda_{X^n}(g; x_1, \dots, x_n) = 1, \qquad \text{for all } g \in \bigcap_{i=1}^{n} \mathrm{Sta}(x_i).$$

Let $\mathcal{R} \subset X_e^n$ be a set of representatives of the regular orbits of $X_e^{\times n}$. Let $\mathcal{S}_{\mathrm{reg}} = \{v_{x_1} \otimes \cdots \otimes v_{x_n} : (x_1, \dots, x_n) \in \mathcal{R}\}$. By Proposition 2.2, the set $\{\mathrm{Av}_G(v) : v \in \mathcal{S}_{\mathrm{reg}}\}$ is a basis of $(V^{\otimes n})_e^G$.

To express the action of the generator $\sigma_i \in \mathcal{B}_n$ in terms of $\{\mathrm{Av}_G(v) : v \in \mathcal{S}_{\mathrm{reg}}\}$, for each $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathcal{R}$ choose $g_{\boldsymbol{x}} \in G$ such that $g_{\boldsymbol{x}} \triangleright \sigma_i'(\boldsymbol{x}) = \boldsymbol{y}$, where $\boldsymbol{y} \in \mathcal{R}$ and $\sigma_i'(\boldsymbol{x}) = (x_1, \dots, x_{i-1}, |x_i|x_{i+1}, x_i, x_{i+2}, \dots, x_n)$. Hence there is $\beta_{i,\boldsymbol{x}} \in \mathbb{C}^\times$ such that $g_{\boldsymbol{x}} \triangleright \sigma_i'(v_{x_1} \otimes \cdots v_{x_n}) = \beta_{i,\boldsymbol{x}} v_{y_1} \otimes \cdots \otimes v_{y_n}$.

Since the action of the generator $\sigma_i \in \mathcal{B}_n$ is given by

$$\begin{aligned}(4)\quad \sigma_i'(v_{x_1} \otimes \cdots \otimes v_{x_n}) = {}&\omega(|x_1| \cdots |x_{i-1}|, |x_i|\,|x_{i+1}|\,|x_i|^{-1}, |x_i|) \\ &\times \lambda_X(|x_i|; x_{i+1})\omega(|x_1| \cdots |x_{i-1}|, |x_i|, |x_{i+1}|)^{-1} \\ &\times v_{x_1} \otimes \cdots \otimes v_{x_{i-1}} \otimes v_{|x_i|x_{i+1}} \otimes v_{x_i} \otimes \cdots \otimes v_{x_n},\end{aligned}$$

we have

$$\begin{aligned}(5)\quad \beta_{i,\boldsymbol{x}} = {}&\omega(|x_1| \cdots |x_{i-1}|, |x_i|\,|x_{i+1}|\,|x_i|^{-1}, |x_i|) \\ &\times \lambda_X(|x_i|; x_{i+1})\omega(|x_1| \cdots |x_{i-1}|, |x_i|, |x_{i+1}|)^{-1}\lambda_{X^n}(g_{\boldsymbol{x}}; \sigma_i'(\boldsymbol{x})).\end{aligned}$$

Theorem 3.4. *Let G be a finite group, $\omega \in Z^3(G, \mathbb{C}^\times)$ and $\boldsymbol{V} = (V, X, (V_x)_{x \in X})$ be a monomial Yetter–Drinfeld module. Let Y be the set of all regular elements in X_e^n and let $\mathcal{R} \subset Y$ be a set of representatives of the G-orbits of Y.*

(a) *The projection $\pi : Y \to \mathcal{R}$ is map of $\mathcal{B}_n$-sets. The image of $\boldsymbol{x} \in \mathcal{R}$ by the generator $\sigma_i \in \mathcal{B}_n$ will be denoted by $\sigma_i \triangleright \boldsymbol{x}$.*

(b) *Let* $\mathcal{S}_{\mathrm{reg}} = \{v_{x_1} \otimes \cdots \otimes v_{x_n} : (x_1, \dots, x_n) \in \mathcal{R}\}$. *The action of the generator* $\sigma_i \in \mathcal{B}_n$ *in the basis* $\{\mathrm{Av}_G(v_{\boldsymbol{x}}) : \boldsymbol{x} \in \mathcal{R}\}$ *is given by*

$$\sigma_i(\mathrm{Av}_G(v_{\boldsymbol{x}})) = \beta_{i,\boldsymbol{x}}\, \mathrm{Av}_G(v_{\sigma_i \triangleright \boldsymbol{x}}),$$

where $\beta_{i,\boldsymbol{x}}$ *was defined in* (5).

Proof. The first part is a consequence of Theorem 3.3.

For the second part, recall that the number $\beta_{i,\boldsymbol{x}}$ and the element $g_{\boldsymbol{x}} \in G$ are such that

$$g_{\boldsymbol{x}} \triangleright \sigma(v_{\boldsymbol{x}}) = \beta_{i,\boldsymbol{x}} v_{\sigma_i \triangleright \boldsymbol{x}}.$$

Hence,

$$\begin{aligned}\sigma_i(\mathrm{Av}_G(v_{\boldsymbol{x}})) &= \mathrm{Av}_G(\sigma_i(v_{\boldsymbol{x}})) \\ &= g_{\boldsymbol{x}} \triangleright \mathrm{Av}_G(\sigma_i(v_{\boldsymbol{x}})) = \mathrm{Av}_G(g_{\boldsymbol{x}} \triangleright \sigma_i(v_{\boldsymbol{x}})) \\ &= \mathrm{Av}_G(\beta_{i,\boldsymbol{x}} v_{\sigma_i \triangleright \boldsymbol{x}}) = \beta_{i,\boldsymbol{x}}\, \mathrm{Av}_G(v_{\sigma_i \triangleright \boldsymbol{x}}). \end{aligned}$$ □

Example 3.5. Let G be a finite group and X be a left crossed G-set. Then the linearization $V_X := \oplus_{x \in X} \mathbb{C}x$ is an (untwisted) Yetter–Drinfeld module in $\mathcal{Z}(\mathrm{Vec}_G)$. Clearly $\lambda_X \equiv 1$, thus every element in $(X^n)_e$ is regular. Hence the canonical projection

$$(X^n)_e \to (X^n)_e // G,$$

is an epimorphism of $\mathcal{B}_n$-sets. In other words, the linear representation of $\mathcal{B}_n$ on $\mathrm{Hom}_{\mathcal{Z}(\mathrm{Vec}_G)}(\mathbb{C}, V_X^{\otimes n})$ is the linearization of the permutation action of $\mathcal{B}_n$ on $(X^n)_e // G$.

4. Braid groups representations associated to Lagrangian algebras

In this section, we prove that every Lagrangian algebra in $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ has a canonical monomial structure. Then the results of Section 3 can be applied to Lagrangian algebras in $\mathcal{Z}(\mathrm{Vec}_G^\omega)$.

4A. ***Lagrangian algebras.*** Following Corollary 3.17 of [Davydov and Simmons 2017], we will describe the Lagrangian algebra on $\mathcal{Z}(G, \omega)$ associated to a pair (H, γ), where $H \subseteq G$ is a subgroup and $\gamma : H \times H \to \mathbb{C}^\times$ is a map such that

$$\frac{\gamma(ab, c)\gamma(a, b)}{\gamma(a, bc)\gamma(b, c)} = \omega(a, b, c), \quad a, b, c \in H.$$

Let $\mathbb{C}_\gamma[H] = \oplus_{h \in H} \mathbb{C}e_h$ be the group algebra of H with the multiplication

$$e_{h_1} e_{h_2} = \gamma(h_1, h_2) e_{h_1 h_2}, \quad h_1, h_2 \in H.$$

The vector space $\mathbb{C}_\gamma[H] = \oplus_{h \in H} \mathbb{C}e_h$, is a commutative algebra in $\mathcal{Z}(\mathrm{Vec}_H^\omega)$, where the H-action is given by

$$h_1 \triangleright e_{h_2} = \epsilon(h_1, h_2) e_{h_1 h_2 h_1^{-1}}, \quad \epsilon(h_1, h_2) := \frac{\gamma(h_1, h_2)}{\gamma({}^{h_1}h_2, h_1)}, \quad h_1, h_2 \in H,$$

and grading $|e_h| = h$ for all $h \in H$.

Let $\mathrm{Map}(G, \mathbb{C}_\gamma[H])$ be the vector space of all set-theoretic maps from G to $\mathbb{C}_\gamma[H]$. With the grading given by

$$|a| = f \quad \Leftrightarrow \quad |a(x)| = x^{-1} f x \quad \text{for all } x \in G,$$

and twisted G-action

$$(g \triangleright a)(x) := \omega(x^{-1}, g^{-1}; |a|)^{-1} a(g^{-1} \triangleright x), \quad g, x \in G,$$

$\mathrm{Map}(G, \mathbb{C}_\gamma[H])$ is a twisted Yetter–Drinfeld module.

The Lagrangian algebra $L(H, \gamma)$ is the Yetter–Drinfeld submodule

$$L(H, \gamma) := \{a \in \mathrm{Maps}(G, \mathbb{C}_\gamma[H]) \mid a(xh) = \omega(h^{-1}, x^{-1}; |a|) h^{-1} \triangleright a(x)\};$$

see [Davydov and Simmons 2017] for more details.

4B. *Monomial structure of the Lagrangian algebras $L(H, \gamma)$.* In this section we will prove that every Lagrangian algebra of the form $L(H, \gamma)$ has a canonical monomial structure.

Let G be a group and $H \subset G$ be a subgroup. We can regard $G \times H$ as a left H-set with actions given by $h \triangleright (g, h') = (gh^{-1}, hh'h^{-1})$. Then we can consider the set of H-orbits that we will denote by $G \times_H H$. The set $G \times_H H$ is equipped with a left G-action given by left multiplication on the first component.

Definition 4.1. Let $L(H, \gamma)$ be a Lagrangian. For each $g \in G$ and $f \in H$, define $\chi_{g,f} \in L(H, \gamma)$ by

$$\chi_{g,f}(x) = \begin{cases} 0, & x \notin gH, \\ \omega(h^{-1}, g^{-1}; {}^{g}f)\epsilon(h^{-1}, f) e_{hfh^{-1}}, & x = gh, \text{ where } h \in H. \end{cases} \tag{6}$$

Remark 4.2. The function $\chi_{g,h}$ can be characterized as the unique map in $L(H, \gamma)$ with support gH and such that $\chi_{g,h}(g) = e_h$.

Lemma 4.3. *Let $L(H, \gamma)$ be a Lagrangian algebra in $\mathcal{Z}(G, \omega)$. Then*

$$\chi_{gh,f} = \omega(h, (gh)^{-1}; {}^{gh}f)\epsilon(h, f)\chi_{g,{}^{h}f}, \qquad g \in G,\ f, h \in H. \tag{7}$$

$$l \triangleright \chi_{g,f} = \omega((lg)^{-1}, l^{-1}; {}^{g}f)\chi_{lg,f}, \qquad g, l \in G, h \in H. \tag{8}$$

Proof. Since the supports of $\chi_{gh,f}$ and $\chi_{g,{}^{h}f}$ are gH, and

$$\chi_{gh,f}(g) = \chi_{gh,f}(ghh^{-1}) = \omega(h, (gh)^{-1}; {}^{gh}f)\epsilon(h, f)\chi_{g,{}^{h}f}(g),$$

we obtain (7).

By the definition of the action of G we have

$$l \triangleright \chi_{g,f}(lg) = \omega((lg)^{-1}, l^{-1}; {}^g f)\chi_{g,f}(g) = \omega((lg)^{-1}, l^{-1}; {}^g f)e_f.$$

Since $l \triangleright \chi_{g,f}$ and $\chi_{gl,f}$ are supported in glH, we get (8). □

It follows from Lemma 4.3 that $\mathbb{C}\chi_{gh,{}^h f} = \mathbb{C}\chi_{g,f}$. Then for any $(g,h) \in G \times_H H$ the space $\mathbb{C}\chi_{g,f}$ is well defined.

Theorem 4.4. *Let $L(H,\gamma)$ be a Lagrangian algebra in $\mathcal{Z}(G,\omega)$. Then $L(H,\gamma)$ with the decomposition*

$$L(H,\gamma) = \bigoplus_{(g,h)\in G\times_H H} \mathbb{C}\chi_{g,h}$$

is a monomial twisted Yetter–Drinfeld module.

Proof. First we will check that in fact the sum $\sum_{(g,h)\in G\times_H H} \mathbb{C}\chi_{g,h}$, is direct. Since $\operatorname{supp}(\chi_{g,f}) = gH$, we have that $\chi_{g,f}$ and $\chi_{g',f}$ are linearly independent if $gH \neq g'H$. Hence it is suffices to check linear independence of the collections $\{\chi_{g,f}\}_{f\in H}$, with g fixed. But if $f \neq f'$, $|\chi_{r,f}| \neq |\chi_{r,f'}|$. It follows that the sum $\sum_{(g,h)\in G\times_H H} \mathbb{C}\chi_{g,h}$ is direct.

In order to see that $L(H,\gamma) = \sum_{(g,h)\in G\times_H H} \mathbb{C}\chi_{g,h}$, fix $\mathcal{R} \subset G$, a set of representatives of the left coset of H in G. Let $a \in L(H,\gamma)$. For each $r \in \mathcal{R}$, suppose

$$a(r) = \sum_{f\in H} \lambda_{r,f} e_f. \tag{9}$$

Then we have

$$a = \sum_{r\in\mathcal{R},\, f\in H} \lambda_{r,f}\chi_{r,f} \in \sum_{(g,h)\in G\times_H H} \mathbb{C}\chi_{g,h}. \tag{10}$$

By (8) and the fact that $|\chi_{g,f}| = gfg^{-1}$, we obtain that $L(H,\gamma)$ is a monomial twisted Yetter–Drinfeld module. □

Corollary 4.5. *Let G be a finite group, $\omega \in Z^3(G,\mathbb{C}^\times)$. If $L(H,\gamma)$ is a Lagrangian algebra in $\mathcal{Z}(\mathrm{Vec}_G^\omega)$, then*

(a) *the action of $\mathcal{B}_n$ on $\operatorname{Hom}_{\mathcal{Z}(\mathrm{Vec}_G^\omega)}(\mathbb{C}, L(H,\gamma)^{\otimes n})$ is monomial,*

(b) *the dimension of $\operatorname{Hom}_{\mathcal{Z}(\mathrm{Vec}_G^\omega)}(\mathbb{C}, L(H,\gamma)^{\otimes n})$ is equal to the number of regular G-orbits under the monomial action of G on*

$$(G\times_H H)^{\times n})_e := \{((g_1,h_1),\dots,(g_n,h_n)) : g_1h_1g_1^{-1}g_2h_2g_2^{-1}\cdots g_nh_ng_n^{-1} = e\}.$$

Proof. This follows from Theorem 4.4 and Theorem 3.3. □

We will fix a set of representatives of the left cosets of G in H, $\mathcal{R} \subset G$. Thus every element $g \in G$ has a unique factorization $g = rh$, $h \in H$, $r \in \mathcal{R}$. We assume $e \in \mathcal{R}$. The uniqueness of the factorization $G = \mathcal{R}H$ implies that there are well defined maps

$$\triangleright : G \times \mathcal{R} \to \mathcal{R}, \quad \kappa : G \times \mathcal{R} \to H,$$

determined by the condition

$$gr = (g \triangleright r)\kappa(g, h), \quad g \in G, r \in \mathcal{R}.$$

As a crossed G-set we can identify $G \times_H H$ with $\mathcal{R} \times H$ with action

$$g \triangleright (r, h) := (g \triangleright r, {}^{\kappa(g,r)}h), \qquad r \in \mathcal{R},\ h \in H,\ g \in G,$$

and grading map

$$|-| : \mathcal{R} \times H \to G \qquad (r, h) \mapsto rhr^{-1}.$$

It follows from Theorem 4.4 that $B_{\mathcal{R}} := \{\chi_{r,h} \mid r \in \mathcal{R},\ h \in H\}$ is a basis for $L(H, \gamma)$.

In order to apply the results of Section 3E, we only need to compute the map $\lambda_{\mathcal{R}\times H} : G \times (\mathcal{R} \times H) \to \mathbb{C}^\times$, such that

$$g \triangleright \chi_{r,h} = \lambda_{\mathcal{R}\times H}(g; r, h)\chi_{g\triangleright r, {}^{\kappa(g,r)}h}, \qquad g \in G, r \in \mathcal{R}, h \in H.$$

Using Lemma 4.3 we obtain

$$\lambda_{\mathcal{R}\times H}(g; r, h) = \omega((gr)^{-1}, g^{-1}; {}^{r}h)\omega(\kappa(g, r), (gr)^{-1}; {}^{gr}h)\epsilon(\kappa(g, r), h), \tag{11}$$

for all $g \in G, r \in \mathcal{R}, h \in H$.

By (3), we have that an element $t = ((r_1, h_1), \ldots, (r_n, h_n)) \in (\mathcal{R} \times H)^n_e$ is regular if and only if

$$\lambda_{(\mathcal{R}\times H)^n}(g; (r_1, h_1), \ldots, (r_n, h_n)) = 1, \quad \text{for all } g \in \bigcap_{i=1}^{n} r_i^{-1} C_H(h_i) r_i, \tag{12}$$

where $\lambda_{(\mathcal{R}\times H)^n}$ was defined in (2) as a function of $\lambda_{\mathcal{R}\times H}$ and ω.

4C. ***Applications and examples.*** In this last section we present some applications of the results of the previous section.

4C1. *Central subgroups.*

Proposition 4.6. *Let G be a finite group and $L(H, \gamma)$ a Lagrangian algebra in $\mathcal{Z}(\mathrm{Vec}_G)$, where $H \subset G$ is a central subgroup. Then*

$$\dim(\mathrm{Hom}_{\mathcal{Z}(\mathrm{Vec}_G)}(\mathbb{C}, L(H, \gamma)^{\otimes n})) = |G|^{n-1}.$$

Moreover, the representation of $\mathcal{B}_n$ is actually a representation of S_n.

Proof. Since H is a central subgroup, $g \triangleright (r, h) = (g \triangleright r, h)$ and

$$|\chi_{r_1,h_1} \otimes \cdots \otimes \chi_{r_k,h_k}| = h_1 \cdots h_k,$$

for any $r_1, \ldots, r_k \in R, h_1, \ldots, h_n \in H$. Hence,

$$|(\mathcal{R} \times H)_e^n| = |(\mathcal{R}^n/G)||H^{n-1}| = [G : H]^n|H|^{n-1} = |G|^{n-1}.$$

To determine the number of orbits, notice that $\epsilon : H \times H \to \mathbb{C}^\times$ is a bicharacter such that $\epsilon(h_1, h_2)\epsilon(h_2, h_1) = 1$. Then, by (12), an element

$$((r_1, h_1), \ldots, (r_n, h_n)) \in (\mathcal{R} \times H)_e^n$$

is regular if and only if

$$\prod_{i=1}^{n} \epsilon(h, h_i) = 1, \quad \text{for all } h \in H.$$

But $\prod_{i=1}^{n} \epsilon(h, h_i) = \epsilon(h, h_1 \cdots h_n) = \epsilon(h', e) = 1$. Hence every element is regular. By Corollary 4.5 the dimension of $\mathrm{Hom}_{\mathcal{Z}(\mathrm{Vec}_G)}(\mathbb{C}, L(H, \gamma)^{\otimes n})$ is $|G|^{n-1}$.

Finally, using (4), we see that

$$\begin{aligned}\sigma_i' \circ \sigma_i'(\chi_{r_1,h_1} \otimes \cdots \otimes \chi_{r_n,h_n}) &= \epsilon(h_i, h_{i+1})\epsilon(h_{i+1}, h_i)(\chi_{r_1,h_1} \otimes \cdots \otimes \chi_{r_n,h_n}) \\ &= \chi_{r_1,h_1} \otimes \cdots \otimes \chi_{r_n,h_n}.\end{aligned}$$

Hence representation of $\mathcal{B}_n$ factors as a representation of S_n. □

4C2. *Lagrangian algebra of the form $L(H, 1)$.* The Lagrangian algebras $L(H, 1)$ as an object in $\mathcal{Z}(\mathrm{Vec}_G)$ are completely determined by the crossed G-set $G \times_H H$, and the monomial representation $\mathrm{Hom}(\mathbb{C}, L(H, 1)^{\otimes n})$ is a permutation representation; see Example 3.5. Let us see some extreme cases:

Case $H = \{e\}$. In this case the crossed G-set is G with the regular action and grading map the constant map e. It is clear that the braiding $c_{G,G}$ is just the flip map

$$(g_1, g_2) \mapsto (g_2, g_1),$$

hence, really the symmetric group S_n acts on G^n.

The set of G-orbits is in bijection with G^{n-1},

$$\begin{aligned}\mathcal{O}(G^n) &\to G^{n-1}, \\ \mathcal{O}_G(g_1, g_2, \ldots, g_n) &\mapsto (e, g_1^{-1}g_2, \ldots, g_1^{-1}g_n).\end{aligned}$$

Using the previous map the action of $\S_n$ is given by

$$\sigma_1(g_1, \ldots, g_{n-1}) = (g_1^{-1}, g_1^{-1}g_2, \ldots, g_1^{-1}g_{n-1})$$

and

$$\sigma_i(g_1, \dots, g_i, g_{i+1}, \dots, g_{n-1}) = (g_1, \dots, g_{i+1}, g_i, \dots, g_{n-1}), \quad 1 < i < n.$$

It is clear that permutation action of S_n on G^{n-1} is faithful; thus the image is isomorphic to S_n.

Case $H = G$. In this case the crossed G-set is G with the action by conjugation and grading map the identity map. Hence, the braiding is given by

$$c_{G,G} : (x, y) \mapsto (y, y^{-1}xy).$$

Note $c_{G,G}$ is symmetric if and only if G is abelian.

If G is abelian, $G^n_e = \{(g_1, \dots, g_{n-1}, (g_1, \dots, g_{n-1})^{-1})\}$ is the set of orbits and, as in the previous example, the group S_n acts faithfully.

4C3. *Dihedral group.* Every time we take H to be a normal subgroup of G, the following proposition provides a way to simplify the situation.

Proposition 4.7. *Let G be a finite group, $H \trianglelefteq G$, and $\mathcal{R}$ a collection of representatives for G/H. Define $B_\gamma[H] \in \mathcal{Z}(\mathrm{Vec}_G)$ as*

$$B(H, \gamma) := \operatorname{span}\{b_{r,h} \mid r \in \mathcal{R}, h \in H\},$$

with grading $|b_{r,h}| = h$ and the G-action

$$g \triangleright b_{r,h} = \epsilon(\kappa(g, r)\, {}^{r^{-1}}h) b_{g \triangleright r, {}^g h}. \tag{13}$$

Then, the mapping

$$B(H, \gamma) \to L(H, \gamma), \quad b_{r,h} \mapsto \chi_{r, {}^{r^{-1}}h}$$

is an isomorphism in $\mathcal{Z}(\mathrm{Vec}_G)$.

Proof. We need to show the map preserves the grading and the G-representation. We have

$$|\chi_{r, {}^{r^{-1}}h}| = {}^{r}({}^{r^{-1}}h) = h = |b_{r,h}|.$$

Now, since

$$g \cdot \chi_{r, {}^{r^{-1}}h} = \epsilon(\kappa(g, r), {}^{r^{-1}}h) \chi_{g \triangleright r, {}^{\kappa(g,r)}({}^{r^{-1}}h)},$$

and

$${}^{\kappa(g,r)}({}^{r^{-1}}h) = {}^{(g \triangleright r)^{-1}}ghg^{-1},$$

we have that

$$g \triangleright b_{r,h} = \epsilon(\kappa(g, r), {}^{r^{-1}}h) a_{g \triangleright r, {}^{(g \triangleright r)^{-1}}({}^g h)}.$$

Hence, by (13) the map is equivariant. $\square$

Proposition 4.7 works particularly well when $\gamma = 1$, since (13) is just

$$g \triangleright b_{r,h} = b_{g \triangleright r, {}^g h}.$$

Thus, the action of G is "decoupled". We use this idea in the following example.

Let $G = D_{2k}$ be the dihedral groups of order $2k$ and $H = \langle r \rangle$. We take $\mathcal{R} = \{e, s\} = \{s^i\}_{i \in \mathbb{Z}/2\mathbb{Z}}$. Then

$$|b_{s^{i_1}, r^{j_1}} \otimes \cdots \otimes b_{s^{i_n}, r^{j_n}}| = r^{\sum_{m=1}^{n} j_m},$$

and

$$\dim(B(H, \gamma)_e^{\otimes n}) = 2^n \times k^{n-1}.$$

Since

$$(s^i r^j)(s^k) = s^{i+k} r^{(-1)^k j},$$

we have

$$(s^i r^j) \triangleright s^k = s^{i+k} \quad \text{and} \quad \kappa(s^i r^j, s^k) = r^{(-1)^k j}.$$

Hence, the action, on the set label is

$$s^i r^j (s^k, r^l) = (s^{i+k}, r^l).$$

It follows that the number of orbits in $(\mathcal{R} \times H)_e^n$ is

$$2^{n-1} \times k^{n-1} = |G|^{n-1}.$$

Since $\gamma = 1$ all orbits are regular and so $\dim(\mathrm{Hom}_{\mathcal{Z}(\mathrm{Vec}_G)}(\mathbb{C}, L(H, 1)^{\otimes n})) = |G|^{n-1}$.

References

[Beigi et al. 2011] S. Beigi, P. W. Shor, and D. Whalen, "The quantum double model with boundary: condensations and symmetries", *Comm. Math. Phys.* **306**:3 (2011), 663–694. MR Zbl

[Bombin and Martin-Delgado 2008] H. Bombin and M. A. Martin-Delgado, "Family of non-abelian Kitaev models on a lattice: topological condensation and confinement", *Phys. Rev. B* **78**:11 (2008), art. id. 115421.

[Bravyi and Kitaev 1998] S. B. Bravyi and A. Y. Kitaev, "Quantum codes on a lattice with boundary", preprint, 1998. arXiv

[Cong et al. 2016] I. Cong, M. Cheng, and Z. Wang, "Topological quantum computation with gapped boundaries", preprint, 2016. arXiv

[Cong et al. 2017a] I. Cong, M. Cheng, and Z. Wang, "Defects between gapped boundaries in two-dimensional topological phases of matter", *Phys. Rev. B* **96**:19 (2017), art. id. 19529.

[Cong et al. 2017b] I. Cong, M. Cheng, and Z. Wang, "Hamiltonian and algebraic theories of gapped boundaries in topological phases of matter", *Comm. Math. Phys.* **355**:2 (2017), 645–689. Correction in **360**:3 (2018), 1189–1190. MR Zbl

[Cong et al. 2017c] I. Cong, M. Cheng, and Z. Wang, "Universal quantum computation with gapped boundaries", *Phys. Rev. Lett.* **119**:17 (2017), art. id. 170504. MR

[Davydov 2010] A. Davydov, "Modular invariants for group-theoretical modular data, I", *J. Algebra* **323**:5 (2010), 1321–1348. MR Zbl

[Davydov and Simmons 2017] A. Davydov and D. Simmons, "On Lagrangian algebras in group-theoretical braided fusion categories", *J. Algebra* **471** (2017), 149–175. MR Zbl

[Davydov et al. 2013] A. Davydov, M. Müger, D. Nikshych, and V. Ostrik, "The Witt group of non-degenerate braided fusion categories", *J. Reine Angew. Math.* **677** (2013), 135–177. MR Zbl

[Dijkgraaf et al. 1991] R. Dijkgraaf, V. Pasquier, and P. Roche, "Quasi Hopf algebras, group cohomology and orbifold models", *Nuclear Phys. B Proc. Suppl.* **18B** (1991), 60–72. MR Zbl

[Etingof et al. 2008] P. Etingof, E. Rowell, and S. Witherspoon, "Braid group representations from twisted quantum doubles of finite groups", *Pacific J. Math.* **234**:1 (2008), 33–41. MR Zbl

[Freyd and Yetter 1989] P. J. Freyd and D. N. Yetter, "Braided compact closed categories with applications to low-dimensional topology", *Adv. Math.* **77**:2 (1989), 156–182. MR Zbl

[Galindo and Rowell 2014] C. Galindo and E. C. Rowell, "Braid representations from unitary braided vector spaces", *J. Math. Phys.* **55**:6 (2014), art. id. 061702. MR Zbl

[Karpilovsky 1985] G. Karpilovsky, *Projective representations of finite groups*, Monographs and Textbooks in Pure and Appl. Math. **94**, Dekker, New York, 1985. MR Zbl

[Kitaev 2003] A. Y. Kitaev, "Fault-tolerant quantum computation by anyons", *Ann. Physics* **303**:1 (2003), 2–30. MR Zbl

[Kitaev and Kong 2012] A. Kitaev and L. Kong, "Models for gapped boundaries and domain walls", *Comm. Math. Phys.* **313**:2 (2012), 351–373. MR Zbl

[Natale 2017] S. Natale, "On the equivalence of module categories over a group-theoretical fusion category", *Symmetry Integrab. Geom. Meth. Appl.* **13** (2017), art. id. 042. MR Zbl

Received July 12, 2017. Revised August 21, 2018.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 300, No. 1, 2019
dx.doi.org/10.2140/pjm.2019.300.17

SPACELIKE HYPERSURFACES WITH CONSTANT CONFORMAL SECTIONAL CURVATURE IN $\mathbb{R}_1^{n+1}$

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Let $f : M^n \to \mathbb{R}_1^{n+1}$ be an n-dimensional umbilic-free spacelike hypersurface in the $(n+1)$-dimensional Lorentzian space $\mathbb{R}_1^{n+1}$. One can define the conformal metric g on f which is invariant under the conformal transformation group of $\mathbb{R}_1^{n+1}$. We classify the n-dimensional spacelike hypersurfaces with constant sectional curvature with respect to the conformal metric g when $n \geq 3$. Such spacelike hypersurfaces are obtained by the standard construction of cylinders, cones or hypersurfaces of revolution over certain spirals in the 2-dimensional Lorentzian space forms $\mathbb{S}_1^2(1)$, $\mathbb{R}_1^2$, $\mathbb{R}_{1+}^2$, respectively.

1. Introduction

Recently the Möbius geometry of submanifolds in Riemannian space forms has been studied extensively and many special hypersurfaces were classified under the Möbius transformation group (see [Guo et al. 2012; Hu and Li 2003; Li et al. 2013; Li and Wang 2003]). As its parallel generalization, the conformal geometry of submanifolds in Lorentzian space forms is another important branch of submanifold theory; however there are fewer results (see [Li and Nie 2013; 2018; Nie 2015]). In this paper, we investigate the spacelike hypersurfaces with constant conformal sectional curvature. Since the conformal geometry of spacelike hypersurfaces in Lorentzian space forms $M_1^{n+1}(c)$ is uniform by the conformal map (2-4) (see Section 2), we only consider the hypersurfaces in $\mathbb{R}_1^{n+1}$.

Let $f : M^n \to \mathbb{R}_1^{n+1}$ be an n-dimensional umbilic-free spacelike hypersurface in the $(n+1)$-dimensional Lorentzian space $\mathbb{R}_1^{n+1}$. Given the first fundamental form $I = df \cdot df$ as well as a local orthonormal basis $\{e_i\}$ and the dual basis $\{\theta_i\}$, we denote $II = \sum_{ij} h_{ij}\theta_i \otimes \theta_j$ the second fundamental form and $H = \frac{1}{n}\sum_i h_{ii}$ the mean curvature. The conformal metric of f,

$$g = \rho^2 df \cdot df = \frac{n}{n-1}(\| II \|^2 - nH^2)I, \tag{1-1}$$

is a Riemannian metric which is invariant under the conformal transformations

MSC2010: 53A30, 53B25.
Keywords: conformal metric, conformal sectional curvature, conformal second fundamental form, curvature-spiral.

of $\mathbb{R}^{n+1}_1$. Together with another quadratic form (called the conformal second fundamental form) they form a complete system of invariants for the spacelike hypersurface ($n \geq 3$) in conformal geometry of the Lorentzian space $\mathbb{R}^{n+1}_1$ (see Section 2). In this framework, a notable class of spacelike hypersurfaces are those with constant conformal sectional curvature (i.e., constant sectional curvature with respect to the conformal metric g). Here we classify them up to a conformal transformation of the Lorentzian space $\mathbb{R}^{n+1}_1$, and our main result is stated below.

Theorem 1.1. *Let $f : M^n \to \mathbb{R}^{n+1}_1$, $n \geq 3$, be an umbilic-free spacelike hypersurface with constant conformal sectional curvature δ. Then locally f is conformally equivalent to one of the following hypersurfaces*:

(i) *A cylinder over a curvature-spiral in a Lorentzian 2-plane $\mathbb{R}^2_1$ (where $\delta \leq 0$).*

(ii) *A cone over a curvature-spiral in a de Sitter 2-sphere $\mathbb{S}^2_1 \subset \mathbb{R}^3_1$ (where $\delta < 0$).*

(iii) *A rotational hypersurface over a curvature-spiral in a Lorentzian hyperbolic 2-plane $\mathbb{R}^2_{1+} \subset \mathbb{R}^2_1$ (the constant curvature δ could be positive, negative or zero).*

(iv) *A cone over the hyperbolic torus $\mathbb{H}^q(-\sqrt{a^2-1}) \times \mathbb{S}^p(a)$, $a > 1$, (where $\delta = 0$).*

The *curvature-spiral* $\gamma(s) \in N^2_1(\epsilon)$ in a 2-dimensional Lorentzian space form $N^2_1(\epsilon)(= \mathbb{S}^2_1(1),\ \mathbb{R}^2_1,\ \mathbb{R}^2_{1+}$ for Gauss curvature $\epsilon = 1, 0, -1$, respectively) is a spacelike curve which is determined by the intrinsic equation

$$\left[\frac{d}{ds}\frac{1}{\kappa}\right]^2 + \epsilon\left[\frac{1}{\kappa}\right]^2 = -\delta, \tag{1-2}$$

where s is the arc-length parameter, and κ denotes the geodesic curvature of the spacelike curve γ, and δ is a real constant. Note that (1-2) is equivalent to the harmonic oscillator equation $(1/\kappa)'' + \epsilon/\kappa = 0$ for the function $\kappa(s)$. It is easy to see that for fixed ϵ and δ the solution curve is unique (because $N^2_1(\epsilon)$ is two-point homogeneous, since any two solutions with arbitrary initial values are congruent to each other).

The Lorentzian hyperbolic 2-plane $\mathbb{R}^2_{1+} \subset \mathbb{R}^2_1$ is defined by

$$\mathbb{R}^2_{1+} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\},$$

endowed with the Lorentzian metric $ds^2 = \frac{1}{y^2}(-dx^2 + dy^2)$. The Gauss curvature of $\mathbb{R}^2_{1+}$ is $\epsilon = -1$ with respect to the Lorentzian metric ds^2. Let $\mathbb{H}^2_1(-1)$ be a 2-dimensional anti-de Sitter sphere; there exists a standard isometric embedding

$$\phi : \mathbb{R}^2_{1+} \to \mathbb{H}^2_1(-1), \quad \phi(x, y) = \left(\frac{y^2 - x^2 + 1}{2y}, \frac{x}{y}, \frac{y^2 - x^2 - 1}{2y}\right). \tag{1-3}$$

The rest of this paper is organized as follows. In Section 2, we study the conformal geometry of spacelike hypersurfaces in Lorentzian space forms $M_1^{n+1}(c)$. In Section 3, we construct some examples of the spacelike hypersurfaces with constant conformal sectional curvature. In Section 4, we give the proof of Theorem 1.1.

2. Conformal geometry of spacelike hypersurfaces

In this section, following Wang [1998], we define some conformal invariants on a spacelike hypersurface and give a congruent theorem of the spacelike hypersurfaces under the conformal group of Lorentzian space forms $M_1^{n+1}(c)$.

Let $\mathbb{R}^{n+2}_s$ be the real vector space $\mathbb{R}^{n+2}$ with the Lorentzian product $\langle\, ,\, \rangle_s$ given by

$$\langle X, Y\rangle_s = -\sum_{i=1}^{s} x_i y_i + \sum_{j=s+1}^{n+2} x_j y_j.$$

For any $a > 0$, the standard sphere $\mathbb{S}^{n+1}(a)$, the hyperbolic space $\mathbb{H}^{n+1}(-a)$, the de Sitter space $\mathbb{S}^{n+1}_1(a)$ and the anti-de Sitter space $\mathbb{H}^{n+1}_1(-a)$ are defined by

$$\begin{aligned}
\mathbb{S}^{n+1}(a) &= \{x \in \mathbb{R}^{n+2} \mid x \cdot x = a^2\},\\
\mathbb{H}^{n+1}(-a) &= \{x \in \mathbb{R}^{n+2}_1 \mid \langle x, x\rangle_1 = -a^2\},\\
\mathbb{S}^{n+1}_1(a) &= \{x \in \mathbb{R}^{n+2}_1 \mid \langle x, x\rangle_1 = a^2\},\\
\mathbb{H}^{n+1}_1(-a) &= \{x \in \mathbb{R}^{n+2}_2 \mid \langle x, x\rangle_2 = -a^2\}.
\end{aligned}$$

Let $M_1^{n+1}(c)$ be a Lorentzian space form. When $c=0$, $M_1^{n+1}(c)=\mathbb{R}^{n+1}_1$; when $c=1$, $M_1^{n+1}(c) = \mathbb{S}^{n+1}_1(1)$; when $c=-1$, $M_1^{n+1}(c) = \mathbb{H}^{n+1}_1(-1)$.

Denoting by C^{n+2} the cone in $\mathbb{R}^{n+3}_2$ and by $\mathbb{Q}^{n+1}_1$ the conformal compactification space in $\mathbb{R}P^{n+3}$,

$$C^{n+2} = \{X \in \mathbb{R}^{n+3}_2 \mid \langle X, X\rangle_2 = 0,\, X \neq 0\}, \qquad \mathbb{Q}^{n+1}_1 = \{[X] \in \mathbb{R}P^{n+2} \mid \langle X, X\rangle_2 = 0\}.$$

Let $O(n+3, 2)$ be the Lorentzian group of $\mathbb{R}^{n+3}_2$ keeping the Lorentzian product $\langle X, Y\rangle_2$ invariant. Then $O(n+3, 2)$ is a transformation group on $\mathbb{Q}^{n+1}_1$ defined by

$$T([X]) = [XT], \quad X \in C^{n+2}, \quad T \in O(n+3, 2).$$

Topologically, $\mathbb{Q}^{n+1}_1$ is identified with the compact space $\mathbb{S}^n \times \mathbb{S}^1/\mathbb{S}^0$, which is endowed by a standard Lorentzian metric

$$h = g_{\mathbb{S}^n} \oplus (-g_{\mathbb{S}^1}),$$

where $g_{\mathbb{S}^k}$ denotes the standard metric of the k-dimensional sphere $\mathbb{S}^k$. Therefore, $\mathbb{Q}^{n+1}_1$ has conformal metric $[h] = \{e^\tau h\}$, $\tau \in C^\infty(\mathbb{Q}^{n+1}_1)$, and $[O(n+3, 2)]$ is the conformal transformation group of $\mathbb{Q}^{n+1}_1$(see [Cahen and Kerbrat 1983; O'Neill 1983]).

Setting $P = \{[X] \in \mathbb{Q}_1^{n+1} \mid x_1 = x_{n+3}\}$, $P_- = \{[X] \in \mathbb{Q}_1^{n+1} \mid x_{n+3} = 0\}$, and $P_+ = \{[X] \in \mathbb{Q}_1^{n+1} \mid x_1 = 0\}$, we can define the following conformal diffeomorphisms

$$\begin{aligned}
&\sigma_0 : \mathbb{R}_1^{n+1} \to \mathbb{Q}_1^{n+1} \backslash P, \quad && u \mapsto \left[\left(\frac{1 + \langle u, u\rangle_1}{2}, u, \frac{\langle u, u\rangle_1 - 1}{2}\right)\right], \\
&\sigma_1 : \mathbb{S}_1^{n+1}(1) \to \mathbb{Q}_1^{n+1} \backslash P_+, \quad && u \mapsto [(1, u)], \\
&\sigma_{-1} : \mathbb{H}_1^{n+1}(-1) \to \mathbb{Q}_1^{n+1} \backslash P_-, \quad && u \mapsto [(u, 1)].
\end{aligned} \tag{2-4}$$

We may regard $\mathbb{Q}_1^{n+1}$ as the common compactification of $\mathbb{R}_1^{n+1}$, $\mathbb{S}_1^{n+1}(1)$, $\mathbb{H}_1^{n+1}(-1)$.

Let $f : M^n \to M_1^{n+1}(c)$ be a spacelike hypersurface. Using σ_c, we obtain the hypersurface $\sigma_c \circ f : M^n \to \mathbb{Q}_1^{n+1}$ in $\mathbb{Q}_1^{n+1}$. From [Cahen and Kerbrat 1983], we have the following theorem:

Theorem 2.1. *Two hypersurfaces $f, \bar{f} : M^n \to M_1^{n+1}(c)$ are conformally equivalent if and only if there exists $T \in O(n+3, 2)$ such that $\sigma_c \circ f = T(\sigma_c \circ \bar{f}) : M^n \to \mathbb{Q}_1^{n+1}$.*

Since $f : M^n \to M_1^{n+1}(c)$ is a spacelike hypersurface, $(\sigma_c \circ f)_*(TM^n)$ is a positive definite subbundle of $T\mathbb{Q}_1^{n+1}$. For any local lift Z of the standard projection $\pi : C^{n+2} \to \mathbb{Q}_1^{n+1}$, we get a local lift $y = Z \circ \sigma_c \circ f : U \to C^{n+1}$ of $\sigma_c \circ f : M \to \mathbb{Q}_1^{n+1}$ in an open subset U of M^n. Thus $\langle \mathrm{d}y, \mathrm{d}y\rangle_2 = \rho^2 \langle df, df\rangle_s$ is a local metric, where $\rho \in C^\infty(U)$. We denote by Δ and κ the Laplacian operator and the normalized scalar curvature with respect to the local positive definite metric $\langle \mathrm{d}y, \mathrm{d}y\rangle_2$, respectively. Much as in the proof of Theorem 1.2 in [Wang 1998], we can get the following theorem:

Theorem 2.2. *Let $f : M^n \to M_1^{n+1}(c)$ be a spacelike hypersurface, then the 2-form $g = -(\langle \Delta y, \Delta y\rangle_2 - n^2\kappa)\langle dy, dy\rangle_2$ is a globally defined conformal invariant. Moreover, g is positive definite at any nonumbilical point of M^n.*

We call g the conformal metric of the spacelike hypersurface f, and there exists a unique lift

$$Y : M^n \to C^{n+2}$$

such that $g = \langle \mathrm{d}Y, \mathrm{d}Y\rangle_2$. We call Y the conformal position vector of the spacelike hypersurface f. Theorem 2.2 implies the following:

Theorem 2.3. *Two spacelike hypersurfaces $f, \bar{f} : M^n \to M_1^{n+1}(c)$ are conformally equivalent if and only if there exists $T \in O(n+3, 2)$ such that $\overline{Y} = YT$, where Y and $\overline{Y}$ are the conformal position vectors of f and $\bar{f}$, respectively.*

Let $\{E_1, \ldots, E_n\}$ be a local orthonormal basis of M^n with respect to g with dual basis $\{\omega_1, \ldots, \omega_n\}$. Denote $Y_i = E_i(Y)$ and define

$$N = -\frac{1}{n}\Delta Y - \frac{1}{2n^2}\langle \Delta Y, \Delta Y\rangle_2 Y,$$

where Δ is the Laplace operator of g, then we have

$$\langle N, Y\rangle_2 = 1,\ \langle N, N\rangle_2 = 0,\ \langle N, Y_k\rangle_2 = 0,\ \langle Y_i, Y_j\rangle_2 = \delta_{ij}, \quad 1 \le i, j, k \le n.$$

We may decompose $\mathbb{R}^{n+3}_2$ such that

$$\mathbb{R}^{n+3}_2 = \mathrm{span}\{Y, N\} \oplus \mathrm{span}\{Y_1, \dots, Y_n\} \oplus \mathbb{V},$$

where $\mathbb{V} \perp \mathrm{span}\{Y, N, Y_1, \dots, Y_n\}$. We call $\mathbb{V}$ the conformal normal bundle of f, which is a linear bundle. Let ξ be a local section of $\mathbb{V}$ and $\langle \xi, \xi\rangle_2 = -1$, then $\{Y, N, Y_1, \dots, Y_n, \xi\}$ forms a moving frame in $\mathbb{R}^{n+3}_2$ along M^n. We write the structure equations as follows:

$$\begin{aligned}
\mathrm{d}Y &= \sum_i \omega_i Y_i, \qquad \mathrm{d}N = \sum_{ij} A_{ij}\omega_j Y_i + \sum_i C_i \omega_i \xi,\\
\mathrm{d}Y_i &= -\sum_j A_{ij}\omega_j Y - \omega_i N + \sum_j \omega_{ij} Y_j + \sum_j B_{ij}\omega_j \xi,\\
\mathrm{d}\xi &= \sum_i C_i \omega_i Y + \sum_{ij} B_{ij}\omega_j Y_i,
\end{aligned} \tag{2-5}$$

where $\omega_{ij} (= -\omega_{ji})$ are the connection 1-forms on M^n with respect to $\{\omega_1, \dots, \omega_n\}$. It is clear that $A = \sum_{ij} A_{ij}\omega_j \otimes \omega_i$, $B = \sum_{ij} B_{ij}\omega_j \otimes \omega_i$, $C = \sum_i C_i \omega_i$ are globally defined conformal invariants. We call A, B and C the Blaschke tensor, the conformal second fundamental form, and the conformal 1-form, respectively. The covariant derivatives of these tensors are defined by

$$\begin{aligned}
\sum_j C_{i,j}\omega_j &= dC_i + \sum_k C_k \omega_{kj},\\
\sum_k A_{ij,k}\omega_k &= dA_{ij} + \sum_k A_{ik}\omega_{kj} + \sum_k A_{kj}\omega_{ki},\\
\sum_k B_{ij,k}\omega_k &= dB_{ij} + \sum_k B_{ik}\omega_{kj} + \sum_k B_{kj}\omega_{ki}.
\end{aligned}$$

By exterior differentiation of the structure equations (2-5), we can get the integrable conditions of the structure equations

$$A_{ij} = A_{ji}, \quad B_{ij} = B_{ji}, \tag{2-6}$$

$$A_{ij,k} - A_{ik,j} = B_{ij}C_k - B_{ik}C_j, \tag{2-7}$$

$$B_{ij,k} - B_{ik,j} = \delta_{ij}C_k - \delta_{ik}C_j, \tag{2-8}$$

$$C_{i,j} - C_{j,i} = \sum_k (B_{ik}A_{kj} - B_{jk}A_{ki}), \tag{2-9}$$

$$R_{ijkl} = B_{il}B_{jk} - B_{ik}B_{jl} + A_{ik}\delta_{jl} + A_{jl}\delta_{ik} - A_{il}\delta_{jk} - A_{jk}\delta_{il}. \tag{2-10}$$

Furthermore, we have

$$\operatorname{tr}(A)=\frac{1}{2n}(n^2\kappa-1),\qquad R_{ij}=\operatorname{tr}(A)\delta_{ij}+(n-2)A_{ij}+\sum_k B_{ik}B_{kj},$$
$$(1-n)C_i=\sum_j B_{ij,j},\qquad \sum_{ij}B_{ij}^2=\frac{n-1}{n},\qquad \sum_i B_{ii}=0, \tag{2-11}$$

where κ is the normalized scalar curvature of g. From (2-11), we see that when $n\geq 3$, all coefficients in the structure equations are determined by the conformal metric g and the conformal second fundamental form B, thus we get the congruent theorem:

Theorem 2.4. *Two spacelike hypersurfaces $f, \bar{f} : M^n \to M_1^{n+1}(c)$, $n \geq 3$, are conformally equivalent if and only if there exists a diffeomorphism $\varphi : M^n \to M^n$ which preserves the conformal metric g and the conformal second fundamental form B.*

The second covariant derivative of the conformal second fundamental form B_{ij} is defined by

$$\sum_m B_{ij,km}\omega_m = dB_{ij,k}+\sum_m B_{mj,k}\omega_{mi}+\sum_m B_{im,k}\omega_{mj}+\sum_m B_{ij,m}\omega_{mk}. \tag{2-12}$$

Thus we have the following Ricci identities

$$B_{ij,kl}-B_{ij,lk}=\sum_m B_{mj}R_{mikl}+\sum_m B_{im}R_{mjkl}. \tag{2-13}$$

Next we give the relations between the conformal invariants and the isometric invariants of a spacelike hypersurface in $\mathbb{R}_1^{n+1}$.

Assume that $f : M^n \to \mathbb{R}_1^{n+1}$ is an umbilic-free spacelike hypersurface. Let $\{e_1, \dots, e_n\}$ be an orthonormal local basis with respect to the induced metric $I=\langle df, df\rangle_1$ with dual basis $\{\theta_1,\dots,\theta_n\}$. Let e_{n+1} be a normal vector field of f, $\langle e_{n+1}, e_{n+1}\rangle_1=-1$. Let $II=\sum_{ij}h_{ij}\theta_i\otimes\theta_j$ denote the second fundamental form, $H=\frac{1}{n}\sum_i h_{ii}$ the mean curvature. Denote by Δ_M the Laplacian operator and κ_M the normalized scalar curvature for I. By the structure equation of $f : M^n \to \mathbb{R}_1^{n+1}$ we get

$$\Delta_M f = nHe_{n+1}. \tag{2-14}$$

There is a local lift of f

$$y : M^n \to C^{n+2},\qquad y=\left(\frac{\langle f, f\rangle_1+1}{2}, f, \frac{\langle f, f\rangle_1-1}{2}\right).$$

It follows from (2-14) that $\langle\Delta y, \Delta y\rangle_2 - n^2\kappa_M = \frac{n}{n-1}(-|II|^2+n|H|^2) = -e^{2\tau}$. Therefore the conformal metric g, the conformal position vector of f, and ξ are expressed as

$$
\begin{aligned}
g &= \frac{n}{n-1}(|II|^2 - n|H|^2)\langle \mathrm{d}f, \mathrm{d}f\rangle_1 = e^{2\tau} I, \quad Y = e^{\tau} y,\\
\xi &= -Hy + (\langle f, e_{n+1}\rangle_1, e_{n+1}, \langle f, e_{n+1}\rangle_1).
\end{aligned} \tag{2-15}
$$

By a direct calculation we get the following expression of the conformal invariants

$$
\begin{aligned}
A_{ij} &= e^{-2\tau}\big[\tau_i\tau_j - h_{ij}H - \tau_{i,j} + \tfrac{1}{2}(-|\nabla\tau|^2 + |H|^2)\delta_{ij}\big],\\
B_{ij} &= e^{-\tau}(h_{ij} - H\delta_{ij}), \quad C_i = e^{-2\tau}\Big(H\tau_i - H_i - \sum_j h_{ij}\tau_j\Big),
\end{aligned} \tag{2-16}
$$

where $\tau_i = e_i(\tau)$ and $|\nabla\tau|^2 = \sum_i \tau_i^2$, and $\tau_{i,j}$ is the Hessian of τ for I and $H_i = e_i(H)$.

3. Typical examples

In this section, we construct some spacelike hypersurfaces with constant conformal sectional curvature. Such spacelike hypersurfaces are obtained by the standard construction of cylinders, cones or hypersurfaces of revolution over curvature-spirals in $N_1^2(\epsilon)$. A key observation is that the conformal metric of those spacelike hypersurfaces constructed over these curvature-spirals are of the form

$$g = \kappa(s)^2(ds^2 + I^{n-1}_{-\epsilon}),$$

where $I^{n-1}_{-\epsilon}$ is the metric of the $(n-1)$-dimensional Riemannian space form of constant curvature $-\epsilon$. For such metric forms we have the following result:

Lemma 3.1. *The metric $g = \kappa(s)^2(ds^2 + I^{n-1}_{-\epsilon})$ given above has constant curvature δ if and only if the function $\kappa(s)$ satisfies* (1-2).

This lemma is easy to prove using exterior differential forms and we omit the proof. Next we give the explicit construction of the spacelike hypersurfaces.

Example 3.2. The cylinder in $\mathbb{R}^{n+1}_1$ over a curve $\gamma(s) \subset \mathbb{R}^2_1$ is defined by

$$f : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n+1}_1, \quad f(s, y) = (\gamma(s), y),$$

where $y \in \mathbb{R}^{n-1}$.

The first and the second fundamental form of the cylinder f are given by

$$I = ds^2 + I_{\mathbb{R}^{n-1}}, \quad II = \kappa ds^2,$$

where $\kappa(s)$ is the geodesic curvature of $\gamma(s) \subset \mathbb{R}^2_1$, and $I_{\mathbb{R}^{n-1}}$ denotes the standard metric of the $(n-1)$-dimensional Euclidean space $\mathbb{R}^{n-1}$. Thus the principal curvatures of the cylinder are $(\kappa, 0, \dots, 0)$, and the mean curvature $H = \frac{\kappa}{n}$. From (2-15), we see that the conformal metric of the cylinder f is $g = \kappa^2(s)(ds^2 + I_{\mathbb{R}^{n-1}})$. By Lemma 3.1 we have the following result:

Proposition 3.3. *The cylinder in $\mathbb{R}^{n+1}_1$ over $\gamma(s) \subset \mathbb{R}^2_1$ as in Example 3.2 is of constant conformal sectional curvature if and only if $\gamma(s)$ is a curvature-spiral in $\mathbb{R}^2_1$.*

Example 3.4. The cone in $\mathbb{R}^{n+1}_1$ over a curve $\gamma(s) \subset \mathbb{S}^2_1(1) \subset \mathbb{R}^3_1$ is defined by

$$f : \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^{n-2} \to \mathbb{R}^{n+1}_1, \quad f(s, t, y) = (t\gamma(s), y),$$

where $y \in \mathbb{R}^{n-2}$ and $\mathbb{R}^+ = \{t \mid t > 0\}$.

The first and the second fundamental form of the cone f are given by

$$I = t^2 ds^2 + I_{\mathbb{R}^{n-1}}, \quad II = t\kappa ds^2,$$

where $\kappa(s)$ is the geodesic curvature of $\gamma(s) \subset \mathbb{S}^2_1(1)$. Thus the principal curvatures of the cone are $(\frac{\kappa}{t}, 0, \dots, 0)$, and the mean curvature $H = \frac{\kappa}{nt}$. From (2-15), we know that the conformal position vector of the cone f is

$$Y = \kappa\left(\frac{t^2 + |y|^2 + 1}{2t},\ \gamma(s),\ \frac{y}{t},\ \frac{t^2 + |y|^2 - 1}{2t}\right).$$

Note that

$$i(t, y) = \left(\frac{t^2+|y|^2+1}{2t}, \frac{y}{t}, \frac{t^2+|y|^2-1}{2t}\right) : \mathbb{R}^+ \times \mathbb{R}^{n-2} = \mathbb{H}^{n-1} \to \mathbb{H}^{n-1} \subset \mathbb{R}^n_1 \tag{3-17}$$

is nothing but the identity map of $\mathbb{H}^{n-1}$, since $\mathbb{R}^+ \times \mathbb{R}^{n-2} = \mathbb{H}^{n-1}$ is the upper half-space endowed with the standard hyperbolic metric. From (2-15), the conformal metric of the cone f is $g = \frac{\kappa^2}{t^2}(t^2 ds^2 + I_{\mathbb{R}^{n-1}}) = \kappa^2(ds^2 + I_{\mathbb{H}^{n-1}})$, where $I_{\mathbb{H}^{n-1}}$ denotes the standard hyperbolic metric of $\mathbb{H}^{n-1}$. By Lemma 3.1 we have the following result:

Proposition 3.5. *The cone in $\mathbb{R}^{n+1}_1$ over $\gamma(s) \subset \mathbb{S}^2_1(1) \subset \mathbb{R}^3_1$ as in Example 3.4 is of constant conformal sectional curvature if and only if $\gamma(s)$ is a curvature-spiral in $\mathbb{S}^2_1$.*

Example 3.6. Let $\mathbb{R}^2_{1+} = \{(x, y) \mid y > 0\}$ be the Lorentzian hyperbolic 2-plane. The rotational hypersurface in $\mathbb{R}^{n+1}_1$ over a curve $\gamma(s) \subset \mathbb{R}^2_{1+}$ is defined by

$$f : \mathbb{R} \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1}_1, \quad f(s, \theta) = (x(s), y(s)\theta),$$

where $\theta \in \mathbb{S}^{n-1}$ is the standard round sphere, and $\gamma(s) = (x(s), y(s)) \subset \mathbb{R}^2_{1+}$.

Denote the covariant differentiation of the metric ds^2 by D in $\mathbb{R}^2_{1+}$. For $\gamma(s) = (x(s), y(s)) \subset R^2_{1+}$, let $\dot{x}$ denote the derivative $\frac{\partial x}{\partial s}$ and so on. Choose the unit tangent vector $\alpha = \frac{1}{y}(\dot{x}, \dot{y})$ and the unit normal vector $\beta = \frac{1}{y}(\dot{y}, \dot{x})$. The geodesic curvature is computed via $\kappa(s) = \langle D_\alpha \alpha, \beta\rangle = \frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{y^2} + \frac{\dot{x}}{y}$. The rotational hypersurface f has the unit normal vector $\eta = \frac{1}{y}(\dot{y}, \dot{x}\theta)$. The first and the second fundamental form of the rotational hypersurface f are given by

$$I = df \cdot df = y^2(ds^2 + I_{\mathbb{S}^{n-1}}), \quad II = -df \cdot d\eta = (y\kappa - \dot{x})ds^2 - \dot{x} I_{\mathbb{S}^{n-1}}.$$

Thus the principal curvatures of the rotational hypersurface f are $\frac{y\kappa-\dot{x}}{y^2}, \frac{-\dot{x}}{y^2}, \dots, \frac{-\dot{x}}{y^2}$. From (2-15), we know that the conformal metric of the rotational hypersurface f is $g = \kappa^2(x)(ds^2 + I_{\mathbb{S}^{n-1}})$. By Lemma 3.1 we have the following result:

Proposition 3.7. *The rotational hypersurface in $\mathbb{R}^{n+1}_1$ over $\gamma(s) \subset \mathbb{R}^2_{1+}$ as in Example 3.6 is of constant conformal sectional curvature if and only if $\gamma(s)$ is a curvature-spiral in $\mathbb{R}^2_{1+}$.*

Example 3.8. Let p, q be any two given natural numbers with $p + q < n$ and let a be a real number $a > 1$. We define the cone over the hyperbolic torus $\mathbb{H}^q(-\sqrt{a^2-1}) \times \mathbb{S}^p(a) \subset \mathbb{S}^{p+q+1}_1(1)$, as follows:

$$f : \mathbb{H}^q(-\sqrt{a^2-1}) \times \mathbb{S}^p(a) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \to \mathbb{R}^{n+1}_1, \quad f(u', u'', t, u''') = (tu', tu'', u'''),$$

where $u' \in \mathbb{H}^q(-\sqrt{a^2-1}),\ u'' \in \mathbb{S}^p(a),\ u''' \in \mathbb{R}^{n-p-q-1}$.

Let $b = \sqrt{a^2-1}$. One of the normal vectors of f can be taken as $e_{n+1} = \left(\frac{a}{b}u', \frac{b}{a}u'', 0\right)$. The first and second fundamental form of f are given by

$$I = t^2(\langle du', du'\rangle_1 + du'' \cdot du'') + dt \cdot dt + du''' \cdot du''',$$
$$II = -\langle dx, de_{n+1}\rangle_1 = -t\left(\frac{a}{b}\langle du', du'\rangle_1 + \frac{b}{a} du'' \cdot du''\right).$$

Thus the mean curvature of f satisfies

$$H = \frac{-pb^2 - qa^2}{nabt},$$

and

$$e^{2\tau} = \frac{n}{n-1}\left[\sum_{ij} h_{ij}^2 - nH^2\right] = \frac{p(n-p)b^4 - 2pqa^2b^2 + q(n-q)a^4}{(n-1)a^2b^2t^2} := \frac{\alpha^2}{t^2}.$$

Let id_k denote the k-dimensional identical mapping. From (2-16), we have

$$B = b_1 \,\mathrm{id}_q \oplus b_2 \,\mathrm{id}_p \oplus b_3 \,\mathrm{id}_{n-q-p}, \quad A = a_1 \,\mathrm{id}_q \oplus a_2 \,\mathrm{id}_p \oplus a_3 \,\mathrm{id}_{n-q-p},$$

where

$$b_1 = \frac{pb^2 - (n-q)a^2}{nab\alpha}, \qquad b_2 = \frac{qa^2 - (n-p)b^2}{nab\alpha}, \qquad b_3 = \frac{pb^2 + qa^2}{nab\alpha},$$

and

$$a_1 = \frac{(pb^2+qa^2)^2-(pb^2+qa^2)2na^2+n^2a^2b^2}{2n^2a^2b^2\alpha^2},$$

$$a_2 = \frac{(pb^2+qa^2)^2-(pb^2+qa^2)2nb^2+n^2a^2b^2}{2n^2a^2b^2\alpha^2},$$

$$a_3 = \frac{(pb^2+qa^2)^2-n^2a^2b^2}{2n^2a^2b^2\alpha^2}.$$

Using these equations and (2-10), it is easy to prove the following result:

Proposition 3.9. *Let $f : M^n \to \mathbb{R}^{n+1}_1$ be a cone over a hyperbolic torus*

$$\mathbb{H}^q(-\sqrt{a^2-1}) \times \mathbb{S}^p(a).$$

If f has constant conformal sectional curvature δ, then $\delta = 0$, $p = q = 1$ and $n = 3$.

A spacelike hypersurface is called a conformal isoparametric spacelike hypersurface if the conformal 1-form vanishes and the eigenvalues of the conformal second fundamental form are constant. In [Li and Nie 2018] and [Nie and Wu 2008], the authors characterized the cone over the hyperbolic torus as follows:

Theorem 3.10 [Li and Nie 2018]. *Let $f : M^n \to M^{n+1}_1(c)$ be a conformal isoparametric spacelike hypersurface with r distinct principal curvatures. If $r \geq 3$, then $r = 3$, and locally f is conformally equivalent to the cone over the hyperbolic torus $\mathbb{H}^q(-\sqrt{a^2-1}) \times \mathbb{S}^p(a)$.*

4. The proof of Theorem 1.1

The hypothesis of constant conformal sectional curvature implies that the spacelike hypersurface is conformally flat. A classical result says that a spacelike hypersurface $f : M^n \to M^{n+1}_1(c)(n \geq 4)$ is conformally flat if and only if there exists a principle curvature which has multiplicity at least $n-1$ everywhere. Since our classification theorem is local, we consider the following two cases:

(1) The spacelike hypersurface has only two distinct principal curvatures.

(2) The 3-dimensional spacelike hypersurface has three distinct principal curvatures.

First, we consider case (1). Let $f : M^n \to \mathbb{R}^{n+1}_1$, $n \geq 3$, be a spacelike hypersurface with two distinct principal curvatures; one of which is simple, while the other has multiplicity $n-1$.

Lemma 4.1. *Let $f : M^n \to \mathbb{R}^{n+1}_1$, $n \geq 3$, be a spacelike hypersurface with two distinct principal curvatures. If the conformal sectional curvature has constant δ, then* *we can choose an orthonormal basis $\{E_1, \dots, E_n\}$ with respect to the conformal metric g such that*

$$(B_{ij}) = \mathrm{diag}\left(\frac{n-1}{n}, \frac{-1}{n}, \dots, \frac{-1}{n}\right); \quad C_2 = \dots = C_n = 0;$$
$$\omega_{1\alpha} = -C_1\omega_\alpha; \quad \delta = C_{1,1} - (C_1)^2; \quad C_{\alpha,\alpha} = -(C_1)^2, \quad \alpha \geq 2. \tag{4-18}$$

Proof. We take a local orthonormal basis $\{E_1, \dots, E_n\}$, with respect to g, under which,

$$(B_{ij}) = \mathrm{diag}(b_1, \underbrace{b_2, \dots, b_2}_{n-1}).$$

From the fourth equation in (2-11), we assume $b_1 = \frac{n-1}{n}$ and $b_2 = -\frac{1}{n}$. Since the spacelike hypersurface has constant conformal curvature δ, by (2-11), we have

$$(A_{ij}) = \mathrm{diag}\left(\frac{\delta}{2} - \frac{2n-1}{2n^2}, \ \frac{\delta}{2} + \frac{1}{2n^2}, \ \dots, \ \frac{\delta}{2} + \frac{1}{2n^2}\right).$$

In this section, we make use of the following convention on the range of indices

$$1 \leq i, j, k \leq n, \quad 2 \leq \alpha, \beta, \gamma \leq n.$$

From $dB_{ij} + \sum_k B_{kj}\omega_{ki} + \sum_k B_{ik}\omega_{kj} = \sum_k B_{ij,k}\omega_k$ and (2-8), we can get

$$B_{1\alpha,\alpha} = -C_1, \quad \omega_{1\alpha} = -C_1\omega_\alpha, \quad C_\alpha = 0, \quad 2 \leq \alpha \leq n,$$
$$(B_{ij,k} = 0 \text{ otherwise}). \tag{4-19}$$

Using $dC_i + \sum_k C_k\omega_{ki} = \sum_k C_{i,k}\omega_k$ and (4-19), we get

$$C_{\alpha,\alpha} = -(C_1)^2; \quad C_{\alpha,k} = 0, \ \alpha \neq k. \tag{4-20}$$

From (4-19), we see that

$$d\omega_{1\alpha} = -dC_1 \wedge \omega_\alpha - C_1^2\omega_1 \wedge \omega_\alpha - C_1 \sum_\gamma \omega_\gamma \wedge \omega_{\gamma\alpha},$$

and

$$d\omega_{1\alpha} - \sum_j \omega_{1j} \wedge \omega_{j\alpha} = -\tfrac{1}{2} \sum_{kl} R_{1\alpha kl}\omega_k \wedge \omega_l.$$

Thus we have

$$R_{1\alpha1\alpha} = C_{1,1} - (C_1)^2, \quad R_{1\alpha\beta\alpha} - C_{1,\beta} = 0. \tag{4-21}$$

□

From Lemma 4.1, we know that the distributions

$$D_1 = \mathrm{span}\{E_1\}, \quad D_2 = \mathrm{span}\{E_2, E_3, \dots, E_n\}$$

are integrable. Any integral submanifold of distribution D_1 is a curve γ, and any integral submanifold of distribution D_2 is an $(n-1)$-dimensional submanifold L. Thus locally, we have

$$M^n = \gamma \times L.$$

Under the orthonormal basis $\{E_1, \dots, E_n\}$ as in Lemma 4.1, $\{Y, N, Y_1, \dots, Y_n, \xi\}$ forms a moving frame in $\mathbb{R}^{n+3}_2$ along M^n. We define

$$F = -\frac{1}{n}Y - \xi, \quad X_1 = -C_1 Y - Y_1, \quad P = -a_2 Y + N + C_1 X_1 - \frac{1}{n}F.$$

Therefore we have

$$\begin{aligned} &\langle F, F\rangle = -1, \quad \langle X_1, X_1\rangle = 1, \quad \langle P, P\rangle = -C_{1,1}, \\ &\langle F, P\rangle = 0, \qquad \langle F, X_1\rangle = 0, \quad \langle X_1, P\rangle = 0. \end{aligned} \tag{4-22}$$

From Lemma 4.1 and the structure equation of f we can derive

$$\begin{aligned} E_1(F) &= X_1, & E_\alpha(F) &= 0, \\ E_1(X_1) &= P + F, & E_\alpha(X_1) &= 0, \\ E_1(P) &= C_1 P + C_{1,1} X_1, & E_\alpha(P) &= 0. \end{aligned} \tag{4-23}$$

Thus the subspace $V_1 = \mathrm{span}\{F, X_1, P\}$ is fixed along M^n. From $\delta = C_{1,1} - (C_1)^2$, we get

$$E_1(C_{1,1}) = 2C_1 C_{1,1}, \quad E_\alpha(C_{1,1}) = 0. \tag{4-24}$$

We define $T = -a_2 Y - N + C_1 Y_1 - \frac{1}{n}\xi$, then we have

$$T \perp V_1, \quad \langle T, T\rangle = C_{1,1}, \quad \langle T, Y_\alpha\rangle = 0, \quad 2 \le \alpha \le n.$$

From (4-24), Lemma 4.1, and the structure equation of f we can derive

$$\begin{aligned} &E_\alpha(T) = -C_{1,1} Y_\alpha, \quad E_1(T) = C_1 T, \quad E_\beta(Y_\alpha) = \sum_\gamma \omega_{\alpha\gamma}(E_\beta) Y_\gamma, \\ &E_\alpha(Y_\alpha) = \sum_\beta \omega_{\alpha\beta}(E_\alpha) Y_\beta + T, \quad E_1(Y_\alpha) = \sum_\beta \omega_{\alpha\beta}(E_1) Y_\beta, \quad \alpha \ne \beta. \end{aligned} \tag{4-25}$$

Thus the subspace $V_2 = \mathrm{span}\{T, Y_2, Y_3, \dots, Y_n\}$ is fixed along M^n, and $V_1 \perp V_2$.

Using theory of linear first-order differential equations for $C_{1,1}$, (4-24) means that $C_{1,1} \equiv 0$ or $C_{1,1} \ne 0$ on an open subset $U \subset M^n$. Thus we need to consider the following three subcases: (1) $C_{1,1} \equiv 0$ on M^n; (2) $C_{1,1} < 0$ on M^n; (3) $C_{1,1} > 0$ on M^n. We will treat them case by case.

Proposition 4.2. *Under the assumptions in Lemma 4.1, if* $C_{1,1} \equiv 0$ *on* M^n, *then* f *is conformally equivalent to a cylinder over a curvature-spiral in* $\mathbb{R}^2_1$.

Proof. Since $C_{1,1} \equiv 0$, we have $\langle P, P\rangle = 0$. From (4-23), we know that P determines a fixed direction. Hence up to a conformal transformation we can write the fixed direction $P \in \mathbb{R}^{n+3}_2$ and constant subspace $V_1 \subset \mathbb{R}^{n+3}_2$ as follows:

$$P = (1, 0, \dots, 0, 1),$$
$$V_1 = \mathrm{span}\{F, X_1, P\} = \mathrm{span}\{(0, 1, 0, \dots, 0),\ (0, 0, 1, 0, \dots, 0),\ (1, 0, \dots, 0, 1)\}.$$

From (4-22), we have

$$\langle F, P\rangle = \langle F, (1, 0, 0, \dots, 0, 1)\rangle = 0, \quad \langle X_1, P\rangle = \langle X_1, (1, 0, 0, \dots, 0, 1)\rangle = 0. \tag{4-26}$$

Let $\{\kappa_1,\ \kappa_2, \dots, \kappa_2\}$ be the principal curvatures of the spacelike hypersurface f, then $e^\tau = |\kappa_1 - \kappa_2|$. We choose an orthonormal basis $\{e_1, \dots, e_n\}$ of TM^n with respect to the first fundamental form $I = df \cdot df$ such that $(h_{ij}) = \mathrm{diag}\{\kappa_1,\ \kappa_2, \dots, \kappa_2\}$; then $\{E_i = e^{-\tau} e_i, 1 \le i \le n\}$ is an orthonormal basis as in Lemma 4.1. From (2-15) and (4-26), we can deduce

$$\kappa_2 = 0, \quad E_1(\tau) = -C_1. \tag{4-27}$$

On the other hand, we have $\langle Y_\alpha, P\rangle = 0$ which implies that

$$E_\alpha(\tau) = 0. \tag{4-28}$$

Let $\{\tilde\omega_1, \dots, \tilde\omega_n\}$ be the dual basis of $\{e_1, \dots, e_n\}$, and $\{\tilde\omega_{ij}\}$ be the corresponding connection forms. Since $\omega_i = e^\tau \tilde\omega_i,\ 1 \le i \le n$, its connections have the relations

$$\omega_{ij} = \tilde\omega_{ij} + e_i(\tau)\tilde\omega_j - e_j(\tau)\tilde\omega_i.$$

Equations (4-27) and (4-28) imply that $\tilde\omega_{1\alpha} = 0$. Thus the spacelike hypersurface f is conformally equivalent to the hypersurface given by Example 3.2. By Proposition 3.3, we finish the proof of Proposition 4.2. □

Proposition 4.3. *Under the assumptions in Lemma 4.1, if $C_{1,1} < 0$ on M^n, then f is conformally equivalent to a cone over a curvature-spiral in $\mathbb{S}^2_1$.*

Proof. Since $C_{1,1} < 0$, by (4-22), the vector field P is a spacelike vector field in $\mathbb{R}^{n+3}_2$. Thus up to a conformal transformation we can write

$$V_1 = \mathrm{span}\{F, X_1, P\} = \mathrm{span}\{(0, 1, 0, \dots, 0),\ (0, 0, 1, 0, \dots, 0),\ (0, 0, 0, 1, \dots, 0)\}.$$

Let f have principal curvatures $\{\kappa_1, \kappa_2, \dots, \kappa_2\}$. Since $e = (1, 0, \dots, 0, 1) \perp V$, we have $\langle F, e\rangle = \langle X_1, e\rangle = 0$ which implies $\kappa_2 = 0,\ E_1(\tau) = -C_1$. By (2-15), we obtain $e^{2\tau} = \kappa_1^2$. Setting

$$\bar P = \frac{P}{\sqrt{-C_{1,1}}}, \quad \theta = \frac{T}{\sqrt{-C_{1,1}}},$$

then $\langle \bar{P}, \bar{P}\rangle = 1$ and $\langle\theta, \theta\rangle = -1$. From (4-23), we know

$$\bar{P} : \gamma \to \mathbb{S}^2_1 \subset R^3_1 = V_1$$

is a curve, and (4-25) gives that

$$\theta : L \to \mathbb{H}^{n-1} \subset \mathbb{R}^n_1$$

is a standard embedding and the sectional curvature of $\theta(L)$ is -1. Since $\dim L = \dim \mathbb{H}^{n-1} = n-1$, we know that $\theta : L \to \mathbb{H}^{n-1}$ is a standard isometric isomorphism. By (3-17), we have the standard isometric isomorphism

$$\theta : L \to \mathbb{H}^{n-1} = \mathbb{R}^+ \times \mathbb{R}^{n-2}.$$

Since $P + T = -C_{1,1}Y$,

$$Y = \frac{1}{\sqrt{-C_{1,1}}}(\bar{P}, \theta) : M^n = \gamma \times L \to \mathbb{S}^2_1 \times \mathbb{H}^{n-1} = \mathbb{S}^2_1 \times \mathbb{R}^+ \times \mathbb{R}^{n-2} \subset \mathbb{R}^{n+3}_1.$$

Therefore

$$g = \langle dY, dY\rangle = -\frac{1}{C_{1,1}}(ds^2 + I_{\mathbb{H}^{n-1}}).$$

Thus the spacelike hypersurface f is conformally equivalent to the hypersurface given by Example 3.4. By Proposition 3.5, we finish the proof of Proposition 4.3. $\square$

Proposition 4.4. *Under the assumptions in Lemma 4.1, if $C_{1,1} > 0$ on M^n, then f is conformally equivalent to a rotational hypersurface over a curvature-spiral in $\mathbb{R}^2_{1+}$.*

Proof. Since $C_{1,1} > 0$, we have $\langle P, P\rangle < 0$. Thus up to a conformal transformation we can write

$$V_1 = \text{span}\{F, X_1, P\} = \text{span}\{(1, 0, \dots, 0),\ (0, \dots, 0, 1),\ (0, 1, 0, \dots, 0)\}.$$

Thus $e = (1, 0, \dots, 0, 1) \in V_1$, and $\langle Y_\alpha, e\rangle = 0,\ 2 \le \alpha \le n$, which imply that $E_\alpha(\tau) = 0,\ 2 \le \alpha \le n$. Setting

$$\bar{P} = \frac{P}{\sqrt{C_{1,1}}}, \quad \theta = \frac{T}{\sqrt{C_{1,1}}},$$

then $\langle \bar{P}, \bar{P}\rangle = -1$ and $\langle\theta, \theta\rangle = 1$. From (4-23), we know

$$\bar{P} : \gamma \to \mathbb{H}^2_1 \subset \mathbb{R}^3_2 = V_1$$

is a curve. From (4-25), we see that

$$\theta : L \to \mathbb{S}^{n-1} \subset \mathbb{R}^n$$

is a standard embedding and the sectional curvature of $\theta(L)$ is 1. Since $\dim L = n-1$,

$\theta : L \to \mathbb{S}^{n-1}$ is a standard isometric isomorphism. Since $P + T = -C_{1,1}Y$,

$$Y = \frac{-1}{\sqrt{C_{1,1}}}(\bar{P}, \theta) : \gamma \times L \to \mathbb{H}^2_1 \times \mathbb{S}^{n-1}.$$

Denote $\bar{P} = (u_1, u_2, u_3) \in \mathbb{H}^2_1$, then

$$Y = \frac{u_3 - u_1}{\sqrt{C_{1,1}}}\left(\frac{u_1}{u_1 - u_3}, \frac{u_2}{u_1 - u_3}, \frac{u_3}{u_1 - u_3}, \frac{\theta}{u_1 - u_3}\right).$$

Thus the hypersurface $f : \mathbb{R} \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1}_1$ is given by $f = \left(\frac{u_2}{u_1-u_3}, \frac{\theta}{u_1-u_3}\right)$. Note that

$$\varphi(u_1, u_2, u_3) = \left(\frac{u_2}{u_1 - u_3}, \frac{1}{u_1 - u_3}\right)$$

is the inverse mapping of the local isometric correspondence $\phi : \mathbb{R}^2_{1+} \to \mathbb{H}^2_1$ (see (1-3)). Thus the spacelike hypersurface f is conformally equivalent to the hypersurface given by Example 3.6. By Proposition 3.7, we finish the proof of Proposition 4.4. ☐

Next we consider case (2). Let $f : M^3 \to \mathbb{R}^4_1$ be a spacelike hypersurface with three distinct principal curvatures.

Proposition 4.5. *Let $f : M^3 \to \mathbb{R}^4_1$ be a spacelike hypersurface with three distinct principal curvatures. If the conformal sectional curvature has constant δ, then $\delta = 0$ and f is conformally equivalent to the spacelike hypersurface defined by Example 3.8.*

To prove Proposition 4.5, we need the following two lemmas.

Lemma 4.6. *Under the same assumptions as in Proposition 4.5, there exist a local orthonormal basis $\{E_1, E_2, E_3\}$, consisting of eigenvectors of B such that*

$$\begin{aligned} B_{11,2} &= \frac{b_3 - b_1}{b_1 - b_2}C_2, \quad B_{11,3} = \frac{b_2 - b_1}{b_1 - b_3}C_3, \quad B_{22,1} = \frac{b_3 - b_2}{b_2 - b_1}C_1, \\ B_{22,3} &= \frac{b_1 - b_2}{b_2 - b_3}C_3, \quad B_{33,1} = \frac{b_2 - b_3}{b_3 - b_1}C_1, \quad B_{33,2} = \frac{b_1 - b_3}{b_3 - b_2}C_2. \end{aligned} \tag{4-29}$$

Proof. Since f is of constant conformal curvature, from (2-11), we have

$$R_{ij} = 2\delta\delta_{ij} = \sum_k B_{ik}B_{kj} + (tr A)\delta_{ij} + A_{ij}. \tag{4-30}$$

Thus we can take a local orthonormal basis $\{E_1, E_2, E_3\}$ such that

$$(B_{ij}) = \mathrm{diag}(b_1, b_2, b_3), \quad (A_{ij}) = \mathrm{diag}(a_1, a_2, a_3). \tag{4-31}$$

Using (4-30) and (4-31), we have

$$B_{ij,k}(b_i+b_j)+A_{ijk}=0, \quad 1\le i,j,k\le 3. \tag{4-32}$$

Using (2-7) and (2-8) and (4-32), we can obtain (4-29) and

$$B_{ij,k}=A_{ij,k}=0, \quad i\neq j,\ i\neq k,\ k\neq j. \tag{4-33}$$

Thus we complete the proof. □

If the conformal 1-form C is equal to 0, by Lemma 4.6, we know that the eigenvalues of the conformal second fundamental form are constant. Thus the spacelike hypersurface is a conformal isoparametric spacelike hypersurface. By Proposition 3.9 and Theorem 3.10, we can prove Proposition 4.5. Next we need to prove $C=0$.

Lemma 4.7. *Under the same assumptions as in Proposition 4.5, the conformal 1-form C is equal to* 0.

Proof. Let $\{\omega_1,\omega_2,\omega_3\}$ be the dual of the local orthonormal basis $\{E_1,E_2,E_3\}$ in Lemma 4.6, and $\{\omega_{ij}\}$ the connection forms. Using covariant derivatives of B_{ij},

$$\omega_{ij}=\frac{B_{ij,i}}{b_i-b_j}\omega_i+\frac{B_{ij,j}}{b_i-b_j}\omega_j, \qquad i\neq j, \quad 1\le i,j\le 3. \tag{4-34}$$

Using (2-8), we have $B_{ij,j}=B_{jj,i}-C_i,\ i\neq j$. Using (2-12), (4-33) and (4-29), we can obtain

$$B_{12,31}=(B_{11,3}-B_{22,3})\frac{B_{12,1}}{b_1-b_2}+(B_{12,1}-B_{32,3})\frac{B_{13,1}}{b_1-b_3},$$

$$\begin{aligned} B_{12,13}=&\frac{3(b_2B_{11,3}-b_1B_{223})}{(b_1-b_2)^2}C_2\\ &+\left(C_{2,3}-\frac{B_{32,3}}{b_3-b_2}C_3\right)\frac{3b_1}{b_2-b_1}+\frac{B_{32,3}B_{13,1}}{b_3-b_2}. \end{aligned} \tag{4-35}$$

From (2-10) and the Ricci identity (2-13), we have $C_{i,j}-C_{j,i}=(b_i-b_j)A_{ij}=0$, and $B_{12,31}=B_{12,13}$. Using (4-35), we can derive

$$b_1C_{2,3}=\frac{b_1b_2+2b_3^2}{(b_2-b_3)(b_3-b_1)}C_3C_2=-C_3C_2,$$

where we use $b_1+b_2+b_3=0$ and $b_1^2+b_2^2+b_3^2=\frac{2}{3}$. Similarly $b_2C_{1,3}=-C_3C_1$ and $b_3C_{1,2}=-C_2C_1$. Thus

$$b_kC_{i,j}=-C_iC_j, \quad i\neq j,\ i\neq k,\ k\neq j. \tag{4-36}$$

Using the covariant derivative of C_i and taking the derivative for (4-36) along E_k, we obtain

$$B_{kk,k}C_{i,j}+b_k\left[C_{i,jk}-C_{k,j}\frac{B_{ki,k}}{b_k-b_i}-C_{i,k}\frac{B_{kj,k}}{b_k-b_j}\right] = -C_i\left[C_{j,k}-C_k\frac{B_{jk,k}}{b_k-b_j}\right]-C_j\left[C_{i,k}-C_k\frac{B_{ik,k}}{b_k-b_i}\right]. \tag{4-37}$$

If $b_1b_2b_3=0$, we can assume that $b_1=0$. From (2-11), we know that $b_2=-b_3=\frac{1}{\sqrt{3}}$. Using (4-29) we have $C=0$.

We assume $b_1b_2b_3\neq 0$. From (4-29), (4-36), (4-37) and $B_{kk,k}=-B_{jj,k}-B_{ii,k}$, we conclude that

$$b_kC_{i,jk}=-\frac{4}{3}\frac{C_iC_jC_k}{b_ib_jb_k}=-\frac{4}{3}\frac{C_1C_2C_3}{b_1b_2b_3}. \tag{4-38}$$

Since $C_{i,jk}=C_{j,ik}=C_{k,ij}$ and $b_i\neq b_j,\ i\neq j$, from (4-38) we get

$$C_{1,23}=0,\quad C_1C_2C_3=0.$$

We can assume that $C_1=0$, and (4-34) can be written as

$$\omega_{12}=\frac{B_{12,1}}{b_1-b_2}\omega_1,\quad \omega_{13}=\frac{B_{13,1}}{b_1-b_3}\omega_1,\quad \omega_{23}=\frac{B_{23,2}}{b_2-b_3}\omega_2+\frac{B_{23,3}}{b_2-b_3}\omega_3. \tag{4-39}$$

Using the covariant derivative of C_i and

$$d\omega_{ij}-\sum_k\omega_{ik}\wedge\omega_{kj}=-\tfrac{1}{2}\sum_{kl}R_{ijkl}\omega_k\wedge\omega_l,$$

we can derive

$$\begin{aligned}
&\frac{3C_3^2[b_2^3-b_3^3-6b_1b_2^2+6b_1^2b_2]}{(b_1-b_3)^2(b_3-b_2)}+\frac{27b_1^2b_3C_2^2}{(b_1-b_2)^2(b_3-b_2)} = 3b_1C_{3,3}+(b_1-b_3)^2\delta,\\
&\frac{3C_2^2[b_3^3-b_2^3-6b_1b_3^2+6b_1^2b_3]}{(b_1-b_2)^2(b_2-b_3)}+\frac{27b_1^2b_2C_3^2}{(b_1-b_3)^2(b_2-b_3)} = 3b_1C_{2,2}+(b_1-b_2)^2\delta,\\
&\frac{3C_3^2[b_1^3-b_3^3+3b_2^3+15b_1b_2^2]}{(b_3-b_2)^2(b_3-b_1)}+\frac{3C_2^2[b_1^3-b_2^3+3b_3^3+15b_1b_3^2]}{(b_3-b_2)^2(b_2-b_1)} = 3b_3C_{2,2}+3b_2C_{3,3}+(b_3-b_2)^2\delta,
\end{aligned} \tag{4-40}$$

where we use $B_{ij,j}=B_{jj,i}-C_i,\ \mathrm{tr}(B)=0$ and $|B|^2=\frac{2}{3}$. We can eliminate $C_{2,2}$

and $C_{3,3}$ in (4-40) and derive

$$\begin{aligned}\frac{3[2b_1^4+2b_2^4+2b_3^4-9b_1^2b_2^2-6b_1^3b_2-12b_1b_2^3]}{(b_1-b_3)^3}C_3^2 \\ +\frac{3[2b_1^4+2b_2^4+2b_3^4-9b_1^2b_3^2-6b_1^3b_3-12b_1b_3^3]}{(b_1-b_2)^3}C_2^2 \\ =-b_1(b_2-b_3)^2\delta.\end{aligned} \tag{4-41}$$

On the other hand, using the covariant derivative of C_i and $C_1=0$, we have

$$\begin{aligned}C_{1,1}&=\frac{3b_1C_2^2}{(b_2-b_1)^2}+\frac{3b_1C_3^2}{(b_3-b_1)^2},\\ C_{2,1}&=C_{1,2}=0,\quad C_{3,1}=C_{1,3}=0,\\ C_{2,2}&=E_2(C_2)+\frac{3b_2C_3^2}{(b_3-b_2)^2},\\ C_{2,3}&=E_3(C_2)-\frac{3b_3C_2C_3}{(b_3-b_2)^2},\\ C_{3,2}&=E_2(C_3)-\frac{3b_2C_2C_3}{(b_3-b_2)^2},\\ C_{3,3}&=E_3(C_3)+\frac{3b_3C_2^2}{(b_3-b_2)^2}.\end{aligned} \tag{4-42}$$

Using the second covariant derivative of the conformal 1-form C defined by

$$\sum_m C_{i,jm}\omega_m=dC_{i,j}+\sum_m C_{m,j}\omega_{mi}+\sum_m C_{i,m}\omega_{mj},$$

and combining (4-29) and (4-42), we can deduce

$$\begin{aligned}C_{3,23}=E_3(E_2(C_3))-3\left[\frac{b_1-b_2}{(b_2-b_3)^3}-\frac{6b_2b_1(b_1-b_2)}{(b_3-b_2)^4(b_1-b_3)}\right]C_3^2C_2\\ +3(C_{2,2}-C_{3,3})\frac{b_3C_2}{(b_3-b_2)^2}\\ -\frac{3b_2}{(b_3-b_2)^2}\left[C_3C_{2,3}+C_2C_{3,3}+\frac{3b_3}{(b_3-b_2)^2}(C_3^2C_2-C_2^3)\right],\end{aligned}$$

$$\begin{aligned}C_{3,32}=E_2(E_3(C_3))+3\left[\frac{b_1-b_3}{(b_3-b_2)^3}-\frac{6b_3b_1(b_1-b_3)}{(b_3-b_2)^4(b_1-b_2)}\right]C_2^3\\ +\frac{6b_3}{(b_3-b_2)^2}C_2C_{2,2}-\frac{18b_3b_2}{(b_3-b_2)^4}C_3^2C_2.\end{aligned}$$

Using the Ricci identity $C_{3,23} - C_{3,32} = \sum_l C_l R_{l323} = \delta C_2$, we obtain

$$
\begin{aligned}
(4\text{-}43)\quad \frac{\delta}{3}C_2 = {} & \frac{3b_1b_3b_2 + b_2b_1^2 + 5b_3b_1^2 - 8b_1b_3^2 - 2b_3b_2^2 - b_1b_2^2 + 2b_2b_3^2}{(b_3-b_2)^4(b_1-b_2)}C_2^3 \\
& + \frac{3b_1b_3b_2 + b_3b_1^2 + 5b_2b_1^2 - 8b_1b_2^2 - 2b_2b_3^2 - b_1b_3^2 + 2b_3b_2^2}{(b_3-b_2)^4(b_1-b_3)}C_2C_3^2 \\
& - \frac{b_3}{(b_3-b_2)^2}C_2C_{2,2} - \frac{b_2}{(b_3-b_2)^2}C_2C_{3,3} + \frac{2b_2}{b_1(b_3-b_2)^2}C_2C_3^2,
\end{aligned}
$$

where we use the equation

$$
\begin{aligned}
& E_3(E_2(C_3)) - E_2(E_3(C_3)) \\
& \quad = [E_3, E_2](C_3) \\
& \quad = (\omega_{23}(E_3)E_3 - \omega_{32}(E_2)E_2)(C_3) \\
& \quad = \frac{3b_3}{(b_3-b_2)^2}C_2C_{3,3} - \frac{9b_3^2}{(b_3-b_2)^4}C_2^3 - \frac{3b_2}{(b_3-b_2)^2}C_3C_{3,2} - \frac{9b_2^2}{(b_3-b_2)^4}C_3^2C_2.
\end{aligned}
$$

From the third equation in (4-40) and (4-43), noting that $b_1b_2b_3 \neq 0$, we can deduce

$$
\frac{2b_2}{b_1(b_3-b_2)^2}C_2C_3^2 = 0. \tag{4-44}
$$

We can assume that $C_2 = 0$. Next we prove that $C_3 = 0$. In fact, if $C_3 \neq 0$, from (4-39), we have

$$
\omega_{12} = 0, \quad \omega_{13} = \frac{B_{13,1}}{b_1 - b_3}\omega_1, \quad \omega_{23} = \frac{B_{23,2}}{b_2 - b_3}\omega_2.
$$

Using

$$
d\omega_{12} - \sum_k \omega_{1k} \wedge \omega_{k2} = -\tfrac{1}{2}\sum_{kl} R_{12kl}\omega_k \wedge \omega_l,
$$

we can derive

$$
\delta = R_{1212} = \frac{-9b_1b_2C_3^2}{(b_1-b_3)^2(b_2-b_3)^2}. \tag{4-45}
$$

Since $C_2 = 0$, (4-41) becomes

$$
\frac{3[2b_1^4 + 2b_2^4 + 2b_3^4 - 9b_1^2b_2^2 - 6b_1^3b_2 - 12b_1b_2^3]}{(b_3-b_1)^2(b_3-b_2)^2}C_3^2 = -b_1(b_1-b_3)\delta. \tag{4-46}
$$

Combining (4-45) and (4-46), we have

$$
[2b_1^4 + 2b_2^4 + 2b_3^4 + 12b_1b_2(b_1b_3 - b_2^2)]C_3^2 = 0. \tag{4-47}
$$

Using $b_1+b_2+b_3=0$ and $b_1^2+b_2^2+b_3^2=\frac{2}{3}$, we see that

$$b_1^4+b_2^4+b_3^4=\tfrac{2}{9} \quad \text{and} \quad b_1b_3-b_2^2=-\tfrac{1}{3}.$$

Thus (4-47) is written as

$$\left(\tfrac{4}{9}-4b_1b_2\right)C_3^2=0.$$

Since $C_3\neq 0$, $\frac{4}{9}-4b_1b_2=0$ which implies that b_1, b_2, b_3 are constant. Thus (4-29) means that $C=0$, which is a contradiction. Thus $C_3=0$ and $C=0$. This completes the proof. □

Combining Propositions 4.2, 4.3, 4.4 and 4.5, we finish the proof of Theorem 1.1.

Acknowledgements

The authors thank the referee for valuable suggestions that improved the clarity and readability of the paper. The authors are supported by the grants No. 11571037 and No. 11471021 of NSFC.

References

[Cahen and Kerbrat 1983] M. Cahen and Y. Kerbrat, "Domaines symétriques des quadriques projectives", *J. Math. Pures Appl.* (9) **62**:3 (1983), 327–348. MR Zbl

[Guo et al. 2012] Z. Guo, T. Li, L. Lin, X. Ma, and C. Wang, "Classification of hypersurfaces with constant Möbius curvature in $\mathbb{S}^{m+1}$", *Math. Z.* **271**:1-2 (2012), 193–219. MR Zbl

[Hu and Li 2003] Z. Hu and H. Li, "Submanifolds with constant Möbius scalar curvature in S^n", *Manuscripta Math.* **111**:3 (2003), 287–302. MR Zbl

[Li and Nie 2013] T. Li and C. Nie, "Conformal geometry of hypersurfaces in Lorentz space forms", *Geometry* **2013** (2013), art. id. 549602. Zbl

[Li and Nie 2018] T. Li and C. Nie, "Spacelike Dupin hypersurfaces in Lorentzian space forms", *J. Math. Soc. Japan* **70**:2 (2018), 463–480. MR Zbl

[Li and Wang 2003] H. Li and C. Wang, "Möbius geometry of hypersurfaces with constant mean curvature and scalar curvature", *Manuscripta Math.* **112**:1 (2003), 1–13. MR Zbl

[Li et al. 2013] T. Li, X. Ma, and C. Wang, "Willmore hypersurfaces with constant Möbius curvature in $\mathbb{R}^{n+1}$", *Geom. Dedicata* **166** (2013), 251–267. MR Zbl

[Nie 2015] C. X. Nie, "Blaschke isoparametric hypersurfaces in the conformal space $\mathbb{Q}_1^{n+1}$, I", *Acta Math. Sin.* (*Engl. Ser.*) **31**:11 (2015), 1751–1758. MR Zbl

[Nie and Wu 2008] C. X. Nie and C. X. Wu, "Space-like hypersurfaces with parallel conformal second fundamental forms in the conformal space", *Acta Math. Sin.* (*Chin. Ser.*) **51**:4 (2008), 685–692. In Chinese. MR

[O'Neill 1983] B. O'Neill, *Semi-Riemannian geometry*, Pure and Applied Math. **103**, Academic Press, New York, 1983. MR Zbl

[Wang 1998] C. Wang, "Moebius geometry of submanifolds in $\mathbb{S}^n$", *Manuscripta Math.* **96**:4 (1998), 517–534. MR Zbl

Received April 11, 2017. Revised May 16, 2018.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 300, No. 1, 2019
dx.doi.org/10.2140/pjm.2019.300.39

CANONICAL FIBRATIONS OF CONTACT METRIC (κ, μ)-SPACES

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We present a classification of the complete, simply connected, contact metric (κ, μ)-spaces as homogeneous contact metric manifolds, by studying the base space of their canonical fibration. According to the value of the Boeckx invariant, it turns out that the base is a complexification or a paracomplexification of a sphere or of a hyperbolic space. In particular, we obtain a new homogeneous representation of the contact metric (κ, μ)-spaces with Boeckx invariant less than -1.

1. Introduction

The study of the curvature tensor of associated metrics to a contact form is a central theme in contact metric geometry. Actually some important classes of contact metric manifolds can be defined using it. We recall for example that Sasakian manifolds, the odd-dimensional analogues of Kähler manifolds, can be characterized by

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

where X, Y are any vector fields and ξ denotes the characteristic vector field of the contact metric manifold. A meaningful generalization of this curvature condition is

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY),$$

where κ, μ are real numbers and $2h$ is the Lie derivative of the structure tensor φ in the direction of the characteristic vector field ξ.

MSC2010: primary 53C25, 53D10; secondary 53C30, 53C35.
Keywords: contact metric (κ, μ)-space, regular contact manifold.

The contact metric manifolds with this property were introduced by Blair, Koufogiorgos and Papantoniou [1995], and are called *contact metric* (κ, μ)*-spaces* in the literature. These spaces have many interesting geometric properties; first of all, they are stable under $\mathcal{D}$-homothetic deformations and moreover in the non-Sasakian case, i.e., when $\kappa \neq 1$, the curvature tensor of the associated metric is completely determined. Looking at contact metric manifolds as strongly pseudoconvex (almost) CR manifolds, it was shown in [Dileo and Lotta 2009] that the (κ, μ) condition is equivalent to the local CR-symmetry with respect to the Webster metric, according to the general notion in [Kaup and Zaitsev 2000]. In this context, another characterization was given by Boeckx and Cho [2008] in terms of the parallelism of the Tanaka–Webster curvature.

Boeckx gave a crucial contribution to the problem of classifying these manifolds; after showing that every non-Sasakian contact (κ, μ)-space is locally homogeneous and strongly locally φ-symmetric [Boeckx 1999], he defined a scalar invariant I_M which completely determines a contact (κ, μ)-space M locally up to equivalence and up to $\mathcal{D}$-homothetic deformations of its contact metric structure [Boeckx 2000].

A standard example is the tangent sphere bundle T_1M of a Riemannian manifold M with constant sectional curvature $c \neq 1$. Being a hypersurface of TM, which is equipped with a natural almost-Kähler structure (J, G), where G is the Sasaki metric, T_1M inherits a standard contact metric structure (for more details, see for instance [Blair 2010]). In particular, the Webster metric g of T_1M is a scalar multiple of G. The corresponding Boeckx invariant is given by

$$I_{T_1M} = \frac{1+c}{|1-c|}.$$

Hence, as c varies in $\mathbb{R} \setminus \{1\}$, I_{T_1M} assumes all real values strictly greater than -1.

The case $I \leqslant -1$ seems to lead to models of different nature. Namely, Boeckx found examples of contact metric (κ, μ)-spaces, for every value of the invariant $I \leqslant -1$, namely a two parameter family of (abstractly constructed) Lie groups with a left-invariant contact metric structure. However, he gave no geometric description of these examples; in particular, to our knowledge, nothing can be found in the literature regarding the topological structure of these manifolds.

One of the first aims of this paper is to fill this gap, showing that simply connected, complete contact metric (κ, μ)-spaces of dimension $2n+1$ (where $n > 1$) with $I < -1$ are exhausted by a one parameter family of invariant contact metric structures on the homogeneous space

$$\mathrm{SO}(n, 2)/\,\mathrm{SO}(n).$$

Actually, we provide a unified treatment of all the models with $I_M \neq \pm 1$. Our classification is accomplished intrinsically, by studying the canonical fibration of non-Sasakian contact metric (κ, μ)-spaces with Boeckx invariant $I_M \neq \pm 1$ and

Boeckx invariant	model space	base space
$I_M > 1$	$\mathrm{SO}(n+2)/\mathrm{SO}(n)$	$\mathrm{SO}(n+2)/(\mathrm{SO}(n) \times \mathrm{SO}(2))$
$-1 < I_M < 1$	$\mathrm{SO}(n+1, 1)/\mathrm{SO}(n)$	$\mathrm{SO}(n+1, 1)/(\mathrm{SO}(n) \times \mathrm{SO}(1, 1))$
$I_M < -1$	$\mathrm{SO}(n, 2)/\mathrm{SO}(n)$	$\mathrm{SO}(n, 2)/(\mathrm{SO}(n) \times \mathrm{SO}(2))$

Table 1. Simply connected complete contact metric (κ, μ)-spaces with $I_M \neq \pm 1$.

endowing the base spaces of a canonical connection. Here we refer to the fibration $M \to M/\xi$ over the leaf space of the foliation determined by the Reeb vector field; as such, it depends only on the contact form of M. First, in Theorem 7, non-Sasakian contact metric (κ, μ)-spaces with Boeckx invariant not equal to ± 1 are characterized by admitting a transitive Lie group of automorphisms whose Lie algebra $\mathfrak{g}$ has a (canonical) symmetric decomposition. This decomposition yields a reductive decomposition for the base space B of the canonical fibration and the associated canonical connection makes B an affine symmetric space (Corollary 8).

Next we show that B admits a uniquely determined standard invariant complex or paracomplex structure, by which it is a complexification or a paracomplexification of the sphere S^n or of the hyperbolic space $\mathbb{H}^n$, according to the value of the Boeckx invariant of the (κ, μ)-space. After identifying the possible base spaces B, in the final section we construct explicitly our models as homogeneous contact metric manifolds fiberings onto them. In conclusion, we obtain the classification list in Table 1. This table also provides a new geometric interpretation of the Boeckx invariant.

2. Preliminaries

Let M be an odd-dimensional smooth manifold. An *almost contact structure* on M is a triple consisting of a $(1, 1)$ tensor field φ, a vector field ξ, and a 1-form η satisfying

$$\varphi^2 = -\operatorname{id} + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

An almost contact manifold always admits a *compatible metric*, namely a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields X, Y on M. If such a metric g satisfies also

$$\mathrm{d}\eta(X, Y) = g(X, \varphi Y),$$

then (φ, ξ, η, g) is called a *contact metric structure* on M. In this case η is a contact form; we shall denote by D the corresponding contact distribution $D = \ker(\eta)$ and by $\mathcal{D}$ the module of smooth sections of D.

A contact metric manifold M is said to be a *K-contact manifold* if its characteristic vector field ξ is Killing. This condition is equivalent to the vanishing of the $(1, 1)$ tensor field

$$h := \tfrac{1}{2}\mathcal{L}_\xi \varphi,$$

where $\mathcal{L}_\xi$ is Lie differentiation in the direction of ξ.

If the curvature tensor R of a contact metric manifold M satisfies the condition

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

for all vector fields X, Y on M, then M is a *Sasakian* manifold. In this case ξ is a Killing vector field and hence M is a K-contact manifold.

A *contact metric (κ, μ)-space* is a contact metric manifold $(M, \varphi, \xi, \eta, g)$ such that

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY),$$

where $X, Y \in \mathfrak{X}(M)$ are arbitrary vector fields and κ, μ are real numbers. The (κ, μ) condition is invariant under D_a-homothetic deformations. We recall that a D_a-homothetic deformation of a contact metric manifold $(M, \varphi, \xi, \eta, g)$ is given by the following changing of the structural tensors of M:

$$\bar{\eta} := a\eta, \quad \bar{\xi} := \tfrac{1}{a}\xi, \quad \bar{g} = ag + a(a-1)\eta \otimes \eta, \tag{1}$$

where a is a positive constant.

By direct computations one can check that a D_a-homothetic deformation transforms a contact metric (κ, μ) space into a contact metric $(\bar{\kappa}, \bar{\mu})$ space where

$$\bar{\kappa} = \frac{\kappa + a^2 - 1}{a^2}, \quad \bar{\mu} = \frac{\mu + 2a - 2}{a}.$$

In particular, a D_a-homothetic deformation of a contact metric manifold $(M, \varphi, \xi, \eta, g)$ satisfying $R(X, Y)\xi = 0$ yields

$$\bar{R}(X, Y)\xi = \frac{a^2 - 1}{a^2}(\bar{\eta}(Y)X - \bar{\eta}(X)Y) + \frac{2a - 2}{a}(\bar{\eta}(Y)\bar{h}X - \bar{\eta}(X)\bar{h}Y).$$

Blair, Koufogiorgos, and Papantoniou [1995] proved the following result.

Theorem 1. *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric (κ, μ) manifold. Then $\kappa \leqslant 1$. Moreover, if $\kappa = 1$ then $h = 0$ and $(M, \varphi, \xi, \eta, g)$ is Sasakian. If $\kappa < 1$, the contact metric structure is not Sasakian and M admits three mutually orthogonal integrable distributions $\mathcal{D}(0)$, $\mathcal{D}(\lambda)$, and $\mathcal{D}(-\lambda)$ corresponding to the eigenspaces of h, where $\lambda = \sqrt{1-\kappa}$.*

The explicit expression of the Riemannian curvature tensor of a non-Sasakian contact metric (κ, μ)-manifold is known (see [Boeckx 1999, Theorem 5]).

Theorem 2. *Let M be a contact metric (κ, μ)-space. If $\kappa \neq 1$, then*

$$\begin{aligned} g(R(X,Y)Z,W) = {} & \bigl(1-\tfrac{1}{2}\mu\bigr)(g(Y,Z)g(X,W)-g(X,Z)g(Y,W)) \\ & + g(Y,Z)g(hX,W)-g(X,Z)g(hY,W) \\ & - g(Y,W)g(hX,Z)+g(X,W)g(hY,Z) \\ & + \frac{1-\mu/2}{1-\kappa}(g(hY,Z)g(hX,W)-g(hX,Z)g(hY,W)) \\ & - \tfrac{1}{2}\mu(g(\varphi Y,Z)g(\varphi X,W)-g(\varphi X,Z)g(\varphi Y,W)) \\ & + \frac{\kappa-\mu/2}{1-\kappa}(g(\varphi hY,Z)g(\varphi hX,W)-g(\varphi hY,W)g(\varphi hX,Z)) \\ & + \mu g(\varphi X,Y)g(\varphi Z,W) \\ & + \eta(X)\eta(W)\bigl(\bigl(\kappa-1+\tfrac{1}{2}\mu\bigr)g(Y,Z)+(\mu-1)g(hY,Z)\bigr) \\ & - \eta(X)\eta(Z)\bigl(\bigl(\kappa-1+\tfrac{1}{2}\mu\bigr)g(Y,W)+(\mu-1)g(hY,W)\bigr) \\ & + \eta(Y)\eta(Z)\bigl(\bigl(\kappa-1+\tfrac{1}{2}\mu\bigr)g(X,W)+(\mu-1)g(hX,W)\bigr) \\ & - \eta(Y)\eta(W)\bigl(\bigl(\kappa-1+\tfrac{1}{2}\mu\bigr)g(X,Z)+(\mu-1)g(hX,Z)\bigr). \end{aligned}$$

The class of non-Sasakian contact metric (κ, μ)-spaces coincides with the class of contact metric manifolds with nonvanishing η-parallel tensor h, according to [Blair, Koufogiorgos, and Papantoniou 1995, Lemma 3.8] and the following result of Boeckx and Cho [2005]:

Theorem 3. *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric manifold which is not K-contact. If $g((\nabla_X h)Y, Z) = 0$ for all vector fields X, Y, Z orthogonal to ξ, then M is a contact metric (κ, μ)-space.*

Finally, we recall also the following characterization in the context of CR geometry (we refer to [Blair 2010, §6.4; Dragomir and Tomassini 2006] for a general reference on this topic):

Theorem 4 [Dileo and Lotta 2009, Theorem 3.2]. *Let (M, HM, J, η) be a pseudo-Hermitian manifold. Assume that the Webster metric g_η is not Sasakian. The following conditions are equivalent*:

(1) *The Webster metric g_η is locally* CR*-symmetric.*

(2) *The underlying contact metric structure satisfies the (κ, μ) condition.*

Non-Sasakian contact metric (κ, μ)-spaces have been completely classified by Boeckx [2000]. In this case $\kappa < 1$ and the real number

$$I_M := \frac{1-\mu/2}{\sqrt{1-\kappa}}$$

is an invariant of the (κ, μ)-structure, which we call *Boeckx invariant*. Indeed we have:

Theorem 5 [Boeckx 2000]. *Let $(M_i, \varphi_i, \xi_i, \eta_i, g_i)$, $i = 1, 2$, be two non-Sasakian (κ_i, μ_i)-spaces of the same dimension. Then $I_{M_1} = I_{M_2}$ if and only if, up to a D-homothetic transformation, the two spaces are locally isometric as contact metric spaces. In particular, if both spaces are simply connected and complete, they are globally isometric up to a D-homothetic deformation.*

Next we recall the notions of straight and twisted complexifications of a Lie triple system (LTS). For more details we refer the reader to [Bertram 2000; 2001]. Given a Lie triple system $(\mathfrak{m}, [\ ,\ ,\ ])$ we shall write as usual

$$R(X, Y)Z := -[X, Y, Z].$$

We shall also write $(\mathfrak{m}, R)$ instead of $(\mathfrak{m}, [\ ,\ ,\ ])$. An *invariant complex structure* on $\mathfrak{m}$ is a complex structure $J : \mathfrak{m} \to \mathfrak{m}$ such that for every $X, Y, Z \in \mathfrak{m}$,

$$[X, Y, JZ] = J[X, Y, Z].$$

An *invariant paracomplex structure* I on $\mathfrak{m}$ is a paracomplex structure on $\mathfrak{m}$ (i.e., an endomorphism of $\mathfrak{m}$ such that $I^2 = \mathrm{id}_{\mathfrak{m}}$ and the ± 1 eigenspaces of I have the same dimension) satisfying

$$[X, Y, IZ] = I[X, Y, Z]$$

for every $X, Y, Z \in \mathfrak{m}$.

For every LTS $\mathfrak{m}$ endowed with an invariant (para-)complex structure, the corresponding simply connected symmetric space G/H is canonically endowed with a G-invariant almost (para-)complex structure and vice versa (see [Bertram 2000, Proposition III.1.4]).

An invariant (para-)complex structure J on a Lie triple system $(\mathfrak{m}, [\ ,\ ,\ ])$ is called *straight* if

$$[JX, Y, Z] = [X, JY, Z]$$

or *twisted* if

$$[JX, Y, Z] = -[X, JY, Z].$$

Accordingly, a *straight* or respectively *twisted* (para-)complex symmetric space is an affine symmetric space $M = G/H$ endowed with an invariant almost (para-)complex structure $\mathcal{J}$ such that

$$R(\mathcal{J}X, Y)Z = R(X, \mathcal{J}Y)Z$$

or respectively

$$R(\mathcal{J}X, Y)Z = -R(X, \mathcal{J}Y)Z,$$

where R is the curvature of M.

A *(para-)complexification* of an LTS $\mathfrak{m}$ is an LTS $(\mathfrak{q}, [\ ,\ ,\ ])$ together with an invariant (para-)complex structure J and an automorphism τ such that $\tau J + J\tau = 0$, $\tau^2 = \mathrm{id}_{\mathfrak{q}}$, and the LTS $\mathfrak{q}^\tau$ given by the space of τ-fixed points of $\mathfrak{q}$ is isomorphic to $\mathfrak{m}$. The (para-)complexification $(\mathfrak{q}, [\ ,\ ,\ ], J, \tau)$ of $\mathfrak{m}$ is called *straight* or *twisted* respectively if J is a straight or twisted.

We recall that every LTS $(\mathfrak{m}, R)$ has a unique straight complexification given by the $\mathbb{C}$-trilinear extension $R_{\mathbb{C}} : \mathfrak{m}_{\mathbb{C}} \times \mathfrak{m}_{\mathbb{C}} \times \mathfrak{m}_{\mathbb{C}} \to \mathfrak{m}_{\mathbb{C}}$ of R [Bertram 2001, Proposition 2.1.4]. The existence of a twisted complexification or paracomplexification of $\mathfrak{m}$ is instead related to the existence of a particular $(1, 3)$-tensor, the *Jordan extension* of R.

Let $M = G/H$ be a symmetric space endowed with an invariant almost (para-)complex structure $\mathcal{J}$. The *structure tensor* of $\mathcal{J}$ is the $(1, 3)$-tensor

$$T(X, Y)Z = -\tfrac{1}{2}(R(X, Y)Z - \mathcal{J}R(X, \mathcal{J}^{-1}Y)Z).$$

This tensor satisfies the following two properties:

(JT1) $T(X, Y)Z = T(Z, Y)X$,

(JT2) $T(U, V)T(X, Y, Z)$
$$= T(T(U, V)X, Y, Z) - T(X, T(U, V)Y, Z) + T(X, Y, T(U, V)Z).$$

Now, a *Jordan triple system* is a pair (V, T), where V is a vector space and $T : V \times V \times V \to V$ is a trilinear map satisfying (JT1), (JT2), called a *Jordan triple product* on V.

Observe that if T is a JT product on V, then

$$[x, y, z] := T(x, y)z - T(y, x)z$$

is a LT product on V.

Let T be a JT product on an LTS $(\mathfrak{m}, R)$. We set

$$R_T(x, y) := -T(x, y) + T(y, x).$$

T is said to be a *Jordan extension* of R if $R = R_T$.

Theorem 6 [Bertram 2000, Theorem III.4.4]. *Let* $(\mathfrak{m}, R)$ *be an LTS. The following objects are in one-to-one correspondence*:

(1) *twisted complexifications of* R,

(2) *twisted paracomplexifications of* R,

(3) *Jordan extensions of* R.

In the next section we shall be concerned with the following basic examples, studying their interplay with the classification of contact metric (κ, μ)-manifolds.

Consider the Lie triple systems $(\mathbb{R}^n, R)$ and $(\mathbb{R}^n, -R)$, associated respectively to the sphere S^n and the hyperbolic space $\mathbb{H}^n$, where R is

$$R(x, y)z := 2(\langle y, z\rangle x - \langle x, z\rangle y).$$

On $(\mathbb{R}^n, R)$ one can consider the following JT product:

$$T(x, y)z = \langle x, z\rangle y - \langle x, y\rangle z - \langle y, z\rangle x.$$

Then, according to Bertram [2000, Proposition IV.1.5], the corresponding twisted complexification and paracomplexification of S^n are the symmetric spaces

$$\mathrm{SO}(n+2)/(\mathrm{SO}(n) \times \mathrm{SO}(2))$$

and

$$\mathrm{SO}(n+1, 1)/(\mathrm{SO}(n) \times \mathrm{SO}(1, 1)).$$

In the case of $\mathbb{H}^n$, one can consider $-T$; the corresponding twisted complexification is (see [Bertram 2000, p. 91])

$$\mathrm{SO}(n, 2)/(\mathrm{SO}(n) \times \mathrm{SO}(2)).$$

3. A characterization of contact metric (κ, μ)-spaces

Let $(M, \varphi, \xi, \eta, g)$ be a connected homogeneous contact metric manifold. Consider a Lie group G acting transitively on M as a group of automorphisms of the contact metric structure, and denote by H the isotropy subgroup of G at $x_o \in M$. The natural map $j : G/H \to M$ given by $j(aH) = ax_o$ is a diffeomorphism. Thus G/H is a homogeneous Riemannian space and in particular it is a reductive homogeneous space (see, e.g., [Tricerri and Vanhecke 1983]). Fix a reductive decomposition of the Lie algebra $\mathfrak{g}$ of G:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \tag{2}$$

where $\mathfrak{h} = \mathrm{Lie}(H)$. The identity component G^o of G acts again transitively on M, and the isotropy subgroup of G^o at x_o is $H \cap G^o$. Let

$$\pi : G^o \to G^o/H \cap G^o \simeq M$$

be the natural fibration of G^o onto the homogeneous space $G^o/H \cap G^o$. Since $\mathrm{Lie}(H) = \mathrm{Lie}(H \cap G^o)$, (2) is also a reductive decomposition for $G^o/H \cap G^o$. Then $\mathfrak{m}$ decomposes into the direct sum of two $H \cap G^o$-invariant subspaces:

$$\mathfrak{m} = \mathbb{R}J \oplus \mathfrak{b},$$

where J is the vector of $\mathfrak{m}$ corresponding to $\xi_{\underline{o}}$ and $\mathfrak{b}$ corresponds to the determination of the contact distribution $D = \ker(\eta)$ at $\underline{o} := \pi(e) \cong x_o$, where e is the neutral element of G.

Now, homogeneity ensures that the contact form η is regular (see [Boothby and Wang 1958, § II]); hence we have a canonical fibration of M, given by (see also [Musso 1991, p. 225])

$$G^o/H \cap G^o \to G^o/S^o(H \cap G^o),$$

where S^o is the identity component of the closed Lie subgroup

$$S := \{h \in G^o \mid \mathrm{Ad}(h)^*\tilde{\eta} = \tilde{\eta}\}$$

of G^o. Here $\tilde{\eta}$ denotes the one form on G^o pull back of η via π. We have that $H \cap G^o \subset S$ [Boothby and Wang 1958, Lemma II.4].

Moreover, the Lie algebra $\bar{\mathfrak{h}}$ of $\overline{H} := S^o(H \cap G^o)$ is given by

$$\bar{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{R}J,$$

and we have the following decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \bar{\mathfrak{h}} \oplus \mathfrak{b}. \tag{3}$$

Our first aim is to characterize the non-Sasakian contact metric (κ, μ)-spaces as homogeneous contact metric manifolds for which decomposition (3) is *symmetric*, i.e.,

$$[\bar{\mathfrak{h}}, \bar{\mathfrak{h}}] \subset \bar{\mathfrak{h}}, \quad [\bar{\mathfrak{h}}, \mathfrak{b}] \subset \mathfrak{b}, \quad [\mathfrak{b}, \mathfrak{b}] \subset \bar{\mathfrak{h}}.$$

Using this, in Corollary 8, we shall be able to endow B of G^o-invariant affine connections making it an affine symmetric space.

Theorem 7. *Let $(M, \varphi, \xi, \eta, g)$ be a simply connected, complete, contact metric manifold. Assume M is not K-contact. Then the following conditions are equivalent*:

(a) *M is a contact metric (κ, μ)-space.*

(b) *M admits a transitive, effective Lie group of automorphisms G whose Lie algebra $\mathfrak{g}$ is a symmetric Lie algebra with symmetric decomposition* (3).

Proof. (a) $\Rightarrow$ (b): According to [Boeckx 1999], $(M, \varphi, \xi, \eta, g)$ is a homogeneous contact metric manifold. Let $G = \mathrm{Aut}(M)$ be the Lie group of all the automorphisms of the contact metric structure of M, and H be the isotropy subgroup of G at $x_o \in M$.

We fix a reductive decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \tag{4}$$

where $\mathfrak{g}$ and $\mathfrak{h}$ are respectively the Lie algebras of G and H. Keeping the notation above we consider also the decompositions

$$\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}J \oplus \mathfrak{b} = \bar{\mathfrak{h}} \oplus \mathfrak{b}.$$

By Theorem 4, for every $x \in M$ there exists a local CR-symmetry at x. Since M is simply connected and complete, the local CR-symmetries are actually globally defined. Let σ be the CR-symmetry at $\underline{o} = eH$. We recall that σ is an isometric CR diffeomorphism of M, whose differential at $\underline{o}$ is $-\mathrm{Id}$ on $D_{\underline{o}}$. In particular, it is an automorphism of the contact metric structure and an affine automorphism of the canonical G-invariant affine connection $\widetilde{\nabla}$ associated to (4). Hence, denoting by $\widetilde{T}$ the torsion of $\widetilde{\nabla}$, we have that, for every $X, Y, Z \in \mathfrak{b} \subset \mathfrak{m}$:

$$\begin{aligned} g_{\underline{o}}(\widetilde{T}(X, Y), Z) &= g_{\underline{o}}(\sigma_\star \widetilde{T}(X, Y), \sigma_\star Z) = g_{\underline{o}}(\widetilde{T}(\sigma_\star X, \sigma_\star Y), \sigma_\star Z) \\ &= -g_{\underline{o}}(\widetilde{T}(X, Y), Z), \end{aligned}$$

which yields that $[X, Y]_{\mathfrak{m}} = -\widetilde{T}_{\underline{o}}(X, Y) \in \mathbb{R}J$, and hence $[\mathfrak{b}, \mathfrak{b}] \subset \bar{\mathfrak{h}}$.

The curvature tensor $\widetilde{R}$ of $\widetilde{\nabla}$ and the Reeb vector field ξ are also preserved by σ. Hence for every $X, Y, Z \in \mathfrak{b}$:

$$\begin{aligned} g_{\underline{o}}(\widetilde{R}(J, X)Y, Z) &= g_{\underline{o}}(\sigma_\star \widetilde{R}(J, X)Y, \sigma_\star Z) = g_{\underline{o}}(\widetilde{R}(\sigma_\star J, \sigma_\star X)\sigma_\star Y, \sigma_\star Z) \\ &= -g_{\underline{o}}(\widetilde{R}(J, X)Y, Z), \end{aligned}$$

moreover, since $\widetilde{\nabla}\mathcal{D} \subset \mathcal{D}$ we have that $\widetilde{R}(J, X)Y \in \mathcal{D}_{\underline{o}}$; thus

$$[[J, X]_{\mathfrak{h}}, Y] = 0$$

for every $X, Y \in \mathfrak{b}$. Since G is effective on M, the adjoint representation $\mathrm{ad} : \mathfrak{h} \to \mathrm{End}(\mathfrak{m})$ is injective; therefore, using also $[\mathfrak{h}, J] = 0$, we conclude that $[J, X]_{\mathfrak{h}} = 0$.

Finally we prove that $[J, X] \in \mathfrak{b}$; indeed we have

$$\begin{aligned} g_{\underline{o}}(\widetilde{T}(J, X), J) &= g_{\underline{o}}(\sigma_\star \widetilde{T}(J, X), \sigma_\star J) = g_{\underline{o}}(\widetilde{T}(\sigma_\star J, \sigma_\star X), \sigma_\star J) \\ &= -g_{\underline{o}}(\widetilde{T}(J, X), J). \end{aligned}$$

This completes the proof of (b).

(b) $\Rightarrow$ (a): Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a reductive decomposition for the homogeneous contact metric space $M = G/H$, where H is the isotropy subgroup of G at a point $x_o \in M$.

Let ∇ and $\widetilde{\nabla}$ respectively the Levi-Civita connection of g and the canonical affine connection on M associated to the fixed reductive decomposition. If we set $A = \nabla - \widetilde{\nabla}$, then

$$(\nabla_X h)Y = (\widetilde{\nabla}_X h)Y + A(X, hY) - hA(X, Y).$$

Now, since the tensor $h = \frac{1}{2}\mathcal{L}_\xi \varphi$ is invariant under automorphisms of the contact metric structure, it is parallel with respect to the canonical connection $\widetilde{\nabla}$ [Kobayashi and Nomizu 1969, p. 193] and hence

$$(\nabla_X h)Y = A(X, hY) - hA(X, Y). \tag{5}$$

Since $\widetilde{\nabla}$ is a metric connection, for $X, Y, Z \in \mathfrak{X}(M)$ we have that

$$g(A(X, Y), Z) + g(Y, A(X, Z)) = 0. \tag{6}$$

Then for every $X, Y, Z \in \mathfrak{X}(M)$,

$$2g(A(X, Y), Z) = -g(\widetilde{T}(X, Y), Z) + g(\widetilde{T}(Y, Z), X) - g(\widetilde{T}(Z, X), Y). \tag{7}$$

Now observe that for every $X, Y \in \mathfrak{b}$,

$$\widetilde{T}_{\underline{o}}(X, Y) = -[X, Y]_{\mathfrak{m}},$$

and

$$[X, Y] \in \mathfrak{h} \oplus \mathbb{R}J,$$

since $\mathfrak{g} = \bar{\mathfrak{h}} \oplus \mathfrak{b}$ a symmetric decomposition by assumption. Thus $\widetilde{T}_{\underline{o}}(X, Y) \in \mathbb{R}J$. Hence for every $X, Y, Z \in \mathcal{D}$,

$$g(\widetilde{T}(X, Y), Z) = 0,$$

and then, by (7),

$$g(A(X, Y), Z) = 0.$$

Thus, using (5), we obtain that

$$g((\nabla_X h)Y, Z) = 0$$

for every $X, Y, Z \in \mathcal{D}$. This implies that M is a contact metric (κ, μ)-space according to Theorem 3. □

Corollary 8. *Let $M = G/H$ be a simply connected, complete, non-Sasakian contact metric (κ, μ)-manifold. Then the base space $B = G^o/\overline{H}$ of the canonical fibration of M is an affine symmetric space.*

Proof. It suffices to prove that $B = G^o/\overline{H}$ is a homogeneous reductive space with respect to decomposition (3); indeed, the associated canonical G^o-invariant connection makes B a locally symmetric affine manifold. Observe that B is simply connected since the fibers of the canonical fibration are connected (see [Boothby and Wang 1958, Theorem II.4]). Since the canonical invariant connection is always complete (see [Kobayashi and Nomizu 1969, Chapter X, Corollary 2.5]), B is actually a symmetric space.

To prove our claim, we recall that $H \cap G^o \subset S$; thus $S^o \subset S^o(H \cap G^o) \subset S$ and $\mathrm{Lie}(S^o) = \bar{\mathfrak{h}}$. Since $[\bar{\mathfrak{h}}, \mathfrak{b}] \subset \mathfrak{b}$ and S^o is connected, it follows that $\mathrm{Ad}(S^o)\mathfrak{b} \subset \mathfrak{b}$ and hence, since also $\mathrm{Ad}(H \cap G^o)(\mathfrak{b}) \subset \mathfrak{b}$, we conclude that $\mathrm{Ad}(\overline{H})\mathfrak{b} \subset \mathfrak{b}$, as claimed. □

We remark that the affine symmetric structure on B thus obtained a priori depends on the initial choice of a reductive decomposition (2) of $\mathfrak{g}$. In the next section, we shall see that actually different choices lead to the same affine symmetric space, up to isomorphism (see Corollary 10).

4. The base space of the canonical fibration

The aim of this section is to give a complete classification of the symmetric base spaces B of the canonical fibrations of simply connected, complete, non-Sasakian contact metric (κ,μ)-manifolds with Boeckx invariant $I_M \neq \pm 1$. We obtain that B is a twisted complexification or paracomplexification of the sphere S^n, or of the hyperbolic space $\mathbb{H}^n$ according to this table:

Boeckx invariant	base space	type
$I_M > 1$	$\mathrm{SO}(n+2)/(\mathrm{SO}(n)\times\mathrm{SO}(2))$	complexification of S^n
$-1 < I_M < 1$	$\mathrm{SO}(n+1,1)/(\mathrm{SO}(n)\times\mathrm{SO}(1,1))$	paracomplexification of S^n
$I_M < -1$	$\mathrm{SO}(n,2)/(\mathrm{SO}(n)\times\mathrm{SO}(2))$	complexification of $\mathbb{H}^n$

Keeping the notations above, we identify the tangent space of B at the base point with the linear subspace $\mathfrak{b} \cong \mathcal{D}_o$. Moreover we denote by $\mathfrak{b}_+$ and $\mathfrak{b}_-$ the subspaces of $\mathfrak{b}$ corresponding respectively to the eigenspaces $\mathcal{D}_o(\lambda)$ and $\mathcal{D}_o(-\lambda)$ of $h_o : \mathfrak{b} \to \mathfrak{b}$.

We start by computing the curvature of B.

Proposition 9. *Let (M,φ,ξ,η,g) be a simply connected, complete, non-Sasakian contact metric (κ,μ)-manifold and B the base space of the canonical fibration of M. If $\overline{\nabla}$ is the canonical affine connection on B associated to any reductive decomposition of type* (3), *then the curvature tensor $\bar{R}$ of $\overline{\nabla}$ at the base point $o \in B$ is given by*

$$\begin{aligned}
\bar{R}_o(X,Y)Z = {} & \big(\big(1-\tfrac{1}{2}\mu\big)g(Y,Z)+g(hY,Z)\big)X \\
& - \big(\big(1-\tfrac{1}{2}\mu\big)g(X,Z)+g(hX,Z)\big)Y \\
& + \Big(\frac{1-\mu/2}{1-\kappa}g(hY,Z)+g(Y,Z)\Big)hX \\
& - \Big(\frac{1-\mu/2}{1-\kappa}g(hX,Z)+g(X,Z)\Big)hY \\
& + \big(\big(1-\tfrac{1}{2}\mu\big)g(\varphi Y,Z)+g(\varphi hY,Z)\big)\varphi X \\
& - \big(\big(1-\tfrac{1}{2}\mu\big)g(\varphi X,Z)+g(\varphi hX,Z)\big)\varphi Y \\
& + \Big(\frac{1-\mu/2}{1-\kappa}g(\varphi hY,Z)+g(\varphi Y,Z)\Big)\varphi hX \\
& - \Big(\frac{1-\mu/2}{1-\kappa}g(\varphi hX,Z)+g(\varphi X,Z)\Big)\varphi hY \\
& + (\mu-2)g(\varphi X,Y)\varphi Z - 2g(\varphi X,Y)\varphi hZ.
\end{aligned} \tag{8}$$

Proof. For every $X, Y, Z \in \mathfrak{b}$ we have

$$\bar{R}_o(X,Y)Z = -[[X,Y]_J + [X,Y]_{\mathfrak{h}}, Z]$$

(see [Kobayashi and Nomizu 1969, Chapter X]), and hence

$$\bar{R}_o(X,Y)Z = \widetilde{R}(X,Y)Z - [[X,Y]_J, Z], \tag{9}$$

where $[X,Y]_J$ and $[X,Y]_{\mathfrak{h}}$ are the components of $[X,Y] \in \mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}J \oplus \mathfrak{b}$ respectively in $\mathbb{R}J$ and $\mathfrak{h}$; $\widetilde{R}$ is the curvature tensor of the canonical connection of the homogeneous reductive space M with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

Let ∇ be the Levi-Civita connection of g and R the curvature tensor of ∇. If we set $A := \widetilde{\nabla} - \nabla$, then a standard computation yields:

$$\begin{aligned}\widetilde{R}(X,Y)Z = R(X,Y)Z - A(X,A(Y,Z)) + A(Y,A(X,Z))&\\ + A(\widetilde{T}(X,Y),Z) + (\widetilde{\nabla}_X A)(Y,Z) - (\widetilde{\nabla}_Y A)(X,Z)&,\end{aligned}$$

for every $X,Y,Z \in \mathfrak{X}(M)$. Moreover, since A is a G-invariant tensor, we have that A is parallel with respect to the canonical connection $\widetilde{\nabla}$ and hence

$$\widetilde{R}(X,Y)Z = R(X,Y)Z - A(X,A(Y,Z)) + A(Y,A(X,Z)) + A(\widetilde{T}(X,Y),Z),$$

and (9) becomes

$$\begin{aligned}\bar{R}_o(X,Y)Z = R(X,Y)Z - A(X,A(Y,Z)) + A(Y,A(X,Z))&\\ + A(\widetilde{T}(X,Y),Z) - [[X,Y]_J,Z]&.\end{aligned}$$

We already observed in the proof of Theorem 7 that for every $X,Y,Z \in \mathcal{D}$,

$$g(A(X,Y),Z) = 0, \quad g(\widetilde{T}(X,Y),Z) = 0;$$

hence

$$A(X,Y) = g(A(X,Y),\xi)\xi, \tag{10}$$

$$\widetilde{T}(X,Y) = g(\widetilde{T}(X,Y),\xi)\xi = -g([X,Y],\xi)\xi = 2g(X,\varphi Y)\xi. \tag{11}$$

In (11) we are using the parallelism of the distributions $\mathcal{D}(\pm\lambda)$ with respect to $\widetilde{\nabla}$, which is a consequence of the fact that $\widetilde{\nabla}h = 0$.

Moreover, we have

$$A(X,\xi) = \widetilde{\nabla}_X\xi - \nabla_X\xi = \varphi X + \varphi hX. \tag{12}$$

Then, using (10), (11), (12), specializing at the point o we obtain

$$\begin{aligned}\bar{R}_o(X,Y)Z &= R(X,Y)Z - g(A(Y,Z),J)A(X,J) + g(A(X,Z),J)A(Y,J)\\ &\qquad + 2g(X,\varphi Y)A(J,Z) + [\widetilde{T}(X,Y),Z]\\ &= R(X,Y)Z - g(A(Y,Z),J)(\varphi X + \varphi hX)\\ &\qquad + g(A(X,Z),J)(\varphi Y + \varphi hY) + 2g(X,\varphi Y)A(J,Z)\\ &\qquad + 2g(X,\varphi Y)[J,Z],\end{aligned} \tag{13}$$

where $X, Y, Z \in \mathfrak{b}$. The $(1, 1)$-tensor $A(X, \cdot)$ is a skew symmetric tensor, since $\widetilde{\nabla} g = 0$. In particular,

$$g(A(X, Y), \xi) = -g(Y, A(X, \xi)),$$

so that, by (12)

$$g(A(X, Y), \xi) = -g(Y, \varphi X + \varphi h X).$$

Thus, (13) becomes

$$\begin{aligned} \bar{R}_o(X, Y)Z = R(X, Y)Z &+ g(Z, \varphi Y + \varphi h Y)(\varphi X + \varphi h X) \\ &- g(\varphi X + \varphi h X, Z)(\varphi Y + \varphi h Y) + 2g(X, \varphi Y)A(J, Z) \\ &+ 2g(X, \varphi Y)[J, Z]. \end{aligned}$$

Now, using Theorem 7,

$$\widetilde{T}_o(J, Z) = -[J, Z]_{\mathfrak{m}} = -[J, Z];$$

on the other hand,

$$\begin{aligned} \widetilde{T}(\xi, W) &= \widetilde{\nabla}_\xi W - \widetilde{\nabla}_W \xi - [\xi, W] = \nabla_\xi W + A(\xi, W) - [\xi, W] \\ &= -\varphi W - \varphi h W + A(\xi, W), \end{aligned}$$

for every W vector field on M. Thus,

$$\begin{aligned} \bar{R}_o(X, Y)Z &= R(X, Y)Z + g(Z, \varphi Y + \varphi h Y)(\varphi X + \varphi h X) \\ &\quad - g(\varphi X + \varphi h X, Z)(\varphi Y + \varphi h Y) + 2g(X, \varphi Y)A(J, Z) \\ &\quad - 2g(X, \varphi Y)(-\varphi Z - \varphi h Z + A(J, Z)) \\ &= R(X, Y)Z + g(Z, \varphi Y + \varphi h Y)(\varphi X + \varphi h X) \\ &\quad - g(\varphi X + \varphi h X, Z)(\varphi Y + \varphi h Y) + 2g(X, \varphi Y)(\varphi Z + \varphi h Z). \end{aligned}$$

Finally, taking into account the explicit expression of the curvature tensor R of M (see Theorem 2), we obtain (8). □

Corollary 10. *The affine base spaces* $(B, \overline{\nabla})$ *of a simply connected, complete, non-Sasakian, contact metric* (κ, μ)*-manifold are all mutually equivalent affine symmetric spaces.*

For a non-Sasakian contact metric (κ, μ)-space the restriction of the $(1, 1)$ tensor φ to the horizontal distribution does not induce a complex structure on the base space, as occurs in the homogeneous Sasakian case, because $h \neq 0$. However, we shall see in the following that B admits a *standard* complex or paracomplex structure, according to the following definition and Theorem 13.

Definition 11. Let $(M, \varphi, \xi, \eta, g)$ be a contact metric (κ, μ)-manifold and $(B, \overline{\nabla})$ the base space of the canonical fibration of M.

A G^o-invariant almost complex structure $\mathcal{J}$ on B will be called *standard complex structure* provided its determination at the base point o is of the form

$$\mathcal{J}_o = \begin{cases} a\varphi & \text{on } \mathfrak{b}_+, \\ \frac{1}{a}\varphi & \text{on } \mathfrak{b}_-, \end{cases} \tag{14}$$

where a is a positive constant.

A *standard paracomplex structure* on B is a G^o-invariant almost paracomplex structure on B whose determination at the base point o is of the form

$$\mathcal{I}_o = \begin{cases} a\varphi & \text{on } \mathfrak{b}_+, \\ -\frac{1}{a}\varphi & \text{on } \mathfrak{b}_-, \end{cases} \tag{15}$$

where a is a positive constant.

Remark 12. A (para-)complex structure J on the vector space $\mathfrak{b}$ defined as in (14) or (15) does not induce in general a G^o-invariant almost complex or paracomplex structure on B.

Theorem 13. *Let $(M, \varphi, \xi, \eta, g)$ be a simply connected, complete, contact metric (κ, μ)-manifold and let $(B, \overline{\nabla})$ be the symmetric base space of the canonical fibration of M. Then:*

(1) $|I_M| > 1$ *if and only if B admits a standard complex structure.*

(2) $|I_M| < 1$ *if and only if B admits a standard paracomplex structure.*

Moreover, in each case such a standard complex or paracomplex structure is uniquely determined; precisely, it corresponds to the following value of the constant a in (14), (15):

$$a = \sqrt{\frac{I_M + 1}{I_M - 1}}$$

when $|I_M| > 1$, and

$$a = \sqrt{-\frac{I_M + 1}{I_M - 1}}$$

when $|I_M| < 1$.

Proof. Let $(\mathfrak{b}, [\ ,\ ,\ ])$ be the Lie triple system associated to the symmetric space $(B, \overline{\nabla})$. The Lie triple product $[\ ,\ ,\ ]$ is given by the curvature $\overline{R}$ of $\overline{\nabla}$ at the base point o:

$$[X, Y, Z] = -\overline{R}_o(X, Y)Z.$$

Let $J : \mathfrak{b} \to \mathfrak{b}$ be a complex structure on $\mathfrak{b}$ of the form

$$J = \begin{cases} a\varphi & \text{on } \mathfrak{b}_+, \\ \frac{1}{a}\varphi & \text{on } \mathfrak{b}_-, \end{cases} \tag{16}$$

where a is a real parameter, $a > 0$.

For every $X_+, Y_+, Z_+ \in \mathfrak{b}_+$ and $X_-, Y_-, Z_- \in \mathfrak{b}_-$, using (8) and (16), by a direct computation, one can check that

$$\begin{gathered} \bar{R}(X_+, Y_+)JZ_+ = J\bar{R}(X_+, Y_+)Z_+, \quad \bar{R}(X_+, Y_+)JZ_- = J\bar{R}(X_+, Y_+)Z_-, \\ \bar{R}(X_-, Y_-)JZ_+ = J\bar{R}(X_-, Y_-)Z_+, \quad \bar{R}(X_-, Y_-)JZ_- = J\bar{R}(X_-, Y_-)Z_-, \\ \bar{R}(X_+, Y_-)JZ_- = \tfrac{1}{a}(2\lambda - \mu + 2)g(\varphi X_+, Y_-)Z_-, \\ J\bar{R}(X_+, Y_-)Z_- = -a(\mu - 2 + 2\lambda)g(\varphi X_+, Y_-)Z_-. \end{gathered}$$

Hence, the condition

$$\bar{R}(X_+, Y_-)JZ_- = J\bar{R}(X_+, Y_-)Z_-$$

is satisfied for every $X_+ \in \mathfrak{b}_+$, $Y_-, Z_- \in \mathfrak{b}_-$ if and only if there exists $a > 0$ such that $2\lambda - \mu + 2 = -a^2(\mu - 2 + 2\lambda)$.

If $\mu - 2 + 2\lambda = 0$ then also $2\lambda - \mu + 2 = 0$. It follows that $\kappa = 1$, but by assumption M is non-Sasakian, then it must be $\mu - 2 + 2\lambda \neq 0$ and

$$-\frac{2\lambda - \mu + 2}{2\lambda + \mu - 2} > 0.$$

This condition is equivalent to requiring that $|I_M| > 1$.

Finally,

$$\begin{aligned} \bar{R}(X_+, Y_-)JZ_+ &= -a(2\lambda + \mu - 2)g(\varphi X_+, Y_-)Z_+, \\ J\bar{R}(X_+, Y_-)Z_+ &= \tfrac{1}{a}(2\lambda - \mu + 2)g(\varphi X_+, Y_-)Z_+. \end{aligned}$$

Thus,

$$\bar{R}(X_+, Y_-)JZ_+ = J\bar{R}(X_+, Y_-)Z_+$$

for every $X_+, Z_+ \in \mathfrak{b}_+$, $Y_- \in \mathfrak{b}_-$ if and only if there exist $a > 0$ such that $2\lambda - \mu + 2 = -a^2(2\lambda + \mu - 2)$.

We conclude that the complex structure J is invariant if and only if $|I_M| > 1$. Moreover, in this case

$$a = \sqrt{\frac{2 - \mu + 2\lambda}{2 - \mu - 2\lambda}}.$$

With analogous considerations, we obtain that the paracomplex structure defined on $\mathfrak{b}$ by

$$I = \begin{cases} a\varphi & \text{on } \mathfrak{b}_+, \\ -\frac{1}{a}\varphi & \text{on } \mathfrak{b}_-, \end{cases} \tag{17}$$

where $a > 0$, is an invariant paracomplex structure if and only if $-1 < I_M < 1$. In this case,

$$a = \sqrt{-\frac{2-\mu+2\lambda}{2-\mu-2\lambda}}. \qquad \square$$

Remark 14. Cappelletti-Montano, Carriazo, and Martín-Molina [2013] showed that every non-Sasakian contact metric (κ, μ)-manifold $(M, \varphi, \xi, \eta, g)$ with $|I_M| > 1$ admits a Sasakian structure $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ obtained by deforming the $(1, 1)$-tensor φ and the Riemannian metric g as

$$\tilde{\varphi} = \epsilon \frac{1}{(1-\kappa)\sqrt{(2-\mu)^2 - 4(1-\kappa)}} \mathcal{L}_\xi h \circ h, \quad \tilde{g} = -\mathrm{d}\eta(\,\cdot\,, \tilde{\varphi}\,\cdot\,) + \eta \otimes \eta,$$

where

$$\epsilon = \begin{cases} 1 & \text{if } I_M > 1, \\ -1 & \text{if } I_M < -1. \end{cases}$$

Moreover, for every point of M there exists a local CR-symmetry [Dileo and Lotta 2009, Theorem 3.2]. Observe that the CR-symmetries preserve the tensor field h, and hence they preserve also $\tilde{\varphi}$ and $\tilde{g}$. Thus, $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is a Sasakian φ-symmetric space [Dileo and Lotta 2009, Proposition 3.3] and fibers over a Kähler manifold $(B, \bar{\mathcal{J}}, \bar{g})$ that is a Hermitian symmetric space [Takahashi 1977]. One can check that $\bar{\mathcal{J}}$ coincides with the standard complex structure $\mathcal{J}$ on B in our sense.

Proposition 15. *The standard (para-)complex structure on the base space $(B, \bar{\bar{\nabla}})$ of a simply connected, complete, non-Sasakian, contact metric (κ, μ)-manifold M with $|I_M| > 1$ $(|I_M| < 1)$ is actually a twisted (para-)complex G^o-invariant structure.*

Proof. This can be easily verified directly using (8). $\square$

Theorem 16. *Let M^{2n+1} be a simply connected, complete, non-Sasakian, contact metric (κ, μ)-manifold. Then:*

(a) *$I_M > 1$ if and only if its twisted complex symmetric base space $(B, \bar{\bar{\nabla}}, \mathcal{J})$ is the complexification $\mathrm{SO}(n+2)/(\mathrm{SO}(n) \times \mathrm{SO}(2))$ of S^n.*

(b) *$-1 < I_M < 1$ if and only if its twisted paracomplex symmetric base space $(B, \bar{\bar{\nabla}}, \mathcal{I})$ is the paracomplexification $\mathrm{SO}(n+1, 1)/(\mathrm{SO}(n) \times \mathrm{SO}(1, 1))$ of S^n.*

(c) *$I_M < -1$ if and only if its twisted complex symmetric base space $(B, \bar{\bar{\nabla}}, \mathcal{J})$ is the complexification $\mathrm{SO}(n, 2)/(\mathrm{SO}(n) \times \mathrm{SO}(2))$ of $\mathbb{H}^n$.*

Proof. Consider the Lie triple system $(\mathfrak{b}, [\ ,\ ,\ ])$ associated to the canonical symmetric base space $(B, \overline{\nabla})$. The Lie triple commutator $[\ ,\ ,\ ] : \mathfrak{b} \times \mathfrak{b} \times \mathfrak{b} \to \mathfrak{b}$ is given by

$$[X, Y, Z] = -\bar{R}_o(X, Y)Z,$$

where $\bar{R}$ is the curvature of $\overline{\nabla}$. By direct computation, using Proposition 9 we see that the linear mapping

$$\tau : X \in \mathfrak{b} \mapsto \tfrac{1}{\lambda} hX \in \mathfrak{b}$$

is an involutive automorphism of the LTS $(\mathfrak{b}, [\ ,\ ,\ ])$. Thus the space $\mathfrak{b}^\tau$ of the τ-fixed elements of $\mathfrak{b}$, together with the induced Lie triple bracket, is a Lie triple system. Actually, since

$$\mathfrak{b}^\tau = \mathfrak{b}_+,$$

and because the restriction $\bar{R}_+$ of $\bar{R}$ to $\mathfrak{b}_+$ is given by

$$\bar{R}_+(X_+, Y_+)Z_+ = (2 - \mu + 2\lambda)(g(Y_+, Z_+)X_+ - g(X_+, Z_+)Y_+),$$

we have that the LTS $(\mathfrak{b}_+, \bar{R}_+)$ is isomorphic to the LTS belonging to the sphere S^n or the hyperbolic space $\mathbb{H}^n$, according to the circumstance that the Boeckx invariant I_M is greater than -1 or less than -1 respectively; indeed we have $2 - \mu + 2\lambda = 2\lambda(I_M + 1)$.

Suppose $|I_M| > 1$. Let J be the twisted complex structure on $\mathfrak{b}$ corresponding to the standard complex structure $\mathcal{J}$ of B. Observe that $J\tau + \tau J = 0$, since $\varphi h + h\varphi = 0$. Then $(\mathfrak{b}, [\ ,\ ,\ ], J, \tau)$ is a twisted complexification of $(\mathfrak{b}_+, \bar{R}_+)$.

We recall that, by definition, the structure tensor T of $\mathcal{J}$ at the base point o is

$$T_o(X, Y)Z = -\tfrac{1}{2}(\bar{R}_o(X, Y)Z + J\bar{R}_o(X, JY)Z),$$

and that its restriction T_+ to $\mathfrak{b}_+$ yields the Jordan extension $(\mathfrak{b}_+, T_+)$ of the LTS $(\mathfrak{b}_+, \bar{R}_+)$, uniquely associated to its twisted complexification $(\mathfrak{b}, [\ ,\ ,\ ], J, \tau)$ (see Theorem 6).

Computing T_+ we obtain

$$\begin{aligned} T_+(X_+, Y_+)Z_+ &= -\tfrac{1}{2}\big(\bar{R}(X_+, Y_+)Z_+ + J\bar{R}(X_+, JY_+)Z_+\big) \\ &= \tfrac{1}{2}(\mu - 2 - 2\lambda)\big(g(Y_+, Z_+)X_+ - g(X_+, Z_+)Y_+ + g(X_+, Y_+)Z_+\big). \end{aligned}$$

Hence, taking into account the complexification diagrams of the sphere and of the hyperbolic space [Bertram 2000, Chapter IV], we obtain assertions (a) and (c).

Now suppose $|I_M| < 1$ and denote by I the twisted paracomplex structure on $\mathfrak{b}$ corresponding to the standard paracomplex structure $\mathcal{I}$ of B at the base point. We have that $I\tau + \tau I = 0$, since $\varphi h + h\varphi = 0$, and hence $(\mathfrak{b}, [\ ,\ ,\ ], I, \tau)$ is a twisted paracomplexification of $(\mathfrak{b}^\tau, \bar{R}_+)$. The structure tensor of $\mathcal{I}$ at the base point o is

$$T_o(X, Y)Z = -\tfrac{1}{2}(\bar{R}_o(X, Y)Z - I\bar{R}_o(X, IY)Z).$$

Then the Jordan extension of $\bar{R}_+$ uniquely associated to the twisted paracomplexification $(\mathfrak{b}, [\ ,\ ,\ ], I, \tau)$ of the LTS $(\mathfrak{b}_+, -\bar{R}_+)$ is

$$\begin{aligned} T(X_+, Y_+)Z_+ &= -\tfrac{1}{2}\big(\bar{R}(X_+, Y_+)Z_+ - I\bar{R}(X_+, IY_+)Z_+\big) \\ &= -\tfrac{1}{2}(2-\mu+2\lambda)\big(g(Y_+, Z_+)X_+ - g(X_+, Z_+)Y_+ + g(X_+, Y_+)Z_+\big). \end{aligned}$$

Then, comparing again with the complexification diagram of the sphere we obtain assertion (b). □

5. Homogeneous model spaces of contact metric (κ, μ)-spaces

In this section we complete our classification, showing that one can actually construct a contact metric (κ, μ)-space with prescribed Boeckx invariant starting from each of the symmetric spaces in the table on page 50. More precisely, we prove

Theorem 17. *The simply connected, complete, contact metric (κ, μ)-spaces of dimension $2n+1$ (where $n > 1$) with Boeckx invariant different from ± 1 can be classified as follows:*

(a) *The homogeneous space $\mathrm{SO}(n, 2)/\mathrm{SO}(n)$ carries a one-parameter family of invariant contact metric (κ, μ)-structures whose Boeckx invariant assumes all the values in $]-\infty, -1[$.*

(b) *The homogeneous space $\mathrm{SO}(n+2)/\mathrm{SO}(n)$ carries a one-parameter family of invariant contact metric (κ, μ)-structures whose Boeckx invariant assumes all the values in $]1, +\infty[$.*

(c) *The homogeneous space $\mathrm{SO}(n+1, 1)/\mathrm{SO}(n)$ carries a one-parameter family of invariant contact metric (κ, μ)-structures whose Boeckx invariant assumes all the values in $]-1, 1[$.*

Proof. Starting from a fixed Hermitian or para-Hermitian symmetric structure on each of the symmetric spaces,

$$\begin{aligned} B_1 &= \mathrm{SO}(n+2)/(\mathrm{SO}(n)\times \mathrm{SO}(2)), \\ B_2 &= \mathrm{SO}(n, 2)/(\mathrm{SO}(n)\times \mathrm{SO}(2)), \\ B_3 &= \mathrm{SO}(n+1, 1)/(\mathrm{SO}(n)\times \mathrm{SO}(1, 1)), \end{aligned}$$

we shall construct explicitly a one-parameter family of invariant contact metric (κ, μ)-structures on the homogeneous spaces

$$\begin{aligned} M_1 &= \mathrm{SO}(n+2)/\mathrm{SO}(n), \\ M_2 &= \mathrm{SO}(n, 2)/\mathrm{SO}(n), \\ M_3 &= \mathrm{SO}(n+1, 1)/\mathrm{SO}(n), \end{aligned}$$

with $I_{M_1} > 1$, $I_{M_2} < -1$, and $-1 < I_{M_3} < 1$.

We first consider the symmetric Lie algebras $\mathfrak{g}_1 := \mathfrak{so}(n+2)$ and $\mathfrak{g}_2 := \mathfrak{so}(n,2)$ with symmetric decompositions

$$\mathfrak{g}_i = \mathfrak{h}_i \oplus \mathfrak{b}_i,$$

where

$$\mathfrak{h}_1 = \mathfrak{h}_2 := \left\{ \left[\begin{array}{cc|c} 0 & -\lambda & \multirow{2}{*}{$\mathbf{0}$} \\ \lambda & 0 & \\ \hline \mathbf{0} & \mathbf{0} & a \end{array}\right] : \lambda \in \mathbb{R},\ a \in \mathfrak{so}(n) \right\} = \mathfrak{so}(2) \oplus \mathfrak{so}(n),$$

$$\mathfrak{b}_1 := \left\{ \left[\begin{array}{c|c} \multirow{2}{*}{$\mathbf{0}$} & -v^T \\ & -w^T \\ \hline v\ w & \mathbf{0} \end{array}\right] : v, w \in \mathbb{R}^n \right\} \simeq T_oB_1,$$

$$\mathfrak{b}_2 := \left\{ \left[\begin{array}{c|c} \multirow{2}{*}{$\mathbf{0}$} & v^T \\ & w^T \\ \hline v\ w & \mathbf{0} \end{array}\right] : v, w \in \mathbb{R}^n \right\} \simeq T_oB_2.$$

The $\mathrm{Ad}(\mathrm{SO}(2) \times \mathrm{SO}(n))$-invariant almost complex structure $J_i : \mathfrak{b}_i \to \mathfrak{b}_i$ defined by

$$J_i(v\ w) = (-1)^i(w\ -v),$$

and the $\mathrm{Ad}(\mathrm{SO}(2) \times \mathrm{SO}(n))$-invariant metric G_i on $\mathfrak{b}_i$

$$G_i((v\ w), (u\ z)) = \langle v, u\rangle + \langle w, z\rangle,$$

determine an invariant Hermitian symmetric structure $(\mathcal{J}_i, \bar{g}_i)$ on B_i; here $\langle\,\rangle$ denotes the standard inner product on $\mathbb{R}^n$ and $(v\ w)$ denotes the matrix

$$\left[\begin{array}{cc|c} 0 & 0 & -w^T \\ 0 & 0 & -v^T \\ \hline v & w & \mathbf{0} \end{array}\right]$$

in the case $i = 1$, and the matrix

$$\left[\begin{array}{cc|c} 0 & 0 & w^T \\ 0 & 0 & v^T \\ \hline v & w & \mathbf{0} \end{array}\right]$$

in the case $i = 2$. Observe that the decomposition of $\mathfrak{g}_i$,

$$\mathfrak{g}_i = \mathfrak{so}(n) \oplus \mathfrak{m}_i, \tag{18}$$

$$\mathfrak{m}_i := \mathbb{R}\xi \oplus \mathfrak{b}_i, \quad \xi := \left[\begin{array}{cc|c} 0 & -1 & \multirow{2}{*}{$\mathbf{0}$} \\ 1 & 0 & \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right],$$

is a reductive decomposition for M_i. Indeed, for every

$$a = \left[\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & a \end{array}\right] \in \mathrm{SO}(n), \qquad X = s\xi + (v\ w) \in \mathfrak{m}_i,$$

we have that $\mathrm{Ad}(a)X = s\xi + (av\ aw)$. In particular, we have $\mathrm{Ad}(a)\xi = \xi$ for every $a \in \mathrm{SO}(n)$.

We have a natural decomposition of $\mathfrak{b}_i$,

$$\mathfrak{b}_i = \mathfrak{p}_i \oplus \mathfrak{q}_i,$$

where

$$\mathfrak{p}_i := \{(v\,0) \mid v \in \mathbb{R}^n\}, \qquad \mathfrak{q}_i := \{(0\,w) \mid w \in \mathbb{R}^n\}.$$

By using this decomposition, we define on $\mathfrak{m}_i$ a $(1, 1)$ tensor φ_i, an inner product g_i, and a 1-form η_i as follows:

$$\varphi_i(Z) := \begin{cases} \alpha JZ & \text{if } Z \in \mathfrak{p}_i, \\ \frac{1}{\alpha} JZ & \text{if } Z \in \mathfrak{q}_i, \\ 0 & \text{if } Z \in \mathbb{R}\xi, \end{cases} \tag{19}$$

$$g_i(X, Y) := st + \tfrac{1}{2}\big(\alpha\langle v, u\rangle + \tfrac{1}{\alpha}\langle w, z\rangle\big), \qquad \eta_i(X) := s,$$

where $\alpha > 0$, and $X = s\xi + (v\ w)$, $Y = t\xi + (u\ z)$ are arbitrary elements of $\mathfrak{m}_i$. These tensors are $\mathrm{Ad}(\mathrm{SO}(n))$-invariant. Indeed for every $a \in \mathrm{SO}(n)$,

$$\begin{aligned} \mathrm{Ad}(a)\varphi_i X &= \mathrm{Ad}(a)\big((-1)^i\big(\alpha(0\ -v) + \tfrac{1}{\alpha}(w\,0)\big)\big) \\ &= (-1)^i\big(\alpha(0\ -av) + \tfrac{1}{\alpha}(aw\,0)\big) \\ &= \varphi_i\,\mathrm{Ad}(a)X, \end{aligned}$$

$$\begin{aligned} g_i(\mathrm{Ad}(a)X, \mathrm{Ad}(a)Y) &= g(s\xi + (av\,aw), t\xi + (au\,az)) \\ &= st + \tfrac{1}{2}\big(\alpha\langle av, au\rangle + \tfrac{1}{\alpha}\langle aw, az\rangle\big) \\ &= st + \tfrac{1}{2}\big(\alpha\langle v, u\rangle + \tfrac{1}{\alpha}\langle w, z\rangle\big) \\ &= g(X, Y). \end{aligned}$$

Finally, since $\mathrm{Ad}(a)\xi = \xi$, we also have that $\mathrm{Ad}(a)^*\eta_i = \eta_i$. Observe that the invariance of η_i implies that, for every $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{X}(M_i)$,

$$0 = (\mathcal{L}_{X^*}\eta_i)Y = X^*(\eta_i Y) - \eta_i([X^*, Y]),$$

where X^* is the fundamental vector field determined by X. Thus, for every $X, Y \in \mathfrak{m}_i$

$$\begin{aligned} 2\mathrm{d}\eta_i(X^*, Y^*) &= X^*(\eta_i Y^*) - Y^*(\eta_i X^*) - \eta_i([X^*, Y^*]) \\ &= -\eta_i([Y^*, X^*]) = -\eta_i([X, Y]^*). \end{aligned}$$

Evaluating this formula at the base point $o \in M_i$ yields

$$2(\mathrm{d}\eta_i)_o(X, Y) = -\eta_i([X, Y]_{\mathfrak{m}_i}). \tag{20}$$

By direct computations, using (19), (20), we obtain that

$$(\mathrm{d}\eta_i)_o(X, Y) = g_i(X, \varphi_i Y), \quad X, Y \in \mathfrak{m}_i.$$

This proves that the invariant tensors $(\varphi_i, \xi, \eta_i, g_i)$ make up a contact metric structure on M_i. Moreover it is a K-contact structure if and only if $\alpha = 1$. Indeed, since ξ and φ_i are invariant tensors on M_i, they are parallel with respect to the canonical connection $\widetilde{\nabla}$ associated to the decomposition (18), hence,

$$\begin{aligned}
(\mathcal{L}_\xi \varphi_i)Y &= [\xi, \varphi_i Y] - \varphi_i[\xi, Y] \\
&= \widetilde{\nabla}_\xi \varphi_i Y - \widetilde{T}(\xi, \varphi_i Y) - \varphi_i(\widetilde{\nabla}_\xi Y - \widetilde{T}(\xi, Y)) \\
&= -\widetilde{T}(\xi, \varphi_i Y) + \varphi_i \widetilde{T}(\xi, Y),
\end{aligned}$$

then

$$\begin{aligned}
2(h_i)_o(v\ w) &= (\mathcal{L}_\xi \varphi_i)_o(v\ w) \\
&= [\xi, \varphi_i(v\ w)] - \varphi_i[\xi, (v\ w)] \\
&= (-1)^i \big[\xi, \big(\tfrac{1}{\alpha} w\ \ -\alpha v\big)\big] - \varphi_i(-w\ v) \\
&= (-1)^i \big(\alpha v\ \tfrac{1}{\alpha} w\big) - (-1)^i \big(\tfrac{1}{\alpha} v\ \alpha w\big) \\
&= (-1)^i \Big(\frac{\alpha^2 - 1}{\alpha} v\ \ -\frac{\alpha^2 - 1}{\alpha} w\Big).
\end{aligned}$$

Applying Theorem 7, we see that $(\varphi_i, \xi, \eta_i, g_i)$ is a contact metric (κ, μ)-structure on M_i for every $\alpha > 0, \alpha \neq 1$; moreover, by construction, $\mathcal{J}_i$ is a standard complex structure on the base space B_i of the canonical fibration of M_i, in the sense of Definition 11. In particular if $0 < \alpha < 1$ then, by the uniqueness result in Theorem 13, we must have

$$\sqrt{\frac{I_{M_1} + 1}{I_{M_1} - 1}} = \frac{1}{\alpha}, \quad \sqrt{\frac{I_{M_2} + 1}{I_{M_2} - 1}} = \alpha,$$

or equivalently

$$I_{M_1} = \frac{1 + \alpha^2}{1 - \alpha^2}, \quad I_{M_2} = -\frac{1 + \alpha^2}{1 - \alpha^2}.$$

Thus, as α varies in $]0, 1[$, I_{M_1} assumes all the values in $]1, +\infty[$ and I_{M_2} assumes all the values in $]-\infty, -1[$.

Now we consider the Lie algebra $\mathfrak{g} := \mathfrak{so}(n+1, 1)$ with symmetric decomposition $\mathfrak{g} = \bar{\mathfrak{h}} \oplus \mathfrak{b}$, where

$$\bar{\mathfrak{h}} := \left\{ \left[\begin{array}{cc|c} 0 & \lambda & \\ \lambda & 0 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & a \end{array}\right] : \lambda \in \mathbb{R},\ a \in \mathfrak{so}(n) \right\} = \mathfrak{so}(1,1) \oplus \mathfrak{so}(n),$$

$$\mathfrak{b} := \left\{ \left[\begin{array}{c|c} & v^T \\ \mathbf{0} & -w^T \\ \hline v\ \ w & \mathbf{0} \end{array}\right] : v, w \in \mathbb{R}^n \right\} \simeq T_o B_3.$$

Let $(\mathcal{I}, \bar{g})$ be the para-Hermitian structure on B_3 determined by the $\mathrm{Ad}(\mathrm{SO}(1,1) \times \mathrm{SO}(n))$-invariant structure (I, G) on $\mathfrak{b}$:

$$I(v\,w) := -(w\,v), \qquad G((v\,w), (u\,z)) := \langle v, u \rangle - \langle w, z \rangle,$$

where $(v\ w)$ denotes the matrix

$$\left[\begin{array}{c|c} & v^T \\ \mathbf{0} & -w^T \\ \hline v\ \ w & \mathbf{0} \end{array}\right] \in \mathfrak{b}.$$

The homogeneous space $\mathrm{SO}(n+1, 1)/\mathrm{SO}(n)$ is reductive with respect to the decomposition

$$\mathfrak{so}(n+1, 1) = \mathfrak{so}(n) \oplus \mathfrak{m},$$

where

$$\mathfrak{m} := \mathfrak{so}(1,1) \oplus \mathfrak{b} = \mathbb{R}\xi \oplus \mathfrak{b},$$

$$\xi := \left[\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right];$$

indeed

$$\mathrm{Ad}(a)(s\xi + (v\,w)) = s\xi + (av\,aw),$$

for every $a \in \mathrm{SO}(n)$, $X = s\xi + (v\ w) \in \mathfrak{m}$.

Now we consider the natural decomposition of $\mathfrak{b}$:

$$\mathfrak{b} = \mathfrak{p} \oplus \mathfrak{q},$$

where

$$\mathfrak{p} := \{(v\,0) \mid v \in \mathbb{R}^n\} \subset \mathfrak{b},$$

$$\mathfrak{q} := \{(0\,w) \mid w \in \mathbb{R}^n\} \subset \mathfrak{b}.$$

Using this decomposition, we define on $\mathfrak{m}$ the following Ad(SO(n))-invariant tensors:

$$\varphi(Z) := \begin{cases} -\alpha I Z & \text{if } Z \in \mathfrak{p}, \\ \frac{1}{\alpha} I Z & \text{if } Z \in \mathfrak{q}, \\ 0 & \text{if } Z \in \mathbb{R}\xi, \end{cases} \tag{21}$$

$$g(X, Y) := st + \tfrac{1}{2}\big(\alpha\langle v, u\rangle + \tfrac{1}{\alpha}\langle w, z\rangle\big), \qquad \eta(X) := s,$$

where $\alpha > 0$ and $X = s\xi + (v\ w)$, $Y = t\xi + (u\ z)$ are any matrices in $\mathfrak{m}$. One checks by the same method used above that (φ, ξ, η, g) is a contact metric (κ, μ)-structure. Moreover

$$2h_o(v\ w) = \Big(-\frac{\alpha^2+1}{\alpha} v\ \ \frac{\alpha^2+1}{\alpha} w\Big).$$

Then applying again Theorem 13 we get

$$I_{M_3} = \frac{\alpha^2 - 1}{\alpha^2 + 1}$$

and hence, as α varies in $\mathbb{R}^*_+$, I_{M_3} assumes all the values in $]-1, 1[$. □

Remark 18. Of course, in the case $I > 1$ we recover, up to isomorphism, the unit tangent sphere bundle $T_1 M$ of a Riemannian manifold (M, g) with constant sectional curvature $c > 0$, $c \neq 1$.

In the case $I < -1$, we obtain a new homogeneous representation of the contact metric (κ, μ)-manifolds M with $I_M < -1$, different from the Lie group representation furnished by Boeckx. Actually these models can be geometrically interpreted also as tangent hyperquadric bundle over Lorentzian space forms, as shown in [Loiudice and Lotta 2018].

Remark 19. The homogeneous model spaces of contact metric (κ, μ)-manifolds here obtained also appear in the classification list of the simply connected sub-Riemannian symmetric spaces carried out by Bieliavsky, Falbel, and Gorodski [1999]. However, in their paper the contact metric structures are not considered.

References

[Bertram 2000] W. Bertram, *The geometry of Jordan and Lie structures*, Lecture Notes in Math. **1754**, Springer, 2000. MR Zbl

[Bertram 2001] W. Bertram, "Complexifications of symmetric spaces and Jordan theory", *Trans. Amer. Math. Soc.* **353**:6 (2001), 2531–2556. MR Zbl

[Bieliavsky, Falbel, and Gorodski 1999] P. Bieliavsky, E. Falbel, and C. Gorodski, "The classification of simply-connected contact sub-Riemannian symmetric spaces", *Pacific J. Math.* **188**:1 (1999), 65–82. MR Zbl

[Blair 2010] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, 2nd ed., Progress in Math. **203**, Birkhäuser, Boston, 2010. MR Zbl

[Blair, Koufogiorgos, and Papantoniou 1995] D. E. Blair, T. Koufogiorgos, and B. J. Papantoniou, "Contact metric manifolds satisfying a nullity condition", *Israel J. Math.* **91**:1-3 (1995), 189–214. MR Zbl

[Boeckx 1999] E. Boeckx, "A class of locally φ-symmetric contact metric spaces", *Arch. Math. (Basel)* **72**:6 (1999), 466–472. MR Zbl

[Boeckx 2000] E. Boeckx, "A full classification of contact metric (k, μ)-spaces", *Illinois J. Math.* **44**:1 (2000), 212–219. MR Zbl

[Boeckx and Cho 2005] E. Boeckx and J. T. Cho, "η-parallel contact metric spaces", *Differential Geom. Appl.* **22**:3 (2005), 275–285. MR Zbl

[Boeckx and Cho 2008] E. Boeckx and J. T. Cho, "Pseudo-Hermitian symmetries", *Israel J. Math.* **166** (2008), 125–145. MR Zbl

[Boothby and Wang 1958] W. M. Boothby and H. C. Wang, "On contact manifolds", *Ann. of Math.* (2) **68** (1958), 721–734. MR Zbl

[Cappelletti-Montano, Carriazo, and Martín-Molina 2013] B. Cappelletti-Montano, A. Carriazo, and V. Martín-Molina, "Sasaki–Einstein and paraSasaki–Einstein metrics from (κ, μ)-structures", *J. Geom. Phys.* **73** (2013), 20–36. MR Zbl

[Dileo and Lotta 2009] G. Dileo and A. Lotta, "A classification of spherical symmetric CR manifolds", *Bull. Aust. Math. Soc.* **80**:2 (2009), 251–274. MR Zbl

[Dragomir and Tomassini 2006] S. Dragomir and G. Tomassini, *Differential geometry and analysis on CR manifolds*, Progress in Math. **246**, Birkhäuser, Boston, 2006. MR Zbl

[Kaup and Zaitsev 2000] W. Kaup and D. Zaitsev, "On symmetric Cauchy–Riemann manifolds", *Adv. Math.* **149**:2 (2000), 145–181. MR Zbl

[Kobayashi and Nomizu 1969] S. Kobayashi and K. Nomizu, *Foundations of differential geometry, II*, Interscience Tracts in Pure and Applied Math. **15**, Interscience, New York, 1969. MR Zbl

[Loiudice and Lotta 2018] E. Loiudice and A. Lotta, "On the classification of contact metric (k, μ)-spaces via tangent hyperquadric bundles", *Math. Nachr.* **291**:11-12 (2018), 1851–1858.

[Musso 1991] E. Musso, "Homogeneous pseudo-Hermitian Riemannian manifolds of Einstein type", *Amer. J. Math.* **113**:2 (1991), 219–241. MR Zbl

[Takahashi 1977] T. Takahashi, "Sasakian ϕ-symmetric spaces", *Tôhoku Math. J.* (2) **29**:1 (1977), 91–113. MR Zbl

[Tricerri and Vanhecke 1983] F. Tricerri and L. Vanhecke, *Homogeneous structures on Riemannian manifolds*, London Math. Soc. Lecture Note Series **83**, Cambridge Univ. Press, 1983. MR Zbl

Received April 9, 2017. Revised January 31, 2018.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 300, No. 1, 2019
dx.doi.org/10.2140/pjm.2019.300.65

THE $\mathrm{SL}_1(D)$-DISTINCTION PROBLEM

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We use the local theta correspondences between the quaternionic Hermitian groups and the quaternionic skew-Hermitian groups to understand the distinction problem for the symmetric pair $\mathrm{SL}_2(E)/\mathrm{SL}_1(D)$, where E is a quadratic field extension of a nonarchimedean local field F and D is a 4-dimensional division quaternion algebra over F.

1. Introduction

Distinction problems are very popular in representation theory. Let F be a finite field extension of $\mathbb{Q}_p$. Let G be a reductive group defined over F. Let H be a closed subgroup of G. Given a smooth representation π of $G(F)$ and a character χ_H of $H(F)$, if $\dim \mathrm{Hom}_{H(F)}(\pi, \chi_H)$ is nonzero, then π is called $(H(F), \chi_H)$-distinguished. Furthermore, if χ_H is a trivial character, then π is called $H(F)$-distinguished. There is a rich literature, such as [Adler and Prasad 2006; Flicker and Hakim 1994; Anandavardhanan and Prasad 2006; Prasad 2015; Anandavardhanan and Prasad 2013], trying to classify all $H(F)$-distinguished representations of $G(F)$. In this paper, we will focus on the case where $G = R_{E/F}\mathrm{SL}_2$, $H = \mathrm{SL}_1(D)$ and χ_H is trivial, where E/F is a quadratic field extension, D is the unique 4-dimensional quaternion division algebra defined over F and $R_{E/F}$ denotes the Weil restriction of scalars.

Let E be a quadratic field extension of a nonarchimedean local field F of characteristic 0. Let W_E (resp. W_F) be the Weil group of E (resp. F) and WD_E (resp. WD_F) be the Weil–Deligne group of E (resp. F). Let G be a quasi-split reductive group defined over F with Langlands dual group $\hat{G}$. Let π be an irreducible smooth representation of $G(E)$ with enhanced Langlands parameter (ϕ_π, λ), where

$$\phi_\pi : WD_E \longrightarrow \hat{G}(\mathbb{C}) \rtimes W_E$$

is the Langlands parameter and λ is a character of the component group $\pi_0(C_{\hat{G}}(\phi_\pi))$, where $C_{\hat{G}}(\phi_\pi)$ is the centralizer of ϕ_π in $\hat{G}$. Dipendra Prasad [2015] formulated a conjectural identity for the multiplicity $\dim \mathrm{Hom}_{G_\alpha(F)}(\pi, \chi_G)$, in terms of the Langlands parameter $\tilde{\phi}$ of G^{op} satisfying $\tilde{\phi}|_{WD_E} = \phi_\pi$, where G^{op} is a quasi-split

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MSC2010: primary 11F27; secondary 22E50.
Keywords: theta lift, distinction problem, division quaternion algebra.

group defined in [Prasad 2015, §9], G_α is a pure inner form of G satisfying $G_\alpha(E)=G(E)$ and χ_G is a quadratic character of $G(F)$ defined in [Prasad 2015, §10].

It is natural to ask what happens if G_α is an inner form of G satisfying $G_\alpha(E)=G(E)$. There is a well-known result of Prasad [1992] and Jeffrey Hakim [1991] about $D^\times$-distinguished representation π of $\mathrm{GL}_2(E)$.

Theorem 1.1 [Prasad 1992, Theorem C]. *Let π be a square-integrable representation of $\mathrm{GL}_2(E)$; then π is $D^\times$-distinguished if and only if π is $\mathrm{GL}_2(F)$-distinguished.*

Remark 1.2. Raphael Beuzart-Plessis [2018] generalizes this result to any inner form G' of a quasi-split reductive group G for the stable square-integrable representations. More precisely, let E be a quadratic field extension of a nonarchimedean local field F. Let $\chi_{G,E}$ (resp. $\chi_{G',E}$) be a quadratic character of $G(F)$ (resp. $G'(F)$). Suppose that the stable square-integrable representations π of $G(E)$ and π' of $G'(E)$ are matching, then there exists an identity

$$\dim \mathrm{Hom}_{G(F)}(\pi, \chi_{G,E}) = \dim \mathrm{Hom}_{G'(F)}(\pi', \chi_{G',E}).$$

Let us fix an element $\epsilon \in F^\times \backslash N_{E/F}E^\times$. Let $\mathrm{SL}_1(D)$ be the inner form of $\mathrm{SL}_2(F)$, which is a non-quasi-split F-group. There exists an embedding

$$\mathrm{SL}_1(D) = \left\{ g = \begin{pmatrix} \bar{x} & \epsilon\bar{y} \\ y & x \end{pmatrix} \;\middle|\; \det(g)=1,\, x, y \in E \right\} \subset \mathrm{SL}_2(E), \tag{1-1}$$

where $\bar{x}=a-b\sqrt{\varpi}$ if $x=a+b\sqrt{\varpi}$ with $a,b\in F$ and $E=F[\sqrt{\varpi}]$, $\varpi\in F^\times\backslash F^{\times 2}$.

Let V_D be an n-dimensional Hermitian right D-vector space with Hermitian form h_D; then

$$\mathrm{Aut}(V_D,h_D)=\{g\in \mathrm{GL}_n(D)\mid h_D(gv_1,gv_2)=h_D(v_1,v_2)\quad \text{for all } v_1,v_2\in V_D\},$$

where $n=\dim_D V_D$. Assume that $R=M_2(E)\cong D\otimes_F E$ is the split quaternion algebra over E. Due to the Morita equivalence, a right Hermitian (resp. left skew-Hermitian) free R-module (V_R,h_R) corresponds to a symplectic (resp. orthogonal) E-vector space $(\mathfrak{W}_E,h_E)$ satisfying

$$\dim_E \mathfrak{W}_E = 2\dim_R V_R,\quad \mathrm{Aut}(V_R,h_R)=\mathrm{Aut}(\mathfrak{W}_E,h_E),$$

see [Scharlau 1985, §10.3]. Let $V_R=V_D\otimes_D R$ be the natural Hermitian free R-module, then $\mathrm{Aut}(V_R,h_R)\cong \mathrm{Sp}_{2n}(E)$ and there exists a canonical embedding

$$\mathrm{Aut}(V_D,h_D)\hookrightarrow \mathrm{Aut}(V_R,h_R)=\mathrm{Sp}_{2n}(E).$$

Letting $n=1$, we obtain a group embedding

$$\mathrm{SL}_1(D)=\mathrm{Aut}(V_D,h_D)\hookrightarrow \mathrm{Aut}(V_R,h_R)=\mathrm{SL}_2(E) \tag{1-2}$$

which is compatible with the embedding (1-1). We will focus on the embedding (1-2) when we use the local theta correspondence over the quaternionic unitary groups to deal with the distinction problem

$$\mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau, \mathbb{C}).$$

Theorem 1.3. *Suppose that τ is an irreducible $\mathrm{SL}_1(D)$-distinguished representation of $\mathrm{SL}_2(E)$.*

(i) *If τ is a square-integrable representation, then*

$$\dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau, \mathbb{C}) = \begin{cases} 2 & \text{if } |\Pi_{\phi_\tau}| = 2, \\ 1 & \text{otherwise}. \end{cases}$$

Here $|\Pi_{\phi_\tau}|$ denotes the size of the L-packet Π_{ϕ_τ}.

(ii) *If $\tau = I(\chi|-|_E^z)$ is a principal series representation*, $\dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau, \mathbb{C}) = 2$.

(iii) *If $\tau \subset I(\omega_{K/E})$, then* $\dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau, \mathbb{C}) = 1$.

Instead of considering each individual dimension, we consider the sum

$$S(\tau) = \sum_{\pi \in \Pi_{\phi_\tau}} \dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\pi, \mathbb{C}),$$

where Π_{ϕ_τ} is the L-packet of representations of $\mathrm{SL}_2(E)$ containing an $\mathrm{SL}_1(D)$-distinguished representation τ.

Theorem 1.4. *Assume that τ is an $\mathrm{SL}_1(D)$-distinguished representation of $\mathrm{SL}_2(E)$ with an L-parameter ϕ_τ.*

(i) *Suppose that τ is a square-integrable representation.*

  (a) *If $|\Pi_{\phi_\tau}| = 1$, i.e., the size of the L-packet Π_{ϕ_τ} is 1, then $S(\tau) = 1$.*

  (b) *If $|\Pi_{\phi_\tau}| = 2$, then only one of them is $\mathrm{SL}_1(D)$-distinguished, the other is not $\mathrm{SL}_1(D)$-distinguished and $S(\tau) = 2$.*

  (c) *If $|\Pi_{\phi_\tau}| = 4$ and $p \neq 2$, then two members inside the L-packet Π_{ϕ_τ} are $\mathrm{SL}_1(D)$-distinguished with the same multiplicity and $S(\tau) = 2$.*

  (d) *If $|\Pi_{\phi_\tau}| = 4$ and $p = 2$, then $S(\tau) = 2$ or $S(\tau) = 4$.*

(ii) *If τ is an irreducible principal series representation, then $S(\tau) = 2$.*

(iii) *If τ is not discrete but tempered and $|\Pi_{\phi_\tau}| = 2$, then $S(\tau) = 2$.*

Remark 1.5. The main contribution in Theorem 1.4 is that not only is the sum $S(\tau)$ known, but also the partition of $S(\tau)$ in one L-packet Π_{ϕ_τ} is given. However, there is no way in terms of the Whittaker datum to specify which member is $\mathrm{SL}_1(D)$-distinguished inside Π_{ϕ_τ} when $S(\tau)$ is nonzero.

We will use the local theta correspondence for the quaternionic groups to prove Theorem 1.3. The basic ideas come from [Lu 2016; 2018]. With the help of the explicit theta correspondences between small groups, we can use the see-saw identities to transfer the distinction problems for $\mathrm{SL}_1(D)$ to another side, which is related to the branching problems for the 1-dimensional quaternionic skew-Hermitian groups (which are nonsplit tori) over a quadratic field extension E/F, and so it becomes easier, see Section 3 for more details.

Remark 1.6. Anandavardhanan and Prasad [2013] discuss the global period problems for $\mathrm{SL}_1(D)$ over a quadratic number field extension $\mathbb{E}/\mathbb{F}$. More precisely, [Anandavardhanan and Prasad 2013, Proposition 9.3] implies that there exists an automorphic representation π of $\mathrm{SL}_1(D)(\mathbb{A}_\mathbb{E})$ which is locally distinguished by $\mathrm{SL}_1(D)(\mathbb{A}_\mathbb{F})$, but not globally distinguished in terms of having nonzero period integral on this subgroup.

Now we briefly describe the contents and the organization of this paper. In Section 2, we set up the notation about the local theta lifts. In Section 2B, we give a careful description for the see-saw identities involving the quaternionic Hermitian groups. In Section 3, the proof of Theorem 1.3 is given and the proof of Theorem 1.4 follows as a result. The identity (3-5) in Lemma 3.3 from the see-saw diagram is the key of the proof, which transfers the $\mathrm{SL}_1(D)$- distinction problems to the branching problems for the 1-dimensional torus. Finally, we give two tables for the multiplicities in one L-packet Π_{ϕ_τ} when τ is $\mathrm{SL}_1(D)$-distinguished.

2. The local theta correspondences

In this section, we will briefly recall some results about the local theta correspondence, following [Mœglin et al. 1987].

Let F be a local field of characteristic zero. Consider the dual pair $\mathrm{O}(m)\times\mathrm{Sp}(2n)$. For simplicity, we may assume that m is even. Fix a nontrivial additive character ψ of F. Let ω_ψ be the Weil representation for $\mathrm{O}(m)\times\mathrm{Sp}(2n)$. If π is an irreducible representation of $\mathrm{O}(m)$ (resp. $\mathrm{Sp}(2n)$), the maximal π-isotypic quotient of ω_ψ has the form

$$\pi \boxtimes \Theta_\psi(\pi)$$

for some smooth representation $\Theta_\psi(\pi)$ of $\mathrm{Sp}(2n)$ (resp. $\mathrm{O}(m)$). We call $\Theta_\psi(\pi)$ the big theta lift of π. It is known that $\Theta_\psi(\pi)$ is of finite length and hence is admissible. Let $\theta_\psi(\pi)$ be the maximal semisimple quotient of $\Theta_\psi(\pi)$, which is called the small theta lift of π. It was conjectured by Roger Howe that

- $\theta_\psi(\pi)$ is irreducible whenever $\Theta_\psi(\pi)$ is nonzero;
- the map $\pi \mapsto \theta_\psi(\pi)$ is injective on its domain.

This has been proved by Waldspurger [1990] when the residual characteristic p of F is not 2. Gan and Takeda [2016a; 2016b] have proved it completely.

Theorem 2.1. *The Howe duality conjecture* (stated on the previous page) *holds.*

Gan and Sun [2017] proved the Howe duality conjecture for the quaternionic unitary groups. More precisely, let D be the unique 4-dimensional quaternion division algebra over F with involution $*$. Let V_D be an n-dimensional Hermitian right D-vector space with quaternionic Hermitian group $U(V_D)$ and Hermitian form $(-,-)_{V_D}$. Let W_D be an m-dimensional skew-Hermitian left D-vector space with quaternionic Hermitian group $U(W_D)$ and skew-Hermitian form $(-,-)_{W_D}$. The tensor product space $V_D \otimes W_D$ admits a symplectic form defined by

$$\begin{aligned}&\langle v \otimes w, v' \otimes w' \rangle \\ &\qquad = \tfrac{1}{2}\big((v, v')_{V_D}(w, w')^*_{W_D} + (w, w')_{W_D}(v, v')^*_{V_D}\big), \quad v, v' \in V_D,\ w, w' \in W_D.\end{aligned}$$

There is an embedding of F-groups

$$U(V_D) \times U(W_D) \longrightarrow \mathrm{Sp}(V_D \otimes W_D) = \mathrm{Sp}_{4mn}(F).$$

We may define the Weil representation ω_ψ on $U(V_D) \times U(W_D)$ similarly. Given an irreducible representation π of $U(V_D)$ (resp. $U(W_D)$), the maximal π-isotypic quotient of ω_ψ has the form $\pi \boxtimes \Theta_\psi(\pi)$ for some smooth representation $\Theta_\psi(\pi)$ of $U(W_D)$ (resp. $U(V_D)$), where $\Theta_\psi(\pi)$ is called the big theta lift and it has an irreducible quotient $\theta_\psi(\pi)$. The map $\pi \mapsto \theta_\psi(\pi)$ is injective on its domain.

2A. ***First occurrence indices for pairs of orthogonal Witt towers.*** Let W_n be the $2n$-dimensional symplectic vector space with associated symplectic group $\mathrm{Sp}(W_n)$ and consider the two towers of orthogonal groups attached to the quadratic spaces with nontrivial discriminant. More precisely, let V_E (resp. ϵV_E) be the 2-dimensional quadratic space with discriminant E and Hasse invariant $+1$ (resp. -1), $\mathbb{H}$ be the 2-dimensional hyperbolic quadratic space over F,

$$V_r^+ = V_E \oplus \mathbb{H}^{r-1} \quad (\text{resp. } V_r^- = \epsilon V_E \oplus \mathbb{H}^{r-1}),$$

and denote the orthogonal groups by $\mathrm{O}(V_r^+)$ (resp. $\mathrm{O}(V_r^-)$). For an irreducible representation π of $\mathrm{Sp}(W_n)$, we may consider the theta lifts $\theta_r^+(\pi)$ and $\theta_r^-(\pi)$ to $\mathrm{O}(V_r^+)$ and $\mathrm{O}(V_r^-)$ respectively, with respect to a fixed nontrivial additive character ψ. Set

$$\begin{cases} r^+(\pi) = \inf\{2r : \theta_r^+(\pi) \neq 0\}, \\ r^-(\pi) = \inf\{2r : \theta_r^-(\pi) \neq 0\}. \end{cases}$$

Then Kudla and Rallis [2005], B. Sun and C. Zhu [2015] showed the following theorem.

Theorem 2.2 (conservation relation). *For any irreducible representation π of* $\mathrm{Sp}(W_n)$, *we have*

$$r^+(\pi)+r^-(\pi)=4n+4=4+2\dim W_n.$$

There is an analogous problem where we fix an irreducible representation of $\mathrm{O}(V_r^+)$ or $\mathrm{O}(V_r^-)$ and consider its theta lifts $\theta_n(\pi)$ to the tower of symplectic groups $\mathrm{Sp}(W_n)$. Then with $n(\pi)$ defined in the analogous fashion, thanks to [Sun and Zhu 2015, Theorem 1.10], we have

$$n(\pi)+n(\pi\otimes\det)=\dim V_r^{\pm}.$$

2B. ***See-saw identities.*** Let V_D be a Hermitian right D-vector space; then $V_D\otimes_D R$ forms a right Hermitian vector space over $R=M_2(E)$ which corresponds to a symplectic E-vector space $\mathfrak{W}_E$ by the Morita equivalence. Let $\mathfrak{V}_E$ be an orthogonal E-vector space corresponding to a skew-Hermitian left free R-module (W_R, h_R). Let $W_D=\mathrm{Res}_{R/D}W_R$ be the same space W_R but now thought of as a D-vector space with skew-Hermitian form $(-,-)_{W_D}$ given by

$$(w_1, w_2)_{W_D}=\mathrm{tr}_{R/D}(h_R(w_1, w_2))/2 \quad \text{for } w_1, w_2\in W_R.$$

Then we have the following isomorphism of symplectic spaces:

$$W_D\otimes_D V_D\cong \mathrm{Res}_{E/F}(\mathfrak{V}_E\otimes_E\mathfrak{W}_E)=\mathbf{W}$$

There is a pair

$$(\mathrm{O}(\mathfrak{V}_E), \mathrm{Sp}(\mathfrak{W}_E)) \text{ and } (U(W_D), U(V_D))$$

of dual reductive pairs in the symplectic group $\mathrm{Sp}(\mathbf{W})$:

$$\begin{tikzcd} \mathrm{Sp}(\mathfrak{W}_E) \arrow[drr, dash] & & U(W_D) \arrow[dll, dash] \\ U(V_D) & & \mathrm{O}(\mathfrak{V}_E) \end{tikzcd}$$

A pair (G, H) and (G', H') of dual reductive pairs in a symplectic group is called a see-saw pair if $H\subset G'$ and $H'\subset G$. Following [Kudla 1994], let us fix the natural splittings

$$i_1:\mathrm{O}(\mathfrak{V}_E)\times\mathrm{Sp}(\mathfrak{W}_E)\longrightarrow \mathrm{Mp}(\mathbf{W}) \quad\text{and}\quad i_2:U(W_D)\times U(V_D)\longrightarrow\mathrm{Mp}(\mathbf{W}).$$

Lemma 2.3 (see-saw identity). *For some see-saw pair of dual reductive pairs* $(\mathrm{Sp}(\mathfrak{W}_E), \mathrm{O}(\mathfrak{V}_E))$ *and* $(U(W_D), U(V_D))$, *let π and π' be irreducible representations of* $\mathrm{O}(\mathfrak{V}_E)$ *and* $U(V_D)$ *respectively. If the splittings i_1 and i_2 satisfy*

(2-1) $$i_1|_{\mathrm{O}(\mathfrak{V}_E)\times U(V_D)}=i_2|_{\mathrm{O}(\mathfrak{V}_E)\times U(V_D)},$$

then we have the isomorphism

$$\mathrm{Hom}_{\mathrm{O}(\mathfrak{V}_E)}(\Theta_\psi(\pi'), \pi) \cong \mathrm{Hom}_{U(V_D)}(\Theta_\psi(\pi), \pi').$$

This follows from [Gurevich and Szpruch 2015, Theorem 8.2]. However, (2-1) may not hold, see [Lu 2016, Lemma 4.3.7]. For our purpose, suppose $\dim_D V_D = 1$ and $\dim_E \mathfrak{V}_E = 2$; then $U(V_D) \cong \mathrm{SL}_1(D)$. Let $\widetilde{\mathrm{O}(\mathfrak{V}_E)}$ denote the preimage of $\mathrm{O}(\mathfrak{V}_E)$ in $\mathrm{Mp}(\mathbf{W})$. Let $\widetilde{\pi}$ be a genuine representation of $\widetilde{\mathrm{O}(\mathfrak{V}_E)}$ associated to π, that is,

$$\widetilde{\pi}(h, \epsilon) = \epsilon \cdot \pi(h) \text{ for } (h, \epsilon) \in \widetilde{\mathrm{O}(\mathfrak{V}_E)}.$$

Observe that $i_1(h) = (h, 1) \in \widetilde{\mathrm{O}(\mathfrak{V}_E)}$ and $i_2(h) = (h, \det(h)) \in \widetilde{\mathrm{O}(\mathfrak{V}_E)}$ for $h \in \mathrm{O}(\mathfrak{V}_E)$. This means that $(i_1^{-1} i_2)|_{\mathrm{O}(\mathfrak{V}_E)}$ corresponds to the quadratic character $\det(\mathrm{O}(\mathfrak{V}_E))$. Hence

$$\mathrm{Hom}_{i_1(\mathrm{O}(\mathfrak{V}_E))}(\omega_\psi, \widetilde{\pi}) \cong \mathrm{Hom}_{i_2(\mathrm{O}(\mathfrak{V}_E))}(\omega_\psi, \widetilde{\pi \otimes \det}).$$

This will be useful in the proof of Theorem 1.3, see Lemma 3.3.

2C. ***Vector spaces.*** Let K/E be a quadratic field extension. Consider K as a 2-dimensional quadratic space $\mathfrak{V}_E$ over E with the norm map $N_{K/E}$. Given a 2-dimensional quadratic E-vector space $\mathfrak{V}_E$ with a nontrivial discriminant $e \in E^\times \setminus E^{\times 2}$, associated with a skew-Hermitian left free R-module (W_R, h_R) by the Morita equivalence, we may construct a skew-Hermitian form $h_D = \mathrm{tr}_{R/D} \circ h_R/2$ on $W_D = \mathrm{Res}_{R/D} W_R$. Then W_D becomes a 2-dimensional skew-Hermitian left D-vector space with discriminant $N_{E/F}(e) \in F^\times/F^{\times 2}$. If $N_{E/F}(e) = 1$, then the skew-Hermitian quaternionic group $U(W_D)$ is denoted by $U_{1,1}(D)$. If $N_{E/F}(e)$ is nontrivial, then the discriminant d of W_D corresponds to a quadratic field extension $L = F(\sqrt{d})$. Moreover, there is a 4-dimensional quaternion division algebra over L such that

$$U(W_D) = \mathrm{GL}_1(D_L)^\natural / F^\times$$

where $\mathrm{GL}_1(D_L)^\natural = \{x \in D_L^\times : N_{D_L/L}(x) \in F^\times\}$, see [Prasad and Takloo-Bighash 2011, §9].

2D. ***Degenerate principal series representations.*** Assume that $U(W_D) = U_{1,1}(D)$. There is a natural group embedding $\mathrm{O}(\mathfrak{V}_E) \hookrightarrow U_{1,1}(D)$. Let P be a Siegel parabolic subgroup of $U_{1,1}(D)$. Assume that

$$\mathcal{I}(s) = \{f : U_{1,1}(D) \to \mathbb{C} \mid f(pg) = |\delta_P(p)|^{s+1/2} f(g),\ p \in P,\ g \in U_{1,1}(D)\}$$

is the degenerate principal series of $U_{1,1}(D)$, where δ_P is the modular character. Let us consider the double coset decomposition $P \backslash U_{1,1}(D)/\mathrm{O}(\mathfrak{V}_E)$.

- If K is a field, then there is only one orbit in $P \backslash U_{1,1}(D)/\mathrm{O}(\mathfrak{V}_E)$.
- If $K = E \oplus E$, then there is one open and one closed orbit in $P \backslash U_{1,1}(D)/\mathrm{O}(\mathfrak{V}_E)$.

Assume that there is a stratification for $U_{1,1}(D)$, i.e., $P\backslash U_{1,1}(D)/\mathrm{O}(\mathfrak{V}_E)=\bigsqcup_{i=0}^{r} X_i$ such that $\bigsqcup_{i=0}^{k} X_i$ is open for each k lying in $\{0, 1, 2, \dots, r\}$. Then there is an $\mathrm{O}(\mathfrak{V}_E)$-equivariant filtration $\{I_i\}_{i=0,1,2,\dots,r}$ of $\mathcal{I}(s)|_{\mathrm{O}(\mathfrak{V}_E)}$ such that

$$0 = I_{-1} \subset I_0 \subset I_1 \subset \cdots \subset I_r = \mathcal{I}(s)|_{\mathrm{O}(\mathfrak{V}_E)}$$

and the smooth functions in the quotient I_i/I_{i-1} are supported on a single orbit X_i in $P\backslash U_{1,1}(D)/\mathrm{O}(\mathfrak{V}_E)$.

Definition 2.4. Given an irreducible representation π of $\mathrm{O}(\mathfrak{V}_E)$, if

$$\mathrm{Hom}_{\mathrm{O}(\mathfrak{V}_E)}(I_{i+1}/I_i, \pi) \neq 0$$

implies that I_{i+1}/I_i is supported on the open orbits in $P\backslash U_{1,1}(D)/\mathrm{O}(\mathfrak{V}_E)$, then we say that the representation π does not occur on the boundary of $\mathcal{I}(s)$.

It is well known that only the open orbits can support supercuspidal representations.

3. Proof of Theorem 1.3

Before we prove Theorem 1.3, let us recall some facts. Let V_D denote the rank 1 Hermitian space over D with quaternionic Hermitian group $U(V_D) = \mathrm{SL}_1(D)$.

Lemma 3.1. *If the discriminant d of $W_D = \mathfrak{V}_E \otimes_E D$ is nontrivial in $F^\times/F^{\times 2}$, let $L = F(\sqrt{d})$, then the theta lift of the trivial representation from $\mathrm{SL}_1(D)$ to $U(W_D) = \mathrm{GL}_1(D_L)^\natural/F^\times$ is a character, i.e.,*

$$\Theta_\psi(\mathbf{1}) = \mathbf{1} \boxtimes \omega_{L/F},$$

where D_L is a quaternion division algebra over L and $\mathrm{GL}_1(D_L)^\natural = \{g \in D_L^\times \mid N_{D_L/L}(g) \in F^\times\}$.

Proof. Following [Gan and Tantono 2014, Proposition 5.1], let $\mathbb{L}/\mathbb{F}$ be a quadratic extension of number fields and $\mathbb{D}$ (resp. $\mathbb{D}_\mathbb{L}$) be a quaternion $\mathbb{F}$-algebra (resp. $\mathbb{L}$-algebra) with involution $*$ such that for some place v_0 of $\mathbb{F}$, we have

$$(\mathbb{L}/\mathbb{F})_{v_0} = L/F \quad \text{and} \quad \mathbb{D}_{v_0} = D \text{ (resp. } (\mathbb{D}_\mathbb{L})_{v_0} = D_L).$$

Let $\mathbb{V}$ denote the rank 1 Hermitian space over $\mathbb{D}$ with hermitian form

$$\langle x, y\rangle = x \cdot y^*$$

and let $\mathbb{W}$ denote the nonsplit rank 2 skew-Hermitian space over $\mathbb{D}$ of discriminant $\mathbb{L}$, such that

$$\mathbb{V}_{v_0} = V_D \text{ and } \mathbb{W}_{v_0} = W_D.$$

Then we have a dual pair $U(\mathbb{V}) \times U(\mathbb{W})$ over $\mathbb{F}$ and we may consider the global theta lift from

$$U(\mathbb{V}) = \mathrm{SL}_1(\mathbb{D})$$

to

$$U(\mathbb{W})^\circ = \mathrm{GL}_1(\mathbb{D}_{\mathbb{L}})^\natural / \mathbb{F}^\times,$$

where $\mathrm{GL}_1(\mathbb{D}_{\mathbb{L}})^\natural = \{g \in \mathbb{D}_{\mathbb{L}}^\times : N_{\mathbb{D}_{\mathbb{L}}/\mathbb{L}}(g) \in \mathbb{F}^\times\}$ and $U(\mathbb{W})^\circ$ is the connected component of $U(\mathbb{W})$ containing the identity. The global theta lift to $U(\mathbb{W})^\circ$ of trivial representation of $\mathrm{SL}_1(\mathbb{D})$ is nonzero since we are in the stable range. Moreover, at the places where $\mathbb{D}$ is unramified, [Lu 2018, Lemma 3.1] implies that the local theta lift of the trivial representation is a character of $U(\mathbb{W}_v)$. By the strong multiplicity one theorem for $\mathrm{GL}_1(\mathbb{D}_{\mathbb{L}})$, we conclude that

$$\Theta(\mathbf{1}) = \mathbf{1} \boxtimes \omega_{\mathbb{L}/\mathbb{F}}.$$

By the local-global compatibility of theta correspondence, we have $\theta_\psi(\mathbf{1}) = \mathbf{1} \boxtimes \omega_{L/F}$. Because $U(W_D)$ is a compact group, the Howe duality theorem implies that

$$\Theta_\psi(\mathbf{1}) = \theta_\psi(\mathbf{1}) = \mathbf{1} \boxtimes \omega_{L/F}. \qquad \square$$

Now we start to prove Theorem 1.3.

Proof of Theorem 1.3. We separate the proof into four cases as follows:

- τ is a supercuspidal representation; see (A).
- τ is an irreducible principal series representation; see (B).
- τ is a Steinberg representation St_E; see (C).
- τ is a constituent of a reducible principle series $I(\chi)$ with $\chi^2 = 1$; see (D).

(A) If τ is supercuspidal, then there exists a character $\mu : K^\times \to \mathbb{C}^\times$ such that $\phi_\tau = \omega_{K/E} \oplus \mathrm{Ind}_{W_K}^{W_E}(\mu^s/\mu)$, where W_K is the Weil group of K, which is a quadratic field extension over E with associated quadratic character $\omega_{K/E}$. In fact, if $\tau = \theta_\psi(\Sigma)$, where Σ is a representation of $\mathrm{O}(\mathfrak{V}_E)$ and $\mathfrak{V}_E$ is a 2-dimensional E-vector space of discriminant K, then the Langlands parameter ϕ of Σ is given by

$$\phi(g) = \begin{cases} \begin{pmatrix} \chi_K(g) & \\ & \chi_K^{-1}(g) \end{pmatrix} & \text{if } g \in W_K, \\[2ex] \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } g = s, \end{cases}$$

where $s \in W_E \setminus W_K$ and the character $\chi_K : W_K \to \mathbb{C}^\times$ is the pull back of a nontrivial character μ_1 of K^1 under the map $K^\times \to K^1$ via $k \mapsto k^s k^{-1}$, i.e.,

$\chi_K(k) = \mu_1(k^s k^{-1})$, see [Kudla 1996, §6.4]. Furthermore, there is an isomorphism between two Langlands parameters of $\mathrm{O}(2)$,

$$\phi \otimes \omega_{K/E} \cong \mathrm{Ind}_{W_K}^{W_E}(\mu^s/\mu).$$

In other words, we have $\chi_K = \mu^s/\mu$ and $\mu_1 = \mu|_{K^1}$ is the restricted character.

Moreover, if $\mu_1^2 \neq \mathbf{1}$, then $\tau = \theta_\psi(\mathrm{Ind}_{\mathrm{SO}(\mathfrak{V}_E)}^{\mathrm{O}(\mathfrak{V}_E)}(\mu_1))$. If $\mu_1^2 = \mathbf{1}$, then there are two extensions of μ_1 from $\mathrm{SO}(\mathfrak{V}_E)$ to $\mathrm{O}(\mathfrak{V}_E)$, denoted by $\mu_1^{\pm}$. The theta lift of μ_1^+ (resp. μ_1^-) from $\mathrm{O}(\mathfrak{V}_E)$ to $\mathrm{SL}_2(E)$ is a tempered representation τ^+ (resp. τ^-). For convenience, if $\mu_1^2 \neq \mathbf{1}$, we use $\mu^+ = \mu^-$ to denote $\mathrm{Ind}_{\mathrm{SO}(\mathfrak{V}_E)}^{\mathrm{O}(\mathfrak{V}_E)}\mu_1$ as well. Assume that $\Theta_\psi(\mu_1^+)$ is a supercuspidal representation of $\mathrm{SL}_2(E)$.

If the discriminant $\mathrm{disc}(\mathrm{Res}_{R/D} W_R) \in F^\times/(F^\times)^2$ is nontrivial, by the see-saw diagram

$$\begin{tikzcd}[column sep=small]
\tau^+ \oplus \tau^- & \mathrm{SL}_2(E) \arrow[dr, dash] & U(W_D) \arrow[dl, dash] & \Theta_\psi(\mathbf{1}) \\
\mathbf{1} & \mathrm{SL}_1(D) & \mathrm{SO}(\mathfrak{V}_E) & \mu_1
\end{tikzcd}$$

where $\tau^- = 0$ if $\mu_1^2 \neq \mathbf{1}$, we have an isomorphism

$$\mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau^+ \oplus \tau^-, \mathbb{C}) \cong \mathrm{Hom}_{\mathrm{SO}(\mathfrak{V}_E)}(\mathbf{1}, \mu_1)$$

which is nonzero if and only if $\mu_1 = \mathbf{1}$. However, $\mathrm{Hom}_{K^1}(\mathbf{1}, \mu_1) = 0$, therefore $\mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau^\pm, \mathbb{C}) = 0$.

If the discriminant of $W_D = \mathrm{Res}_{R/D} W_R$ is $1 \in F^\times/(F^\times)^2$, we denote by $\mathcal{I}(s)$ the degenerate principal series of $U_{1,1}(D)$. Further, we assume that $F^\times/(F^\times)^2 \supset \{1, u, \varpi, u\varpi\}$ and $E = F[\sqrt{\varpi}\,]$ with associated Galois group $\mathrm{Gal}(E/F) = \langle \sigma \rangle$, where $K = E\big[\sqrt{u}\big]$. Then (3-5) (which will be proved later) implies

$$\mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau^+, \mathbb{C}) = \mathrm{Hom}_{\mathrm{O}(\mathfrak{V}_E)}\big(\mathcal{I}\big(\tfrac{1}{2}\big), \mu_1^-\big) \cong \mathrm{Hom}_{U(W')}((\mu_1^-)^{-1}, \mathbb{C}), \tag{3-1}$$

where K is a quadratic unramified extension over E, W' is a one-dimensional skew-Hermitian left D-vector space with discriminant u. Here we use the fact that there is only one orbit for the double coset $P \backslash U_{1,1}(D)/\mathrm{O}(\mathfrak{V}_E)$, whose stabilizer is isomorphic to $U(W')$. In this case, (3-1) can be rewritten as the identity

$$\dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau^+, \mathbb{C}) = \dim \mathrm{Hom}_{U(W')}(\mu_1^-, \mathbb{C}), \tag{3-2}$$

which is nonzero if and only if

$$\mu\big(x - y\sqrt{u}\,\big) = \mu\big(x + y\sqrt{u}\,\big) \tag{3-3}$$

for $x, y \in F$. Similarly, if $\mu_1^2 = \mathbf{1}$, we have

$$\dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau^-, \mathbb{C}) = \dim \mathrm{Hom}_{U(W')}(\mu_1^+, \mathbb{C}).$$

Remark 3.2. If the Hasse-invariant of $\mathfrak{V}_E$ is -1 and the discriminant of $\mathfrak{V}_E$ is K, then

$$\dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau^+, \mathbb{C}) \neq 0$$

if and only if

$$\mu(x - y\sqrt{u\varpi}\,) = \mu(x + y\sqrt{u\varpi}\,) \tag{3-4}$$

for $x, y \in F$. If both (3-3) and (3-4) hold, then $\mu^s/\mu = \chi_F \circ N_{K/F}$ with $\chi_F^2 = \mathbf{1}$. Moreover, if p is odd, then μ^s/μ is trivial. Because $\mu^s \neq \mu$, (3-3) and (3-4) can not hold at the same time unless $p = 2$.

Lemma 3.3. *Let $\mathfrak{V}_E$ be a 2-dimensional quadratic E-vector space associated with a skew-Hermitian free R-module W_R by the Morita equivalence. Assume that $W_D = \mathrm{Res}_{R/D} W_R$ is a 2-dimensional skew-Hermitian left D-vector space with trivial discriminant and π is an irreducible representation of $\mathrm{O}(\mathfrak{V}_E)$, then*

$$\dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\Theta_\psi(\pi \otimes \det), \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{O}(\mathfrak{V}_E)}(\mathcal{I}(1/2), \pi), \tag{3-5}$$

where $\mathcal{I}(s)$ is the degenerate principal series of $U(W_D)$, and the big theta lift $\Theta_\psi(\pi \otimes \det)$ is under the splitting $i_1 : \mathrm{SL}_2(E) \times \mathrm{O}(\mathfrak{V}_E) \to \mathrm{Mp}_8(F)$.

Proof. Let us fix the splitting $i_2 : \mathrm{SL}_1(D) \times U(W_D) \to \mathrm{Mp}(\mathbf{W})$; then [Yamana 2011, Theorem 1.3] implies that $\Theta_\psi(\mathbf{1}) = \mathcal{I}(1/2)$ is an irreducible representation of $U(W_D)$. The splitting from $\mathrm{SL}_2(E)$ to $\mathrm{Mp}_8(F)$ is unique, so $i_1 i_2^{-1}$ is a quadratic character on $\mathrm{SL}_1(D) \times \mathrm{O}(\mathfrak{V}_E)$ and trivial on $\mathrm{SL}_1(D)$. Thus,

$$\begin{aligned} \dim \mathrm{Hom}_{\mathrm{SL}_1(D)}&(\Theta_\psi(\pi \otimes \det), \mathbb{C}) \\ &= \dim \mathrm{Hom}_{i_1(\mathrm{SL}_1(D)\times \mathrm{O}(\mathfrak{V}_E))}(\omega_\psi, \widetilde{\mathbb{C}} \otimes \widetilde{(\pi \otimes \det)}) \\ &= \dim \mathrm{Hom}_{i_2(\mathrm{SL}_1(D)\times \mathrm{O}(\mathfrak{V}_E))}(\omega_\psi, \widetilde{\mathbb{C}} \otimes \widetilde{\pi}) \\ &= \dim \mathrm{Hom}_{\mathrm{O}(\mathfrak{V}_E)}(\Theta_\psi(\mathbf{1}), \pi) \\ &= \dim \mathrm{Hom}_{\mathrm{O}(\mathfrak{V}_E)}(\mathcal{I}(1/2), \pi), \end{aligned} \tag{3-6}$$

where $\widetilde{\pi}(h, \epsilon) = \epsilon \cdot \pi(h)$ for $(h, \epsilon) \in \widetilde{\mathrm{O}(W)}$. □

Now we continue with the proof of Theorem 1.3, recalling that

$$\dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau^+, \mathbb{C}) = \dim \mathrm{Hom}_{U(W')}(\mu_1^-, \mathbb{C})$$

is nonzero if and only if $\mu(x - y\sqrt{u}\,) = \mu(x + y\sqrt{u}\,)$ for $x, y \in F$, where $\mathrm{disc}(W')$ is $u \in F^\times/F^{\times 2}$.

Suppose that $p \neq 2$ and $\mu_1^2 = \mathbf{1}$. If $\mu(x - u\sqrt{u}\,) = \mu(x + y\sqrt{u}\,)$, then τ^+ is $\mathrm{SL}_1(D)$-distinguished. Moreover,

$$\dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau^+, \mathbb{C}) = 1 = \dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau^-, \mathbb{C}).$$

In the L-packet containing an $\mathrm{SL}_1(D)$-distinguished representation τ, half members in Π_{ϕ_τ} are $\mathrm{SL}_1(D)$-distinguished and

$$\sum_{\tau'\in\Pi_{\phi_\tau}} \dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau', \mathbb{C}) = 2.$$

If $p \neq 2$ and $\mu_1^2 \neq \mathbf{1}$, then $\dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{O}(\mathfrak{V}_E)}(\mathcal{I}(1/2), \mu_1^+)$, which is equal to the sum

$$\dim \mathrm{Hom}_{U(W')}(\mu_1, \mathbb{C}) + \dim \mathrm{Hom}_{U(W')}(\mu_1^{-1}, \mathbb{C}) = \begin{cases} 2 & \text{if } \mu|_{E'} = \chi_F \circ N_{E'/F},\ E' \neq E, \\ 0 & \text{otherwise.} \end{cases}$$

If $p = 2$, there are two more cases.

(i) Suppose that there are two distinct quadratic fields E' and E'' over F such that $\mu|_{E'} = \chi'_F \circ N_{E'/F}$ and $\mu|_{E''} = \chi''_F \circ N_{E''/F}$. Furthermore, χ'_F/χ''_F is a quadratic character of $F^\times$ that is not trivial restricted on the Weil group W_K of K, i.e., χ'_F/χ''_F is different from three quadratic characters $\omega_{E/F}$, $\omega_{E'/F}$ and $\omega_{E''/F}$,

$$\mu(t) = \mu^s(t) \cdot (\chi'_F/\chi''_F)|_{W_K}(t), \quad t \in W_K$$

which may happen only when $p = 2$. We obtain $\dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau^+, \mathbb{C}) = 1$ by the identity (3-2).

Suppose that τ is $\mathrm{SL}_1(D)$-distinguished, then the set $\{\dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau', \mathbb{C}) : \tau' \in \Pi_{\phi_{\tau^+}}\}$ is $\{1, 1, 1, 1\}$ and

$$\sum_{\tau'\in\Pi_{\phi_\tau}} \dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau', \mathbb{C}) = 4.$$

Remark 3.4. For the $\mathrm{SL}_2(F)$-distinction problem, the set of the multiplicities in the L-packet Π_{ϕ_τ} is $\{4, 0, 0, 0\}$, see [Anandavardhanan and Prasad 2003; Lu 2018].

(ii) A supercuspidal representation π of $\mathrm{GL}_2(E)$, which is not dihedral with respect to any quadratic extension K over E, is irreducible when restricted to $\mathrm{SL}_2(E)$. Suppose that $\tau = \pi|_{\mathrm{SL}_2(E)}$ is irreducible. If we consider a 2-dimensional skew-Hermitian left D-vector space X with trivial discriminant, then $U(X) = U_{1,1}(D)$ can be naturally embedded into the special orthogonal group $\mathrm{SO}(2, 2)(E)$. Let $\pi \boxtimes \pi$ be the irreducible representation of the similitude special orthogonal group

$$\mathrm{GSO}(2, 2)(E) = \frac{\mathrm{GL}_2(E) \times \mathrm{GL}_2(E)}{\{(t, t^{-1}) : t \in E^\times\}}.$$

Observe that

$$(\pi \boxtimes \pi)|_{\mathrm{SO}(2,2)(E)} = \Theta(\pi)|_{\mathrm{SO}(2,2)(E)} = \Theta(\pi|_{\mathrm{SL}_2(E)}) = \Theta(\tau)$$

is irreducible since τ is supercuspidal. Suppose that Y is a 2-dimensional Hermitian right D-vector space. Let $\mathfrak{I}(s)$ be the degenerate principal series of $U(Y)$. Considering the see-saw diagram

$$\begin{tikzcd} \mathfrak{I}(1/2) & U(Y) \arrow[dr, no head] & \mathrm{SO}(2,2)(E) \arrow[dl, no head] & (\pi \boxtimes \pi) \\ \pi|_{\mathrm{SL}_2(E)} & \mathrm{SL}_2(E) & U_{1,1}(D) & \mathbb{C} \end{tikzcd}$$

due to the structure of $\mathfrak{I}(1/2)$ in [Yamana 2011, Theorem 1.4], we can get an equality

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(E)}(\mathfrak{I}(1/2), \pi) = \dim \mathrm{Hom}_{U_{1,1}(D)}((\pi \boxtimes \pi)|_{\mathrm{SO}(2,2)(E)}, \mathbb{C}).$$

The supercuspidal representation $\pi|_{\mathrm{SL}_2(E)}$ does not occur on the boundary of $\mathfrak{I}(1/2)$, thus

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(E)}(\mathfrak{I}(1/2), \pi) = \dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\pi^\vee, \mathbb{C}).$$

Hence

$$\begin{aligned} \text{(3-7)} \quad \dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\pi^\vee, \mathbb{C}) &= \dim \mathrm{Hom}_{U_{1,1}(D)}((\pi \boxtimes \pi)|_{\mathrm{SO}(2,2)(E)}, \mathbb{C}) \\ &= \dim \mathrm{Hom}_{GU_{1,1}(D)}(\pi \boxtimes \pi, \mathbb{C}) + \dim \mathrm{Hom}_{GU_{1,1}(D)}(\pi \boxtimes \pi, \omega_{E/F}) \\ &= \dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi, \mathbb{C}) \dim \mathrm{Hom}_{D^\times}(\pi, \mathbb{C}) \\ &\qquad + \dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi, \omega_{E/F}) \dim \mathrm{Hom}_{D^\times}(\pi, \omega_{E/F}), \end{aligned}$$

where

$$GU_{1,1}(D) \cong \frac{\mathrm{GL}_2(F) \times D^\times}{\{(t, t^{-1}) : t \in F^\times\}}.$$

Therefore, if π is not dihedral with respect to any quadratic field extension K over E and so $\tau = \pi|_{\mathrm{SL}_2(E)}$ is irreducible, then the following are equivalent:

- The Langlands parameter ϕ_π is conjugate-self-dual in the sense of [Gan et al. 2012, §3].
- $\dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau, \mathbb{C}) = 1$.

Remark 3.5. This method can be used to deal with the case when τ is the Steinberg representation St_E of $\mathrm{SL}_2(E)$, which implies $\dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\mathrm{St}_E, \mathbb{C}) = 1$ directly.

Remark 3.6. When we consider the distinction problem for the symmetric pair $\mathrm{SL}_2(E)/\mathrm{SL}_2(F)$ in [Lu 2018], instead of $U_{1,1}(D)$, we use $\mathrm{SO}_{3,1}(F) = \mathrm{GL}_2(E)^\natural / F^\times$, where

$$\mathrm{GL}_2(E)^\natural = \{g \in \mathrm{GL}_2(E) \mid \det(g) \in F^\times\} \cong \mathrm{GSpin}_{3,1}(F).$$

We use the big theta lift of the trivial representation from $\mathrm{SO}_{3,1}(F)$ to $\mathrm{Sp}_4(F)$ to deduce the multiplicity $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\pi, \mathbb{C})$. The see-saw identity implies

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(E)}(I(1/2, \omega_{E/F}), \pi) = \dim \mathrm{Hom}_{\mathrm{O}_{3,1}(F)}((\pi \boxtimes \pi)^+, \mathbb{C}), \tag{3-8}$$

where $P = MN$ is the Siegel parabolic subgroup of $\mathrm{Sp}_4(F)$ and $M \cong \mathrm{GL}_2(F)$.

$$I(1/2, \omega_{E/F}) = \bigl\{ f : \mathrm{Sp}_4(F) \longrightarrow \mathbb{C} \mid f(mng) = |\det(m)|^{s+3/2} \omega_{E/F}(\det m) f(g) \\ \text{for } mn \in P,\, g \in \mathrm{Sp}_4(F) \bigr\}$$

since the big theta lift of the trivial representation equals $I(1/2, \omega_{E/F})$, see [Gan and Ichino 2014, Proposition 7.2]. Due to the fact that the supercuspidal representation π does not occur on the boundary, (3-8) implies that

$$\begin{aligned} \dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\pi^\vee, \mathbb{C}) &= \dim \mathrm{Hom}_{\mathrm{SL}_2(E)}(I(1/2, \omega_{E/F}), \pi) \\ &= \dim \mathrm{Hom}_{\mathrm{O}_{3,1}}((\pi \boxtimes \pi)^+, \mathbb{C}) \\ &= \dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi^\sigma, \pi^\vee). \end{aligned} \tag{3-9}$$

(B) Let χ be a unitary character of $E^\times$. Since there is only one orbit for $D^\times$-action on the projective variety $P(E)\backslash \mathrm{GL}_2(E) \cong B(E)\backslash \mathrm{SL}_2(E)$, where $P(E)$ is the Borel subgroup of $\mathrm{GL}_2(E)$, its stabilizer is isomorphic to $E^\times$ and $B(E)\backslash \mathrm{SL}_2(E) \cong E^\times \backslash D^\times$. There are two orbits for $\mathrm{SL}_1(D)$-action on $B(E)\backslash \mathrm{SL}_2(E)$. If $\tau = I(z, \chi) = \mathrm{Ind}_{B(E)}^{\mathrm{SL}_2(E)} \chi |-|_E^z$ (normalized induction) is an irreducible principal series, due to the double coset decomposition

$$\mathrm{SL}_2(E) = B(E)\mathrm{SL}_1(D) \sqcup B(E)\eta \mathrm{SL}_1(D),$$

where $\eta = \left(\begin{smallmatrix} z_1 & \bar{z}_2 \\ z_2 & \bar{z}_1/\epsilon \end{smallmatrix}\right)$, $d = z_1 + z_2 j$, $z_1, z_2 \in E$ and $N_{D/F}(d) = \epsilon \in F^\times \setminus N_{E/F}E^\times$, there is an exact sequence

$$0 \to \mathrm{Hom}_{E^1}(\chi, \mathbb{C}) \to \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau, \mathbb{C}) \to \mathrm{Hom}_{E^1}(\chi, \mathbb{C}) \to 0, \tag{3-10}$$

where $E^1 = \ker N_{E/F}$. Then $\dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau, \mathbb{C}) = 2$ if and only if $\chi = \chi_F \circ N_{E/F}$.

(C) If $\tau = \mathrm{St}_E$ is a Steinberg representation of $\mathrm{SL}_2(E)$, then the exact sequence (3-10) implies that

$$\dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(I(|-|_E), \mathbb{C}) = 2,$$

so that $\dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\mathrm{St}_E, \mathbb{C}) = 2 - 1 = 1$.

(D) Assume that τ is tempered. If $\tau \subset I(\omega_{K/E})$ is an irreducible constituent of a reducible principal series, set $\chi = \omega_{K/E}$, $\chi^+(\omega) = 1$, $\omega = \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$; then from [Kudla 1996, page 86], we can see that

$$I(\omega_{K/E}) = \theta_\psi(\chi^+) \oplus \theta_\psi(\chi^-), \quad \text{where } \chi^- = \chi^+ \otimes \det$$

and $\tau^+ = \theta_\psi(\chi^+) = \Theta_\psi(\chi^+)$, $\tau^- = \theta_\psi(\chi^-)$, where $\theta_\psi(\chi^\pm)$ is the theta lift of $\chi^\pm$ from $\mathrm{O}_{1,1}(E)$ to $\mathrm{SL}_2(E)$. By (3-5) and the see-saw diagram

$$\begin{tikzcd}[column sep=large]
\tau^+ & \mathrm{SL}_2(E) \arrow[dr, dash] & U_{1,1}(D) \arrow[dl, dash] & \mathcal{I}(1/2) \\
\mathbb{C} & \mathrm{SL}_1(D) & \mathrm{O}_{1,1}(E) & \chi^+ \otimes \det
\end{tikzcd}$$

where $\mathcal{I}(s)$ is the principal series of $U_{1,1}(D)$, we have an identity

$$\dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau^+, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{O}_{1,1}(E)}\big(\mathcal{I}(\tfrac{1}{2}), \chi^+ \otimes \det\big),$$

which is equal to

$$\dim \mathrm{Hom}_{E^1}(\chi, \mathbb{C}) = \begin{cases} 1 & \text{if } \chi = \chi_F \circ N_{E/F}, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we can prove $\dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau^+, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau^-, \mathbb{C})$.

This finishes the proof of Theorem 1.3. □

Corollary 3.7. *Let τ be an $\mathrm{SL}_1(D)$-distinguished ψ-generic representation of $\mathrm{SL}_2(E)$. If the representation τ' lies in the L-packet Π_{ϕ_τ} and is ψ_a-generic for some $a \in E^1$, then*

$$\dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau', \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau, \mathbb{C}).$$

Proof. Let U_2 be the non-quasi-split unitary group contained in $\mathrm{GL}_2(E)$. Thanks to the isomorphism

$$\mathrm{SU}_2 \cong \mathrm{SL}_1(D),$$

if τ' is ψ_a-generic for $a \in E^1$, then $\tau' = \tau^a$ where

$$\tau^a(g) = \tau\left(\begin{pmatrix} \bar{a} & \\ & 1 \end{pmatrix} g \begin{pmatrix} a & \\ & 1 \end{pmatrix}\right)$$

and so

$$\dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau', \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau^a, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau, \mathbb{C}).$$

Thus the multiplicity $\dim \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau, \mathbb{C})$ is stable under the inner-conjugation action of U_2. □

Remark 3.8. In [Anandavardhanan and Prasad 2003, Lemma 3.2], there is an analogous result

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau^x, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbb{C}) \quad \text{for } x \in F^\times,$$

which implies that $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbb{C})$ is stable under the inner-conjugation action of $\mathrm{GL}_2(F)$.

In fact, Theorem 1.4 follows from the above arguments as well.

The tables below show the multiplicities for the L-packet Π_{ϕ_τ} containing an $\mathrm{SL}_1(D)$-distinguished or $\mathrm{SL}_2(F)$-distinguished representation τ of $\mathrm{SL}_2(E)$.

	$\mathrm{SL}_1(D)$-distinguished	$\mathrm{SL}_2(F)$-distinguished	the character μ of $K^\times$, $K=E[\sqrt{u}]$
$\lvert\Pi_{\phi_\tau}\rvert=1$	$\{1\}$	$\{1\}$	N.A.
$\lvert\Pi_{\phi_\tau}\rvert=2$	$\{2,0\}$	$\{2,0\}$	$\mu(x-y\sqrt{u})=\mu(x+y\sqrt{u}),\mu_1^2\neq\mathbf{1}$
$\lvert\Pi_{\phi_\tau}\rvert=4$	$\{1,0,1,0\}$	$\{1,1,0,0\}$	$\mu^s=-\mu$ and $\mu_1^2=\mathbf{1}$
	$\{1,1,1,1\}$	$\{4,0,0,0\}$	$\mu\chi'_F/\chi''_F=\mu^s\neq\pm\mu$ and $\mu_1^2=\mathbf{1}$

Multiplicities for the L-packet Π_{ϕ_τ} assuming τ is square-integrable.
The case shown on the last row of the first table occurs only when $p=2$ and τ is a supercuspidal representation of $\mathrm{SL}_2(E)$.

	$\mathrm{SL}_1(D)$-distinguished	$\mathrm{SL}_2(F)$-distinguished	the character χ_E of $E^\times$
$\lvert\Pi_{\phi_\tau}\rvert=1$	$\{2\}$	$\{2\}$	$\chi_E=\mathbf{1}$
	$\{2\}$	$\{2\}$	$\chi_E=\chi_F\circ N_{E/F}$ and $\chi_E^2\neq\mathbf{1}$
	$\{0\}$	$\{1\}$	$\chi_E\vert_{F^\times}=\mathbf{1}$ and $\chi_E^2\neq\mathbf{1}$
$\lvert\Pi_{\phi_\tau}\rvert=2$	$\{1,1\}$	$\{1,1\}$	$\omega_{K/E}=\chi_F\circ N_{E/F}$ with $\chi_F^2=\omega_{E/F}$
	$\{1,1\}$	$\{3,0\}$	$\omega_{K/E}=\chi_F\circ N_{E/F}$ with $\chi_F^2=\mathbf{1}$

Multiplicities for the L-packet Π_{ϕ_τ} assuming τ is not square-integrable.

Remark 3.9. If $\tau=I(\chi_E)$ is an irreducible principle representation of $\mathrm{SL}_2(E)$, where χ_E is a unitary character of $E^\times$ with $\chi_E^2\neq\mathbf{1}$ and $\chi_E\vert_{F^\times}=\mathbf{1}$, then $I(\chi_E)$ is not $\mathrm{SL}_1(D)$-distinguished but $\mathrm{SL}_2(F)$-distinguished. It corresponds to the case where the representation

$$\pi=\pi(\chi,\chi\chi_E)$$

of $\mathrm{GL}_2(E)$ with $\chi\vert_{F^\times}=\mathbf{1}$, is not $\mathrm{GL}_1(D)$-distinguished but $\mathrm{GL}_2(F)$-distinguished.

Remark 3.10. Assume that $\tau\subset I(\omega_{K/E})$, where K is a quadratic field extension over E associated with a quadratic character $\omega_{K/E}$ by the local class field theory. If $\omega_{K/E}\vert_{F^\times}=\mathbf{1}$, then $\omega_{K/E}^\sigma=\omega_{K/E}$, and so $\omega_{K/E}$ must factor through the norm map $N_{E/F}$. The third case of D from [Lu 2018, Page 490] does not exist, i.e., the set $\{1,0\}$ does not appear in the above tables when τ is $\mathrm{SL}_2(F)$-distinguished.

Acknowledgments

The author thanks Wee Teck Gan for his guidance and numerous discussions when he was doing his Ph.D. study at the National University of Singapore. He would like to thank Dipendra Prasad for useful discussions as well, especially for his comments on the first version of this paper. He is also grateful to the anonymous referee for the careful reading of the manuscript and numerous suggestions which improved the exposition of this paper.

References

[Adler and Prasad 2006] J. D. Adler and D. Prasad, "On certain multiplicity one theorems", *Israel J. Math.* **153** (2006), 221–245. MR Zbl

[Anandavardhanan and Prasad 2003] U. K. Anandavardhanan and D. Prasad, "Distinguished representations for SL(2)", *Math. Res. Lett.* **10**:6 (2003), 867–878. MR Zbl

[Anandavardhanan and Prasad 2006] U. K. Anandavardhanan and D. Prasad, "On the SL(2) period integral", *Amer. J. Math.* **128**:6 (2006), 1429–1453. MR Zbl

[Anandavardhanan and Prasad 2013] U. K. Anandavardhanan and D. Prasad, "A local-global question in automorphic forms", *Compos. Math.* **149**:6 (2013), 959–995. MR Zbl

[Beuzart-Plessis 2018] R. Beuzart-Plessis, "On distinguished square-integrable representations for Galois pairs and a conjecture of Prasad", *Invent. Math.* **214**:1 (2018), 437–521. MR

[Flicker and Hakim 1994] Y. Z. Flicker and J. L. Hakim, "Quaternionic distinguished representations", *Amer. J. Math.* **116**:3 (1994), 683–736. MR Zbl

[Gan and Ichino 2014] W. T. Gan and A. Ichino, "Formal degrees and local theta correspondence", *Invent. Math.* **195**:3 (2014), 509–672. MR Zbl

[Gan and Sun 2017] W. T. Gan and B. Sun, "The Howe duality conjecture: quaternionic case", pp. 175–192 in *Representation theory, number theory, and invariant theory* (New Haven, CT, 2015), edited by J. Cogdell et al., Progr. Math. **323**, Birkhäuser, Cham, 2017. MR

[Gan and Takeda 2016a] W. T. Gan and S. Takeda, "On the Howe duality conjecture in classical theta correspondence", pp. 105–117 in *Advances in the theory of automorphic forms and their L-functions* (Vienna, 2013), edited by D. Jiang et al., Contemp. Math. **664**, Amer. Math. Soc., Providence, RI, 2016. MR Zbl

[Gan and Takeda 2016b] W. T. Gan and S. Takeda, "A proof of the Howe duality conjecture", *J. Amer. Math. Soc.* **29**:2 (2016), 473–493. MR Zbl

[Gan and Tantono 2014] W. T. Gan and W. Tantono, "The local Langlands conjecture for GSp(4), II: The case of inner forms", *Amer. J. Math.* **136**:3 (2014), 761–805. MR Zbl

[Gan et al. 2012] W. T. Gan, B. H. Gross, and D. Prasad, "Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups", pp. 1–109 in *Sur les conjectures de Gross et Prasad, I*, Astérisque **346**, 2012. MR Zbl arXiv

[Gurevich and Szpruch 2015] N. Gurevich and D. Szpruch, "The non-tempered θ_{10} Arthur parameter and Gross–Prasad conjectures", *J. Number Theory* **153** (2015), 372–426. MR Zbl

[Hakim 1991] J. Hakim, "Distinguished p-adic representations", *Duke Math. J.* **62**:1 (1991), 1–22. MR Zbl

[Kudla 1994] S. S. Kudla, "Splitting metaplectic covers of dual reductive pairs", *Israel J. Math.* **87**:1-3 (1994), 361–401. MR Zbl

[Kudla 1996] S. S. Kudla, "Notes on the local theta correspondence", lecture notes, 1996, available at http://www.math.toronto.edu/skudla/castle.pdf.

[Kudla and Rallis 2005] S. S. Kudla and S. Rallis, "On first occurrence in the local theta correspondence", pp. 273–308 in *Automorphic representations, L-functions and applications: progress and prospects* (Columbus, OH, 2003), edited by J. W. Cogdell et al., Ohio State Univ. Math. Res. Inst. Publ. **11**, de Gruyter, Berlin, 2005. MR Zbl

[Lu 2016] H. Lu, GSp(4)*-period problems over a quadratic field extension*, Ph.D. thesis, National University of Singapore, 2016, available at http://scholarbank.nus.edu.sg/handle/10635/135863.

[Lu 2018] H. Lu, "Theta correspondence and the Prasad conjecture for SL(2)", *Pacific J. Math.* **295**:2 (2018), 477–498. MR Zbl

[Mœglin et al. 1987] C. Mœglin, M.-F. Vignéras, and J.-L. Waldspurger, *Correspondances de Howe sur un corps p-adique*, Lecture Notes in Math. **1291**, Springer, 1987. MR Zbl

[Prasad 1992] D. Prasad, "Invariant forms for representations of GL_2 over a local field", *Amer. J. Math.* **114**:6 (1992), 1317–1363. MR Zbl

[Prasad 2015] D. Prasad, "A 'relative' local Langlands correspondence", preprint, 2015. arXiv

[Prasad and Takloo-Bighash 2011] D. Prasad and R. Takloo-Bighash, "Bessel models for GSp(4)", *J. Reine Angew. Math.* **655** (2011), 189–243. MR Zbl

[Scharlau 1985] W. Scharlau, *Quadratic and Hermitian forms*, Grundlehren der Math. Wissenschaften **270**, Springer, 1985. MR Zbl

[Sun and Zhu 2015] B. Sun and C.-B. Zhu, "Conservation relations for local theta correspondence", *J. Amer. Math. Soc.* **28**:4 (2015), 939–983. MR Zbl

[Waldspurger 1990] J.-L. Waldspurger, "Démonstration d'une conjecture de dualité de Howe dans le cas p-adique, $p \neq 2$", pp. 267–324 in *Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, I* (Tel Aviv, 1989), edited by R. Howe and P. Sarnak, Israel Math. Conf. Proc. **2**, Weizmann Sci., Jerusalem, 1990. MR Zbl

[Yamana 2011] S. Yamana, "Degenerate principal series representations for quaternionic unitary groups", *Israel J. Math.* **185** (2011), 77–124. MR Zbl

Received May 6, 2018. Revised September 12, 2018.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 300, No. 1, 2019
dx.doi.org/10.2140/pjm.2019.300.83

EIGENVALUE ASYMPTOTICS AND BOHR'S FORMULA FOR FRACTAL SCHRÖDINGER OPERATORS

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For a Schrödinger operator defined by a fractal measure with a continuous potential and a coupling parameter, we obtain an analog of a semiclassical asymptotic formula for the number of bound states as the parameter tends to infinity. We also study Bohr's formula for fractal Schrödinger operators on blowups of self-similar sets. For a locally bounded potential that tends to infinity, we derive an analog of Bohr's formula under various assumptions. We demonstrate how this result can be applied to self-similar measures with overlaps, including the infinite Bernoulli convolution associated with the golden ratio, a family of convolutions of Cantor-type measures, and a family of measures that are essentially of finite type.

1. Introduction

Negative eigenvalues of a Schrödinger operator are referred to by physicists as *bound state energies*. Let Δ be the Dirichlet Laplacian on $\mathbb{R}^n$ and let $N^-(V)$ be

The authors are supported in part by the National Natural Science Foundation of China, Grants 11771136 and 11271122, and Construct Program of the Key Discipline in Hunan Province. Ngai is also supported in part by the Hunan Province Hundred Talents Program, the Center of Mathematical Sciences and Applications (CMSA) of Harvard University, and a Faculty Research Scholarly Pursuit Award from Georgia Southern University.

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MSC2010: primary 28A80, 35J10; secondary 35J05, 35P20.

Keywords: fractal, Schrödinger operator, Bohr's formula, Laplacian, self-similar measure with overlaps.

the number of negative eigenvalues of the Schrödinger operator $-\Delta + V$, where V is a potential. In the early 1970s, Birman and Borzov [1972], Martin [1972], and Tamura [1974] proved various forms of the semiclassical asymptotic formula

$$N^-(\beta V) \sim \frac{\omega_n}{(4\pi)^{n/2}} \beta^{n/2} \int_{D_n^-(V)} (-V(x))^{n/2}\, dx \qquad \text{as } \beta \to \infty, \tag{1-1}$$

where V is a continuous and compactly supported potential, β is called a *coupling parameter*, $D_n^-(V) := \{x \in \mathbb{R}^n : V(x) \le 0\}$, ω_n is the volume of the unit ball in $\mathbb{R}^n$ and, throughout this paper, $f \sim g$ means $\lim_{x\to\infty} f(x)/g(x) = 1$. The main ingredients are the *Dirichlet–Neumann bracketing technique* [Reed and Simon 1978; Kigami and Lapidus 1993; Hambly and Nyberg 2003] and the *Weyl law* [Weyl 1912], which are basic and useful techniques for deriving various asymptotic formulas of Laplace and Schrödinger operators. When computing spectral asymptotics, it is often necessary to decompose a domain into a finite union of subdomains. Using the idea of Dirichlet–Neumann bracketing, one can bound the Laplacian on the domain by those obtained by imposing Dirichlet or Neumann boundary conditions on the common boundary of the subdomains (see, e.g., [Reed and Simon 1978, Section XIII.15]). We use this technique in the proof of Theorem 1.2.

For fractal sets, Strichartz [2009] studied the counting function for the negative eigenvalues of the Schrödinger operator $-\Delta + V$ on the product of two copies of an infinite blowup of the Sierpiński gasket, where Δ is the Laplacian on the product and V is a Coulomb potential. He showed that the number of eigenvalues that are less than $-\epsilon$ is of the order $\epsilon^{-\delta}$ as $\epsilon \to 0^+$, where $\delta = (\ln(25/9)\ln 9)/(\ln(9/5)\ln 5)$. A main goal of this paper is to obtain a crude analog of (1-1) for Schrödinger operators $-\Delta_\mu + \beta V$ defined on domains by a measure μ (see Theorem 1.2).

Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$ that is semibounded below. If A has compact resolvent, then the number of negative eigenvalues, counting multiplicity, is finite; moreover, each eigenspace is finite-dimensional. We define the *eigenvalue counting function* as

$$N(\lambda, A) := \#\{n : \lambda_n(A) \le \lambda\}, \tag{1-2}$$

where $\lambda_n(A)$ is the n-th eigenvalue of A counted according to their multiplicities, and $\#F$ denotes the cardinality of a finite set F. Furthermore, we define the *lower* and *upper spectral dimensions* of A, respectively, as

$$\underline{d}_s(A) := \varliminf_{\lambda\to\infty} \frac{2\ln N(\lambda, A)}{\ln \lambda} \quad \text{and} \quad \bar{d}_s(A) := \varlimsup_{\lambda\to\infty} \frac{2\ln N(\lambda, A)}{\ln \lambda}.$$

If $\underline{d}_s(A) = \bar{d}_s(A)$, the common value, denoted by $d_s(A)$, is called the *spectral dimension* of A; it measures the asymptotic growth rate of the eigenvalue counting function.

Let $E \subseteq \mathbb{R}^n$. We denote by $\overline{E}$, ∂E, $|E|$, E°, and $\mathcal{L}^n(E)$ the closure, boundary, diameter, interior, and n-dimensional Lebesgue measure of E, respectively. For a real-valued function f on E, we define $f_+ := \max\{f, 0\}$ and $f_- := -\min\{f, 0\}$, and let $f|_F$ denote the restriction of the function f to $F \subseteq E$. For a positive measure ν on E, we denote by $\nu|_F$ the restriction of ν to $F \subseteq E$, and let $\|u\|_{p,\nu} := \|u\|_{L^p(E,\nu)}$ denote the norm in $L^p(E, \nu)$, where $1 \le p \le \infty$.

The classical one-dimensional Bohr's formula states that, under suitable conditions,

$$N(\lambda, -\Delta + V) \sim \frac{1}{\pi} \int_0^\infty (\lambda - V(x))_+^{1/2} \, dx \qquad \text{as } \lambda \to \infty,$$

where Δ is the Laplacian in $L^2([0, \infty), dx)$, and $V(x) \to \infty$ as $x \to \infty$ (see [Holt and Molchanov 2005]). In the classical setting, various forms of Bohr's formula have been obtained and studied extensively (see, e.g., [Reed and Simon 1978]). In the fractal setting, Bohr's formula has been obtained by Chen et al. [2015] for some unbounded potentials V on several types of unbounded fractal spaces K_∞ supporting a measure μ_∞ and having a well-defined Laplacian Δ_{μ_∞}. K_∞ is obtained by blowing up some fractal K. In [Chen et al. 2015], sufficient conditions for the following Bohr's formula to hold are obtained:

$$N(\lambda, -\Delta_{\mu_\infty} + V) \sim g(V, \lambda) \qquad \text{as } \lambda \to \infty,$$

where

$$g(V, \lambda) := \int_{K_\infty} ((\lambda - V(x))_+)^{d_s/2} G\big(\tfrac{1}{2} \ln((\lambda - V(x))_+)\big) \, d\mu_\infty(x), \tag{1-3}$$

$d_s = d_s(-\Delta_{\mu_\infty})$, and $G(\,\cdot\,)$ is a periodic function. Moreover, these conditions are verified for fractafolds and fractal fields based on nested fractals. A key condition in [Chen et al. 2015] is

$$N(\lambda, -\Delta^b_{\mu_\infty|_K}) = \lambda^{d_s/2}\big(G\big(\tfrac{1}{2} \ln \lambda\big) + R^b(\lambda)\big) \qquad \text{as } \lambda \to \infty, \tag{1-4}$$

where $b \in \{D, N\}$, $R^b(\lambda)$ is a remainder term of order $o(1)$, and $-\Delta^D_{\mu_\infty|_K}$ and $-\Delta^N_{\mu_\infty|_K}$ are Dirichlet and Neumann Laplacians in $L^2(K, \mu_\infty|_K)$, respectively. Unfortunately, fractals with overlaps usually do not, or are not known to, satisfy this condition. Thus it is another main goal of this paper to derive an analog of Bohr's formula for such fractals by modifying (1-4).

In the rest of this section, we let $X \subseteq \mathbb{R}^n$ be a compact subset with nonempty interior and μ be a positive finite Borel measure on X with $\mu(X^\circ) > 0$ and $\mathrm{supp}(\mu) \subseteq X$. It is known that μ defines a Dirichlet Laplace operator Δ^D_μ (or denoted simply by Δ_μ) provided the following *Poincaré inequality for a measure* (*MPI*) holds:

There exists a constant $C > 0$ such that

$$\int_{X^\circ} |u|^2 \, d\mu \le C \int_{X^\circ} |\nabla u|^2 \, dx \qquad \text{for all } u \in C_c^\infty(X^\circ); \tag{1-5}$$

see, e.g., [Mazya 1985; Naimark and Solomyak 1995; Hu et al. 2006]. By assuming some regularity conditions of the boundary of X, one can define a Neumann Laplacian as in [Hu et al. 2006]. We say that $f \in C^\infty(X)$ if $f \in C^\infty(X^\circ)$ and all of whose partial derivatives can be extended continuously to X. Assume X° has the extension property. The following analog of MPI, which we call *Poincaré inequality* for measures* (*MPI**) is crucial: There exists a constant $C > 0$ such that

$$\int_{X^\circ} |u|^2 \, d\mu \le C\biggl(\int_{X^\circ} |\nabla u|^2 \, dx + \int_{X^\circ} |u|^2 \, dx\biggr) \qquad \text{for all } u \in C^\infty(X). \tag{1-6}$$

We remark that MPI* is stronger than MPI. We need one additional inequality, namely, *Poincaré inequality* (*PI*), i.e., there exists some constant $C > 0$ such that

$$\int_{X^\circ} |u - u^*|^2 \, dx \le C \int_{X^\circ} |\nabla u|^2 \, dx \qquad \text{for all } u \in H^1(X^\circ), \tag{1-7}$$

where $u^* := (1/\mathcal{L}^n(X^\circ)) \cdot \int_{X^\circ} u \, dx$ (see, e.g., [Lieb and Loss 2001, Theorem 8.11]). If MPI* and PI hold, then using the same procedure for constructing Δ_μ^D (see [Hu et al. 2006]), one can obtain a Neumann Laplace operator Δ_μ^N defined by μ. For convenience, we summarize the definitions of Δ_μ^D and Δ_μ^N in Section 2B and Section 2C, respectively. Also, in the rest of this section, we assume that μ satisfies MPI.

The first part of this paper studies Schrödinger operators $-\Delta_\mu + \beta V$ in $L^2(X, \mu)$ with a continuous potential V and a coupling parameter β, focusing on self-similar measures. Throughout this paper, we let $D^-(V) := \{x \in X : V(x) \le 0\}$ and $N_\mu^-(V)$ be the number of negative eigenvalues of $-\Delta_\mu + V$, where V is a real-valued continuous function V on X.

Before stating the main results, we introduce some definitions that will be used. We call a μ-measurable closed subset B of X a *cell* (*in* X) if $\mu(B^\circ) > 0$. Clearly, X itself is a cell.

Definition 1.1. We say that a cell B in X satisfies condition (N) if

(1) B° has the extension property and satisfies PI;

(2) $\mu|_B$ satisfies MPI*;

(3) $-\Delta_{\mu|_B}^N$ has compact resolvent.

Conditions (1) and (2) ensure that $-\Delta_{\mu|_B}^N$ is well defined, and condition (3) implies that $N(\lambda, -\Delta_{\mu|_B}^N)$ is well defined for $\lambda > 0$. We call a finite family $\mathcal{P}$ of interior disjoint cells a *μ-partition* of X if $\mu(X) = \sum_{B \in \mathcal{P}} \mu(B)$. Let ν be a positive finite Borel measure on X. Roughly speaking, a sequence of μ-partitions $(\mathcal{P}_k)_{k \ge 1}$

is said to be *refining* with respect to ν if each member of $\mathcal{P}_{k+1}$ is a subset of some member of $\mathcal{P}_k$, and $\max\{\nu(B) : B \in \mathcal{P}_k\} \to 0$ as $k \to \infty$.

Theorem 1.2. *Let $X \subseteq \mathbb{R}^n$ be a compact subset with nonempty interior and μ be a positive finite Borel measure on $\mathbb{R}^n$ with* $\operatorname{supp}(\mu) \subseteq X$ *and $\mu(X^\circ) > 0$. Assume that μ satisfies MPI and V is a real-valued continuous function on X. Let ν be a positive Borel measure on X.*

(a) *If there exist positive constants C and α, and a refining μ-partition $(\mathcal{P}_k)_{k\geq 1}$ of X with respect to ν such that for all $B \in \bigcup_{k=1}^{\infty} \mathcal{P}_k$,*

$$N(\lambda, -\Delta^D_{\mu|_B}) \geq \lambda^{\alpha/2}(C\nu(B) + o(1)) \qquad \text{as } \lambda \to \infty, \tag{1-8}$$

then

$$N^-_\mu(\beta V) \geq \beta^{\alpha/2}\left(C \int_{D^-(V)} (-V)^{\alpha/2}\, d\nu + o(1)\right) \qquad \text{as } \beta \to \infty. \tag{1-9}$$

(b) *If there exist positive constants C and α, and a refining μ-partition $(\mathcal{P}_k)_{k\geq 1}$ of X with respect to ν such that each $B \in \bigcup_{k=1}^{\infty} \mathcal{P}_k$ satisfies condition* (N), *and*

$$N(\lambda, -\Delta^N_{\mu|_B}) \leq \lambda^{\alpha/2}(C\nu(B) + o(1)) \qquad \text{as } \lambda \to \infty, \tag{1-10}$$

then the reverse inequality in (1-9) *holds.*

We remark that (1-8) and (1-10) are more general than the Weyl law. In the proof of Theorem 1.2, we use a similar method as in [Reed and Simon 1978, Theorem XIII.79] with (1-8) and (1-10) replacing the Weyl law. We illustrate Theorem 1.2 by a family of self-similar measures that are said to be *essentially of finite type* (*EFT*) (see Section 3).

An iterated function system (IFS) $\{S_i\}_{i=1}^m$ on $\mathbb{R}^n$ is said to satisfy the *open set condition* (*OSC*) if there exists a nonempty bounded open set $U \subset \mathbb{R}^n$ such that $\bigcup_{i=1}^m S_i(U) \subseteq U$ and $S_i(U) \cap S_j(U) = \varnothing$ if $i \neq j$. An IFS that does not satisfy OSC is said to have *overlaps*; in this case, we also say that an associated self-similar measure has overlaps. The second part of this paper studies Bohr's formula for the Schrödinger operator on blowups of compact subsets with locally bounded nonnegative piecewise continuous potentials that tend to infinity, focusing on fractals defined by IFS with overlaps. We first state some Weyl asymptotic properties for Δ_μ, which will be used in Section 4.

Definition 1.3. Let $X \subseteq \mathbb{R}^n$ be a compact subset with nonempty interior and μ be a positive finite Borel measure on X with $\mu(X^\circ) > 0$ and $\operatorname{supp}(\mu) \subseteq X$. Assume that μ satisfies MPI. Let Δ_μ be the associated Dirichlet Laplacian (see definition in Section 2B), and assume that the spectral dimension d_s of $-\Delta_\mu$ exists. Define the following two Weyl asymptotic properties.

(W1) There exist positive constants C_1, C_2 such that

$$C_1\lambda^{d_s/2} \le N(\lambda, -\Delta_\mu) \le C_2\lambda^{d_s/2} \qquad \text{for all sufficiently large } \lambda.$$

(W2) There exists a finite collection of closed subsets $\{Y_j\}_{j\in J}$ of X with $\mu(Y_j^\circ) > 0$ satisfying the following conditions:

(1) There exist positive constants C_0 and $(\xi_{j,k})_{j\in J}$, $k = 1, 2$, such that for all $\lambda > 0$,

$$\sum_{j\in J} N(\xi_{j,1}\lambda, -\Delta_{\mu|_{Y_j}}) - C_0 \le N(\lambda, -\Delta_\mu) \le \sum_{j\in J} N(\xi_{j,2}\lambda, -\Delta_{\mu|_{Y_j}}) + C_0. \tag{1-11}$$

(2) For each $j \in J$, there exists a periodic or constant function $G_j : \mathbb{R} \to \mathbb{R}^+$ such that $0 < \inf G_j \le \sup G_j < \infty$, and

$$N(\lambda, -\Delta_{\mu|_{Y_j}}) = \lambda^{d_s/2}(G_j(\ln\lambda) + R_j(\lambda)) \qquad \text{as } \lambda \to \infty, \tag{1-12}$$

where $R_j(\lambda)$ is a remainder term of order $o(1)$.

We remark that (W2) is stronger than (W1). Condition (2) of (W2) means that $-\Delta_{\mu|_{Y_j}}$ satisfies (1-4) for all $j \in J$. Consequently, (W2) is more general than (1-4), which corresponds to (W2) with $J = \{1\}$, $Y_1 = X$, and $G_1(\cdot)$ being a periodic function. Weyl asymptotic properties of fractal Laplacians have been studied in [Kigami and Lapidus 1993; Hambly and Nyberg 2003; Ngai 2011; Ngai et al. 2018; Naimark and Solomyak 1995]. If no confusion is possible, we also call $d_s(-\Delta_\mu)$ the *spectral dimension of* μ.

We extend X to an unbounded space X_∞ as follows. Let $X_\infty := \bigcup_{i\in I} X_i$, where

(C1) I is a countably infinite index set containing 0;

(C2) for each $i \in I$ there corresponds a similitude $\tau_i : X \to X_i$ of the form $\tau_i(x) = x + b_i$, with $b_i \in \mathbb{R}^n$ such that τ_0 is the identity map on $\mathbb{R}^n$ and $\tau_i(X) = X_i$;

(C3) for any distinct $i, j \in I$, $X_i \cap X_j = \partial X_i \cap \partial X_j$.

Since each τ_i is an isometry, $|X_i| = |X|$ for all $i \in I$. Condition (C3) implies that the interiors of any two distinct X_i are disjoint. For each $i \in I$, $\mu_i := \mu \circ \tau_i^{-1}$ defines a positive finite Borel measure on X_i. Intuitively, μ_i and μ have the same measure structure. Also, $\mu_0 = \mu$. In a natural way, we can define a glued measure μ_∞ on X_∞ by

$$\mu_\infty(E) := \sum_{i\in I} \mu_i(E \cap X_i) \qquad \text{for all Borel subsets } E \subseteq X_\infty. \tag{1-13}$$

Throughout this paper, we assume that $\mu_\infty(X_i \cap X_j) = 0$ for any distinct $i, j \in I$. For a real-valued function f on X_∞ and $\lambda > 0$, we define the *distribution function* *of* f with respect to μ_∞ as:

$$F(\lambda, f) := \mu_\infty\big(\{x \in X_\infty : f(x) \le \lambda\}\big). \tag{1-14}$$

Assume (W2) holds. For any $j \in J$, define

$$X_{\infty,j} := \bigcup_{i\in I} \tau_i(Y_j) \quad \text{and} \quad \mu_{\infty,j} := \mu_\infty|_{X_{\infty,j}}. \tag{1-15}$$

In order to state the precise results, we introduce the following associated *Bohr's asymptotic function*: for any $j \in J$, $\lambda > 0$, and $f \in L^1_{\mathrm{loc}}(X_{\infty,j}, \mu_{\infty,j})$, define

$$g_j(\lambda, f) := \frac{1}{\mu(Y_j)} \int_{X_{\infty,j}} ((\lambda - f(x))_+)^{d_s/2} G_j(\ln((\lambda - f(x))_+))\, d\mu_{\infty,j}(x), \tag{1-16}$$

where $G_j(\cdot)$ is given in (W2). We remark that $g_j(\cdot,\cdot)$ is an analog of the $g(\cdot,\cdot)$ in (1-3), which appears in [Chen et al. 2015], but is slightly different because it is assumed in [Chen et al. 2015] that $\mu(K) = 1$. Let V be a locally bounded nonnegative piecewise continuous function on X_∞ such that $V(x) \to \infty$ as $|x| \to \infty$. Also, let $V^\wedge$ (resp. $V^\vee$) be the piecewise constant function which takes the value $\sup_{x\in X_i} V(x)$ (resp. $\inf_{x\in X_i} V(x)$) on X_i. Theorem 1.4 gives the eigenvalue asymptotics of $N(\lambda, -\Delta_{\mu_\infty} + V)$, where $-\Delta_{\mu_\infty} := \bigoplus_{i\in I}(-\Delta_{\mu_i})$.

Theorem 1.4. *Use the notation above. Let V be a locally bounded nonnegative piecewise continuous function on X_∞ such that $V(x) \to \infty$ as $|x| \to \infty$. Assume MPI and* (W2) *hold. Let $F(\cdot,\cdot)$ and $g_j(\cdot,\cdot)$ be defined as in* (1-14) *and* (1-16) *for $j \in J$, respectively. Assume that*

$$F(\lambda, V^\vee)/F(\lambda, V^\wedge) = 1 + o(1) \qquad \text{as } \lambda \to \infty, \tag{1-17}$$

and that there exists some $C > 0$ such that $F(2\lambda, V^\vee) \le C F(\lambda, V^\wedge)$ for all sufficiently large $\lambda > 0$. Then as $\lambda \to \infty$,

$$(1+o(1))\sum_{j\in J} g_j(\xi_{j,1}\lambda, \xi_{j,1}V) \le N(\lambda, -\Delta_{\mu_\infty} + V) \le (1+o(1))\sum_{j\in J} g_j(\xi_{j,2}\lambda, \xi_{j,2}V),$$

where $(\xi_{j,k})_{j\in J}$, $k = 1, 2$, are the constants in (1-11).

We remark that Theorem 1.4 cannot be deduced from [Chen et al. 2015, Theorem 2.11], since (W2) is more general than (1-4), which is a key assumption in [Chen et al. 2015, Theorem 2.11]. Theorem 1.4 allows us to obtain eigenvalue asymptotics of Schrödinger operators in the absence of condition (1-4), as illustrated in the examples of IFSs with overlaps in Section 5. It also enables us to draw conclusions on $N(\lambda, -\Delta_{\mu_\infty} + V)$ even though we only have information about the Weyl asymptotics of the Laplacian on proper subsets of X.

In Section 5, we apply Theorem 1.4 to three classes of self-similar measures. The infinite Bernoulli convolution associated with the golden ratio and a class of convolutions of Cantor-type measures have been studied very extensively (see [Lau and Ngai 1998; 2000; Lau and Wang 2005; Feng and Olivier 2003; Ngai 2011; Gu et al. 2016]). They define Laplacians that exhibit many behaviors analogous to Laplacians on post-critically finite fractals, such as sub-Gaussian heat kernel estimates [Gu et al. 2016] and infinite wave propagation speed [Ngai et al. 2019]. The third class is used in [Ngai et al. 2018] to illustrate self-similar measures satisfying EFT. We show that all these three classes of measures satisfy (W2). However, it is not clear whether they satisfy (1-4).

The rest of this paper is organized as follows. Section 2 summarizes some of the definitions and results that will be needed throughout the paper. In Section 3, we prove Theorem 1.2, and apply it to a class of self-similar measures satisfying EFT. In Section 4, we study Bohr's formula for Schrödinger operators defined by measures and nonnegative locally bounded potentials, and prove Theorem 1.4. Finally, in Section 5, we illustrate Theorem 1.4 by the three classes of self-similar measures with overlaps mentioned above.

2. Preliminaries

Let $(\mathcal{H}_1, \|\cdot\|_1)$ and $(\mathcal{H}_2, \|\cdot\|_2)$ be Hilbert spaces. Let A_1, A_2 be linear operators in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. A_1 and A_2 are said to be *unitarily equivalent*, denoted $A_1 \approx A_2$, if there exists a unitary operator $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\varphi(\operatorname{dom} A_1) = \operatorname{dom} A_2 \quad \text{and} \quad \varphi(A_1(u)) = A_2(\varphi(u)) \quad \text{for all } u \in \operatorname{dom} A_1.$$

Note that u is a λ-eigenvector of A_1 if and only if $\varphi(u)$ is a λ-eigenvector of A_2. In particular, unitarily equivalent operators have the same set of eigenvalues.

Let $(\mathcal{H}_i)_{i\in I}$ be a finite or countably infinite family of Hilbert spaces. Define a Hilbert space

$$\mathcal{H} = \bigoplus_{i\in I} \mathcal{H}_i := \Big\{ u = (u_i)_{i\in I} : u_i \in \mathcal{H}_i \text{ for all } i \in I \text{ and } \|u\|_{\mathcal{H}}^2 := \sum_{i\in I} \|u_i\|_{\mathcal{H}_i}^2 < \infty \Big\}.$$

Assume that each A_i is a self-adjoint operator in $\mathcal{H}_i$. We write $A := \bigoplus_{i\in I} A_i$ if $Au := (A_i u_i)_{i\in I}$ with domain $\operatorname{dom} A := \{u = (u_i)_{i\in I} \in \mathcal{H} : u_i \in \operatorname{dom} A_i \text{ for all } i \in I$ and $Au \in \mathcal{H}\}$ (see [Reed and Simon 1972]). We remark that $(A, \operatorname{dom} A)$ is a self-adjoint operator in $\mathcal{H}$.

2A. ***Quadratic forms.*** Let $\mathcal{H}$ be a (real or complex) Hilbert space with inner product $(\cdot\,,\cdot)$ and norm $\|\cdot\|$. We call a symmetric densely defined bilinear form $\mathcal{E}$ in $\mathcal{H}$ a *quadratic form* in $\mathcal{H}$. A quadratic form $(\mathcal{E}, \operatorname{dom} \mathcal{E})$ is said to be (a) *semibounded below* if there exists some constant $M \geq 0$ such that

$$\mathcal{E}(u, u) \geq -M\|u\|^2 \qquad \text{for all } u \in \operatorname{dom} \mathcal{E}; \tag{2-1}$$

(b) *nonnegative* if we may take $M = 0$ in (2-1) (see [Reed and Simon 1972]); and
(c) *closed* if it is semibounded below and $(\mathcal{E}_{M+1}, \operatorname{dom}\mathcal{E})$ is a Hilbert space, where $\mathcal{E}_{M+1}(u, v) := \mathcal{E}(u, v) + (M+1)(u, v)$ for all $u, v \in \operatorname{dom}\mathcal{E}$.

A self-adjoint operator $(A, \operatorname{dom} A)$ in $\mathcal{H}$ is said to be *semibounded below* if there exists some constant $C \geq 0$ such that $(Au, u) \geq -C\|u\|^2$ for all $u \in \operatorname{dom} A$. It is well known that a closed quadratic form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ corresponds to a unique self-adjoint operator A, that is semibounded below, such that $\operatorname{dom} A \subseteq \operatorname{dom}\mathcal{E}$, and

$$\mathcal{E}(u, v) = (Au, v) \qquad \text{for all } u \in \operatorname{dom} A \text{ and } v \in \operatorname{dom}\mathcal{E};$$

see, e.g., [Fukushima et al. 2010, Section 1.3]. In this case, A is called the *generator* of $(\mathcal{E}, \operatorname{dom}\mathcal{E})$. On the other hand, any self-adjoint operator $(A, \operatorname{dom} A)$ in $\mathcal{H}$ determines a quadratic form $(\mathcal{E}, \operatorname{dom} A)$ by $\mathcal{E}(u, v) := (Au, v)$ for all $u, v \in \operatorname{dom} A$. Moreover, if A is semibounded below, then $(\mathcal{E}, \operatorname{dom} A)$ is closable, and its closure $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is called the *closed quadratic form associated with* A. We let $\operatorname{dom}_F(A) := \operatorname{dom}\mathcal{E}$ and call it the *form domain of* A. Furthermore, if A is nonnegative, then $A^{1/2}$ is well defined, and

$$\mathcal{E}(u, v) = (A^{1/2}u, A^{1/2}v) \quad \text{and} \quad \operatorname{dom}\mathcal{E} = \operatorname{dom}(A^{1/2});$$

see, e.g., [Fukushima et al. 2010, Theorem 1.3.1]. Moreover, for $u \in \operatorname{dom}\mathcal{E}$, we have $u \in \operatorname{dom} A$ if and only if there exists a unique $f \in \mathcal{H}$ such that $\mathcal{E}(u, v) = (f, v)$ for all $v \in \operatorname{dom}\mathcal{E}$. In this case, $Au = f$.

For $i = 1, 2$, let $(\mathcal{E}_i, \operatorname{dom}\mathcal{E}_i)$ be a closed quadratic form in a Hilbert space $\mathcal{H}$ with generator A_i. If $\operatorname{dom}\mathcal{E}_1 \cap \operatorname{dom}\mathcal{E}_2$ is dense in $\mathcal{H}$, then we denote the generator of the closure of $(\mathcal{E}_1 + \mathcal{E}_2, \operatorname{dom}\mathcal{E}_1 \cap \operatorname{dom}\mathcal{E}_2)$ by $A_1 + A_2$, and say that $A_1 + A_2$ is an operator *defined as a sum of quadratic forms*.

Let E be a subset of $\mathbb{R}^n$ and ν be a positive σ-finite Borel measure on E. For any $V \in L^1_{\mathrm{loc}}(E, \nu)$, the quadratic form $\mathcal{E}_V$ given by

$$\mathcal{E}_V(u, v) = \int_E uvV\,d\nu \qquad \text{for all } u, v \in C_c^\infty(E),$$

is closable on $L^2(E, \nu)$. In this case, we denote the closure of $(\mathcal{E}_V, C_c^\infty(E))$ by $(\mathcal{E}_V, \operatorname{dom}\mathcal{E}_V)$ and regard V as the generator (see [Davies 1989; Reed and Simon 1972]).

Definition 2.1. For $i = 1, 2$, let A_i be a self-adjoint operator in a Hilbert space $\mathcal{H}_i$ that is semibounded below, and $(\mathcal{E}_i, \operatorname{dom}\mathcal{E}_i)$ be the associated closed quadratic form. We say $A_1 \preccurlyeq A_2$ (in the sense of quadratic forms) if $\mathcal{H}_2 \subseteq \mathcal{H}_1$, $\operatorname{dom}\mathcal{E}_2 \subseteq \operatorname{dom}\mathcal{E}_1$, and $\mathcal{E}_1(u, u) \leq \mathcal{E}_2(u, u)$ for all $u \in \operatorname{dom}\mathcal{E}_2$.

We state a simple proposition. A proof can be found in [Reed and Simon 1978, Section XIII].

Proposition 2.2. *For $i = 1, 2$, let A_i be a self-adjoint operator in a Hilbert space $\mathcal{H}_i$ that is semibounded below. Assume $A_1 \preccurlyeq A_2$. If A_1 has compact resolvent, then so does A_2; moreover, $N(\lambda, A_1) \geq N(\lambda, A_2)$ for all $\lambda \in \mathbb{R}$.*

2B. ***Dirichlet Laplacian defined by a measure.*** For convenience, we summarize the definition of the Dirichlet Laplacian on a bounded domain defined by a measure; details can be found in [Hu et al. 2006]. Let $U \subseteq \mathbb{R}^n$ be a bounded open subset and μ be a positive finite Borel measure with $\operatorname{supp}(\mu) \subseteq \overline{U}$ and $\mu(U) > 0$. We assume that μ satisfies MPI (see (1-5)). MPI implies that each equivalence class $u \in H_0^1(U)$ contains a unique (in the $L^2(U, \mu)$ sense) member $\bar{u}$ that belongs to $L^2(U, \mu)$ and satisfies both conditions below:

(1) There exists a sequence $\{u_n\}$ in $C_c^\infty(U)$ such that $u_n \to \bar{u}$ in $H_0^1(U)$ and $u_n \to \bar{u}$ in $L^2(U, \mu)$.

(2) $\bar{u}$ satisfies inequality (1-5).

We call $\bar{u}$ the *$L^2(U, \mu)$-representative* of u. Define a mapping $\iota : H_0^1(U) \to L^2(U, \mu)$ by $\iota(u) = \bar{u}$. ι is a bounded linear operator, but not necessarily injective. Consider the subspace $\mathcal{N}$ of $H_0^1(U)$ defined as $\mathcal{N} := \{u \in H_0^1(U) : \|\iota(u)\|_{2,\mu} = 0\}$. Now let $\mathcal{N}^\perp$ be the orthogonal complement of $\mathcal{N}$ in $H_0^1(U)$. Then $\iota : \mathcal{N}^\perp \to L^2(U, \mu)$ is injective. Unless explicitly stated otherwise, we will denote the $L^2(U, \mu)$-representative $\bar{u}$ simply by u.

Consider the nonnegative bilinear form $\mathcal{E}_D(\cdot, \cdot)$ in $L^2(U, \mu)$ given by

$$\mathcal{E}_D(u, v) := \int_U \nabla u \cdot \nabla v \, dx \tag{2-2}$$

with domain $\operatorname{dom} \mathcal{E}_D = \mathcal{N}^\perp$, or more precisely, $\iota(\mathcal{N}^\perp)$. MPI implies $(\mathcal{E}_D, \operatorname{dom} \mathcal{E}_D)$ is a nonnegative closed quadratic form in $L^2(U, \mu)$. We use $-\Delta_\mu^D$ (or simply $-\Delta_\mu$) to denote the generator of $(\mathcal{E}_D, \operatorname{dom} \mathcal{E}_D)$, and call it the (*Dirichlet*) *Laplacian* with respect to μ.

Some sufficient conditions for MPI and the existence of an orthonormal basis $\{\varphi_n\}_{n=1}^\infty$ of $L^2(U, \mu)$ consisting of the eigenfunctions of $-\Delta_\mu$ can be found in [Hu et al. 2006; Davies 1995; Mazya 1985]. We remark that if $n = 1$, then MPI holds for any such μ, and thus Δ_μ is well defined; moreover, $-\Delta_\mu$ has compact resolvent.

2C. ***Neumann Laplacian defined by a measure.*** We state a result below that is sufficient for the purpose of this paper. Let U be a bounded open subset of $\mathbb{R}^n$. Suppose U is a bounded open subset in $\mathbb{R}^n$ that has the extension property. Then $C^\infty(\overline{U})$ is dense in $H^1(U)$. All bounded regions in $\mathbb{R}^n$ with piecewise smooth or Lipschitz boundaries have the extension property. Let μ be a finite positive Borel measure on U with $\operatorname{supp}(\mu) \subseteq \overline{U}$ and $\mu(U) > 0$. Assume that μ satisfies MPI* (see (1-6)). As in the construction of the Dirichlet Laplacian, MPI* implies that each equivalence class $u \in H^1(U)$ contains a unique (in the $L^2(U,\mu)$ sense) member $\hat{u}$ that belongs to $L^2(U,\mu)$ and satisfies both conditions below:

(1) There exists a sequence $\{u_n\}$ in $C^\infty(\overline{U})$ such that $u_n \to \hat{u}$ in $H^1(U)$ and $u_n \to \hat{u}$ in $L^2(U,\mu)$.

(2) $\hat{u}$ satisfies inequality (1-6).

Define a quadratic form $\mathcal{E}_N(\cdot,\cdot)$ in $L^2(U,\mu)$ by

$$\mathcal{E}_N(u,v) := \int_U \nabla u \cdot \nabla v \, dx,$$

with domain $\operatorname{dom}\mathcal{E}_N := \iota(\mathcal{N}^\perp)$, where $\iota : H^1(U) \to L^2(U,\mu)$ and $\mathcal{N}$ are analogs of those in Section 2B. Assume, in addition, U satisfies PI (see (1-7)). Then MPI* and PI imply that $(\mathcal{E}_N, \operatorname{dom}\mathcal{E}_N)$ is a nonnegative closed quadratic form in $L^2(U,\mu)$ (see [Hu et al. 2006; Lau and Ngai $\geq$ 2019]). We denote the generator of $(\mathcal{E}_N, \operatorname{dom}\mathcal{E}_N)$ by $-\Delta_\mu^N$, and call it the *Neumann Laplacian* with respect to μ. We remark that $-\Delta_\mu^N \preccurlyeq -\Delta_\mu^D$.

We remark that if $n = 1$ and $U = (a,b)$, then MPI* holds for any such μ and PI holds, and thus Δ_μ^N is well defined. Moreover, Δ_μ^N has compact resolvent.

3. Fractal analog of a semiclassical asymptotic formula for the number of bound states

In this section, we prove Theorem 1.2 and illustrate it by a class of self-similar measures with overlaps.

Let $X \subseteq \mathbb{R}^n$ be a compact subset with nonempty interior and μ be a positive finite Borel measure on $\mathbb{R}^n$ with $\operatorname{supp}(\mu) \subseteq X$ and $\mu(X^\circ) > 0$. We say that two cells B and B' are *μ-equivalent*, denoted by $B \simeq_{\mu,\tau,w} B'$ (or simply $B \simeq_\mu B'$), if there exist some similitude $\tau : B \to B'$ of the form $\tau(x) = rx + b$, $r > 0$, $b \in \mathbb{R}^n$, and some constant $w > 0$ such that $\tau(B) = B'$ and

$$\mu|_{B'} = w \cdot \mu|_B \circ \tau^{-1}. \tag{3-1}$$

It is easy to check that $\simeq_\mu$ is an equivalence relation.

Let $(\mathcal{P}_k)_{k\geq 1}$ be a sequence of μ-partitions of X, and let ν be a positive finite Borel measure on X. For each $k \geq 1$, let $\overline{m}_k = \overline{m}_k(\mathcal{P}_k) := \max\{\nu(B) : B \in \mathcal{P}_k\}$. We say that $(\mathcal{P}_k)_{k\geq 1}$ is *refining* with respect to ν if it satisfies the following conditions:

(1) $\{\overline{m}_k\}$ is nonincreasing and $\lim_{k\to\infty} \overline{m}_k = 0$.

(2) For any $B \in \mathcal{P}_k$ and any $B' \in \mathcal{P}_{k+1}$, either $B' \subseteq B$ or $(B')^\circ \cap B^\circ = \varnothing$.

Condition (2) means that each member of $\mathcal{P}_{k+1}$ is a subset of some member of $\mathcal{P}_k$.

3A. ***Proof of Theorem 1.2.*** We now prove Theorem 1.2 by modifying a method in [Reed and Simon 1978, Theorem XIII 79].

Proof of Theorem 1.2. Since X is compact and V is continuous, $-\Delta_\mu + \beta V$ has discrete spectrum on the negative real line for any $\beta > 0$, i.e., $N_\mu^-(\beta V)$ is finite for any $\beta > 0$. In fact, $N_\mu^-(\beta V) \le N_\mu^-(\beta V_{\min}) = N(-\beta V_{\min}, -\Delta_\mu) < \infty$, where $V_{\min} := \min\{V(x) : x \in X\}$. For each $k \ge 1$, let $\mathcal{P}_k := \{B_{k,\ell}\}_{\ell\in\Pi_k}$ and define $V_k^\vee$ (resp. $V_k^\wedge$) to be the piecewise constant function over each $B_{k,\ell}$ with the value $V_{k,\ell}^\vee := \min\{V(x) : x \in B_{k,\ell}\}$ (resp. $V_{k,\ell}^\wedge := \max\{V(x) : x \in B_{k,\ell}\}$).

(a) For $k \ge 1$, let $-\Delta_\mu^{k,D}$ be the Dirichlet Laplacian on the union of the interiors of the cells in $\mathcal{P}_k$. Since $C_c^\infty\big(\bigcup_{B\in\mathcal{P}_k} B^\circ\big) \subseteq C_c^\infty(X^\circ)$, we have $-\Delta_\mu \preccurlyeq -\Delta_\mu^{k,D}$ for $k \ge 1$. Combining this inequality with $V \le V_k^\wedge$, we have $-\Delta_\mu + \beta V \preccurlyeq -\Delta_\mu^{k,D} + \beta V_k^\wedge$ for $k \ge 1$ and $\beta > 0$. It follows from Proposition 2.2 that for all $k \ge 1$ and $\beta > 0$,

$$
\begin{aligned}
\text{(3-2)}\quad N_\mu^-(\beta V) \ge N(0, -\Delta_\mu^{k,D} + \beta V_k^\wedge) &= \sum_{\ell\in\Pi_k} N(0, -\Delta^D_{\mu|_{B_{k,\ell}}} + \beta V_{k,\ell}^\wedge) \\
&= \sum_{\ell\in\Pi_k} N(-\beta V_{k,\ell}^\wedge, -\Delta^D_{\mu|_{B_{k,\ell}}}) = \sum_{\{\ell\in\Pi_k : V_{k,\ell}^\wedge \le 0\}} N(-\beta V_{k,\ell}^\wedge, -\Delta^D_{\mu|_{B_{k,\ell}}}).
\end{aligned}
$$

Combining (1-8) and (3-2) yields, for each $k \ge 1$,

$$
\text{(3-3)}\qquad N_\mu^-(\beta V) \ge \beta^{\alpha/2}\Bigg(C \sum_{\{\ell\in\Pi_k : V_{k,\ell}^\wedge \le 0\}} (-V_{k,\ell}^\wedge)^{\alpha/2}\nu(B_{k,\ell}) + o(1)\Bigg) \quad \text{as } \beta \to \infty.
$$

The definition of refining implies that $\lim_{k\to\infty} \max\{\nu(B_{k,\ell}) : \ell \in \Pi_k\} = 0$. Moreover, it follows from the continuity of V that

$$
\text{(3-4)}\quad \lim_{k\to\infty} \sum_{\{\ell\in\Pi_k : V_{k,\ell}^b \le 0\}} (-V_{k,\ell}^b)^{\alpha/2}\nu(B_{k,\ell}) = \int_{D^-(V)} (-V)^{\alpha/2}\, d\nu \quad \text{for } b \in \{\vee, \wedge\},
$$

which, together with (3-3), yields the desired inequality.

(b) The proof is similar to that of part (a). Condition (N) implies that $-\Delta^N_{\mu|_B}$ is well defined for all $B \in \bigcup_{k\ge1}\mathcal{P}_k$. Thus the Neumann Laplacian $-\Delta_\mu^{k,N}$ on the union of the interiors of the cells in $\mathcal{P}_k$ is well defined for all $k \ge 1$. We note that $-\Delta_\mu^{k,N} \preccurlyeq -\Delta_\mu$ for $k \ge 1$. Hence $-\Delta_\mu^{k,N} + \beta V_k^\vee \preccurlyeq -\Delta_\mu + \beta V$ for all $k \ge 1$ and $\beta > 0$, which, together with Proposition 2.2, yields

$$
\begin{aligned}
N_\mu^-(\beta V) \le N(0, -\Delta_\mu^{k,N} + \beta V_k^\vee) &= \sum_{\ell\in\Pi_k} N(0, -\Delta^N_{\mu|_{B_{k,\ell}}} + \beta V_{k,\ell}^\vee) \\
&= \sum_{\ell\in\Pi_k} N(-\beta V_{k,\ell}^\vee, -\Delta^N_{\mu|_{B_{k,\ell}}}) = \sum_{\{\ell\in\Pi_k : V_{k,\ell}^\vee \le 0\}} N(-\beta V_{k,\ell}^\vee, -\Delta^N_{\mu|_{B_{k,\ell}}}).
\end{aligned}
$$

Thus (1-10) implies the following analog of (3-3), which holds for all $k \geq 1$,

$$N_\mu^-(\beta V) \leq \beta^{\alpha/2}\Bigg(C \sum_{\{\ell\in\Pi_k : V_{k,\ell}^\vee \leq 0\}} (-V_{k,\ell}^\vee)^{\alpha/2}\nu(B_{k,\ell}) + o(1)\Bigg) \quad \text{as } \beta \to \infty. \tag{3-5}$$

Hence, the assertion follows from (3-4) and (3-5). □

Let $X \subseteq \mathbb{R}^n$ be a compact subset with nonempty interior and μ be a positive finite Borel measure on $\mathbb{R}^n$ with $\operatorname{supp}(\mu) \subseteq X$ and $\mu(X^\circ) > 0$. It is well known that if $n = 1$ and a cell B is a closed interval, then B satisfies condition (N), and $N(\lambda, -\Delta_{\mu|_B}^D) \leq N(\lambda, -\Delta_{\mu|_B}^N) \leq N(\lambda, -\Delta_{\mu|_B}^D) + 2$ for all $\lambda \geq 0$ (see, for example, [Ngai 2011]). Thus $N(\lambda, -\Delta_{\mu|_B}^D)$ and $N(\lambda, -\Delta_{\mu|_B}^N)$ have the same asymptotic behavior as $\lambda \to \infty$. Consequently, the following remark holds.

Remark 3.1. Let $X = [a, b]$. If there exist positive constants C and α, and a refining μ-partition $(\mathcal{P}_k)_{k\geq 1}$ of X such that each $B \in \bigcup_{k=1}^\infty \mathcal{P}_k$ is a closed interval, and satisfies the reverse inequality in (1-8), then the conclusion of Theorem 1.2(b) holds.

A sufficient condition for condition (N) can be found in [Lau and Ngai $\geq$ 2019] for $n \geq 2$.

Let (E, ν) be a measure space with ν being a σ-finite Borel measure, and let $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ be a nonnegative closed quadratic form in $L^2(E, \nu)$ with generator A. By assuming that Sobolev's inequality holds for some $q > 2$, namely, there exists some constant $C > 0$ such that $\|u\|_{q,\nu}^2 \leq C\mathcal{E}(u, u)$ for all $u \in \operatorname{dom}\mathcal{E}$, Levin and Solomyak [1997, Theorem 1.2] proved the following *general Cwikel–Lieb–Rosenbljum* (*CLR*) *inequality*:

$$N(0, A - \beta V) \leq e^p C^p \beta^p \int_E V^p \, d\nu \qquad \text{for all } \beta > 0, \tag{3-6}$$

where $0 \leq V \in L^p(E, \nu)$ and $p := q/(q-2) > 1$. In the case $E = \mathbb{R}^n$, $n \geq 3$, μ is Lebesgue measure on $\mathbb{R}^n$, and the generator A is the Dirichlet Laplacian $-\Delta$ on $\mathbb{R}^n$, then (3-6) holds with $p = n/2$ and $C^{-1} = (n(n-2)/4)^{n/2}\omega_{n-1}$, where ω_{n-1} is the volume of the unit $(n-1)$-sphere in $\mathbb{R}^n$. In this case, (3-6) is called the *classical CLR inequality* (see [Rozenbljum 1972; Cwikel 1977; Lieb 1976; Reed and Simon 1978; Li and Yau 1983]). We give a simple corollary of the general CLR inequality (3-6).

Corollary 3.2. *Suppose μ is a continuous Borel probability measure on $\mathbb{R}$ with* $\operatorname{supp}(\mu) \subseteq [a, b]$, *and that* $(\mathcal{E}_D, \operatorname{dom}\mathcal{E}_D)$ *is defined as in* (2-2). *Assume* $0 \leq V \in L^p((a, b), \mu)$ *for some* $p > 1$. *Then,*

$$N(0, -\Delta_\mu - \beta V) \leq e^p (b-a)^p \beta^p \int_a^b V^p \, d\mu \qquad \textit{for all } \beta > 0.$$

Proof. For all $u \in H_0^1(a,b)$ and $x \in [a,b]$,

$$|u(x)| = |u(x) - u(a)| = \left| \int_a^x u'(t)\,dt \right| \le (b-a)^{1/2} \mathcal{E}_D(u,u)^{1/2}.$$

It follows that for all $q > 0$,

$$\left(\int_a^b |u(x)|^q \,d\mu \right)^{2/q} \le (b-a)\mathcal{E}_D(u,u),$$

and thus Sobolev's inequality holds with $C := b-a$. Setting $q := 2p/(p-1)$, and using the discussion above or [Levin and Solomyak 1997, Theorem 1.2], the desired inequality holds. □

We remark that Theorem 1.2(b) does not follow from [Levin and Solomyak 1997, Theorem 1.2], which requires Sobolev's inequality. For $n=1$, Corollary 3.2 implies that the general CLR inequality (3-6) holds for all $p > 1$. However, Theorem 1.2(b) does not follow from [Levin and Solomyak 1997, Theorem 1.2] in this case either, since the constant α in Theorem 1.2(b), which corresponds to the constant p in the general CLR inequality (3-6), could be less than or equal to 1. Precisely, we would like to have $\alpha = d_s(-\Delta_\mu) \le 1$ if $d_s(-\Delta_\mu)$ exists and $n = 1$, as in our examples below.

For the convenience of the reader, we state a slightly modified version of [Ngai 2011, Proposition 2.2(b)] below, which will be used later in this paper.

Proposition 3.3 [Ngai 2011, Proposition 2.2]. *Let $S : \mathbb{R} \to \mathbb{R}$ be a similitude, with Lipschitz constant r, such that $S[a,b] = [c,d]$, $S(a) = c$, and $S(b) = d$. Let ν be a continuous positive finite Borel measure on $[a,b]$ with* $\mathrm{supp}(\nu) \subseteq [a,b]$. *Assume that $[a,b] \simeq_{\nu,w,S} [c,d]$. Then $-\Delta_{\nu|_{[c,d]}} \approx (rw)^{-1} \cdot \big(-\Delta_{\nu|_{[a,b]}}\big)$.*

We now apply Theorem 1.2 to self-similar measures on $\mathbb{R}$. Let $\{S_i\}_{i=1}^m$, $m \ge 2$, be an IFS on $\mathbb{R}$, and let μ be a self-similar measure defined by $\{S_i\}_{i=1}^m$ and a probability vector $(p_i)_{i=1}^m$. For $k \ge 0$ and

$$\boldsymbol{i} = (i_1, \dots, i_k) \in \{1, \dots, m\}^k := \big\{(i_1, \dots, i_k) : i_j \in \{1, \dots, m\} \quad \text{for } j = 1, \dots, k\big\},$$

we use the standard notation

$$S_{\boldsymbol{i}} := S_{i_1} \circ \cdots \circ S_{i_k}, \qquad r_{\boldsymbol{i}} := r_{i_1} \cdots r_{i_k}, \qquad p_{\boldsymbol{i}} := p_{i_1} \cdots p_{i_k}$$

with $S_\varnothing := \mathrm{id}$, $r_\varnothing = p_\varnothing := 1$, where id is the identity map on $\mathbb{R}$. Assume that $\{S_i\}_{i=1}^m$ satisfies OSC with respect to an open set (a,b). Let $X = [a,b]$, and d_s be the unique solution of

$$\sum_{i=1}^m (p_i r_i)^{d_s/2} = 1,$$

where r_i is the contraction ratio of S_i. Solomyak and Verbitsky [1995] studied the asymptotic behavior of the eigenvalue counting function $N(\lambda, -\Delta_\mu)$ as $\lambda \to \infty$. They proved that there exist positive constants C_1, C_2 such that

$$C_1\lambda^{d_s/2} \le N(\lambda, -\Delta_\mu) \le C_2\lambda^{d_s/2} \qquad \text{for all sufficiently large } \lambda. \tag{3-7}$$

In particular, if at least one of the ratios $\ln(r_k p_k)/\ln(r_\ell p_\ell)$ is irrational for $k, \ell \in \{1, \dots, m\}$, then there exists some constant $C > 0$ such that $N(\lambda, -\Delta_\mu) \sim C\lambda^{d_s/2}$. The same holds for the Neumann Laplacian with the same constant C. We note that $d_s = d_s(-\Delta_\mu)$.

Proposition 3.4. *Use the notation above and let $\{S_i\}_{i=1}^m$ be an IFS on $\mathbb{R}$ satisfying OSC. Let ν be the self-similar measure defined by $\{S_i\}_{i=1}^m$ together with the probability vector $((p_i r_i)^{d_s/2})_{i=1}^m$. Then for any continuous function V on X,*

(a) *there exist positive constants C_1, C_2 such that for all sufficiently large β,*

$$C_1 \int_{D^-(V)} (-V)^{d_s/2}\,d\nu \le \frac{N_\mu^-(\beta V)}{\beta^{d_s/2}} \le C_2 \int_{D^-(V)} (-V)^{d_s/2}\,d\nu; \tag{3-8}$$

(b) *if at least one of the ratios $\ln(r_k p_k)/\ln(r_\ell p_\ell)$ is irrational for $k, \ell \in \{1, \dots, m\}$, then one may take $C_1 = C_2$ in* (3-8).

Proof. Using the discussion above, we see that (b) follows from (a). Thus, we only prove (a). For $k \ge 1$, let $\mathcal{P}_k := \{S_{\boldsymbol{i}}([a, b]) : \boldsymbol{i} \in \{1, \dots, m\}^k\}$. It is easy to see that $(\mathcal{P}_k)_{k\ge1}$ is a refining μ-partition of $[a, b]$ with respect to ν, and all cells in $\bigcup_{k\ge1} \mathcal{P}_k$ are closed intervals. Fix any $k \ge 1$ and any $\boldsymbol{i} \in \{1, \dots, m\}^k$. OSC implies that $\mu|_{S_{\boldsymbol{i}}([a,b])} = p_{\boldsymbol{i}}\mu|_{[a,b]} \circ S_{\boldsymbol{i}}^{-1}$ on $S_{\boldsymbol{i}}([a, b])$. It follows that $[a, b] \simeq_{\mu, p_{\boldsymbol{i}}, S_{\boldsymbol{i}}} S_{\boldsymbol{i}}([a, b])$ and $\mu(S_{\boldsymbol{i}}([a, b])) = p_{\boldsymbol{i}}$. In view of Proposition 3.3, we get $N(\lambda, -\Delta_{\mu|_{S_{\boldsymbol{i}}([a,b])}}) = N(r_{\boldsymbol{i}} p_{\boldsymbol{i}}\lambda, -\Delta_\mu)$. Combining this with (3-7), we see that there exist positive constants C_1, C_2 such that, for all sufficiently large λ,

$$C_1(r_{\boldsymbol{i}} p_{\boldsymbol{i}})^{d_s/2}\lambda^{d_s/2} \le N(\lambda, -\Delta_{\mu|_{S_{\boldsymbol{i}}([a,b])}}) \le C_2(r_{\boldsymbol{i}} p_{\boldsymbol{i}})^{d_s/2}\lambda^{d_s/2}.$$

Since $(r_{\boldsymbol{i}} p_{\boldsymbol{i}})^{d_s/2} = \nu(S_{\boldsymbol{i}}([a, b]))$, for all sufficiently large λ,

$$C_1\nu(S_{\boldsymbol{i}}([a, b]))\lambda^{d_s/2} \le N\big(\lambda, -\Delta_{\mu|_{S_{\boldsymbol{i}}([a,b])}}\big) \le C_2\nu(S_{\boldsymbol{i}}([a, b]))\lambda^{d_s/2},$$

which, together with Theorem 1.2 and Remark 3.1, implies the desired result. □

3B. ***A class of self-similar measures satisfying EFT.*** In this subsection, we consider the following family of IFSs:

$$S_1(x) = r_1x, \quad S_2(x) = r_2x + r_1(1 - r_2), \quad S_3(x) = r_2x + 1 - r_2, \tag{3-9}$$

where the contraction ratios $r_1, r_2 \in (0, 1)$ satisfy $r_1 + 2r_2 - r_1r_2 \le 1$; that is, $S_2(1) \le S_3(0)$. The Hausdorff dimension of the self-similar sets is computed in

[Lau and Wang 2004]. The multifractal properties and spectral dimension of the corresponding self-similar measures were recently studied in [Deng and Ngai 2017; Ngai et al. 2018].

Let μ be a self-similar measure defined by an IFS in (3-9) and a probability vector $(p_i)_{i=1}^3$, and $-\Delta_\mu$ be the associated Dirichlet Laplacian with respect to μ. We note that $X := \operatorname{supp}(\mu) \subseteq [0, 1]$. Let d_s be the unique solution of

$$(1-(p_2r_2)^{d_s/2})(1-(p_3r_2)^{d_s/2})\sum_{k=0}^{\infty}\bigl(w_1(k)r_1r_2^k\bigr)^{d_s/2}+\bigl(p_2^{d_s/2}+p_3^{d_s/2}\bigr)r_2^{d_s/2}=1, \tag{3-10}$$

where $w_1(k) := p_1 \sum_{i=0}^{k} p_2^{k-i} p_3^i$. [Ngai et al. 2018, Theorem 1.2] implies that there exist some positive constants C_1, C_2 such that, for $k = 0, 1$,

$$C_1\lambda^{d_s/2} \le N(\lambda, -\Delta_{\mu|_{B_{1,k}}}) \le C_2\lambda^{d_s/2} \qquad \text{for all sufficiently large } \lambda, \tag{3-11}$$

where $B_{1,1} := S_1(X) \cup S_2(X)$ and $B_{1,0} := S_3(X)$. In particular, $d_s = d_s(-\Delta_\mu)$.

In order to define a sequence of refining μ-partitions of [0, 1] with respect to μ, we adopt the definition of an island from [Ngai et al. 2018]. Let $\mathcal{M}_k := \{1, 2, 3\}^k$ for $k \ge 1$ and $\mathcal{M}_0 := \varnothing$. A closed subset $B \subseteq [0, 1]$ is called a *level-k island* with respect to $\{\mathcal{M}_k\}$ if the following conditions hold:

(a) There exists a finite sequence of indexes $\boldsymbol{i}_0, \boldsymbol{i}_1, \dots, \boldsymbol{i}_n$ in $\mathcal{M}_k$ such that $S_{\boldsymbol{i}_k}(0, 1) \cap S_{\boldsymbol{i}_{k+1}}(0, 1) \neq \varnothing$ for all $k = 0, \dots, n-1$, and $B = \bigcup_{k=0}^{n} S_{\boldsymbol{i}_k}([0, 1])$.

(b) $S_{\boldsymbol{j}}(0, 1) \cap S_{\boldsymbol{i}_k}(0, 1) = \varnothing$ for any $\boldsymbol{j} \in \mathcal{M}_k \setminus \{\boldsymbol{i}_0, \dots, \boldsymbol{i}_n\}$ and any $k \in \{0, \dots, n\}$.

Intuitively, for each level-k island B, B° is a connected component of $S_{\mathcal{M}_k}(0, 1) := \bigcup_{\boldsymbol{i}\in\mathcal{M}_k} S_{\boldsymbol{i}}(0, 1)$ (see Figure 1). For $k \ge 1$, define

$$\mathcal{P}_k := \{B : B \text{ is a level-}k \text{ island with respect to } \{\mathcal{M}_k\}\}. \tag{3-12}$$

We note that $\mathcal{P}_1 = \{B_{1,1}, B_{1,0}\}$ (see Figure 1). It is easy to see that $(\mathcal{P}_k)_{k\ge1}$ is a sequence of μ-partitions of [0, 1]. By the proof of [Ngai et al. 2018, Example 3.3], $(\mathcal{P}_k)_{k\ge1}$ is refining with respect to μ; moreover, for any $k \ge 1$ and any $B \in \mathcal{P}_k$, if B is not μ-equivalent to $B_{1,i}$, $i = 0, 1$, then for any $\ell \ge 1$, there exists some subset

$$\mathcal{B}_\ell := \Bigl(\bigcup_{i=1}^{\ell}\bigl\{B_{i,0}^*, B_{i,1}^*\bigr\}\Bigr) \cup \bigl\{B_\ell^*\bigr\}, \tag{3-13}$$

of $\bigcup_{k\ge1}\mathcal{P}_k$ satisfying the following conditions:

(i) $B_\ell^* \in \mathcal{P}_{k+\ell}$ is not μ-equivalent to $B_{1,j}$, $j = 0, 1$, and $\mu(B_\ell^*) \to 0$ as $\ell \to \infty$.

(ii) For $1 \le i \le \ell$, $\{B_{i,0}^*, B_{i,1}^*\} \subseteq \mathcal{P}_{k+i}$, and for $m \in \{0, 1\}$, there exists some $j \in \{0, 1\}$ such that $B_{i,m}^* \simeq_\mu B_{1,j}$.

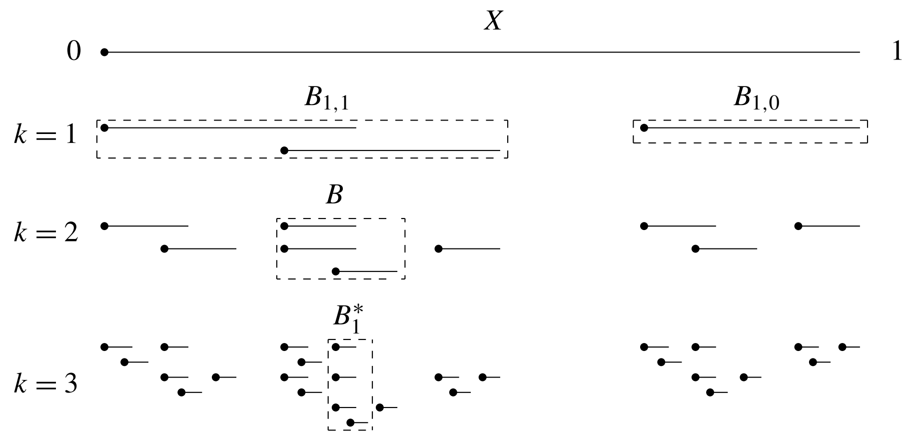

Figure 1. μ-partitions $\mathcal{P}_k$ for $k = 1, 2, 3$, $B := S_2(B_{1,1})$, and $B_\ell^* := S_{2^{\ell+1}}(B_{1,1})$, where $\mathcal{P}_k$ is defined as in (3-12). Cells that are labeled consist of line segments enclosed by a box. The figure is drawn with $r_1 = 1/3$ and $r_2 = 2/7$.

For example, if $B := S_2(B_{1,1})$, then B is not μ-equivalent to $B_{1,i}$, $i = 0$, 1. Moreover, for any $\ell \geq 1$,

$$\mathcal{B}_\ell := \left(\bigcup_{i=1}^{\ell} \bigl\{ S_{2^i 1}(B_{1,1}),\ S_{2^{i+1}}(B_{1,0}) \bigr\} \right) \cup \{ S_{2^{\ell+1}}(B_{1,1}) \}$$

satisfies conditions (i) and (ii) (see Figure 1).

Proposition 3.5. *Use the notation above. Let ν be a positive finite Borel measure on $\mathbb{R}$ and assume that* $\max\{\nu(B) : B \in \mathcal{P}_k\} \to 0$ *as $k \to \infty$. Let d_s be defined as in* (3-10) *and let* $\mathcal{P}_* := \{B \in \mathcal{P}_k : k \geq 1$ *and* $B \simeq_\mu B_{1,i}$ *for some* $i \in \{0, 1\}\}$.

(a) *If there exists some constant $c > 0$ such that*

$$(|B|\,\mu(B))^{d_s/2} \geq c\nu(B) \qquad \text{for all } B \in \mathcal{P}_*, \tag{3-14}$$

then there exists some constant $C > 0$ such that

$$N_\mu^-(\beta V) \geq C\beta^{d_s/2} \left(\int_{D^-(V)} (-V)^{d_s/2}\, d\nu + o(1) \right) \qquad \text{as } \beta \to \infty. \tag{3-15}$$

(b) *The reverse inequality in* (3-15) *holds if* (3-14) *holds with the inequality being reversed.*

Proof. (a) Since $(\mathcal{P}_k)_{k\geq 1}$ is refining with respect to μ and $\max\{\nu(B) : B \in \mathcal{P}_k\} \to 0$ as $k \to \infty$, $(\mathcal{P}_k)_{k\geq 1}$ is refining with respect to ν. In view of Theorem 1.2(a), it suffices to show that for all $B \in \bigcup_{k\geq 1} \mathcal{P}_k$,

$$N(\lambda, -\Delta_{\mu|_B}) \geq \lambda^{d_s/2}(C\nu(B) + o(1)) \qquad \text{as } \lambda \to \infty. \tag{3-16}$$

Assume that $B \in \mathcal{P}_*$. Then there exists a unique number $w > 0$, a unique $\kappa \in \{0, 1\}$, and a unique similitude τ with contraction ratio r_τ such that $B_{1,\kappa} \simeq_{\mu,w,\tau} B$. Thus $\mu(B) = w\mu(B_{1,\kappa})$, $|B| = r_\tau |B_{1,\kappa}|$, and Proposition 3.3 implies that $N(\lambda, -\Delta_{\mu|_B}) = N(wr_\tau\lambda, -\Delta_{\mu|_{B_{1,\kappa}}})$. Combining these equalities with (3-11) and (3-14), we obtain positive constants C_1, C_2 such that

$$\begin{aligned} N(\lambda, -\Delta_{\mu|_B}) &\geq C_1(wr_\tau)^{d_s/2}\lambda^{d_s/2} = C_1\left(\frac{\mu(B)}{\mu(B_{1,\kappa})} \cdot \frac{|B|}{|B_{1,\kappa}|}\right)^{d_s/2} \cdot \lambda^{d_s/2} \\ &\geq C_2\lambda^{d_s/2}\nu(B) \quad \text{for sufficiently large } \lambda, \end{aligned} \tag{3-17}$$

proving (3-16) for $B \in \mathcal{P}_*$.

On the other hand, assume $B \in \bigcup_{k\geq 1} \mathcal{P}_k$ but $B \notin \mathcal{P}_*$. Let $\mathcal{B}_\ell$ be defined as in (3-13) satisfying conditions (i) and (ii) in the paragraph preceding this proposition for $\ell \geq 1$. By assumption, we have $\nu(B_\ell^*) \to 0$ as $\ell \to \infty$, and thus

$$\nu(B) = \sum_{i=1}^{\infty}\sum_{j=0}^{1} \nu(B_{i,j}^*). \tag{3-18}$$

Using calculations from [Ngai et al. 2018, Sections 4 and 5], we get, as $\lambda \to \infty$,

$$N(\lambda, -\Delta_{\mu|_B}) = \sum_{i=1}^{\infty}\sum_{j=0}^{1} N(\lambda, -\Delta_{\mu|_{B_{i,j}^*}}) + \lambda^{d_s/2}o(1). \tag{3-19}$$

Combining (3-19) with (3-17) and (3-18), we obtain, as $\lambda \to \infty$,

$$N(\lambda, -\Delta_{\mu|_B}) \geq C_2\lambda^{d_s/2}\left(\sum_{i=1}^{\infty}\sum_{j=0}^{1} \nu(B_{i,j}^*) + o(1)\right) = C_2\lambda^{d_s/2}(\nu(B) + o(1)).$$

Finally, (3-16) holds for all $B \in \bigcup_{k\geq 1} \mathcal{P}_k$, which completes the proof.

(b) The proof is similar to that of part (a). If (3-14) holds with the inequality being reversed, then the same is true for (3-17). Consequently, the desired inequality holds. □

We now give a sufficient condition for the reverse inequality in (3-14) to hold.

Remark 3.6. Use the notation in Proposition 3.5. If $(r_1 p_1)^{d_s/2} \leq p_1$, $p_2 = p_3$ and $(r_2 p_2)^{d_s/2} \leq p_2$, then there exists some constant $c > 0$ such that

$$(|B|\,\mu(B))^{d_s/2} \leq c\mu(B) \qquad \text{for all } B \in \mathcal{P}_*.$$

Proof. Let c be a positive constant such that

$$(|B|\,\mu(B))^{d_s/2} \le c\mu(B) \qquad \text{for } B \in \{B_{1,0}, B_{1,1}\}. \tag{3-20}$$

By assumption, $w_1(j) = p_1 \sum_{i=0}^{j} p_2^{j-i} p_3^i = p_1(j+1)p_2^j$ for $j \ge 0$. Using the assumptions $(r_1p_1)^{d_s/2} \le p_1$ and $(r_2p_2)^{d_s/2} \le p_2$, we have

$$\begin{aligned}(r_1r_2^j w_1(j))^{d_s/2} &= (r_1p_1)^{d_s/2} \cdot ((j+1)(p_2r_2)^j)^{d_s/2} \le p_1(j+1)^{d_s/2} p_2^j \\ &\le p_1(j+1)p_2^j = w_1(j),\end{aligned} \tag{3-21}$$

where the last inequality uses the fact $d_s/2 < 1$. Fix any $B \in \mathcal{P}_*$. By the definition of $\mathcal{P}_*$, there exist a unique $k_0 \in \{0, 1\}$, $w > 0$, and $\boldsymbol{i} \in \bigcup_{k\ge0}\{1, 2, 3\}^k$ such that $B_{1,k_0} \simeq_{\mu,w,S_{\boldsymbol{i}}} B$. Let $r_{\boldsymbol{i}}$ be the contraction ratio of $S_{\boldsymbol{i}}$. By the definition of $\simeq_\mu$, $|B| = r_{\boldsymbol{i}}|B_{1,k_0}|$ and $\mu(B) = w\mu(B_{1,k_0})$. From the proofs of [Ngai et al. 2018, Lemma 3.5 and Example 3.3], we see that w can be expressed as $w = w_1(i)p_1^j p_2^\ell p_3^{k-1-i-j-\ell} = w_1(i)p_1^j p_2^{k-1-i-j}$ for some $i, j, \ell \in \{0, 1, \dots, k\}$. In this particular case, $r_{\boldsymbol{i}} = r_1^{j+1} r_2^{k-j-1}$. Hence,

$$\begin{aligned}(|B|\,\mu(B))^{d_s/2} &= (r_{\boldsymbol{i}}|B_{1,k_0}| \cdot w\mu(B_{1,k_0}))^{d_s/2} \\ &= \big(|B_{1,k_0}|\mu(B_{1,k_0}) \cdot r_1r_2^i w_1(i) \cdot (p_1r_1)^j \cdot ((p_2r_2)^{k-1-i-j})\big)^{d_s/2} \\ &\le c\mu(B_{1,k_0})w_1(i)p_1^j p_2^{k-1-i-j} = cw\mu(B_{1,k_0}) = c\mu(B),\end{aligned}$$

where we have used (3-20), (3-21), and the assumptions to get the inequality. This completes the proof. □

4. Bohr's formula for Schrödinger operators with locally bounded potentials

Let $X \subseteq \mathbb{R}^n$ $(n \ge 1)$ be a compact subset with nonempty interior, and μ be a positive finite Borel measure on X such that $\mu(X^\circ) > 0$ and $\operatorname{supp}(\mu) \subseteq X$. We extend X to $X_\infty := \bigcup_{i\in I} X_i$ as described in Section 1 so that conditions (C1)–(C3) are satisfied. For each $i \in I$, let $\tau_i(x) = x + b_i$, $b_i \in \mathbb{R}^n$, be the similitude in condition (C2), and $\mu_i := \mu \circ \tau_i^{-1}$. Also, let μ_∞ be a positive measure on X_∞ defined as in (1-13). Assume that $\mu_\infty(X_i \cap X_j) = 0$ for any distinct $i, j \in I$.

We first give a simple proposition.

Proposition 4.1. *Let $(\mu_i)_{i\in I}$, $(X_i)_{i\in I}$, X_∞, and μ_∞ be defined as above. Assume that μ satisfies MPI, and let $-\Delta_\mu$ be the Dirichlet Laplacian with respect to μ. Then*

(a) *for any $i \in I$, the Dirichlet Laplacian $-\Delta_{\mu_i}$ with respect to μ_i is well defined and $-\Delta_{\mu_i} \approx -\Delta_\mu$;*

(b) *$-\Delta_{\mu_\infty} := \bigoplus_{i\in I}(-\Delta_{\mu_i})$ is a nonnegative self-adjoint operator in $L^2(X_\infty, \mu_\infty)$.*

Proof. Part (a) can be proved by verifying MPI and using a similar argument as that in [Ngai 2011, Lemma 2.1]. Part (b) follows from the facts that $L^2(X_\infty, \mu_\infty) = \bigoplus_{i\in I} L^2(X_i, \mu_i)$ and that $-\Delta_{\mu_i}$ is a nonnegative self-adjoint operator in $L^2(X_i, \mu_i)$ for all $i \in I$. We omit the details. □

In the rest of this section, we assume that μ satisfies MPI, and let $-\Delta_{\mu_\infty}$ be as in Proposition 4.1(b).

Theorem 4.2. *Use the notation in Proposition 4.1 and assume that V is a locally bounded nonnegative piecewise continuous function on X_∞ so that $V(x) \to \infty$ as $|x| \to \infty$. Then the Schrödinger operator $-\Delta_{\mu_\infty} + V$, defined as a sum of quadratic forms, is a nonnegative self-adjoint operator in $L^2(X_\infty, \mu_\infty)$ and has compact resolvent.*

Proof. Let $\mathcal{D} := \{(u_i)_{i\in I} \in L^2(X_\infty, \mu_\infty) : u_i \in C_c^\infty(X_i^\circ) \text{ for all } i \in I\}$. It follows from the fact

$$\mathcal{D} \subseteq \mathrm{dom}_F(-\Delta_{\mu_\infty}) \cap \mathrm{dom}_F(V)$$

is dense in $L^2(X_\infty, \mu_\infty)$ that $-\Delta_{\mu_\infty} + V$, defined as a sum of quadratic forms, is a nonnegative self-adjoint operator in $L^2(X_\infty, \mu_\infty)$. The remaining assertion holds by using the proof of [Reed and Simon 1978, Theorem XIII.16] and [Reed and Simon 1978, Theorem XIII.64]. □

Let V be a locally bounded nonnegative piecewise continuous function on X_∞ such that $V(x) \to \infty$ as $|x| \to \infty$. Then $-\Delta_{\mu_i} + V|_{X_i}$ is a nonnegative self-adjoint operator in $L^2(X_i, \mu_i)$ for all $i \in I$. Proposition 4.1(b) implies that $-\Delta_{\mu_\infty} + V = \bigoplus_{i\in I}(-\Delta_{\mu_i} + V|_{X_i})$. It follows that

$$N(\lambda, -\Delta_{\mu_\infty} + V) = \sum_{i\in I} N(\lambda, -\Delta_{\mu_i} + V|_{X_i}) \qquad \text{for all } \lambda > 0. \tag{4-1}$$

Let $V^\wedge$ (resp. $V^\vee$) be the piecewise constant function which takes the value $\sup_{x\in X_i} V(x)$ (resp. $\inf_{x\in X_i} V(x)$) on X_i. By applying Theorem 4.2 to V^b for $b \in \{\vee, \wedge\}$, we see that $-\Delta_{\mu_\infty} + V^b$ is a nonnegative self-adjoint operator in $L^2(X_\infty, \mu_\infty)$. Note that σ is an eigenvalue of $-\Delta_{\mu_i} + V^b|_{X_i}$ with eigenfunction φ if and only if $\sigma - V^b|_{X_i}$ is an eigenvalue of $-\Delta_{\mu_i}$ with the same eigenfunction. Hence,

$$N(\lambda, -\Delta_{\mu_i} + V^b|_{X_i}) = N(\lambda - V^b|_{X_i}, -\Delta_{\mu_i}). \tag{4-2}$$

This allows us to relate the eigenvalue counting function of the Schrödinger operator to that of the Laplacian. Since $0 \le V^\vee \le V \le V^\wedge$, we have $-\Delta_{\mu_\infty} + V^\vee \preccurlyeq -\Delta_{\mu_\infty} + V \preccurlyeq -\Delta_{\mu_\infty} + V^\wedge$, and thus, for all $\lambda > 0$,

$$N(\lambda, -\Delta_{\mu_\infty} + V^\wedge) \le N(\lambda, -\Delta_{\mu_\infty} + V) \le N(\lambda, -\Delta_{\mu_\infty} + V^\vee). \tag{4-3}$$

As in (4-1), for $b \in \{\vee, \wedge\}$,

$$\begin{aligned}(4\text{-}4)\quad N(\lambda, -\Delta_{\mu_\infty}+V^b) &= \sum_{i\in I} N(\lambda, -\Delta_{\mu_i}+V^b|_{X_i}) = \sum_{i\in I} N(\lambda - V^b|_{X_i}, -\Delta_{\mu_i}) \\ &= \sum_{i\in I} N(\lambda - V^b|_{X_i}, -\Delta_\mu) \\ &= \sum_{\{i\in I: V^b|_{X_i}\le\lambda\}} N(\lambda - V^b|_{X_i}, -\Delta_\mu),\end{aligned}$$

where (4-2) and Proposition 4.1(a) are used in the second and third equality, respectively.

Define $B(x,r) := \{y \in X_\infty : |x-y| < r\}$. The following theorem gives the existence of spectral dimension of $-\Delta_{\mu_\infty} + V$. A similar result was obtained by Chen et al. [2015]. We replace their assumption on the Ahlfors-regularity of μ_∞ by a more general condition.

Theorem 4.3. *Use the notation in Theorem 4.2. Let $B(x,r)$ and V^b, $b \in \{\vee, \wedge\}$, be defined as above. Assume that* (W1) *holds, and that there exist positive constants c_1, c_2, c_3, θ such that*

$$(4\text{-}5)\qquad c_1|x|^\theta \le V(x) \le c_2|x|^\theta \qquad \textit{for all } x \in X_\infty \textit{ with sufficiently large } |x|,$$

and that $\mu_\infty(B(0,2r)) \le c_3\mu_\infty(B(0,r))$ for all sufficiently large r. Then there exist positive constants C, C_1, C_2 such that for all sufficiently large λ,

$$F(2\lambda, V^\vee) \le C F(\lambda, V^\wedge)$$

and

$$C_1\lambda^{d_s/2} F(\lambda, V) \le N(\lambda, -\Delta_{\mu_\infty} + V) \le C_2\lambda^{d_s/2} F(\lambda, V),$$

where $F(\cdot,\cdot)$ is defined as in (1-14) *and d_s comes from* (W1).

Proof. Fix any $b \in \{\vee, \wedge\}$. Since $V^b|_{X_i}$ is a constant for any $i \in I$, we see that

$$(4\text{-}6)\quad F(\lambda, V^b) = \sum_{\{i\in I: V^b|_{X_i}\le\lambda\}} \mu_\infty(X_i) = \mu(X)\cdot \#\{i \in I : V^b|_{X_i} \le \lambda\} \quad \text{for } \lambda > 0.$$

By (W1), there exist positive constants c_4, c_5, M_0 such that $c_4\lambda^{d_s/2} \le N(\lambda, -\Delta_\mu) \le c_5\lambda^{d_s/2}$ for all $\lambda > M_0$. Thus,

$$N(\lambda - V^b|_{X_i}, -\Delta_\mu) \le N(\lambda, -\Delta_\mu) \le c_5\lambda^{d_s/2} \quad \text{for all } \lambda > M_0 \text{ and any } i \in I,$$

while for all $\lambda > 2M_0$ and $i \in I$ such that $V^b|_{X_i} \le \lambda/2$,

$$N(\lambda - V^b|_{X_i}, -\Delta_\mu) \ge N(\lambda/2, -\Delta_\mu) \ge (c_4 2^{-d_s/2})\cdot\lambda^{d_s/2}.$$

Combining these estimates with (4-4) and (4-6), we get

$$\begin{aligned}N(\lambda, -\Delta_{\mu_\infty} + V^b) &\le c_5 \lambda^{d_s/2} \#\{i \in I : V^b|_{X_i} \le \lambda\}\\ &= (c_5/\mu(X)) \cdot \lambda^{d_s/2} F(\lambda, V^b) \qquad \text{for all } \lambda > M_0, \text{ and,}\\ N(\lambda, -\Delta_{\mu_\infty} + V^b) &\ge \sum_{\{i \in I : V^b|_{X_i} \le \lambda/2\}} N(\lambda - V^b|_{X_i}, -\Delta_\mu)\\ &\ge (c_4 2^{-d_s/2}) \cdot \lambda^{d_s/2} \#\{i \in I : V^b|_{X_i} \le \lambda/2\}\\ &= (c_4 2^{-d_s/2}/\mu(X)) \cdot \lambda^{d_s/2} F(\lambda/2, V^b) \qquad \text{for all } \lambda > 2M_0.\end{aligned}$$

It follows that there exist constants $c_6, c_7 > 0$ such that for all $\lambda > 2M_0$,

$$c_6 \lambda^{d_s/2} F(\lambda/2, V^b) \le N(\lambda, -\Delta_{\mu_\infty} + V^b) \le c_7 \lambda^{d_s/2} F(\lambda, V^b). \tag{4-7}$$

By the definition of $F(\cdot\,,\cdot)$, $F(\lambda/2, V^\wedge) \le F(\lambda, V) \le F(\lambda, V^\vee)$ for all $\lambda > 0$. Using (4-3), we have

$$\frac{N(\lambda, -\Delta_{\mu_\infty} + V^\wedge)}{\lambda^{d_s/2} F(\lambda, V^\vee)} \le \frac{N(\lambda, -\Delta_{\mu_\infty} + V)}{\lambda^{d_s/2} F(\lambda, V)} \le \frac{N(\lambda, -\Delta_{\mu_\infty} + V^\vee)}{\lambda^{d_s/2} F(\lambda/2, V^\wedge)} \qquad \text{for all } \lambda > 0,$$

which, together with (4-7), gives

$$c_6 \frac{F(\lambda/2, V^\wedge)}{F(\lambda, V^\vee)} \le \frac{N(\lambda, -\Delta_{\mu_\infty} + V)}{\lambda^{d_s/2} F(\lambda, V)} \le c_7 \frac{F(\lambda, V^\vee)}{F(\lambda/2, V^\wedge)} \qquad \text{for all } \lambda > 2M_0. \tag{4-8}$$

Using (4-5), we obtain positive constants r_0, c_8, c_9 such that

$$c_8 |x|^\theta \le V^\vee(x) \le V^\wedge(x) \le c_9 |x|^\theta \qquad \text{for all } x \in X_\infty \text{ with } |x| > r_0.$$

Define $D_0 := \sup\{V^\wedge(x) : x \in X_\infty \text{ such that } |x| \le r_0\}$. Then for all $\lambda > 2D_0$,

$$\begin{aligned}F(\lambda/2, V^\wedge) &\ge \mu_\infty\bigl(\{x \in X_\infty : c_9|x|^\theta \le \lambda/2\}\bigr) = \mu_\infty(B(0, c_{10}\lambda^{1/\theta})), \text{ and}\\ F(\lambda, V^\vee) &\le \mu_\infty\bigl(\{x \in X_\infty : c_8|x|^\theta \le \lambda\}\bigr) = \mu_\infty(B(0, c_{11}\lambda^{1/\theta})),\end{aligned}$$

where $c_{10} := (2c_9)^{-1/\theta}$ and $c_{11} := c_8^{-1/\theta}$. Moreover, in view of the assumption that $\mu_\infty(B(0, 2r)) \le c_3 \mu_\infty(B(0, r))$ for all sufficiently large r, we have

$$\mu_\infty(B(0, c_{11}\lambda^{1/\theta})) \le c_3^{m_0} \mu_\infty(B(0, 2^{-m_0} c_{11}\lambda^{1/\theta})) \le c_3^{m_0} \mu_\infty(B(0, c_{10}\lambda^{1/\theta}))$$

for all sufficiently large λ, where $m_0 := \min\{i \in \mathbb{Z} : i \ge \ln(c_{11}/c_{10})/\ln 2\}$. Thus $F(\lambda, V^\vee) \le c_3^{m_0} F(\lambda/2, V^\wedge)$ for all sufficiently large λ. Combining this inequality with (4-8), we get, for all sufficiently large λ,

$$c_3^{-m_0} c_6 \lambda^{d_s/2} F(\lambda, V) \le N(\lambda, -\Delta_{\mu_\infty} + V) \le c_3^{m_0} c_7 \lambda^{d_s/2} F(\lambda, V),$$

which completes the proof. □

Assume (W2) holds with a finite collection of closed subsets $\{Y_j\}_{j\in J}$ and spectral dimension d_s. Hence, we can obtain the following analogs of Proposition 4.1 and Theorem 4.2.

Remark 4.4. Use the notation in Proposition 4.1. Assume that μ satisfies MPI and (W2). Let $X_{\infty,j}$ and $\mu_{\infty,j}$ be defined as in (1-15) for $j \in J$. Then for all $j \in J$,

(a) $-\Delta_{\mu_i|_{\tau_i(Y_j)}} \approx -\Delta_{\mu|_{Y_j}}$ for any $i \in I$;

(b) the operator $-\Delta_{\mu_{\infty,j}} := \bigoplus_{i\in I}(-\Delta_{\mu_i|_{\tau_i(Y_j)}})$ is nonnegative and self-adjoint in $L^2(X_{\infty,j}, \mu_{\infty,j})$;

(c) $-\Delta_{\mu_{\infty,j}} + V|_{X_{\infty,j}}$ is a nonnegative self-adjoint operator in $L^2(X_{\infty,j}, \mu_{\infty,j})$ with compact resolvent, where V is given as in Theorem 4.2.

Replacing Proposition 4.1(b) and Theorem 4.2 by Remark 4.4 (b) and (c), respectively, we can also obtain analogs of (4-1) and (4-4) as follows. For all $j \in J$ and $b \in \{\vee, \wedge\}$,

$$N(\lambda, -\Delta_{\mu_{\infty,j}} + V|_{X_{\infty,j}}) = \sum_{i\in I} N(\lambda, -\Delta_{\mu_i|_{\tau_i(Y_j)}} + V|_{\tau_i(Y_j)}), \tag{4-9}$$

$$N(\lambda, -\Delta_{\mu_{\infty,j}} + V^b|_{X_{\infty,j}}) = \sum_{\{i\in I : V^b|_{X_i}\le\lambda\}} N(\lambda - V^b|_{X_i}, -\Delta_{\mu|_{Y_j}}). \tag{4-10}$$

Fix $j \in J$ and $b \in \{\vee, \wedge\}$. Define

$$R_j(\lambda, V^b) := \sum_{\{i\in I : V^b|_{X_i}\le\lambda\}} (\lambda - V^b|_{X_i})^{d_s/2} R_j(\lambda - V^b|_{X_i}), \tag{4-11}$$

where $R_j(\,\cdot\,)$ is the remainder term in (1-12). Let $g_j(\,\cdot\,,\cdot\,)$ be defined as in (1-16) for $j \in J$. We first observe that

$$g_j(\lambda, V^b) = \sum_{\{i\in I : V^b|_{X_i}\le\lambda\}} (\lambda - V^b|_{X_i})^{d_s/2} G_j(\ln(\lambda - V^b|_{X_i})). \tag{4-12}$$

Thus $\lim_{\lambda\to\infty} R_j(\lambda, V^b)/g_j(\lambda, V^b) = 0$, and using (4-10) and (1-12), we have $N(\lambda, -\Delta_{\mu_{\infty,j}} + V^b|_{X_{\infty,j}}) = g_j(\lambda, V^b) + R_j(\lambda, V^b)$ as $\lambda \to \infty$. It follows that

$$\lim_{\lambda\to\infty} \frac{N(\lambda, -\Delta_{\mu_{\infty,j}} + V^b|_{X_{\infty,j}})}{g_j(\lambda, V^b)} = \lim_{\lambda\to\infty} \frac{g_j(\lambda, V^b) + R_j(\lambda, V^b)}{g_j(\lambda, V^b)} = 1. \tag{4-13}$$

The following theorem is slightly modified from a similar one in [Chen et al. 2015], in order to suit our purpose. We include a proof for completeness.

Theorem 4.5 [Chen et al. 2015, Theorem 2.11]. *Use the notation in Remark 4.4. Let V be a locally bounded nonnegative piecewise continuous function on X_∞ such that $V(x) \to \infty$ as $|x| \to \infty$. Assume that* (W2) *and* (1-17) *hold. Let $F(\,\cdot\,,\cdot\,)$*

and $g_j(\cdot,\cdot)$, $j \in J$, *be defined as in* (1-14) *and* (1-16), *respectively. Then for each* $j \in J$,

$$N(\lambda, -\Delta_{\mu_{\infty,j}} + V|_{X_{\infty,j}}) \sim g_j(\lambda, V) \qquad \text{as } \lambda \to \infty. \tag{4-14}$$

Proof. Fix any $j \in J$. We claim that

$$g_j(\lambda, V^\vee)/g_j(\lambda, V^\wedge) = 1 + o(1) \qquad \text{as } \lambda \to \infty. \tag{4-15}$$

Define $F_j(\lambda, V^b) := \mu_\infty(\{x \in X_{\infty,j} : V^b(x) \le \lambda\})$ for $b \in \{\vee, \wedge\}$. Similar to (4-6), we get $F_j(\lambda, V^b) = \mu(Y_j) \cdot \#\{i \in I : V^b|_{X_i} \le \lambda\}$ for $b \in \{\vee, \wedge\}$ and $\lambda > 0$. This, together with (4-6), yields $F_j(\lambda, V^\vee)/F_j(\lambda, V^\wedge) = F(\lambda, V^\vee)/F(\lambda, V^\wedge)$. By [Chen et al. 2015, Proposition 4.2], if $F_j(\lambda, V^\vee) \sim F_j(\lambda, V^\wedge)$ as $\lambda \to \infty$, then (4-15) holds. The claim follows by combining these observations with (1-17). Combining (4-15) and (4-13), we get

$$\lim_{\lambda\to\infty} \frac{N(\lambda, -\Delta_{\mu_{\infty,j}} + V^\vee|_{X_{\infty,j}})}{g_j(\lambda, V^\wedge)} = \lim_{\lambda\to\infty} \frac{N(\lambda, -\Delta_{\mu_{\infty,j}} + V^\wedge|_{X_{\infty,j}})}{g_j(\lambda, V^\vee)} = 1. \tag{4-16}$$

We note that $h_j(\lambda) := \lambda^{d_s/2} G_j(\ln\lambda)$ is nondecreasing on (M, ∞) for some constant $M > 0$. Hence, by the definition of $g_j(\cdot,\cdot)$ in (1-16),

$$g_j(\lambda, V^\wedge) \le g_j(\lambda, V) \le g_j(\lambda, V^\vee) \qquad \text{for all sufficiently large } \lambda. \tag{4-17}$$

As in (4-3), for all $\lambda > 0$,

$$N(\lambda, -\Delta_{\mu_{\infty,j}} + V^\wedge|_{X_{\infty,j}}) \le N(\lambda, -\Delta_{\mu_{\infty,j}} + V|_{X_{\infty,j}}) \le N(\lambda, -\Delta_{\mu_{\infty,j}} + V^\vee|_{X_{\infty,j}}).$$

It follows that, for all sufficiently large λ,

$$\frac{N(\lambda, -\Delta_{\mu_{\infty,j}} + V^\wedge|_{X_{\infty,j}})}{g_j(\lambda, V^\vee)} \le \frac{N(\lambda, -\Delta_{\mu_{\infty,j}} + V|_{X_{\infty,j}})}{g_j(\lambda, V)} \le \frac{N(\lambda, -\Delta_{\mu_{\infty,j}} + V^\vee|_{X_{\infty,j}})}{g_j(\lambda, V^\wedge)},$$

which, together with (4-16), yields (4-14). □

We now prove Theorem 1.4.

Proof of Theorem 1.4. Proposition 4.1(a) and Remark 4.4(a) imply $N(\lambda, -\Delta_{\mu_i}) = N(\lambda, -\Delta_\mu)$ and $N(\lambda, -\Delta_{\mu_i|_{\tau_i(Y_j)}}) = N(\lambda, -\Delta_{\mu|_{Y_j}})$ for all $i \in I$, all $j \in J$, and $\lambda > 0$. Also, (1-11) holds by (W2). Thus, for all $i \in I$ and all $\lambda > 0$,

$$\begin{aligned} \sum_{j\in J} N(\xi_{j,1}\lambda, -\Delta_{\mu_i|_{\tau_i(Y_j)}}) - C_0 \\ \le N(\lambda, -\Delta_{\mu_i}) \le \sum_{j\in J} N(\xi_{j,2}\lambda, -\Delta_{\mu_i|_{\tau_i(Y_j)}}) + C_0. \end{aligned} \tag{4-18}$$

For all $i \in I$, since $-\Delta_{\mu_i} + V^\vee|_{X_i} \preccurlyeq -\Delta_{\mu_i} + V|_{X_i} \preccurlyeq -\Delta_{\mu_i} + V^\wedge|_{X_i}$, Proposition 2.2 and (4-2) give

$$\begin{aligned} N(\lambda - V^\wedge|_{X_i}, -\Delta_{\mu_i}) = N(\lambda, -\Delta_{\mu_i} + V^\wedge|_{X_i}) &\le N(\lambda, -\Delta_{\mu_i} + V|_{X_i}) \\ &\le N(\lambda, -\Delta_{\mu_i} + V^\vee|_{X_i}) = N(\lambda - V^\vee|_{X_i}, -\Delta_{\mu_i}), \end{aligned}$$

which, together with (4-18), yields

$$\begin{aligned} &\sum_{j \in J} N\big(\xi_{j,1}(\lambda - V^\wedge|_{X_i}), -\Delta_{\mu_i|_{\tau_i(Y_j)}}\big) - C_0 \\ &\qquad \le N(\lambda, -\Delta_{\mu_i} + V|_{X_i}) \le \sum_{j \in J} N\big(\xi_{j,2}(\lambda - V^\vee|_{X_i}), -\Delta_{\mu_i|_{\tau_i(Y_j)}}\big) + C_0. \end{aligned}$$

It follows that, for all $\lambda > 0$,

$$\begin{aligned} &\sum_{i \in I} \sum_{j \in J} N(\xi_{j,1}(\lambda - V^\wedge|_{X_i}), -\Delta_{\mu_i|_{\tau_i(Y_j)}}) - C_0 \cdot \#\{i \in I : V^\wedge|_{X_i} \le \lambda\} \\ &\qquad \le N(\lambda, -\Delta_{\mu_\infty} + V) = \sum_{i \in I} N(\lambda, -\Delta_{\mu_i} + V|_{X_i}) \\ &\qquad \le \sum_{i \in I} \sum_{j \in J} N(\xi_{j,2}(\lambda - V^\vee|_{X_i}), -\Delta_{\mu_i|_{\tau_i(Y_j)}}) + C_0 \cdot \#\{i \in I : V^\vee|_{X_i} \le \lambda\}. \end{aligned}$$

Combining this equality with (4-6) and (4-10), we get, for all $\lambda > 0$,

$$\begin{aligned} &\sum_{j \in J} N(\xi_{j,1}\lambda, -\Delta_{\mu_{\infty,j}} + \xi_{j,1} V^\wedge|_{X_{\infty,j}}) - C_1 F(\lambda, V^\wedge) \\ &\qquad \le N(\lambda, -\Delta_{\mu_\infty} + V) \\ &\qquad \le \sum_{j \in J} N(\xi_{j,2}\lambda, -\Delta_{\mu_{\infty,j}} + \xi_{j,2} V^\vee|_{X_{\infty,j}}) + C_2 F(\lambda, V^\vee), \end{aligned} \tag{4-19}$$

where $C_i,\ i = 1, 2$ are positive constants. We observe that (4-12) and (4-6) imply that for $b \in \{\vee, \wedge\}$, all $c > 0$, and all $\lambda > 0$,

$$\begin{aligned} g_j(c\lambda, cV^b) &\ge (c^{d_s/2} \inf G_j) \cdot \sum_{\{i \in I : V^b|_{X_i} \le \lambda\}} (\lambda - V^b|_{X_i})^{d_s/2} \\ &\ge (c^{d_s/2} \inf G_j)(\lambda/2)^{d_s/2} \cdot \#\{i \in I : V^b|_{X_i} \le \lambda/2\} \\ &\ge C_3 \lambda^{d_s/2} F(\lambda/2, V^b), \qquad \text{(by (4-6))}, \end{aligned} \tag{4-20}$$

where $C_3 > 0$ is a constant. In view of (4-20) and the assumption $F(2\lambda, V^\vee) \le CF(\lambda, V^\wedge)$ for all sufficiently large λ, there exists a constant $C_4 > 0$ such that

$$g_j(c\lambda, cV^b) \ge C_4 \lambda^{d_s/2} F(\lambda, V^b) > 0 \qquad \text{for all sufficiently large } \lambda. \tag{4-21}$$

Consequently, combining the above estimates, we have

$$\begin{aligned}
&\lim_{\lambda\to\infty} \frac{N(\lambda, -\Delta_{\mu_\infty}+V)}{\sum_{j\in J} g_j(\xi_{j,1}\lambda, \xi_{j,1}V)} \\
&\qquad \geq \lim_{\lambda\to\infty} \frac{\sum_{j\in J} N(\xi_{j,1}\lambda, -\Delta_{\mu_{\infty,j}} + \xi_{j,1}V^\wedge|_{X_{\infty,j}})}{\sum_{j\in J} g_j(\xi_{j,1}\lambda, \xi_{j,1}V^\vee)} \\
&\qquad\qquad\qquad - \lim_{\lambda\to\infty} \frac{C_1F(\lambda, V^\wedge)}{\sum_{j\in J} g_j(\xi_{j,1}\lambda, \xi_{j,1}V^\vee)} \qquad \text{(by (4-17) and (4-19))} \\
&\qquad \geq 1 - \lim_{\lambda\to\infty} \frac{C_1F(\lambda, V^\wedge)}{g_{j_0}(\xi_{j_0,1}\lambda, \xi_{j_0,1}V^\vee)} \qquad \text{(by (4-16))} \\
&\qquad = 1 - \lim_{\lambda\to\infty} \frac{C_1F(\lambda, V^\wedge)}{g_{j_0}(\xi_{j_0,1}\lambda, \xi_{j_0,1}V^\wedge)} \qquad \text{(by (4-15))} \\
&\qquad \geq 1 - \lim_{\lambda\to\infty} \frac{C_1}{C_4}\lambda^{-d_s/2} \qquad \text{(by (4-21))} \\
&\qquad = 1 - 0 = 1,
\end{aligned}$$

where j_0 is any index in J. Similarly, we have

$$\lim_{\lambda\to\infty} \frac{N(\lambda, -\Delta_{\mu_\infty}+V)}{\sum_{j\in J} g_j(\xi_{j,2}\lambda, \xi_{j,2}V)} \leq 1,$$

which completes the proof. □

A sufficient condition for (1-17) is given in [Chen et al. 2015, Remark 2.9]. We now give a simple sufficient condition for (1-17), which is needed in Section 5.

Proposition 4.6. *Use the notation in Proposition 4.1, and let $X := [0, a]$. Assume that V is a locally bounded nonnegative piecewise continuous function on X_∞ and assume that there exist positive constants β and c such that $V(x) = c|x|^\beta$ for all $x \in X_\infty$ with $|x|$ sufficiently large. Let $F(\,\cdot\,,\cdot\,)$ be defined as in* (1-14)*. Then* (1-17) *holds.*

Proof. By the assumptions on V, there exists some $r_0 > a$ such that for all $x \in X_\infty$ with $|x| > r_0$, we have

$$V(x) = c|x|^\beta \quad\text{and}\quad c(|x|-a)^\beta \leq V^\vee(x) \leq V^\wedge(x) \leq c(|x|+a)^\beta. \tag{4-22}$$

Let $M := \max\{V^\wedge(x) : x \in X_\infty \text{ with } |x| \leq r_0\}$. For $\lambda > 0$, let $W_\lambda := \{x \in X_\infty : V^\vee(x) \leq \lambda < V^\wedge(x)\}$. We claim that for all $\lambda > M$, $\mu_\infty(W_\lambda) \leq 4\mu(X)$. To see this, we first notice that for all $\lambda > M$,

$$\{x \in X_\infty : |x| \leq r_0\} \subseteq \{x \in X_\infty : V^\wedge(x) \leq \lambda\} \subseteq \mathbb{R} \setminus W_\lambda.$$

Next, it follows from (4-22) that for $x \in W_\lambda$,

$$V^\vee(x + 2\mathrm{sgn}(x)a) \geq c(|x + 2\mathrm{sgn}(x)a| - a)^\beta = c(|x| + a)^\beta \geq V^\wedge(x) > \lambda,$$

and hence $x + 2\mathrm{sgn}(x)a \notin W_\lambda$. Finally, we observe that $V^b(x)$, $b \in \{\vee, \wedge\}$, is nondecreasing (resp. nonincreasing) on $(r_0, +\infty)$ (resp. $(-\infty, -r_0)$). We conclude that W_λ intersects with at most four translates of X in X_∞. This proves the claim. Using the claim and the definition of $F(\cdot, \cdot)$, we obtain

$$\begin{aligned} 1 \leq \lim_{\lambda\to\infty} \frac{F(\lambda, V^\vee)}{F(\lambda, V^\wedge)} &= \lim_{\lambda\to\infty} \frac{\mu_\infty(\{x \in X_\infty : V^\vee(x) \leq \lambda\})}{\mu_\infty(\{x \in X_\infty : V^\wedge(x) \leq \lambda\})} \\ &= \lim_{\lambda\to\infty} \frac{\mu_\infty(\{x \in X_\infty : V^\wedge(x) \leq \lambda\}) + \mu_\infty(W_\lambda)}{\mu_\infty(\{x \in X_\infty : V^\wedge(x) \leq \lambda\})} \\ &\leq \lim_{\lambda\to\infty} \frac{\mu_\infty(\{x \in X_\infty : V^\wedge(x) \leq \lambda\}) + 4\mu(X)}{\mu_\infty(\{x \in X_\infty : V^\wedge(x) \leq \lambda\})} = 1. \end{aligned}$$

This completes the proof. □

5. Examples: self-similar measures on ℝ with overlaps

In this section, we apply Theorem 1.4 to self-similar measures on ℝ with overlaps. We first prove a simple proposition, which leads to a sufficient condition for Theorem 4.3.

Proposition 5.1. *Let $X := [0, a]$ and μ be a positive finite Borel measure with* $\mathrm{supp}(\mu) \subseteq [0, a]$ *and $\mu(0, a) > 0$. Let $X_\infty := \bigcup_{i\in I} X_i$ and μ_∞ be defined as in Section 1 with $I = \mathbb{Z}$, $\tau_i(x) = x + b_i$, and $b_i = a + b_{i-1}$. Then $X_\infty = \mathbb{R}$, and there exist positive constants C_1, C_2 such that $C_1 r \leq \mu_\infty(B(x, r)) \leq C_2 r$ for all $x \in \mathbb{R}$ and $r \geq 2a$, where $B(x, r) := \{x \in \mathbb{R} : |x| < r\}$. Consequently, under the assumptions of* (W1) *and* (4-5), *the conclusions of Theorem 4.3 hold.*

Proof. By assumption, $\tau_i(0) = \tau_{i-1}(a)$ and $|X_i| = |\tau_i(X)| = a$ for all $i \in \mathbb{Z}$. Thus $X_\infty = \mathbb{R}$. Fix any $x \in \mathbb{R}$ and $r > 2a$. Then there exist positive integers m_0, m_1 such that $m_1 - m_0 \geq 2$ and $\bigcup_{i=m_0}^{m_1} X_i \subseteq B(x, r) \subseteq \bigcup_{i=m_0-1}^{m_1+1} X_i$. Thus

$$a(m_1 - m_0) = \sum_{i=m_0}^{m_1} |X_i| \leq 2r \leq \sum_{i=m_0-1}^{m_1+1} |X_i| = a(m_1 - m_0 + 2) \leq 2a(m_1 - m_0).$$

It follows that

$$\frac{\mu(X)}{a} r \leq (m_1 - m_0)\mu(X) \leq \mu_\infty(B(x, r)) \leq 2(m_1 - m_0)\mu(X) \leq \frac{4\mu(X)}{a} r,$$

where the fact $\mu_\infty(X_i) = \mu(X)$ for all $i \in I$ is used. Hence the assertion holds. □

The spectral dimension of the examples in Sections 5A and 5B are computed in [Ngai 2011]. We will compute the spectral dimension of the example in Section 5C by using a similar method. The technique is to apply a vector-valued renewal theorem [Lau et al. 1995, Theorem 4.2] by deriving a system of renewal equations for the eigenvalue counting functions, and express them in vector form as

$$\boldsymbol{f} = \boldsymbol{f} * \boldsymbol{M}_\alpha + \boldsymbol{z}, \tag{5-1}$$

where $\alpha \geq 0$, and

$$\begin{aligned} \boldsymbol{f} = \boldsymbol{f}^\alpha(t) &:= \big[f_1^{(\alpha)}(t), \ldots, f_n^{(\alpha)}(t)\big], \quad t \in \mathbb{R}; \\ \boldsymbol{M}_\alpha &:= \big[\mu_{\ell m}^{(\alpha)}\big] \text{ is an } n \times n \text{ matrix of Radon measures on } \mathbb{R}; \\ \boldsymbol{z} := \boldsymbol{z}^{(\alpha)}(t) &= \big[z_1^{(\alpha)}(t), \ldots, z_n^{(\alpha)}(t)\big] \text{ is some error function.} \end{aligned} \tag{5-2}$$

Let

$$\boldsymbol{M}_\alpha(\infty) := \big[\mu_{\ell m}^{(\alpha)}(\mathbb{R})\big]_{\ell,m=1}^n. \tag{5-3}$$

If the error functions decay exponentially to 0 as $t \to \infty$, then $d_s(-\Delta_\mu)$ is given by the unique α such that the spectral radius of $\boldsymbol{M}_\alpha(\infty)$ is equal to 1.

For the examples in this section, the functions G_j in condition (W2) tend to either a constant or a (nonconstant) periodic function as $\lambda \to \infty$. This dichotomy is determined by whether a set $\mathbb{R}_{\boldsymbol{M}}$ in [Lau et al. 1995] is arithmetic or nonarithmetic, where $\boldsymbol{M} := \boldsymbol{M}_\alpha = \big[\mu_{\ell m}^{(\alpha)}\big]_{\ell,m=1}^n$ is given as in (5-2). Precisely, $\mathbb{R}_{\boldsymbol{M}}$ is the closed subgroup of $(\mathbb{R}, +)$ generated by $\mathcal{G} := \bigcup\{\operatorname{supp}(\mu_\gamma) : \gamma \text{ is a simple cycle on } \{1, \ldots, n\}\}$ (see [Lau et al. 1995, Lemma 2.3]), where $\mu_\gamma = \mu_{i_1 i_2}^{(\alpha)} * \mu_{i_2 i_3}^{(\alpha)} * \cdots * \mu_{i_{k-1} i_k}^{(\alpha)}$ for any path $\gamma = (i_1, i_2, \ldots, i_k)$.

5A. ***Infinite Bernoulli convolution associated with the golden ratio.*** In this subsection, we consider the infinite Bernoulli convolution associated with the golden ratio:

$$\mu = \tfrac{1}{2}\mu \circ S_1^{-1} + \tfrac{1}{2}\mu \circ S_2^{-1}, \tag{5-4}$$

where $S_1(x) = \rho x$, $S_2(x) = \rho x + (1 - \rho)$, and $\rho = (\sqrt{5} - 1)/2$. We note that $\operatorname{supp}(\mu) = [0, 1] =: X$. Strichartz et al. [1995] showed that μ satisfies a family of second-order identities with respect to the following auxiliary IFS:

$$T_0(x) := \rho^2 x, \qquad T_1(x) := \rho^3 x + \rho^2, \qquad T_2(x) := \rho^2 x + \rho. \tag{5-5}$$

For any integer $k \geq 0$ and any index $\boldsymbol{j} = (j_1, \ldots, j_k) \in \{0, 2\}^k$, define

$$c_{\boldsymbol{j}} := \frac{1}{2 \cdot 4^{k+1}} \begin{bmatrix} 1 & 1 \end{bmatrix} P_{\boldsymbol{j}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad P_{\boldsymbol{j}} := P_{j_1} \cdots P_{j_k}, \quad P_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad P_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The vector-valued renewal equation (5-1) reduces to the following scalar-valued equation:

$$f(t)=\sum_{k=0}^{\infty}\sum_{\boldsymbol{j}\in\{0,2\}^k}(\rho^{2k+3}c_{\boldsymbol{j}})^{\alpha}f(t+\ln(\rho^{2k+3}c_{\boldsymbol{j}}))+z^{\alpha}(t),$$

where $f(t)=e^{-\alpha t}N(e^t,-\Delta_{\mu|_{T_1(X)}})$ and $z^{\alpha}(t)=o(e^{-\sigma t})$ as $t\to\infty$ for some $\sigma>0$ (see [Ngai 2011, Section 5]). Moreover, $\boldsymbol{M}=[\mu^{(\alpha)}]$ is a 1×1 matrix-valued Radon measure, where $\mu^{(\alpha)}$ is a discrete measure with $\operatorname{supp}\mathcal{G}:=\{-\ln(\rho^{2k+3}c_{\boldsymbol{j}}):k\ge 0,\ \boldsymbol{j}\in\{0,2\}^k\}$. Thus $\mathbb{R}_{\boldsymbol{M}}$ is the closed subgroup of $(\mathbb{R},+)$ generated by $\mathcal{G}$.

[Ngai 2011, Theorem 1.2] shows that $d_s(-\Delta_\mu)=d_s$, and (W1) holds, where d_s is the unique positive solution of

$$\sum_{k=0}^{\infty}\sum_{\boldsymbol{j}\in\{0,2\}^k}(\rho^{2k+3}c_{\boldsymbol{j}})^{d_s/2}=1. \tag{5-6}$$

Proposition 5.2. *Let μ be the self-similar measure defined as in* (5-4), *and $-\Delta_\mu$ be the associated Dirichlet Laplacian with respect to μ. Then* (1-11) *holds with $J=\{1\}$ and $Y_1:=T_1(X)$, where T_1 is defined as in* (5-5). *Moreover*, (*W*2) *holds*; *in particular*, *the nonarithmetic case holds*: *there exists a constant $G_1>0$ such that*

$$N\big(\lambda,-\Delta_{\mu|_{T_1(X)}}\big)=\lambda^{d_s/2}(G_1+o(1))\qquad \text{as } \lambda\to\infty,$$

where d_s is defined as in (5-6).

Proof. From the paragraph following Proposition 3.2 in [Ngai 2011], we see that there exists a constant $\xi>0$ such that

$$N\big(\lambda,-\Delta_{\mu|_{T_1(X)}}\big)\le N(\lambda,-\Delta_{\mu|_X})\le N\big(\xi\lambda,-\Delta_{\mu|_{T_1(X)}}\big)+1\qquad \text{for all } \lambda>0, \tag{5-7}$$

and hence (1-11) holds with $J=\{1\}$ and $Y_1:=T_1(X)$. Condition (2) of (W2) holds by using [Ngai 2011, Theorems 1.2 and 4.1]. We now use [Ngai 2011, Theorem 4.1] again to show that the nonarithmetic case holds by verifying that $\mathbb{R}_{\boldsymbol{M}}=\mathbb{R}$. Suppose, on the contrary, that $\mathbb{R}_{\boldsymbol{M}}\ne\mathbb{R}$. Letting $k=0$ and 1, we obtain the elements $a:=-\ln(\rho^3/4)$ and $b:=-\ln(3\rho^5/32)$ in $\mathcal{G}$. Hence $b/a=1-\ln(3\rho^2/8)/a\in\mathbb{Q}$ and thus there exist $m,n\in\mathbb{Z}$ such that $-\ln(3\rho^2/8)/a=n/m$. Consequently, $3^m=2^{3m-2n}\beta^{2m-3n}$, where $\beta=2/(\sqrt{5}-1)=1/\rho$. Without loss of generality, we assume that $2m-3n>0$. Define $h(x):=2^{3m-2n}x^{2m-3n}-3^m$. Then $h(\beta)=0$. Since β is an algebraic integer with x^2-x-1 being its minimal polynomial, x^2-x-1 divides $h(x)$, a contradiction. Hence, $\mathbb{R}_{\boldsymbol{M}}=\mathbb{R}$, which implies the desired result. □

Define $X_\infty:=\bigcup_{i\in\mathbb{Z}}\tau_i(X)$, where $\tau_i(x)=x+i$ for $i\in\mathbb{Z}$. Thus $X_\infty=\mathbb{R}$, and conditions (C1)–(C3) in Section 1 hold. For each $i\in\mathbb{Z}$, let $\mu_i:=\mu\circ\tau_i^{-1}$ be the induced positive finite Borel measure on $X_i:=\tau_i(X)$. Then we can define a glued measure μ_∞ on X_∞ as in (1-13). Define $X_{\infty,1} := \bigcup_{i\in\mathbb{Z}} \tau_i(T_1(X))$ and $\mu_{\infty,1} := \mu_\infty|_{X_{\infty,1}}$.

Corollary 5.3. *Let X_∞, μ_∞, $X_{\infty,1}$ and $\mu_{\infty,1}$ be defined as above. Assume the same hypotheses on V as in Proposition 4.6. Let d_s be defined as in* (5-6). *Then*

(a) *there exist positive constants C_1, C_2 such that, for all sufficiently large λ,*

$$C_1\lambda^{d_s/2}F(\lambda, V) \le N(\lambda, -\Delta_{\mu_\infty} + V) \le C_2\lambda^{d_s/2}F(\lambda, V),$$

where $F(\,\cdot\,,\cdot\,)$ is defined as in (1-14);

(b) *as $\lambda \to \infty$,*

$$(1+o(1))g_1(\lambda, V) \le N(\lambda, -\Delta_{\mu_\infty} + V) \le (1+o(1))g_1(\xi\lambda, \xi V),$$

where ξ comes from (5-7), *and $g_1(\,\cdot\,,\cdot\,)$ is defined as in* (1-16) *with $G_1(\cdot)$ being a constant function in Proposition 5.2.*

Proof. Part (a) follows from Proposition 5.1, the fact that (W1) holds, and the assumptions on V. Part (b) follows by combining Theorems 1.4 and 4.3 with Propositions 4.6, 5.1 and 5.2. □

5B. ***A class of convolutions of Cantor-type measures.*** The m-fold convolution μ_m of the standard Cantor measure is the self-similar measure defined by the following IFS with overlaps (see [Lau and Ngai 2000; Ngai 2011]):

$$S_i(x) = \frac{1}{m}x + \frac{m-1}{m}i, \qquad i = 0, 1, \dots, m,$$

together with probability weights $w_i := \binom{m}{i}/2^m$, $i = 0, 1, \dots, m$; that is,

$$\mu_m = \sum_{i=0}^{m} w_i \cdot \mu_m \circ S_i^{-1}. \tag{5-8}$$

We will assume that m is an odd integer and $m \ge 3$. Note that $\operatorname{supp}(\mu_m) = [0, m] =: X$. It is shown in [Lau and Ngai 2000] that μ_m satisfies a family of second-order identities with respect to the IFS

$$T_j(x) = \frac{1}{m}x + j, \qquad j = 0, 1, \dots, m-1. \tag{5-9}$$

The vector-valued renewal equation (5-1) is given in [Ngai 2011, Section 6]. In particular, $\boldsymbol{M} := \boldsymbol{M}_\alpha = \big[\mu_{k\ell}^{(\alpha)}\big]_{k,\ell=1}^{m-2}$ is an $(m-2)\times(m-2)$ matrix of Radon measures. By the proof of [Ngai 2011, Proposition 6.2], we have

$$\operatorname{supp}(\mu_{11}^{(\alpha)}) = \left\{\ln(2^m), -\ln\!\left(\frac{m-1}{2^{m+1}}\right)\right\} \cup \left\{-\ln\!\left(\frac{c_{\boldsymbol{j}}}{m^{k+2}}\right) : k \ge 0,\ \boldsymbol{j} \in \{0, m-1\}^k\right\},$$

where for any integer $k \geq 0$ and any index $\boldsymbol{j} = (j_1, \dots, j_k) \in \{0, m-1\}^k$,

$$P_{\boldsymbol{j}} := P_{j_1} \cdots P_{j_k}, \quad c_{\boldsymbol{j}} := \frac{1}{2^{2m+mk}} \begin{bmatrix} \binom{m}{2} & \binom{m}{1} \end{bmatrix} P_{\boldsymbol{j}} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$P_0 := \begin{bmatrix} 1 & 0 \\ 1 & m \end{bmatrix}, \quad P_{m-1} := \begin{bmatrix} m & 1 \\ 0 & 1 \end{bmatrix}. \tag{5-10}$$

By the definition of $\mathcal{G}$ and [Lau et al. 1995, Lemma 2.3], we get $\mathcal{G}^* := \operatorname{supp}(\mu_{11}^{(\alpha)}) \subseteq \mathcal{G}$. In particular, the equation holds if $m = 3$.

If no confusion is possible, we denote μ_m simply by μ. An explicit formula for the spectral dimension of $-\Delta_\mu$ is given in [Ngai 2011, Theorem 1.3], which also shows that (W1) holds.

Proposition 5.4. *Let $\mu := \mu_m$ be defined as in* (5-8), *and $-\Delta_\mu$ be the associated Dirichlet Laplacian. Then* (1-11) *holds with $J = \{j\}$ and $Y_j := T_j(X)$ for any $j = 1, \dots, m-2$, where T_j is defined as in* (5-9). *Moreover*, (W2) *holds. In particular*, *the nonarithmetic case holds*: *for any $j = 1, \dots, m-2$, there exists a constant $G_j > 0$ such that*

$$N(\lambda, -\Delta_{\mu|_{T_j(X)}}) = \lambda^{d_s/2}(G_j + o(1)) \qquad \text{as } \lambda \to \infty,$$

where d_s is the spectral dimension of $-\Delta_\mu$.

Proof. As in the proof of Proposition 5.2, using the discussion in the paragraph following Proposition 3.2 in [Ngai 2011], we see that there exist positive constants $(\xi_j)_{j=1}^{m-2}$ such that for each $j = 1, \dots, m-2$,

$$N(\lambda, -\Delta_{\mu|_{T_j(X)}}) \leq N(\lambda, -\Delta_{\mu|_X}) \leq N(\xi_j \lambda, -\Delta_{\mu|_{T_j(X)}}) + 1. \tag{5-11}$$

Hence, (1-11) holds. Condition (2) of (W2) follows from [Ngai 2011, Section 6 and Theorem 4.1]. As in Proposition 5.2, we show that $\mathbb{R}_{\boldsymbol{M}} = \mathbb{R}$. Letting $k = 0$, we get

$$a := -\ln((m+1)/(2m)) + 2\ln(2^m) \in \mathcal{G}^* \subseteq \mathbb{R}_{\boldsymbol{M}}.$$

Suppose $\mathbb{R}_{\boldsymbol{M}} \neq \mathbb{R}$. Since $\ln(2^m) \in \mathcal{G}^*$, we have $-a/\ln(2^m) + 2 = \ln((m+1)/(2m))/\ln(2^m) = t/s$ for some $s, t \in \mathbb{Z}$. Thus $(m+1)^s/m^s = 2^{mt+s}$, a contradiction, and the assertion follows. □

Let $Y_j := T_j(X)$ for $j = 1, \dots, m-2$. Let $I := \mathbb{Z}$ and define $\tau_i(x) = x + mi$ for all $i \in I$. Define $X_\infty := \bigcup_{i \in I} \tau_i(X)$ and let μ_∞ be defined as in (1-13). Then $X_\infty = \mathbb{R}$. Define $X_{\infty,j} := \bigcup_{i \in I} \tau_i(Y_j)$ and $\mu_{\infty,j} := \mu_\infty|_{X_{\infty,j}}$ for $j = 1, \dots, m-2$.

Corollary 5.5. *Let X_∞, μ_∞, $(X_{\infty,j})_{j=1}^{m-1}$ and $(\mu_{\infty,j})_{j=1}^{m-1}$ be defined as above. Assume the same hypotheses on V as in Proposition 4.6. Let d_s be the spectral dimension of $-\Delta_\mu$. Then the following hold*:

(a) *There exist positive constants* C_1, C_2 *such that, for all sufficiently large* λ,

$$C_1 \lambda^{d_s/2} F(\lambda, V) \le N(\lambda, -\Delta_{\mu_\infty} + V) \le C_2 \lambda^{d_s/2} F(\lambda, V)$$

where $F(\cdot, \cdot)$ *is defined as in* (1-14).

(b) *As* $\lambda \to \infty$,

$$(1+o(1)) \sum_{j=1}^{m-2} g_j(\lambda, V) \le N(\lambda, -\Delta_{\mu_\infty} + V) \le (1+o(1)) \sum_{j=1}^{m-2} g_j(\xi_j \lambda, \xi_j V),$$

where ξ_j *comes from* (5-11), *and* $g_j(\cdot, \cdot)$ *is defined as in* (1-16) *with* $G_j(\cdot)$ *being the constant function in Proposition 5.4.*

Proof. The proof is similar to that of Corollary 5.3 with Proposition 5.4 replacing Proposition 5.2. □

5C. ***A class of graph-directed self-similar measures satisfying EFT.*** The purpose of this subsection is to illustrate the arithmetic case by constructing a special graph-directed self-similar measure.

A *graph-directed iterated function system* (*GIFS*) of contractive similitudes is an ordered pair $G = (V, E)$ described as follows (see [Mauldin and Williams 1988]). $V := \{1, \dots, q\}$ is the set of *vertices* and E is the set of *directed edges* with each edge beginning and ending at a vertex. It is possible for an edge to begin and end at the same vertex and we allow more than one edge between two vertices. Let E_{ij} denote the set of all edges that begin at vertex i and end at vertex j. We call $\boldsymbol{e} = e_1 \dots e_k$ a *path* with length k if the terminal vertex of each edge e_i $(1 \le i \le k-1)$ equals the initial vertex of the edge e_{i+1}.

Consider the GIFS $G = (V, E)$ with $V = \{1, 2\}$ and $E = \{e_i : 1 \le i \le 5\}$, where $e_1, e_2 \in E_{11}$, $e_3 \in E_{12}$, $e_4 \in E_{21}$, $e_5 \in E_{22}$. The five similitudes associated with E are defined by

$$S_{e_1}(x) = \tfrac{1}{4}x, \quad S_{e_2}(x) = \tfrac{1}{4}x + \tfrac{3}{4}, \quad S_{e_3}(x) = \tfrac{1}{4}x - \tfrac{5}{16}, \quad S_{e_4}(x) = \tfrac{1}{4}x + 2, \quad S_{e_5}(x) = \tfrac{1}{4}x + \tfrac{9}{4}.$$

The GIFS $G = (V, E)$ is used in [Das and Ngai 2004] as a basic example for the graph finite type condition. It is known (see [Falconer 1997; Mauldin and Williams 1988]) that if for each edge $e \in E$ there corresponds a *transition probability* p_e, then for each $i \in V$ there exists a unique Borel probability measure μ_i such that

$$\mu_i = \sum_{j=1}^{2} \sum_{e \in E_{ij}} p_e \cdot \mu_j \circ S_e^{-1}.$$

We note that $\mathrm{supp}(\mu_1) = [0, 1]$ and $\mathrm{supp}(\mu_2) = [2, 3]$.

Define $\mu(E) := \mu_1(E \cap [0, 1]) + \mu_2(E \cap [2, 3])$ for all measurable subsets $E \subseteq \mathbb{R}$. We call μ the *graph-directed self-similar measure* defined by $G = (V, E)$

and probability matrix $(p_e)_{e\in E}$. Since μ satisfies EFT (see [Ngai et al. 2018, Example 3.6]), we can derive a vector-valued renewal equation by the method in [Ngai et al. 2018, Section 4] as follows. Let $Y_1 := S_{e_1}([0,1]) \cup S_{e_3}([2,3])$ and $Y_2 := S_{e_2}([0,1])$. For $\alpha \geq 0$ and $j = 1, 2$, define $f_j(t) = f_j^{(\alpha)}(t) := e^{-\alpha t} N(e^t, -\Delta_{\mu|_{Y_j}})$. Thus, combining the proof of [Ngai et al. 2018, Example 3.6] and the process of deriving the vector-valued renewal equation in [Ngai et al. 2018, Section 4], we see that (5-1) can be written as

$$\begin{aligned} f_1(t) = {} & \left(\frac{p_{e_1}}{4}\right)^{\alpha} f_1\left(t + \ln\left(\frac{p_{e_1}}{4}\right)\right) \\ & + \left(\frac{p_{e_1}}{4} + \frac{p_{e_3}p_{e_4}}{4p_{e_2}}\right)^{\alpha} f_2\left(t + \ln\left(\frac{p_{e_1}}{4} + \frac{p_{e_3}p_{e_4}}{4p_{e_2}}\right)\right) \\ & + \sum_{k=1}^{\infty} \left(\frac{p_{e_3}p_{e_4}p_{e_5}^k}{4^{k+1}p_{e_2}}\right)^{\alpha} f_2\left(t + \ln\left(\frac{p_{e_3}p_{e_4}p_{e_5}^k}{4^{k+1}p_{e_2}}\right)\right) + z_1^{(\alpha)}(t), \\ f_2(t) = {} & \left(\frac{p_{e_2}}{4}\right)^{\alpha} f_1\left(t + \ln\left(\frac{p_{e_2}}{4}\right)\right) + \left(\frac{p_{e_2}}{4}\right)^{\alpha} f_2\left(t + \ln\left(\frac{p_{e_2}}{4}\right)\right) + z_2^{(\alpha)}(t), \end{aligned} \tag{5-12}$$

where $z_1^{(\alpha)}(t) := e^{-\alpha t}(N(e^t, -\Delta_{\mu|_{B_{n_t}}}) + \varepsilon(n_t, 1))$, $B_{n_t} := S_{e_3 e_5^{n_t-1}}(S_{e_5}[2,3])$, and $z_2^{(\alpha)}(t) := e^{-\alpha t}\varepsilon(2,2)$.

For $j, k \in \{1, 2\}$, let $\mu_{\ell m}^{(\alpha)}$ be the discrete measure such that

$$\begin{aligned} \mu_{11}^{(\alpha)}\left(-\ln\left(\frac{p_{e_1}}{4}\right)\right) &:= \left(\frac{p_{e_1}}{4}\right)^{\alpha}, \\ \mu_{21}^{(\alpha)}\left(-\ln\left(\frac{p_{e_2}}{4}\right)\right) &:= \left(\frac{p_{e_2}}{4}\right)^{\alpha}, \\ \mu_{12}^{(\alpha)}\left(-\ln\left(\frac{p_{e_1}}{4} + \frac{p_{e_3}p_{e_4}}{4p_{e_2}}\right)\right) &:= \left(\frac{p_{e_1}}{4} + \frac{p_{e_3}p_{e_4}}{4p_{e_2}}\right)^{\alpha}, \\ \mu_{12}^{(\alpha)}\left(-\ln\left(\frac{p_{e_3}p_{e_4}p_{e_5}^k}{4^{k+1}p_{e_2}}\right)\right) &:= \left(\frac{p_{e_3}p_{e_4}p_{e_5}^k}{4^{k+1}p_{e_2}}\right)^{\alpha} \qquad \text{for } k \geq 1, \\ \mu_{22}^{(\alpha)}\left(-\ln\left(\frac{p_{e_2}}{4}\right)\right) &:= \left(\frac{p_{e_2}}{4}\right)^{\alpha}. \end{aligned} \tag{5-13}$$

Let $\boldsymbol{M}_\alpha(\infty)$ be defined as in (5-3). Since $\mu_{\ell m}^{(\alpha)}(\mathbb{R}) > 0$ for all $\ell, m \in \{1, 2\}$, $\boldsymbol{M}_\alpha(\infty)$ is irreducible. The remaining conditions of [Ngai et al. 2018, Theorem 1.1(b)] can be easily checked by using the same method as in [Ngai et al. 2018, Propositions 5.2 and 5.4]. Finally, It follows from [Ngai et al. 2018, Theorem 1.1(b)] that the spectral dimension of $-\Delta_\mu$ exists, and (W1) holds.

Proposition 5.6. *Let μ be the graph-directed self-similar measure defined by the above GIFS with probability vector $(p_e)_{e\in E}$, and $-\Delta_\mu$ be the associated Dirichlet Laplacian. Also, let Y_1 and Y_2 be defined as above. Then* (1-11) *holds with $J=\{j\}$ and $Y_j := T_j(X)$ for any $j=1,2$. Moreover,* (W2) *holds. In particular, if $p_{e_2}=p_{e_3}=1/4$ and $p_{e_1}=p_{e_4}=p_{e_5}=1/2$, then the arithmetic case holds: there exist nonconstant period functions $G_1(\cdot)$ and $G_2(\cdot)$ such that for $j=1,2$,*

$$N(\lambda, -\Delta_{\mu|_{Y_j}}) = \lambda^{d_s/2}(G_j(\ln\lambda)+o(1)) \qquad \text{as } \lambda\to\infty, \tag{5-14}$$

where d_s is the spectral dimension of $-\Delta_\mu$.

Proof. Combining [Ngai et al. 2018, Example 3.6] and [Ngai et al. 2018, Proposition 4.5], we see that for each $j=1,2$, there exists some constant $\xi_j>0$ such that

$$N(\lambda, -\Delta_{\mu|_{Y_j}}) \le N(\lambda, -\Delta_\mu) \le N\big(\xi_j\lambda, -\Delta_{\mu|_{Y_j}}\big). \tag{5-15}$$

Hence, (1-11) holds with $J=\{j\}$ and $Y_j := T_j(X)$ for any $j=1,2$. Since all conditions of [Ngai et al. 2018, Theorem 1.1(b)] hold, condition (2) of (W2) follows from [Ngai 2011, Theorem 4.1]. Hence, (W2) holds. Assume that $p_{e_2}=p_{e_3}=1/4$ and $p_{e_1}=p_{e_4}=p_{e_5}=1/2$. Using [Ngai 2011, Theorem 4.1] again, we show that the arithmetic case holds by verifying that $\mathbb{R}_{\boldsymbol{M}}$ can be generated by a real number $a\in\mathbb{R}$. By (5-12), $\boldsymbol{M} := \boldsymbol{M}_\alpha = \big[\mu_{ij}^{(\alpha)}\big]$ is a 2×2 matrix-valued Radon measure, where $\mu_{ij}^{(\alpha)}$ is defined as in (5-13). It follows from [Lau et al. 1995, Lemma 2.3] that $\mathbb{R}_{\boldsymbol{M}}$ is the closed subgroup generated by $\operatorname{supp}(\mu_{11}^{(\alpha)})$, $\operatorname{supp}(\mu_{22}^{(\alpha)})$, and the closure of $\operatorname{supp}(\mu_{12}^{(\alpha)})+\operatorname{supp}(\mu_{21}^{(\alpha)})$. Combining (5-13) and the assumptions on $(p_e)_{e\in E}$ shows that $\operatorname{supp}(\mu_{11}^{(\alpha)})=\{\ln(8)\}$, $\operatorname{supp}(\mu_{21}^{(\alpha)})=\operatorname{supp}(\mu_{22}^{(\alpha)})=\{\ln(16)\}$, and $\operatorname{supp}(\mu_{12}^{(\alpha)})=\{\ln(4)\}\cup\{\ln(2^{3k+3}):k\ge 1\}$. Consequently, $\mathbb{R}_{\boldsymbol{M}}$ can be generated by $\ln(2)$, which completes the proof. □

Let $X=[0,3]$ and $I:=\mathbb{Z}$. Define $\tau_i(x)=x+3i$ for all $i\in I$. Define $X_\infty := \bigcup_{i\in I}\tau_i(X)$ and let μ_∞ be defined as in (1-13). Then $X_\infty=\mathbb{R}$. Define

$$X_{\infty,j} := \bigcup_{i\in I}\tau_i(Y_j) \quad\text{and}\quad \mu_{\infty,j} := \mu_\infty|_{X_{\infty,j}}$$

for $j=1,2$.

Corollary 5.7. *Let X_∞, μ_∞, $(X_{\infty,j})_{j=1}^2$ and $(\mu_{\infty,j})_{j=1}^2$ be defined as above. Assume the same hypotheses on V as in Proposition 4.6. Let d_s be the spectral dimension of $-\Delta_\mu$. Then the following hold.*

(a) *There exist positive constants C_1, C_2 such that, for all sufficiently large λ,*

$$\begin{aligned} C_1\lambda^{d_s/2}F(\lambda,V) &\le N(\lambda, -\Delta_{\mu_\infty}+V) \\ &\le C_2\lambda^{d_s/2}F(\lambda,V), \end{aligned}$$

where $F(\cdot,\cdot)$ is defined as in (1-14).

(b) *As* $\lambda \to \infty$,

$$(1+o(1))\sum_{j=1}^{2} g_j(\lambda, V) \le N(\lambda, -\Delta_{\mu_\infty}+V) \le (1+o(1))\sum_{j=1}^{2} g_j(\xi_j\lambda, \xi_j V),$$

where ξ_j *comes from* (5-15), *and* $g_j(\,\cdot\,,\,\cdot\,)$ *is defined as in* (1-16) *with the nonconstant period function* $G_j(\,\cdot\,)$ *in* (5-14).

Proof. The proof is similar to that of Corollary 5.3 with Proposition 5.6 replacing Proposition 5.2. □

Acknowledgements

Part of this work was carried out while Ngai was visiting the Center of Mathematical Sciences and Applications of Harvard University. He expresses his gratitude to Professor Shing-Tung Yau for making this possible and thanks the center for its hospitality and support. The authors are very grateful to the anonymous referee for some valuable suggestions and comments.

References

[Birman and Borzov 1972] M. S. Birman and V. V. Borzov, "On the asymptotic formula for the discrete spectrum of certain singular differential operators", pp. 19–30 in *Spectral theory*, edited by M. S. Birman, Topics in Math. Phys. **5**, Consultants Bureau, New York, 1972.

[Chen et al. 2015] J. P. Chen, S. Molchanov, and A. Teplyaev, "Spectral dimension and Bohr's formula for Schrödinger operators on unbounded fractal spaces", *J. Phys. A* **48**:39 (2015), art. id. 395203. MR Zbl

[Cwikel 1977] M. Cwikel, "Weak type estimates for singular values and the number of bound states of Schrödinger operators", *Ann. of Math.* (2) **106**:1 (1977), 93–100. MR Zbl

[Das and Ngai 2004] M. Das and S.-M. Ngai, "Graph-directed iterated function systems with overlaps", *Indiana Univ. Math. J.* **53**:1 (2004), 109–134. MR Zbl

[Davies 1989] E. B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Math. **92**, Cambridge Univ. Press, 1989. MR Zbl

[Davies 1995] E. B. Davies, *Spectral theory and differential operators*, Cambridge Studies in Adv. Math. **42**, Cambridge Univ. Press, 1995. MR Zbl

[Deng and Ngai 2017] G. Deng and S.-M. Ngai, "Differentiability of L^q-spectrum and multifractal decomposition by using infinite graph-directed IFSs", *Adv. Math.* **311** (2017), 190–237. MR Zbl

[Falconer 1997] K. Falconer, *Techniques in fractal geometry*, Wiley, Chichester, 1997. MR Zbl

[Feng and Olivier 2003] D.-J. Feng and E. Olivier, "Multifractal analysis of weak Gibbs measures and phase transition-application to some Bernoulli convolutions", *Ergodic Theory Dynam. Systems* **23**:6 (2003), 1751–1784. MR Zbl

[Fukushima et al. 2010] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet forms and symmetric Markov processes*, 2nd revised ed., de Gruyter Studies in Math. **19**, de Gruyter, Berlin, 2010. MR Zbl

[Gu et al. 2016] Q. Gu, J. Hu, and S.-M. Ngai, "Two-sided sub-Gaussian estimates of heat kernels on intervals for self-similar measures with overlaps", preprint, 2016, available at https://tinyurl.com/qgupreprint. To appear in *Comm. Pure. Appl. Anal.*

[Hambly and Nyberg 2003] B. M. Hambly and S. O. G. Nyberg, "Finitely ramified graph-directed fractals, spectral asymptotics and the multidimensional renewal theorem", *Proc. Edinb. Math. Soc.* (2) **46**:1 (2003), 1–34. MR Zbl

[Holt and Molchanov 2005] J. Holt and S. Molchanov, "On the Bohr formula for the one-dimensional Schrödinger operator with increasing potential", *Appl. Anal.* **84**:6 (2005), 555–569. MR Zbl

[Hu et al. 2006] J. Hu, K.-S. Lau, and S.-M. Ngai, "Laplace operators related to self-similar measures on $\mathbb{R}^d$", *J. Funct. Anal.* **239**:2 (2006), 542–565. MR Zbl

[Kigami and Lapidus 1993] J. Kigami and M. L. Lapidus, "Weyl's problem for the spectral distribution of Laplacians on p.c.f. self-similar fractals", *Comm. Math. Phys.* **158**:1 (1993), 93–125. MR Zbl

[Lau and Ngai 1998] K.-S. Lau and S.-M. Ngai, "L^q-spectrum of the Bernoulli convolution associated with the golden ratio", *Studia Math.* **131**:3 (1998), 225–251. MR Zbl

[Lau and Ngai 2000] K.-S. Lau and S.-M. Ngai, "Second-order self-similar identities and multifractal decompositions", *Indiana Univ. Math. J.* **49**:3 (2000), 925–972. MR Zbl

[Lau and Ngai $\geq$ 2019] K.-S. Lau and S.-M. Ngai, "Fractal Laplacians with mixed boundary conditions and outer approximation", in preparation.

[Lau and Wang 2004] K.-S. Lau and X.-Y. Wang, "Iterated function systems with a weak separation condition", *Studia Math.* **161**:3 (2004), 249–268. MR Zbl

[Lau and Wang 2005] K.-S. Lau and X.-Y. Wang, "Some exceptional phenomena in multifractal formalism, I", *Asian J. Math.* **9**:2 (2005), 275–294. MR Zbl

[Lau et al. 1995] K.-S. Lau, J. Wang, and C.-H. Chu, "Vector-valued Choquet–Deny theorem, renewal equation and self-similar measures", *Studia Math.* **117**:1 (1995), 1–28. MR Zbl

[Levin and Solomyak 1997] D. Levin and M. Solomyak, "The Rozenblum–Lieb–Cwikel inequality for Markov generators", *J. Anal. Math.* **71** (1997), 173–193. MR Zbl

[Li and Yau 1983] P. Li and S. T. Yau, "On the Schrödinger equation and the eigenvalue problem", *Comm. Math. Phys.* **88**:3 (1983), 309–318. MR Zbl

[Lieb 1976] E. Lieb, "Bounds on the eigenvalues of the Laplace and Schrödinger operators", *Bull. Amer. Math. Soc.* **82**:5 (1976), 751–753. MR Zbl

[Lieb and Loss 2001] E. H. Lieb and M. Loss, *Analysis*, 2nd ed., Graduate Studies in Math. **14**, Amer. Math. Soc., Providence, RI, 2001. MR Zbl

[Martin 1972] A. Martin, "Bound states in the strong coupling limit", *Helv. Phys. Acta* **45** (1972), 140–148.

[Mauldin and Williams 1988] R. D. Mauldin and S. C. Williams, "Hausdorff dimension in graph directed constructions", *Trans. Amer. Math. Soc.* **309**:2 (1988), 811–829. MR Zbl

[Mazya 1985] V. G. Mazya, *Sobolev spaces*, Springer, 1985. MR Zbl

[Naimark and Solomyak 1995] K. Naimark and M. Solomyak, "The eigenvalue behaviour for the boundary value problems related to self-similar measures on $\mathbb{R}^d$", *Math. Res. Lett.* **2**:3 (1995), 279–298. MR Zbl

[Ngai 2011] S.-M. Ngai, "Spectral asymptotics of Laplacians associated with one-dimensional iterated function systems with overlaps", *Canad. J. Math.* **63**:3 (2011), 648–688. MR Zbl

[Ngai et al. 2018] S.-M. Ngai, W. Tang, and Y. Xie, "Spectral asymptotics of one-dimensional fractal Laplacians in the absence of second-order identities", *Discrete Contin. Dyn. Syst.* **38**:4 (2018), 1849–1887. MR Zbl

[Ngai et al. 2019] S.-M. Ngai, W. Tang, and Y. Xie, "Wave propagation speed on fractals", preprint, 2019. To appear in *J. Fourier Anal. Appl.*

[Reed and Simon 1972] M. Reed and B. Simon, *Methods of modern mathematical physics, I: Functional analysis*, Academic Press, New York, 1972. MR Zbl

[Reed and Simon 1978] M. Reed and B. Simon, *Methods of modern mathematical physics, IV: Analysis of operators*, Academic Press, New York, 1978. MR Zbl

[Rozenbljum 1972] G. V. Rozenbljum, "Distribution of the discrete spectrum of singular differential operators", *Dokl. Akad. Nauk SSSR* **202** (1972), 1012–1015. In Russian; translated in *Soviet Math. Dokl.* **13** (1972), 245–249. MR

[Solomyak and Verbitsky 1995] M. Solomyak and E. Verbitsky, "On a spectral problem related to self-similar measures", *Bull. London Math. Soc.* **27**:3 (1995), 242–248. MR Zbl

[Strichartz 2009] R. S. Strichartz, "A fractal quantum mechanical model with Coulomb potential", *Commun. Pure Appl. Anal.* **8**:2 (2009), 743–755. MR Zbl

[Strichartz et al. 1995] R. S. Strichartz, A. Taylor, and T. Zhang, "Densities of self-similar measures on the line", *Experiment. Math.* **4**:2 (1995), 101–128. MR Zbl

[Tamura 1974] H. Tamura, "The asymptotic eigenvalue distribution for non-smooth elliptic operators", *Proc. Japan Acad.* **50** (1974), 19–22. MR Zbl

[Weyl 1912] H. Weyl, "Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)", *Math. Ann.* **71**:4 (1912), 441–479. MR Zbl

Received November 27, 2017. Revised July 30, 2018.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 300, No. 1, 2019
dx.doi.org/10.2140/pjm.2019.300.121

ADIABATIC LIMIT AND THE FRÖLICHER SPECTRAL SEQUENCE

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Motivated by our conjecture of an earlier work predicting the degeneration at the second page of the Frölicher spectral sequence of any compact complex manifold supporting an SKT metric ω (i.e., such that $\partial\bar\partial\omega = 0$), we prove degeneration at E_2 whenever the manifold admits a Hermitian metric whose torsion operator τ and its adjoint vanish on Δ''-harmonic forms of positive degrees up to $\dim_{\mathbb{C}} X$. Besides the pseudodifferential Laplacian inducing a Hodge theory for E_2 that we constructed in earlier work and Demailly's Bochner–Kodaira–Nakano formula for Hermitian metrics, a key ingredient is a general formula for the dimensions of the vector spaces featuring in the Frölicher spectral sequence in terms of the asymptotics, as a positive constant h decreases to zero, of the small eigenvalues of a rescaled Laplacian Δ_h, introduced here in the present form, that we adapt to the context of a complex structure from the well-known construction of the adiabatic limit and from the analogous result for Riemannian foliations of Álvarez López and Kordyukov.

1. Introduction

Let X be a compact complex manifold of dimension n. It is well known that the existence of a Kähler metric ω on X implies the degeneration at E_1 of the Frölicher spectral sequence that relates the complex structure of X (encapsulated in the Dolbeault, i.e., the $\bar\partial$-, cohomology $H^{p,q}(X, \mathbb{C})$, the start page of this spectral sequence) to the differential structure of X (encapsulated in the de Rham, i.e., the d-, cohomology $H^k(X, \mathbb{C})$, the limiting page of this spectral sequence). However, since Kähler metrics exist only rarely when $n \geq 3$, it is natural to search for weaker metric conditions on X that ensure a (possibly weaker) degeneration property of the algebrogeometric object that is the Frölicher spectral sequence of X. The best we can hope for in the non-Kähler context is the degeneration at the second page. To this end, we proposed the following conjecture in [Popovici 2016]:

MSC2010: 14C30, 14F40, 32W05, 53C55.
Keywords: Frölicher spectral sequence, Hodge theory, Hermitian metrics, elliptic differential and pseudodifferential operators, spectral theory.

Conjecture 1.1. *If a compact complex manifold X admits an **SKT metric** ω (i.e., a Hermitian metric ω such that $\partial\bar\partial\omega = 0$), the Frölicher spectral sequence of X degenerates at E_2.*

There is evidence that this ought to be true. The statement holds true on all the examples of compact complex manifolds that we are aware of, namely all the 3-dimensional nilmanifolds, the 3-dimensional solvmanifolds that are currently classified, the Calabi–Eckmann manifold $S^3 \times S^3$, etc. We proved this statement under the extra assumption that the SKT metric ω which is supposed to exist has a *small torsion* in the sense that the upper bound of its torsion operator of type $(0, 0)$ (defined in a precise way) does not exceed a third of the spectral gap of the elliptic, self-adjoint and nonnegative, differential operator $\Delta' + \Delta''$ in every bidegree (p, q) [Popovici 2016]. As usual, $\Delta' = \Delta'_\omega = \partial\partial^\star_\omega + \partial^\star_\omega\partial$ and $\Delta'' = \Delta''_\omega = \bar\partial\bar\partial^\star_\omega + \bar\partial^\star_\omega\bar\partial$ are the ∂-, resp. $\bar\partial$-Laplacians on smooth differential forms on X.

While Conjecture 1.1 remains elusive at the moment, we give in this paper a different sufficient metric condition for degeneration at E_2 that does not assume the fixed Hermitian metric ω to be SKT. As usual (see, e.g., [Demailly 1986; Demailly 2012, VII, §.1]), we consider the torsion operator $\tau = \tau_\omega := [\Lambda_\omega, \partial\omega \wedge \cdot\,]$ of type $(1, 0)$ defined on smooth differential forms on X, where Λ_ω is the adjoint of the multiplication by ω w.r.t. the inner product defined by ω, while $[A, B] = AB - (-1)^{ab}BA$ is the graded commutator of any two endomorphisms A, B of respective degrees a, b of the bigraded algebra $C^\infty_{\bullet,\bullet}$ of smooth differential forms on X. Specifically, we prove:

Theorem 1.2. *Let (X, ω) be a compact Hermitian manifold with $\dim_{\mathbb{C}} X = n$ such that the inclusion of kernels*

$$\ker\Delta'' \subset \ker[\tau, \tau^\star] \tag{1}$$

holds for the operators Δ'', $[\tau, \tau^\star] : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C})$ in every degree $k \in \{1, \dots, n\}$.

Then, the Frölicher spectral sequence of X degenerates at the second page E_2.

Hypothesis (1) is of a qualitative nature and it is comparatively easy to check on concrete examples of compact Hermitian manifolds (X, ω) whether it holds or not. For example, $S^3 \times S^3$ equipped with the Calabi–Eckmann complex structure and the Iwasawa manifold do not satisfy it when they are given the natural non-Kähler metrics (easy verifications that are left to the reader). Intuitively, (1) requires the torsion of ω to be “small” since, for nonnegative operators, the smaller one has a larger kernel. (We will use throughout the paper the usual order relation for linear operators A, B: $A \geq B$ will mean that $\langle\langle Au, u\rangle\rangle \geq \langle\langle Bu, u\rangle\rangle$ for all forms u, where $\langle\langle\, ,\, \rangle\rangle$ stands for the L^2 inner product induced by the fixed Hermitian metric ω on X.) Hypothesis (1) is obviously satisfied if ω is Kähler since $\tau = 0$ in that case. We do not know whether there exist compact complex non-Kähler manifolds that satisfy hypothesis (1).

Inspired by the extensive literature on the adiabatic limit associated with a Riemannian foliation (see, e.g., [Witten 1985; Mazzeo and Melrose 1990; Forman 1995; Álvarez López and Kordyukov 2000]), we adapt that construction to the case of the splitting $d = \partial + \bar\partial$ defining the complex structure of X. Thus, for every constant $h > 0$ that is eventually let to converge to 0, we define in Section 2 two *rescalings* of the usual d-Laplacian $\Delta = dd^\star + d^\star d$ acting on the smooth differential forms on an arbitrary compact Hermitian manifold (X, ω):

$$\Delta_h := d_h d_h^\star + d_h^\star d_h,$$

where $d_h := h\partial + \bar\partial$ modifies d by *rescaling* ∂ while keeping $\bar\partial$ fixed, but its formal adjoint $d_h^\star$ is computed w.r.t. the given Hermitian metric ω, and

$$\Delta_{\omega_h} := dd_{\omega_h}^\star + d_{\omega_h}^\star d,$$

where $d = \partial + \bar\partial$ is kept unchanged, but its formal adjoint $d_{\omega_h}^\star$ is computed w.r.t. a *rescaled metric* ω_h that modifies the original ω by multiplying the pointwise inner product of (p, q)-forms by h^{2p}. So, the antiholomorphic degree q of (p, q)-forms does not contribute to the definition of ω_h. Although strongly inspired by the adiabatic limit construction in the presence of a Riemannian foliation, this partial rescaling of a Hermitian metric seems to be new and to hold further promise for the future.

In Section 2, we study these two *rescaled Laplacians* and the relationships between them. As in the foliated case of [Álvarez López and Kordyukov 2000], Δ_h and Δ_{ω_h} are seen to have the same spectrum and to have eigenspaces that are obtained from each other via a *rescaling isometry*.

A key ingredient in the proof of Theorem 1.2 is the following formula for the dimensions of the vector spaces featuring on each page of the Frölicher spectral sequence of X in terms of the number of small eigenvalues of the rescaled Laplacian Δ_h (or, equivalently, Δ_{ω_h}). "Small" refers to the eigenvalues' decay rate to zero as $h \downarrow 0$. This result and its proof are strongly inspired by the analogous result for foliations proved by Álvarez López and Kordyukov [2000]. However, to our knowledge, this particular form of the result in the context of the Frölicher spectral sequence seems new and is of independent interest.

Theorem 1.3. *Let (X, ω) be a compact Hermitian manifold with* $\dim_{\mathbb{C}} X = n$. *For every $r \in \mathbb{N}^\star$ and every $k = 0, \dots, 2n$, the following identity holds*:

$$\dim_{\mathbb{C}} E_r^k = \sharp\{i \mid \lambda_i^k(h) \in O(h^{2r}) \text{ as } h \downarrow 0\}, \tag{2}$$

where $E_r^k := \bigoplus_{p+q=k} E_r^{p,q}$ *is the direct sum of the spaces of total degree* k *on the* r*-th page of the Frölicher spectral sequence of* X, *while* $0 \leq \lambda_1^k(h) \leq \lambda_2^k(h) \leq \cdots \leq \lambda_i^k(h) \leq \cdots$ *are the eigenvalues, counted with multiplicities, of the rescaled Laplacian* $\Delta_h : C_k^\infty(X, \mathbb{C}) \to C_k^\infty(X, \mathbb{C})$ (*equal to those of* $\Delta_{\omega_h} : C_k^\infty(X, \mathbb{C}) \to C_k^\infty(X, \mathbb{C})$) *acting on* k*-forms. As usual,* $\sharp$ *stands for the cardinal of a set.*

The proof of this statement proceeds along the lines of the one given in [Álvarez López and Kordyukov 2000] for the analogous statement in the foliated case with some simplifications, adjustments and inevitable differences in detail. We spell it out in Section 4. In the proof of Theorem 1.3, we also use our pseudodifferential Laplacian $\widetilde{\Delta} = \partial p''\partial^\star + \partial^\star p''\partial + \Delta'' : C_{p,q}^\infty(X, \mathbb{C}) \to C_{p,q}^\infty(X, \mathbb{C})$ (where p'' is the orthogonal projection onto $\ker \Delta''$) constructed in every bidegree (p, q) in [Popovici 2016] and shown there to induce a Hodge isomorphism between its kernel and the space $E_2^{p,q}$ of bidegree (p, q) featuring on the second page of the Frölicher spectral sequence.

Along with Theorem 1.3 and the pseudodifferential Laplacian $\widetilde{\Delta}$, the third main ingredient in the proof of Theorem 1.2 is the following formula of the Bochner–Kodaira–Nakano type for Hermitian (not necessarily Kähler) metrics ω established by Demailly [1986] (see also [Demailly 2012, VII, §1]), originating in [Griffiths 1969] and also much related to [Ohsawa 1982, Chapter 1, §1]:

$$\Delta'' = \Delta'_\tau + \left[\Lambda, \left[\Lambda, \tfrac{1}{2} i\partial\bar{\partial}\omega\right]\right] - [\partial\omega \wedge \cdot\,, (\partial\omega \wedge \cdot)^\star], \tag{3}$$

where $[\,\cdot, \cdot\,]$ is the usual graded commutator (see, e.g., Notation 1.4 below), $\Lambda = \Lambda_\omega$ is the adjoint of the multiplication operator $\omega \wedge \cdot$, $\tau = \tau_\omega := [\Lambda, \partial\omega \wedge \cdot\,]$ is the torsion operator of ω and $\Delta'_\tau := [\partial + \tau, (\partial + \tau)^\star]$. This formula enables us to compare various Laplacians and finish the proof of Theorem 1.2 in Section 6.

This paper owes much to the ideas and techniques in our main source of inspiration [Álvarez López and Kordyukov 2000] and to the treatment given to the Leray spectral sequence in [Mazzeo and Melrose 1990; Forman 1995], although the setting and the objectives are different.

In the Appendix, we give an estimate of the discrepancy between the Laplacians Δ' and Δ'' under the SKT assumption on the metric ω (see Lemma A.1). This is of independent interest and leads to the lower bound $-Ch^2$ for the operator $\Delta_h - h^2\Delta$ for all $0 < h < 1$ when ω is SKT, where $C \geq 0$ is a constant independent of h that can be chosen to be any upper bound of the nonnegative *bounded* torsion operator $[\bar{\tau}, \bar{\tau}^\star]$ (see Lemma 6.2). In view of Theorem 1.3 and some minor extra arguments, if the lower bound $-Ch^2$ could be improved to 0, Conjecture 1.1 would be solved, but at the moment we are unfortunately short of arguments to perform this improvement.

Notation 1.4. For a given Hermitian metric ω on a given compact complex manifold X, $\langle\langle\ ,\ \rangle\rangle = \langle\langle\ ,\ \rangle\rangle_\omega$ will stand for the L^2 inner product defined by ω on the spaces

$C^\infty_{p,q}(X,\mathbb{C})$ (resp. $C^\infty_k(X,\mathbb{C})$) of smooth differential (p,q)-forms (resp. k-forms) on X, while $\|\ \| = \|\ \|_\omega$ will denote the corresponding L^2-norm. For self-adjoint linear operators A, B on the bigraded algebra $\bigoplus_{p,q} C^\infty_{p,q}(X,\mathbb{C})$, by $A \geq B$ we shall mean (as is the standard convention) that $\langle\langle Au, u\rangle\rangle \geq \langle\langle Bu, u\rangle\rangle$ for every form u lying in the space on which A and B are defined. We shall also use the usual bracket $[A, B] := AB - (-1)^{ab}BA$ for graded linear operators A, B of respective degrees a, b on the algebra $\bigoplus_k \Lambda^k T^\star X$ of differential forms on X.

2. Rescaled Laplacians

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. We fix a Hermitian metric ω on X.

2.1. ***Rescaling the metric.*** The first operation we will consider is a *partial rescaling of* ω in a way that depends solely on the *holomorphic* degree of forms.

Definition 2.1. For all $p, q \in \{0, \ldots, n\}$, all (p,q)-forms u, v and every constant $h > 0$, we define the *pointwise inner product*

$$\langle u, v\rangle_{\omega_h} := h^{2p}\langle u, v\rangle_\omega$$

where $\langle\ ,\ \rangle_\omega$ stands for the pointwise inner product defined by the original Hermitian metric ω.

Note that, for every $h > 0$, we obtain in this way a Hermitian metric ω_h on every vector bundle $\Lambda^{p,q}T^\star X$ of (p,q)-forms on X. The maps

$$\theta_h : \Lambda^{p,q}T^\star X \to \Lambda^{p,q}T^\star X, \quad u \mapsto \theta_h u := h^p u,$$

induce an *isometry* of Hermitian vector bundles $\theta_h : (\Lambda T^\star X, \omega_h) \to (\Lambda T^\star X, \omega)$ since

$$\langle u, v\rangle_{\omega_h} = \langle h^p u, h^p v\rangle_\omega = \langle \theta_h u, \theta_h v\rangle_\omega \quad \text{for all } u, v \in \Lambda^{p,q}T^\star X.$$

In particular, we have defined a Hermitian metric

$$\omega_h = \frac{1}{h^2}\omega, \quad h > 0,$$

on the holomorphic tangent bundle $T^{1,0}X$ of vector fields of type $(1,0)$, or equivalently, a rescaled C^∞ positive-definite $(1,1)$-form $\omega_h = h^{-2}\omega$ on X. This induces a C^∞ positive volume form

$$dV_{\omega_h} := \frac{\omega_h^n}{n!} = \frac{1}{h^{2n}}\frac{\omega^n}{n!} = \frac{1}{h^{2n}}dV_\omega$$

on X, which in turn gives rise, in conjunction with the above pointwise inner product $\langle\ ,\ \rangle_{\omega_h}$, to the following L^2 *inner product*:

$$\langle\langle u,v\rangle\rangle_{\omega_h} := \int_X \langle u,v\rangle_{\omega_h} dV_{\omega_h} = \frac{1}{h^{2n}}\int_X \langle\theta_h u,\theta_h v\rangle_\omega dV_\omega = \frac{1}{h^{2n}}\langle\langle\theta_h u,\theta_h v\rangle\rangle_\omega$$

for all forms $u, v \in C^\infty_{p,q}(X,\mathbb{C})$ and all bidegrees (p,q).

Formula 2.2. For all (p,q)-forms u, v, we have

$$\langle\langle u,v\rangle\rangle_{\omega_h} = \frac{1}{h^{2(n-p)}}\langle\langle u,v\rangle\rangle_\omega, \quad \text{hence} \quad \|u\|_{\omega_h} = h^{-(n-p)}\|u\|_\omega.$$

Proof. The formula follows at once from the last identity and from the fact that $\theta_h u = h^p u$ for all (p,q)-forms u. □

Definition 2.3. Let (X,ω) be a compact Hermitian manifold with $\dim_{\mathbb{C}} X = n$. For every $k = 0,\dots,2n$ and every constant $h>0$, we consider the *d-Laplacian w.r.t. the rescaled metric ω_h* acting on C^∞ k-forms on X:

$$\Delta_{\omega_h} : C^\infty_k(X,\mathbb{C}) \to C^\infty_k(X,\mathbb{C}), \quad \Delta_{\omega_h} := dd^\star_{\omega_h} + d^\star_{\omega_h} d,$$

where $d^\star_{\omega_h}$ is the formal adjoint of d w.r.t. $\langle\langle\ ,\ \rangle\rangle_{\omega_h}$ and $\langle\langle\ ,\ \rangle\rangle_{\omega_h}$ has been extended from the spaces $C^\infty_{p,q}(X,\mathbb{C})$ to $C^\infty_k(X,\mathbb{C}) = \bigoplus_{p+q=k} C^\infty_{p,q}(X,\mathbb{C})$ by sesquilinearity and by imposing that $\langle\langle u,v\rangle\rangle_{\omega_h} = 0$ whenever $u \in C^\infty_{p,q}(X,\mathbb{C})$ and $v \in C^\infty_{r,s}(X,\mathbb{C})$ with $(p,q)\neq(r,s)$.

2.2. *Rescaling the differential.* The second operation we will consider is a *partial rescaling of $d = \partial + \bar\partial$* that applies solely to its component of type $(1,0)$.

Definition 2.4. Let X be a compact complex manifold, $\dim_{\mathbb{C}} X = n$. For every constant $h>0$, let

$$d_h := h\partial + \bar\partial : C^\infty_k(X,\mathbb{C}) \to C^\infty_{k+1}(X,\mathbb{C}), \quad k\in\{0,\dots,2n\}.$$

Some basic properties of the rescaled differential d_h are summed up in the following:

Lemma 2.5. (i) *The operators d and d_h are related by the identity*

$$d_h = \theta_h d \theta_h^{-1}.$$

(ii) $d_h^2 = 0$ *and the d- and d_h-cohomologies are related by the* ***isomorphism***

$$H^k_d(X,\mathbb{C}) \xrightarrow{\simeq} H^k_{d_h}(X,\mathbb{C}), \quad \{u\}_d \mapsto \{\theta_h u\}_{d_h};$$

here $H^k_d(X,\mathbb{C}) = H^k_{DR}(X,\mathbb{C})$ stands for the usual de Rham cohomology groups, and $H^k_{d_h}(X,\mathbb{C})$ for the d_h-cohomology groups

$$\ker(d_h : C^\infty_k(X,\mathbb{C}) \to C^\infty_{k+1}(X,\mathbb{C}))/\operatorname{Im}(d_h : C^\infty_{k-1}(X,\mathbb{C}) \to C^\infty_k(X,\mathbb{C})).$$

Proof. (i) If u is a (p, q)-form, we have

$$\begin{aligned}(\theta_h d\theta_h^{-1})(u) &= \theta_h d(h^{-p}u) \\ &= h^{-p}\theta_h(\partial u) + h^{-p}\theta_h(\bar\partial u) \\ &= h^{-p}h^{p+1}\partial u + h^{-p}h^{p}\bar\partial u \\ &= h\partial u + \bar\partial u = d_h u.\end{aligned}$$

Thus, $d_h = \theta_h d\theta_h^{-1}$ on pure-type forms, so this identity extends to arbitrary forms by linearity.

(ii) On the one hand, $d_h^2 = \theta_h d^2\theta_h^{-1} = 0$; on the other hand, $d_h(\theta_h u) = \theta_h du$, so we have the equivalence: $\theta_h u \in \ker(d_h) \iff u \in \ker d$; $\theta_h u = d_h v$ if and only if $u = d(\theta_h^{-1}v)$, so we have the equivalence: $\theta_h u \in \mathrm{Im}(d_h) \iff u \in \mathrm{Im}\, d$. These equivalences show that the linear map $H^k_d(X, \mathbb{C}) \ni \{u\}_d \mapsto \{\theta_h u\}_{d_h} \in H^k_{d_h}(X, \mathbb{C})$ is well defined and bijective. $\square$

In particular, the spectral sequences induced by the pairs of differentials $(\partial, \bar\partial)$ and $(h\partial, \bar\partial)$ are *isomorphic*, so degenerate at the same page. The first of them is the Frölicher spectral sequence of X.

Definition 2.6. Let (X, ω) be a compact Hermitian manifold with $\dim_{\mathbb{C}} X = n$. For every constant $h > 0$ and every degree $k \in \{0, \dots, 2n\}$, we consider the *d_h-Laplacian w.r.t. the given metric ω* acting on C^∞ k-forms on X:

$$\Delta_h : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C}), \quad \Delta_h := d_h d_h^\star + d_h^\star d_h,$$

where $d_h^\star$ is the formal adjoint of d_h w.r.t. the L^2 inner product induced by ω.

2.3. *Comparison of the two rescaled Laplacians.* We now bring together the above two operations by comparing the corresponding Laplace-type operators. Note that Δ_{ω_h} was defined by the rescaled differential d_h and the original metric ω, while Δ_h was induced by the rescaled metric ω_h and the original differential d.

Lemma 2.7. (i) *If $\theta_h^\star$ and $d_h^\star$ stand for the formal adjoints of θ_h, resp. d_h, w.r.t. the pointwise, resp. L^2, inner product induced by ω, we have*

$$\theta_h^\star = \theta_h \quad \textit{and} \quad d_h^\star = \theta_h^{-1} d^\star \theta_h.$$

(ii) *The adjoints $\partial^\star_{\omega_h}$, $\bar\partial^\star_{\omega_h}$ of ∂, $\bar\partial$ w.r.t. to the metric ω_h, as well as the adjoints $\partial^\star_\omega = \partial^\star$ and $\bar\partial^\star_\omega = \bar\partial^\star$ of ∂, $\bar\partial$ w.r.t. to the metric ω, are related by the formulae*

$$\partial^\star_{\omega_h} = h^2\partial^\star \quad \textit{and} \quad \bar\partial^\star_{\omega_h} = \bar\partial^\star.$$

Consequently, we get

$$\begin{aligned}\Delta_{\omega_h} &= h^2\Delta' + \Delta'' + [\partial, \bar\partial^\star] + h^2[\bar\partial, \partial^\star]\\ &= h^2\Delta' + \Delta'' - [\partial, \bar\tau^\star] - h^2[\bar\tau, \partial^\star] = h^2\Delta' + \Delta'' - [\tau, \bar\partial^\star] - h^2[\bar\partial, \tau^\star],\end{aligned}$$

and

$$\begin{aligned}\Delta_h &= h^2\Delta' + \Delta'' + h[\partial, \bar\partial^\star] + h[\bar\partial, \partial^\star]\\ &= h^2\Delta' + \Delta'' - h[\partial, \bar\tau^\star] - h[\bar\tau, \partial^\star] = h^2\Delta' + \Delta'' - h[\tau, \bar\partial^\star] - h[\bar\partial, \tau^\star],\end{aligned}$$

where the adjoints $\partial^\star$, $\bar\partial^\star$, $\tau^\star$, $\bar\tau^\star$ *and the Laplacians* Δ', Δ'' *are computed w.r.t. the metric* ω, *while*

$$\tau = \tau_\omega := [\Lambda_\omega, \partial\omega\wedge\cdot\,] : C^\infty_{p,q}(X, \mathbb{C}) \to C^\infty_{p+1,q}(X, \mathbb{C})$$

is the ***torsion operator*** (*of type* $(1, 0)$ *and order zero*, *acting on smooth forms of any bidegree* (p, q), *where* Λ_ω *is the adjoint of the multiplication operator* $\omega\wedge\cdot$) *associated with the metric* ω *as defined in* [Demailly 1986] (see also [Demailly 2012, VII, §1]).

In particular, *the second-order Laplacians* Δ_{ω_h} *and* Δ_h *are* ***elliptic*** *since the second-order Laplacians* Δ' *and* Δ'' *are and the deviation terms* $-[\partial, \bar\tau^\star] - h^2[\bar\tau, \partial^\star]$ *and* $-h[\partial, \bar\tau^\star] - h[\bar\tau, \partial^\star]$ *are only of order* 1.

Note that $\langle\langle[\partial, \bar\partial^\star]u, u\rangle\rangle = \langle\langle[\bar\partial, \partial^\star]u, u\rangle\rangle = 0$ *whenever the form u is* ***of pure type*** *and whatever metric is used to define* $\langle\langle\ ,\ \rangle\rangle$ (*because pure-type forms of different bidegrees are orthogonal w.r.t. any metric*), *so*

$$\langle\langle\Delta_{\omega_h}u, u\rangle\rangle = \langle\langle\Delta_h u, u\rangle\rangle = h^2\langle\langle\Delta' u, u\rangle\rangle + \langle\langle\Delta'' u, u\rangle\rangle \tag{4}$$

for every ***pure-type*** *form u*.

(*This fails*, *in general*, *if u is not of pure type*, *unless the metric* ω *is Kähler.*)

(*iii*) *The rescaled Laplacians* Δ_{ω_h} *and* Δ_h *are related by the formula*

$$\Delta_h = \theta_h\Delta_{\omega_h}\theta_h^{-1}. \tag{5}$$

Proof. (i) For any k-forms $u = \sum_{p+q=k} u^{p,q}$ and $v = \sum_{p+q=k} v^{p,q}$, we have

$$\langle\theta_h u, v\rangle_\omega = \sum_{p+q=k}\langle h^p u^{p,q}, v^{p,q}\rangle_\omega = \sum_{p+q=k}\langle u^{p,q}, h^p v^{p,q}\rangle_\omega = \langle u, \theta_h v\rangle_\omega,$$

so $\theta_h^\star = \theta_h$. The second identity in (i) follows by taking conjugates in $d_h = \theta_h d\theta_h^{-1}$.

(ii) For any forms $\alpha \in C^\infty_{p-1,q}(X, \mathbb{C})$ and $\beta \in C^\infty_{p,q}(X, \mathbb{C})$, we have

$$\begin{aligned}
\langle\langle \alpha, \partial^\star_\omega \beta\rangle\rangle_\omega = \langle\langle \partial\alpha, \beta\rangle\rangle_\omega &= \int_X \langle \partial\alpha, \beta\rangle_\omega \, dV_\omega \\
&= \int_X \frac{1}{h^{2p}} \langle \partial\alpha, \beta\rangle_{\omega_h} h^{2n} \, dV_{\omega_h} = h^{2(n-p)} \langle\langle \partial\alpha, \beta\rangle\rangle_{\omega_h} \\
&= h^{2(n-p)} \langle\langle \alpha, \partial^\star_{\omega_h} \beta\rangle\rangle_{\omega_h} = h^{2(n-p)} \int_X h^{2(p-1)} \langle \alpha, \partial^\star_{\omega_h} \beta\rangle_\omega \frac{1}{h^{2n}} \, dV_\omega \\
&= \frac{1}{h^2} \langle\langle \alpha, \partial^\star_{\omega_h} \beta\rangle\rangle_\omega.
\end{aligned}$$

We get $\partial^\star_\omega = h^{-2}\partial^\star_{\omega_h}$, which is the first identity under (ii).

The identity $\bar\partial^\star_{\omega_h} = \bar\partial^\star_\omega$ is proved in the same way by using the fact that $\bar\partial$ acts only on the antiholomorphic degree of forms which is unaffected by the change of metric from ω to ω_h.

Using these formulae, we get

$$\begin{aligned}
\Delta_{\omega_h} &= [\partial + \bar\partial, \partial^\star_{\omega_h} + \bar\partial^\star_{\omega_h}] = [\partial, h^2\partial^\star] + [\bar\partial, \bar\partial^\star] + [\partial, \bar\partial^\star] + [\bar\partial, h^2\partial^\star] \\
&= h^2\Delta' + \Delta'' + [\partial, \bar\partial^\star] + h^2[\bar\partial, \partial^\star]
\end{aligned}$$

and

$$\begin{aligned}
\Delta_h &= [h\partial + \bar\partial, h\partial^\star + \bar\partial^\star] = h^2[\partial, \partial^\star] + [\bar\partial, \bar\partial^\star] + h[\partial, \bar\partial^\star] + h[\bar\partial, \partial^\star] \\
&= h^2\Delta' + \Delta'' + h[\partial, \bar\partial^\star] + h[\bar\partial, \partial^\star].
\end{aligned}$$

On the other hand, we know from [Demailly 1986] (or [Demailly 2012, VII, §1]) that

$$[\partial, \bar\partial^\star] = -[\partial, \bar\tau^\star] = -[\tau, \bar\partial^\star]$$

and, by conjugation, we get

$$[\bar\partial, \partial^\star] = -[\bar\partial, \tau^\star] = -[\bar\tau, \partial^\star].$$

So, the terms measuring the deviations of Δ_{ω_h} and Δ_h from $h^2\Delta' + \Delta''$ are of order 1 and we get the alternative formulae for Δ_{ω_h} and Δ_h spelt out in the statement.

(iii) For any smooth (p, q)-form α, we have

$$\begin{aligned}
(\theta_h \Delta_{\omega_h} \theta_h^{-1})\alpha = \frac{1}{h^p}\theta_h \Delta_{\omega_h}\alpha &= \frac{1}{h^p}\theta_h(h^2\Delta'\alpha) + \frac{1}{h^p}\theta_h(\Delta''\alpha) \\
&\qquad + \frac{1}{h^p}\theta_h([\partial, \bar\partial^\star]\alpha) + \frac{1}{h^p}\theta_h(h^2[\bar\partial, \partial^\star]\alpha) \\
&= \frac{h^2 h^p}{h^p}\Delta'\alpha + \frac{h^p}{h^p}\Delta''\alpha + \frac{h^{p+1}}{h^p}[\partial, \bar\partial^\star]\alpha + \frac{h^2 h^{p-1}}{h^p}[\bar\partial, \partial^\star]\alpha
\end{aligned}$$

$$= h^2\Delta'\alpha + \Delta''\alpha + h[\partial, \bar\partial^\star]\alpha + h[\bar\partial, \partial^\star]\alpha = \Delta_h\alpha.$$

Thus, $\theta_h\Delta_{\omega_h}\theta_h^{-1} = \Delta_h$ on pure-type forms and this identity extends to arbitrary forms by linearity. □

Corollary 2.8. *Let (X, ω) be a compact Hermitian manifold with $\dim_{\mathbb{C}} X = n$. For every constant $h > 0$ and every degree $k \in \{0, \ldots, 2n\}$, the **spectra** of the rescaled Laplacians $\Delta_h, \Delta_{\omega_h} : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C})$ **coincide**, i.e.,*

$$\mathrm{Spec}(\Delta_h) = \mathrm{Spec}(\Delta_{\omega_h}), \tag{6}$$

*and their respective **eigenspaces** are obtained from each other via the rescaling isometry θ_h:*

$$\theta_h(E_{\Delta_{\omega_h}}(\lambda)) = E_{\Delta_h}(\lambda) \quad \textit{for every } \lambda \in \mathrm{Spec}(\Delta_h) = \mathrm{Spec}(\Delta_{\omega_h}), \tag{7}$$

where $E_{\Delta_{\omega_h}}(\lambda)$, resp. $E_{\Delta_h}(\lambda)$, stands for the eigenspace corresponding to the eigenvalue λ of the operator Δ_{ω_h}, resp. Δ_h.

*Thus, Δ_h and Δ_{ω_h} have the **same eigenvalues** with the **same multiplicities**.*

Proof. Let $\lambda \in \mathrm{Spec}(\Delta_{\omega_h})$ and let $\alpha \in E_{\Delta_{\omega_h}}(\lambda) \subset C^\infty_k(X, \mathbb{C})$. So $\Delta_{\omega_h}\alpha = \lambda\alpha$, hence

$$\Delta_h(\theta_h\alpha) = (\theta_h\Delta_{\omega_h}\theta_h^{-1})(\theta_h\alpha) = \theta_h(\lambda\alpha) = \lambda(\theta_h\alpha).$$

Thus, $\lambda \in \mathrm{Spec}(\Delta_h)$ and $\theta_h\alpha \in E_{\Delta_h}(\lambda)$. These implications also hold in reverse order, so we get the equivalences

$$\lambda \in \mathrm{Spec}(\Delta_h) \iff \lambda \in \mathrm{Spec}(\Delta_{\omega_h}) \quad \text{and} \quad \alpha \in E_{\Delta_{\omega_h}}(\lambda) \iff \theta_h\alpha \in E_{\Delta_h}(\lambda).$$

These equivalences amount to (6) and (7). □

Another consequence of the above discussion is a Hodge theory for the d_h-cohomology and the resulting equidimensionality of the kernels of Δ and Δ_h in every degree.

Corollary 2.9. *Let (X, ω) be a compact Hermitian manifold with $\dim_{\mathbb{C}} X = n$. For every constant $h > 0$ and every degree $k \in \{0, \ldots, 2n\}$, the operator $d_h : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C})$ induces the following L^2_ω-orthogonal direct-sum decomposition:*

$$C^\infty_k(X, \mathbb{C}) = \mathcal{H}^k_{\Delta_h}(X, \mathbb{C}) \oplus \mathrm{Im}\, d_h \oplus \mathrm{Im}\, d_h^\star,$$

*where $\mathcal{H}^k_{\Delta_h}(X, \mathbb{C})$ is the kernel of $\Delta_h : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C})$ and $\ker d_h = \mathcal{H}^k_{\Delta_h}(X, \mathbb{C}) \oplus \mathrm{Im}\, d_h$. The vector space $\mathcal{H}^k_{\Delta_h}(X, \mathbb{C})$ is **finite-dimensional**, while $\mathrm{Im}\, d_h$ and $\mathrm{Im}\, d_h^\star$ are **closed** subspaces of $C^\infty_k(X, \mathbb{C})$.*

*This, in turn, induces the **Hodge isomorphism***

$$\mathcal{H}^k_{\Delta_h}(X, \mathbb{C}) \simeq H^k_{d_h}(X, \mathbb{C}), \quad \alpha \mapsto \{\alpha\}_{d_h}.$$

Since $H^k_d(X, \mathbb{C})$ *and* $H^k_{d_h}(X, \mathbb{C})$ *are isomorphic* (*via* θ_h, *see Lemma 2.5*) *and* $\mathcal{H}^k_\Delta(X, \mathbb{C}) \simeq H^k_d(X, \mathbb{C})$ (*by standard Hodge theory*), *we infer that* $\mathcal{H}^k_\Delta(X, \mathbb{C})$ *and* $\mathcal{H}^k_{\Delta_h}(X, \mathbb{C})$ *are* ***isomorphic*** (*although the isomorphism need not be defined by* θ_h). *In particular*,

$$\dim \mathcal{H}^k_{\Delta_h}(X, \mathbb{C}) = \dim \mathcal{H}^k_\Delta(X, \mathbb{C}) \quad \textit{for all } h > 0.$$

Proof. Since X is compact and Δ_h is elliptic and self-adjoint, a standard consequence of Gårding's inequality (see, e.g., [Demailly 2012, VI]) yields the two-space orthogonal decomposition $C^\infty_k(X, \mathbb{C}) = \mathcal{H}^k_{\Delta_h}(X, \mathbb{C}) \oplus \operatorname{Im} \Delta_h$, while this, together with the integrability property $d_h^2 = 0$, further induces the orthogonal splitting $\operatorname{Im} \Delta_h = \operatorname{Im} d_h \oplus \operatorname{Im} d_h^\star$. The same consequence of Gårding's inequality ensures that $\ker \Delta_h$ is finite-dimensional and that the images in $C^\infty_k(X, \mathbb{C})$ of d_h and $d_h^\star$ are closed. □

3. The differentials in the Frölicher spectral sequence

We begin by recalling the well-known construction of the Frölicher spectral sequence in order to fix the notation and to point out the key features for us.

Let X be a compact complex manifold with $\dim_\mathbb{C} X = n$. Recall that the zeroth page E_0 of the Frölicher spectral sequence of X consists of the spaces $E_0^{p,q} := C^\infty_{p,q}(X, \mathbb{C})$ of smooth pure-type forms on X and of the type-$(0, 1)$ differentials $d_0 := \bar\partial$ forming the Dolbeault complex

$$\cdots \xrightarrow{d_0} E_0^{p,q-1} \xrightarrow{d_0} E_0^{p,q} \xrightarrow{d_0} E_0^{p,q+1} \xrightarrow{d_0} \cdots$$

Thus, in every bidegree (p, q), the inclusions $\operatorname{Im} d_0^{p,q-1} \subset \ker d_0^{p,q} \subset E_0^{p,q}$ induce (infinitely many, noncanonical) isomorphisms

$$C^\infty_{p,q}(X, \mathbb{C}) \simeq \operatorname{Im} d_0^{p,q-1} \oplus E_1^{p,q} \oplus (E_0^{p,q} / \ker d_0^{p,q}), \tag{8}$$

where $d_0 = d_0^{p,q} : E_0^{p,q} \to E_0^{p,q+1}$ is the differential d_0 acting in bidegree (p, q) and the $E_1^{p,q} := \ker d_0^{p,q} / \operatorname{Im} d_0^{p,q-1} = H^{p,q}_{\bar\partial}(X, \mathbb{C})$ are the Dolbeault cohomology groups of X.

The first page E_1 of the Frölicher spectral sequence consists of the spaces $E_1^{p,q}$ (i.e., the cohomology of the zeroth page) and of the type-$(1, 0)$ differentials d_1:

$$\cdots \xrightarrow{d_1} E_1^{p-1,q} \xrightarrow{d_1} E_1^{p,q} \xrightarrow{d_1} E_1^{p+1,q} \xrightarrow{d_1} \cdots$$

induced in cohomology by ∂ (i.e., $d_1([\alpha]_{\bar\partial}) := [\partial\alpha]_{\bar\partial}$). Thus, in every bidegree (p, q), the inclusions $\operatorname{Im} d_1^{p-1,q} \subset \ker d_1^{p,q} \subset E_1^{p,q}$ induce (infinitely many, noncanonical) isomorphisms

$$E_1^{p,q} \simeq \operatorname{Im} d_1^{p-1,q} \oplus E_2^{p,q} \oplus (E_1^{p,q} / \ker d_1^{p,q}), \tag{9}$$

where $d_1^{p,q}$ is d_1 acting in bidegree (p,q), while the spaces

$$E_2^{p,q} := \ker d_1^{p,q} / \operatorname{Im} d_1^{p-1,q}$$

form the cohomology of the page E_1.

The remaining pages are constructed inductively: the differentials $d_r = d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ are of type $(r, -r+1)$ for every r, while the spaces $E_r^{p,q} := \ker d_{r-1}^{p,q} / \operatorname{Im} d_{r-1}^{p-r+1,q+r-2}$ on the r-th page are defined as the cohomology of the previous page E_{r-1}. On every page E_r and in every bidegree (p,q), the inclusions $\operatorname{Im} d_r^{p-r,q+r-1} \subset \ker d_r^{p,q} \subset E_r^{p,q}$ induce (infinitely many, noncanonical) isomorphisms

$$E_r^{p,q} \simeq \operatorname{Im} d_r^{p-r,q+r-1} \oplus E_{r+1}^{p,q} \oplus (E_r^{p,q} / \ker d_r^{p,q}), \tag{10}$$

where $E_{r+1}^{p,q} := \ker d_r^{p,q} / \operatorname{Im} d_r^{p-r,q+r-1}$.

It is worth stressing that (8), (9) and (10) only assert that the vector spaces on either side of $\simeq$ are isomorphic, but no choice of preferred isomorphism is possible at this stage.

A classical result of Frölicher [1955] asserts that this spectral sequence converges to the de Rham cohomology of X and degenerates after finitely many steps. This means that there are (noncanonical) isomorphisms

$$H^k_{DR}(X,\mathbb{C}) \simeq \bigoplus_{p+q=k} E_\infty^{p,q}, \quad k = 0, \dots, 2n, \tag{11}$$

where $E_\infty^{p,q} = \cdots = E_{N+2}^{p,q} = E_{N+1}^{p,q} = E_N^{p,q}$ for all p, q and where $N \geq 1$ is the positive integer such that the spectral sequence degenerates at E_N.

3.1. ***Identification of the d_r with restrictions of d.*** Summing up (8), (9), (10) over $r = 0, \dots, N-1$, we get (infinitely many, noncanonical) isomorphisms

$$C^\infty_{p,q}(X,\mathbb{C}) \simeq \bigoplus_{r=0}^{N-1} \operatorname{Im} d_r^{p-r,q+r-1} \oplus E_\infty^{p,q} \oplus \bigoplus_{r=0}^{N-1} (E_r^{p,q} / \ker d_r^{p,q})$$

for every bidegree (p,q). Note that the isomorphisms (8), (9), (10) identify the spaces $\operatorname{Im} d_r^{p-r,q+r-1}$, $E_r^{p,q}$ (including for $r = \infty$) and $E_r^{p,q} / \ker d_r^{p,q}$ with certain subspaces of $C^\infty_{p,q}(X,\mathbb{C})$. However, these subspaces have not been specified yet since multiple choices (and no canonical choice) are possible for the isomorphisms (8), (9), (10). These choices can only be made unique once a Hermitian metric has been fixed on X. (See Section 3.2.)

Now, since $C_k^\infty(X,\mathbb{C}) = \bigoplus_{p+q=k} C_{p,q}^\infty(X,\mathbb{C})$ for all $k = 0, \ldots, 2n$, we get

$$\begin{array}{l} C_k^\infty(X,\mathbb{C}) \simeq \bigoplus\limits_{\substack{0\le r\le N-1\\ p+q=k}} \operatorname{Im} d_r^{p-r,q+r-1} \oplus \bigoplus\limits_{p+q=k} E_\infty^{p,q} \oplus \bigoplus\limits_{\substack{0\le r\le N-1\\ p+q=k}} (E_r^{p,q}/\ker d_r^{p,q}) \\ \Big\downarrow d \\ C_{k+1}^\infty(X,\mathbb{C}) \simeq \bigoplus\limits_{\substack{0\le r\le N-1\\ p+q=k}} \operatorname{Im} d_r^{p,q} \oplus \bigoplus\limits_{p'+q'=k+1} E_\infty^{p',q'} \\ \qquad\qquad\qquad \oplus \bigoplus\limits_{\substack{0\le r\le N-1\\ p+q=k}} (E_r^{p+r,q-r+1}/\ker d_r^{p+r,q-r+1}). \end{array}$$

Thus, under these isomorphisms, the operator $d = d^{(k)} : C_k^\infty(X,\mathbb{C}) \to C_{k+1}^\infty(X,\mathbb{C})$ identifies as

$$d^{(k)} \simeq \bigoplus_{\substack{0\le r\le N-1\\ p+q=k}} d_r^{p,q}, \tag{12}$$

where the isomorphism $d_r^{p,q} : E_r^{p,q}/\ker d_r^{p,q} \to \operatorname{Im} d_r^{p,q}$ is the restriction of $d_r = d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ to the third piece on the right-hand side of (10). The fact that d_r is of type $(r, -r+1)$ will play a key role in the sequel.

On the other hand, summing up the splittings of $C_{p,q}^\infty(X,\mathbb{C})$ over $p \ge s$ for any given s, we get

$$\begin{aligned} \mathcal{A}_s^k &:= \bigoplus_{\substack{p\ge s\\ p+q=k}} C_{p,q}^\infty(X,\mathbb{C}) \\ &\simeq \bigoplus_{\substack{p\ge s\\ p+q=k}} \Bigg[\bigoplus_{r=0}^{N-1} \operatorname{Im} d_r^{p-r,q+r-1} \oplus E_\infty^{p,q} \oplus \bigoplus_{r=0}^{N-1} (E_r^{p,q}/\ker d_r^{p,q}) \Bigg]. \end{aligned}$$

Lemma 3.1. (i) *For every r and every k, let $E_r^k := \bigoplus_{p+q=k} E_r^{p,q}$. Then*

$$\dim E_r^k = \sum_{p+q=k} \dim E_r^{p,q} = b_k + m_r^{k-1} + m_r^k, \quad 0 \le r \le N, \quad 0 \le k \le 2n, \tag{13}$$

where we set $m_r^k := \sum_{l\ge r} \sum_{p+q=k} \dim(E_l^{p,q}/\ker d_l^{p,q})$.

(ii) *For every r and every k, let*

$$L_r^{p,q} := \bigoplus_{l\ge r} (E_l^{p,q}/\ker d_l^{p,q}) \quad \textit{and} \quad L_r^k := \bigoplus_{p+q=k} L_r^{p,q}.$$

Then, $\dim L_r^k = m_r^k$ (*obvious*) *and*, *under the identifications defined by the isomorphisms* (8), (9), (10), *the following inclusions hold*:

$$d(L_r^{p,q}) \subset \mathcal{A}_{p+r}^{p+q+1}, \quad 0 \le r \le N, \quad 0 \le p, q \le n, \tag{14}$$

where $d(L_r^{p,q}) := \bigoplus_{l\geq r} d_l^{p,q}(E_l^{p,q}/\ker d_l^{p,q})$ *in keeping with identification* (12).

Proof. (i) For every fixed r, summing up the splittings (10) with l in place of r over $l \geq r$ and then summing up over $p+q=k$ for every fixed k, we get

$$E_r^k \simeq \bigoplus_{p+q=k} E_\infty^{p,q} \oplus \bigoplus_{l\geq r}\bigoplus_{p+q=k} \operatorname{Im} d_l^{p-l,q+l-1} \oplus \bigoplus_{l\geq r}\bigoplus_{p+q=k}(E_l^{p,q}/\ker d_l^{p,q}).$$

Since $\operatorname{Im} d_l^{p-l,q+l-1} \simeq E_l^{p-l,q+l-1}/\ker d_l^{p-l,q+l-1}$ for all p, q, l, if we set $p' := p-l$ and $q' := q+l-1$, we have $p'+q' = k-1$ when $p+q=k$ and the above isomorphism translates to

$$E_r^k \simeq \bigoplus_{p+q=k} E_\infty^{p,q} \oplus \bigoplus_{l\geq r}\bigoplus_{p'+q'=k-1}(E_l^{p',q'}/\ker d_l^{p',q'}) \oplus \bigoplus_{l\geq r}\bigoplus_{p+q=k}(E_l^{p,q}/\ker d_l^{p,q})$$

for every k. Now, $\dim \bigoplus_{p+q=k} E_\infty^{p,q} = b_k$ (the k-th Betti number of X) thanks to (11), so taking dimensions in the above isomorphism, we get (13).

(ii) Since $d_l^{p,q} : E_l^{p,q}/\ker d_l^{p,q} \to \operatorname{Im} d_l^{p,q}$ is an isomorphism of type $(l, -l+1)$ for all l, p, q, we get for all $l \geq r$,

$$d(L_r^{p,q}) = \bigoplus_{l\geq r} d_l^{p,q}(E_l^{p,q}/\ker d_l^{p,q})$$

and

$$d_l^{p,q}(E_l^{p,q}/\ker d_l^{p,q}) \subset E_l^{p+l,q-l+1} \subset C^\infty_{p+l,q-l+1} \subset \mathcal{A}^{p+q+1}_{p+r}$$

under the identification of each space $E_l^{p+l,q-l+1}$ with a subspace of $C^\infty_{p+l,q-l+1}$ defined by the isomorphisms (8), (9), (10). This proves (14). $\square$

3.2. *Explicit description of the above identifications.* We take this opportunity to point out an explicit description of the differentials d_r in cohomology and of their unique realisations induced by a given Hermitian metric on X.

Lemma 3.2. *Let X be a compact complex manifold with* $\dim_{\mathbb{C}} X = n$.

(i) *For every r and every bidegree (p, q), the vector space of type (p, q) featuring on the r-th page of the Frölicher spectral sequence of X can be explicitly described as the following set of multicohomology classes* (*i.e.*, *each of these is the d_{r-1}-class of a d_{r-2}-class … of a d_1-class of a $\bar\partial$-class*):

(15) $$E_r^{p,q} = \big\{[\dots[[\alpha]_{\bar\partial}]_{d_1}\dots]_{d_{r-1}} \mid \alpha \in C^\infty_{p,q}(X,\mathbb{C}) \text{ such that } \alpha \text{ satisfies condition } (P_r)\big\},$$

where condition (P_r) on α requires the existence of forms $u_l \in C^\infty_{p+l,q-l}(X,\mathbb{C})$ for $l \in \{1, \dots, r-1\}$ such that

(16) $$\bar\partial\alpha = 0, \quad \partial\alpha = \bar\partial u_1, \quad \partial u_1 = \bar\partial u_2, \dots, \partial u_{r-2} = \bar\partial u_{r-1}.$$

(ii) *For every r and every bidegree (p,q), the differential $d_r = d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ featuring on the r-th page of the Frölicher spectral sequence of X is explicitly described as*

$$d_r([\dots[[\alpha]_{\bar\partial}]_{d_1}\dots]_{d_{r-1}}) = [\dots[[\partial u_{r-1}]_{\bar\partial}]_{d_1}\dots]_{d_{r-1}}, \tag{17}$$

for every $[\dots[[\alpha]_{\bar\partial}]_{d_1}\dots]_{d_{r-1}} \in E_r^{p,q}$. Moreover, this description of d_r is independent of the choice of forms $u_l \in C^\infty_{p+l,q-l}(X,\mathbb{C})$ in (16) (*which are unique only modulo* $\ker\bar\partial$).

Proof. These facts are well-known (see [Cordero et al. 1997]). We will only explain the well-definedness of Formula (17) for d_r. Let $(u_1,\dots,u_{r-1})$ and $(u_1 + \zeta_1,\dots,u_{r-1}+\zeta_{r-1})$ be two sets of forms satisfying (16), i.e., $\bar\partial\alpha = 0$, $\partial\alpha = \bar\partial u_1 = \bar\partial(u_1+\zeta_1)$ and

$$\begin{aligned}\partial u_1 &= \bar\partial u_2,\\ \partial(u_1+\zeta_1) &= \bar\partial(u_2+\zeta_2),\dots,\partial u_{r-2} = \bar\partial u_{r-1},\\ \partial(u_{r-2}+\zeta_{r-2}) &= \bar\partial(u_{r-1}+\zeta_{r-1}).\end{aligned}$$

These identities imply the identities

$$\bar\partial\zeta_1 = 0,\quad \partial\zeta_1 = \bar\partial\zeta_2,\dots,\partial\zeta_{r-2} = \bar\partial\zeta_{r-1},$$

which, in turn, imply that ζ_1 satisfies condition (P_{r-1}) (hence defines a multicohomology class lying in $E_{r-1}^{p+1,q-1}$) and that

$$d_{r-1}([\dots[[\zeta_1]_{\bar\partial}]_{d_1}\dots]_{d_{r-2}}) = [\dots[[\partial\zeta_{r-1}]_{\bar\partial}]_{d_1}\dots]_{d_{r-2}} \in \operatorname{Im} d_{r-1}.$$

Consequently,

$$[[\dots[[\partial\zeta_{r-1}]_{\bar\partial}]_{d_1}\dots]_{d_{r-2}}]_{d_{r-1}} = 0,$$

so

$$[\dots[[\partial(u_{r-1}+\zeta_{r-1})]_{\bar\partial}]_{d_1}\dots]_{d_{r-1}} = [\dots[[\partial u_{r-1}]_{\bar\partial}]_{d_1}\dots]_{d_{r-1}}.$$

Thus, the result we get by (17) for $d_r([\dots[[\alpha]_{\bar\partial}]_{d_1}\dots]_{d_{r-1}})$ is the same whether we work with the choices $(u_1,\dots,u_{r-1})$ or $(u_1+\zeta_1,\dots,u_{r-1}+\zeta_{r-1})$. □

Thus, $d\alpha = \partial\alpha$ induces the multicohomology class $d_r([\dots[[\alpha]_{\bar\partial}]_{d_1}\dots]_{d_{r-1}})$. This helps to explain that, intuitively, d acts as d_r on representatives of E_r-classes (see (12)).

Now, recall that infinitely many choices are possible for the isomorphisms (8), (9) and (10). However, any fixed Hermitian metric ω on X selects a unique realisation of each of these isomorphisms and, implicitly, identifies each space $E_r^{p,q}$ with a precise subspace $\mathcal{H}_r^{p,q}$ (depending on ω) of $C^\infty_{p,q}(X,\mathbb{C})$ via an isomorphism $E_r^{p,q} \simeq \mathcal{H}_r^{p,q}$ depending on ω. These *harmonic* subspaces $\mathcal{H}_r^{p,q} \subset C^\infty_{p,q}(X,\mathbb{C})$ are constructed by induction on $r \geq 1$ as follows:

Definition 3.3. Let $\mathcal{H}_1^{p,q} \subset C^\infty_{p,q}(X, \mathbb{C})$ be the orthogonal complement for the L^2_ω-norm of $\operatorname{Im} d_0^{p,q-1}$ in $\ker d_0^{p,q}$. Due to (8), $\mathcal{H}_1^{p,q}$ is isomorphic to $E_1^{p,q}$. In every bidegree (p, q), the linear map $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ induces a linear map (denoted by the same symbol) $d_1^{p,q} : \mathcal{H}_1^{p,q} \to \mathcal{H}_1^{p+1,q}$ via the isomorphisms $\mathcal{H}_1^{p,q} \simeq E_1^{p,q}$ and $\mathcal{H}_1^{p+1,q} \simeq E_1^{p+1,q}$. Let $\mathcal{H}_2^{p,q} \subset \mathcal{H}_1^{p,q} \subset C^\infty_{p,q}(X, \mathbb{C})$ be the orthogonal complement for the L^2_ω-norm of $\operatorname{Im} d_1^{p-1,q}$ in $\ker d_1^{p,q}$ (viewed as subspaces of $\mathcal{H}_1^{p,q}$). Due to (9), $\mathcal{H}_2^{p,q}$ is isomorphic to $E_2^{p,q}$. Continuing inductively, when the linear maps $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ have induced counterparts (denoted by the same symbol) $d_r^{p,q} : \mathcal{H}_r^{p,q} \to \mathcal{H}_r^{p+r,q-r+1}$ between the already constructed subspaces $\mathcal{H}_r^{p,q} \subset C^\infty_{p,q}(X, \mathbb{C})$ and $\mathcal{H}_r^{p+r,q-r+1} \subset C^\infty_{p+r,q-r+1}(X, \mathbb{C})$, we let $\mathcal{H}_{r+1}^{p,q} \subset \mathcal{H}_r^{p,q} \subset C^\infty_{p,q}(X, \mathbb{C})$ be the orthogonal complement for the L^2_ω-norm of $\operatorname{Im} d_r^{p-r,q+r-1}$ in $\ker d_r^{p,q}$ (viewed as subspaces of $\mathcal{H}_r^{p,q}$). Due to (10), $\mathcal{H}_{r+1}^{p,q}$ is isomorphic to $E_{r+1}^{p,q}$.

Note that we have

$$
\begin{aligned}
(18) \quad \mathcal{H}_1^{p,q} &= \ker(\Delta'' : C^\infty_{p,q}(X, \mathbb{C}) \to C^\infty_{p,q}(X, \mathbb{C})) \\
&= \{u \in C^\infty_{p,q}(X, \mathbb{C}) \mid \bar\partial u = 0 \text{ and } \bar\partial^\star u = 0\}, \\
\mathcal{H}_2^{p,q} &= \ker(\widetilde{\Delta} : C^\infty_{p,q}(X, \mathbb{C}) \to C^\infty_{p,q}(X, \mathbb{C})) \\
&= \{u \in C^\infty_{p,q}(X, \mathbb{C}) \mid \bar\partial u = 0, \bar\partial^\star u = 0, p''(\partial u) = 0 \text{ and } p''\partial^\star u = 0\},
\end{aligned}
$$

where $\widetilde{\Delta} = \partial p'' \partial^\star + \partial^\star p'' \partial + \Delta''$ is the pseudodifferential Laplacian constructed in [Popovici 2016].

Also note that standard Hodge theory (for the elliptic differential operator Δ'') is used to ensure that $\operatorname{Im} d_0^{p,q-1}$ is closed in $C^\infty_{p,q}(X, \mathbb{C})$ and that $\mathcal{H}_1^{p,q}$ is finite-dimensional. However, all the other images $\operatorname{Im} d_r^{p-r,q+r-1}$ are automatically closed since they are (necessarily finite-dimensional) vector subspaces of a finite-dimensional vector space. It is also possible to construct pseudodifferential operators $\widetilde{\Delta}_{(r)} : C^\infty_{p,q}(X, \mathbb{C}) \to C^\infty_{p,q}(X, \mathbb{C})$ whose kernels are isomorphic to the spaces $\mathcal{H}_r^{p,q}$ (see forthcoming joint work of the author with L. Ugarte, where the Hodge theory found in [Popovici 2016] for the second page of the Frölicher spectral sequence is extended to all the pages), making these spaces into *harmonic spaces* for these *pseudodifferential Laplacians*, but the mere spaces $\mathcal{H}_r^{p,q}$ suffice for our purposes here.

When the vector space $C^\infty_{p,q}(X, \mathbb{C})$ is endowed with the L^2-norm induced by ω, every subspace $\mathcal{H}_r^{p,q}$ inherits the restricted norm. On the other hand, every space $E_r^{p,q}$ has a quotient norm induced by the L^2_ω-norm. The isomorphisms $E_r^{p,q} \simeq \mathcal{H}_r^{p,q}$ constructed above are isometries when $E_r^{p,q}$ and $\mathcal{H}_r^{p,q}$ are endowed with the quotient, resp. L^2 norms.

Conclusion 3.4. Let X be a compact complex manifold and let ω be any Hermitian metric on X. Let $\cdots \subset \mathcal{H}_{r+1}^{p,q} \subset \mathcal{H}_r^{p,q} \subset \cdots \subset \mathcal{H}_1^{p,q} \subset C^\infty_{p,q}(X, \mathbb{C})$ be the subspaces of Definition 3.3 induced by ω.

For every r and every bidegree (p, q), each class $[\dots[[\alpha]_{\bar\partial}]_{d_1}\dots]_{d_{r-1}} \in E_r^{p,q}$ contains a unique representative $\alpha \in \mathcal{H}_r^{p,q}$ (necessarily satisfying condition (P_r)). For $l \in \{1, \dots, r-1\}$, let $u_l \in C^\infty_{p+l,q-l}(X, \mathbb{C})$ be the *unique* solutions with *minimal* L^2_ω-norms of the equations

$$\bar\partial\alpha = 0, \quad \partial\alpha = \bar\partial u_1, \quad \partial u_1 = \bar\partial u_2, \dots, \partial u_{r-2} = \bar\partial u_{r-1}$$

constructed inductively from one another. The well-known Neumann formula yields

$$u_1 = \Delta''^{-1}\bar\partial^\star(\partial\alpha) \quad \text{and} \quad u_l = \Delta''^{-1}\bar\partial^\star(\partial u_{l-1}) \quad \text{for } l \in \{2, \dots, r-1\}.$$

In particular, the maps $\alpha \mapsto u_1$ and $u_{l-1} \mapsto u_l$ are linear.

For all r, p, q, we define the linear operator

$$T_r = T_r^{p,q} : \mathcal{H}_r^{p,q} \to C^\infty_{p+r,q-r+1}(X, \mathbb{C}), \quad \alpha \mapsto T_r(\alpha) := \partial u_{r-1}.$$

Since $\mathcal{H}_r^{p,q}$ is finite-dimensional, T_r is bounded, so there exists a constant $C_r^{p,q} > 0$ such that

$$\|T_r(\alpha)\| = \|\partial u_{r-1}\| \le C_r^{p,q}\|\alpha\| \quad \text{for all } \alpha \in \mathcal{H}_r^{p,q}.$$

It is easy to see that $T_r(\alpha)$ need not belong to $\mathcal{H}_r^{p+r,q-r+1}$ when $\alpha \in \mathcal{H}_r^{p,q}$. If we let $P_r^{p,q} : C^\infty_{p,q}(X, \mathbb{C}) \to \mathcal{H}_r^{p,q}$ be the L_ω-orthogonal projection onto $\mathcal{H}_r^{p,q}$, we get

$$\|(P_r^{p,q} \circ T_r)(\alpha)\| = \|P_r^{p,q}(\partial u_{r-1})\| \le \|\partial u_{r-1}\| \le C_r^{p,q}\|\alpha\| \quad \text{for all } \alpha \in \mathcal{H}_r^{p,q}.$$

4. Use of the rescaled Laplacians in the study of the Frölicher spectral sequence

In this section, we prove Theorem 1.3.

As in [Efremov and Shubin 1989; Gromov and Shubin 1991; Álvarez López and Kordyukov 2000], we consider the *spectrum distribution function* associated with any of the rescaled Laplacians Δ_h, Δ_{ω_h} in our context. Its definition and its study are made far simpler in this setting than in those references by the manifold X being *compact* and by the Laplacians Δ', Δ'' being *elliptic*.

Notation 4.1. Let (X, ω) be a compact Hermitian manifold with $\dim_{\mathbb{C}} X = n$. For every $k \in \{0, \dots, n\}$ and every constant $\lambda \ge 0$, let $N_h^k(\lambda)$ stand for the number of eigenvalues (counted with multiplicities) of Δ_h that are $\le \lambda$.

Replacing Δ_h with Δ_{ω_h} does not change the spectrum distribution function $N_h^k : [0, +\infty) \to \mathbb{N}$ since Δ_h and Δ_{ω_h} have the same eigenvalues with the same multiplicities (see Corollary 2.8). Theorem 1.3 can be reworded to ensure the existence of a constant $C > 0$ independent of h such that, for all r and k, we have

$$\dim E_r^k = N_h^k(Ch^{2r}) \quad \text{when } 0 < h \ll 1. \tag{19}$$

4.1. *The Efremov–Shubin variational principle.* The main technical ingredient we will need is the following variant of the *variational principle* proved in a more general context in [Efremov and Shubin 1989] and used extensively thereafter (e.g., [Gromov and Shubin 1991; Álvarez López and Kordyukov 2000]) in settings different from ours. We adapt to our situation the result of Efremov and Shubin.

Proposition 4.2 [Efremov and Shubin 1989]. *Let* (X, ω) *be a compact Hermitian manifold with* $\dim_{\mathbb{C}} X = n$. *For every* $k = 0, \ldots, 2n$ *and every* $\lambda \geq 0$, *the following identity holds*:

$$N_h^k(\lambda) = F_h^{k-1}(\lambda) + b_k + F_h^k(\lambda), \tag{20}$$

where b_k *is the* k*-th Betti number of* X *and the function* $F_h^k : [0, +\infty) \to \mathbb{N}$ *is defined by*

$$F_h^k(\lambda) = \sup_L \dim L, \tag{21}$$

where L *ranges over the* ***closed*** *vector subspaces of the quotient space*

$$C_k^\infty(X, \mathbb{C}) / \ker d$$

on which the operator

$$d : C_k^\infty(X, \mathbb{C}) / \ker d \to C_{k+1}^\infty(X, \mathbb{C})$$

induced by $d : C_k^\infty(X, \mathbb{C}) \to C_{k+1}^\infty(X, \mathbb{C})$ *satisfies the following* $L^2_{\omega_h}$*-norm estimate*:

$$\|d\zeta\|_{\omega_h} \leq \sqrt{\lambda}\|\zeta\|_{\omega_h} \quad \text{for every } \zeta \in L. \tag{22}$$

(*The understanding is that* $\|d\zeta\|_{\omega_h}$ *stands for the usual* L^2*-norm induced by the metric* ω_h, *while* $\|\zeta\|_{\omega_h}$ *stands for the* ***quotient norm*** *induced on* $C_k^\infty(X, \mathbb{C}) / \ker d$ *by the* $L^2_{\omega_h}$*-norm.*)

We will present a detailed proof of this statement along the lines of [Efremov and Shubin 1989] with a few minor simplifications afforded by our special setting where the manifold X is ***compact*** and the operator Δ_h is ***elliptic***. While a more general version for unbounded operators on L^2 spaces was needed in [Álvarez López and Kordyukov 2000], we stress that, in this context, we can confine ourselves to the case of operators on spaces of C^∞ differential forms.

The main step is the following statement (a version of the classical min-max principle) that was proved in a more general setting in [Efremov and Shubin 1989].

Proposition 4.3. *Let* (X, ω) *be a* ***compact*** *Hermitian manifold with* $\dim_{\mathbb{C}} X = n$. *For an arbitrary* $k \in \{0, \ldots, 2n\}$, *let* $P : C_k^\infty(X, \mathbb{C}) \to C_k^\infty(X, \mathbb{C})$ *be an* ***elliptic***, *self-adjoint and nonnegative differential operator of order* ≥ 1.

Then, for every $\lambda \geq 0$, the ***spectrum distribution function*** N_k ***of*** P (*i.e.*, $N_k(\lambda)$ *is defined to be the number of eigenvalues of* P, *counted with multiplicities*, *that are less than or equal to* λ) *is given by the following identities* (*in which the suprema are actually maxima*):

$$N_k(\lambda) = \sup_{L \in \mathcal{L}_\lambda^{(k)}} \dim L = \sup_{E \in \mathcal{P}_\lambda^{(k)}} \operatorname{Tr} E, \tag{23}$$

where $\mathcal{L}_\lambda^{(k)}$ *stands for the set of* ***closed*** *vector subspaces* $L \subset C_k^\infty(X, \mathbb{C})$ *such that*

$$\langle\langle Pu, u\rangle\rangle \leq \lambda \|u\|^2 \quad \textit{for all } u \in L,$$

while $\mathcal{P}_\lambda^{(k)}$ *stands for the set of all bounded linear operators* $E : C_k^\infty(X, \mathbb{C}) \to C_k^\infty(X, \mathbb{C})$ *satisfying the conditions*:

(i) $E^2 = E = E^\star$ (*i.e.*, E *is an* ***orthogonal projection*** *w.r.t. the* L^2_ω *inner product*);

(ii) $\langle\langle Pu, u\rangle\rangle \leq \lambda \|u\|^2$ *for all* $u \in \operatorname{Im} E$. (*In other words*, E *is the orthogonal projection onto one of the subspaces* $L \in \mathcal{L}_\lambda^{(k)}$, *so* $L = \operatorname{Im} E$ *for some* $L \in \mathcal{L}_\lambda^{(k)}$.)

Proof. The second identity in (23) follows at once from the fact that the dimension of any closed subspace $L \subset C_k^\infty(X, \mathbb{C})$ equals the trace of the orthogonal projection onto L. So, we only have to prove the first identity in (23).

Since X is compact and P is elliptic, self-adjoint and nonnegative, the spectrum of P is discrete and consists of nonnegative eigenvalues, while there exists a countable orthonormal (w.r.t. the L^2_ω inner product) basis of $C_k^\infty(X, \mathbb{C})$ (and of the Hilbert space $L_k^2(X, \mathbb{C})$ of L^2 k-forms) consisting of eigenvectors of P. For every $\mu \geq 0$, let $E_P(\mu) \subset C_k^\infty(X, \mathbb{C})$ be the eigenspace of P corresponding to the eigenvalue μ (with the understanding that $E_P(\mu) = \{0\}$ if μ is not an actual eigenvalue). The spaces $E_P(\mu)$ are finite-dimensional and consist of C^∞ forms since P is assumed to be elliptic (hence also hypoelliptic) and X is compact.

For every $\lambda \geq 0$, let $L_\lambda := \bigoplus_{0 \leq \mu \leq \lambda} E_P(\mu) \subset C_k^\infty(X, \mathbb{C})$. Thus, L_λ is finite-dimensional and $\dim L_\lambda = N_k(\lambda)$, while $\langle\langle Pu, u\rangle\rangle \leq \lambda \|u\|^2$ for all $u \in L_\lambda$. Hence $L_\lambda \in \mathcal{L}_\lambda^{(k)}$, so $N_k(\lambda) \leq \sup_{L \in \mathcal{L}_\lambda^{(k)}} \dim L$.

To prove the reverse inequality, let $\lambda \geq 0$ and let $L \in \mathcal{L}_\lambda^{(k)}$. The existence of an orthonormal basis of eigenvectors implies the orthogonal direct-sum decomposition

$$C_k^\infty(X, \mathbb{C}) = \bigoplus_{0 \leq \mu \leq \lambda} E_P(\mu) \oplus \bigoplus_{\mu > \lambda} E_P(\mu).$$

In particular, $\bigoplus_{\mu > \lambda} E_P(\mu) = \ker E_\lambda$, where E_λ is the orthogonal projection onto $\bigoplus_{0 \leq \mu \leq \lambda} E_P(\mu)$.

Now, $\langle\langle Pu, u\rangle\rangle > \lambda \|u\|^2$ for all $u \in \bigoplus_{\mu > \lambda} E_P(\mu) \setminus \{0\}$, while $\langle\langle Pu, u\rangle\rangle \leq \lambda \|u\|^2$ for all $u \in L$. So, $L \cap \ker E_\lambda = L \cap \bigoplus_{\mu > \lambda} E_P(\mu) = \{0\}$. This implies that the restriction

$$E_{\lambda|L} : L \to \operatorname{Im} E_\lambda = \bigoplus_{0\leq\mu\leq\lambda} E_P(\mu)$$

is *injective*. In particular, $\dim L \leq \dim \bigoplus_{0\leq\mu\leq\lambda} E_P(\mu) = N_k(\lambda)$. Since L has been chosen arbitrarily in $\mathcal{L}_\lambda^{(k)}$, we conclude that $\sup_{L\in\mathcal{L}_\lambda^{(k)}} \dim L \leq N_k(\lambda)$ and we are done. □

The second step towards proving Proposition 4.2 is the standard 3-space decomposition used in Hodge theory. For every $k = 0, \dots, 2n$, the operator $\Delta_{\omega_h} : C_k^\infty(X, \mathbb{C}) \to C_k^\infty(X, \mathbb{C})$ is elliptic and since the manifold X is compact and $d^2 = 0$, we have the $L^2_{\omega_h}$-orthogonal decomposition

$$C_k^\infty(X, \mathbb{C}) = \mathcal{H}^k_{\Delta_{\omega_h}}(X, \mathbb{C}) \oplus E_k(X, \mathbb{C}) \oplus E_k^\star(X, \mathbb{C}), \tag{24}$$

where $\ker d = \mathcal{H}^k_{\Delta_{\omega_h}}(X, \mathbb{C}) \oplus E_k(X, \mathbb{C})$ and $\mathcal{H}^k_{\Delta_{\omega_h}}(X, \mathbb{C})$ is the kernel of $\Delta_{\omega_h} : C_k^\infty(X, \mathbb{C}) \to C_k^\infty(X, \mathbb{C})$, $E_k(X, \mathbb{C}) := \operatorname{Im}(d : C_{k-1}^\infty(X, \mathbb{C}) \to C_k^\infty(X, \mathbb{C}))$ and $E_k^\star(X, \mathbb{C}) := \operatorname{Im}(d^\star_{\omega_h} : C_{k+1}^\infty(X, \mathbb{C}) \to C_k^\infty(X, \mathbb{C}))$.

Moreover, each of the three subspaces into which $C_k^\infty(X, \mathbb{C})$ splits in (24) is Δ_{ω_h}-*invariant*, i.e.,

$$\Delta_{\omega_h}(\mathcal{H}^k_{\Delta_{\omega_h}}(X, \mathbb{C})) \subset \mathcal{H}^k_{\Delta_{\omega_h}}(X, \mathbb{C}), \quad \Delta_{\omega_h}(E_k(X, \mathbb{C})) \subset E_k(X, \mathbb{C}),$$
$$\Delta_{\omega_h}(E_k^\star(X, \mathbb{C})) \subset E_k^\star(X, \mathbb{C}),$$

because Δ_{ω_h} commutes with d and with $d^\star_{\omega_h}$. The invariance implies that an $L^2_{\omega_h}$-orthonormal basis $\{e_i^k(h)\}_{i\in\mathbb{N}^\star}$ of $C_k^\infty(X, \mathbb{C})$ consisting of eigenvectors for Δ_{ω_h} (whose existence follows from the standard elliptic theory) can be chosen such that each $e_i^k(h)$ belongs to one and only one of the subspaces $\mathcal{H}^k_{\Delta_{\omega_h}}(X, \mathbb{C})$, $E_k(X, \mathbb{C})$ and $E_k^\star(X, \mathbb{C})$. Let $0 \leq \lambda_1^k(h) \leq \cdots \leq \lambda_i^k(h) \leq \cdots$ be the corresponding eigenvalues, counted with multiplicities, of the rescaled Laplacian $\Delta_h : C_k^\infty(X, \mathbb{C}) \to C_k^\infty(X, \mathbb{C})$ (equal to those of $\Delta_{\omega_h} : C_k^\infty(X, \mathbb{C}) \to C_k^\infty(X, \mathbb{C})$). Thus, $\Delta_{\omega_h} e_i^k(h) = \lambda_i^k(h) e_i^k(h)$ for all i.

Consequently, we can define functions $F_h^k : [0, +\infty) \to \mathbb{N}$ and $G_h^k : [0, +\infty) \to \mathbb{N}$ by

$$F_h^k(\lambda) := \sharp\{i \mid e_i^k(h) \in E_k^\star(X, \mathbb{C}) \quad \text{and} \quad \lambda_i^k(h) \leq \lambda\}$$

and

$$G_h^k(\lambda) := \sharp\{i \mid e_i^k(h) \in E_k(X, \mathbb{C}) \quad \text{and} \quad \lambda_i^k(h) \leq \lambda\}.$$

These definitions of F_h^k and $G_h^k(\lambda)$ are independent of the choice of orthonormal basis $\{e_i^k(h)\}_{i\in\mathbb{N}^\star}$ of $C_k^\infty(X, \mathbb{C})$ satisfying the above properties.

Lemma 4.4. *The functions F_h^k and G_h^k are the spectrum distribution functions of the restrictions $\Delta_{\omega_h|E_k^\star(X,\mathbb{C})} : E_k^\star(X, \mathbb{C}) \to E_k^\star(X, \mathbb{C})$, resp. $\Delta_{\omega_h|E_k(X,\mathbb{C})} : E_k(X, \mathbb{C}) \to E_k(X, \mathbb{C})$.*

In other words, they are described as follows:

$$F_h^k(\lambda) = \sup_{L\in\mathcal{L}_\lambda^{''(k)}} \dim L, \quad G_h^k(\lambda) = \sup_{L\in\mathcal{L}_\lambda^{'(k)}} \dim L, \tag{25}$$

where $\mathcal{L}_\lambda^{''(k)}$ *stands for the set of* ***closed*** *vector subspaces* $L \subset E_k^\star(X, \mathbb{C})$ *such that*

$$\|du\|^2_{\omega_h} \leq \lambda\|u\|^2_{\omega_h} \quad \textit{for all } u \in L, \tag{26}$$

and $\mathcal{L}_\lambda^{'(k)}$ *stands for the set of* ***closed*** *vector subspaces* $L \subset E_k(X, \mathbb{C})$ *such that*

$$\|d^\star_{\omega_h} u\|^2_{\omega_h} \leq \lambda\|u\|^2_{\omega_h} \quad \textit{for all } u \in L. \tag{27}$$

Proof. This is an immediate application of the *variational principle* of Proposition 4.3 to the restrictions $\Delta_{\omega_h|E_k^\star(X,\mathbb{C})} : E_k^\star(X, \mathbb{C})) \to E_k^\star(X, \mathbb{C})$ and

$$\Delta_{\omega_h|E_k(X,\mathbb{C})} : E_k(X, C)) \to E_k(X, \mathbb{C}).$$

Estimates (26) and (27) are consequences of the identity $\langle\langle\Delta_{\omega_h}u, u\rangle\rangle_{\omega_h} = \|du\|^2_{\omega_h} + \|d^\star_{\omega_h}u\|^2_{\omega_h}$ and of the fact that $d^\star_{\omega_h}u = 0$ whenever $u \in E_k^\star(X, \mathbb{C})$ (since $\operatorname{Im} d^\star_{\omega_h} \subset \ker d^\star_{\omega_h}$) and that $du = 0$ whenever $u \in E_k(X, \mathbb{C})$ (since $\operatorname{Im} d \subset \ker d$). □

The last ingredient we need is the following very simple observation.

Lemma 4.5. *For every* $\lambda \geq 0$ *and every* $k \in \{-1, 0, \dots, 2n\}$, *we have*

$$F_h^k(\lambda) = G_h^{k+1}(\lambda)$$

with the understanding that

$$F_h^{-1}(\lambda) = G_h^{2n+1}(\lambda) = 0.$$

Proof. We know from the orthogonal decompositions (24) that the restriction of d to $E_k^\star(X, \mathbb{C})$ is injective, so

$$d_{|E_k^\star(X,\mathbb{C})} : E_k^\star(X, \mathbb{C}) \to E_{k+1}(X, \mathbb{C})$$

is an isomorphism. Moreover, $d\Delta_{\omega_h} = \Delta_{\omega_h}d$, so whenever $\Delta_{\omega_h}u_i = \lambda_i^k(h)u_i$, we get $\Delta_{\omega_h}(du_i) = \lambda_i^k(h)(du_i)$. Combined with the above isomorphism, with the invariance of $E_k^\star(X, \mathbb{C})$ under Δ_{ω_h} and with the definitions of $F_k^h(\lambda)$ and $G_{k+1}^h(\lambda)$, this implies the contention. □

Proof of Proposition 4.2. Putting together (24), the definitions of $F_h^k(\lambda)$ and $G_h^k(\lambda)$ and the fact that the Hodge isomorphism $\mathcal{H}^k_{\Delta_{\omega_h}} \simeq H^k_{DR}(X, \mathbb{C})$ (which follows at once from (24)) implies $b_k = \dim \mathcal{H}^k_{\Delta_{\omega_h}}$, we get

$$N_h^k(\lambda) = b_k + G_h^k(\lambda) + F_h^k(\lambda)$$

for all k and all $\lambda \geq 0$. Using Lemma 4.5, this is equivalent to (20).

On the other hand, the descriptions (25) and (26) of $F_h^k(\lambda)$ coincide with the descriptions (21) and (22) thanks to the isomorphism $E_k^\star(X, \mathbb{C}) \simeq C_k^\infty(X, \mathbb{C})/\ker d$, which is another consequence of the decompositions (24). $\square$

4.2. ***Metric independence of asymptotics.*** Although the following statement has no impact on either the statement of Theorem 1.3 or its proof, we pause briefly to show, exactly as in the foliated case of [Álvarez López and Kordyukov 2000], that the asymptotics of the eigenvalues $\lambda_i^k(h)$ and of the spectrum distribution function N_h^k as $h \downarrow 0$ depend only on the complex structure of X. The proof is an easy application of the variational principle of Proposition 4.2.

Proposition 4.6. *The asymptotics of the* $\lambda_i^k(h)$ *and of* N_h^k *as* $h \downarrow 0$ *are independent of the choice of Hermitian metric* ω.

Proof. We adapt to our setting the proof of the corresponding result in [Álvarez López and Kordyukov 2000]. Let ω and ω' be two Hermitian metrics on X. They induce, respectively, rescaled metrics $(\omega_h)_{h>0}$ and $(\omega'_h)_{h>0}$. Let $N_h^{'k}(\lambda) = F_h^{'k-1}(\lambda) + b_k + F_h^{'k}(\lambda)$ be the spectrum distribution function associated with the rescaled Laplacian $\Delta_{\omega'_h} : C_k^\infty(X, \mathbb{C}) \to C_k^\infty(X, \mathbb{C})$, written as in (20).

Since X is compact, there exists a constant $C > 0$ such that the respective L^2-norms satisfy the following inequalities in every bidegree (p, q):

$$\frac{1}{C}\| \ \|_\omega \le \| \ \|_{\omega'} \le C\| \ \|_\omega,$$

hence

$$\frac{1}{C}\| \ \|_{\omega_h} \le \| \ \|_{\omega'_h} \le C\| \ \|_{\omega_h} \quad \text{on } L^2_{p,q}(X, \mathbb{C}) \text{ for every } h > 0.$$

The constant C is independent of $h > 0$ thanks to Formula 2.2.

Hence, for every $\zeta \in C_k^\infty(X, \mathbb{C})/\ker d$ such that $\|d\zeta\|_{\omega_h} \le \sqrt{\lambda}\|\zeta\|_{\omega_h}$, we get $\|d\zeta\|_{\omega'_h} \le \sqrt{C^4\lambda}\|\zeta\|_{\omega'_h}$. Thanks to Proposition 4.2, this implies

$$F_h^k(\lambda) \le F_h^{'k}(C^4\lambda), \quad \lambda \ge 0, \quad h > 0.$$

By symmetry, we also get $F_h^{'k}(\lambda) \le F_h^k(C^4\lambda)$, so putting the last two inequalities together, we get

$$F_h^{'k}(C^{-4}\lambda) \le F_h^k(\lambda) \le F_h^{'k}(C^4\lambda), \quad \lambda \ge 0, \quad h > 0. \qquad \square$$

4.3. ***Proof of the inequality "≤" in Theorem 1.3.*** We are now in a position to prove the following:

Theorem 4.7. *Let* (X, ω) *be a compact Hermitian manifold with* $\dim_{\mathbb{C}} X = n$. *For every* r *and every* $k = 0, \dots, 2n$, *the following inequality holds*:

$$\dim E_r^k \le \sharp\{i \mid \lambda_i^k(h) \in O(h^{2r}) \quad \text{as } h \downarrow 0\}. \tag{28}$$

Proof. We have to prove the existence of a uniform constant $C > 0$ such that $\dim E_r^k \leq N_h^k(Ch^{2r})$ for all r, k and all $0 < h \ll 1$. Recall the following facts:

(i) $\dim E_r^k = b_k + m_r^{k-1} + m_r^k$, where $m_r^k := \dim L_r^k$ and

$$L_r^k := \bigoplus_{p+q=k} L_r^{p,q} = \bigoplus_{p+q=k} \bigoplus_{l \geq r} (E_l^{p,q} / \ker d_l^{p,q})$$

(proved in (13) of Lemma 3.1).

(ii) $N_h^k(\lambda) = b_k + F_h^{k-1}(\lambda) + F_h^k(\lambda)$ for all $\lambda \geq 0$ (see (20) of Proposition 4.2).

Thus, it suffices to prove that

$$m_r^k \leq F_h^k(Ch^{2r}) \quad \text{for all } 0 < h \ll 1, \tag{29}$$

for a uniform constant $C > 0$ and for all r and k.

Now, thanks to the definition (21) of F_h^k, to prove (29) it suffices to prove that L_r^k is one of the subspaces of $C_k^\infty(X, \mathbb{C}) / \ker d$ contributing to the definition of $F_h^k(Ch^{2r})$ for some uniform constant $C > 0$. In other words, it suffices to prove that there exists $C > 0$ such that

$$\|d\zeta\|_{\omega_h} \leq \sqrt{C} h^r \|\zeta\|_{\omega_h} \quad \text{for all } \zeta \in L_r^k \text{ and all } 0 < h \ll 1. \tag{30}$$

Meanwhile, every $\zeta \in L_r^k = \bigoplus_{p+q=k} L_r^{p,q}$ splits uniquely as $\zeta = \sum_{p+q=k} \zeta^{p,q}$ with $\zeta^{p,q} \in L_r^{p,q}$ for all p, q. Thus, it suffices to prove that, for a uniform constant $C > 0$, we have

$$\|d\zeta^{p,q}\|_{\omega_h} \leq \sqrt{C} h^r \|\zeta^{p,q}\|_{\omega_h} \text{ for all } p, q, \text{ all } \zeta^{p,q} \in L_r^{p,q} \text{ and all } 0 < h \ll 1. \tag{31}$$

This holds mainly because d_r is of type $(r, -r+1)$, so d_r increases the holomorphic degree by r and thus the norm $|\ |_{\omega_h}$ brings out an extra factor h^r. Specifically, for every $\zeta^{p,q} \in L_r^{p,q}$, (14) of Lemma 3.1 yields $d\zeta^{p,q} \in d(L_r^{p,q}) \subset \mathcal{A}_{p+r}^{p+q-1}$. Therefore, the holomorphic degree of $d\zeta^{p,q}$ is $\geq p + r$, so from Formula 2.2 we get

$$\|d\zeta^{p,q}\|_{\omega_h} \leq \frac{h^{p+r}}{h^n} \|d\zeta^{p,q}\|_\omega \quad \text{for all } p, q, \text{ all } \zeta^{p,q} \in L_r^{p,q} \text{ and all } 0 < h < 1.$$

Now, $L_r^{p,q}$ is a *finite-dimensional* vector subspace of $C_k^\infty(X, \mathbb{C}) / \ker d$, so there exists a constant $C_r > 0$ (depending on r, p, q, but independent of h) such that $\|d\zeta^{p,q}\|_\omega \leq C_r \|\zeta^{p,q}\|_\omega$ for all $\zeta^{p,q} \in L_r^{p,q}$. Meanwhile, Formula 2.2 tells us again that $\|\zeta^{p,q}\|_\omega = (h^n / h^p) \|\zeta^{p,q}\|_{\omega_h}$, so putting the last three relations together, we get

$$\|d\zeta^{p,q}\|_{\omega_h} \leq C_r h^r \|\zeta^{p,q}\|_{\omega_h} \quad \text{for all } p, q, \text{ all } \zeta^{p,q} \in L_r^{p,q} \text{ and all } 0 < h < 1.$$

This proves (31) after setting $C := \max_{\substack{0 \leq r \leq N \\ 0 \leq p,q \leq n}} C_r^2 > 0$. □

Note that L_r^k is a vector space of classes of cohomology classes, rather than of differential forms, so what is meant by L_r^k in the above proof is its image in $C_k^\infty(X,\mathbb{C})/\ker d$ under the isometries explained in Section 3.2. We can use these isometries, the identification of d acting on $\mathcal{H}_r^{p,q}$ with d_r and Conclusion 3.4 in the following way to make the above proof even more explicit. If we choose $\zeta^{p,q}$ to be the ω_h-*harmonic* representative of its class (also denoted by $\zeta^{p,q}$) and to play the role of α of Conclusion 3.4, we can rewrite the above inequalities in a more detailed form as follows:

$$\begin{aligned}\|d\zeta^{p,q}\|_{\omega_h} = \|(P(\partial u_{r-1})\|_{\omega_h} &\leq \frac{h^{p+r}}{h^n}\|(P\circ T)(\zeta^{p,q})\|_\omega \\ &\leq \frac{h^{p+r}}{h^n}C_r\|\zeta^{p,q}\|_\omega = C_r h^r\|\alpha\|_{\omega_h},\end{aligned}$$

where P and T are the linear maps $P_r^{p,q}$ and $T_r^{p,q}$ (with indices removed) of Conclusion 3.4 that was used above, while $\|\ \|_{\omega_h}$ stands for the $L^2_{\omega_h}$-norm when applied to a form and for the induced quotient norm when applied to a class.

4.4. ***Preliminaries to the proof of the inequality "≥" in Theorem 1.3.*** We will need a few simple observations.

Lemma 4.8. *Let* (X,ω) *be a compact Hermitian manifold with* $\dim_{\mathbb{C}} X = n$. *For every bidegree* (p,q) *and every* (p,q)*-form* u *on* X, *the following identities hold*:

$$\langle\langle \Delta_h u, u\rangle\rangle_\omega = h^{2(n-p)}\langle\langle \Delta_{\omega_h} u, u\rangle\rangle_{\omega_h} = h^{2(n-p)}(\|du\|^2_{\omega_h} + \|d^\star_{\omega_h}u\|^2_{\omega_h}). \tag{32}$$

Proof. The latter identity is obvious, so we will only prove the former one. Since u is of pure type, (4) yields the first identity below, while the second identity follows from Formula 2.2:

$$\begin{aligned}\langle\langle \Delta_h u, u\rangle\rangle_\omega &= h^2\langle\langle \Delta' u, u\rangle\rangle_\omega + \langle\langle \Delta'' u, u\rangle\rangle_\omega \\ &= h^2 h^{2(n-p)}\langle\langle \Delta' u, u\rangle\rangle_{\omega_h} + h^{2(n-p)}\langle\langle \Delta'' u, u\rangle\rangle_{\omega_h} \\ &= h^{2(n-p)}\langle\langle \Delta_{\omega_h} u, u\rangle\rangle_{\omega_h}.\end{aligned}$$

The last identity follows again from (4). □

Lemma 4.9. *Let* $u \in C^\infty_{p,q}(X,\mathbb{C})$ *be an arbitrary form. Considering the splitting*

$$d = d^{(k)} = \bigoplus_{\substack{0\leq r\leq N-1\\ p+q=k}} d_r^{p,q} : C_k^\infty(X,\mathbb{C}) \to C_{k+1}^\infty(X,\mathbb{C})$$

of the operator d (see (12)) *and the splitting*

$$u = \sum_{r=0}^{N-1} u_r + \ker d, \quad \textit{implying} \quad du = \sum_{r=0}^{N-1} d_r u_r,$$

with $u_r \in E_r^{p,q}/\ker d_r^{p,q}$ (see Section 3 and recall that $d_r : E_r^{p,q}/\ker d_r^{p,q} \to \operatorname{Im} d_r^{p,q} \subset C^\infty_{p+r,q-r+1}(X,\mathbb{C})$ is an isomorphism), *the following identity holds*:

$$h^{2(n-p)}\|du\|^2_{\omega_h} = \sum_{r=0}^{N-1} h^{2r}\|d_r u_r\|^2_\omega \quad \textit{for all } h > 0. \tag{33}$$

Proof. Since d_r is of type $(r,-r+1)$, $d_r u_r$ is of type $(p+r, q-r+1)$, so the $d_r u_r$ are mutually orthogonal (w.r.t. any metric) when r varies. We get

$$\|du\|^2_{\omega_h} = \sum_{r=0}^{N-1}\|d_r u_r\|^2_{\omega_h} = \sum_{r=0}^{N-1}\frac{h^{2(p+r)}}{h^{2n}}\|d_r u_r\|^2_\omega,$$

where for the last identity we used Formula 2.2. □

Lemma 4.10. *For every r and every bidegree (p,q), the formal adjoints of d_r w.r.t. the metrics ω_h and ω compare as follows*:

$$(d_r)^\star_{\omega_h} = h^{2r}(d_r)^\star_\omega. \tag{34}$$

Thus, for every form $u \in C^\infty_{p,q}(X,\mathbb{C})$, the following counterpart of Lemma 4.9 for the adjoints holds. Considering the splitting

$$(d^{(k)})^\star_{\omega_h} = \bigoplus_{\substack{0\le r\le N-1\\ p+q=k}} (d_r^{p,q})^\star_{\omega_h} : C^\infty_{k+1}(X,\mathbb{C}) \to C^\infty_k(X,\mathbb{C})$$

of the operator $d^\star$ and the splitting

$$u = \sum_{r=0}^{N-1} v_r + \ker d^\star_{\omega_h}, \quad \textit{implying} \quad d^\star_{\omega_h}u = \sum_{r=0}^{N-1}(d_r)^\star_{\omega_h} v_r,$$

with $v_r \in \operatorname{Im} d_r^{p-r,q+r-1}$ (see Section 3.1), *the following identity holds*:

$$h^{2(n-p)}\|d^\star_{\omega_h}u\|^2_{\omega_h} = \sum_{r=0}^{N-1} h^{2r}\|(d_r)^\star_\omega v_r\|^2_\omega \quad \textit{for all } h > 0. \tag{35}$$

Proof. For every (p,q)-form v and every $(p-r, q+r-1)$-form u, we have

$$\begin{aligned}\frac{h^{2(p-r)}}{h^{2n}}\langle\langle (d_r)^\star_{\omega_h} v, u\rangle\rangle_\omega &= \langle\langle (d_r)^\star_{\omega_h} v, u\rangle\rangle_{\omega_h}\\ &= \langle\langle v, d_r u\rangle\rangle_{\omega_h} = \frac{h^{2p}}{h^{2n}}\langle\langle v, d_r u\rangle\rangle_\omega = \frac{h^{2p}}{h^{2n}}\langle\langle (d_r)^\star_\omega v, u\rangle\rangle_\omega.\end{aligned}$$

This proves (34). Using the mutual orthogonality of the $(d_r)^\star_{\omega_h} v_r$ (due to bidegree reasons) and Formula 2.2, we get

$$\|d^\star_{\omega_h}u\|^2_{\omega_h}=\sum_{r=0}^{N-1}\|(d_r)^\star_{\omega_h}v_r\|^2_{\omega_h}=\sum_{r=0}^{N-1}\frac{h^{2(p-r)}}{h^{2n}}\|(d_r)^\star_{\omega_h}v_r\|^2_{\omega}$$
$$=\sum_{r=0}^{N-1}\frac{h^{2(p-r)}}{h^{2n}}h^{4r}\|(d_r)^\star_{\omega}v_r\|^2_{\omega}.$$

This proves (35). □

Putting together (32), (33) and (35), we get:

Corollary 4.11. *Let (X,ω) be a compact Hermitian manifold with $\dim_{\mathbb{C}}X=n$. For every bidegree (p,q) and every (p,q)-form u on X, the following identity holds*:

$$\langle\langle\Delta_h u,u\rangle\rangle_\omega=\sum_{r'=0}^{N-1}h^{2r'}\|d_{r'}u_{r'}\|^2_\omega+\sum_{r'=0}^{N-1}h^{2r'}\|(d_{r'})^\star_\omega v_{r'}\|^2_\omega,$$

where u splits uniquely (see Section 3.1) *as*

$$u=\sum_{r'=0}^{N-1}u_{r'}+\ker d=\sum_{r'=0}^{N-1}v_{r'}+\ker d^\star=\sum_{r'=0}^{N-1}u_{r'}+\sum_{r'=0}^{N-1}v_{r'}+w$$

with $u_{r'}\in E^{p,q}_{r'}/\ker d^{p,q}_{r'}$, $v_{r'}\in\operatorname{Im}d^{p-r',q+r'-1}_{r'}$ and $w\in E^{p,q}_\infty$.

4.5. ***Proof of the inequality "≥" in Theorem 1.3.*** Following again the analogy with the foliated case of [Álvarez López and Kordyukov 2000], we will actually prove a stronger statement from which the following result will follow as a corollary.

Theorem 4.12. *Let (X,ω) be a compact Hermitian manifold with $\dim_{\mathbb{C}}X=n$. For every r and every $k=0,\dots,2n$, the following inequality holds*:

$$\dim E^k_r\geq\sharp\{i\mid\lambda^k_i(h)\in O(h^{2r})\quad\textit{as}\ \ h\downarrow 0\}.\tag{36}$$

The first main ingredient we will use is the *pseudodifferential Laplacian*

$$\widetilde{\Delta}=\partial p''\partial^\star+\partial^\star p''\partial+\Delta'':C^\infty_{p,q}(X,\mathbb{C})\to C^\infty_{p,q}(X,\mathbb{C})$$

defined in arbitrary bidegree (p,q) and introduced in [Popovici 2016], where $p'':C^\infty_{p,q}(X,\mathbb{C})\to\ker\Delta''$ is the orthogonal projection (w.r.t. the L^2_ω-norm) onto the Δ''-harmonic subspace of $C^\infty_{p,q}(X,\mathbb{C})$. The pseudodifferential Laplacian $\widetilde{\Delta}$ gives a Hodge theory for the second page of the Frölicher spectral sequence in the sense that there is a *Hodge isomorphism*

$$E^{p,q}_2\xrightarrow{\simeq}\mathcal{H}^{p,q}_{\widetilde{\Delta}}(X,\mathbb{C})\tag{37}$$
$$:=\ker(\widetilde{\Delta}:C^\infty_{p,q}(X,\mathbb{C})\to C^\infty_{p,q}(X,\mathbb{C}))\quad\text{for all}\ \ p,q=0,\dots,n.$$

Note that $(p'')^2 = p'' = (p'')^\star$, so $\partial p'' \partial^\star = (p''\partial^\star)^\star (p''\partial^\star)$ and $\partial^\star p'' \partial = (p''\partial)^\star (p''\partial)$. Thus, $\widetilde{\Delta}$ is a sum of nonnegative operators, so its kernel is the intersection of the respective kernels. Since $\ker(A^\star A) = \ker A$ for any operator A, we get

$$\ker \widetilde{\Delta} = \ker(p''\partial) \cap \ker(p''\partial^\star) \cap \ker \bar{\partial} \cap \ker \bar{\partial}^\star.$$

The second main ingredient we will use is the following lower estimate of the rescaled Laplacian Δ_h. It is the analogue in our context of a result in [Álvarez López and Kordyukov 2000].

Lemma 4.13. *Let (X, ω) be a compact Hermitian manifold with* $\dim_{\mathbb{C}} X = n$. *There exists a constant $C > 0$ such that the following inequality of linear operators* (see Notation 1.4) *holds on differential forms of any degree $k = 0, \ldots, 2n$:*

$$\Delta_h \geq \tfrac{3}{4}\Delta'' + h^2 \Delta' - C h^2 \quad \textit{for all } h > 0,$$

where $\Delta'' = \bar{\partial}\bar{\partial}^\star + \bar{\partial}^\star\bar{\partial}$ and $\Delta' = \partial\partial^\star + \partial^\star\partial$ are the usual $\bar{\partial}$- and ∂-Laplacians.

The coefficients $\frac{3}{4}$ and 1 are not optimal, but they suffice for our purposes and the proof provided below shows that they can be made optimal if this is desired.

Proof of Lemma 4.13. We know from (i) of Lemma 2.7 that

$$\Delta_h = \Delta'' + h^2 \Delta' - h([\tau, \bar{\partial}^\star] + [\tau^\star, \bar{\partial}]), \tag{38}$$

where $\tau = \tau_\omega := [\Lambda, \partial\omega \wedge \cdot\,]$ is the zeroth order *torsion operator* of type $(1, 0)$ associated with ω.

For any form u, the first-order terms on the right-hand side of (38) are easily estimated using the Cauchy–Schwarz inequality as follows:

$$\begin{aligned}
h|\langle\langle [\tau, \bar{\partial}^\star]u + [\tau^\star, \bar{\partial}]u, u\rangle\rangle| \qquad&\\
&= h|\langle\langle \bar{\partial}^\star u, \tau^\star u\rangle\rangle + \langle\langle \tau u, \bar{\partial} u\rangle\rangle + \langle\langle \bar{\partial} u, \tau u\rangle\rangle + \langle\langle \tau^\star u, \bar{\partial}^\star u\rangle\rangle| \\
&\leq 2h\|\tau u\|\,\|\bar{\partial} u\| + 2h\|\tau^\star u\|\,\|\bar{\partial}^\star u\| \\
&\leq \tfrac{1}{4}(\|\bar{\partial} u\|^2 + \|\bar{\partial}^\star u\|^2) + 4h^2(\|\tau u\|^2 + \|\tau^\star u\|^2) \\
&\leq \tfrac{1}{4}\langle\langle \Delta'' u, u\rangle\rangle + C h^2 \|u\|^2,
\end{aligned}$$

where the constant $C > 0$ exists because the linear operators τ and $\tau^\star$ are of order zero, hence bounded. In particular, we get the operator inequality

$$-h([\tau, \bar{\partial}^\star] + [\tau^\star, \bar{\partial}]) \geq -\frac{1}{4}\Delta'' - C h^2,$$

which, alongside (38), proves the contention. $\square$

We are now ready to state and prove a general result that will imply Theorem 4.12.

Theorem 4.14. *Let (X,ω) be a compact Hermitian manifold with $\dim_{\mathbb{C}} X = n$. Let $k \in \{0,\dots,2n\}$ and $r \geq 1$ be fixed integers. Suppose there exist a sequence $(h_i)_{i\in\mathbb{N}}$ of constants $h_i > 0$ such that $h_i \downarrow 0$ and a sequence $(u_i)_{i\in\mathbb{N}}$ of k-forms $u_i \in C^\infty_k(X,\mathbb{C})$ such that $\|u_i\|_\omega = 1$ for every i and*

$$\langle\langle \Delta_{h_i} u_i, u_i\rangle\rangle_\omega \in o(h_i^{2(r-1)}) \quad \text{as } i \to +\infty. \tag{39}$$

Then, there exists a subsequence $(u_{i_l})_{l\in\mathbb{N}}$ of $(u_i)_{i\in\mathbb{N}}$ such that $(u_{i_l})_{l\in\mathbb{N}}$ converges in the L^2_ω-topology to some k-form $u \in \mathcal{H}^k_r := \bigoplus_{p+q=k} \mathcal{H}^{p,q}_r \simeq E^k_r$, where the $\mathcal{H}^{p,q}_r \subset C^\infty_{p,q}(X,\mathbb{C})$ are the "harmonic" vector subspaces of Definition 3.3 induced by the metric ω.

Proof. Case $r=1$. In this case, hypothesis (39) means that $\langle\langle \Delta_{h_i} u_i, u_i\rangle\rangle_\omega \to 0$ as $i \to +\infty$. Then also $\langle\langle \Delta_{h_i} u_i, u_i\rangle\rangle_\omega + Ch_i^2 \to 0$ as $i \to +\infty$. Since, by Lemma 4.13, we have

$$\langle\langle \Delta_{h_i} u_i, u_i\rangle\rangle_\omega + Ch_i^2 \geq \tfrac{3}{4}\langle\langle \Delta'' u_i, u_i\rangle\rangle_\omega + h_i^2 \langle\langle \Delta' u_i, u_i\rangle\rangle_\omega \geq 0 \quad \text{for all } i \in \mathbb{N},$$

we get

$$\begin{aligned} &\text{(i)} && \langle\langle \Delta'' u_i, u_i\rangle\rangle_\omega \to 0 \quad \text{as } i \to +\infty, \\ &\text{(ii)} && h_i^2\langle\langle \Delta' u_i, u_i\rangle\rangle_\omega \to 0 \quad \text{as } i \to +\infty. \end{aligned} \tag{40}$$

Meanwhile, the $\bar\partial$-Laplacian Δ'' is *elliptic* and the manifold X is *compact*, so the Gårding inequality yields constants $\delta_1, \delta_2 > 0$ such that the first inequality below holds:

$$\delta_2 \|u_i\|_{W^1} \leq \langle\langle \Delta'' u_i, u_i\rangle\rangle_\omega + \delta_1 \|u_i\|_\omega \leq C_1 \quad \text{for all } i \in \mathbb{N},$$

where $\|\ \|_{W^1}$ stands for the Sobolev norm W^1 induced by the metric ω. The second inequality above holds for some constant $C_1 > 0$ since the quantity $\langle\langle \Delta'' u_i, u_i\rangle\rangle_\omega$ converges to zero (see (40)), hence is bounded, and $\|u_i\|_\omega = 1$ by the hypothesis of Theorem 4.14.

Consequently, the sequence $(u_i)_{i\in\mathbb{N}}$ is bounded in the Sobolev space W^1 (a Hilbert space), so by the Banach–Alaoglu theorem there exists a subsequence $(u_{i_l})_{l\in\mathbb{N}}$ that converges in the weak topology of W^1 to some k-form $u \in W^1$. In particular, the following convergences hold in the weak topology of distributions:

$$\bar\partial u_{i_l} \to \bar\partial u \quad \text{and} \quad \bar\partial^\star u_{i_l} \to \bar\partial^\star u \quad \text{as } l \to +\infty.$$

On the other hand, $\|\bar\partial u_i\|^2 + \|\bar\partial^\star u_i\|^2 = \langle\langle \Delta'' u_i, u_i\rangle\rangle_\omega \to 0$ as $i \to +\infty$, so $\bar\partial u_i \to 0$ and $\bar\partial^\star u_i \to 0$ in the L^2-topology as $i \to +\infty$. Comparing this with the above convergences in the weak topology of distributions, we get

$$\bar\partial u = 0 \quad \text{and} \quad \bar\partial^\star u = 0,$$

which, by (18), is equivalent to $u \in \ker(\Delta'' : C^\infty_k(X,\mathbb{C}) \to C^\infty_k(X,\mathbb{C})) = \mathcal{H}^k_1 \simeq E^k_1$.

Note that by the Rellich lemma (asserting the compactness of the inclusion $W^1 \hookrightarrow L^2$), the convergence of $(u_{i_l})_{l\in\mathbb{N}}$ to u in the weak topology of W^1 implies that $(u_{i_l})_{l\in\mathbb{N}}$ also converges in the L^2-topology to u. Moreover, the ellipticity of Δ'' and the relation $u \in \ker \Delta''$ imply that u is C^∞.

Case $r = 2$. In this case, hypothesis (39) means that $\langle\langle \Delta_{h_i} u_i, u_i \rangle\rangle_\omega \in o(h_i^2)$ as $i \to +\infty$. Since $\langle\langle \Delta_{h_i} u_i, u_i \rangle\rangle_\omega = \|d_{h_i} u_i\|^2 + \|d_{h_i}^\star u_i\|^2 = \|h_i \partial u_i + \bar\partial u_i\|^2 + \|h_i \partial^\star u_i + \bar\partial^\star u_i\|^2$, this implies that

$$\text{(41)}\quad \partial u_i + \frac{1}{h_i}\bar\partial u_i \to 0 \quad \text{and} \quad \partial^\star u_i + \frac{1}{h_i}\bar\partial^\star u_i \to 0 \text{ in the } L^2\text{-topology as } i \to +\infty.$$

Since the orthogonal projection p'' onto $\ker \Delta''$ is continuous w.r.t. the L^2-topology and since $p''\bar\partial = 0$ and $p''\bar\partial^\star = 0$ (because $\operatorname{Im}\bar\partial \perp \ker \Delta''$ and $\operatorname{Im}\bar\partial^\star \perp \ker \Delta''$), an application of p'' to (41) yields

$$\text{(42)}\qquad p''\partial u_i \to 0 \quad \text{and} \quad p''\partial^\star u_i \to 0 \quad \text{in the } L^2\text{-topology as } i \to +\infty.$$

On the other hand, we know from the discussion of the case $r = 1$ (whose weaker assumption is still valid in the case $r = 2$) that there exists a subsequence $(u_{i_l})_{l\in\mathbb{N}}$ that converges in the weak topology of W^1 to some k-form $u \in W^1$. Thus, $\partial u_{i_l} \to \partial u \in L^2$ in the weak topology of L^2 as $l \to +\infty$. This means that

$$\langle\langle \partial u_{i_l}, v \rangle\rangle_\omega \to \langle\langle \partial u, v \rangle\rangle_\omega \quad \text{for all } v \in L^2,$$

hence

$$\langle\langle \partial u_{i_l}, p''v \rangle\rangle_\omega \to \langle\langle \partial u, p''v \rangle\rangle_\omega \quad \text{for all } v \in L^2,$$

as $l \to +\infty$. (The second convergence follows from the first since $\|p''v\| \le \|v\|$ for all $v \in L^2$, so $p''(L^2) \subset L^2$.) Now, p'' is self-adjoint, so the last convergence translates to

$$\langle\langle p''\partial u_{i_l}, v \rangle\rangle_\omega \to \langle\langle p''\partial u, v \rangle\rangle_\omega \quad \text{as } l \to +\infty \quad \text{for all } v \in L^2.$$

This means that $p''\partial u_{i_l}$ converges to $p''\partial u$ in the weak topology of L^2 as $l \to +\infty$. However, we know from (42) that $p''\partial u_{i_l}$ converges to 0 in the L^2-topology. Hence $p''\partial u = 0$. The same argument run with $\partial^\star$ in place of ∂ yields that $p''\partial^\star u = 0$. On the other hand, we know from the discussion of the case $r = 1$ that $u \in \ker\bar\partial \cap \ker\bar\partial^\star = \ker\Delta''$, so we get

$$u \in \ker(p''\partial) \cap \ker(p''\partial^\star) \cap \ker\bar\partial \cap \ker\bar\partial^\star = \mathcal{H}_2^k \simeq E_2^k$$

after recalling the description (18) of the spaces $\mathcal{H}_2^{p,q}$ and that $\mathcal{H}_2^k = \bigoplus_{p+q=k} \mathcal{H}_2^{p,q}$.

Case $r \ge 3$. Using the information from the first two cases and from subsection Section 4.4, this last case can easily be dealt with as follows.

For each of the k-forms u_i given by the hypotheses of Theorem 4.14, we consider the splitting

$$u_i = \sum_{r'=0}^{N-1} u_{r'}^{(i)} + \sum_{r'=0}^{N-1} v_{r'}^{(i)} + w_i,$$

with $u_{r'}^{(i)} \in E_{r'}^{p,q} / \ker d_{r'}^{p,q}$, $v_{r'}^{(i)} \in \operatorname{Im} d_{r'}^{p-r',q+r'-1}$ and $w_i \in E_\infty^{p,q}$, and the corresponding splitting

$$\langle\langle \Delta_{h_i} u_i, u_i \rangle\rangle_\omega = \sum_{r'=0}^{N-1} h_i^{2r'} \|d_{r'} u_{r'}^{(i)}\|_\omega^2 + \sum_{r'=0}^{N-1} h_i^{2r'} \|(d_{r'})_\omega^\star v_{r'}^{(i)}\|_\omega^2$$

obtained in Corollary 4.11.

On the other hand, (39) ensures that $\langle\langle \Delta_{h_i} u_i, u_i \rangle\rangle_\omega \in o(h_i^{2(r-1)})$ as $i \to +\infty$. Together with the above identity, this implies the following convergences in the L^2_ω-norm as $i \to +\infty$:

$$d_{r'} u_{r'}^{(i)} \to 0 \quad \text{and} \quad (d_{r'})_\omega^\star v_{r'}^{(i)} \to 0 \quad \text{for every } r' \in \{0, \dots, r-1\}.$$

We even get

$$\frac{1}{h_i^{r-r'-1}} d_{r'} u_{r'}^{(i)} \to 0 \quad \text{and} \quad \frac{1}{h_i^{r-r'-1}} (d_{r'})_\omega^\star v_{r'}^{(i)} \to 0 \quad \text{for every } r' \in \{0, \dots, r-1\}.$$

Defining in an ad hoc way a "formal" Laplacian by $\Delta_{r'}^{\text{formal}} := d_{r'} (d_{r'})_\omega^\star + (d_{r'})_\omega^\star d_{r'}$, we get that the limit u of a subsequence of $(u_i)_{i\in\mathbb{N}}$ lies in

$$\ker\Bigg(\Delta_{r-1}^{\text{formal}} : \bigoplus_{p+q=k} E_{r-1}^{p,q} \to \bigoplus_{p+q=k} E_{r-1}^{p,q}\Bigg) \simeq \mathcal{H}_r^k \simeq E_r^k$$

and we are done. □

Proof of Theorem 4.12. It is an immediate consequence of Theorem 4.14. Indeed, fix any $r \in \mathbb{N}^\star$ and $k \in \{0, \dots, 2n\}$ and suppose that inequality (36) does not hold. Then, the reverse strict inequality holds, so there exists a sequence $(h_i)_{i\in\mathbb{N}}$ of positive constants such that $h_i \downarrow 0$ when $i \to +\infty$ and a sequence $(u_i)_{i\in\mathbb{N}}$ of eigenvectors for the Laplacians Δ_{h_i} acting on k-forms such that $\|u_i\|_\omega = 1$, $u_i \perp \mathcal{H}_r^k$ for all i and $\langle\langle \Delta_{h_i} u_i, u_i \rangle\rangle \in o(h_i^{2(r-1)})$ as $i \to +\infty$.

Thanks to Theorem 4.14, there exists a subsequence $(u_{i_l})_{l\in\mathbb{N}}$ of $(u_i)_{i\in\mathbb{N}}$ such that $(u_{i_l})_{l\in\mathbb{N}}$ converges in the L^2_ω-topology to some k-form $u \in \mathcal{H}_r^k \simeq E_r^k$. However, the form u is orthogonal to $\mathcal{H}_r^k$ since $u_i \perp \mathcal{H}_r^k$ for all i and the orthogonality property is preserved in the limit. Since $\|u\|_\omega = 1$ (because $\|u_i\|_\omega = 1$ for all i), $u \neq 0$, so u cannot be at once orthogonal to and a member of $\mathcal{H}_r^k$. This is a contradiction. □

5. Consequences of Theorem 1.3

The following consequences of Theorem 1.3 are of independent interest.

Proposition 5.1. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. For every $r \in \mathbb{N}^\star$ and every $k = 0, \dots, 2n$, the following identity (a kind of numerical Poincaré duality extended to all the pages of the spectral sequence) holds:*

$$\dim_{\mathbb{C}} E_r^k = \dim_{\mathbb{C}} E_r^{2n-k},$$

where, as usual, $E_r^k = \sum_{p+q=k} E_r^{p,q}$ is the direct sum of the spaces of total degree k on the r-th page of the Frölicher spectral sequence of X.

This is an immediate consequence of Theorem 1.3 and of the following:

Proposition 5.2. *Let (X, ω) be an n-dimensional compact complex Hermitian manifold. Fix an arbitrary constant $h > 0$.*

(i) *If $d_h^\star$, resp. $\star$, are the formal adjoint of d_h, resp. the Hodge star operator induced by ω, then*

$$d_h^\star = -\star \bar{d}_h \star.$$

(ii) *If, for every $h > 0$, every $k = 0, \dots, 2n$ and every $\lambda \geq 0$, $E_{\Delta_h}^k(\lambda)$ stands for the λ-eigenspace of $\Delta_h : C_k^\infty(X, \mathbb{C}) \to C_k^\infty(X, \mathbb{C})$, the linear map*

$$E_{\Delta_h}^k(\lambda) \to E_{\Delta_h}^{2n-k}(\lambda), \quad u \mapsto \star\bar{u},$$

is well defined and an isomorphism.

In particular, the operators $\Delta_h : C_k^\infty(X, \mathbb{C}) \to C_k^\infty(X, \mathbb{C})$ and

$$\Delta_h : C_{2n-k}^\infty(X, \mathbb{C}) \to C_{2n-k}^\infty(X, \mathbb{C})$$

have the same spectra and their corresponding eigenvalues have the same multiplicities for all $h > 0$ and all $k = 0, \dots, 2n$.

Proof. (i) We have $d_h^\star = h\partial^\star + \bar\partial^\star = -h\star\bar\partial\star - \star\partial\star = -\star(h\bar\partial + \partial)\star = -\star\bar{d}_h\star$ thanks to the standard formulae $\partial^\star = -\star\bar\partial\star$ and $\bar\partial^\star = -\star\partial\star$.

(ii) Using the formula under (i) and $\star\star = (-1)^k$ on k-forms, we get the following equivalences:

$$\begin{aligned} u \in E_{\Delta_h}^k(\lambda) &\iff -d_h\star\bar{d}_h\star u - \star\bar{d}_h\star d_h u = \lambda u \\ &\overset{(a)}{\iff} (-\star\bar{d}_h\star)d_h(\star\bar{u}) - (-1)^{\deg u}\star\star d_h\star\bar{d}_h\star\star\bar{u} = \lambda(\star\bar{u}) \\ &\iff d_h^\star d_h(\star\bar{u}) + d_h d_h^\star(\star\bar{u}) = \lambda(\star\bar{u}) \iff \star\bar{u} \in E_{\Delta_h}^{2n-k}(\lambda), \end{aligned}$$

where (a) was obtained by conjugating and then applying the isomorphism $\star$.

This shows the well-definedness of the linear map under consideration. Both the conjugation and $\star$ are isomorphisms, hence so is that linear map. □

Proof of Proposition 5.1. By Theorem 1.3, $\dim_{\mathbb{C}} E_r^k$, resp. $\dim_{\mathbb{C}} E_r^{2n-k}$, is the number of eigenvalues $\lambda_i^k(h) \in \mathcal{O}(h^{2r})$, resp. $\lambda_i^{2n-k}(h) \in \mathcal{O}(h^{2r})$, counted with multiplicities, of Δ_h in degree k, resp. $2n-k$. Since, by Proposition 5.2, $\lambda_i^k(h) = \lambda_i^{2n-k}(h)$ for all $i \in \mathbb{N}^\star$ and all $h > 0$, the statement follows. □

The last consequence of Theorem 1.3 that we notice in this section is the following degeneration criterion for the Frölicher spectral sequence.

Proposition 5.3. *Let (X, ω) be an n-dimensional compact complex Hermitian manifold. For every constant $h > 0$, let $\delta_h^{(k)} > 0$ be the smallest positive eigenvalue of $\Delta_h : C_k^\infty(X, \mathbb{C}) \to C_k^\infty(X, \mathbb{C})$.*

Then, for every $r \in \mathbb{N}^\star$, the Frölicher spectral sequence of X degenerates at E_r if and only if

$$\lim_{h\to 0} \frac{\delta_h^{(k)}}{h^{2r}} = +\infty \quad \textit{for all } k \in \{1, \dots, n\}.$$

Proof. The multiplicity of 0 as an eigenvalue of $\Delta_h : C_k^\infty(X, \mathbb{C}) \to C_k^\infty(X, \mathbb{C})$ is the k-th Betti number b_k of X (see Corollary 2.9), so the degeneration at E_r of the Frölicher spectral sequence (known to be equivalent to the identities $b_k = \dim E_r^k$ for all $k = 0, 1, \dots, 2n$) amounts, thanks to Theorem 1.3, to $\delta_h^{(k)}$ converging to zero (if it does converge to zero at all as $h \downarrow 0$) strictly less fast than Ch^{2r} for all $k = 0, 1, \dots, 2n$. On the other hand, the numerical duality statement of Proposition 5.1 reduces the verification of this property to the cases $k = 1, \dots, n$. □

6. Degeneration at E_2 of the Frölicher spectral sequence

In this section, we prove Theorem 1.2.

We start off by noticing a lower estimate for $\Delta_h - h^2\Delta$ that holds for any Hermitian metric.

Lemma 6.1. *Let (X, ω) be a compact complex manifold. For every $0 < h < 1$, the following inequality of operators holds on smooth differential forms of all degrees:*

$$\Delta_h - h^2\Delta \geq (1-h)h(\Delta'' - h[\tau, \tau^\star]). \tag{43}$$

Proof. We know from Lemma 2.7 that $\Delta_h = h^2\Delta' + \Delta'' - h[\tau, \bar\partial^\star] - h[\bar\partial, \tau^\star]$ for any Hermitian metric ω, while $\Delta = [\partial + \bar\partial, \partial^\star + \bar\partial^\star] = \Delta' + \Delta'' - [\tau, \bar\partial^\star] - [\bar\partial, \tau^\star]$. Thus, we get

$$\begin{aligned} \Delta_h - h^2\Delta &= (1-h^2)\Delta'' + h(h-1)([\bar\partial, \tau^\star] + [\bar\partial^\star, \tau]) \\ &= (1-h)((1+h)\Delta'' - h[\bar\partial, \tau^\star] - h[\bar\partial^\star, \tau]). \end{aligned} \tag{44}$$

We shall now estimate the signless terms on the right-hand side of (44). For any form u, we have

$$\begin{aligned}\langle\langle[\bar\partial, \tau^\star]u, u\rangle\rangle + \langle\langle[\bar\partial^\star, \tau]u, u\rangle\rangle &= \langle\langle\tau^\star u, \bar\partial^\star u\rangle\rangle + \langle\langle\bar\partial u, \tau u\rangle\rangle + \langle\langle\tau u, \bar\partial u\rangle\rangle + \langle\langle\bar\partial^\star u, \tau^\star u\rangle\rangle \\ &= 2\,\mathrm{Re}\langle\langle\bar\partial^\star u, \tau^\star u\rangle\rangle + 2\,\mathrm{Re}\langle\langle\bar\partial u, \tau u\rangle\rangle.\end{aligned}$$

Thus, for any Hermitian metric ω, we have

$$\begin{aligned}h|\langle\langle([\bar\partial, \tau^\star] + [\bar\partial^\star, \tau])u, u\rangle\rangle| &\le 2h|\langle\langle\bar\partial u, \tau u\rangle\rangle| + 2h|\langle\langle\bar\partial^\star u, \tau^\star u\rangle\rangle| \\ &\le (\|\bar\partial u\|^2 + \|\bar\partial^\star u\|^2) + h^2(\|\tau u\|^2 + \|\tau^\star u\|^2) \\ &= \langle\langle\Delta'' u, u\rangle\rangle + h^2\langle\langle[\tau, \tau^\star]u, u\rangle\rangle.\end{aligned}$$

Using this last estimate in (44), we get $\Delta_h - h^2\Delta \ge (1-h)(h\Delta'' - h^2[\tau, \tau^\star])$ in the sense of operators. This is precisely (43).

Note that we can also write

$$|\langle\langle([\bar\partial, \tau^\star] + [\bar\partial^\star, \tau])u, u\rangle\rangle| \le \langle\langle\Delta'' u, u\rangle\rangle + \langle\langle[\tau, \tau^\star]u, u\rangle\rangle$$

for every form u, which, alongside (44), yields $\Delta_h - h^2\Delta \ge (1-h)(\Delta'' - h[\tau, \tau^\star])$. This is slightly better than (43) if the right-hand side is nonnegative, but worse otherwise. □

We shall now give a sufficient condition for the right-hand side of (43) to be nonnegative.

Lemma 6.2. *Let (X, ω) be a compact Hermitian manifold with $\dim_{\mathbb{C}} X = n$ such that the inclusion of kernels*

$$\ker \Delta'' \subset \ker[\tau, \tau^\star]$$

holds for the operators $\Delta'', [\tau, \tau^\star] : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C})$ in a fixed degree $k \in \{1, \dots, n\}$.

Then, there exists a constant $h_0(k) \in (0, 1]$ such that the following inequality of operators holds in degree k:

$$\Delta'' \ge h[\tau, \tau^\star] \quad \text{for all } 0 < h < h_0(k).$$

Proof. Let $\delta''_k > 0$ be the smallest positive eigenvalue of the elliptic, self-adjoint and nonnegative differential operator $\Delta'' : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C})$.

On the other hand, the operator $[\tau, \tau^\star] : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C})$ is of order zero, hence bounded, so the constant $C_k := \sup_{\|u\|\le 1}\langle\langle[\tau, \tau^\star]u, u\rangle\rangle$ is finite.

We put $h_0(k) := \min\{\delta''_k/C_k, 1\}$ and will prove that $\langle\langle\Delta'' u, u\rangle\rangle \ge h\langle\langle[\tau, \tau^\star]u, u\rangle\rangle$ for all $u \in C^\infty_k(X, \mathbb{C})$ and all $h \in (0, h_0(k))$. Let us fix a form $u \in C^\infty_k(X, \mathbb{C})$.

Since Δ'' is elliptic and preserves bidegrees, the orthogonal splitting

$$C^\infty_k(X, \mathbb{C}) = \ker \Delta'' \oplus \mathrm{Im}\, \Delta''$$

holds and induces a unique splitting $u = u_h + u_{h^\perp}$ with $u_h \in \ker\Delta''$ and $u_{h^\perp} \in \operatorname{Im}\Delta''$. In particular, $u_h \in \ker[\tau, \tau^\star]$ thanks to our assumption.

We get

$$\langle\langle \Delta'' u, u\rangle\rangle = \langle\langle \Delta'' u_{h^\perp}, u_h + u_{h^\perp}\rangle\rangle = \langle\langle \Delta'' u_{h^\perp}, u_{h^\perp}\rangle\rangle \geq \delta_k'' \|u_{h^\perp}\|^2 \tag{45}$$

since $u_{h^\perp} \perp \ker\Delta''$, so $u_{h^\perp}$ lies in the orthogonal direct sum of the eigenspaces of Δ'' corresponding to positive eigenvalues (equal to eigenvalues greater than or equal to δ_k'').

On the other hand,

$$\begin{aligned}\langle\langle [\tau, \tau^\star] u, u\rangle\rangle &\overset{\text{(a)}}{=} \langle\langle [\tau, \tau^\star] u_{h^\perp}, u_h + u_{h^\perp}\rangle\rangle \\ &\overset{\text{(b)}}{=} \langle\langle u_{h^\perp}, [\tau, \tau^\star] u_h\rangle\rangle + \langle\langle [\tau, \tau^\star] u_{h^\perp}, u_{h^\perp}\rangle\rangle \\ &\overset{\text{(c)}}{=} \langle\langle [\tau, \tau^\star] u_{h^\perp}, u_{h^\perp}\rangle\rangle \\ &\overset{\text{(d)}}{\leq} C_k \|u_{h^\perp}\|^2,\end{aligned} \tag{46}$$

where for (a) we used the fact that $u_h \in \ker[\tau, \tau^\star]$, for (b) we used the self-adjointness of $[\tau, \tau^\star]$, (c) follows from $u_h \in \ker[\tau, \tau^\star]$, while (d) follows from the definition of C_k.

Since $h_0(k) = \min\{\delta_k''/C_k, 1\}$, inequalities (45) and (46) imply that

$$h\langle\langle [\tau, \tau^\star] u, u\rangle\rangle \leq C_k h \|u_{h^\perp}\|^2 \leq \frac{C_k h}{\delta_k''}\langle\langle \Delta'' u, u\rangle\rangle \leq \langle\langle \Delta'' u, u\rangle\rangle$$

for all $h \in (0, h_0(k))$. □

Corollary 6.3. *Let (X, ω) be a compact Hermitian manifold such that* $\ker\Delta'' \subset \ker[\tau, \tau^\star]$ *in a fixed degree k. Then, there exists a constant $h_0(k) \in (0, 1]$ such that the following inequality of operators holds in degree k:*

$$\Delta_h \geq h^2 \Delta \quad \textit{for all } 0 < h < h_0(k).$$

Proof. This is an immediate consequence of Lemmas 6.1 and 6.2. □

We can now prove the spectral sequence degeneration statement of this paper.

Proof of Theorem 1.2. Let us fix an arbitrary $k \in \{1, \dots, n\}$. Hypothesis (1) and Corollary 6.3 imply that $\ker\Delta_h \subset \ker\Delta$ for all $0 < h < h_0(k)$ since $\langle\langle \Delta u, u\rangle\rangle \geq 0$ for every u and $u \in \ker\Delta$ if and only if $\langle\langle \Delta u, u\rangle\rangle = 0$. Meanwhile, we know from Corollary 2.9 that $\ker\Delta_h$ and $\ker\Delta$ are finite-dimensional vector spaces of *equal dimensions*, so for all $0 < h < h_0(k)$ we get

$$\ker\Delta_h = \ker\Delta. \tag{47}$$

For every $h > 0$, let $\delta_h^{(k)} > 0$ be the smallest positive eigenvalue of the elliptic operator $\Delta_h : C_k^\infty(X, \mathbb{C}) \to C_k^\infty(X, \mathbb{C})$ and let $u_h \in C_k^\infty(X, \mathbb{C})$ be a corresponding unitary eigenvector, i.e.,

$$\|u_h\| = 1 \quad \text{and} \quad \Delta_h u_h = \delta_h^{(k)} u_h.$$

Now, u_h is orthogonal to $\ker \Delta_h$, hence, thanks to (47), u_h is also orthogonal to $\ker \Delta$ for every $0 < h < h_0(k)$. Consequently, $\langle\langle \Delta u_h, u_h \rangle\rangle \geq \delta_k \|u_h\|^2 = \delta_k$, where $\delta_k > 0$ is the smallest positive eigenvalue of $\Delta : C_k^\infty(X, \mathbb{C}) \to C_k^\infty(X, \mathbb{C})$.

Using this and Corollary 6.3, we get

$$\delta_h^{(k)} = \langle\langle \Delta_h u_h, u_h \rangle\rangle \geq h^2 \langle\langle \Delta u_h, u_h \rangle\rangle \geq \delta_k h^2 \quad \text{for all } 0 < h < h_0(k).$$

In particular, $\lim_{h\to 0}(\delta_h^{(k)}/h^4) = +\infty$.

As in the proof of Proposition 5.3, this and Theorem 1.3 imply that $\dim E_2^k = b_k$ for the degree $k \in \{1, \dots, n\}$ that was arbitrarily fixed in the beginning. By the duality statement of Proposition 5.1, this also yields $\dim E_2^{2n-k} = b_k = b_{2n-k}$. Since this holds for all $k \in \{1, \dots, n\}$, the Frölicher spectral sequence of X degenerates at E_2. □

Appendix: Comparison of Laplacians when the metric is SKT

In this section, we come within an ε (equal to Ch^2) of solving Conjecture 1.1 as an application of Theorem 1.3 and of a comparison of the Laplacians Δ' and Δ'' defined by an arbitrary SKT metric ω supposed to exist on a given compact complex manifold X. Recall that an SKT metric ω is a C^∞ positive definite $(1, 1)$-form ω such that $\partial\bar\partial\omega = 0$ on X.

Lemma A.1. *Let X be a compact complex manifold on which an **SKT metric** ω exists.*

(i) *The usual ∂- and $\bar\partial$-Laplacians $\Delta' = [\partial, \partial^\star]$ and $\Delta'' = [\bar\partial, \bar\partial^\star]$ induced by ω satisfy the following inequalities on differential forms of all bidegrees*:

$$(1+\delta)\Delta'' + \Big(1 + \frac{1}{\delta}\Big)[\bar\tau, \bar\tau^\star] \geq \Delta' \geq \frac{1}{1+\delta}\Delta'' - \frac{1}{\delta}[\tau, \tau^\star] \quad \textit{for all } \delta > 0, \tag{48}$$

where $\tau = \tau_\omega := [\Lambda_\omega, \partial\omega \wedge \cdot\,]$ is the torsion operator of type $(1, 0)$ and $\bar\tau^\star$ is the formal adjoint w.r.t. the L^2_ω-inner product of its complex conjugate.

(ii) *The following inequality also holds*:

$$\Delta'' \geq h\Delta' + \Big(h\bar{X}_\omega - \frac{h}{1-h}[\bar\tau, \bar\tau^\star]\Big) \quad \textit{for all } 0 < h < 1, \tag{49}$$

where $X_\omega := [\partial\omega \wedge \cdot\,, (\partial\omega \wedge \cdot\,)^\star]$. Implicitly, we have

$$\Delta_h - h\Delta \geq h((1-h)\bar{X}_\omega - [\bar\tau, \bar\tau^\star]) \quad \textit{for all } 0 < h < 1. \tag{50}$$

Since $\overline{X}_\omega$ and $[\bar\tau, \bar\tau^\star]$ are zeroth order operators, they are bounded, so (50) *implies the existence of a constant* $C > 0$ *independent of* h *such that*

$$\Delta_h - h\Delta \geq -Ch \quad \textit{for all } 0 < h < 1. \tag{51}$$

Proof. (i) Demailly's formula (see [Demailly 1986; 2012, VII, §.1]) of the Bochner–Kodaira–Nakano type for arbitrary Hermitian metrics ω (originating in [Griffiths 1969] and also contributed to in [Ohsawa 1982]) reads

$$\Delta' = \Delta''_{\bar\tau} - \overline{X}_\omega + \big[\Lambda_\omega, \big[\Lambda_\omega, \tfrac{1}{2} i\partial\bar\partial\omega\big]\big],$$

where $\Delta''_{\bar\tau} := [\bar\partial + \bar\tau, (\bar\partial + \bar\tau)^\star]$ and $\overline{X}_\omega := [\bar\partial\omega \wedge \cdot, (\bar\partial\omega \wedge \cdot)^\star]$. The last term on the right-hand side above vanishes if ω is SKT, so we get

$$\Delta'' + ([\bar\partial, \bar\tau^\star] + [\bar\tau, \bar\partial^\star]) + [\bar\tau, \bar\tau^\star] = \Delta' + \overline{X}_\omega \quad \text{if } \partial\bar\partial\omega = 0. \tag{52}$$

Now, the signless terms can be easily estimated using the elementary inequality $2|ab| \leq \delta a^2 + (1/\delta)b^2$ for arbitrary $a, b \in \mathbb{C}$ and $\delta > 0$. For every differential form u of any degree, we get

$$\begin{aligned} |\langle\langle[\bar\partial, \bar\tau^\star]u, u\rangle\rangle + \langle\langle[\bar\tau, \bar\partial^\star]u, u\rangle\rangle| &= |2\,\mathrm{Re}\langle\langle\bar\partial u, \bar\tau u\rangle\rangle + 2\,\mathrm{Re}\langle\langle\bar\partial^\star u, \bar\tau^\star u\rangle\rangle| \\ &\leq 2|\langle\langle\bar\partial u, \bar\tau u\rangle\rangle| + 2|\langle\langle\bar\partial^\star u, \bar\tau^\star u\rangle\rangle| \\ &\leq \delta\|\bar\partial u\|^2 + \frac{1}{\delta}\|\bar\tau u\|^2 + \delta\|\bar\partial^\star u\|^2 + \frac{1}{\delta}\|\bar\tau^\star u\|^2 \\ &= \delta\langle\langle\Delta'' u, u\rangle\rangle + \frac{1}{\delta}\langle\langle[\bar\tau, \bar\tau^\star]u, u\rangle\rangle. \end{aligned} \tag{53}$$

Together with (52), this implies that $(1+\delta)\Delta'' + (1 + 1/\delta)[\bar\tau, \bar\tau^\star] \geq \Delta' + \overline{X}_\omega$ if ω is SKT. This is essentially an upper estimate for Δ' whose conjugate yields a lower estimate for $\Delta' = \overline{\Delta''}$. Putting these upper and lower estimates together, we get

$$(1+\delta)\Delta'' + \Big(1 + \frac{1}{\delta}\Big)[\bar\tau, \bar\tau^\star] - \overline{X}_\omega \geq \Delta' \geq \frac{1}{1+\delta}\Delta'' + \frac{1}{1+\delta}X_\omega - \frac{1}{\delta}[\tau, \tau^\star], \tag{54}$$

for all $\delta > 0$. Since X_ω and $\overline{X}_\omega$ are nonnegative operators, ignoring them weakens these inequalities to (48).

(ii) After dividing by $1+\delta$, the left-hand side inequality in (54) translates to

$$\Delta'' \geq \frac{1}{1+\delta}\Delta' + \frac{1}{1+\delta}\overline{X}_\omega - \frac{1}{\delta}[\bar\tau, \bar\tau^\star].$$

This is precisely (49) if we put $h := 1/(1+\delta) \in (0, 1)$ since in this case $\delta = (1-h)/h$.

To get (50) from (49), it suffices to notice that $\Delta_h - h\Delta = h(h-1)\Delta' + (1-h)\Delta'' = (1-h)(\Delta'' - h\Delta')$. □

We now observe an analogue of inequality (50) for $\Delta_h - h^2\Delta$.

Lemma 6.2. *Let X be a compact complex manifold on which an **SKT metric** ω exists. The following inequalities of operators hold*:

$$\Delta_h - h^2\Delta \geq h^2((1-h)\overline{X}_\omega - [\bar{\tau}, \bar{\tau}^\star]) \geq -Ch^2 \quad \textit{for all } 0 < h < 1, \tag{55}$$

where $\overline{X}_\omega := [\bar\partial\omega \wedge \cdot\,, (\bar\partial\omega \wedge \cdot)^\star]$ *and* $C \geq 0$ *is a constant independent of* h.

Proof. Since $\Delta_h = h^2\Delta' + \Delta'' + hA$ and $\Delta = \Delta' + \Delta'' + A$, where $A := [\partial, \bar\partial^\star] + [\bar\partial, \partial^\star]$, we get

$$\Delta_h - h^2\Delta = (1-h)((1+h)\Delta'' + hA).$$

On the other hand, the signless operator A can be estimated in the same way as a similar operator was estimated in the proof of Lemma A.1. We get $\langle\langle Au, u\rangle\rangle = 2\,\mathrm{Re}\langle\langle \partial u, \bar\partial u\rangle\rangle + 2\,\mathrm{Re}\langle\langle \partial^\star u, \bar\partial^\star u\rangle\rangle$, hence

$$h|\langle\langle Au, u\rangle\rangle| \leq h^2\|\partial u\|^2 + \|\bar\partial u\|^2 + h^2\|\partial^\star u\|^2 + \|\bar\partial^\star u\|^2 = h^2\langle\langle \Delta' u, u\rangle\rangle + \langle\langle \Delta'' u, u\rangle\rangle$$

for any form u. Consequently, $(1+h)\Delta'' + hA \geq h\Delta'' - h^2\Delta'$ as operators, so we get

$$\Delta_h - h^2\Delta \geq h(1-h)(\Delta'' - h\Delta').$$

(Note that we can also write $|\langle\langle Au, u\rangle\rangle| \leq \langle\langle \Delta' u, u\rangle\rangle + \langle\langle \Delta'' u, u\rangle\rangle$ and we get $\Delta_h - h^2\Delta = (1-h)((1+h)\Delta'' + hA) \geq (1-h)(\Delta'' - h\Delta')$ for every form u.)

Meanwhile, from (49) we know that $(1-h)(\Delta'' - h\Delta') \geq h((1-h)\overline{X}_\omega - [\bar\tau, \bar\tau^\star])$ for all $0 < h < 1$. Together with the last inequality, this proves the first inequality in (55).

The second inequality in (55) follows at once from the first since $\overline{X}_\omega \geq 0$ and the nonnegative operator $[\bar\tau, \bar\tau^\star]$ is of order zero, hence bounded, so we can choose $C := \sup_{\|u\|=1}\langle\langle [\bar\tau, \bar\tau^\star]u, u\rangle\rangle < +\infty$.

(Using the alternative lower estimate $\Delta_h - h^2\Delta \geq (1-h)(\Delta'' - h\Delta')$ noticed above, the inequalities in (55) get replaced by $\Delta_h - h^2\Delta \geq h((1-h)\overline{X}_\omega - [\bar\tau, \bar\tau^\star]) \geq -Ch$.) □

If the lower bound $-Ch^2$ in (55) could be improved to 0, then we would have $\Delta_h \geq h^2\Delta$ for all $0 < h \ll 1$ (as in Corollary 6.3) and Conjecture 1.1 would follow by the argument spelt out at the end of section Section 6.

Acknowledgments

The author wishes to thank L. Ugarte for useful discussions about the content of this paper and for suggestions for Section 5. Thanks are also due to S. Rao and Q. Zhao for stimulating discussions.

References

[Álvarez López and Kordyukov 2000] J. A. Álvarez López and Y. A. Kordyukov, "Adiabatic limits and spectral sequences for Riemannian foliations", *Geom. Funct. Anal.* **10**:5 (2000), 977–1027. MR Zbl

[Cordero et al. 1997] L. A. Cordero, M. Fernández, L. Ugarte, and A. Gray, "A general description of the terms in the Frölicher spectral sequence", *Differential Geom. Appl.* **7**:1 (1997), 75–84. MR Zbl

[Demailly 1986] J.-P. Demailly, "Sur l'identité de Bochner–Kodaira–Nakano en géométrie hermitienne", pp. 88–97 in *Séminaire d'analyse P. Lelong–P. Dolbeault–H. Skoda, années 1983/1984*, edited by P. Lelong et al., Lecture Notes in Math. **1198**, Springer, 1986. MR Zbl

[Demailly 2012] J.-P. Demailly, "Complex analytic and differential geometry", preprint, 2012, https://tinyurl.com/complexjpd.

[Efremov and Shubin 1989] D. V. Efremov and M. A. Shubin, "Spectrum distribution function and variational principle for automorphic operators on hyperbolic space", exposé 8 in *Séminaire sur les Équations aux Dérivées Partielles, 1988–1989*, École Polytech., Palaiseau, France, 1989. MR Zbl

[Forman 1995] R. Forman, "Spectral sequences and adiabatic limits", *Comm. Math. Phys.* **168**:1 (1995), 57–116. MR Zbl

[Frölicher 1955] A. Frölicher, "Relations between the cohomology groups of Dolbeault and topological invariants", *Proc. Nat. Acad. Sci. U.S.A.* **41** (1955), 641–644. MR Zbl

[Griffiths 1969] P. A. Griffiths, "Hermitian differential geometry, Chern classes, and positive vector bundles", pp. 185–251 in *Global analysis*, edited by D. C. Spencer and S. Iyanaga, Univ. Tokyo Press, 1969. MR Zbl

[Gromov and Shubin 1991] M. Gromov and M. A. Shubin, "Von Neumann spectra near zero", *Geom. Funct. Anal.* **1**:4 (1991), 375–404. MR Zbl

[Mazzeo and Melrose 1990] R. R. Mazzeo and R. B. Melrose, "The adiabatic limit, Hodge cohomology and Leray's spectral sequence for a fibration", *J. Differential Geom.* **31**:1 (1990), 185–213. MR Zbl

[Ohsawa 1982] T. Ohsawa, "Isomorphism theorems for cohomology groups of weakly 1-complete manifolds", *Publ. Res. Inst. Math. Sci.* **18**:1 (1982), 191–232. MR Zbl

[Popovici 2016] D. Popovici, "Degeneration at E_2 of certain spectral sequences", *Internat. J. Math.* **27**:14 (2016), art. id. 1650111. MR Zbl

[Witten 1985] E. Witten, "Global gravitational anomalies", *Comm. Math. Phys.* **100**:2 (1985), 197–229. MR Zbl

Received October 10, 2017. Revised July 15, 2018.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 300, No. 1, 2019
dx.doi.org/10.2140/pjm.2019.300.159

ON A COMPLEX HESSIAN FLOW

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We consider a gradient flow generated by a complex Hessian functional which is defined on compact Kähler manifolds. By setting up the a priori estimates of the admissible solutions, we prove the long-time existence of the solution to the flow and its convergence. Thus we show the functional admits a local minimal point in the space of admissible functions. As its application, we show the solvability of a class of complex Hessian equations.

1. Introduction

Let (M, ω) be a compact Kähler manifold of dimension $n \geq 2$. For convenience, we write $\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ in a local coordinate chart. Let u be a smooth function on M. We denote $\omega_u = \omega + \sqrt{-1}\partial\bar{\partial}u$. Locally it can be written as $\sqrt{-1}(g_{i\bar{j}} + u_{i\bar{j}}) dz^i \wedge d\bar{z}^j$. We formulate the complex Hessian equations as follows.

Let $\sigma_k(\lambda_1, \ldots, \lambda_n)$ be the k-th elementary symmetric function, i.e.,

$$\sigma_k(\lambda_1, \ldots, \lambda_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k},$$

where $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, and $1 \leq k \leq n$. Let $\lambda_\omega\{a_{i\bar{j}}\}$ denote the eigenvalues of the Hermitian symmetric matrix $\{a_{i\bar{j}}\}$ with respect to the Kähler form ω. We define

$$\sigma_k(a_{i\bar{j}}) = \sigma_k(\lambda_\omega\{a_{i\bar{j}}\}).$$

By a simple calculation, it can be shown that

$$\sigma_k(\omega_u) = \binom{n}{k} \frac{\omega_u^k \wedge \omega^{n-k}}{\omega^n}.$$

When $k = 1$, it is a quasilinear operator; when $k = n$, the n-Hessian operator corresponds to the complex Monge–Ampére operator, which plays a central role in Kähler geometry. We require the function u to satisfy a natural admissible condition,

$$u \in SH_k(\omega) = \{u \in C^{2,\alpha}(M) \mid \sigma_j(\omega + \sqrt{-1}\partial\bar{\partial}u) > 0,\ 1 \leq j \leq k\}. \tag{1-1}$$

MSC2010: 35K55, 53C44, 53C55.
Keywords: fully nonlinear flow, complex Hessian equation.

In this case, the operator $\log\sigma_k$ is concave. We consider the following parabolic complex Hessian equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \log\sigma_k(g_{i\bar{j}}+u_{i\bar{j}}) - \log f(x,u) = \log\dbinom{n}{k}\dfrac{\omega_u^k\wedge\omega^{n-k}}{\omega^n} - \log f(x,u), \\ u|_{M\times\{0\}} = u_0, \end{cases} \tag{1-2}$$

where $f(x,z)\in C^\infty(M\times\mathbb{R})$ is a given strictly positive function and $[0,T)$ is the maximal time interval in which the solution exists. We are going to study its existence and convergence of the solution to the flow (1-2). In order to obtain the L^∞-estimate of the solutions, we further require the function f to satisfy the following conditions: $\frac{\partial f(x,z)}{\partial z} > 0$ for each $x\in M$, and there exists a positive constant $z_0\in\mathbb{R}$ such that $f(x,z) > \binom{n}{k}$ at each point when $z > z_0$, and $f(x,z) < \binom{n}{k}$ when $z < -z_0$. This requirement is natural in real fully nonlinear type Yamabe problems; see [Li and Sheng 2011], for example. When $k=n$ and $f(x,z) = f(x)e^z$, the corresponding flow is the Kähler Ricci flow over manifolds with negative first Chern class, which has been solved by H. D. Cao [1985]. Our research can be viewed as a generalization of Cao's work.

Lu [2013] discussed the complex Hessian equation

$$\sigma_k(g_{i\bar{j}}+u_{i\bar{j}}) = f(x,u) = \binom{n}{k}\frac{(\omega+\sqrt{-1}\partial\bar{\partial}u)^k\wedge\omega^{n-k}}{\omega^n}, \tag{1-3}$$

and proved the uniqueness and existence of the solution in a weak sense under some integrability condition. In the case that f does not depend on u, the complex Hessian equation has been studied extensively in recent years. Hou [2009] proved an a priori C^0 estimate and solved it on manifolds with nonnegative holomorphic bisectional curvature. Hou et al. [2010] obtained the second-order estimate which depends on the square of the gradient estimates. By a blow-up argument and a Liouville type theorem, Dinew and Kolodziej [2017] further obtained the gradient estimate. W. Sun [2017] obtained the L^∞ estimate under the assumption of cone condition. D. H. Phong et al. [2016] studied more general complex Hessian equations for f depending not only on u but also on Du, and got the C^2 estimates under the assumption that the solutions are $(k+1)$-admissible. Recently Lu and Nguyen [2015] studied the complex Hessian functional

$$H_k(u) = \frac{\binom{n}{k}}{k+1}\sum_{j=0}^{n}\int_M u(\omega+\sqrt{-1}\partial\bar{\partial}u)^j\wedge\omega^{n-j}.$$

It can be easily shown that its gradient operator is

$$\binom{n}{k}\frac{(\omega+\sqrt{-1}\partial\bar{\partial}u)^k\wedge\omega^{n-k}}{\omega^n}.$$

In this paper we consider the complex Hessian functional

$$J_k(u) = \int_M F(x,u)\omega^n - \frac{\binom{n}{k}}{k+1} \sum_{j=0}^{k} \int_M u(\omega + \sqrt{-1}\partial\bar{\partial}u)^j \wedge \omega^{n-j} \tag{1-4}$$

where $F(x,u(x)) = \int_{t_0}^{u(x)} f(x,z)dz$. By use of the observation that (1-2) is the gradient flow of (1-4), we can prove the long-time existence of the solution to (1-2) and its convergence in the space of admissible functions. As its application, we show the regularity of the solution in [Lu 2013] under a slight restriction on f.

Recently, the complex Hessian equations (1-2) and (1-3) have attracted much interest because of their relations with many geometric problems, including the J-flow [Song and Weinkove 2008], quaternion geometry [Alesker and Verbitsky 2010] and Fu–Yau equations [Fu et al. 2012], etc. The real counterpart of the Hessian equations have been studied intensively in the literature; see the survey paper [Wang 2009].

Our main result in this paper can be stated as follows:

Theorem 1.1. *Let (M,ω) be a closed Kähler manifold of dimension $n \geq 2$. Assume that $f(x,z)$ satisfies the particular monotonicity conditions,*

$$0 < f \in C^\infty(M\times\mathbb{R}), \qquad \frac{\partial f(x,z)}{\partial z} > 0, \quad \textit{and}$$

$$\textit{there exists } z_0 \textit{ such that } \begin{cases} f(x,z) > \binom{n}{k} & \text{if } z > z_0, \\ \text{and } f(x,z) < \binom{n}{k} & \text{if } z < -z_0, \end{cases} \tag{1-5}$$

then (1-2) *admits a unique solution $u(x,t)$ defined on $M \times [0,T]$ for any $T > 0$, and*

$$\|u\|_{C^{2,\alpha}(M\times[0,T])} \leq C, \tag{1-6}$$

where C depends only on $\|f\|_{C^{2,\alpha}(M\times[-\|u\|_{C^0},\|u\|_{C^0}])}$, $\|u_0\|_{C^{2,\alpha}}$ and the metric ω on M. Further, $u(x,t)$ converges to the solution of (1-3) *uniformly when $t\to\infty$.*

In Section 2 we calculate the first and second variations of the Hessian functional (1-4), and show the parabolic equation (1-2) is a decreasing gradient flow, which would converge to the critical point if it is convergent and achieves a local minima under the monotonicity conditions (1-5) of $f(x,z)$. We also give a proof of the short-time existence of the parabolic equation, which follows from the standard parabolic methods.

In Section 3, we obtain:

Theorem 1.2. *Let (M,ω) be a closed Kähler manifold of dimension $n \geq 2$. Assume that $f(x,z)$ satisfies* (1-5). *Let u be an admissible solution to* (1-2). *Then*

$|u(x,t)| \le z_0 + 1$ on $M \times [0,T]$ for any $T > 0$. Furthermore, $\left|\frac{\partial}{\partial t}u(x,t)\right| \le e^{-H_1 t}\|u_t(x,0)\|_{C^0(M)}$, where the constant $H_1 = \inf_{M\times[-L,L]} h_z(x,z)$ depends only on z_0 and $f(x,z)$, and $h(x,z) = \log f(x,z)$.

We note that the estimate on u_t implies the convergence of the flow in the space of the admissible functions. We also remark here that if we drop out the monotonicity condition (1-5) on f, it would be much more complicated to obtain the lower bound of u_t, whereas the uniform upper bound exists by a direct calculation.

Next we denote $\|u(x,t)\|_{C^0(M\times[0,T])} := L$. In Section 5 we show:

Theorem 1.3. *Let (M,ω) be a closed Kähler manifold of dimension $n \ge 2$. Assume that $f(x,z)$ satisfies $0 < f(x,z) \in C^\infty(M \times \mathbb{R})$. Let u be an admissible solution to* (1-2). *Then*

$$\sup_{M\times\{t\}} |\partial\bar{\partial}u| \le C\Big(\sup_{M\times[0,T)} |Du|^2 + 1\Big),$$

where the positive constant C depends only on L, $\|\log f(x,z)\|_{C^2(M\times[-L,L])}$ and $\sup|u_t|$, *and is independent of t.*

Our argument comes from [Chou and Wang 2001] (see [Sheng et al. 2004] and [Hou et al. 2010] also).

By use of a standard blow-up argument, we reduce the gradient estimate of u, i.e., an estimate on $\sup_{M\times[0,T]}\|Du\|$ independent of T, to a Liouville type theorem which has been set up in [Dinew and Kołodziej 2017].

Theorem 1.4. *Let (M,ω) be a closed Kähler manifold of dimension $n \ge 2$. Assume that $f(x,z)$ satisfies $0 < f(x,z) \in C^\infty(M \times \mathbb{R})$. Let u be an admissible solution to* (1-2). *Then* $\sup_{M\times\{t\}}\|Du\|^2 \le K$, *where K is a constant which depends only on L, $\|\log f(x,z)\|_{C^2(M\times\{t\})}$, and* $\sup_{M\times\{t\}}|u_t|$, *and is independent of t.*

In the final section, we describe how to get the higher-order estimates, which follows from the classic Evans–Krylov type local estimate. We state our result as follows:

Let $x \in M$, $t \in [0,T]$, $T > 0$, and $R \in \mathbb{R}$, $R > 0$. Denote

$$Q(x,t,R) := \{(y,s) \in M\times[0,T] \mid \operatorname{dist}(y,x) \le R, s \in (t-R,t)\},$$

$$\tilde{C}^{2,\alpha}(M\times(a,b)) := \{u(x,t) \in C^{2,\alpha}(M) \text{ if fix } t; u(x,t) \in C^{1,\frac{1}{2}\alpha}(a',b') \text{ if fix } x\},$$

$$\sup_{M\times[0,T]} |\partial\bar{\partial}u| := S.$$

Theorem 1.5. *Let*

$$F(u) - \frac{\partial u}{\partial t} = h(x,u)$$

be a fully nonlinear parabolic equation on a closed Kähler manifold M of dimension $n \ge 2$, where $F(A)$ is monotone and concave on admissible space. Then at each

$(x, t) \in M \times [0, T]$, *there exists* R_0, *such that*

$$\|u(x,t)\|_{\tilde{C}^{2,\alpha}(Q(x,t,R_0))} \leq C(\|u\|_{\tilde{C}^2(Q(x,t,R_0))}).$$

2. Discussion on the functional

Throughout the paper, we denote $A = \{a_{i\bar{j}}\}$ an Hermitian (1,1)-tensor, $\lambda_\omega(A)$ its eigenvalues with respect to the metric ω, and $F(A) := \sigma_k(\lambda_\omega(a_{i\bar{j}}))$. We also write

$$F^{i\bar{j}} := \frac{\partial F}{\partial a_{i\bar{j}}}, \quad F^{i\bar{j},p\bar{q}} := \frac{\partial^2 F}{\partial a_{i\bar{j}} \partial a_{p\bar{q}}}.$$

By a standard calculation (see [Wang 2009]) we can see when $g_{i\bar{j}} = \delta_{ij}$ and $a_{i\bar{j}}$ is diagonal,

$$F^{i\bar{j}} = \sigma_{k-1;i}\delta_{ij}, \tag{2-1}$$

and

$$F^{i\bar{j},p\bar{q}} = \begin{cases} F^{i\bar{i},p\bar{p}} = \frac{\sigma_{k-2;i,p}}{\sigma_k} - \frac{\sigma_{k-1;i}\sigma_{k-1;p}}{(\sigma_k)^2} & i = j,\ p = q,\ i \neq p, \\ F^{i\bar{j},j\bar{i}} = -\frac{\sigma_{k-2;i,j}}{\sigma_k} & i = p,\ j = q,\ i \neq j, \\ 0 & \text{otherwise}, \end{cases} \tag{2-2}$$

where $\sigma_{k;i} := \sigma_k(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n)$.

We consider the following functional which was studied by Lu and Nguyen [2015]:

$$J_k(u) = \int_M F(x,u)\omega^n - \frac{\binom{n}{k}}{k+1} \sum_{j=0}^{k} \int_M u(\omega + \sqrt{-1}\partial\bar{\partial}u)^j \wedge \omega^{n-j}. \tag{2-3}$$

Its Euler–Lagrange equation is

$$\sigma_k(g_{i\bar{j}} + u_{i\bar{j}}) = f(x,u) = \binom{n}{k} \frac{(\omega + \sqrt{-1}\partial\bar{\partial}u)^k \wedge \omega^{n-k}}{\omega^n}, \tag{2-4}$$

where

$$F(x, u(x)) = \int_0^{u(x)} f(x,z)dz.$$

Note that $F(x,z) > 0$ when $z > 0$, and $F \leq 0$ otherwise.

For convenience, we list calculations on the first and second variations as follows.

$$\begin{aligned} \delta J(u)\phi = \int_M f(x,u)\phi\omega^n - \frac{\binom{n}{k}}{k+1} \sum_{j=0}^{k} \int_M \phi(\omega_u)^j \wedge \omega^{n-j} \\ - \frac{\binom{n}{k}}{k+1} \sum_{j=0}^{k} \int_M j\,u(\sqrt{-1}\partial\bar{\partial}\phi) \wedge (\omega_u)^{j-1} \wedge \omega^{n-j}, \end{aligned}$$

with the third term

$$-\frac{\binom{n}{k}}{k+1}\sum_{j=1}^{k}\int_M j\,u(\sqrt{-1}\partial\bar\partial\phi)\wedge(\omega_u)^{j-1}\wedge\omega^{n-j}$$
$$=-\frac{\binom{n}{k}}{k+1}\sum_{j=1}^{k}\int_M j\,\phi(\sqrt{-1}\partial\bar\partial u)\wedge(\omega_u)^{j-1}\wedge\omega^{n-j}$$
$$=-\frac{\binom{n}{k}}{k+1}\sum_{j=1}^{k}\int_M j\,\phi(\sqrt{-1}\partial\bar\partial u+\omega-\omega)\wedge(\omega_u)^{j-1}\wedge\omega^{n-j}$$
$$=-\frac{\binom{n}{k}}{k+1}\sum_{j=1}^{k}\int_M j\,\phi(\omega_u)^{j}\wedge\omega^{n-j}+\frac{\binom{n}{k}}{k+1}\sum_{j=0}^{k-1}\int_M (j+1)\,\phi(\omega_u)^{j}\wedge\omega^{n-j}$$
$$=\frac{\binom{n}{k}}{k+1}\int_M\phi\omega^n+\frac{\binom{n}{k}}{k+1}\sum_{j=1}^{k-1}\int_M\phi(\omega_u)^{j}\wedge\omega^{n-j}-\frac{k\binom{n}{k}}{k+1}\int_M\phi(\omega_u)^{k}\wedge\omega^{n-k}.$$

Hence the original formula turns out to be

$$\delta J(u)\phi=\int_M\phi f(x,u)-\binom{n}{k}\int_M\phi(\omega+\sqrt{-1}\partial\bar\partial u)^k\wedge\omega^{n-k}. \tag{2-5}$$

Now we calculate the second variations. It is easy to get

$$\delta^2 J_k(u)\phi^2=\int_M\frac{\partial}{\partial z}f(x,u(x))\phi^2\omega^n +\sqrt{-1}\int_M\partial\phi\wedge\bar\partial\phi\wedge(\omega+\sqrt{-1}\partial\bar\partial u)^{k-1}\wedge\omega^{n-k}. \tag{2-6}$$

Under the normal coordinates $\omega+\sqrt{-1}\partial\bar\partial u=\sqrt{-1}\sum_i(1+u_{i\bar i})dz^i\wedge d\bar z^i$, we compute the second term

$$\sqrt{-1}d\phi\wedge\bar\partial\psi\wedge(\omega+\sqrt{-1}\partial\bar\partial u)^{k-1}\wedge\omega^{n-k}$$
$$=(n-k)!(k-1)!(\sqrt{-1})^n\phi_{z^i}\psi_{\bar z^i}\sigma_{k-1;i}(\omega+\sqrt{-1}\partial\bar\partial u)dz^1\wedge d\bar z^1\wedge\cdots\wedge d\bar z^n.$$

Therefore, under the monotonicity condition (1-5) on $f(x,z)$, the second variation (2-6) is positive for nonzero C^2 function ϕ, and the functional J_k would achieve a local minimum at each critical point.

In the rest of this section, we show the solution to the flow (1-2) exists in a short-time interval. Consider the following map $\mathbb{F}$ which is defined from $C^{2,\alpha}(M\times[0,T))$ to $C^{0,\alpha}(M\times[0,T))$:

$$\mathbb{F}(w)=\log\sigma_k(g_{i\bar j}+w_{i\bar j})-\log f(x,w)-w_t.$$

At first, for any initial function $u_0 \in SH_k(\omega)$, we extend u_0 to a function Φ in $C^{2,\alpha}(M \times [0, T))$ so that

$$\frac{\partial^j}{(\partial t)^j}\bigg|_{t=0} (\mathbb{F}(\Phi)) = 0,$$

where $j = 0, 1$. We write $\Phi(x, t) = u_0(x) + u_1(x)t + \frac{1}{2}u_2(x)t^2$, where

$$\begin{aligned} u_1(x) &= \log \sigma_k(g_{i\bar{j}} + (u_0)_{i\bar{j}}) - \log f(x, u_0), \\ u_2(x) &= F^{i\bar{j}}(\omega + dd^c u_0)(u_1)_{i\bar{j}} - \frac{f_z}{f}(x, u_0)u_1. \end{aligned}$$

The linearization of the map $\mathbb{F}$ at Φ is

$$D\mathbb{F}(\Phi)w = F^{i\bar{j}}(\Phi)w_{i\bar{j}} - \frac{f_z}{f}(x, \Phi)w - w_t.$$

Thus, by the invertibility of the Fréchet derivative $D\mathbb{F}(\Phi)$ and the implicit function theorem, we can choose t to be suitably small such that $\Phi \in SH_k(\omega, t)$ and $\|\mathbb{F}(\Phi)\|_{C^{0,\alpha}(M\times[0,t))}$ are sufficiently small, therefore $\mathbb{F}(\Phi + w) = 0$ is solvable. Therefore we get:

Proposition 2.1. Let (M, ω) be a closed Kähler manifold of dimension $n \geq 2$ and $u_0 \in SH_k(\omega)$. Then the flow (1-2) has a unique solution on the interval $[0, \epsilon)$ for some $\epsilon > 0$.

3. Estimates on $|u(x, z)|$, $|u_t(x, z)|$ and convergence of the flow

Let x_t be the maximal point of u at time t, and M_t the maximum. It is easy to see that M_t varies continuously. At x_t, by (1-2), we have

$$\frac{du}{dt} = \log \sigma_k - \log f(x_t, M_t) \leq \log \binom{n}{k} - \log f(x_t, M_t).$$

Suppose $M_t > z_0 + 1$, then on $(t - \epsilon, t + \epsilon)$, by the monotonicity condition (1-5), we have $f(x_t, M_t) > \binom{n}{k}$, which implies

$$\frac{du}{dt} \leq 0.$$

Similarly, let y_t the minimal point of u at time t with its value m_t. At y_t, if $m_t < -z_0$, then by (1-5)

$$\frac{du}{dt} = \log \sigma_k - \log f(y_t, m_t) \geq \log \binom{n}{k} - \log f(y_t, m_t) \geq 0.$$

So $|u(x, z)| \leq z_0 + 1$.

Next we estimate on $|u_t|$. Differentiating (1-2) on both sides simultaneously at t, we obtain

$$u_{tt} = F^{i\bar{j}} u_{i\bar{j}t} - h_z u_t = F^{i\bar{j}} u_{ti\bar{j}} - h_z u_t,$$

where we denote $h(x,z) = \log f(x,z)$. Since $h_z > 0$, let $H_1 = \inf_{M\times[-L.L]} h_z(x,z)$ be its positive lower bound. Let $t \in (0,T)$ be an arbitrary time. Suppose u_t achieves its maximum M_t at x_t. Without loss of generality, we may suppose $M_t > 0$. Then at x_t,

$$u_{tt} = F^{i\bar{i}} u_{ti\bar{i}} - h_z(x,u) u_t \le -H_1 u_t.$$

It follows that $u_t \le e^{-H_1 t}\|u_t(x,0)\|$. Similarly, we can get $u_t \ge -e^{-H_1 t}\|u_t(x,0)\|$. Therefore we have $\|u_t\| \le e^{-H_1 t}\|u_t(x,0)\|$. Moreover, by use of the second-order derivative and gradient estimates in the next two sections, we can get the $C^{2,\alpha}$-estimate of the solutions to (1-2) in the last section. Then there is a subsequence $t_i \to \infty$ such that $u(x,t_i) \in C^2(M) \bigcap \mathcal{SH}_k(M,\omega)$ converges, which can be obtained directly by the fact $\lim_{t\to\infty} |u_t(x,t)| = 0$. So the limit of the solutions $\lim_{t\to\infty} u(x,t)$ is a critical point of $J_k(u)$ which solves the corresponding elliptic equation.

4. Second-order estimates

As we denoted above, $h(x,z) := \log f(x,z)$. Our arguments are a parabolic version of those in [Hou et al. 2010], through a careful calculation so as to make sure the second-order estimate is controlled by the square of the gradient estimate linearly. We use the following conventions in this section:

$$\begin{aligned} K &= \max\{\sup|Du|^2, 1\},\\ L &= \sup|u|,\\ H &= \|h(x,z)\|_{C^2(M\times[-L,L])},\\ U &= \sup|u_t|. \end{aligned}$$

Consider

$$W(x,t,\xi) = (1 + u_{i\bar{j}}\xi^i\bar{\xi}^j)\exp(\phi(\|Du\|^2) + \psi(u)), \tag{4-1}$$

where ξ is a unit tangent vector at the corresponding point. We define

$$\phi(z) = -\frac{1}{2}\log\Big(1 - \frac{z}{2K}\Big),$$

and

$$\psi(z) = -A\log\Big(1 + \frac{z}{2L}\Big).$$

Then we have

$$\phi'(z) = \frac{1}{2}\frac{1}{2K-z},$$
$$\phi''(z) = \frac{1}{2}\frac{1}{(2K-z)^2} = 2(\phi'(z))^2,$$
$$\psi'(z) = -A\frac{1}{2L+z},$$
$$\psi''(z) = A\frac{1}{(2L+z)^2} > 0,$$

and $\phi'(\|Du\|^2) \in \left[\frac{1}{4K}, \frac{1}{2K}\right]$, $\psi'(u) \in \left[-\frac{A}{L}, -\frac{1}{3L}A\right]$. We choose A as

$$A = 6L \sup \|Rm\| - \frac{L}{2} + 1.$$

For any $T > 0$, we choose the coordinates around the maximum point of W on $M \times (0, T] \times \mathbb{S}^{2n-1}$ such that the matrix $\{g_{i\bar{j}} + u_{i\bar{j}}\}$ is diagonal at the point, and satisfies

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n,$$

where

$$\lambda_i = 1 + u_{i\bar{i}}.$$

After a straightforward calculation, we see

$$F^{i\bar{i}}(x_0, t_0) = \frac{\sigma_{k-1;i}}{\sigma_k},$$
$$\mathcal{F} := \sum_{i=1}^{n} F^{i\bar{i}} = (n-k+1)\frac{\sigma_{k-1}}{\sigma_k},$$
$$F^{i\bar{i}}\lambda_i = k.$$

Since $\xi = \frac{\partial}{\partial z^1}$, we may extend ξ to the chart by letting $\xi = (g_{1\bar{1}})^{-\frac{1}{2}}\frac{\partial}{\partial z^1}$. Using these coordinates, at (x_0, t_0), we have

$$G(x,t) := \log W(x,t,\xi) = \log\left(1 + \frac{u_{1\bar{1}}}{g_{1\bar{1}}}\right) + \phi(|Du|^2) + \psi(u), \tag{4-2}$$

$$0 \leq G_t = \frac{u_{1\bar{1}t}}{1+u_{1\bar{1}}} + \phi' \sum_p \{u_{pt}u_{\bar{p}} + u_p u_{\bar{p}t}\} + \psi' u_t, \tag{4-3}$$

$$0 = G_i = \frac{u_{1\bar{1}i}}{1+u_{1\bar{1}}} + \phi'\{u_{pi}u_{\bar{p}} + u_i u_{\bar{i}i}\} + \psi' u_i, \tag{4-4}$$

and

$$0 \geq F^{i\bar{i}} G_{i\bar{i}} - G_t = F^{i\bar{i}} \frac{u_{1\bar{1}i\bar{i}}}{1+u_{1\bar{1}}} - F^{i\bar{i}} \Big\| \frac{u_{1\bar{1}i}}{1+u_{1\bar{1}}} \Big\|^2 + \phi' F^{i\bar{i}} u_{pi} u_{\bar{p}\bar{i}} + \phi' F^{i\bar{i}} \|u_{i\bar{i}}\|^2 + \phi' F^{i\bar{i}} \{u_{pi\bar{i}} u_{\bar{p}} + u_p u_{\bar{p}i\bar{i}}\} + \phi'' F^{i\bar{i}} \|u_{pi} u_{\bar{p}} + u_i u_{\bar{i}i}\|^2 + \psi' F^{i\bar{i}} u_{i\bar{i}} + \psi'' F^{i\bar{i}} \|u_i\|^2 - G_t. \tag{4-5}$$

Taking the derivative of (1-2) on both sides in the $\partial/\partial z^p$ direction, we have

$$u_{tp} = u_{pt} = F^{i\bar{i}} u_{i\bar{i}p} - h_z u_p - h_p. \tag{4-6}$$

By commuting the covariant derivatives

$$F^{i\bar{i}} u_{i\bar{i}p} = F^{i\bar{i}} u_{pi\bar{i}} + F^{i\bar{i}} R_{\bar{i}pi\bar{q}} u_q,$$

where we have used the curvature tensor $R(\partial/\partial z^i, \partial/\partial \bar{z}^j)\partial/\partial z^k = R_{i\bar{j}k}{}^l \partial/\partial z^l$. We obtain

$$F^{i\bar{i}} u_{pi\bar{i}} = u_{pt} + h_z u_p - F^{i\bar{i}} R_{\bar{i}pi\bar{q}} u_q + h_p.$$

Similarly

$$F^{i\bar{i}} u_{\bar{p}i\bar{i}} = u_{\bar{p}t} + h_z u_{\bar{p}} + h_{\bar{p}}.$$

Hence we have

$$\begin{aligned} \phi' \sum F^{i\bar{i}} \{u_{pi\bar{i}} u_{\bar{p}} + u_p u_{\bar{p}i\bar{i}}\} - \phi' \sum \{u_{pt} u_{\bar{p}} + u_p u_{\bar{p}t}\} &= \phi' \big\{2h_z |Du|^2 - F^{i\bar{i}} R_{\bar{i}pi\bar{q}} u_q u_{\bar{p}} + h_p u_{\bar{p}} + h_{\bar{p}} u_p\big\} \\ &\geq -\frac{\sup |R_m|}{2} \mathcal{F} - \frac{3}{2} H. \end{aligned} \tag{4-7}$$

Taking the second derivative of (4-6) on both sides, we have

$$u_{tk\bar{l}} = u_{k\bar{l}t} = F^{i\bar{j}} u_{i\bar{j}k\bar{l}} + F^{i\bar{j},p\bar{q}} u_{i\bar{j}k} u_{p\bar{q}\bar{l}} - h_{k\bar{l}} - h_{kz} u_{\bar{l}} - h_z u_{k\bar{l}} - h_{z\bar{l}} u_k - h_{zz} u_k u_{\bar{l}}.$$

Since

$$u_{i\bar{j}k\bar{l}} = u_{k\bar{l}i\bar{j}} + u_{p\bar{j}} R_{i}{}^{p}{}_{k\bar{l}} - u_{p\bar{l}} R_{i}{}^{p}{}_{k\bar{j}}, \tag{4-8}$$

by choosing $k = 1$, $\bar{l} = \bar{1}$, and taking the value at the maximal point, we get

$$F^{i\bar{i}} u_{i\bar{i}1\bar{1}} = F^{i\bar{i}} u_{1\bar{1}i\bar{i}} + F^{i\bar{i}} R_{i\bar{1}1\bar{i}} (u_{i\bar{i}} - u_{1\bar{1}}).$$

Hence we have

$$
\begin{aligned}
(4\text{-}9)\quad & F^{i\bar{i}}\frac{u_{1\bar{1}i\bar{i}}}{1+u_{1\bar{1}}}-\frac{u_{1\bar{1}t}}{1+u_{1\bar{1}}}\\
&=\lambda_1^{-1}\big\{-F^{i\bar{j},p\bar{q}}u_{i\bar{j}1}u_{p\bar{q}\bar{1}}+h_{1\bar{1}}+h_{1z}u_{\bar{1}}+h_zu_{1\bar{1}}+h_{z\bar{1}}u_1+h_{zz}|u_1|^2\\
&\qquad\qquad+F^{i\bar{i}}R_{i\bar{1}1\bar{i}}(u_{1\bar{1}}-u_{i\bar{i}})\big\}\\
&\geq\lambda_1^{-1}\big\{-F^{i\bar{j},p\bar{q}}u_{i\bar{j}1}u_{p\bar{q}\bar{1}}-2H-2H\sqrt{K}-H\lambda_1-HK\\
&\qquad\qquad-\sup\|Rm\|(\lambda_1\mathcal{F}-k)\big\}.
\end{aligned}
$$

We may suppose $\lambda_1\geq K$, and $K\geq 1$, otherwise we are done. Thus

$$\frac{-2H\sqrt{K}-HK}{\lambda_1}\geq -3H.$$

Then (4-9) becomes

$$
\begin{aligned}
(4\text{-}10)\quad & F^{i\bar{i}}\frac{u_{1\bar{1}i\bar{i}}}{1+u_{1\bar{1}}}-\frac{u_{1\bar{1}t}}{1+u_{1\bar{1}}}\\
&\geq -F^{i\bar{j},p\bar{q}}u_{i\bar{j}1}u_{p\bar{q}\bar{1}}-\sup\|Rm\|\mathcal{F}-5H-k\sup\|Rm\|.
\end{aligned}
$$

The third term of (4-3) satisfies

$$-\psi' u_t\geq -\frac{A}{L}U. \tag{4-11}$$

The seventh term of (4-5) satisfies

$$\psi' F^{i\bar{i}}u_{i\bar{i}}=\psi' F^{i\bar{i}}\lambda_i-\psi'\sum F^{i\bar{i}}\geq -\frac{A}{L}k+\frac{A}{3L}\mathcal{F}. \tag{4-12}$$

The forth term of (4-5) satisfies

$$
\begin{aligned}
(4\text{-}13)\quad & \phi' F^{i\bar{i}}|u_{i\bar{i}}|^2=\phi' F^{i\bar{i}}(\lambda_i-1)^2\\
&=\phi' F^{i\bar{i}}\lambda_i^2-2\phi' F^{i\bar{i}}\lambda_i+\phi'\mathcal{F}\geq\frac{1}{4K}F^{i\bar{i}}\lambda_i^2+\frac{1}{4K}\mathcal{F}-\frac{k}{K}.
\end{aligned}
$$

Substituting (4-7)–(4-13) into (4-5), we obtain

$$
\begin{aligned}
(4\text{-}14)\quad 0\geq & -F^{i\bar{j},p\bar{q}}u_{i\bar{j}1}u_{p\bar{q}\bar{1}}-F^{i\bar{i}}\left|\frac{u_{1\bar{1}i}}{1+u_{1\bar{1}}}\right|^2+\phi'' F^{i\bar{i}}\left|u_{pi}u_{\bar{p}}+u_iu_{\bar{i}i}\right|^2\\
&+\psi'' F^{i\bar{i}}|u_i|^2+\left\{-2\sup|R_m|+\frac{1}{4K}+\frac{A}{3L}\right\}\mathcal{F}\\
&-\left\{k\sup|R_m|+5H+\frac{A}{L}U+\frac{A}{L}k+\frac{k}{K}\right\}+\frac{1}{4K}F^{i\bar{i}}\lambda_i^2.
\end{aligned}
$$

Now we set

$$\delta=\frac{1}{1+2A}=\frac{1}{6L\sup\|Rm\|+2}.$$

We separate the rest of the calculations into two cases.

Case 1: $\lambda_n \le -\delta\lambda_1$. The first term of (4-14) is

$$-F^{i\bar{j},p\bar{q}} u_{i\bar{j}1} u_{p\bar{q}\bar{1}} \ge 0. \tag{4-15}$$

The second term is

$$\begin{aligned} -F^{i\bar{i}} \left| \frac{u_{1\bar{1}i}}{1+u_{1\bar{1}}} \right| &= -F^{i\bar{i}} \left| \phi'(u_{pi}u_{\bar{p}} + u_i u_{\bar{i}i}) + \psi' u_i \right|^2 \\ &\ge -2(\phi')^2 F^{i\bar{i}} |u_{pi}u_{\bar{p}} + u_i u_{\bar{i}i}|^2 - 2(\psi')^2 F^{i\bar{i}} |u_i|^2. \end{aligned} \tag{4-16}$$

Since $2\phi'^2 = \phi''$, the first term of (4-16) cancels out the third term of (4-14). The second term of (4-16) can be estimated as

$$-2(\psi')^2 F^{i\bar{i}} |u_i|^2 \ge -2(\psi')^2 \mathcal{F} K \ge -A^2 \frac{8}{9L^2} \mathcal{F} K. \tag{4-17}$$

Substituting (4-15)–(4-17) into (4-14), we obtain

$$\begin{aligned} 0 \ge &\left\{ -2 \sup \|Rm\| + \frac{1}{4K} + \frac{A}{3L} - A^2 \frac{8}{9L^2} K \right\} \mathcal{F} \\ &- \left\{ k \sup \|Rm\| + 5H + \frac{A}{L} U + \frac{A}{L} k + \frac{k}{K} \right\} + \frac{1}{4K} F^{i\bar{i}} \lambda_i^2. \end{aligned} \tag{4-18}$$

In the final term of (4-18), since $\lambda_n^2 \ge \delta^2 \lambda_1^2$, and

$$F^{n\bar{n}} \ge \cdots \ge F^{1\bar{1}}$$

imply

$$F^{n\bar{n}} \ge \frac{\mathcal{F}}{n},$$

we can estimate $F^{n\bar{n}}$ as

$$\frac{1}{4K} F^{i\bar{i}} \lambda_i^2 \ge \frac{\delta^2}{4Kn} \mathcal{F} \lambda_1^2.$$

Therefore, (4-18) becomes

$$\begin{aligned} &\left\{ 2 \sup \|Rm\| - \frac{1}{4K} - \frac{A}{3L} + A^2 \frac{8}{9L^2} K \right\} \mathcal{F} \\ &\qquad + \left\{ k \sup \|Rm\| + 5H + \frac{A}{L} U + \frac{A}{L} k + \frac{k}{K} \right\} \ge \frac{\delta^2}{4Kn} \mathcal{F} \lambda_1^2. \end{aligned}$$

The choice of A implies that the coefficient of $\mathcal{F}$ is negative. So we have

$$\lambda_1^2 \le C(K^2 + 1).$$

Case 2: $\lambda_n > -\delta\lambda_1$. The argument comes from [Chou and Wang 2001] in real Hessian equations (see also [Hou et al. 2010] in complex Hessian equations) where the authors proved the inequality $F^{1\bar{1}} = \frac{\sigma_{k-1;1}}{\sigma_k} \geq \frac{k}{n}\frac{1}{\lambda_1}$. Thus, $1/F^{1\bar{1}} \leq \frac{n}{k}\lambda_1$. Let

$$\mathbb{I} = \left\{ j \;\middle|\; F^{j\bar{j}}(x_0,t_0) \geq \frac{1}{\delta}F^{1\bar{1}}(x_0,t_0), \quad i.e., \quad \sigma_{k-1;j} \geq \frac{1}{\delta}\sigma_{k-1;1} \right\}.$$

It is easy to verify $1 \notin \mathbb{I}$. Then we separate the second term of (4-14) into two parts; one part is

$$-\sum_{\mathbb{I}^c} F^{i\bar{i}} \left| \frac{u_{1\bar{1}i}}{1+u_{1\bar{1}}} \right|^2 = -\sum_{\mathbb{I}^c} F^{i\bar{i}} |\phi'\{u_{pi}u_{\bar{p}} + u_i u_{\bar{i}i}\} + \psi' u_i|^2$$
$$\geq -2(\phi')^2 \sum_{\mathbb{I}^c} F^{i\bar{i}} |u_{pi}u_{\bar{p}} + u_i u_{\bar{i}i}|^2 - 2(\psi')^2 \sum_{\mathbb{I}^c} F^{i\bar{i}} |u_i|^2. \tag{4-19}$$

The final term in (4-19) can be estimated by

$$-2(\psi')^2 \sum_{\mathbb{I}^c} F^{i\bar{i}} |u_i|^2 \geq -A^2 \frac{2n}{L^2\delta^2} F^{1\bar{1}} K. \tag{4-20}$$

We claim the other part of the second term of (4-14) satisfies

$$-\sum_{\mathbb{I}} F^{i\bar{i}} \left| \frac{u_{1\bar{1}i}}{1+u_{1\bar{1}}} \right|^2 - F^{i\bar{j},p\bar{q}} \frac{u_{i\bar{j}1}u_{p\bar{q}\bar{1}}}{1+u_{1\bar{1}}}$$
$$+\phi'' \sum_{\mathbb{I}} F^{i\bar{i}} |u_{pi}u_{\bar{p}} + u_i u_{\bar{i}i}|^2 + \sum \psi'' F^{i\bar{i}} |u_i|^2 \geq 0. \tag{4-21}$$

Then, by (4-19)–(4-21), (4-14) becomes

$$0 \geq \left\{ -2\sup\|Rm\| + \frac{1}{4K} + \frac{A}{3L} \right\} \mathcal{F}$$
$$-\left\{ k\sup\|Rm\| + 5H + \frac{A}{L}U + \frac{A}{L}k + \frac{k}{K} \right\}$$
$$+\frac{1}{4K} F^{i\bar{i}} \lambda_i^2 - A^2 \frac{2n}{L^2\delta^2} F^{1\bar{1}} K. \tag{4-22}$$

Since $A \geq 6L\sup\|Rm\| - \frac{L}{2K} + 1$, the coefficient of $\mathcal{F}$ is positive. We denote it as C_1, which depends only on L. We then have

$$k\sup\|Rm\| + 5H + \frac{A}{L}U + \frac{A}{L}k + \frac{k}{K} \geq C_1\mathcal{F} + \frac{1}{4K} F^{1\bar{1}} \lambda_1^2 - 2nA^2 \frac{1}{L^2\delta^2} F^{1\bar{1}} K. \tag{4-23}$$

We may suppose

$$F^{1\bar{1}} \lambda_1^2 \frac{1}{8K} \geq 2nA^2 \frac{1}{L^2\delta^2} F^{1\bar{1}} K, \tag{4-24}$$

otherwise,

$$\lambda_1 < 5nA\frac{K}{L\delta},$$

and we are done. Now by (4-24), (4-23) becomes

$$k\sup|R_m| + 5H + \frac{A}{L}U + \frac{A}{L}k + \frac{k}{K} \geq C_1\mathcal{F} + \frac{1}{8K}F^{1\bar{1}}\lambda_1^2.$$

By [Chou and Wang 2001], we have $F^{1\bar{1}} = \sigma_{k-1;1}/\sigma_k \geq (k/n)(1/\lambda_1)$. Therefore

$$\lambda_1 \leq C(K+1).$$

Now in order to finish the proof of Theorem 1.3, we need to prove the claim (4-21). In fact it was proved in [Hou et al. 2010], so we omit it here.

5. A blow up argument

In order to obtain the C^1-estimate, we employ a blow-up method analogous to that in [Dinew and Kołodziej 2017] and reduce the problem to a Liouville type theorem which was proved in [Dinew and Kołodziej 2017]. Thus we obtain a contradiction.

Suppose on a compact Kähler manifold (M, ω), there exists a positive function $f(x,z) \in C^\infty(M\times\mathbb{R})$ and $u(x,t)$ on $M\times[0,T]$, such that for each t, $u(x,t) \in SH_k(\omega)$ and

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^k \wedge \omega^{n-k} = f(x,u)e^{\frac{\partial u}{\partial t}}\omega^n, \tag{5-1}$$

but there exists a sequence of t_j,

$$\sup_M \|du(x,t_j)\| = \|du(x_j,t_j)\| = C_j \to +\infty, \quad \text{as } j \to +\infty, \tag{5-2}$$

where x_j is the maximal point of $\|du(x,t_j)\| =: \|du_j(x)\|$.

It follows from Theorem 1.3 that

$$\sup\|\sqrt{-1}\partial\bar{\partial}u_j\| \leq C(C_j^2+1) = O(C_j^2).$$

Without loss of generality, we suppose $x_j \to x_0$, $t_j \to t_0$ as $j \to +\infty$, and all x_j are contained in a geodesic ball $\mathbb{B}_\epsilon(x_0)$, which is of radius ϵ and centered at x_0, where $\epsilon > 0$ is a fixed constant. Choose coordinates $(z^1,\dots,z^n)$, briefly denoted by z, and centered at x_0, and $x_j = (z_j^1,\dots,z_j^n)$, briefly z_j. By the definition of Kähler metric, ω can be approximated by a standard Euclidean type metric β up to order 2, i.e.,

$$\omega = \beta + O(|z|^2), \ |z| \to 0.$$

Moreover by the $\partial\bar{\partial}$-lemma, the metric can be locally written as the potential of a function v,

$$\omega = \sqrt{-1}\partial\bar{\partial}v(z^1,\dots,z^n).$$

Now we consider the functions

$$\hat{u}_j(z) = u\Big(z_j + \frac{z}{C_j}, t_j\Big).$$

By the construction, we conclude that $\hat{u}_j \in SH_k(\omega)$ on the ball

$$\mathbb{B}_{\epsilon C_j}(0) = \{z \in \mathbb{C}^n, |z| < \epsilon C_j\},$$

and

$$d\hat{u}_j(z) = \frac{1}{C_j} du(z, t_j), \qquad \sqrt{-1}\partial\bar{\partial}\hat{u}_j(z) = \frac{1}{C_j^2} dd^c u(z, t_j).$$

Hence we have

$$\sup_M \|d\hat{u}_j\| = 1, \qquad \sup_M \|\sqrt{-1}\partial\bar{\partial}\hat{u}_j\| < C.$$

This yields that $\hat{u}_j$ is contained in the Hölder space $C^{1,\gamma}$ with a uniform bound. Along with a standard application of the Azela–Ascoli theorem, we may suppose $\hat{u}_j$ has a limit $\hat{u}$ on the complex Euclidean space $\mathbb{C}^n$. However by (5-1),

$$(\sqrt{-1}\partial\bar{\partial}(v + u)(z))^k \wedge (\sqrt{-1}\partial\bar{\partial}v(z))^{n-k} = f(z, u(z, t_j))e^{u_{t_j}}(dd^c v(z))^n.$$

Hence $\hat{u}_j$ satisfies

$$\left(\frac{1}{C_j^2}\sqrt{-1}\partial\bar{\partial}v\Big(z_j + \frac{z}{C_j}\Big) + \sqrt{-1}\partial\bar{\partial}\hat{u}_j(z)\right)^k \wedge \left(\frac{1}{C_j^2}\sqrt{-1}\partial\bar{\partial}v\Big(z_j + \frac{z}{C_j}\Big)\right)^{n-k} = O\Big(\frac{1}{C_j^{2n}}\Big)\left(\sqrt{-1}\partial\bar{\partial}v\Big(z_j + \frac{z}{C_j}\Big)\right)^n \tag{5-3}$$

Since $\omega = \beta + O(|z|^2)$ near the origin, we have for j sufficiently large,

$$\left(\frac{1}{C_j^2}\beta + \sqrt{-1}\partial\bar{\partial}\hat{u}_j\right)^k \wedge \left(\frac{1}{C_j^2}\beta\right)^{n-k} = O\Big(\frac{1}{C_j^{2n}}\Big)\beta^n. \tag{5-4}$$

Taking the limits on both sides simultaneously,

$$(\sqrt{-1}\partial\bar{\partial}\hat{u}(z))^k \wedge \beta^{n-k} = 0, \tag{5-5}$$

where $\hat{u}$ is defined on $\mathbb{C}^n$. When $1 \le m < k$, similar arguments show that

$$(\sqrt{-1}\partial\bar{\partial}\hat{u})^m \wedge \beta^{n-m} \ge 0.$$

Then by the Liouville type theorem proved in [Dinew and Kołodziej 2017], $\hat{u}$ must be a constant. Since $\|d\hat{u}(x_j, t_j)\| = 1$, there is a sufficiently small neighborhood of x_0, $\|d\hat{u}(x, t_j)\| > 1 - \epsilon$. Therefore $\|d\hat{u}(x_0)\| \ge 1 - \epsilon$. This is a contradiction.

6. An Evans–Krylov type estimate of parabolic version

We give an Evans–Krylov's type local estimate (Theorem 1.5) in this section.

At any point (x, t), choose local coordinates on its neighborhood $U \times [0, T)$. First by differentiating the equation on both sides of (1-2) twice simultaneously, we can obtain a uniform bound at each point in this neighborhood on $\left\{\frac{\partial}{\partial t} - \triangle_{\mathrm{real}}\right\} u_{k\bar{l}}$ and $\left\{\frac{\partial}{\partial t} - \triangle_{\mathrm{real}}\right\} u_t$.

$$F^{i\bar{i}} u_{k\bar{l}i\bar{i}} - u_{k\bar{l}t} = -F^{i\bar{j},p\bar{q}} u_{i\bar{j}k} u_{p\bar{q}\bar{l}} + h_{k\bar{l}} + h_{kz} u_{\bar{l}} + h_z u_{k\bar{l}} + h_{z\bar{l}} u_k + h_{zz} u_k u_{\bar{l}} + F^{i\bar{i}} R_{i\bar{k}k\bar{i}} (u_{k\bar{k}} - u_{i\bar{i}}), \tag{6-1}$$

where we have used the Ricci identities (4-8). Here we recall $h(x, z) = \log f(x, z)$. Since

$$F^{i\bar{i}} R_{i\bar{k}k\bar{i}} (u_{k\bar{k}} - u_{i\bar{i}}) = F^{i\bar{i}} R_{i\bar{k}k\bar{i}} (\lambda_k - \lambda_i) \geq -2S \sup \|Rm\| \sum_i F^{i\bar{i}},$$

we then choose

$$\tilde{\psi}(z) = 6LS \sup \|Rm\| \log\left(1 + \frac{z}{2L}\right).$$

Note that

$$\tilde{\psi}'(z) = \frac{6LS \sup \|Rm\|}{2L + z}.$$

We have

$$\begin{aligned} F^{i\bar{i}} (u_{k\bar{l}}(x, t) - \tilde{\psi}(u))_{i\bar{i}} - (u_{k\bar{l}}(x, t) - \tilde{\psi}(u))_t \\ = -F^{i\bar{j},p\bar{q}} u_{i\bar{j}k} u_{p\bar{q}\bar{l}} + h_{k\bar{l}} + h_{kz} u_{\bar{l}} + h_z u_{k\bar{l}} + h_{z\bar{l}} u_k + h_{zz} u_k u_{\bar{l}} \\ + F^{i\bar{i}} R_{i\bar{k}k\bar{i}} (u_{k\bar{k}} - u_{i\bar{i}}) - F^{i\bar{i}} \tilde{\psi}' u_{i\bar{i}} - F^{i\bar{i}} \tilde{\psi}'' |u_i|^2 + \tilde{\psi}' u_t. \end{aligned}$$

Since

$$\begin{aligned} F^{i\bar{i}} R_{i\bar{k}k\bar{i}} (u_{k\bar{k}} - u_{i\bar{i}}) - F^{i\bar{i}} \tilde{\psi}' u_{i\bar{i}} \\ \geq -2 \sup \|Rm\| S \sum F^{i\bar{i}} - 6kLS \sup \|Rm\| + 2S \sup \|Rm\| \sum F^{i\bar{i}} \\ \geq -6kLS \sup \|Rm\|, \end{aligned}$$

we have

$$F^{i\bar{i}} (u_{k\bar{l}}(x, t) - \tilde{\psi}(u))_{i\bar{i}} - (u_{k\bar{l}}(x, t) - \tilde{\psi}(u))_t \geq -C_1. \tag{6-2}$$

By (1-2), we have

$$F^{i\bar{i}} u_{ti\bar{i}} - u_{tt} = h_z(x, u) \geq -C_1. \tag{6-3}$$

Furthermore, by a simple calculation, we can find $0 < \lambda \leq \Lambda$, such that for any smooth function w,

$$\frac{\Lambda}{4}\triangle_{\mathrm{real}} w = \Lambda \triangle_{\bar\partial} w \geq F^{i\bar i} w_{i\bar i} \geq \lambda \triangle_{\bar\partial} w = \frac{\lambda}{4}\triangle_{\mathrm{real}} w. \tag{6-4}$$

With the bounds we obtained before, there is a useful weak Harnack inequality which we state as follows.

Lemma 6.1. *(see* [*Lieberman 1996, Theorem* 6.18]*) If $w_t - \Lambda\triangle_{\mathrm{real}} w \leq C_1$, then there exist positive constants p, C, k such that*

$$\left(\frac{1}{|Q(x,t,R)|}\int_{Q(x,t,R)} \big(\sup_{Q(x,t,2R)} w - w\big)^p\right)^{\frac{1}{p}} \leq C\left(\sup_{Q(x,t,2R)} w - \sup_{Q(x,t,R)} w + kR\right),$$

Since $\tilde\psi(u)$ has bounds depending only on $\sup|R_m|$ and $\sup|dd^c u|$, and $\|u\|_{C^0}$ and $\{u_{k\bar l}\}$ are Hermitian matrices, we can apply these to $u_{k\bar k}$, u_t to obtain the inequalities

$$\left(\frac{1}{|Q(x,t,R)|}\int_{Q(x,t,R)} \big(\sup_{Q(x,t,2R)} u_{k\bar k} - u_{k\bar k}\big)^p\right)^{\frac{1}{p}} \leq C\left(\sup_{Q(x,t,2R)} u_{k\bar k} - \sup_{Q(x,t,R)} u_{k\bar k} + kR\right),$$

$$\left(\frac{1}{|Q(x,t,R)|}\int_{Q(x,t,R)} \big(\sup_{Q(x,t,2R)} u_t - u_t\big)^p\right)^{\frac{1}{p}} \leq C\left(\sup_{Q(x,t,2R)} u_t - \sup_{Q(x,t,R)} u_t + kR\right),$$

where C does not depend on $u_{k\bar l}$ or u_t.

By concavity, for any two points (x,t), (y,s),

$$\begin{aligned} -C_2 &\leq h(x,u(x,t)) + u_t(x,t) - h(y,u(y,s)) - u_t(y,s)\\ &= F(x,u(x,t)) - F(y,u(y,s))\\ &\leq F^{i\bar j}(x,t)(u_{i\bar j}(x,t) - u_{i\bar j}(y,s)) \leq \Lambda\sum_{i=1}^n (u_{i\bar i}(x,t) - u_{i\bar i}(y,s)). \end{aligned}$$

Hence

$$\sum_{k=1}^n (u_{k\bar k}(y,s) - u_{k\bar k}(x,t)) \leq C_2\Lambda,$$

where Λ represents the maximal eigenvalue of $F^{i\bar j}$.

Next we choose (y_1,s_1) in $Q(x,t,2R)$, and (y_2,s_2) in $Q(x,t,R)$,

$$u_{i\bar i}(y_2,s_2) - u_{i\bar i}(y_1,s_1) \leq C_2\Lambda + \sum_{k\neq i}(u_{k\bar k}(y_1,s_1) - u_{k\bar k}(y_2,s_2)).$$

Take the supremum on (y_1,s_1) on both sides simultaneously:

$$0 \leq u_{i\bar i}(y_2,s_2) - \inf_{Q(x,t,2R)} u_{i\bar i}(y_1,s_1) \leq C_2\Lambda + \sum_{k\neq i}\big(\sup_{Q(x,t,2R)} u_{k\bar k}(y_1,s_1) - u_{k\bar k}(y_2,s_2)\big).$$

Let $\omega(R) := \sum_k \big\{\sup_{Q(x,t,R)} u_{k\bar k} - \inf_{Q(x,t,R)} u_{k\bar k}\big\}$. We have

$$
\begin{aligned}
&\sup_{Q(2R)} u_{k\bar{k}} - \inf_{Q(2R)} u_{k\bar{k}} \\
&= \left\{ \frac{1}{|Q(R)|} \int_{Q(R)} | \sup_{Q(2R)} u_{k\bar{k}} - \inf_{Q(2R)} u_{k\bar{k}}|^p \right\}^{\frac{1}{p}} \\
&\leq \left\{ \frac{1}{|Q(R)|} \int_{Q(R)} |u_{k\bar{k}} - \inf_{Q(2R)} u_{k\bar{k}}|^p \right\}^{\frac{1}{p}} + \left\{ \frac{1}{|Q(R)|} \int_{Q(R)} | \sup_{Q(2R)} u_{k\bar{k}} - u_{k\bar{k}}|^p \right\}^{\frac{1}{p}} \\
&\leq C_2\Lambda + \sum_{i\neq k} \left\{ \frac{1}{|Q(R)|} \int_{Q(R)} | \sup_{Q(2R)} u_{i\bar{i}} - u_{i\bar{i}}|^p \right\}^{\frac{1}{p}} + \left\{ \frac{1}{|Q(R)|} \int_{Q(R)} | \sup_{Q(2R)} u_{k\bar{k}} - u_{k\bar{k}}|^p \right\}^{\frac{1}{p}} \\
&\leq C_2\Lambda + C\{\omega(2R) - \omega(R) + kR\},
\end{aligned}
$$

where the last step is established by Lemma 6.1 and the fact that

$$
\sum_i \big(\sup_{Q_{2R}} u_{i\bar{i}} - \sup_{Q_R} u_{i\bar{i}} \big) \leq \omega(2R) - \omega(R).
$$

We finally get $\omega(R) \leq \big(1 - \frac{1}{C}\big)\omega(2R) + \frac{k}{C} + \frac{C_2\Lambda}{C}$. Thus it follows from [Gilbarg and Trudinger 1977, Lemma 8.23] that there are positive constants α, R_0 such that

$$
\|u_{k\bar{k}}\|_{C^{0,\alpha}(Q(x,t,R_0))} \leq \sup_{R\leq R_0} \frac{\omega(R)}{R^\alpha} \leq C(\alpha, R_0, \sup_{Q(x,t,R_0)} |u_{k\bar{k}}|).
$$

Therefore for each point (p, t), there exists a neighborhood U of p and a uniform constant C which is independent of the choice of points such that for $k = 1, \dots, n$,

$$
\|u_{k\bar{k}}\|_{C^{2,\alpha}(U\times[0,T])} \leq C,
$$

and thus we obtain the Hölder estimate for $\{u_{k\bar{l}}\}(p, t)$ if we choose the normal coordinates at that point. Consequently, around each point (x, t), there is a neighborhood on which the required estimate holds.

Acknowledgements

The authors would like to thank Professor Xu-Jia Wang for his suggestion. They also would like to thank Dr. Bin Zhou for his helpful discussion.

References

[Alesker and Verbitsky 2010] S. Alesker and M. Verbitsky, "Quaternionic Monge–Ampère equation and Calabi problem for HKT-manifolds", *Israel J. Math.* **176**:1 (2010), 109–138. MR Zbl

[Cao 1985] H. D. Cao, "Deformation of Kähler metrics to Kähler–Einstein metrics on compact Kähler manifolds", *Invent. Math.* **81**:2 (1985), 359–372. MR Zbl

[Chou and Wang 2001] K.-S. Chou and X.-J. Wang, "A variational theory of the Hessian equation", *Comm. Pure Appl. Math.* **54**:9 (2001), 1029–1064. MR Zbl

[Dinew and Kołodziej 2017] S. Dinew and S. Kołodziej, "Liouville and Calabi–Yau type theorems for complex Hessian equations", *Amer. J. Math.* **139**:2 (2017), 403–415. MR Zbl

[Fu et al. 2012] J. Fu, J. Li, and S.-T. Yau, "Balanced metrics on non-Kähler Calabi–Yau threefolds", *J. Differential Geom.* **90**:1 (2012), 81–129. MR Zbl

[Gilbarg and Trudinger 1977] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Grundlehren der Math. Wissenschaften **224**, Springer, 1977. MR Zbl

[Hou 2009] Z. Hou, "Complex Hessian equation on Kähler manifold", *Int. Math. Res. Not.* **2009**:16 (2009), 3098–3111. MR Zbl

[Hou et al. 2010] Z. Hou, X.-N. Ma, and D. Wu, "A second order estimate for complex Hessian equations on a compact Kähler manifold", *Math. Res. Lett.* **17**:3 (2010), 547–561. MR Zbl

[Li and Sheng 2011] Q.-R. Li and W. Sheng, "Some Dirichlet problems arising from conformal geometry", *Pacific J. Math.* **251**:2 (2011), 337–359. MR Zbl

[Lieberman 1996] G. M. Lieberman, *Second order parabolic differential equations*, World Sci., River Edge, NJ, 1996. MR Zbl

[Lu 2013] H. C. Lu, "Solutions to degenerate complex Hessian equations", *J. Math. Pures Appl.* (9) **100**:6 (2013), 785–805. MR Zbl

[Lu and Nguyen 2015] C. H. Lu and V.-D. Nguyen, "Degenerate complex Hessian equations on compact Kähler manifolds", *Indiana Univ. Math. J.* **64**:6 (2015), 1721–1745. MR Zbl

[Phong et al. 2016] D. Phong, S. Picard, and X. Zhang, "A second order estimate for general complex Hessian equations", *Anal. PDE* **9**:7 (2016), 1693–1709. MR Zbl

[Sheng et al. 2004] W. Sheng, J. Urbas, and X.-J. Wang, "Interior curvature bounds for a class of curvature equations", *Duke Math. J.* **123**:2 (2004), 235–264. MR Zbl

[Song and Weinkove 2008] J. Song and B. Weinkove, "On the convergence and singularities of the J-flow with applications to the Mabuchi energy", *Comm. Pure Appl. Math.* **61**:2 (2008), 210–229. MR Zbl

[Sun 2017] W. Sun, "On a class of fully nonlinear elliptic equations on closed Hermitian manifolds, II: L^∞ estimate", *Comm. Pure Appl. Math.* **70**:1 (2017), 172–199. MR Zbl

[Wang 2009] X.-J. Wang, "The k-Hessian equation", pp. 177–252 in *Geometric analysis and PDEs* (Cetraro, 2007), edited by A. Ambrosetti et al., Lecture Notes in Math. **1977**, Springer, 2009. MR Zbl

Received January 6, 2018. Revised August 14, 2018.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 300, No. 1, 2019
dx.doi.org/10.2140/pjm.2019.300.179

FULLY NONLINEAR PARABOLIC DEAD CORE PROBLEMS

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We establish geometric regularity estimates for diffusive models driven by fully nonlinear second-order parabolic operators with measurable coefficients under a strong absorption condition as follows:

$$\mathscr{F}(x, t, Du, D^2u) - \partial_t u = \lambda_0(x, t)u^{\mu}\chi_{\{u>0\}} \quad \text{in } \Omega_T := \Omega \times (0, T),$$

where $\Omega \subset \mathbb{R}^n$ is a bounded and smooth domain, $0 \le \mu < 1$ and λ_0 is bounded away from zero and infinity. Such models arise in applied sciences and become mathematically interesting because they permit the formation of dead-core zones, i.e., regions where nonnegative solutions vanish identically. Our main result gives sharp and improved $C^{2/(1-\mu)}$ parabolic regularity estimates along the free boundary $\partial\{u > 0\}$. In addition, we derive weak geometric and measure-theoretic properties of solutions and their free boundaries as: nondegeneracy, porosity, uniform positive density and finite speed of propagation. As an application, we prove a Liouville type result for entire solutions and we carry out a blow-up analysis. Finally, we prove the finiteness of parabolic $(n+1)$-Hausdorff measure of the free boundary for a particular class of operators.

1. Introduction

Throughout this article, we are interested in sharp and improved geometric regularity estimates for diffusive models with strong absorption as follows:

$$\textbf{(DCP)}\quad \mathscr{F}(x, t, Du, D^2u) - \partial_t u = \lambda_0(x, t).u^{\mu}\chi_{\{u>0\}}(x, t) \quad \text{in } \Omega_T := \Omega \times (0, T),$$

with continuous and nonnegative boundary data, where $\Omega \subset \mathbb{R}^n$ is a bounded and smooth domain, $0 \le \mu < 1$ is the order of reaction, λ_0 is bounded away from zero and infinity and it is known as the *Thiele modulus*. Moreover, $\mathscr{F} : \Omega_T \times \mathbb{R}^n \times \mathrm{Sym}(n) \to \mathbb{R}$ is a fully nonlinear, second-order uniformly elliptic operator with Lipschitz character: there exist constants $\Lambda \ge \lambda > 0$ (*ellipticity parameters*) and $\kappa \ge 0$ such that

$$\text{(1-1)}\quad \lambda\|Y\| - \kappa|\varsigma| \le \mathscr{F}(x, t, \xi, X) - \mathscr{F}(x, t, \xi + \varsigma, X + Y) \le \Lambda\|Y\| + \kappa|\varsigma|$$

MSC2010: 35B65, 35K55.
Keywords: dead-core problems, fully nonlinear parabolic equations, sharp and improved regularity estimates, parabolic Hausdorff measure estimates.

for every $X, Y \in \mathrm{Sym}(n)$ with $Y \geq 0$, $(x, t, \xi, \varsigma) \in \Omega_T \times \mathbb{R}^n \times \mathbb{R}^n$. It is worth mentioning that $\mathscr{F}$ is assumed to have bounded measurable coefficients. For this reason, bounded viscosity solutions to (**DCP**) have, as the best a priori estimates, a universal Hölder modulus of continuity according to celebrated Krylov–Safonov parabolic estimates; see [Krylov and Safonov 1980] for more details.

Another fundamental aspect of such models is that if $f(u) = \lambda_0 u^\mu \chi_{\{u>0\}}$ is not Lipschitz, then the maximum principle is not applicable. Consequently, nonnegative solutions may create plateau regions, which are known in the literature as *dead-cores* and represent regions where no reaction takes place in the diffusion process from (**DCP**). Solutions of this class are currently called dead-core solutions and appear in a number of physical-mathematical models; see, for example, [Antontsev et al. 2002; Bandle and Stakgold 1984; Choe and Weiss 2003; da Silva et al. 2018; Guo and Souplet 2005].

The main first result of our manuscript concerns sharp and improved regularity of dead-core solutions along their free boundaries. We refer the reader to Section 2 for the employed notation.

Theorem 1.1 (improved regularity at free boundary points). *Let u be a nonnegative and bounded viscosity solution to* (**DCP**), *so that $\partial_t u \geq -\mathfrak{c}_0(x, t)u^\mu \chi_{\{u>0\}}$ (in the viscosity sense)*[1] *for $\mathfrak{c}_0$ a nonnegative bounded function and $\mathfrak{K} \Subset \Omega_T$ a compact set. Then there exists a universal constant*[2] *$\mathfrak{C} > 0$ such that for all $(x_0, t_0) \in \partial\{u > 0\} \cap \mathfrak{K}$,*

$$u(x, t) \leq \mathfrak{C} \cdot \|u\|_{L^\infty(\Omega_T)} \mathrm{dist}_p((x, t), (x_0, t_0))^{\frac{2}{1-\mu}},$$

for all (x, t) sufficiently close to (x_0, t_0).

We shall also provide how dead-core solutions leave their free boundaries.

Theorem 1.2 (nondegeneracy). *There exists a constant $\mathfrak{C}_0^* = \mathfrak{C}_0^*(n, \lambda, \Lambda, \kappa, \mathfrak{m}, \mu) > 0$ such that any viscosity subsolution to* (**DCP**) *satisfies*

$$\sup_{\mathscr{C}_r^-(x_0,t_0)} u(x, t) \geq \mathfrak{C}_0^* \cdot r^{\frac{2}{1-\mu}}, \tag{1-2}$$

for any $(x_0, t_0) \in \overline{\{u > 0\}} \cap \Omega_T$ and $\mathscr{C}_r(x_0, t_0) \subset \Omega \times (0, \infty)$.

Finally, our last main result concerns the parabolic Hausdorff measure of the free boundary.

[1] Notice that such an assumption is weaker than those imposed in [Choe and Weiss 2003; Shahgholian 2003; 2008]. It means that solutions can decrease in time direction, but with an appropriate lower bound control.

[2] Throughout this manuscript *universal constants* are those which depend only on dimension and structural parameters of the problem, namely λ, Λ (ellipticity constants of the operator), $\mathfrak{m}$, $\mathfrak{M}$ (bound of λ_0), κ (bounds for the gradient variable of $\mathfrak{F}$), $\mathrm{dist}(\mathfrak{K}, \partial_p \Omega_T)$ and μ.

Theorem 1.3 (Hausdorff measure estimates). *Let u be a viscosity solution to* (**DCP**) *with $\mathscr{F}$ a concave operator. There exists a universal constant $\mathfrak{C} > 0$ such that for all $(x_0, t_0) \in \partial\{u > 0\} \cap \mathscr{C}_{1/2}$,*

$$\mathscr{H}^{n+1}_{\text{par}}(\partial\{u > 0\} \cap \mathscr{C}_\rho(x_0, t_0)) \leq \mathfrak{C}\rho^{n+1},$$

for all $\rho \ll 1$, where $\mathscr{H}^{n+1}_{\text{par}}$ is the $(n+1)$-dimensional Hausdorff measure with respect to the parabolic metric.

1A. ***Motivations, state of the art and overview.*** Throughout the last four decades parabolic PDEs with strong absorption conditions have received much attention due to their connections with the modeling of several phenomena in pure and applied sciences (see [Antontsev et al. 2002; Bandle and Stakgold 1984; Díaz 2001; Stakgold 1986]). An illustrative example coming from isothermal, catalytic reaction-diffusion processes is

$$\begin{cases} \Delta u - \partial_t u = u^\mu \chi_{\{u>0\}} & \text{in } \Omega_T, \\ u(x, t) = g(x, t) & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \overline{\Omega}, \end{cases}$$

where the boundary data satisfy

$$0 < u_0 \in C^0(\overline{\Omega}), \quad g(x, t) = \mathfrak{k} > 0 \quad \text{and} \quad u(x, 0) = \mathfrak{k} \quad \text{for all } x \in \partial\Omega.$$

In this context, u represents the concentration of a (gas-liquid) reactant over a diffusing material evolving in time. Hence, the development of dead-core regions occurs precisely when the reactant becomes inactive. Notice that the boundary condition means that the reactant is injected with a fixed isothermal flux on the boundary. From a chemical engineering point of view, to understand the dead-core phenomenon is crucial, since the catalytic material is wasted precisely along the dead-core zone.

Other insights for our study come from the theory of nonlinear geometric free boundary problems (see [Apushkinskaya et al. 2002; Caffarelli et al. 2004; Shahgholian 2003; Teixeira 2016] for some enlightening examples). In this direction, we cite the class of "pseudo" free boundary problems

$$\textbf{(FBP)} \qquad \begin{cases} \max\{\mathfrak{P}[u] - \partial_t u, -u\} = 0 & \text{in } \Omega_T, \\ u(x, t) = g(x, t) & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{on } \overline{\Omega}, \end{cases}$$

where $\mathfrak{P}[u] := \mathscr{F}(x, t, Du, D^2u) - \lambda_0(x, t)u^\mu \chi_{\{u>0\}}(x, t)$ for μ, g, u_0 and $\mathscr{F}$ as before. As a particular application of (**FBP**) to financial markets, we may consider the operator

$$\mathfrak{P}[\mathscr{V}] = \tfrac{1}{2}(\sigma_1\mathfrak{S}_1, \sigma_2\mathfrak{S}_2)^T \begin{pmatrix} \frac{\partial^2\mathscr{V}}{\partial(\mathfrak{S}_1)^2} & \frac{\partial^2\mathscr{V}}{\partial\mathfrak{S}_1\partial\mathfrak{S}_2} \\ \frac{\partial^2\mathscr{V}}{\partial\mathfrak{S}_2\partial\mathfrak{S}_1} & \frac{\partial^2\mathscr{V}}{\partial(\mathfrak{S}_2)^2} \end{pmatrix} \cdot (\sigma_1\mathfrak{S}_1, \sigma_2\mathfrak{S}_2) + \sum_{i=1}^{2}(r_0 - \sigma_i)\mathfrak{S}_i \frac{\partial\mathscr{V}}{\partial\mathfrak{S}_i} - r_0\mathscr{V}_+^{1-\mu},$$

where $r_0 > 0$ is the interest rate, σ_i is the volatility of the price of the corresponding asset, $\mathscr{V}$ is the American option and $(\mathfrak{S}_1, \mathfrak{S}_2)$ is the price vector of underlying assets. Such nonlinear obstacle problems can be interpreted as the extended model (with zero constraint) in pricing of American options in financial mathematics, which precisely deals with the case $\mu = 0$ (an obstacle type problem). The interested reader may see the references [Blanchet et al. 2005; 2006; Petrosyan and Shahgholian 2007; Shahgholian 2008] for a more complete treatment.

Despite the fact that a number of qualitative and quantitative features for linear models in divergence form like

$$\Delta u - \partial_t u = \lambda_0 u^\mu \chi_{\{u>0\}}(x, t) \quad \text{in } \Omega_T \tag{1-3}$$

have been extensively studied by many authors by using variational approaches in the last four decades (see [Choe and Weiss 2003] for a seminal treatment, see also [Antontsev et al. 2002] and [Díaz 2001] for classical references on this theme), many pivotal issues have not been established for a general model (**DCP**), until now, due to the rigidity of the structure of such operators. For this reason, the treatment of such free boundary problems in nondivergence form requires the development of new approaches and modern techniques.

We are particularly interested in the smoothness and weak geometric properties around free boundary points of viscosity solutions of models like (**DCP**). Such issues were our impetus for researching parabolic dead-core problems via a modern, nonvariational and systematic approach based on geometric regularity theory (compare with [Choe and Weiss 2003] for a dead core problem ruled by a heat operator and [da Silva et al. 2018] for its extension to degenerate evolution operators).

Beyond the several applications, the topics treated in this article help to understand general issues in free boundary problems. This fact is illustrated by Theorem 1.1, which shows that the dead-core's analysis brings to light an impressive feature: better regularity estimates (at free boundary points) than those currently available. In effect, in our approach we impose just bounded measurable coefficients for $\mathscr{F}$. Notwithstanding, the modulus of continuity improves upon the expected Hölder regularity coming from the classical Krylov–Safonov regularity estimate (see [Crandall et al. 2000, Section 5; Krylov and Safonov 1980; Wang 1992a, Section 4.4]). Furthermore, even for constant coefficient problems, $\mathscr{F}(Du, D^2u) - \partial_t u = f(x, t, u) \in L^\infty$, our result is surprising, because in this setting, the $C^{1+\alpha,(1+\alpha)/2}$-estimate is the available regularity; see [Wang 1992b, Section 1.2] and [da Silva and Teixeira 2017, Sections 4 and 6]. Finally, we must compare our estimates with ones coming from Schauder type estimates (see [Tian and Wang 2013] and [Wang 1992b, Section 1.1]). For simplicity, let us suppose that $\mathscr{F}(D^2u) = \Delta u$ and $\lambda_0 > 0$ is constant. Notice that $\lambda_0 u^\mu \in C^{\mu,\mu/2}(\mathscr{C}_1)$. Therefore, the classical Schauder theory implies that $u \in C^{2+\mu,(2+\mu)/2}_{\text{loc}}(\mathscr{C}_1)$. On the other hand, the estimates from Theorem 1.1 tell us that u has $\omega(s) = s^{2/(1-\mu)}$ as modulus of continuity at points on the free boundary. Finally, the main point is that

$$\frac{2}{1-\mu} > 2+\mu$$

for any $\mu \in (0, 1)$. In other words, we obtain an improved decay estimate (at free boundary points for a right-hand side which is not not necessarily Hölder) when compared with classical Schauder estimates. Taking into account the previous statements, our results are new even for linear parabolic problems with nondivergence structure and bounded and merely measurable coefficients.

The insight for the proof of Theorem 1.1 is inspired by techniques from regularity theory of fully nonlinear equations and free boundary problems (see, e.g., [da Silva et al. 2017; da Silva and Teixeira 2017; Shahgholian 2003; 2008; Teixeira 2016]). It consists of a finer geometric decay throughout an iterative process, which is based on the sharp scaling of the equation and maximum principle tools for a limiting caloric profile via a contradiction reasoning. It is worth mentioning that a difficulty in our studies is the absence of a strong maximum principle for $\mathscr{F}$-caloric functions, i.e., $\mathscr{F}(D^2\mathfrak{h}) - \partial_t \mathfrak{h} = 0$ (a viscosity solution to homogeneous problem with constant coefficients). For this very reason, the assumption of control in time variable will play an essential role in our analysis in order to overcome such an obstacle, since in such a limit configuration solutions will become nondecreasing in time direction. This will enable us to apply a strong maximum principle for fully nonlinear equations; see [Da Lio 2004].

Finally, it is worth highlighting that our article extends, as well as generalizes to some extent, the previous seminal results (sharp regularity and weak geometric properties) from [Choe and Weiss 2003] and [Teixeira 2016] by using different approaches and techniques adapted to the general framework of the fully nonlinear parabolic operators (compare also with [Caffarelli et al. 2004] and [Shahgholian 2008]).

The paper is organized as follows: The reader will find the main definitions and assumptions in Section 2. Afterwards, we will present the proofs of the improved regularity and nondegeneracy Theorems 1.1 and 1.2 in Section 3. In Section 4 we put forward a number of consequences of these results. Section 5 treats global analysis results of Liouville and blow-up type. Finally, the Hausdorff estimates in Theorem 1.3 and related results will be delivered in Section 6.

2. Preliminaries and main assumptions

Let us start with some standard parabolic notation. By Ω we shall denote a bounded, open and smooth set in $\mathbb{R}^n$. For $x_0 \in \mathbb{R}^n$ and $r > 0$, we denote by $B_r(x_0)$ the Euclidean open ball with center x_0 and radius r. Also, for a point $(x_0, t_0) \in \Omega \times \mathbb{R}$ and $r > 0$, we consider three kinds of parabolic cylinders:

$$\begin{aligned}
\mathscr{C}_r(x_0, t_0) &:= B_r(x_0) \times (t_0 - r^2, t_0 + r^2) && \text{(whole cylinder)}\\
\mathscr{C}_r^+(x_0, t_0) &:= B_r(x_0) \times [t_0, t_0 + r^2) && \text{(the upper semi-cylinder)}\\
\mathscr{C}_r^-(x_0, t_0) &:= B_r(x_0) \times (t_0 - r^2, t_0] && \text{(the lower semicylinder).}
\end{aligned}$$

Moreover, we will omit the center of the cylinder as $(x_0, t_0) = (0, 0)$.

For a parabolic cylinder $\mathscr{C} = \Omega \times \mathscr{I}$, where $\mathscr{I}$ is a closed interval with endpoints $\mathfrak{a} < \mathfrak{b}$, we define the parabolic boundary by: $\partial_p \mathscr{C} := (\overline{\Omega} \times \{\mathfrak{a}\}) \cup (\partial\Omega \times \mathscr{I})$.

Given $(x, t), (y, s) \in \Omega \times \mathbb{R}$ the parabolic distance (or metric) between (x, t) and (y, s) is given by

$$\operatorname{dist}_p((x, t), (y, s)) := \sqrt{|x - y|^2 + |t - s|}.$$

For $r > 0$ and $\mathscr{O} \subset \mathbb{R}^{n+1}$, we let $\mathscr{N}_r(\mathscr{O}) := \{(x, t) \in \mathbb{R}^{n+1} : \operatorname{dist}_p((x, t), \mathscr{O}) > r\}$ for the parabolic tubular neighborhood of radius r of $\mathscr{O}$.

By $\operatorname{Sym}(n)$ we mean the set of symmetric real matrices of size $n \times n$. If M is a given matrix, we shall use $\|M\|$ to denote any matrix norm.

Throughout this manuscript $\mathscr{F} : \mathscr{C}_1 \times \mathbb{R}^n \times \operatorname{Sym}(n) \to \mathbb{R}$ is a second-order fully nonlinear operator satisfying the structural condition

$$\textbf{(SC)} \qquad \begin{aligned} \mathscr{P}^-_{\lambda,\Lambda}(M - P) - \kappa|\vec{p} - \vec{q}\,| &\le \mathscr{F}(x, t, \vec{p}, M) - \mathscr{F}(x, t, \vec{q}, P)\\ &\le \mathscr{P}^+_{\lambda,\Lambda}(M - P) + \kappa|\vec{p} - \vec{q}\,|, \end{aligned}$$

for any $M, P \in \operatorname{Sym}(n)$ and $\vec{p}, \vec{q} \in \mathbb{R}^n$, where $\mathscr{P}^{\pm}_{\lambda/n,\Lambda}$ denote the *Pucci's extremal operators*

$$\mathscr{P}^+_{\lambda,\Lambda}(M) := \lambda \cdot \sum_{e_i<0} e_i + \Lambda \cdot \sum_{e_i>0} e_i \quad \text{and} \quad \mathscr{P}^-_{\lambda,\Lambda}(M) := \lambda \cdot \sum_{e_i>0} e_i + \Lambda \cdot \sum_{e_i<0} e_i$$

and $\{e_i : 1 \le i \le n\}$ are the eigenvalues of M.

For a fixed $(x_0, t_0) \in \mathscr{C}_1$, we will measure the oscillation of the coefficients of $\mathscr{F}$ around (x_0, t_0) by the quantity

$$\text{(2-1)} \qquad \Theta_{\mathscr{F}}(x_0, t_0, x, t) := \sup_{M \in \operatorname{Sym}(n) \setminus \{0\}} \frac{|\mathscr{F}(x, t, 0, M) - \mathscr{F}(x_0, t_0, 0, M)|}{\|M\|}.$$

For notation purposes, we shall often write $\Theta_{\mathscr{F}}(0, 0, x, t) = \Theta_{\mathscr{F}}(x, t)$. Hence, the coefficients of the operator are merely measurable.

In the following definition, we provide the class of solutions that we consider in this work.

Definition 2.1 (viscosity solutions). We say that a function $u \in C^0(\mathscr{C}_1)$ is a viscosity subsolution (resp. supersolution) to

$$\mathscr{F}(x, t, Du(x, t), D^2u(x, t)) - \partial_t u - f(x, t, u) = 0 \quad \text{in } \mathscr{C}_1 \tag{2-2}$$

if for all $\varphi \in C^{2,1}(\mathscr{C}_1)$ whenever $u - \varphi$ has a local minimum (resp. maximum) at $(x_0, t_0) \in \mathscr{C}_1$ then

$$\mathscr{F}(x, t, D\varphi(x, t), D^2\varphi(x, t)) - \partial_t \varphi - f(x, t, \varphi) \geq 0 \quad (resp. \ \leq 0).$$

Finally we say that u is a viscosity solution to (2-2) if it is both a viscosity subsolution and a supersolution.

We recall the existence of a universal constant $\mathfrak{p}_0$, satisfying $\frac{n+2}{2} \leq \mathfrak{p}_0 < n+1$, for which Harnack inequality (resp. Hölder regularity) holds for viscosity solutions with RHS in L^p, provided $p > \mathfrak{p}_0$; see for instance [Crandall et al. 2000, Section 5]. The following compactness result then becomes available:

Proposition 2.2 (compactness of solutions). *Let u be a viscosity solution to*

$$\partial_t u - \mathscr{F}(x, t, Du, D^2u) = \mathfrak{f}(x, t, u) \quad \text{in } \mathscr{C}_r, \tag{2-3}$$

under the assumption $f \in L^p$ with $p > \mathfrak{p}_0$. Then u is locally of class $C^{0,\beta}$ for some $0 < \beta < 1$ and

$$\|u\|_{C^\beta(\mathscr{C}_r)} \leq C(n, \lambda, \Lambda, \kappa) r^{-\beta} (\|u\|_{L^\infty(\mathscr{C}_r)} + r^{2-\frac{n+2}{p}} \|f\|_{L^p(\mathscr{C}_r)}).$$

Another piece of information we need in our approach concerns the stability of the notion of viscosity solutions, i.e., the limit of a sequence of viscosity solutions turns out to be a viscosity solution of the limiting equation. More precisely, we refer to the following lemma, whose proof and general form can be found in [Crandall et al. 2000, Theorem 6.1].

Lemma 2.3 (continuity with respect to the equation). *Let $\mathscr{F}_j, \mathscr{F}$ be normalized $(\lambda, \Lambda, \kappa)$-operators, $p > \mathfrak{p}_0$, with $f, f_j \in L^p(\mathscr{C}_1) \cap C^0(\mathscr{C}_1)$ and let u_j be viscosity solutions to*

$$\partial_t u_j - \mathscr{F}_j(x, t, Du_j, D^2u_j) = f_j(x, t) \quad in\ \mathscr{C}_1$$

for all $j \in \mathbb{N}$. Assume that $u_j \to u$ locally uniformly as $j \to \infty$. Moreover, for all $\mathscr{C}_r(x_0, t_0) \subset \mathscr{C}_1$ and all $\varphi \in C^{2,1}(\mathscr{C}_r(x_0, t_0))$ (test function), assume that

$$g_j(x, t) := \mathscr{F}_j(x, t, D\varphi(x, t), D^2\varphi(x, t)) - f_j(x, t)$$

and

$$g(x, t) := \mathscr{F}(x, t, D\varphi(x, t), D^2\varphi(x, t)) - f(x, t)$$

satisfy

$$\|g - g_j\|_{L^p(\mathscr{C}_r(x_0,t_0))} \to 0 \quad \text{as } j \to \infty. \tag{2-4}$$

Then, u is a viscosity solution to

$$\partial_t u - \mathscr{F}(x, t, Du, D^2u) = f(x, t) \quad \text{in } \mathscr{C}_r(x_0, t_0).$$

Proposition 2.4 (gradient estimates [Crandall et al. 2000, Remark 7.7; Wang 1992b, Section 4.2). *Let u be a viscosity solution to* (2-3) *with $\mathscr{F}$ a normalized* $(\lambda, \Lambda, \kappa)$-*operator and $f \in L^p(\mathscr{C}_1)$. If*

$$\lim_{r\to 0+} \sup_{(y,s)\in\mathscr{C}_{\frac{1}{2}}} \left(\fint_{\mathscr{C}_r(y,s)} \Theta^p_{\mathscr{F}}(y, s, x, t) \right)^{\frac{1}{p}} = 0,$$

then $u \in C^{1+\alpha,(1+\alpha)/2}(\overline{\mathscr{C}_{1/2}})$ for some universal $0 < \alpha < 1$. Furthermore, there exists a universal constant $\mathfrak{C} = \mathfrak{C}(n, \lambda, \Lambda, \kappa) > 0$ such that

$$\|Du\|_{L^\infty(\overline{\mathscr{C}_{1/2}})} \le \mathfrak{C}(n, \lambda, \Lambda, \kappa)(\|u\|_{L^\infty(\mathscr{C}_1)} + \|f\|_{L^\infty(\mathscr{C}_1)}).$$

The next result can be proved in a similar way to one in [Crandall et al. 1992, Theorem 8.3].

Theorem 2.5 (comparison principle). *Let u_1 and u_2 be continuous functions in $\overline{\mathscr{C}_1}$ so that*

$$\begin{aligned} \mathscr{F}(x, t, Du_1, D^2u_1) - \partial_t u_1 - \lambda_0(x, t)(u_1)^\mu_+ \\ \le 0 \le \mathscr{F}(x, t, Du_2, D^2u_2) - \partial_t u_2 - \lambda_0(x, t)(u_2)^\mu_+ \quad \text{in } \mathscr{C}_1 \end{aligned} \tag{2-5}$$

in the viscosity sense. If $u_1 \ge u_2$ on $\partial_p\mathscr{C}_1$, then $u_1 \ge u_2$ in $\mathscr{C}_1$.

In the next theorem, we shall state the existence of solutions to the problem

$$\begin{cases} \mathscr{F}(x, t, Du, D^2u) - \partial_t u = \lambda_0(x, t)u^\mu \chi_{\{u>0\}}(x, t) & \text{in } \mathscr{C}_1, \\ u(x, t) = g(x, t) & \text{on} \partial B_1 \times (-1, 1), \\ u(x, 0) = u_0(x) & \text{in} \overline{B_1} \times \{-1\}, \end{cases} \tag{2-6}$$

for continuous functions g and u_0 satisfying the compatibility condition $g(x, 0) = u_0(x)$ for $x \in \partial B_1$. The existence is achieved by the celebrated Perron's method combined with the previous comparison principle, Theorem 2.5.

Theorem 2.6 (existence of dead core solutions). *Suppose that assumption* (**SC**) *holds for $\mathscr{F}$, and that λ_0 is continuous. If there exist a viscosity subsolution $u_\sharp$ to* (2-6) *and a viscosity supersolution $u^\sharp$ to* (2-6) *such that*

$$u_\sharp = u^\sharp \qquad \text{on } \partial_p\mathscr{C}_1,$$

then there exists a viscosity solution u to problem (2-6). *Furthermore, such a u is nonnegative provided the data are nonnegative.*

3. Improved regularity estimates and nondegeneracy of solutions

In this section we will prove our main results, Theorem 1.1 and 1.2. We start by deriving the improved regularity estimate in a normalized class of viscosity solutions defined in the unit cylinder, and then we extend the results to general dead-core viscosity solutions.

Definition 3.1. For any fully nonlinear operator $\mathscr{F}$ fulfilling (**SC**) we say that $u \in \mathfrak{J}(\mathscr{F}, \lambda_0, \mu)(\mathscr{C}_1)$ if:

- $\mathscr{F}(x, t, Du, D^2u) - \partial_t u = \lambda_0(x, t)u^{\mu}\chi_{\{u>0\}}(x, t)$ in $\mathscr{C}_1$ (in the viscosity sense) with $\|\lambda_0 u^{\mu}\|_{L^{\infty}(\mathscr{C}_1)} \ll 1$.
- $0 \le u \le 1$, $0 < \mathfrak{m} \le \lambda_0 \le \mathfrak{M}$ in $\mathscr{C}_1$.
- $\partial_t u \ge -\mathfrak{c}_0(x, t)u^{\mu}\chi_{\{u>0\}}(x, t)$ in $\mathscr{C}_1$ (in the viscosity sense) for a nonnegative and bounded function $\mathfrak{c}_0$.
- $u(0, 0) = 0$.

Hereafter, we shall adopt the notation $\mathscr{S}_{(r,x_0,t_0)}[u] := \sup_{\mathscr{C}_r^-(x_0,t_0)} u(x, t)$.

In the next, we shall define for $u \in \mathfrak{J}(\mathscr{F}, \lambda_0, \mu)(\mathscr{C}_1)$ the set

$$\mathbb{V}[u] := \left\{ j \in \mathbb{N} \cup \{0\};\ \mathscr{S}_{\frac{1}{2^j}}[u] \le 2^{\frac{2}{1-\mu}} \max\left\{1, \frac{1}{\mathfrak{C}_0^*}\right\} \cdot \mathscr{S}_{\frac{1}{2^{j+1}}}[u] \right\},$$

where $\mathfrak{C}_0^* > 0$ is the nondegeneracy constant from Theorem 1.2. Notice that $\mathbb{V}[u]$ is not empty. Indeed, $j = 0 \in \mathbb{V}[u]$ since, in view of Theorem 1.2,

$$\mathscr{S}_{\frac{1}{2}}[u] \ge \mathfrak{C}_0^*\left(\tfrac{1}{2}\right)^{\frac{2}{1-\mu}} \ge \mathfrak{C}_0^*\left(\tfrac{1}{2}\right)^{\frac{2}{1-\mu}} \mathscr{S}_1[u],$$

which implies that

$$\mathscr{S}_1[u] \le 2^{\frac{2}{1-\mu}} \max\left\{1, \frac{1}{\mathfrak{C}_0^*}\right\} \mathscr{S}_{\frac{1}{2}}[u].$$

We now present a key lemma for proving Theorem 1.1, which provides the sharp growth rate for functions on $\mathfrak{J}(\mathscr{F}, \lambda_0, \mu)(\mathscr{C}_1)$.

Lemma 3.2. *There exists a positive constant $\mathfrak{C}_0 = \mathfrak{C}_0(n, \lambda, \Lambda, \mu, \mathfrak{M})$ such that*

$$\mathscr{S}_{\frac{1}{2^{j+1}}}[u] \le \mathfrak{C}_0 \cdot \left(\frac{1}{2^j}\right)^{\frac{2}{1-\mu}} \tag{3-1}$$

for all $u \in \mathfrak{J}(\mathscr{F}, \lambda_0, \mu)(\mathscr{C}_1)$ and $j \in \mathbb{V}[u]$.

Proof. The proof will hold by reductio ad absurdum argument. Let us suppose that the thesis of the lemma fails to hold. Then, for each $k \in \mathbb{N}$ we might find

$u_k \in \mathfrak{J}(\mathscr{F}, \lambda_0, \mu)(\mathscr{C}_1)$ and $j_k \in \mathbb{V}[u_k]$ such that

$$\mathscr{S}_{\frac{1}{2^{j_k+1}}}[u_k] > k\left(\frac{1}{2^{j_k}}\right)^{\frac{2}{1-\mu}}. \tag{3-2}$$

Now, we define the auxiliary function:

$$v_k(x,t) := \frac{u_k\left(\frac{1}{2^{j_k}}x, \frac{1}{2^{2j_k}}t\right)}{\mathscr{S}_{\frac{1}{2^{j_k+1}}}[u_k]} \quad \text{in } \mathscr{C}_1.$$

Thus, v_k fulfills

- $0 \le v_k(x,t) \le \frac{\mathscr{S}_{1/2^{j_k}}[u_k]}{\mathscr{S}_{1/2^{j_k+1}}[u_k]} \le \mathfrak{A} := 2^{2/(1-\mu)} \max\{1, 1/\mathfrak{C}_0^*\}$ in $\mathscr{C}_1^-$ and $v_k(0,0) = 0$.
- $\mathscr{S}_{\frac{1}{2}}[v_k] \ge 1$.
- $\partial_t v_k \ge -\mathfrak{c}_0\left(\frac{1}{2^{j_k}}x, \frac{1}{2^{2j_k}}t\right) \frac{2^{-2j_k}}{\mathscr{S}^{1-\mu}_{1/2^{j_k+1}}[u_k]} v_k^{\mu}(x,t)$ in $\mathscr{C}_1^-$.
- $\mathscr{F}_k(x,t,Dv_k,D^2v_k) - \partial_t v_k = \frac{1}{2^{2\cdot j_k}} \frac{1}{\mathscr{S}^{1-\mu}_{1/2^{j_k+1}}[u_k]} \lambda_0\left(\frac{1}{2^{j_k}}x, \frac{1}{2^{2\cdot j_k}}t\right)(v_k)_+^{\mu}(x,t)$ in $\mathscr{C}_1$ in the viscosity sense, where

$$\mathscr{F}_k(x,t,\vec{p},M) := \frac{1}{2^{2j_k}\mathscr{S}_{1/2^{j_k+1}}[u_k]} \mathscr{F}\left(\frac{1}{2^{j_k}}x, \frac{1}{2^{2\cdot j_k}}t, 2^{j_k}\mathscr{S}_{1/2^{j_k+1}}[u_k]\cdot\vec{p}, 2^{2j_k}\mathscr{S}_{1/2^{j_k+1}}[u_k]\cdot M\right).$$

Notice that the operator $\mathscr{F}_k$ fulfills (**SC**). Moreover,

$$\left\| \frac{1}{2^{2\cdot j_k}} \frac{1}{\mathscr{S}^{1-\mu}_{1/2^{j_k+1}}[u_k]} \lambda_0\left(\frac{1}{2^{j_k}}x, \frac{1}{2^{2\cdot j_k}}t\right)(v_k)_+^{\mu}(x,t) \right\|_{L^\infty(\mathscr{C}_1)} \le \mathfrak{A}^{\mu}\cdot\mathfrak{M}\cdot\left(\frac{1}{k}\right)^{1-\mu} \to 0 \quad \text{as } k\to\infty.$$

The previous sentences together with standard compactness arguments for fully nonlinear parabolic equations (see Proposition 2.2) imply that, up to a subsequence, $v_k \to v$ locally uniformly in $\overline{\mathscr{C}_{4/5}^-}$ and $\mathscr{F}_k \to \mathscr{F}_0$. Furthermore, by stability results (see Lemma 2.3) we have

- $\mathscr{F}_0(D^2v) - \partial_t v = 0$ in $\mathscr{C}_{4/5}^-$.
- $0 \le v \le \mathfrak{A}$ and $\frac{\partial v}{\partial t} \ge 0$ in $\mathscr{C}_{4/5}^-$.
- $v(0,t) = 0$ for all $t \in \left(-\left(\frac{4}{5}\right)^2, 0\right]$.
- $\mathscr{S}_{1/2}[v] \ge 1$.

Therefore, according to the strong minimum principle (see [Da Lio 2004]) $v \equiv 0$, which contradicts the previous sentence. □

In the next result, we state a version of Theorem 1.1 for the class $\mathfrak{J}(\mathscr{F}, \lambda_0, \mu)(\mathscr{C}_1)$.

Theorem 3.3. *There exists a positive constant $\mathfrak{C} = \mathfrak{C}(n, \lambda, \Lambda, \mu, \mathfrak{M})$ such that for all $u \in \mathfrak{J}(\mathscr{F}, \lambda_0, \mu)(\mathscr{C}_1)$*

$$u(x,t) \le \mathfrak{C} \cdot \mathfrak{d}(x,t)^{\frac{2}{1-\mu}} \quad \textit{for all } (x,t) \in \mathscr{C}_{\frac{1}{2}},$$

where

$$\mathfrak{d}(x,t) := \begin{cases} \sup\{r \ge 0; \mathscr{C}_r(x,t) \subset \{u > 0\}\} & \textit{for } (x,t) \in \{u > 0\}, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. The proof will be by induction. First of all, we claim that

$$\mathscr{S}_{\frac{1}{2^j}}[u] \le \mathfrak{C}_0 \cdot \left(\frac{1}{2^{j-1}}\right)^{\frac{2}{1-\mu}} \quad \text{for all } j \in \mathbb{N}, \tag{3-3}$$

where $\mathfrak{C}_0$ is the constant coming from Lemma 3.2. Note that if $\mathfrak{C}_0 \ge 1$, which we can suppose without loss of generality, then (3-3) holds for $j = 0$. Suppose now that (3-3) holds for some $j \in \mathbb{N}$. We will verify the $(j+1)$-th step of induction. In fact, if $j \in \mathbb{V}[u]$, the result holds directly by Lemma 3.2. On the other hand, if (3-3) fails, by using the induction hypothesis we obtain

$$\mathscr{S}_{\frac{1}{2^{j+1}}}[u] \le \left(\frac{1}{2}\right)^{\frac{2}{1-\mu}} \cdot \mathscr{S}_{\frac{1}{2^j}}[u] \le \mathfrak{C}_0 \cdot \left(\frac{1}{2}\right)^{\frac{2}{1-\mu}} \left(\frac{1}{2^{j-1}}\right)^{\frac{2}{1-\mu}} = \mathfrak{C}_1 \cdot \left(\frac{1}{2^j}\right)^{\frac{2}{1-\mu}}$$

Therefore, (3-3) holds for all $j \in \mathbb{N}$.

In order to finish the proof for a continuous parameter, for $r \in (0,1)$ let $j \in \mathbb{N}$ be the greatest integer such that $1/2^{j+1} \le r < 1/2^j$. Then,

$$\mathscr{S}_r[u] \le \mathscr{S}_{\frac{1}{2^j}}[u] \le \mathfrak{C}_0 \cdot \left(\frac{1}{2^{j-1}}\right)^{\frac{2}{1-\mu}} \le \mathfrak{C}(n, \lambda, \Lambda, \mu, \mathfrak{M}) \cdot r^{\frac{2}{1-\mu}}.$$

Finally, in order to obtain an estimate for u over the whole cylinder we will use a suitable barrier function from above. Let us define

$$\mathfrak{c} := \left(\frac{\mathfrak{m}(1-\mu)^2}{2[2\Lambda\mu + \kappa(1-\mu)]}\right)^{\frac{1}{1-\mu}}$$

and $\zeta(x,t) := \mathfrak{c} \cdot (|x|^2 + 2n\Lambda \cdot t)^{1/1-\mu}$. Then, we have

$$\begin{aligned} &\mathscr{F}(x,t,D\zeta,D^2\zeta) - \frac{\partial \zeta}{\partial t} - \lambda_0(x,t)\zeta_+^{\mu} \\ &\qquad \le 0 \le \mathscr{F}(x,t,Du,D^2u) - \frac{\partial u}{\partial t} - \lambda_0(x,t)u_+^{\mu} \quad \text{in } \mathscr{C}_1^+. \end{aligned}$$

Moreover, we have that $\zeta \ge u$ on $\partial_p \mathscr{C}_1^+$, where we have used $\mathscr{S}_r[u] \le \mathfrak{c} \cdot r^{2/(1-\mu)}$ for the estimate on $\{t = 0\}$. Consequently, the comparison principle for viscosity solutions in Theorem 2.5 implies that $\zeta \geq u$ in $\mathscr{C}_1^+$. Therefore,

$$\sup_{\mathscr{C}_r} u(x,t) \leq \mathfrak{C}(n, \lambda, \Lambda, \mu, \kappa, \mathfrak{m}) \cdot r^{\frac{2}{1-\mu}}. \qquad \square$$

Remark 3.4. Following the same arguments as in the proof of Theorem 1.1, it is possible to obtain similar regularity estimates for a family of problems with a general (not μ-homogeneous) nonnegative and nonlinear absorption term $\mathfrak{f}$: $\mathscr{C}_1 \times [0, \|u\|_\infty] \to \mathbb{R}_+$, i.e.,

$$\mathscr{F}(x, t, Du, D^2u) - \partial_t u = \mathfrak{f}(x, t, u) \quad \text{in } \Omega_T,$$

provided $\mathfrak{f}(x, t, r) \leq \mathfrak{C}_* r^\mu$, for all $0 < r \ll 1$ and for some constant $\mathfrak{C}_* > 0$. Some interesting examples include

$$\mathfrak{f}(u) = \begin{cases} \lambda_0(x,t)(e^{u_+^s} - 1) & \text{for } s \geq \mu > 0, \\ \lambda_0(x,t)\ln(u_+^s + 1) & \text{for } s \geq \mu > 0, \\ \lambda_0(x,t)u_+^\mu \ln(u^s + 1) & \text{for } s > 0, \\ \lambda_0(x,t)u_+^\mu/(1+u^s)^m & \text{for } s > 0 \text{ and } 0 < m \leq \mu. \end{cases}$$

We have decided to treat the case $\mathfrak{f}(x, t, u) = \lambda_0(x, t)u_+^\mu(x, t)$ in order to introduce the main novelties in our approach.

Remark 3.5. Notice that Lemma 3.2 ensures that there exists a universal constant $0 < \tau_0 \ll 1$ (small enough) such that if $u \in \mathfrak{J}(\mathscr{F}, \lambda_0, \mu)(\mathscr{C}_1)$ with

$$\|\mathscr{F}(x, t, Du, D^2u) - \partial_t u\|_{L^\infty(\mathscr{C}_1)} \leq \tau_0,$$

then

$$\mathscr{S}_{\frac{1}{2^{j+1}}}[u] \leq \mathfrak{C}_0 \cdot \left(\frac{1}{2^j}\right)^{\frac{2}{1-\mu}}.$$

We now are ready to prove the main result of the article.

Proof of Theorem 1.1. In order to prove Theorem 1.1, we have to reduce the hypothesis presented on it to the framework of Theorem 3.3. We assume without loss of generality that $\mathfrak{K} = \overline{\mathscr{C}_1} \subset \Omega_T$. For $(x, t) \in \{u > 0\} \cap \mathfrak{K}$ let $\mathfrak{d}(x, t)$ the distance comes from Theorem 3.3. For $(x_0, t_0) \in \partial\{u > 0\} \cap \mathfrak{K}$ let us define

$$v(x,t) := \frac{u(x_0 + \mathfrak{R}_0 x, t_0 + \mathfrak{R}_0^2 t)}{\kappa_0} \quad \text{in } \mathscr{C}_1$$

for constants $\kappa_0, \mathfrak{R} > 0$ to be determined universally a posteriori.

From the equation satisfied by u, we easily verify that, in the viscosity sense, v fulfills

$$\mathscr{G}(x, t, Dv, D^2v) - \partial_t v = \hat{\lambda}_0(x,t) \cdot v_+^\mu(x,t), \tag{3-4}$$

for

$$\mathscr{G}(x, t, \vec{p}, \mathscr{X}) := \frac{\mathfrak{R}_0^2}{\kappa_0}\mathscr{F}\left(x_0 + \mathfrak{R}_0 x, t_0 + \mathfrak{R}_0^2, \frac{\kappa_0}{\mathfrak{R}_0}\vec{p}, \frac{\kappa_0}{\mathfrak{R}_0^2}\mathscr{X}\right)$$

and

$$\hat{\lambda}_0(x, t) := \frac{\mathfrak{R}_0^2}{\kappa_0^{1-\mu}}\lambda_0(x_0 + \mathfrak{R}_0 x, t_0 + \mathfrak{R}_0^2 t).$$

Observe that $\mathscr{G}$ satisfies assumption (**SC**) from Section 2 with the same ellipticity constants λ and Λ, Lipschitz constant κ, and a modulus of continuity $\hat{\omega}(s) = \omega(\mathfrak{R}s)$.

Now, let $\tau_0 > 0$ be the greatest universal constant, from Remark 3.5 such that Lemma 3.2 holds provided $\|\mathscr{G}(x, t, Dv, D^2u) - \partial_t v\|_{L^\infty(\mathscr{C}_1)} \le \tau_0$. By choosing

$$\kappa_0 := \|u\|_{L^\infty(\Omega_T)} \quad \text{and} \quad 0 < \mathfrak{R} < \min\left\{1, \frac{\operatorname{dist}(\mathfrak{K}, \partial_p \Omega_T)}{2}, \sqrt{\frac{\tau_0 \kappa_0^{1-\mu}}{\mathfrak{M}}}\right\},$$

v fits into the framework of Theorem 3.3. Hence, there exists a constant $\mathfrak{C} = \mathfrak{C}(n, \lambda, \Lambda, \mathfrak{M})$ so that

$$v(x, t) \le \mathfrak{C} \cdot \mathfrak{d}(x, t)^{\frac{2}{1-\mu}},$$

where

$$\mathfrak{d}(x, t) := \begin{cases} \sup\{r \ge 0; \mathscr{C}_r(x, t) \subset \{v > 0\}\} & \text{for } (x, t) \in \{v > 0\}, \\ 0 & \text{otherwise.} \end{cases}$$

By scaling back, we obtain the conclusion of Theorem 1.1. □

This final part is devoted to proving Theorem 1.2 which tells us how dead-core solutions detach their free boundaries. As a byproduct, we shall also give important consequences of such a nondegeneracy property, including uniform Lebesgue density of positive sets $\{u > 0\}$, porosity of t-level sets of the free boundary and finite speed propagation of $\{u > 0\}$; see Section 4 for more details.

Proof of Theorem 1.2. Firstly, notice that by continuity of viscosity solutions, it is sufficient to show that (1-2) holds for $(x_0, t_0) \in \{u > 0\}$. To this end, fix such a point and take $r > 0$ so that $\mathscr{C}_r(x_0, t_0) \subset \Omega \times (0, \infty)$. Now, define the comparison function

$$\Psi(x, t) := \mathfrak{c}_1[|x - x_0|^2 + \mathfrak{c}_2(t_0 - t)]^{\frac{1}{1-\mu}}, \quad (x, t) \in \mathscr{C}_r^-(x_0, t_0),$$

where $\mathfrak{c}_1, \mathfrak{c}_2$ are positive constants satisfying $\mathfrak{c}_2 < \frac{\mathfrak{m}(1-\mu)}{2} c_1^{\mu-1}$ and

$$\mathfrak{c}_1 := \left\{\frac{\mathfrak{m}(1-\mu)^2}{4[2\mu\Lambda + (n\Lambda + \kappa)(1-\mu)]}\right\}^{\frac{1}{1-\mu}},$$

where $0 < \mathfrak{m} \le \lambda_0$. Hence, it follows that

$$\mathscr{F}(x, t, D\Psi, D^2\Psi) - \partial_t \psi - \lambda_0(x, t)\psi^\mu \le 0 \quad \text{in } \mathscr{C}_r^-(x_0, t_0).$$

Now, if $\Psi \ge u$ on whole $\partial_p \mathscr{C}_r^-(x_0, t_0)$, then the comparison principle Theorem 2.5 would imply that $u \le \Psi$ in $\mathscr{C}_r^-(x_0, t_0)$. However, this is a contradiction with the fact that $\Psi(x_0, t_0) = 0 < u(x_0, t_0)$. Therefore, there exists a point $(x', t') \in \partial_p \mathscr{C}_r^-(x_0, t_0)$ such that $u(x', t') \ge \Psi(x', t')$. Since $\Psi(x', t') = \mathfrak{c} r^{2/(1-\mu)}$ for $\mathfrak{c}$ independently of (x', t'), we have completed the proof of the theorem. □

4. Applications

Using Theorem 1.1 we are able to prove a similar growth rate for the gradient of dead-core solutions provided that the coefficients of $\mathscr{F}$ are VMO (see Proposition 2.4).

Corollary 4.1 (growth estimates for gradient). *Suppose that the assumptions of Proposition 2.4 are in force. Then a positive constant $\mathfrak{C}_1 = \mathfrak{C}_1(n, \lambda, \Lambda, \mu, \mathfrak{M})$ exists such that for all $u \in \mathfrak{J}(\mathscr{F}, \lambda_0, \mu)(\mathscr{C}_1)$:*

$$\|Du(x, t)\| \le \mathfrak{C}_1 \cdot \mathfrak{d}(x, t)^{\frac{1+\mu}{1-\mu}} \quad \textit{for all } \mathscr{C}_{\frac{1}{2}}.$$

Proof. As before, it is enough to prove the estimate

$$\mathscr{S}_{\frac{1}{2^{j+1}}}[\|Du\|] \le \max\left\{\mathfrak{C}_2 \cdot \left(\frac{1}{2^j}\right)^{\frac{1+\mu}{1-\mu}}, \left(\frac{1}{2}\right)^{\frac{1+\mu}{1-\mu}} \mathscr{S}_{\frac{1}{2^j}}[\|Du\|]\right\}, \tag{4-1}$$

for all $j \in \mathbb{N}$ and a constant $\mathfrak{C}_2 = \mathfrak{C}_2(n, \lambda, \Lambda, \mu, \mathfrak{M})$.

Let us suppose that (4-1) does not hold. Then, there exists $u_j \in \mathfrak{J}(\mathscr{F}, \lambda_0, \mu)(\mathscr{C}_1)$ such that

$$\mathscr{S}_{\frac{1}{2^{j+1}}}[\|Du_j\|] \ge \max\left\{j\left(\frac{1}{2^j}\right)^{\frac{1+\mu}{1-\mu}}, \left(\frac{1}{2}\right)^{\frac{1+\mu}{1-\mu}} \mathscr{S}_{\frac{1}{2^j}}[\|Du_j\|]\right\}. \tag{4-2}$$

Next, we define the auxiliary normalized and scaled function

$$v_j(x, t) := \frac{2^j u_j\left(\frac{1}{2^j}x, \frac{1}{2^{2j}}t\right)}{\mathscr{S}_{\frac{1}{2^{j+1}}}[\|Du_j\|]}.$$

Notice that using (3-1) and (4-2) we obtain

$$\begin{aligned} 0 \le v_j(x) &\le \frac{2^j \mathfrak{C}(2^{-j})^{\frac{2}{1-\mu}}}{\mathscr{S}_{\frac{1}{2^{j+1}}}[\|Du_j\|]} \\ &\le \frac{\mathfrak{C}_0}{j^{1-\mu}} \quad \text{for } (x, t) \in \mathscr{C}_1. \end{aligned}$$

Furthermore:

- $\mathscr{F}_j(x, t, Dv_j, D^2v_j) - \partial_t v_j = \hat{\lambda_0}(x, t)(v_j)_+^{\mu}(x, t)$ in $\mathscr{C}_1$ in the viscosity sense, where

$$\mathscr{F}_j(x, t, \vec{p}, M) \\ := \frac{1}{2^j \mathscr{S}_{\frac{1}{2^{j+1}}}[\|Du_j\|]} \mathscr{F}\left(\frac{1}{2^j}x, \frac{1}{2^{2j}}t, \mathscr{S}_{\frac{1}{2^{j+1}}}[\|Du_j\|] \cdot \vec{p}, 2^j \mathscr{S}_{\frac{1}{2^{j+1}}}[\|Du_j\|] \cdot M\right)$$

and

$$\hat{\lambda_0}(x, t) := \frac{1}{2^{j(1+\mu)}} \frac{1}{\mathscr{S}^{1-\mu}_{\frac{1}{2^{j+1}}}[\|Du_j\|]} \lambda_0\left(\frac{1}{2^j}x, \frac{1}{2^{2j}}t\right).$$

- $\mathscr{S}_{\frac{1}{2}}[\|Dv_j\|] \geq 1$.

Consequently,

$$\|\hat{\lambda_0} \cdot (v_j)_+^{\mu}\|_{L^\infty(\mathscr{C}_1)} \leq \mathfrak{M} \cdot \mathfrak{C}^{\mu} \cdot \frac{1}{j}.$$

Finally, by using the a priori gradient estimate from Proposition 2.4 we obtain

$$\begin{aligned} 1 \leq \mathscr{S}_{\frac{1}{2}}[\|Dv_j\|] &\leq \mathfrak{C}(n, \lambda, \Lambda)[\|v_j\|_{L^\infty(\mathscr{C}_1)} + \|\hat{\lambda_0}(v_j)_+^{\mu}\|_{L^\infty(\mathscr{C}_1)}] \\ &\leq \mathfrak{C}^* \cdot \frac{1}{j} \to 0 \quad \text{as } j \to \infty, \end{aligned}$$

which is a contradiction. Therefore the proof is ended. □

An interesting piece of information coming from our technique is that by using again the previous iterative geometric argument and supposing better assumptions on $\mathscr{F}$ (to be clarified soon) we will be able to access an improved growth rate for the higher derivatives of u (namely, the temporal derivative and the Hessian of u, respectively) according to Schauder type estimates.

Corollary 4.2 (growth estimates for higher derivatives). *Assume that* (**DCP**) *has locally $C^{2,1}$ a priori estimates, i.e., there exists a universal constant $\mathfrak{C}_\sharp > 0$ such that*

$$\|D^2u\|_{L^\infty(\mathscr{C}_r)}, |\partial_t u|_{L^\infty(\mathscr{C}_r)} \leq \mathfrak{C}_\sharp \quad \textit{for all } r \ll 1.$$

Then, there exists a positive constant $\mathfrak{C}_2 = \mathfrak{C}_1(n, \lambda, \Lambda, \mu, \mathfrak{M})$ such that for all $u \in \mathfrak{J}(\mathscr{F}, \lambda_0, \mu)(\mathscr{C}_1)$,

$$\|D^2u(x,t)\| \leq \mathfrak{C}_2 \cdot \mathfrak{d}(x,t)^{\frac{2\mu}{1-\mu}} \quad (\textit{resp. } |\partial_t u(x,t)| \leq \mathfrak{C}_2 \cdot \mathfrak{d}(x,t)^{\frac{\mu}{1-\mu}}) \quad \textit{for all } (x,t) \in \mathscr{C}_{\frac{1}{2}},$$

Remark 4.3. An interesting class of operators for which we can apply Corollary 4.2 is the class of convex (or concave) operators (recall that such a family of operators enjoy local $C^{2+\alpha,(2+\alpha)/2}$ a priori estimates; see [Krylov 1983; Wang 1992b, Section 4.3]), because under such assumptions on $\mathscr{F}$ it is possible to develop a Schauder type estimate provided the source term enjoys a universal modulus of continuity, suitably integrable at origin (see [Tian and Wang 2013] and [Wang 1992b, Section 1.1] for more details and compare with [da Silva and Teixeira 2017, Section 6] and [da Silva and dos Prazeres 2019, Section 6] for results when such an assumption fails). Finally, in [da Silva and dos Prazeres 2019], Schauder type estimates were proved for flat C^0-viscosity solutions, i.e., solutions with very small oscillation, provided $\mathscr{F}$ is in $C^{1,\tau}(\mathrm{Sym}(n))$ and has Dini continuous coefficients. Therefore, such a family of solutions and operators is an interesting class where Corollary 4.2 holds true.

Remark 4.4 (dead core solutions vs. flat solutions). In view of previous results, we must highlight the relationship between regularity coming from dead-core solutions and that coming from the classical Schauder theory. To this end, let us suppose that u is a flat C^0-viscosity solution to

$$\mathscr{F}(x,t,D^2u)-\partial_t u=\lambda_0(x,t)u_+^{\mu}(x,t)\quad \text{in } \mathscr{C}_1, \tag{4-3}$$

where $0<\mu<1$, $\lambda_0\in C^{0,\mu}(\mathscr{C}_1)$ and $\mathscr{F}$ is subject to the assumptions in, for example, [da Silva and dos Prazeres 2019]; see Remark 4.3. Under such assumptions, the Schauder type estimates from [da Silva and dos Prazeres 2019] ensure that solutions to (4-3) are $C^{2+\mu,(2+\mu/2)}_{\mathrm{loc}}(\mathscr{C}_1)$ (particularly at free boundary points). On the other hand, our main theorem, Theorem 1.1, claims that u is $C^{\varsigma+\alpha,(\varsigma+\alpha)/2}$ at free boundary points, where

$$\varsigma:=\left\lfloor\frac{2}{1-\mu}\right\rfloor \quad\text{and}\quad \alpha:=\frac{2}{1-\mu}-\left\lfloor\frac{2}{1-\mu}\right\rfloor.$$

Nevertheless, notice that for any $0<\mu<1$ we have

$$\varsigma+\alpha=\frac{2}{1-\mu}>2+\mu,$$

which means that dead-core solutions are more regular, along free boundary points, than the best regularity result coming from classical regularity theory in [da Silva and dos Prazeres 2019].

Remark 4.5 (regularity in some problems from geometry). Over the last decades the study of geometric flows has proved to be extremely effective in solving some of the most important problems in topology, differential geometry and geometric analysis. Geometric considerations lead to equations of the form

$$\begin{aligned}\mathscr{F}(x,t,Du,D^2u)-\partial_t u&\\ &=\mathfrak{f}\left(x,t,u,\int_{B_1}\mathscr{G}(Du,D^2u)\,dx\right)\chi_{\{u>0\}}\quad \text{in } \mathscr{M}\subset\mathbb{R}^{n+1},\end{aligned} \tag{4-4}$$

where $\mathscr{F}$ is a convex (concave) operator and $\mathscr{G} \in C^\infty(\mathbb{R}^n \times \mathrm{Sym}(n), \mathbb{R}^m)$ is a vector field (see [Tian and Wang 2013] for more detail on these topics). Such equations appear in many applications of parabolic PDEs in curvature and gradient flows. For this reason, our work has also been motivated by the study of such equations coming from differential geometry and geometric analysis in order to establish high order estimates to solutions along their free boundaries.

Next, we will comment on interior regularity results for general nonlinear curvature and gradient flows (4-4) (at free boundary points); they provide an interesting application in the geometric setting for our main theorem. We consider $\mathscr{M}$ to be a closed manifold without boundary under a volume constraint assumption; thus interior regularity is sufficient. For (4-4), one can to obtain $C^{2,1}$ estimates for solutions via maximum principle

$$\|u\|_{C^{2,1}(\mathscr{C}_r)} \leq C \quad \text{for all } r \ll 1.$$

Furthermore, such an estimate implies that (4-4) is uniformly elliptic; thus the structural condition is satisfied. Notice that such an estimate also implies that the RHS is C^2 in x and bounded and measurable in t. Hence, we fall into the assumptions of Remark 3.4. As a result, Theorem 1.1 can be applied for viscosity solutions to equations of form (4-4).

Finally, this result can be further applied to equations of the form

$$\mathscr{F}(D\mathfrak{h}, D^2\mathfrak{h}) - \partial_t\mathfrak{h} - \mathfrak{h}_+^\mu|\mathfrak{A}|^2 = 0,$$

where $\mathfrak{h}$ is the inwards mean curvature vector of the surface at position x and time t and $|\mathfrak{A}|$ represents the norm of the second fundamental form. This equation is the extended version for models describing the mean curvature hypersurface in the Euclidean space $\mathbb{R}^{n+1}$; see [Sheng and Wang 2010].

Next, we will establish a finer control for dead-core solutions close to free boundary points. In brief, any viscosity solution to (**DCP**) will be "trapped" between the graph of two suitable multiples of $\mathrm{dist}_p(\,\cdot\,, \partial\{u > 0\})^{2/(1-\mu)}$.

Corollary 4.6. *Let u be a nonnegative, bounded viscosity solution to* (**DCP**) *and $\Omega' \Subset \Omega_T$. Given $(x_0, t_0) \in \{u > 0\} \cap \Omega'$, there exists a universal constant $\mathfrak{C}_\sharp > 0$ such that*

$$\mathfrak{C}_\sharp \mathrm{dist}_p((x_0, t_0), \partial\{u > 0\})^{\frac{2}{1-\mu}} \leq u(x_0, t_0) \leq \mathfrak{C}^\sharp \mathrm{dist}_p((x_0, t_0), \partial\{u > 0\})^{\frac{2}{1-\mu}}.$$

Proof. The upper estimate is an immediate consequence of Theorem 1.1. Next, let us suppose that such a $\mathfrak{C}_\sharp > 0$ does not exist. Then there exists a sequence $P_k := (x_k, t_k) \in \{u > 0\} \cap \Omega'$ with

$$d_k := \mathrm{dist}_p(P_k, \partial\{u > 0\} \cap \Omega') \to 0 \quad \text{as } k \to \infty \quad \text{and} \quad u(P_k) \leq k^{-1} d_k^{\frac{2}{1-\mu}}.$$

Now, let us define the auxiliary function $v_k : \mathscr{C}_1 \to \mathbb{R}$ by

$$v_k(y, s) := \frac{u(P_k + (d_k y, d_k^2 s))}{d_k^{\frac{2}{1-\mu}}}.$$

It is easy to verify that:

(1) $\mathscr{F}_k(x, t, Dv_k, D^2 v_k) - \partial_t v_k = \widehat{\lambda}_k(x, t) \cdot (v_k)_+^\mu$ in $\mathscr{C}_{1/2}$ in the viscosity sense where

$$\mathscr{F}_k(y, s, \vec{p}, \mathscr{X}) := d_k^{-\frac{2\mu}{1-\mu}} \mathscr{F}\Big(P_k + (d_k y, d_k^2 s), d_k^{\frac{1+\mu}{1-\mu}} \vec{p}, d_k^{\frac{2\mu}{1-\mu}} \mathscr{X}\Big)$$

and

$$\widehat{\lambda}_k(y, s) := \lambda_0(P_k + (d_k y, d_k^2 s)).$$

(2) $u(P_k + (d_k y, d_k^2 s)) \le \sup_{\mathscr{C}^+_{d_k}(\widehat{P}_k)} u(y, s) \le \mathfrak{C} d_k^{2/(1-\mu)}$ according to Theorem 1.1, where $\widehat{P}_k$ is such that $d_k = \operatorname{dist}_p(P_k, \widehat{P}_k)$. Hence, v_k is nonnegative and uniformly bounded.

(3) $v_k(y, s) \le C \cdot d_k^\alpha + \frac{1}{k}$ for all $(y, s) \in \mathscr{C}^-_{1/2}$ due to local Hölder regularity of solutions; see [Crandall et al. 2000, Section 5; Krylov and Safonov 1980; Wang 1992a, Section 4.4].

From the nondegeneracy property, Theorem 1.2, and the last sentence we obtain

$$0 < \mathfrak{C}_0 \cdot \left(\frac{1}{2}\right)^{\frac{2}{1-\mu}} \le \sup_{\partial_p \mathscr{C}^-_{\frac{1}{2}}} v_k(y, s) \le \sup_{\overline{\mathscr{C}^-_{\frac{1}{2}}}} v_k(y, s) \le C d_k^\alpha + \frac{1}{k} \to 0 \quad \text{as } k \to \infty. \tag{4-5}$$

Such a contradiction finishes the proof. □

As an another application, we establish an average control for the μ-power of dead-core solutions. Such an estimate will be useful in order to prove Hausdorff measure estimates.

Corollary 4.7. *Let u be a nonnegative, bounded viscosity solution to* (**DCP**) *and $\Omega' \Subset \Omega_T$. For all $(x_0, t_0) \in \partial\{u > 0\} \cap \Omega'$, there exist universal constants $\mathfrak{C}_\sharp > 0$ and*

$$0 < r_0 \ll \min\left\{1, \frac{\operatorname{dist}_p(\Omega', \partial_p \Omega_T)}{2}\right\}$$

such that

$$\fint_{\mathscr{C}^-_r(x_0, t_0)} u^\mu(x, t) \ge \mathfrak{C}_\sharp r^{\frac{2\mu}{1-\mu}},$$

for any $r \le r_0$.

Proof. Once more, we will proceed via a contradiction argument. If such a $\mathfrak{C}_* > 0$ does not exist, then there would exist a sequence $P_k := (x_k, t_k) \in \partial\{u > 0\} \cap \Omega'$ such that for any sequence $r_k \to 0+$ as $k \to \infty$ we would have

$$\fint_{\mathscr{C}^-_{r_k}(P_k)} u^\mu(x,t) < k^{-1} r_k^{\frac{2\mu}{1-\mu}}.$$

Now, define the function $v_k : \mathscr{C}^-_1 \to \mathbb{R}$ by

$$v_k(y,s) := \frac{u(P_k + (r_k y, r_k^2 s))}{r_k^{\frac{2}{1-\mu}}}.$$

It is easy to verify

$$\mathscr{G}_k(x,t,Dv_k,D^2v_k) - \partial_t v_k = \lambda_k(x,t)\cdot(v_k)^\mu_+ \quad \text{in } \mathscr{C}^-_1$$

in the viscosity sense, where

$$\mathscr{G}_k(y,s,\vec{p},\mathscr{X}) := r_k^{-\frac{2\mu}{1-\mu}} \mathscr{F}(P_k + (r_k y, r_k^2 s), r_k^{\frac{1+\mu}{1-\mu}}\vec{p}, r_k^{\frac{2\mu}{1-\mu}}\mathscr{X})$$

and

$$\lambda_k(y,s) := \lambda_0(P_k + (r_k y, r_k^2 s)).$$

On the one hand, using the contradiction assumption,

$$\fint_{\mathscr{C}^-_{\frac{1}{2}}(0,0)} v_k^\mu(y,s) = 2^{n+2} \fint_{\mathscr{C}^-_{r_k}(P_k)} \frac{u^\mu(x,t)}{r_k^{\frac{2\mu}{1-\mu}}} < \frac{2^{n+2}}{k} \to 0 \quad \text{as } k \to \infty. \tag{4-6}$$

On the other hand, using Corollary 4.6 we obtain

$$\begin{aligned}\fint_{\mathscr{C}^-_{\frac{1}{2}}(0,0)} v_k^\mu(y,s) &= \fint_{\mathscr{C}^-_{\frac{1}{2}}(0,0)} \frac{u^\mu(P_k + (r_k y, r_k^2 s))}{r_k^{\frac{2\mu}{1-\mu}}} \\ &\geq \mathfrak{C}_\sharp^\mu \fint_{\mathscr{C}^-_{\frac{1}{2}}(0,0)} \left(\frac{\operatorname{dist}_p(P_k + (r_k y, r_k^2 s), \partial\{u>0\})}{r_k}\right)^{\frac{2\mu}{1-\mu}}.\end{aligned}$$

Now, let us denote $d_k(y,s) := \operatorname{dist}_p(P_k + (r_k y, r_k^2 s), \partial\{u > 0\} \cap \Omega')$. Under such a notation we define

$$\mathfrak{D}_k := \{(y,s) \in \mathscr{C}^-_{\frac{1}{2}}(0,0) \mid d_k(y,s) < a_k r_k\},$$

where

$$a_k := \left(\frac{1}{k}\right)^{\frac{1-\mu}{2\mu}} |\mathscr{C}^-_{\frac{1}{2}}(0,0)|^{\frac{1-\mu}{2\mu}} (\mathfrak{C}_\sharp^{-\mu})^{\frac{1-\mu}{2\mu}} (2^{\alpha(n+2)})^{\frac{1-\mu}{2\mu}}$$

with $\alpha > 0$ chosen such that

$$\frac{2^{\alpha(n+2)}}{10}|\mathscr{C}^-_{\frac{1}{2}}(0,0)| > 2^{n+2}.$$

Notice that for $k \gg 1$ large enough, $\mathfrak{D}_k \cap \mathscr{C}^-_{r_k}(P_k) \cap \{u > 0\} \neq \varnothing$. Moreover, since $a_k \to 0$ as $k \to \infty$, we have, for $k \gg 1$ large enough, that $|\mathfrak{D}^c_k| \geq \frac{1}{10}|\mathscr{C}^-_{1/2}(0,0)|$. Therefore, we can estimate for $k \gg 1$

$$\begin{aligned}\fint_{\mathscr{C}^-_{\frac{1}{2}}(0,0)} v^\mu_k(y,s) &\geq \frac{\mathfrak{C}^\mu_\sharp}{|\mathscr{C}^-_{\frac{1}{2}}(0,0)|}\int_{\mathfrak{D}^c_k}\left(\frac{d_k(y,s)}{r_k}\right)^{\frac{2\mu}{1-\mu}} \\ &\geq \frac{2^{\alpha(N+2)}}{k}|\mathfrak{D}^c_k| \geq \frac{2^{\alpha(n+2)}}{10k}|\mathscr{C}^-_{\frac{1}{2}}(0,0)| > \frac{2^{n+2}}{k},\end{aligned}$$

which contradicts (4-6). This finishes the proof of the corollary. □

The nondegeneracy property and the growth rate for viscosity solutions to (**DCP**) will lead us to establish some measure-theoretic properties of the free boundary. We start by showing a property of positive density.

Corollary 4.8 (positive Lebesgue density of $\{u > 0\}$). *Let u be as in Theorem 1.1. Then, there exists a positive constant $\zeta = \zeta(n, \lambda, \Lambda, \mathfrak{M}, \|u\|_{L^\infty(\mathscr{C}_1)})$ such that for all $(x_0, t_0) \in \overline{\{u > 0\}}$ and $0 < r < 1$ such that $\mathscr{C}_r(x_0, t_0) \subset \mathscr{C}_{1/2}$, the inclusion*

$$\mathscr{C}_{\zeta r}(x', t') \subset \mathscr{C}_r(x_0, t_0) \cap \{u > 0\},$$

holds for some $(x', t') \in \mathscr{C}^-_r(x_0, t_0)$.

Proof. Let $(x_0, t_0) \in \overline{\{u > 0\}} \cap \overline{\mathscr{C}_{1/2}}$. For r small enough, we have by Theorem 1.2 that there exists $(x', t') \in \mathscr{C}^-_{r/2}(x_0, t_0)$ such that

$$u(x', t') \geq \mathfrak{c} \cdot \left(\frac{r}{2}\right)^{\frac{2}{1-\mu}}. \tag{4-7}$$

Suppose that for all $0 < \mathfrak{d} < 1$ small, there exists a point $(x, t) \in \partial\{u > 0\} \cap \overline{\mathscr{C}_{1/2}}$ satisfying

$$(x', t') \in \mathscr{C}_{\mathfrak{d}}(x, t) \subset \mathscr{C}_r(x_0, t_0). \tag{4-8}$$

Now, according to (4-7), (4-8) and Theorem 1.1, it follows

$$\mathfrak{c} \cdot \left(\frac{r}{2}\right)^{\frac{2}{1-\mu}} \leq u(x', t') \leq \sup_{\mathscr{C}_{\mathfrak{d}}(x,t)} u \leq \mathfrak{C}, \mathfrak{d}^{\frac{2}{1-\mu}}.$$

This clearly does not hold for $\mathfrak{d} < \Xi \cdot \frac{r}{2}$, where $\Xi := \left(\frac{\mathfrak{c}}{\mathfrak{C}}\right)^{(1-\mu)/2} < 1$. Hence $\mathscr{C}_{\Xi/4\, r}(x', t') \subset \mathscr{C}_r(x_0, t_0) \cap \{u > 0\}$. This ends the proof of the theorem. □

Remark 4.9. Notice that Corollary 4.8 ensures that the free boundary cannot have Lebesgue points. Consequently, for any compact set $\mathfrak{K} \subset \mathscr{C}_1$, we have

$$\mathscr{L}^{n+1}(\partial\{u > 0\} \cap \mathfrak{K}) = 0.$$

Next, we shall prove, as an easy consequence of the above result, that the free boundary is a porous set. We recall the definition of this notion.

Definition 4.10 (porous set). A set $\mathscr{E} \in \mathbb{R}^n$ is said to be porous with porosity constant $0 < \zeta \leq 1$ if there exists $\mathfrak{R} > 0$ such that for each $x_0 \in \mathscr{E}$ and $0 < \mathfrak{r} < \mathfrak{R}$ there is a point x' so that $B_{\mathfrak{r}\zeta}(x') \subset B_{\mathfrak{r}}(x_0) \setminus \mathscr{E}$.

Observe that a porous set has Hausdorff dimension at most $n - c_0\zeta^n$, where $c_0 = c_0(n) > 0$.

Corollary 4.11 (porosity for t-level of free boundary). *Let u be a viscosity solution to* (**DCP**). *For every compact set $\mathscr{K} \subset \mathscr{C}_1$,*

$$\mathscr{H}^{n-\delta}(\partial\{u > 0\} \cap \mathscr{K} \cap \{t = t_0\}) < \infty$$

for a constant $0 < \delta = \delta(n, \lambda, \Lambda, \mu, \mathfrak{M}, \|u\|_{L^\infty(\mathscr{C}_1)}, \operatorname{dist}(\mathscr{K}, \partial\mathscr{C}_1)) \leq 1$.

Proof. The proof is standard (see, for instance, [Choe and Weiss 2003]). However, we quote full details for completeness. Without loss of generality we can suppose that $\mathscr{K} = \overline{\mathscr{C}_{1/2}}$. Let $(z, t_0) \in \partial\{u > 0\} \cap \overline{\mathscr{C}_{1/2}}$; then for $0 < r \ll 1$, according to the nondegeneracy property, there exists $x' \in \partial B_r(z)$ such that

$$u(x', t_0) \geq \mathfrak{C}_0^* \cdot r^{\frac{2}{1-\mu}}.$$

On the other hand, from Theorem 3.3,

$$u(x', t_0) \leq \mathfrak{C} \cdot \mathfrak{d}(x', t_0)^{\frac{2}{1-\mu}}.$$

Consequently,

$$\mathfrak{c} \cdot r^{\frac{2}{1-\mu}} \leq u(x', t_0) \leq \mathfrak{C} \cdot \mathfrak{d}(x', t_0)^{\frac{2}{1-\mu}}.$$

Next, by selecting $\delta = \left(\frac{\mathfrak{c}}{\mathfrak{C}}\right)^{(1-\mu)/2}$, then $\mathfrak{d}(x', t_0) \geq \delta \cdot r$. Therefore

$$B_{\delta \cdot r}(x') \cap B_r(z) \subset \{u(\,\cdot\,, t_0) > 0\}.$$

Now, choose $y \in [z, x']$ such that $|y - x'| = \frac{\delta r}{2}$. Note that for any $y_0 \in B_{(\delta r)/2}(y)$,

$$|y_0 - x'| \leq |y_0 - y| + |y - x'| = \delta r.$$

Moreover, since $|z - x'| = |y - z| + |y - x'|$,

$$|y_0 - z| \leq |y_0 - y| + (|z - x'| - |y - x'|) \leq \frac{\delta r}{2} + \left(r - \frac{\delta r}{2}\right) = r,$$

so we conclude that $B_{(\delta r)/2}(y) \subset B_{\delta r}(x') \cap B_r(z) \subset B_r(z) \setminus (\partial\{u > 0\} \cap \{t = t_0\})$. Therefore, $\partial\{u > 0\} \cap \{t = t_0\} \cap \overline{\mathscr{C}_{1/2}}$ is porous with porosity constant $\frac{\delta}{2}$. $\square$

In contrast to one of the most known properties of heat equations, namely the infinity speed of propagation, fully nonlinear parabolic dead core problems have the property of *finite speed propagation*. Such a property supports the physical soundness of the equation to diffusive models. Moreover, the occurrence of this phenomenon is a consequence of loss of diffusivity of the equation at the level set $u = 0$. The proof will be based on [Choe and Weiss 2003, Corollary 4.4].

Corollary 4.12 (finite speed propagation of $\{u > 0\}$). *There exists a constant* $\mathfrak{c}(n, \lambda, \Lambda, \mu) \geq 1$ *such that, for any solution to* (**DCP**), *with nonnegative and bounded time derivative, and any* $\mathscr{C}_r^+(x_0, t_0) \subset \Omega \times (0, \infty)$, *the implication*

$$u(\,\cdot\,, t_0) = 0 \quad \textit{in } B_r(x_0) \Rightarrow u(\,\cdot\,, t_0 + \mathfrak{s}^2) = 0 \quad \textit{in } B_{\max\{0, r - \mathfrak{c}\mathfrak{s}\}}(x_0)$$

holds.

Proof. Let us suppose that for $0 < \mathfrak{s}_1 < \frac{r}{\mathfrak{c}}$ there exists a point $x_1 \in B_{r - \mathfrak{c}\mathfrak{s}_1}(x_0)$ such that $u(x_1, t_0 + \mathfrak{s}_1^2) > 0$. The nondegeneracy property (Theorem 1.2) implies

$$u(x_2, \varsigma) \geq \mathfrak{C}_0^* \mathfrak{s}_1^{\frac{2}{1-\mu}}$$

for some $(x_2, \varsigma) \in \overline{\mathscr{C}_{\mathfrak{s}_1}^-(x_1, t_0 + \mathfrak{s}_1^2)}$. Moreover, since $\frac{\partial u}{\partial t}$ is nonnegative and bounded, we deduce that there exist $0 < \tau(n, \mu) < 1$ and $(x_2, t_0 + \mathfrak{s}_2^2)$ satisfying

$$u(x_2, t_0 + \mathfrak{s}_2^2) > 0, \quad \text{with} \quad 0 \leq \mathfrak{s}_2 \leq (1 - \tau)\mathfrak{s}_1 \quad \text{and} \quad |x_2 - x_1| \leq \mathfrak{s}_1.$$

By iterating the previous reasoning we can obtain a point $(x_k, t_0 + \mathfrak{s}_k^\theta)$ such that

$$u(x_k, t_0 + \mathfrak{s}_k^2) > 0, \quad \text{with } 0 \leq \mathfrak{s}_k \leq (1 - \tau)^{k-1}\mathfrak{s}_1 \text{ and } |x_k - x_1| \leq \frac{\mathfrak{s}_1[1 - (1 - \tau)^{k-1}]}{\tau}.$$

Finally, up to a subsequence, $x_k \to x_\infty$ as $k \to \infty$; thus we obtain a point $(x_\infty, t_0) \in \overline{\{u > 0\}}$ fulfilling $|x_\infty - x_1| < \mathfrak{s}_1/\tau$. However, this contradicts our assumptions provided $\mathfrak{c} \geq \frac{4}{\tau}$. This contradiction proves the corollary. $\square$

5. Global analysis results

5A. ***Blow-up analysis.*** Throughout this subsection we shall study the blow-up analysis over free boundary points (interior touching points). Historically, such a procedure provides a powerful device in order to study certain one-dimensional profiles in several free boundary problems (see [Apushkinskaya et al. 2002] and [Caffarelli et al. 2004] for some enlightenment). Thus, let u be a solution to (**DCP**), $\mathfrak{p}_0 := (x_0, t_0) \in \partial\{u > 0\} \cap \mathscr{C}_{1/2}$ and $(\mathfrak{p}_k = (x_k, t_k))_{k \in \mathbb{N}} \in \partial\{u > 0\} \cap \mathscr{C}_{1/2}$ such that

$\mathfrak{p}_k \to \mathfrak{p}_0$. Now, consider, for each $\varepsilon_k \searrow 0$, the *blow-up sequence* $u_{\varepsilon_k} : \mathscr{C}_{1/2} \to \mathbb{R}$ given by

$$u_{\varepsilon_k}^{\mathfrak{p}_k}(x,t) := \frac{u(x_k + \varepsilon_k x, t_k + \varepsilon_k^2 t)}{\varepsilon_k^{\frac{2}{1-\mu}}}.$$

We must stress that this sequence is indeed an ε_k-zoom-in of u (on the free boundary points) rescaled in a suitable way. The next result analyses the "limiting profile" for any blow-up sequence.

Theorem 5.1 (blow-up limit). *Let* $\mathfrak{p}_0 = (x_0, t_0) \in \partial\{u > 0\}$ *be a free boundary point,* $(\mathfrak{p}_k)_{k\in\mathbb{N}} \in \partial\{u > 0\}$ *such that* $\mathfrak{p}_k \to \mathfrak{p}_0$ *and a blow-up sequence* $(u_{\varepsilon_k}^{\mathfrak{p}_k})_{k\in\mathbb{N}}$. *Then, up to a subsequence*

$$u_{\varepsilon_k}^{\mathfrak{p}_k} \to u_0^{\mathfrak{p}_0} \ \textit{uniformly in every compact } \mathfrak{K} \subset \mathbb{R}^n \times \mathbb{R}.$$

Furthermore, $u_0^{\mathfrak{p}}$ *is a nonnegative viscosity solution to the following global parabolic dead-core problem with constant coefficients*:

$$\mathscr{F}(x_0, t_0, D^2 u_0^{\mathfrak{p}_0}(x,t)) - \partial_t u_0^{\mathfrak{p}_0}(x,t) = \lambda_0(x_0,t_0)\cdot (u_0^{\mathfrak{p}_0})_+^{\mu}(x,t) \quad \textit{in } \mathbb{R}^n \times \mathbb{R}. \tag{5-1}$$

Finally, $(0,0) \in \partial\{u_0^{\mathfrak{p}_0} > 0\}$.

Proof. Note that $u_{\varepsilon_k}^{\mathfrak{p}_k}$ fulfills, in the viscosity sense,

$$\mathscr{F}_{\varepsilon_k}(x,t,Du_{\varepsilon_k}^{\mathfrak{p}_k}, D^2 u_{\varepsilon_k}^{\mathfrak{p}_k}) - \partial_t u_{\varepsilon_k}^{\mathfrak{p}_k} = \widehat{\lambda}_k(x,t)\cdot (u_{\varepsilon_k}^{\mathfrak{p}_k})_+^{\mu} \quad \text{in } \mathscr{C}_{\frac{1}{2\varepsilon_k}}$$

where

$$\mathscr{F}_{\varepsilon_k}(x,t,\vec{p},\mathscr{X}) := \varepsilon_k^{-\frac{2\mu}{1-\mu}} \mathscr{F}(x_k + \varepsilon_k x, t_k + \varepsilon_k^2 t, \varepsilon_k^{\frac{1+\mu}{1-\mu}}\vec{p}, \varepsilon_k^{\frac{2\mu}{1-\mu}}\mathscr{X})$$

and

$$\widehat{\lambda}_k(x,t) := \lambda_0(x_k + \varepsilon_k x, t_k + \varepsilon_k^2 t).$$

Observe that $\mathscr{F}_{\varepsilon_k}$ satisfies assumption (**SC**) from Section 2, with the same structural parameters. As a consequence, from Theorem 1.1 we have

$$u_{\varepsilon_k}^{\mathfrak{p}_k}(x,t) \le C(n,\lambda,\Lambda,\mathfrak{M},\mu) \quad \text{for all } (x,t) \in \mathscr{C}_{\frac{1}{2\varepsilon_k}}.$$

Particularly, $u_{\varepsilon_k}^{\mathfrak{p}_k}$ is locally bounded in $\mathscr{C}_{1/(2\varepsilon_k)}$. From universal Hölder regularity, see for instance Proposition 2.2, up to a subsequence, $u_{\varepsilon_k}^{\mathfrak{p}_k} \to u_0^{\mathfrak{p}_0}$ locally uniformly to an entire function. Furthermore, $\partial\{u_0^{\mathfrak{p}_0} > 0\}$ has zero $(n+1)$-Lebesgue measure and by stability results (see Lemma 2.3) $u_0^{\mathfrak{p}_0}$ fulfills

$$\mathscr{F}(x_0, t_0, D^2 u_0^{\mathfrak{p}_0}(x,t)) - \partial_t u_0^{\mathfrak{p}_0}(x,t) = \lambda_0(x_0,t_0)\cdot (u_0^{\mathfrak{p}_0})_+^{\mu}(x,t) \quad \text{in } \mathbb{R}^n \times \mathbb{R}.$$

Obviously, $u_0^{\mathfrak{p}_0}$ is a global nonnegative solution. By nondegeneracy (Theorem 1.2)

$$\sup_{\mathscr{C}_r^-(0,0)} u_0^{\mathfrak{p}_0}(x,t) = \lim_{k\to\infty} \sup_{\mathscr{C}_r^-(0,0)} u_{\varepsilon_k}^{\mathfrak{p}_k}(x,t) \ge \mathfrak{C}_0^* \cdot r^{\frac{2}{1-\mu}},$$

which ensures that $(0,0) \in \partial\{u_0^{\mathfrak{p}_0} > 0\}$. ☐

From now on, for our purposes, $u_0^{\mathfrak{p}_0}$ will always denote a limiting function, using the previous reasoning. For this reason, we will label it as the *blow-up solution* at (x_0, t_0).

The next result establishes a quantitative control profile at infinity for a class of entire solutions to the dead-core problem, namely, blow-up solutions.

Theorem 5.2 (behavior of blow-up solutions). *Let u_0 be a blow-up solution at $(0, 0) \in \partial\{u > 0\}$. Then, there exist universal constants $\mathfrak{c}_0, \mathfrak{C}_0 > 0$ such that*

$$\begin{aligned} \mathfrak{c}_0 &\le \liminf_{\mathrm{dist}_p((x,t),(0,0))\to\infty} \frac{u_0(x,t)}{\mathrm{dist}_p((x,t),(0,0))^{\frac{2}{1-\mu}}} \\ &\le \limsup_{\mathrm{dist}_p((x,t),(0,0))\to\infty} \frac{u_0(x,t)}{\mathrm{dist}_p((x,t),(0,0))^{\frac{2}{1-\mu}}} \le \mathfrak{C}_0. \end{aligned} \tag{5-2}$$

Proof. Such a lower and upper control at infinity are a consequence of Theorem 1.1 and Theorem 1.2, respectively. ☐

Remark 5.3. Note that (5-2) implies that blow-up solutions are nontrivial. Moreover, (5-2) says that blow-up solutions satisfy

$$c_0 \cdot r^{\frac{2}{1-\mu}} \le \mathscr{S}_r[u_0] \le C_0 \cdot r^{\frac{2}{1-\mu}},$$

for values of r large enough.

Remark 5.4. In view of Theorem 5.2, the nontrivial space-independent blow-up solution $u = u(t)$ to

$$\mathscr{F}(x_0, t_0, D^2u) - \partial_t u = \lambda_0(x_0, t_0)u_+^{\mu}(x, t) \quad \text{in } \mathbb{R}^n \times \mathbb{R}$$

is given by $u(t) = [-(1-\mu)\lambda_0(x_0, t_0)t]_+^{1/(1-\mu)}$. On the other hand, when $\mathscr{F}(\cdot) = \mathrm{Tr}(\cdot)$, nontrivial time-independent blow-up solutions $u = u(x)$ are of the form

$$u(x) = \{C_{n,\mu}\cdot(x_i)_+^{\frac{2}{1-\mu}}, C_{n,\mu}\cdot(x_i)_-^{\frac{2}{1-\mu}}, C_{n,\mu}\cdot(|x-x_0|-\mathfrak{R}_0)_+^{\frac{2}{1-\mu}}\},$$

for any $i = 1, \dots, n$, where

$$C_{n,\mu} := \left(\frac{\lambda_0(1-\mu)^2}{2(1+\mu)}\right)^{\frac{1}{1-\mu}}.$$

Notice that the first blow-up type solutions are half-space solutions and the last one is a radial solution with dead core being precisely $B_{\mathfrak{R}_0}(x_0)$; see [da Silva et al. 2017, Section 6] for an analysis about radial solutions of fully nonlinear elliptic dead core problems. Finally, another interesting example of blow-up solutions are those with independent variables, i.e., $u(x, t) = v(x) + w(t)$.

5B. ***A Liouville-type theorem.*** In this section we are concerned with proving a Liouville-type result for global dead-core solutions. In summary, we show that a global viscosity solution must grow faster than $(|x|+|t|^{1/2})^{2/(1-\mu)}$ as $|x|+|t|^{1/2}\to\infty$, unless it is identically zero. The proof will be based on [Teixeira 2016, Theorem 8].

Theorem 5.5. *Let u be an entire viscosity solution to*

$$\mathscr{F}(x,t,D^2u)-\partial_t u(x,t)=\lambda_0(x,t)\cdot u_+^{\mu}(x,t)$$

with $u(0,0)=0$. If $u(x,t)=\mathrm{o}(\mathrm{dist}_p((x,t),(0,0))^{\frac{2}{1-\mu}})$ as $\mathrm{dist}_p((x,t),(0,0))\to\infty$, then $u\equiv 0$.

Proof. For each positive number $r\gg 1$, let us define

$$u_r(x,t):=\frac{u(r\,x,r^2t)}{r^{\frac{2}{1-\mu}}}.$$

Thus, it is easy to check that

$$\mathscr{F}_r(x,t,D^2u_r)-\partial_t u_r=\lambda_0(rx,r^2t)(u_r)_+^{\mu}\quad\text{in }\mathscr{C}_1$$

and $u_r(0,0)=0$, where $\mathscr{F}_r(x,t,\mathscr{X}):=r^{-2\mu/(1-\mu)}\mathscr{F}(rx,r^2t,r^{2\mu/(1-\mu)}\mathscr{X})$. Moreover, we note that $\|u_r\|_{L^\infty(\mathscr{C}_1)}=\mathrm{o}(1)$. In fact, for each $r\in\mathbb{N}$, let $(x_r,t_r)\in\mathbb{R}^n\times\mathbb{R}$ be such that

$$u_r(x_r,t_r)=\sup_{\mathscr{C}_1}u_r(x,t).$$

We must consider two possibilities:

(1) If $\lim\limits_{r\to\infty}\mathrm{dist}_p((rx_r,r^2t_r),(0,0))=\infty$, we get

$$\begin{aligned}u_r(x_r,t_r)&=\mathrm{dist}_p((rx_r,r^2t_r),(0,0))^{-\frac{2}{1-\mu}}u(rx_r,r^2t_r)\mathrm{dist}_p((rx_r,r^2t_r),(0,0))^{\frac{2}{1-\mu}}\\&\le C(n,\lambda,\Lambda,\mu)\cdot o(1)\to 0\quad\text{as }r\to\infty.\end{aligned}$$

(2) On the other hand, if $\lim\limits_{r\to\infty}\mathrm{dist}_p((rx_r,r^2t_r),(0,0))<\infty$, the conclusion is immediate, because u is a continuous function.

Therefore, applying Theorem 1.1 we have

$$u_r(x,t)\le\mathrm{o}(1)\cdot\mathrm{dist}_p((x,t),(0,0))^{\frac{2}{1-\mu}}\quad\text{in }\mathscr{C}_{\frac{1}{2}}.\tag{5-3}$$

Now, if we assume that there exists $(\hat{x},\hat{t})\in(\mathbb{R}^n\times\mathbb{R})\setminus\{(0,0)\}$ such that $u(\hat{x},\hat{t})>0$, we deduce from (5-3) that

$$\sup_{\mathscr{C}_{\frac{1}{2}}}\frac{u_r(x,t)}{\mathrm{dist}_p((x,t),(0,0))^{\frac{2}{1-\mu}}}\le\frac{u(\hat{x},\hat{t})}{100\,\mathrm{dist}_p((\hat{x},\hat{t}),(0,0))^{\frac{2}{1-\mu}}},\tag{5-4}$$

provided $r \gg 1$. We now estimate, for $r \gg \max\{2|\hat{x}|, \sqrt{2}|\hat{t}|\}$:

$$\begin{aligned}\frac{u(\hat{x},\hat{t})}{\mathrm{dist}_p((\hat{x},\hat{t}),(0,0))^{\frac{2}{1-\mu}}} &\leq \sup_{\mathscr{C}_{\frac{r}{2}}} \frac{u(x,t)}{\mathrm{dist}_p((x,t),(0,0))^{\frac{2}{1-\mu}}}\\ &\leq \sup_{\mathscr{C}_{\frac{1}{2}}} \frac{u_r(x,t)}{\mathrm{dist}_p((x,t),(0,0))^{\frac{2}{1-\mu}}}\\ &\leq \frac{u(\hat{x},\hat{t})}{100\mathrm{dist}_p((\hat{x},\hat{t}),(0,0))^{\frac{2}{1-\mu}}},\end{aligned}$$

which finally drives us to a contradiction, completing the proof of theorem. □

6. Hausdorff measure estimates

In this section, we will proceed to estimate the parabolic Hausdorff measure (i.e., the Hausdorff measure with respect to the parabolic distance) of the free boundary set of dead-core solutions u. To this end, we need first some preliminary results, which are based on the set of assumptions in Section 2 on the operator $\mathscr{F}$, together with the following additional hypothesis:

(C) (($\mathbb{M}$, $\mathfrak{b}$)-concavity) There exist a constant $\hat{\mathfrak{C}} > 0$ and a bounded symmetric positive definite Lipschitz matrix $\mathbb{M} : \mathscr{C}_1 \to \mathrm{Sym}(n)$, $\mathbb{M} = [m_{ij}]$, so that

$$\mathscr{F}(x, t, \vec{p}, \mathscr{X}) \leq \mathrm{Tr}(\mathbb{M} \cdot \mathscr{X}) + \mathfrak{b}|\vec{p}|,$$

in the viscosity sense, where $\mathfrak{b} \geq 0$ and $0 \leq \mathfrak{c}(x, t) \leq \mathfrak{c}_0^*$. We further assume that there exists a constant $\beta > 0$ such that $m_{ij} \geq \beta$ for all i, j (see [Ricarte et al. 2017, Section 5] for a similar property).

(T) (lower bound for $\partial_t u$) There is a constant $\mathfrak{c}_0 \geq 0$ such that $\partial_t u \geq -\mathfrak{c}_0 u_+^{\mu}$ in the viscosity sense.

Before discussing the main result of this section, let us present some useful notions and preliminary results.

Definition 6.1. Let $\mathscr{A}$ be a subset of a parabolic domain $\mathscr{C}$. We say that $\mathscr{A}$ has the (δ, ζ)-density property if there is $\delta \in (0, 1)$ so that there corresponds $\zeta > 0$ with the property

$$\frac{\mathscr{L}^{n+1}(\mathscr{C}_\delta(x,t) \cap \mathscr{A})}{\mathscr{L}^{n+1}(\mathscr{C}_\delta(x,t))} \geq \zeta, \tag{6-1}$$

for all $(x, t) \in \partial A \cap \mathscr{C}$. If (6-1) holds for all δ in (0, 1), then we say that $\mathscr{A}$ has uniform density in $\mathscr{C}$ along $\partial\mathscr{A}$.

As a consequence of the above definition, we derive the following facts which will be used in our proof of Hausdorff estimates for the free boundary.

Proposition 6.2. *Let $\mathscr{A} \subset \mathscr{C}$ be open. Then*:

- *If $\mathscr{A}$ has the (δ, ζ)-density property, there is a constant $M = M(n)$ such that*
$$\mathscr{L}^{n+1}(\mathscr{N}_\delta(\partial\mathscr{A}) \cap \mathscr{C}_\rho(x_0, t_0)) \le \frac{M(n)}{\zeta} \mathscr{L}^{n+1}(\mathscr{N}_\delta(\partial\mathscr{A}) \cap \mathscr{C}_\rho(x_0, t_0) \cap \mathscr{A})$$
for $(x_0, t_0) \in \partial\mathscr{A} \cap \mathscr{C}$ and $\delta \ll \rho$.
- *If $\mathscr{A}$ has uniform density in $\mathscr{C}$ along $\partial\mathscr{A}$, then $\mathscr{L}^{n+1}(\partial\mathscr{A} \cap \mathscr{C}) = 0$.*

Proof. We prove the first part. Let $(x_0, t_0) \in \partial\mathscr{A} \cap \mathscr{C}$ and $\delta < \rho$. Consider a covering of $\mathscr{N}_\delta(\partial\mathscr{A}) \cap \mathscr{C}_\rho(x_0, t_0)$ by cylinders $\mathscr{C}_\delta(x, t)$ centered at points on $\partial\mathscr{A} \cap \mathscr{C}_\rho(x_0, t_0)$. By [Lieberman 1996, Lemma 7.8], we may extract a countable and disjoint subfamily of cylinders $\{\mathscr{C}_\delta(x_i, t_i)\}_i$ so that $\{\mathscr{C}_{5\delta}(x_i, t_i)\}_i$ covers $\mathscr{N}_\delta(\partial\mathscr{A}) \cap \mathscr{C}_\rho(x_0, t_0)$. Hence:

$$\begin{aligned}\mathscr{L}^{n+1}(\mathscr{N}_\delta(\partial\mathscr{A}) \cap \mathscr{C}_\rho(x_0, t_0)) &\le \sum \mathscr{L}^{n+1}(\mathscr{C}_{5\delta}(x_i, t_i)) \\ &\le \frac{M(n)}{\zeta} \mathscr{L}^{n+1}(\mathscr{C}_\delta(x_i, t_i) \cap \mathscr{A}) \\ &\le \frac{M(n)}{\zeta} \mathscr{L}^{n+1}(\mathscr{N}_\delta(\partial\mathscr{A}) \cap \mathscr{C}_{2\rho}(x_0, t_0) \cap \mathscr{A}),\end{aligned}$$

where we have used the (δ, ζ)-density property of $\mathscr{A}$ and the fact that $\mathscr{C}_\delta(x_i, t_i) \subset \mathscr{N}_\delta(\partial\mathscr{A}) \cap \mathscr{C}_{2\rho}(x_0, t_0)$ □

We start with the series of preliminary results needed in the proof of Theorem 1.3. The first lemma contains an L^2 estimate on the gradient near free boundary points.

Lemma 6.3. *There exists a constant $C > 0$ such that for all $(x_0, t_0) \in \partial\{u > 0\} \cap \mathscr{C}_{1/2}$ and $\rho < \frac{1}{4}$, the following holds*:

$$\int\limits_{\mathscr{C}_\rho(x_0,t_0) \cap \{0 < u < \epsilon^{\frac{2}{1-\mu}}\}} |\nabla u|^2 \le C\epsilon\rho^{n+1}.$$

Proof. Define

$$\Phi(x, t) := u(x, t)\chi_{\{0 < u \le \epsilon^{\frac{2}{1-\mu}}\}}(x, t) + \epsilon\chi_{\{u > \epsilon^{\frac{2}{1-\mu}}\}}(x, t). \tag{6-2}$$

Integration by parts gives

$$\begin{aligned}&\int\limits_{\mathscr{C}_\rho(x_0,t_0)} \Phi \cdot m_{ij} D_{ij} u \\ &\quad = \int_{t_0-\rho^2}^{t_0+\rho^2} \Bigg[\frac{1}{\rho} \int\limits_{\partial B_\rho(x_0)} \Phi \cdot m_{ij} \cdot D_j u \cdot (x^i - x_0^i)\, d\mathscr{H}^{n-1} - \int\limits_{B_\rho(x_0)} D_i(\Phi \cdot m_{ij}) D_j u\, dx\Bigg] dt\end{aligned}$$

In view of the assumptions **(C)** and **(T)**, nondegeneracy on average for dead core solutions (Corollary 4.7), the growth rate on gradient, as well as that $\frac{2\mu}{1-\mu} \le \frac{1+\mu}{1-\mu}$ for any $0 < \mu < 1$, we conclude

$$\begin{aligned}\int_{\mathscr{C}_\rho(x_0,t_0)} \operatorname{Tr}(\mathbb{M}D^2u) &\ge \int_{\mathscr{C}_\rho(x_0,t_0)} (\mathfrak{m}-\mathfrak{c}_0)u_+^\mu(x,t) - \mathfrak{b}|Du| \\ &\ge \omega_n\rho^{n+2}[\mathfrak{C}_*(\mathfrak{m}-\mathfrak{c}_0)\rho^{\frac{2\mu}{1-\mu}} - \mathfrak{b}\mathfrak{C}_1\rho^{\frac{1+\mu}{1-\mu}}] \\ &\ge \omega_n\rho^{n+2}\rho^{\frac{2\mu}{1-\mu}}[\mathfrak{C}_*(\mathfrak{m}-\mathfrak{c}_0) - \mathfrak{b}\mathfrak{C}_1] \ge 0.\end{aligned}$$

In particular, we derive

$$\int_{\mathscr{C}_\rho(x_0,t_0)} \Phi\cdot m_{ij}D_{ij}u \ge 0.$$

Hence

$$\begin{aligned}\int_{\mathscr{C}_\rho(x_0,t_0)\cap\{0<u<\epsilon^{\frac{2}{1-\mu}}\}} m_{ij}D_iu\cdot D_ju \le \frac{1}{\rho}\int_{t_0-\rho^2}^{t_0+\rho^2}\int_{\partial B_\rho(x_0)} \Phi\cdot m_{ij}\cdot D_ju\cdot(x^i-x_0^i)\,d\mathscr{H}^{n-1} \\ -\int_{\mathscr{C}_\rho(x_0,t_0)\cap\{0<u<\epsilon^{\frac{2}{1-\mu}}\}} \Phi D_im_{ij}\cdot D_ju.\end{aligned}$$

Using regularity of Du and that $0 < \rho \ll 1$, we conclude the proof of the lemma. □

The above gradient estimate may be applied to get bounds on the Lebesgue measure of the set $\{0 < u < \epsilon^{2/(1-\mu)}\}$ near the free boundary, in terms of the upper bound ϵ. Precisely, we have the next lemma.

Lemma 6.4. *There exists a constant $C > 0$ such that for any $\epsilon > 0$ small enough, any $(x_0, t_0) \in \partial\{u > 0\} \cap \mathscr{C}_{1/2}$ and any ρ small, the estimate*

$$\mathscr{L}^{n+1}(\mathscr{C}_\rho(x_0,t_0)\cap\{0<u<\epsilon^{\frac{2}{1-\mu}}\}) \le C\rho^{n+1}\epsilon,$$

holds.

Proof. From a Vitali covering theorem for parabolic cylinders, see [Lieberman 1996, Lemma 7.8], consider $\{\mathscr{C}_j\}$, a finite covering by cylinders of $\partial\{u > 0\} \cap \mathscr{C}_\rho(x_0, t_0)$, with center at $(x_j, t_j) \in \partial\{u > 0\}$ and radius $C^*\epsilon$, for a constant $C^* > 0$ to be determined a posteriori. Moreover, we require that

$$\bigcup_j \mathscr{C}_j \subset \mathscr{N}_{\frac{1}{4}}(\mathscr{C}_{\frac{1}{2}}) \cap \mathscr{C}_\rho(x_0,t_0)$$

Observe that, from the Heine–Borel lemma there exists a constant $\mathfrak{l} > 0$ (with dimensional dependence) such that

$$\sum_j \chi_{\mathscr{C}_j}(x,t) \le \mathfrak{l}. \tag{6-3}$$

We first prove the estimate

$$\int_{\{0<u<\epsilon^{\frac{2}{1-\mu}}\}\cap\mathscr{C}_j} |\nabla u|^2 \geq C\mathscr{L}^{n+1}(\mathscr{C}_j), \tag{6-4}$$

for some $C > 0$ and C^* to be chosen large enough. Indeed, in view of the nondegeneracy property, there is $(x_j^1, t_j^1) \in \frac{1}{4}\mathscr{C}_j$ such that

$$u(x_j^1, t_j^1) = \sup_{\frac{1}{4}\mathscr{C}_j} u \geq \mathfrak{C}_0^* \cdot \left(\frac{1}{4}C^*\epsilon\right)^{\frac{2}{1-\mu}}.$$

Next, choose $C^* > 0$ large enough so that

$$K := \sup_{\mathscr{N}_{\frac{1}{4}}\left(\mathscr{C}_{\frac{1}{2}}\right)} |\nabla u| \geq \frac{1}{C^*} \qquad \text{and} \qquad \mathfrak{C}_0^* \cdot (C^*)^{\frac{2}{1-\mu}} > 4^{\frac{2}{1-\mu}}.$$

Next, we choose $\epsilon > 0$ small enough so that if $r_j^1 = \frac{\epsilon}{K}$ and $r_j^2 = \frac{\epsilon}{K}$, then

$$\Phi \geq \frac{3\epsilon}{4} \quad \text{in } \mathscr{C}_j^1 := \mathscr{C}_{r_j^1}(x_j^1, t_j^1) \tag{6-5}$$

and

$$\Phi < \frac{2\epsilon}{3} < \epsilon \quad \text{in } \mathscr{C}_j^2 := \mathscr{C}_{r_j^2}(x_j, t_j), \tag{6-6}$$

where Φ is defined in (6-2). We claim that if $m_j := \fint_{\mathscr{C}_j} \Phi$, then $|\Phi - m_j| > \varsigma\epsilon$ for some $\varsigma > 0$ and for at least one of the cylinders $\mathscr{C}_j^1$ and $\mathscr{C}_j^2$. In fact, if this is not the case, then we can find sequences $(x_k, t_k) \in \mathscr{C}_j^1$ and $(y_k, s_k) \in \mathscr{C}_j^2$ such that

$$\frac{|\Phi(x_k, t_k) - m_j|}{\epsilon} < \frac{1}{k} \quad \text{and} \quad \frac{|\Phi(y_k, s_k) - m_j|}{\epsilon} < \frac{1}{k}.$$

Letting $k \to \infty$, we obtain

$$\frac{|\Phi(x_k, t_k) - \Phi(y_k, s_k)|}{\epsilon} \to 0.$$

This contradicts (6-5) and (6-6). Thus, by Poincaré inequality, we have

$$\varsigma^2\epsilon^2 \leq \fint_{\mathscr{C}_j} |\Phi - m_j| \leq (C^*\epsilon)^2 \fint_{\mathscr{C}_j} |\nabla\Phi|^2$$

and hence, for a universal constant $C_2 > 0$, we conclude that

$$\int_{\{0<u<\epsilon\}\cap\mathscr{C}_j} |\nabla u|^2 \geq C_2\mathscr{L}^{n+1}(\mathscr{C}_j).$$

Moreover, by Corollary 4.6, if $(y,s) \in \{0 < u < \epsilon^{2/(1-\mu)}\} \cap \mathscr{C}_\rho(x_0,t_0)$, then

$$\mathfrak{C}_0^* \mathrm{dist}((y,s), \partial\{u>0\})^{\frac{2}{1-\mu}} \le u(y,s) < \epsilon^{\frac{2}{1-\mu}}.$$

Thus,

$$\{0 < u < \epsilon^{\frac{2}{1-\mu}}\} \cap \mathscr{C}_\rho(x_0,t_0) \subset \mathscr{N}_{\left(\frac{\epsilon}{\mathfrak{C}_0^*}\right)^{\frac{1-\mu}{2}}}(\partial\{u>0\} \cap \mathscr{C}_{2\rho}(x_0,t_0)).$$

Therefore, by enlarging C^* and diminishing ϵ if necessary, we conclude that

$$\{0 < u < \epsilon^{\frac{2}{1-\mu}}\} \cap \mathscr{C}_\rho(x_0,t_0) \subset \bigcup_j 2\mathscr{C}_j \subset \mathscr{C}_{4\rho}.$$

Appealing to Lemma 6.3 and the estimate (6-4), we conclude

$$\begin{aligned} C\epsilon\rho^{n+1} \ge \int_{\{0<u<\epsilon^{2/(1-\mu)}\}\cap\mathscr{C}_j} |\nabla u|^2 &\ge \frac{1}{\mathfrak{l}} \sum_j \int_{2\mathscr{C}_j \cap \{0<u<\epsilon^{2/(1-\mu)}\}} |\nabla u|^2 \\ &\ge \frac{C_2}{\mathfrak{l}} \sum_j \mathscr{L}^{n+1}(\mathscr{C}_j) \\ &\ge \frac{C_2}{\mathfrak{l}} \mathscr{L}^{n+1}(\{0 < u < \epsilon^{\frac{2}{1-\mu}}\} \cap \mathscr{C}_\rho(x_0,t_0)). \qquad \square \end{aligned}$$

Theorem 6.5. *There exists a constant* $C > 0$ *such that*

$$\mathscr{L}^{n+1}(\mathscr{N}_\epsilon(\{u>0\} \cap \mathscr{C}_\rho(x_0,t_0))) \le C\epsilon\rho^{n+1},$$

for $(x_0,t_0) \in \partial\{u>0\} \cap \mathscr{C}_{1/2}$.

Proof. First, observe that

$$[\mathscr{N}_\delta(\partial\{u>0\}) \cap \mathscr{C}_\rho(x_0,t_0) \cap \{u>0\}] \subset [\{0 < u < \mathfrak{C}\delta^{\frac{2}{1-\mu}}\} \cap \mathscr{C}_\rho(x_0,t_0)] \tag{6-7}$$

for $\mathfrak{C} > 0$ as in Theorem 1.1. Indeed, if $(x,t) \in \mathscr{N}_\delta(\partial\{u>0\}) \cap \mathscr{C}_\rho(x_0,t_0) \cap \{u>0\}$ and $(y,s) \in \partial\{u>0\}$, then

$$u(x,t) \le \mathfrak{C}(|x-y| + |t-s|^{\frac{1}{2}})^{\frac{2}{1-\mu}} \le \mathfrak{C}\delta^{\frac{2}{1-\mu}}.$$

By the uniform positive Lebesgue density of the positive set of u (see Corollary 4.8), we have that there exists a constant ζ so that

$$\frac{\mathscr{L}^{n+1}(\mathscr{C}_\delta(x_0,t_0)) \cap \{u>0\}}{\mathscr{L}^{n+1}(\mathscr{C}_\delta(x_0,t_0))} \ge \zeta.$$

Hence, the set $\{u>0\}$ has the (δ,ζ)-density property, and then by Proposition 6.2, there is a constant $M > 0$

$$\begin{aligned} &\mathscr{L}^{n+1}(\mathscr{N}_\delta(\partial\{u>0\}) \cap \mathscr{C}_\rho(x_0,t_0)) \\ &\qquad \le C_2 \mathscr{L}^{n+1}(\mathscr{N}_\delta(\partial\{u>0\}) \cap \mathscr{C}_\rho(x_0,t_0) \cap \{u>0\}) + M\delta\rho^{n+1}. \end{aligned}$$

Thus, by appealing to (6-7), we derive

$$\mathscr{L}^{n+1}(\mathscr{N}_\delta(\partial\{u>0\})\cap\mathscr{C}_\rho(x_0,t_0))\le C_2\mathscr{L}^{n+1}(\{0<u<\mathfrak{C}\delta^{\frac{2}{1-\mu}}\}\cap\mathscr{C}_\rho(x_0,t_0))+M\delta\rho^{n+1}.$$

From Lemma 6.4, we get for δ small enough that

$$\mathscr{L}^{n+1}(\mathscr{N}_\delta(\partial\{u>0\}\cap\mathscr{C}_\rho(x_0,t_0)))\le C\delta\rho^{n+1},$$

for some universal $C>0$. □

Remark 6.6. It will be useful to introduce the notion of *parabolic Hausdorff dimension* for a set $\Sigma_0\subseteq\mathbb{R}^{n+1}$.

$$\mathscr{H}_{\text{par}}(\Sigma_0):=\inf\Bigg\{0\le s<\infty:\text{for all }\gamma>0\text{ there exists }\{\mathscr{C}_{r_j}(x_j,t_j)\}_{j\ge1}\\ \text{such that }\Sigma_0\subseteq\bigcup_{j\ge1}\mathscr{C}_{r_j}(x_j,t_j)\text{ and }\sum_{j\ge1}r_j^s<\gamma\Bigg\}.$$

We will finish this section with the proof of the Hausdorff estimate for the free boundary.

Proof of Theorem 1.3. Let $0<\delta<\rho<\frac14$, and consider a covering $\mathscr{C}_j$ by cylinders of the set $\partial\{u>0\}\cap\mathscr{C}_\rho(x_0,t_0)$ centered at points in $\partial\{u>0\}\cap\mathscr{C}_\rho(x_0,t_0)$ and with radius δ. Hence,

$$\bigcup_j\mathscr{C}_j\subset\mathscr{N}_\delta(\partial\{u>0\})\cap\mathscr{C}_{\rho+\delta}(x_0,t_0).$$

Thus, we derive

$$\begin{aligned}\text{(6-8)}\quad \mathscr{H}^{n+1}_{\text{par},\delta}(\partial\{u>0\}\cap\mathscr{C}_\rho(x_0,t_0))&\le C\sum_j\delta^{n+1}=C\sum_j\frac{1}{\delta}\mathscr{L}^{n+1}(\mathscr{C}_j)\\&\le\frac{C}{\delta}\mathscr{L}^{n+1}(\mathscr{N}_\delta(\partial\{u>0\})\cap\mathscr{C}_{\rho+\delta}(x_0,t_0))\\&\le C(\rho+\delta)^{n+1},\end{aligned}$$

where we have used Theorem 6.5 to obtain the last inequality. Hence, the conclusion is reached by letting $\delta\to0$. □

Remark 6.7. We must highlight that the parabolic Hausdorff dimension and classical Hausdorff dimension have the relationship given by

$$2\mathscr{H}(\Sigma_0)-n\le\mathscr{H}_{\text{par}}(\Sigma_0)\le\mathscr{H}(\Sigma_0)+1.$$

Therefore, $\mathscr{H}((\partial\{u>0\}\cap\mathfrak{K}))\le n+\frac12$, for any $\mathfrak{K}\Subset\Omega_T$.

We end this section by providing a t-integral estimate on lower-dimensional Hausdorff measure of the free boundary. This is a direct consequence of the previous arguments.

Lemma 6.8. *Consider the set-valued mapping $\mathfrak{F}(s) := \partial\{u > 0\} \cap \{t = s\}$. Hence the mapping $s \to \mathscr{H}^{n-1}(\mathfrak{F}(s) \cap \mathscr{C}_\rho(x_0, t_0))$, for $(x_0, t_0) \in \{u > 0\}$, is Borel measurable in $(t_0 - \rho^2, t_0 + \rho^2)$ and the estimate*

$$\int_{t_0-\rho^2}^{t_0+\rho^2} \mathscr{H}^{n-1}(\mathscr{F}(s) \cap \mathscr{C}_\rho(x_0, t_0))\, ds \leq \hat{\mathfrak{C}} \rho^{n+1}, \tag{6-9}$$

holds for a universal constant $\hat{\mathfrak{C}}$ depending only on dimension.

Proof. We first show that $\mathfrak{F}$ is upper semicontinuous in $(t_0 - \rho^2, t_0 + \rho^2)$. Indeed, if $\mathscr{U}$ is an open set containing $\mathfrak{F}(s_0) \cap \mathfrak{K}$, $\mathfrak{K} \subset \mathscr{C}_{1/2}$ compact, then $\mathfrak{F}(s) \cap \mathfrak{K} \subset \mathscr{U}$ for all s sufficiently close to s_0. In fact, if this is not the case, then we can build up a sequence $(y_k, s_k) \in \mathfrak{F}(s_k) \cap (\mathfrak{K} \setminus \mathscr{U})$ with $s_k \to s_0$. Passing to a subsequence, we have $(y_k, s_k) \to (y_0, s_0) \in \{u > 0\} \cap (\mathfrak{K} \setminus \mathscr{U})$. This contradicts that $\mathfrak{F}(s_0) \subset \mathscr{U}$. Therefore, the mappings $s \to \mathscr{H}_\delta^{n-1}(\mathfrak{F}(s) \cap \overline{\mathscr{C}_\rho(x_0, t_0)})$ and $s \to \mathscr{H}^{n-1}(\mathfrak{F}(s) \cap \overline{\mathscr{C}_\rho(x_0, t_0)})$ are Borel measurable.

Consider now the covering by cylinders from Lemma 6.4. Hence by (6-8), we obtain

$$\begin{aligned}
\mathfrak{C} \cdot (\rho + \delta)^{n+1} &\geq \frac{1}{\delta} \int_{\bigcup_j \mathscr{C}_j(x_j, t_j)} d\mathscr{L}^{n+1} \\
&= \frac{1}{\delta} \int_{-(\delta+\rho)^2}^{(\delta+\rho)^2} \mathscr{L}^n \Big(\bigcup_j \mathscr{C}_j(x_j, t_j) \cap \{t = s\} \Big)\, ds \qquad \text{(by coarea formula)} \\
&\geq \frac{\omega_n}{\mathfrak{l}} \int_{-(\delta+\rho)^2}^{(\delta+\rho)^2} \delta^{n-1} \sum_j \chi_{\{\mathscr{C}_j(x_j,t_j) \cap \{t=s\}\}}\, ds \qquad (\mathfrak{l} \text{ as in (6-3)}) \\
&\geq \frac{\omega_n}{\mathfrak{l}\omega_{n-1}} \int_{-(\delta+\rho)^2}^{(\delta+\rho)^2} \mathscr{H}_\delta^{n-1}(\mathfrak{F}(s) \cap \mathscr{C}_\rho(x_0, t_0))\, ds.
\end{aligned}$$

By letting $\delta \to 0$, we derive the estimate (6-9). □

Remark 6.9. We highlight that Lemma 6.8 implies particularly, from geometric measure theory results (see [Federer 1969, Theorems 4.5.6 and 4.5.11]), that $\chi_{\{u>0\}}(\cdot, t)$ is a function of bounded variation for almost every $t \in (0, T)$. Moreover, for any $\varphi \in C^{0,1}(\Omega : \mathbb{R}^n)$ it follows that

$$\int_{\{u(\cdot,t)>0\}} \operatorname{div}\varphi\, dx = \int_{\partial_{\text{red}}\{u(\cdot,t)>0\}} \varphi \cdot \vec{\nu}\, d\mathscr{H}^{n-1} \quad \text{for almost every } t \in (0, T),$$

where $\vec{\nu}$ is the normal vector in the measure theoretic sense. Nevertheless, such a previous sentence does not yield any additional information on the singular set of free boundary, because neither the $\chi_{\{u>0\}}(\cdot, t)$ nor the reduced boundary $\partial_{\text{red}}\{u(\cdot, t) > 0\}$ detect the singular set.

Acknowledgments

The authors would like to thank Eduardo V. Teixeira for the comments and suggestions which improved the outcome of this article. The authors thank the anonymous referee for valuable comments. This work has been partially supported by Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET-Argentina) PIP GI No. 11220150100036CO. P. Ochoa has been partially supported by grant PICT 2015-1701 and SECTyP B051. J. V. da Silva would like to thank the Research Group on Partial Differential Equations, Dept. of Math and FCEyN from Universidad de Buenos Aires for fostering a pleasant and productive scientific atmosphere. The authors are members of CONICET.

References

[Antontsev et al. 2002] S. N. Antontsev, J. I. Díaz, and S. Shmarev, *Energy methods for free boundary problems*, Progr. Nonlinear Diff. Equations Appl. **48**, Birkhäuser, Boston, 2002. MR Zbl

[Apushkinskaya et al. 2002] D. E. Apushkinskaya, N. N. Ural'tseva, and H. Shahgholian, "On global solutions of a parabolic obstacle problem", *Algebra i Analiz* **14**:1 (2002), 3–25. In Russian; translated in *St. Petersburg Math. J.* **14**:1 (2003), 1–17. MR

[Bandle and Stakgold 1984] C. Bandle and I. Stakgold, "The formation of the dead core in parabolic reaction-diffusion problems", *Trans. Amer. Math. Soc.* **286**:1 (1984), 275–293. MR Zbl

[Blanchet et al. 2005] A. Blanchet, J. Dolbeault, and R. Monneau, "On the one-dimensional parabolic obstacle problem with variable coefficients", pp. 59–66 in *Elliptic and parabolic problems* (Gaeta, Italy, 2004), edited by C. Bandle et al., Progr. Nonlinear Diff. Equations Appl. **63**, Birkhäuser, Basel, 2005. MR Zbl

[Blanchet et al. 2006] A. Blanchet, J. Dolbeault, and R. Monneau, "On the continuity of the time derivative of the solution to the parabolic obstacle problem with variable coefficients", *J. Math. Pures Appl.* (9) **85**:3 (2006), 371–414. Correction in **94**:4 (2010), 447–449. MR Zbl

[Caffarelli et al. 2004] L. Caffarelli, A. Petrosyan, and H. Shahgholian, "Regularity of a free boundary in parabolic potential theory", *J. Amer. Math. Soc.* **17**:4 (2004), 827–869. MR Zbl

[Choe and Weiss 2003] H. J. Choe and G. S. Weiss, "A semilinear parabolic equation with free boundary", *Indiana Univ. Math. J.* **52**:1 (2003), 19–50. MR Zbl

[Crandall et al. 1992] M. G. Crandall, H. Ishii, and P.-L. Lions, "User's guide to viscosity solutions of second order partial differential equations", *Bull. Amer. Math. Soc.* (*N.S.*) **27**:1 (1992), 1–67. MR Zbl

[Crandall et al. 2000] M. G. Crandall, M. Kocan, and A. Święch, "L^p-theory for fully nonlinear uniformly parabolic equations", *Comm. Partial Diff. Equations* **25**:11-12 (2000), 1997–2053. MR Zbl

[Da Lio 2004] F. Da Lio, "Remarks on the strong maximum principle for viscosity solutions to fully nonlinear parabolic equations", *Commun. Pure Appl. Anal.* **3**:3 (2004), 395–415. MR Zbl

[Díaz 2001] J. I. Díaz, "Qualitative study of nonlinear parabolic equations: an introduction", *Extracta Math.* **16**:3 (2001), 303–341. MR Zbl

[Federer 1969] H. Federer, *Geometric measure theory*, Grundlehren der Math. Wissenschaften **153**, Springer, 1969. MR Zbl

[Guo and Souplet 2005] J.-S. Guo and P. Souplet, "Fast rate of formation of dead-core for the heat equation with strong absorption and applications to fast blow-up", *Math. Ann.* **331**:3 (2005), 651–667. MR Zbl

[Krylov 1983] N. V. Krylov, "Boundedly nonhomogeneous elliptic and parabolic equations in a domain", *Izv. Akad. Nauk SSSR Ser. Mat.* **47**:1 (1983), 75–108. In Russian; translated in *Math. USSR-Izv.* **22**:1 (1984), 67–97. MR Zbl

[Krylov and Safonov 1980] N. V. Krylov and M. V. Safonov, "A certain property of the solutions of parabolic equations with measurable coefficients", *Izv. Akad. Nauk SSSR Ser. Mat.* **44**:1 (1980), 161–175. In Russian; translated in *Math. USSR-Izv.* **16**:1 (1981), 151–164. MR

[Lieberman 1996] G. M. Lieberman, *Second order parabolic differential equations*, World Sci., River Edge, NJ, 1996. MR Zbl

[Petrosyan and Shahgholian 2007] A. Petrosyan and H. Shahgholian, "Parabolic obstacle problems applied to finance", pp. 117–133 in *Recent developments in nonlinear partial differential equations* (West Lafayette, IN, 2004), edited by D. Danielli, Contemp. Math. **439**, Amer. Math. Soc., Providence, RI, 2007. MR Zbl

[Ricarte et al. 2017] G. C. Ricarte, J. V. da Silva, and R. Teymurazyan, "Cavity type problems ruled by infinity Laplacian operator", *J. Diff. Equations* **262**:3 (2017), 2135–2157. MR Zbl

[Shahgholian 2003] H. Shahgholian, "Analysis of the free boundary for the p-parabolic variational problem ($p \geq 2$)", *Rev. Mat. Iberoamericana* **19**:3 (2003), 797–812. MR Zbl

[Shahgholian 2008] H. Shahgholian, "Free boundary regularity close to initial state for parabolic obstacle problem", *Trans. Amer. Math. Soc.* **360**:4 (2008), 2077–2087. MR Zbl

[Sheng and Wang 2010] W. Sheng and X.-J. Wang, "Regularity and singularity in the mean curvature flow", pp. 399–436 in *Trends in partial differential equations* (Hangzhou, 2008), edited by B. Bian et al., Adv. Lect. Math. **10**, Int. Press, Somerville, MA, 2010. MR Zbl

[da Silva and dos Prazeres 2019] J. V. da Silva and D. dos Prazeres, "Schauder type estimates for "flat" viscosity solutions to non-convex fully nonlinear parabolic equations and applications", *Potential Anal.* **50**:2 (2019), 149–170. MR

[da Silva and Teixeira 2017] J. V. da Silva and E. V. Teixeira, "Sharp regularity estimates for second order fully nonlinear parabolic equations", *Math. Ann.* **369**:3-4 (2017), 1623–1648. MR Zbl

[da Silva et al. 2017] J. V. da Silva, R. A. Leitão, and G. Ricarte, "Geometric regularity estimates for fully nonlinear elliptic equations with free boundaries", submitted, 2017, Available at https://www.researchgate.net/publication/316283716/.

[da Silva et al. 2018] J. V. da Silva, P. Ochoa, and A. Silva, "Regularity for degenerate evolution equations with strong absorption", *J. Diff. Equations* **264**:12 (2018), 7270–7293. MR Zbl

[Stakgold 1986] I. Stakgold, "Reaction-diffusion problems in chemical engineering", pp. 119–152 in *Nonlinear diffusion problems* (Montecatini Terme, Italy, 1985), edited by A. Fasano and M. Primicerio, Lecture Notes in Math. **1224**, Springer, 1986. MR Zbl

[Teixeira 2016] E. V. Teixeira, "Regularity for the fully nonlinear dead-core problem", *Math. Ann.* **364**:3-4 (2016), 1121–1134. MR Zbl

[Tian and Wang 2013] G. Tian and X.-J. Wang, "A priori estimates for fully nonlinear parabolic equations", *Int. Math. Res. Not.* **2013**:17 (2013), 3857–3877. MR Zbl

[Wang 1992a] L. Wang, "On the regularity theory of fully nonlinear parabolic equations, I", *Comm. Pure Appl. Math.* **45**:1 (1992), 27–76. MR Zbl

[Wang 1992b] L. Wang, "On the regularity theory of fully nonlinear parabolic equations, II", *Comm. Pure Appl. Math.* **45**:2 (1992), 141–178. MR Zbl

Received October 30, 2017. Revised May 14, 2018.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 300, No. 1, 2019
dx.doi.org/10.2140/pjm.2019.300.215

HOMOTOPY DECOMPOSITIONS OF THE CLASSIFYING SPACES OF POINTED GAUGE GROUPS

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Let G be a topological group and let $\mathcal{G}^*(P)$ be the pointed gauge group of a principal G-bundle $P \to M$. We prove that if G is homotopy commutative then the homotopy type of the classifying space $B\mathcal{G}^*(P)$ can be completely determined for certain M. This also works p-locally, and valid choices of M include closed simply connected four-manifolds when localised at an odd prime p. In this case, an application is to calculate part of the mod-p homology of the classifying space of the full gauge group.

1. Introduction

Let G be a topological group and let M be a pointed space. Let $P \to M$ be a principal G-bundle over M. The *gauge group* $\mathcal{G}(P)$ is the group of G-equivariant automorphisms of P that fix M. The *pointed gauge group* $\mathcal{G}^*(P)$ is the subgroup of $\mathcal{G}(P)$ that fixes the fibre over the basepoint in M. Gauge groups are of wide interest due to their prominent role in both mathematical physics, Donaldson theory, and the study of semistable holomorphic vector bundles and their related moduli spaces. Important problems are to calculate the mod-p homology and cohomology of the classifying spaces $B\mathcal{G}(P)$ and $B\mathcal{G}^*(P)$ for a prime p when M is a closed simply connected four-manifold, and to determine the integral homotopy types of various spaces related to $B\mathcal{G}^*(P)$ when M is an orientable closed Riemann surface.

In this paper, assume that the topological groups have the homotopy type of connected, finite type CW-complexes. We show that if G is homotopy commutative then for certain spaces M there is a homotopy decomposition of $B\mathcal{G}^*(P)$ as recognisable factors. This also works p-locally. Two applications are given. The first is in the case when G is a simply connected, simple compact Lie group and M is a closed simply connected four-manifold. For appropriate primes p, a p-local homotopy decomposition of $B\mathcal{G}^*(P)$ holds and this is used to determine a large split subalgebra of the mod-p cohomology of the full gauge group $B\mathcal{G}(P)$.

The author would like to thank the referee for making many valuable comments that helped improve the clarity of the paper.

MSC2010: primary 55P15, 55R35; secondary 54C35, 81T13.

Keywords: gauge group, mapping space, homotopy type, homology.

The second is in the case when G is the infinite unitary group and M is a closed orientable Riemann surface. A homotopy decomposition of $B\mathcal{G}^*(P)$ is used to determine the homotopy type of the space $\mathrm{Hom}(\pi_1(\Sigma_g), U)$ of homomorphisms from the fundamental group of the Riemann surface to the infinite unitary group.

The key result is a decomposition of certain pointed mapping spaces. Consider adjunction spaces of the form

$$N = \left(\bigvee_{i=1}^{m} \Sigma A_i\right) \cup_a e^n,$$

where $\bigvee_{i=1}^m \Sigma A_i$ is a CW-complex of dimension strictly less than n, $a\colon S^{n-1} \to \bigvee_{i=1}^m \Sigma A_i$ is the attaching map of the n-cell, and $m \geq 2$. For $1 \leq i \leq m$, let $\iota_j\colon \Sigma A_j \to \bigvee_{i=1}^m \Sigma A_i$ be the inclusion of the j-th wedge summand. Let $\mathcal{N}$ be the collection of all such adjunction spaces N with the additional property that the attaching map a factors through a map a' which is a wedge sum of some of the Whitehead products $\Sigma A_j \wedge A_k \xrightarrow{[\iota_j,\iota_k]} \bigvee_{i=1}^m \Sigma A_i$.

Observe that there is a cofibration

$$\bigvee_{i=1}^{m} \Sigma A_i \xrightarrow{b} N \xrightarrow{q} S^n,$$

where b is the inclusion and q collapses $\bigvee_{i=1}^m \Sigma A$ to a point. Let G be a topological group and let BG be its classifying space. Then the cofibration sequence induces a fibration sequence

$$\mathrm{Map}^*(N, BG) \xrightarrow{b^*} \mathrm{Map}^*\left(\bigvee_{i=1}^{m} \Sigma A_i, BG\right) \xrightarrow{a^*} \mathrm{Map}^*(S^{n-1}, BG). \tag{1}$$

Theorem 1.1. *Let $N \in \mathcal{N}$ and let G be a topological group whose multiplication is homotopy commutative. Then the map b^* in* (1) *has a right inverse and there is a homotopy equivalence*

$$\mathrm{Map}^*(N, BG) \simeq \mathrm{Map}^*\left(\bigvee_{i=1}^{m} \Sigma A_i, BG\right) \times \mathrm{Map}^*(S^n, BG).$$

A p-local version of Theorem 1.1 also holds if the multiplication on G is only homotopy commutative at p. This is particularly relevant since James and Thomas [1962a] showed that no simply connected, simple compact Lie group has its standard multiplication being homotopy commutative, but McGibbon [1984] showed that after localising at an odd prime there are cases when the multiplication is homotopy commutative and he classified these. The classification is given in Section 2.

The connection with gauge groups comes from work of Gottlieb [1972] or Atiyah and Bott [1983]. They showed that if M is a pointed space and $P \to M$ is a principal G-bundle then there is a homotopy equivalence $B\mathcal{G}^*(P) \simeq \mathrm{Map}^*{}_P(M, BG)$, where $\mathrm{Map}^*{}_P(M, BG)$ is the component of $\mathrm{Map}^*(M, BG)$ that contains the map inducing P. Consider two cases. First, let M be a closed simply connected four-manifold and let G be a simply connected simple compact Lie group. By [Milnor 1958], M is homotopy equivalent to a CW-complex $\left(\bigvee_{i=1}^m S^2\right) \cup_a e^4$. Second, let M be an orientable closed Riemann surface of genus g and let $G = U(n)$. Classically (see [Hatcher 2002] for instance), M is homotopy equivalent to a CW-complex $\left(\bigvee_{i=1}^{2g} S^1\right) \cup_a e^2$. In either case, $[M, BG] \cong \mathbb{Z}$ so there is a component of $\mathrm{Map}^*(M, BG)$ for each integer k, and this integer determines a corresponding equivalence class of principal G-bundles $P \to M$. Write P_k for the equivalence class corresponding to k and let $\mathcal{G}_k^*(M) = \mathcal{G}^*(P_k)$.

Let $\Omega_0^3 G$ be the component of $\Omega^3 G$ containing the basepoint. Write $X_{(p)}$ for a space X localised at the prime p.

Corollary 1.2. *Let M be a closed simply connected Spin four-manifold with m two-cells, $m \geq 2$, and let G be a simply connected simple compact Lie group whose multiplication is homotopy commutative when localised at p. Then there is a p-local homotopy equivalence*

$$B\mathcal{G}_k^*(M)_{(p)} \simeq \left(\prod_{i=1}^m \Omega G_{(p)}\right) \times \Omega_0^3 G_{(p)}.$$

In the second case, stabilise by considering the infinite unitary group U. Since U is an infinite loop space its loop multiplication is homotopy commutative. Write Σ_g for the surface of genus g, and let $\Omega_0 U$ be the component of ΩU containing the basepoint.

Corollary 1.3. *Let Σ_g be a closed orientable closed Riemann surface of genus $g \geq 1$. Then there is an integral homotopy equivalence*

$$B\mathcal{G}_k^*(\Sigma_g) \simeq \left(\prod_{i=1}^{2g} U\right) \times \Omega_0 U.$$

Corollaries 1.2 and 1.3 are the first systematic decompositions of the classifying spaces of pointed gauge groups. In the context of Corollary 1.2, Masbaum [1991] proved the $G = SU(2)$ case earlier but by using different methods that depended on the specific group. Also, while a great deal of work has been done recently to identify the p-local homotopy types of gauge groups [Kishimoto and Kono 2010; Kishimoto et al. 2013b; 2014; Theriault 2010] and study their properties [Kishimoto et al. 2013a], nothing has been done for their classifying spaces.

Applications of these decompositions to the mod-p homology of gauge groups and the homotopy type of $\mathrm{Hom}(\pi_1(\Sigma_g), U)$ will be discussed in the final section of the paper.

2. Preliminary homotopy theory

In this section we discuss some notions from homotopy theory involving Whitehead products and the homotopy commutativity of topological groups. As we are building towards a strictly commutative diagram in (6) rather than a homotopy commutative diagram, some extra care will be taken along the way.

Let G be a topological group and let

$$ev\colon \Sigma\Omega BG \to BG$$

be the evaluation map. Let $i_L\colon \Sigma\Omega BG \to \Sigma\Omega BG \vee \Sigma\Omega BG$ and $i_R\colon \Sigma\Omega BG \to \Sigma\Omega BG \vee \Sigma\Omega BG$ be the inclusions of the left and right wedge summands respectively and let

$$[i_L, i_R]\colon \Sigma\Omega BG \wedge \Omega BG \to \Sigma\Omega BG \vee \Sigma\Omega BG$$

be the Whitehead product of i_L and i_R. By [Arkowitz 1962] there is a homotopy equivalence

$$(\Sigma\Omega BG \vee \Sigma\Omega BG) \cup_{[i_L,i_R]} C(\Sigma\Omega BG \wedge \Omega BG) \simeq \Sigma\Omega BG \times \Sigma\Omega BG,$$

where $C(\Sigma\Omega BG \wedge \Omega BG)$ is the reduced cone on $\Sigma\Omega BG \wedge \Omega BG$. Let t be the composite

$$t\colon \Sigma\Omega BG \vee \Sigma\Omega BG \xrightarrow{ev\vee ev} BG \vee BG \xrightarrow{\nabla} BG,$$

where ∇ is the folding map and let

$$[ev, ev]\colon \Sigma\Omega BG \wedge \Omega BG \to BG$$

be the Whitehead product of ev with itself. Note that $[ev, ev]$ is homotopic to $\nabla \circ [i_L, i_R]$. The following proposition connects the homotopy commutativity of G to the existence of a certain extension.

Proposition 2.1. *Let G be a topological group. Then the following are equivalent*:

(a) *G is homotopy commutative.*

(b) *The Whitehead product $[ev, ev]$ is null homotopic.*

(c) *There is a strictly commutative diagram*

$$\begin{tikzcd} \Sigma\Omega BG \vee \Sigma\Omega BG \arrow[r, "t"] \arrow[d] & BG \\ (\Sigma\Omega BG \vee \Sigma\Omega BG) \cup_{[i_L, i_R]} C(\Sigma\Omega BG \wedge \Omega BG) \arrow[ur, "e"'] & \end{tikzcd}$$

for some map e.

Proof. The equivalence of parts (a) and (b) was proved by James and Thomas [1962b] and the equivalence of parts (b) and (c) was proved by Arkowitz [1962]. □

Remark 2.2. It should be noted that the homotopy commutativity condition in Proposition 2.1 is fairly restrictive. For example, there are no simply connected, simple compact Lie groups which are homotopy commutative [James and Thomas 1962a]. However, obstructions to homotopy commutativity may vanish when localised at a prime p (see [Hilton et al. 1975] for a good discussion of localisation). McGibbon [1984] classified those simply connected, simple compact Lie groups G which are homotopy commutative at p. To describe these, recall that G is rationally homotopy equivalent to a product of spheres, $G \simeq_{\mathbb{Q}} \prod_{i=1}^{l} S^{2n_i-1}$. The *type* of G is defined to be $\{n_1, \dots, n_l\}$. The loop multiplication on G is homotopy commutative when localised at p in precisely the following cases:

$$p > 2n_l; \quad G = Sp(2) \text{ and } p = 3; \quad G = G_2 \text{ and } p = 5. \tag{2}$$

On the other hand, Bott periodicity implies that the infinite matrix groups U, SU, SO, and Sp are all infinite loop spaces and so are integrally homotopy commutative.

Next, we generalise the (a) implies (c) part of Proposition 2.1. Let $X_1, \dots, X_m$ be path-connected, pointed spaces and consider the wedge $\bigvee_{i=1}^{m} \Sigma X_i$. For $1 \le j \le m$, let $\iota_j \colon \Sigma X_j \to \bigvee_{i=1}^{m} \Sigma X_i$ be the inclusion of the j-th wedge summand. Let

$$f \colon \bigvee_{1 \le j < k \le m} \Sigma X_j \wedge X_k \to \bigvee_{i=1}^{m} \Sigma X_i$$

be the wedge sum of the Whitehead products $[\iota_j, \iota_k]$. Let

$$T(\Sigma X_1, \dots, \Sigma X_m) = \Bigl(\bigvee_{i=1}^{m} \Sigma X_i\Bigr) \cup_f C\Bigl(\bigvee_{1 \le j < k \le m} \Sigma X_j \wedge X_k\Bigr).$$

Observe that there is a homotopy equivalence

$$T(\Sigma X_1, \dots, \Sigma X_m) \simeq \bigcup_{1 \le j < k \le m} \Sigma X_j \times \Sigma X_k.$$

To be clear, $T(\Sigma X_1, \dots, \Sigma X_m)$ is a subspace of $\Sigma X_1 \times \cdots \times \Sigma X_m$, each term $\Sigma X_j \times \Sigma X_k$ in the union is regarded as including into the (j, k) coordinates of $\Sigma X_1 \times \cdots \times \Sigma X_m$, and intersections are identified.

This construction is natural. Suppose that there are maps $g\colon \Sigma A \to Z$, $h\colon \Sigma B \to Z$, and $t\colon Z \to Z'$. Represent the homotopy class $[g, h]$ as the adjoint of the Samelson product $\langle g', h'\rangle$, where $g'\colon A \to \Omega Z$ and $h'\colon B \to \Omega Z$ are the adjoints of g and h respectively. The Samelson product is defined by the pointwise commutator in ΩZ, which commutes with any loop map $\Omega Z \xrightarrow{\Omega t} \Omega Z'$. Thus we obtain $t \circ [g, h] = [t \circ g, t \circ h]$ on the nose. Hence, given maps $f_i\colon \Sigma X_i \to \Sigma X'_i$ for $1 \le i \le m$, we obtain a strictly commutative diagram

$$
\begin{tikzcd}
\bigvee_{i=1}^m \Sigma X_i \arrow[r, "\bigvee_{i=1}^m f_i"] \arrow[d] & \bigvee_{i=1}^m \Sigma X'_i \arrow[d] \\
T(\Sigma X_1, \dots, \Sigma X_m) \arrow[r, "T(f_1, \dots, f_m)"] & T(\Sigma X'_1, \dots, \Sigma X'_m).
\end{tikzcd}
\tag{3}
$$

In our case, for $1 \le i \le m$, let $X_i = \Omega BG$. Write $T(\Sigma\Omega BG)$ for $T(\Sigma\Omega BG, \dots, \Sigma\Omega BG)$. Let t_m be the composite

$$t_m\colon \bigvee_{i=1}^{m} \Sigma\Omega BG \xrightarrow{\bigvee_{i=1}^m ev} \bigvee_{i=1}^{m} BG \xrightarrow{\nabla_m} BG,$$

where ∇_m is the m-fold folding map. By Proposition 2.1, if G is homotopy commutative then the restriction of t_m to any pair $\Sigma\Omega BG \vee \Sigma\Omega BG$ extends to a map

$$(\Sigma\Omega BG \vee \Sigma\Omega BG) \cup_{[i_L, i_R]} C(\Sigma\Omega BG \wedge \Omega BG) \to BG.$$

Construct an extension for all pairs of wedge summands indexed by (j, k) for $1 \le j < k \le m$. Observe that the extensions are compatible because they intersect only on the wedge summands. Thus they may be assembled to produce a map $T(\Sigma\Omega BG) \to BG$ extending t_m. This is recorded as follows.

Lemma 2.3. *Let G be a topological group whose loop multiplication is homotopy commutative. Then there is a strictly commutative diagram*

$$
\begin{tikzcd}
\bigvee_{i=1}^m \Sigma\Omega BG \arrow[r, "t_m"] \arrow[d] & BG \\
T(\Sigma\Omega BG) \arrow[ur, "e_m"'] &
\end{tikzcd}
$$

for some map e_m.

We close this section with one more observation about $T(\Sigma X_1, \dots, \Sigma X_m)$. Let $X \xrightarrow{E} \Omega\Sigma X$ be the suspension map, defined by sending $x \in X$ to the loop ω_x on ΣX, where ω_x is characterised by $\omega_x(t) = (t, x)$. The evaluation map $\Sigma\Omega Y \xrightarrow{ev} Y$ is defined by sending (s, ω) to $\omega(s)$. The definitions imply that the composite $\Sigma X \xrightarrow{\Sigma E} \Sigma\Omega\Sigma X \xrightarrow{ev} \Sigma X$ is the identity map on ΣX. Now suppose that there is a map $f : \Sigma X \to Y$. The naturality of the evaluation map implies that there is a strictly commutative diagram

$$\begin{tikzcd}
 & \Sigma\Omega\Sigma X \arrow[r, "\Sigma\Omega f"] \arrow[d, "ev"] & \Sigma\Omega Y \arrow[d, "ev"] \\
\Sigma X \arrow[ur, "\Sigma E"] \arrow[r, equal] & \Sigma X \arrow[r, "f"] & Y.
\end{tikzcd}$$

Thus, if $\bar{f} = (\Sigma\Omega f) \circ \Sigma E$, then we obtain a lift

$$\begin{tikzcd}
 & \Sigma\Omega Y \arrow[d, "ev"] \\
\Sigma X \arrow[ur, "\bar{f}"] \arrow[r, "f"] & Y.
\end{tikzcd} \tag{4}$$

Combining this with (3) we obtain the following:

Lemma 2.4. *Suppose that for $1 \le i \le m$ there are maps $f_i : \Sigma X_i \to Y$. Then there is a strictly commutative diagram*

$$\begin{tikzcd}
\bigvee_{i=1}^m \Sigma X_i \arrow[r, "\bigvee_{i=1}^m \bar{f}_i"] \arrow[d] & \bigvee_{i=1}^m \Sigma\Omega Y \arrow[d] \\
T(\Sigma X_1, \dots, \Sigma X_m) \arrow[r, "T(\bar{f}_1, \dots, \bar{f}_m)"] & T(\Sigma\Omega Y, \dots, \Sigma\Omega Y).
\end{tikzcd}$$

3. The class $\mathcal{N}$

Recall from Section 1 that $\mathcal{N}$ is the class of adjunction spaces

$$N = \left(\bigvee_{i=1}^m \Sigma A_i\right) \cup_a e^n,$$

where $\bigvee_{i=1}^m \Sigma A_i$ is a CW-complex of dimension strictly less than n, the attaching map a factors through a map a' which is a wedge sum of some of the Whitehead products $\Sigma A_j \wedge A_k \xrightarrow{[\iota_j, \iota_k]} \bigvee_{i=1}^m \Sigma A_i$, and $m \ge 2$. The factorisation condition on a can be restrictive. In the context of gauge groups, one typically wants to work with an N that is homotopy equivalent to a manifold. Most manifolds do not satisfy the attaching map condition. However, there are some very interesting families of manifolds that do. For example,

(a) if M is a simply connected Spin four-manifold with $H^2(M;\mathbb{Z})$ of rank $m \geq 2$, then M is homotopy equivalent to a CW-complex $\left(\bigvee_{i=1}^m S^2\right) \cup_a e^4 \in \mathcal{N}$;

(b) if Σ_g is a closed orientable surface of genus $g \geq 1$, then Σ_g is homotopy equivalent to a CW-complex $\left(\bigvee_{i=1}^{2g} S^1\right) \cup_a e^2 \in \mathcal{N}$;

(c) if M is a simply connected Spin five-manifold then M is homotopy equivalent to a CW-complex $\left(\bigvee_{i=1}^m \Sigma A_i\right) \cup_a e^5$, where each ΣA_i is either S^2, S^3, or a mod-p^r Moore space of dimension three, and if $m \geq 2$ then this CW-complex is in $\mathcal{N}$.

The CW-structure for M in (a) is due to Milnor [1958]; the CW-structure for Σ_g in (b) is commonly known, one reference is [Hatcher 2002]; the CW-structure for M in (c) is given in [Stöcker 1982]. Other examples exist, such as certain $(n-1)$-connected $2n$-dimensional manifolds [Wall 1962] and the connected sum of products of two spheres.

The property that is needed for the spaces in $\mathcal{N}$ is the following. Recall that there is a homotopy cofibration $S^{n-1} \xrightarrow{a} \bigvee_{i=1}^m \Sigma A_i \xrightarrow{b} N$, where b is the inclusion.

Lemma 3.1. *Let $N \in \mathcal{N}$. Then there is an extension*

$$\begin{tikzcd} \bigvee_{i=1}^m \Sigma A_i \arrow[r, "b"] \arrow[d] & N \arrow[dl, "e_N"] \\ T(\Sigma A_1, \dots, \Sigma A_m) & \end{tikzcd}$$

for some map e_N.

Proof. Since $N = \left(\bigvee_{i=1}^m \Sigma A_i\right) \cup_a e^n$, to show that the extension e_N exists it is equivalent to show that the composite $S^{n-1} \xrightarrow{a} \bigvee_{i=1}^m \Sigma A_i \to T(\Sigma A_1, \dots, \Sigma A_m)$ is null homotopic. By definition, $T(\Sigma A_1, \dots, \Sigma A_m)$ is the adjunction space formed from coning off the sum of all the Whitehead products $[\iota_j, \iota_k]$ for $1 \leq j < k \leq m$. In particular, each composition $\Sigma A_j \wedge A_k \xrightarrow{[\iota_j,\iota_k]} \bigvee_{i=1}^m \Sigma A_i \to T(\Sigma A_1, \dots, \Sigma A_m)$ is null homotopic. Thus, as a factors through a wedge sum of some of the Whitehead products $[\iota_j, \iota_k]$, the composite $S^{n-1} \xrightarrow{a} \bigvee_{i=1}^m \Sigma A_i \to T(\Sigma A_1, \dots, \Sigma A_m)$ is also null homotopic. □

4. A decomposition of Map*(N, BG)

Let $N \in \mathcal{N}$. In the sequence of maps

$$S^{n-1} \xrightarrow{a} \bigvee_{i=1}^m \Sigma A_i \xrightarrow{b} N \xrightarrow{q} S^n,$$

the maps a and b form a homotopy cofibre sequence, while b and q form a cofibre sequence on the nose. If G is a topological group then there is an induced sequence

$$\text{(5)}\quad \mathrm{Map}^*(S^n, BG) \xrightarrow{q^*} \mathrm{Map}^*(N, BG) \xrightarrow{b^*} \mathrm{Map}^*\Big(\bigvee_{i=1}^m \Sigma A_i, BG\Big) \xrightarrow{a^*} \mathrm{Map}^*(S^{n-1}, BG),$$

where the maps q^* and b^* form a fibre sequence on the nose while b^* and a^* form a homotopy fibre sequence. We will show that if the multiplication on G is homotopy commutative then the map b^* has a right inverse.

Let $f\colon \bigvee_{i=1}^m \Sigma A_i \to BG$ be a pointed map. Universally, a map out of a wedge is determined by its restrictions to the wedge summands, so $f = \bigvee_{i=1}^m f_i$, where $f_i\colon \Sigma A_i \to BG$ is the restriction of f to ΣA_i. By (4), each f_i lifts through $\Sigma\Omega BG \xrightarrow{ev} BG$ to a map $\bar{f}_i = (\Sigma\Omega f_i) \circ \Sigma E$. So if $N \in \mathcal{N}$ and the multiplication on G is homotopy commutative, we may combine the diagrams in Lemmas 2.3, 2.4, and 3.1 to obtain a strictly commutative diagram

$$\text{(6)}\quad \begin{tikzcd} & \bigvee_{i=1}^m \Sigma A_i \arrow[r, "\bigvee_{i=1}^m \bar{f}_i"] \arrow[dl, "b"'] \arrow[d] & \bigvee_{i=1}^m \Sigma\Omega BG \arrow[r, "t_m"] \arrow[d] & BG \\ N \arrow[r, "e_N"] & T(\Sigma A_1, \dots, \Sigma A_m) \arrow[r, "T(\bar{f}_1,\dots,\bar{f}_m)"] & T(\Sigma\Omega BG). \arrow[ur, "e_m"'] & \end{tikzcd}$$

By the definitions of t_m and each $\bar{f}_i$, we have $t_m \circ \big(\bigvee_{i=1}^m \bar{f}_i\big) = \bigvee_{i=1}^m f_i$. So (6) lets us define a map

$$\theta\colon \mathrm{Map}^*\Big(\bigvee_{i=1}^m \Sigma A_i, BG\Big) \to \mathrm{Map}^*(N, BG)$$

by $\theta(f) = \theta\big(\bigvee_{i=1}^m f_i\big) = e_m \circ T(\bar{f}_1, \dots, \bar{f}_m) \circ e_N$. We wish to show that θ is continuous and that $b^* \circ \theta$ is the identity map.

Lemma 4.1. *The map θ is continuous.*

Proof. The map θ is defined as the composite of the continuous maps e_m and e_N and the continuous functor $T(\bar{f}_1, \dots, \bar{f}_m)$. Note that if Y is a locally compact Hausdorff space then the composition $\mathrm{Map}^*(Y, Z) \times \mathrm{Map}^*(X, Y) \to \mathrm{Map}^*(X, Z)$ is continuous with respect to the compact open topology. Therefore θ is continuous. □

Lemma 4.2. *The composite of continuous maps*

$$\mathrm{Map}^*\Big(\bigvee_{i=1}^m \Sigma A_i, BG\Big) \xrightarrow{\theta} \mathrm{Map}^*(N, BG) \xrightarrow{b^*} \mathrm{Map}^*\Big(\bigvee_{i=1}^m \Sigma A_i, BG\Big)$$

is equal to the identity map.

Proof. By definition, b^* sends a map $\phi\colon N \to BG$ to the composite

$$\bigvee_{i=1}^{m} \Sigma A_i \xrightarrow{b} N \xrightarrow{\phi} BG.$$

Therefore, by the definition of θ, we have

$$\begin{aligned} b^* \circ \theta(f) &= b^* \circ \theta\left(\bigvee_{i=1}^{m} f_i\right) \\ &= b^*(e_m \circ T(\bar{f}_1, \dots, \bar{f}_m) \circ e_N) = e_m \circ T(\bar{f}_1, \dots, \bar{f}_m) \circ e_N \circ b. \end{aligned}$$

By (6) and the definition of t_m, we have

$$e_m \circ T(\bar{f}_1, \dots, \bar{f}_m) \circ e_N \circ b = t_m \circ \left(\bigvee_{i=1}^{m} \bar{f}_i\right) = \bigvee_{i=1}^{m} f_i = f.$$

Thus $b^* \circ \theta(f) = f$. □

Proof of Theorem 1.1. In general, suppose that $\Omega B \xrightarrow{\partial} F \xrightarrow{r} E \xrightarrow{s} B$ is a homotopy fibration sequence and r has a right homotopy inverse $t\colon E \to F$. Then s is null homotopic because

(i) $r \circ t \simeq 1_E$ implies that $s \simeq s \circ r \circ t$, and

(ii) $s \circ r$ is null homotopic as it is the composition of two consecutive maps in a homotopy fibration.

The null homotopy for s implies that $F \simeq E \times \Omega B$. In our case, consider the homotopy fibration sequence (5). By Lemma 4.2, the map b^* has a right inverse. Therefore there is a homotopy equivalence

$$\mathrm{Map}^*(N, BG) \simeq \mathrm{Map}^*\left(\bigvee_{i=1}^{m} \Sigma A_i, BG\right) \times \mathrm{Map}^*(S^n, BG). \qquad \square$$

To illustrate Theorem 1.1 we consider two cases of interest. Note that

$$\mathrm{Map}^*(S^t, BG) \simeq \Omega^{t-1} G.$$

Example 4.3. Let M be a simply connected Spin four-manifold with m two-cells, where $m \geq 2$. As in Section 3, there is a homotopy equivalence $M \simeq \left(\bigvee_{i=1}^{m} S^2\right) \cup_a e^4$. Let G be a simply connected, simple compact Lie group listed in (2), whose multiplication is homotopy commutative when localised at p. By [Hilton et al. 1975], p-localisation commutes with mapping spaces in the context of simply connected (and more generally, nilpotent) spaces, so we have $\mathrm{Map}^*(M, BG)_{(p)} \simeq$

$\mathrm{Map}^*(M_{(p)}, BG_{(p)})$. Thus Theorem 1.1 implies that there is a homotopy equivalence

$$\mathrm{Map}^*(M, BG)_{(p)} \simeq \left(\prod_{i=1}^{m} \Omega G_{(p)}\right) \times \Omega^3 G_{(p)}.$$

Example 4.4. Let Σ_g be a close orientable surface of genus $g \geq 1$. As in Section 3, $\Sigma_g \simeq \left(\bigvee_{i=1}^{2g} S^1\right) \cup_a e^2 \in \mathcal{N}$. Let $G = U$, the infinite unitary group. Since U is an infinite loop space it is homotopy commutative so by Theorem 1.1 there is a homotopy equivalence

$$\mathrm{Map}^*(\Sigma_g, BU) \simeq \left(\prod_{i=1}^{2g} U\right) \times \Omega U.$$

We close this section by proving Corollaries 1.2 and 1.3.

Proof of Corollary 1.2. Recall from Section 1 that if G is a simply connected simple compact Lie group, M is a simply connected four-manifold, and $P_k \to M$ is a principal G-bundle induced by the homotopy class in $[M, BG] \cong \mathbb{Z}$ corresponding to k, then there is a homotopy equivalence $B\mathcal{G}_k^*(M) \simeq \mathrm{Map}^*{}_k(M, BG)$. By Example 4.3, there is a p-local homotopy equivalence $\mathrm{Map}^*{}_k(M, BG)_{(p)} \simeq \left(\prod_{i=1}^{m} \Omega G_{(p)}\right) \times \Omega_k^3 G_{(p)}$, where $\Omega_k^3 G$ is the connected component of $\Omega^3 G$ that contains the map $S^3 \to G$ of degree k in the third homology group. Since $\pi_0(\Omega^3 G)$ is a group, there is a homotopy equivalence $\Omega_k^3 G \simeq \Omega_0^3 G$. Therefore $B\mathcal{G}_k^*(M)_{(p)} \simeq \left(\prod_{i=1}^{m} \Omega G_{(p)}\right) \times \Omega_0^3 G_{(p)}$. □

Proof of Corollary 1.3. Again, recall from Section 1 that if $G = U$, Σ_g is a closed orientable surface of genus g, and $P_k \to \Sigma_g$ is a principal G-bundle induced by the homotopy class in $[\Sigma_g, BU] \cong \mathbb{Z}$ corresponding to k, then there is a homotopy equivalence $B\mathcal{G}_k(\Sigma_g) \simeq \mathrm{Map}^*{}_k(\Sigma_g, BU)$. By Example 4.4, there is a homotopy equivalence $\mathrm{Map}^*{}_k(\Sigma_g, BU) \simeq \left(\prod_{i=1}^{2g} U\right) \times \Omega_k U$, where $\Omega_k U$ is the connected component of ΩU that contains the map $S^1 \to U$ of degree k in the first homology group. Since $\pi_0(\Omega U)$ is a group, there is a homotopy equivalence $\Omega_k U \simeq \Omega_0 U$. Therefore there is a homotopy equivalence $B\mathcal{G}_k(\Sigma_g) \simeq \left(\prod_{i=1}^{2g} U\right) \times \Omega U$. □

5. Applications

In this section we give two applications, one to the calculation of the mod-p homology or cohomology of the classifying space of certain full gauge groups, and the other to the homotopy type of a certain group of homomorphisms.

First, return to the case when G is a simply connected simple compact Lie group, M is a simply connected four-manifold, and $P_k \to M$ is a principal G-bundle induced by the homotopy class in $[M, BG] \cong \mathbb{Z}$ corresponding to k. By [Atiyah and Bott 1983] there is a homotopy commutative diagram

$$
\begin{tikzcd}
B\mathcal{G}_k^*(M) \arrow[r] \arrow[d, "\psi^*"] & B\mathcal{G}_k(M) \arrow[d, "\psi"] \\
\mathrm{Map}^*{}_k(M, BG) \arrow[r] & \mathrm{Map}_k(M, BG),
\end{tikzcd}
\tag{7}
$$

where ψ^* and ψ are homotopy equivalences. Observe also that there is a fibration

$$\mathrm{Map}^*{}_k(M, BG) \to \mathrm{Map}_k(M, BG) \xrightarrow{ev} BG,$$

where ev evaluates a map at the basepoint of M. Stated in terms of gauge groups, up to homotopy equivalences, there is a fibration

$$B\mathcal{G}_k^*(M) \to B\mathcal{G}_k(M) \to BG.$$

Take homology and cohomology with mod-p coefficients. Corollary 1.2 immediately implies that if G is homotopy commutative when localised at p then there is a coalgebra isomorphism

$$H_*(B\mathcal{G}_k^*(M)) \cong \left(\bigotimes_{i=1}^m H_*(\Omega G)\right) \otimes H_*(\Omega_0^2 G)$$

and an algebra isomorphism

$$H^*(B\mathcal{G}_k^*(M)) \cong \left(\bigotimes_{i=1}^m H^*(\Omega G)\right) \otimes H^*(\Omega_0^2 G).$$

We aim to prove the following:

Theorem 5.1. *Let M be a closed simply connected Spin four-manifold and let G be a simply connected simple compact Lie group whose multiplication is homotopy commutative when localised at p. Then the composite of coalgebras*

$$\bigotimes_{i=1}^m H_*(\Omega G) \to H_*(B\mathcal{G}_k^*(M)) \to H_*(B\mathcal{G}_k(M))$$

has a left inverse, and the composite of algebras

$$H^*(B\mathcal{G}_k(M)) \to H^*(B\mathcal{G}_k^*(M)) \to \bigotimes_{i=1}^m H^*(\Omega G)$$

has a right inverse.

For example, let $G = SU(2)$, in which case G is homeomorphic to S^3 and $H^*(\Omega S^3)$ is well known. This case is of key interest in Donaldson theory and a major open problem is calculating the mod-p homology of $B\mathcal{G}_k(M)$. As $SU(2)$ is homotopy commutative when localised at primes $p \geq 5$, Theorem 5.1 applies for any such prime, giving significant information about $H_*(B\mathcal{G}_k(M))$.

To prove Theorem 5.1, we begin by recalling some general facts about mapping spaces. Let $X_1, \ldots, X_m$ and Y be Hausdorff spaces, and let $\coprod_{i=1}^m X_i$ be their disjoint union. Then there is a homeomorphism

$$\mathrm{Map}\Big(\coprod_{i=1}^{m} X_i, Y\Big) \cong \prod_{i=1}^{m} \mathrm{Map}(X_i, Y).$$

Further, if each of $X_1, \ldots, X_m$ and Y are pointed, then there is a homeomorphism

$$\mathrm{Map}^*\Big(\bigvee_{i=1}^{m} X_i, Y\Big) \cong \prod_{i=1}^{m} \mathrm{Map}^*(X_i, Y).$$

These two decompositions are compatible in the following sense. There is a quotient map

$$\mathfrak{q}\colon \coprod_{i=1}^{m} X_i \to \bigvee_{i=1}^{m} X_i$$

which identifies the basepoints in each space X_i to a common point. So there is an induced map

$$\mathfrak{q}^*\colon \mathrm{Map}\Big(\bigvee_{i=1}^{m} X_i, Y\Big) \to \mathrm{Map}\Big(\coprod_{i=1}^{m} X_i, Y\Big).$$

The two homeomorphisms above are compatible via a strictly commutative diagram

$$\begin{tikzcd}
\mathrm{Map}^*\big(\bigvee_{i=1}^m X_i, Y\big) \arrow[r, "\mathrm{incl}"] \arrow[d, "\cong"] & \mathrm{Map}\big(\bigvee_{i=1}^m X_i, Y\big) \arrow[r, "\mathfrak{q}^*"] & \mathrm{Map}\big(\coprod_{i=1}^m X_i, Y\big) \arrow[d, "\cong"] \\
\prod_{i=1}^m \mathrm{Map}^*(X_i, Y) \arrow[rr, "\prod_{i=1}^m \mathrm{incl}"] & & \prod_{i=1}^m \mathrm{Map}(X_i, Y).
\end{tikzcd} \tag{8}$$

Returning to the case of interest, as in Section 3, if M is any closed simply connected Spin four-manifold then there is a space $N = \big(\bigvee_{i=1}^m S^2\big) \cup_a e^4 \in \mathcal{N}$. The inclusion $\bigvee_{i=1}^m S^2 \xrightarrow{b} N$ induces a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}^*(N, BG) \arrow[r] \arrow[d, "b^*"] & \mathrm{Map}(N, BG) \arrow[d, "b^*"] \\
\mathrm{Map}^*\big(\bigvee_{i=1}^m S^2, BG\big) \arrow[r] & \mathrm{Map}\big(\bigvee_{i=1}^m S^2, BG\big).
\end{tikzcd} \tag{9}$$

Localising at p, the fact that mapping spaces commute with localisation of nilpotent spaces [Hilton et al. 1975] implies that there is a homotopy commutative diagram

$$
\begin{tikzcd}
\mathrm{Map}^*(M, BG)_{(p)} \arrow[r] \arrow[d, "\simeq"] & \mathrm{Map}(M, BG)_{(p)} \arrow[d, "\simeq"] \\
\mathrm{Map}^*(N, BG)_{(p)} \arrow[r] & \mathrm{Map}(N, BG)_{(p)}.
\end{tikzcd}
\tag{10}
$$

Juxtaposing the diagrams (7), (8), (9), and (10) we obtain a p-local homotopy commutative diagram

$$
\begin{tikzcd}
B\mathcal{G}^*_k(M)_{(p)} \arrow[r] \arrow[d, "\psi^*"] & B\mathcal{G}_k(M)_{(p)} \arrow[d, "\psi"] \\
\mathrm{Map}^*{}_k(M, BG)_{(p)} \arrow[r] \arrow[d, "\simeq"] & \mathrm{Map}_k(M, BG)_{(p)} \arrow[d, "\simeq"] \\
\mathrm{Map}^*{}_k(N, BG)_{(p)} \arrow[r] \arrow[d, "b^*"] & \mathrm{Map}_k(N, BG)_{(p)} \arrow[d, "b^*"] \\
\mathrm{Map}^*\big(\bigvee_{i=1}^m S^2, BG\big)_{(p)} \arrow[r] \arrow[dd, "\cong"] & \mathrm{Map}\big(\bigvee_{i=1}^m S^2, BG\big)_{(p)} \arrow[d, "q^*"] \\
 & \mathrm{Map}\big(\coprod_{i=1}^m S^2, BG\big)_{(p)} \arrow[d, "\cong"] \\
\prod_{i=1}^m \mathrm{Map}^*(S^2, BG)_{(p)} \arrow[r, "\prod_{i=1}^m \mathrm{incl}"] & \prod_{i=1}^m \mathrm{Map}(S^2, BG)_{(p)}.
\end{tikzcd}
$$

By Lemma 4.2, the map b^* has a right inverse. Lifting this, up to homotopy, through the homotopy equivalences $B\mathcal{G}^*_k(M)_{(p)} \xrightarrow{\psi^*} \mathrm{Map}^*{}_k(M, BG)_{(p)} \xrightarrow{\simeq} \mathrm{Map}^*{}_k(N, BG)_{(p)}$, we obtain the following:

Lemma 5.2. *Let M be a closed simply connected Spin four-manifold and let G be a simply connected simple compact Lie group whose multiplication is homotopy commutative when localised at a prime p. Then there is a homotopy commutative diagram*

$$
\begin{tikzcd}
\mathrm{Map}^*{}_k\big(\bigvee_{i=1}^m S^2, BG\big)_{(p)} \arrow[r] \arrow[dr, "\simeq"] & B\mathcal{G}^*_k(M)_{(p)} \arrow[r] \arrow[d] & B\mathcal{G}_k(M)_{(p)} \arrow[d] \\
 & \prod_{i=1}^m \mathrm{Map}^*(S^2, BG)_{(p)} \arrow[r, "\prod_{i=1}^m \mathrm{incl}"] & \prod_{i=1}^m \mathrm{Map}(S^2, BG)_{(p)}.
\end{tikzcd}
$$

Lemma 5.2 is used to extract information about $H_*(B\mathcal{G}_k(M))$ and $H^*(B\mathcal{G}_k(M))$.

Proof of Theorem 5.1. Consider the map $\mathrm{Map}^*(S^2, BG) \xrightarrow{\text{incl}} \mathrm{Map}(S^2, BG)$ whose p-localisation appears in the bottom row of the diagram in Lemma 5.2. The inclusion is the fibre of the evaluation map $\mathrm{Map}(S^2, BG) \xrightarrow{ev} BG$, which sends a map $f : S^2 \to BG$ to $f(*)$. Also, we have $\mathrm{Map}^*(S^2, BG) = \Omega G$. So there is a fibration

$$\Omega G \to \mathrm{Map}(S^2, BG) \xrightarrow{ev} BG. \tag{11}$$

By (2), the cases when the multiplication on G is homotopy commutative when localised at p are known. In each such case, $H^*(G)$ is an exterior algebra on odd degree generators, so by [Borel 1953] $H^*(BG)$ is a polynomial algebra on even degree generators. Since cohomology is with mod-p coefficients, we can dualise to see that $H_*(BG)$ is also concentrated in even degrees. Further, by [Bott 1956], the integral cohomology of ΩG is concentrated in even degrees, and therefore so is the mod-p cohomology. Therefore the homology Serre spectral sequence for the fibration (11) collapses at the E^2-term and there are no extension issues. Hence

$$H_*(\mathrm{Map}(S^2, BG)) \cong H_*(BG) \otimes H_*(\Omega G).$$

Consequently, taking homology for the diagram in Lemma 5.2, we see that the composite

$$\bigotimes_{i=1}^{m} H_*(\Omega G) \to H_*(B\mathcal{G}_k^*(M)) \to H_*(B\mathcal{G}_k(M))$$

has a left inverse.

Similarly,

$$H^*(\mathrm{Map}(S^2, BG)) \cong H^*(BG) \otimes H^*(\Omega G)$$

and the composite

$$H^*(B\mathcal{G}_k(M)) \to H^*(B\mathcal{G}_k^*(M)) \to \bigotimes_{i=1}^{m} H^*(\Omega G)$$

has a right inverse. □

We now turn to the second application. Let K and L be topological groups, and let $\mathrm{Hom}(K, L)$ be the set of homomorphisms from K to L, topologised as a subspace of the mapping space $\mathrm{Map}(K, L)$. If BK, BL are the classifying spaces of K and L respectively, there is a natural map

$$B : \ \mathrm{Hom}(K, L) \to \mathrm{Map}^*(BK, BL).$$

This map has been a subject of intense study due to its connections with the Sullivan conjecture in homotopy theory, to the moduli space of representations in algebraic geometry, and to the space of flat connections modulo gauge equivalence in Yang–Mills theory. Consider the special case

$$\mathrm{Hom}(\pi_1(\Sigma_g), U(n)) \to \mathrm{Map}^*(B\pi_1(\Sigma_g), BU(n)).$$

Since the universal cover of Σ_g is contractible there is a homotopy equivalence $\Sigma_g \simeq B\pi_1(\Sigma_g)$. So up to a homotopy equivalence we may regard the preceding map as

$$\mathrm{Hom}(\pi_1(\Sigma_g), U(n)) \to \mathrm{Map}^*(\Sigma_g, BU(n)).$$

Ramras [2011, Theorem 3.4] used gauge theoretic methods to show that this map is an injection on π_0 and an isomorphism on π_m for $m \leq 2g(n-1)+1$. Stabilising to the infinite unitary group, we obtain a map

$$\mathrm{Hom}(\pi_1(\Sigma_g), U) \to \mathrm{Map}^*(\Sigma_g, BU),$$

which is an injection on π_0 and an isomorphism on π_m for every $m \geq 1$. Thus if $\mathrm{Hom}_I(\pi_1(\Sigma_g), U))$ is the component of $\mathrm{Hom}(\pi_1(\Sigma_g), U))$ containing the identity map, from Corollary 1.3 we obtain homotopy equivalences

$$\mathrm{Hom}_I(\pi_1(\Sigma_g), U)) \xrightarrow{\simeq} \mathrm{Map}^*{}_0(\Sigma_g, BU) \xrightarrow{\simeq} \left(\prod_{i=1}^{2g} U\right) \times \Omega_0 U,$$

which lets one easily identify $\pi_m(\mathrm{Hom}(\pi_1(\Sigma_g), U))$ for $m \geq 1$.

References

[Arkowitz 1962] M. Arkowitz, "The generalized Whitehead product", *Pacific J. Math.* **12** (1962), 7–23. MR Zbl

[Atiyah and Bott 1983] M. F. Atiyah and R. Bott, "The Yang–Mills equations over Riemann surfaces", *Philos. Trans. Roy. Soc. London Ser. A* **308**:1505 (1983), 523–615. MR Zbl

[Borel 1953] A. Borel, "Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts", *Ann. of Math.* (2) **57** (1953), 115–207. MR Zbl

[Bott 1956] R. Bott, "An application of the Morse theory to the topology of Lie-groups", *Bull. Soc. Math. France* **84** (1956), 251–281. MR Zbl

[Gottlieb 1972] D. H. Gottlieb, "Applications of bundle map theory", *Trans. Amer. Math. Soc.* **171** (1972), 23–50. MR Zbl

[Hatcher 2002] A. Hatcher, *Algebraic topology*, Cambridge Univ. Press, 2002. MR Zbl

[Hilton et al. 1975] P. Hilton, G. Mislin, and J. Roitberg, *Localization of nilpotent groups and spaces*, North-Holland Math. Studies **15**, North-Holland Publ., Amsterdam, 1975. MR Zbl

[James and Thomas 1962a] I. James and E. Thomas, "Homotopy-abelian topological groups", *Topology* **1** (1962), 237–240. MR Zbl

[James and Thomas 1962b] I. James and E. Thomas, "On homotopy-commutativity", *Ann. of Math.* (2) **76** (1962), 9–17. MR Zbl

[Kishimoto and Kono 2010] D. Kishimoto and A. Kono, “Splitting of gauge groups”, *Trans. Amer. Math. Soc.* **362**:12 (2010), 6715–6731. MR Zbl

[Kishimoto et al. 2013a] D. Kishimoto, A. Kono, and S. Theriault, “Homotopy commutativity in p-localized gauge groups”, *Proc. Roy. Soc. Edinburgh Sect. A* **143**:4 (2013), 851–870. MR Zbl

[Kishimoto et al. 2013b] D. Kishimoto, A. Kono, and M. Tsutaya, “Mod p decompositions of gauge groups”, *Algebr. Geom. Topol.* **13**:3 (2013), 1757–1778. MR Zbl

[Kishimoto et al. 2014] D. Kishimoto, A. Kono, and S. Theriault, “Refined gauge group decompositions”, *Kyoto J. Math.* **54**:3 (2014), 679–691. MR Zbl

[Masbaum 1991] G. Masbaum, “On the cohomology of the classifying space of the gauge group over some 4-complexes”, *Bull. Soc. Math. France* **119**:1 (1991), 1–31. MR Zbl

[McGibbon 1984] C. A. McGibbon, “Homotopy commutativity in localized groups”, *Amer. J. Math.* **106**:3 (1984), 665–687. MR Zbl

[Milnor 1958] J. Milnor, “On simply connected 4-manifolds”, pp. 122–128 in *Symposium internacional de topología algebraica*, Universidad Nacional Autónoma de México, Mexico City, 1958. MR Zbl

[Ramras 2011] D. A. Ramras, “The stable moduli space of flat connections over a surface”, *Trans. Amer. Math. Soc.* **363**:2 (2011), 1061–1100. MR Zbl

[Stöcker 1982] R. Stöcker, “On the structure of 5-dimensional Poincaré duality spaces”, *Comment. Math. Helv.* **57**:3 (1982), 481–510. MR Zbl

[Theriault 2010] S. D. Theriault, “Odd primary homotopy decompositions of gauge groups”, *Algebr. Geom. Topol.* **10**:1 (2010), 535–564. MR Zbl

[Wall 1962] C. T. C. Wall, “Classification of $(n-1)$-connected $2n$-manifolds”, *Ann. of Math.* (2) **75** (1962), 163–189. MR Zbl

Received March 2, 2017. Revised August 6, 2018.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 300, No. 1, 2019
dx.doi.org/10.2140/pjm.2019.300.233

LOCAL SOBOLEV CONSTANT ESTIMATE FOR INTEGRAL BAKRY–ÉMERY RICCI CURVATURE

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We extend several geometrical results in Dai et al. (2018) for Riemannian manifolds with integral curvature to complete smooth metric measure spaces with integral Bakry–Émery Ricci curvature.

1. Introduction

Sobolev inequalities not only encode rich analytical and geometrical information about manifold, but have wide applications in differential geometry. A useful method to estimate the Sobolev constant is to estimate the isoperimetric constant since they are equivalent [Cheeger 1970; Chavel 1993; Li 2012]. A key issue for the isoperimetric constant study is the volume control, which is given by the Ricci curvature lower bound. After Petersen and Wei [1997] generalized the classical Laplacian and volume comparison to the integral Ricci curvature bound, many results for the pointwise Ricci lower bound have been extended to the integral Ricci curvature bound, see, e.g., [Petersen and Wei 2001; Aubry 2007; 2009; Dai et al. 2000; Tian and Zhang 2016; Dai et al. 2018; Zhang and Zhu 2017; 2018; Rose 2017; Rose and Stollmann 2017]. In particular, Dai, Wei and Zhang [Dai et al. 2018] obtained the local isoperimetric constant estimate for integral Ricci curvature.

An n-dimensional smooth metric measure space, denoted by $M^n_f := (M^n, g, e^{-f} d\,\mathrm{vol})$, is a complete n-dimensional Riemannian manifold (M^n, g) coupled with a weighted volume $e^{-f} d\,\mathrm{vol}$ for some $f \in C^\infty(M)$, where $d\,\mathrm{vol}$ is the usual Riemannian volume element on M. For a smooth metric measure space M^n_f, a natural generalization of the Ricci curvature is the Bakry–Émery Ricci curvature [Bakry and Émery 1985] defined by

$$\mathrm{Ric}_f := \mathrm{Ric} + \mathrm{Hess}\, f.$$

A great deal of effort has been devoted to the study of smooth metric measure spaces with Bakry–Émery Ricci curvature bounded below, and some of the earlier works are [Lott 2003; Wei and Wylie 2009; Cao and Zhou 2010; Munteanu and

MSC2010: 53C23, 53C20.
Keywords: integral Bakry–Émery Ricci curvature, isoperimetric constant estimate, Sobolev constant.

Wang 2011]. However, limited work has been done for the integral Bakry–Émery Ricci curvature.

Recently, Wu [2018] has extended the volume comparison in [Petersen and Wei 1997] to the integral Bakry–Émery Ricci curvature case. In this paper we extend the local isoperimetric constant estimate in [Dai et al. 2018] to integral Bakry–Émery Ricci curvature and give some applications.

To state the results, we fix some notations. Given $x \in M^n_f$, let $\rho_f(x)$ be the smallest eigenvalue of $\mathrm{Ric}_f : T_x M \to T_x M$, and

$$\mathrm{Ric}^H_{f-} = ((n-1)H - \rho_f(x))_+ = \max\{0, (n-1)H - \rho_f(x)\}.$$

Denote $B_x(R) \subset M$ by the ball with radius R, centered at x. Various weighted L^p norms of the function h on a smooth metric measure space M^n_f are

$$\|h\|_{p,f,B_x(R)} = \left(\int_{B_x(R)} |h|^p e^{-f} \, d\,\mathrm{vol} \right)^{\frac{1}{p}},$$

$$\|h\|_{p,f,a}(R) = \sup_{x \in M^n_f} \left[\int_0^R \int_{\mathbb{S}^{n-1}} |h|^p e^{-at} \mathcal{A}_f(t,\theta) \, d\theta \, dt \right]^{\frac{1}{p}}.$$

Here $d\theta$ is the volume element of the unit sphere $\mathbb{S}^{n-1}$ and $\mathcal{A}_f(t,\theta)$ is the volume element of weighted measure $e^{-f} d\,\mathrm{vol} = \mathcal{A}_f(t,\theta)\, d\theta \wedge dt$. Clearly, $\|\mathrm{Ric}^H_{f-}\|_{p,f,B_x(R)} \equiv 0$ if and only if $\mathrm{Ric}_f \geq (n-1)H$. For convenience, we assume $H = 0$ and write $\mathrm{Ric}^0_{f-} = \mathrm{Ric}_{f-}$ in the whole paper. Then Ric^0_{f-} is the negative part of ρ_f, denoted by ρ_{f-}. The following scale-invariant curvature quantity will be useful:

$$\bar{\kappa}(p,f,a,R) = R^2 \sup_{x \in M^n_f} \left[\fint_{B_x(R)} \rho^p_{f-} e^{-at} \mathcal{A}_f(t,\theta) \, d\theta \wedge dt \right]^{\frac{1}{p}},$$

where

$$\mathrm{vol}_f B_x(R) = \int_{B_x(R)} e^{-f} \, d\mathrm{vol} \quad \text{and} \quad \fint_{B_x(R)} = \frac{1}{\mathrm{vol}_f B_x(R)} \int_{B_x(R)}.$$

For a fixed point $x \in M$, let $r(y) := d(y,x)$ be a distance function from x to y. In geodesic polar coordinates at x, let $\nabla r = \partial_r$. Our first result gives an estimation of the local normalized Dirichlet isoperimetric constant under a lower bound of $\partial_r f$.

Theorem 1.1. *Let M^n_f be a complete smooth metric measure space. Assume that $\partial_r f \geq -a$ along all minimal geodesic segments for some constant $a \geq 0$. For $p > \frac{1}{2}n$, there exists $\varepsilon = \varepsilon(n,p,a) > 0$ such that if $\bar{\kappa}(p,f,a,1) \leq \varepsilon$, then for any $x \in M^n_f$, $\partial B_x(R) \neq \varnothing$, $R \leq 1$, we have the estimate*

$$\mathrm{ID}^*_{n,f} B_x(R) \geq 10^{-2n} e^{-2a} R^{-1}, \tag{1-1}$$

where

$$\mathrm{ID}^*_{n,f} B_x(R) = \mathrm{vol}_f B_x(R)^{-\frac{1}{n}} \inf_{\Omega} \left\{ \frac{\mathrm{vol}_f \partial\Omega}{(\mathrm{vol}_f \Omega)^{\frac{n-1}{n}}} \right\}.$$

Here the infimum runs over all subdomains $\Omega \subset B_x(R)$ *with smooth boundary and* $\partial\Omega \cap \partial B_x(R) = \varnothing$.

Remark 1.2. Clearly, the local normalized Dirichlet constant has explicit and accurate dependency of the growth of f; Theorem 1.1 will recover to [Dai et al. 2018, Theorem 1.1] when f is a constant. The smallness of $\bar{\kappa}(p, f, a, 1)$ is necessary; see the counterexample in [Dai et al. 2018, §6] when f is constant and $\bar{\kappa}(p, f, a, 1)$ is bounded. Also the result is not true when $p \le \frac{1}{2}n$; see details in [Aubry 2007].

It is well known that the classical Dirichlet isoperimetric and Sobolev constants are equivalent; see, e.g., [Li 2012]. In Section 3, we introduce their weighted versions for smooth metric measure spaces; see Definitions 3.1, 3.2. A similar proof shows that they are also equivalent; see Theorem A.1. Hence we have:

Proposition 1.3. *With the assumptions of Theorem 1.1, the Sobolev inequality*

$$\fint_{B_x(R)} |\nabla h| e^{-f} \, d\,\mathrm{vol} \ge 10^{-2n} e^{-2a} R^{-1} \left(\fint_{B_x(R)} h^{\frac{n}{n-1}} e^{-f} \, d\,\mathrm{vol} \right)^{\frac{n-1}{n}} \tag{1-2}$$

holds for all $h \in C_0^\infty(B_x(R))$.

Recall that the f-Laplacian of M^n_f is

$$\Delta_f = \Delta - \nabla f \cdot \nabla.$$

Given the normalized form of integral in Proposition 1.3, we denote the normalized L^p norm for function h by

$$\|h\|^*_{p,f,B_x(R)} = \|h\|_{p,f,B_x(R)} (\mathrm{vol}_f B_x(R))^{-\frac{1}{p}},$$

and

$$\|h\|^*_{p,f,a}(R) = \sup_{x \in M^n_f} \left(\fint_{B_x(R)} h e^{-at} \mathcal{A}_f(t,\theta) \, d\theta \wedge dt \right)^{\frac{1}{p}}$$

with the same $\mathcal{A}_f(t,\theta)\, d\theta \wedge dt$ as above. It is easy to observe that

$$\|h\|^*_{p,f,B_x(R)} \le e^{\frac{aR}{p}} \|h\|^*_{p,f,a}(R), \tag{1-3}$$

and the normalized L^∞ norm is independent of f satisfying $\|h\|^*_{\infty,f,B_x(R)} = \sup_{B_x(R)} h$. By employing the above Sobolev inequality (1-2), we extend the maximum principle in [Petersen and Wei 2001; Dai et al. 2018] to the integral Bakry–Émery Ricci curvature situation.

Theorem 1.4. *Let M^n_f be a complete smooth metric measure space. Assume that $\partial_r f \geq -a$ along all minimal geodesic segments for some constant $a \geq 0$. For $p > \frac{1}{2}n$, there exists an $\varepsilon = \varepsilon(n, p, a) > 0$ and $C = C(n, p, a) > 1$ such that if $\bar{\kappa}(p, f, a, 1) \leq \varepsilon$ and $R \leq 1$, then for any function $u : \Omega(\subset B_x(R)) \subset M^n_f \to \mathbb{R}$ with $\Delta_f u \geq h$, we have*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + C \cdot R^2 \cdot \|h_-\|^*_{p,f,\Omega},$$

where h_- denotes the negative part of the function h.

Also we have the gradient estimate.

Theorem 1.5. *Let M^n_f be a complete smooth metric measure space. Assume that $\partial_r f \geq -a$ along all minimal geodesics for some constant $a \geq 0$. For $p > \frac{1}{2}n$, there exists an $\varepsilon = \varepsilon(n, p, a) > 0$ and $C(n, p, a) > 1$ such that if $\bar{\kappa}(p, f, a, 1) \leq \varepsilon$ and $R \leq 1$ and u is a function on $B_x(R)$ satisfying*

$$\Delta_f u = h,$$

then

$$\sup_{B_x(\frac{R}{2})} |\nabla u|^2 \leq C(n, p, a) R^{-2} [(\|h\|^*_{2p,f,B_x(R)})^2 + (\|u\|^*_{2,f,B_x(R)})^2].$$

An outline of this paper is as follows. In Section 2, we review the Laplacian and volume comparison for the integral Bakry–Émery Ricci curvature. In Section 3, we define local Dirichlet isoperimetric and Sobloev constants, as well as their normalized form in smooth metric measure spaces. Moreover, we estimate the normalized isoperimetric constant for integral Bakry–Émery Ricci curvature; see Theorem 1.1. In Section 4, as applications, we establish the maximum principle (Theorem 1.4) and the gradient estimate (Theorem 1.5) in a complete smooth metric measure space with integral Bakry–Émery Ricci curvature. In the Appendix, we give the proof of equivalence between the two constants defined in Section 3.

2. Preliminaries

In this section, we review the Laplacian and volume comparison for smooth metric measure spaces. Let M^n_f be a complete smooth metric measure space and $r(y) = d(y, x)$ be a distance function from $x \in M^n_f$. Assume that f satisfies $\partial_r f \geq -a$ along all minimal geodesic segments for some constant $a \geq 0$. By choosing the Euclidean space with a weighted function as the model space, that is $\mathbb{R}^n_a = (\mathbb{R}^n, g_{\mathbb{R}^n}, e^{-h} d\,\mathrm{vol})$ with $h(x) = -a|x|$ for $x \in \mathbb{R}^n$, then the f-Laplacian error term is

$$\psi(y) = \left(\Delta_f r - \frac{n-1}{r} - a\right)_+.$$

Wei and Wylie [2009] have proved that $\mathrm{Ric}_f \geq 0$ yields $\Delta_f r \leq \dfrac{n-1}{r} + a$; that is,

$\|\mathrm{Ric}_{f-}\|_{p,f,a}(r) \equiv 0$ implies that $\psi \equiv 0$. In [Wu 2018], this has been extended to integral Bakry–Émery Ricci curvature bound following the work of Petersen and Wei [1997] for integral Ricci curvature.

Theorem 2.1 [Wu 2018, Theorem 1.1]. *For any $p > \frac{1}{2}n$, we have*

$$\|\psi\|_{2p,f,a}(r) \le C(n,p)[\|\mathrm{Ric}_{f-}\|_{p,f,a}(r)]^{\frac{1}{2}}, \tag{2-1}$$

with

$$C(n,p) = \Big(\frac{(n-1)(2p-1)}{2p-n}\Big)^{\frac{1}{2}}.$$

Moreover, letting $B_x(r_2)$ and $B_x(r_1)$ be geodesic balls centered at x with radius $r_2 \ge r_1 > 0$, we have

$$\left(\frac{\mathrm{vol}_f B_x(r_2)}{V(n,a,r_2)}\right)^{\frac{1}{2p}} - \left(\frac{\mathrm{vol}_f B_x(r_1)}{V(n,a,r_1)}\right)^{\frac{1}{2p}} \le C(n,p,a,r_2)(\|\mathrm{Ric}_{f-}\|_{p,f,a}(r_2))^{\frac{1}{2}}, \tag{2-2}$$

where $V(n,a,t)$ denotes the volume of geodesic ball $B_0(t)$ in the model space $\mathbb{R}^n_a$.

Remark 2.2. In fact, the proof of [Wu 2018, Theorem 1.1] gives the following normalized form of Laplacian comparison:

$$\begin{aligned} \|\psi\|^*_{2p,f,a}(r) &\le C(n,p)(\|\mathrm{Ric}_{f-}\|^*_{p,f,a}(r))^{\frac{1}{2}} \\ &= C(n,p)r^{-1}(\kappa(p,f,a,r))^{\frac{1}{2}}. \end{aligned} \tag{2-3}$$

Remark 2.3. Here we choose the power $2p$ rather than $2p-1$, so the explicit expression of $C(n,p,a,r_2)$ is similar to the one in [Petersen and Wei 1997, Lemma 2.1] rather than the one in [Wu 2018]. If we denote the volume of $(n-1)$-dimensional unit ball in $\mathbb{R}^n$ and the weighted volume of geodesic sphere $\partial B_0(t)$ by ω_n and $A(n,a,t)$, respectively, then $A(n,a,t) = \omega_n t^{n-1} e^{at}$ and

$$\begin{aligned} C(n,p,a,r_2) &= C(n,p)\int_0^{r_2} tA(n,a,t)\Big(\frac{1}{V(n,a,t)}\Big)^{1+\frac{1}{2p}}\,dt \\ &\le C(n,p)\int_0^{r_2} tA(n,a,t)\bigg(\int_0^t A(n,0,s)\,ds\bigg)^{-(1+\frac{1}{2p})}\,dt \\ &= C(n,p)e^{ar_2} r_2^{1-\frac{n}{2p}}. \end{aligned}$$

Hence, (2-2) implies that

$$\left(\frac{\mathrm{vol}_f B_x(r_1)}{\mathrm{vol}_f B_x(r_2)}\right)^{\frac{1}{2p}} \geq \left(\frac{V(n,a,r_1)}{V(n,a,r_2)}\right)^{\frac{1}{2p}} \big(1 - C(n,p)e^{ar_2} r_2^{1-\frac{n}{2p}} \cdot (V(n,a,r_2))^{\frac{1}{2p}} (\|\mathrm{Ric}_{f\_}\|^*_{p,f,a}(r_2))^{\frac{1}{2}}\big) \geq e^{-\frac{ar_2}{2p}} \left(\frac{r_1}{r_2}\right)^{\frac{n}{2p}} \big(1 - C(n,p)e^{(1+\frac{1}{2p})ar_2} \bar{\kappa}^{\frac{1}{2}}(p,f,a,r_2)\big), \tag{2-4}$$

where $C(n,p)$ is a constant depending on n and p. Hence, there exists a constant $\varepsilon_0 = \varepsilon_0(n,p,a,r_0) > 0$ such that if $\bar{\kappa}(p,f,a,r_0) \leq \varepsilon_0$, then

$$\frac{\mathrm{vol}_f B_x(r)}{\mathrm{vol}_f B_x(r_0)} \geq \tfrac{1}{2} e^{-ar_0} \left(\frac{r}{r_0}\right)^n \quad \forall r \leq r_0. \tag{2-5}$$

For $r_0 \leq 1$, from (2-4), it is easy to observe that there exists a $\varepsilon_0 = \varepsilon_0(n,p,a)$, independent of r, such that (2-5) holds for $\bar{\kappa}(p,f,a,r_0) \leq \varepsilon_0$.

Remark 2.4. The scale invariant $\bar{\kappa}(p,f,a,r)$ has curvature inequalities. For any $r_1 \leq r_2$, and $\bar{\kappa}(p,f,a,r_2) \leq \varepsilon_0$, on one hand,

$$\begin{aligned} \bar{\kappa}(p,f,a,r_1) &\leq r_1^2 \left(\frac{\mathrm{vol}_f B_x(r_2)}{\mathrm{vol}_f B_x(r_1)} \cdot \frac{1}{\mathrm{vol}_f B_x(r_2)} \int_0^{r_2} \int_{S^{n-1}} \rho_{f\_}^p e^{-at} \mathcal{A}_f(t,\theta)\, d\theta\, dt\right)^{\frac{1}{p}} \\ &\leq 2^{\frac{1}{p}} e^{\frac{ar_2}{p}} \left(\frac{r_1}{r_2}\right)^{2-\frac{n}{p}} \bar{\kappa}(p,f,a,r_2). \end{aligned} \tag{2-6}$$

Hence, $\bar{\kappa}(p,f,a,r_1) \leq \varepsilon_0$ holds for $r_1 \leq 2^{-1/(2p-n)} e^{-ar_2/(2p-n)} r_2$. On the other hand, if $\bar{\kappa}(p,f,a,r_1) \leq \varepsilon_0$, using the same method as in [Petersen and Wei 2001, §2.3] and the volume doubling property (2-5), we have

$$\begin{aligned} \bar{\kappa}(p,f,a,r_2) &= r_2^2 \sup_{x \in M_f^n} \left(\frac{1}{\mathrm{vol}_f B_x(r_2)} \int_0^{r_2} \int_{S^{n-1}} \rho_{f\_}^p e^{-at} \mathcal{A}_f(t,\theta)\, d\theta\, dt\right)^{\frac{1}{p}} \\ &\leq \left(\frac{r_2}{r_1}\right)^2 2^{\frac{n+1}{p}} e^{\frac{ar_1}{p}} \bar{\kappa}(p,f,a,r_1). \end{aligned} \tag{2-7}$$

Hence, it is sufficient to work with the case where $\bar{\kappa}(p,f,a,1)$ is small and then scale the metric to obtain the curvature condition of $\bar{\kappa}(p,f,a,r)$.

3. Local Dirichlet isoperimetric constant estimate

In this section, we introduce the local isoperimetric and Sobolev constants in smooth metric measure spaces motivated by the classical ones in [Cheeger 1970; Li 2012]. Furthermore, we estimate the normalized form of these constants.

Definition 3.1. Let $B_x(r)$ be a geodesic ball with $\partial B_x(r) \neq \varnothing$ in a complete smooth metric measure space M^n_f. For $n \leq \alpha \leq \infty$, the Dirichlet α-isoperimetric constant of $B_x(r)$ is defined by

$$\mathrm{ID}_{\alpha,f}\, B_x(r) = \inf_{\Omega} \frac{\mathrm{vol}_f\, \partial\Omega}{(\mathrm{vol}_f\, \Omega)^{1-\frac{1}{\alpha}}},$$

where Ω is an open submanifold of $B_x(r)$ with $\partial\Omega \cap \partial B_x(r) = \varnothing$.

Clearly, $\mathrm{ID}_{n,f}\, B_x(r)$ is a scale invariant and $\mathrm{ID}_{\infty,f}\, B_x(r)$ is a weighted Cheeger constant.

Definition 3.2. The Dirichlet α-Sobolev constant of $B_x(r) \subset M^n_f$ is defined by

$$\mathrm{SD}_{\alpha,f}\, B_x(r) = \inf_h \frac{\|\nabla h\|_{1,f,B_x(r)}}{\|h\|_{\frac{\alpha}{\alpha-1},f,B_x(r)}},$$

where the infimum is taken over all $h \in C_0^\infty(B_x(r))$.

For convenience, we normalize the two kinds of constants above thus:

$$\begin{aligned} \mathrm{ID}^*_{\alpha,f}\, B_x(r) &= \mathrm{ID}_{\alpha,f}\, B_x(r)(\mathrm{vol}_f\, B_x(r))^{-\frac{1}{\alpha}}, \\ \mathrm{SD}^*_{\alpha,f}\, B_x(r) &= \mathrm{SD}_{\alpha,f}\, B_x(r)(\mathrm{vol}_f\, B_x(r))^{-\frac{1}{\alpha}}. \end{aligned} \tag{3-1}$$

To estimate $\mathrm{ID}^*_{\alpha,f}\, B_x(R)$, we need several lemmas. A suitable modification of Gromov's observation [Gromov 1979; Cheeger and Colding 1996] yields the first:

Lemma 3.3. *Let M^n_f be a complete smooth metric measure space. Assume that $\partial_r f \geq -a$ along all minimal geodesic segments for some constant $a \geq 0$. Let S be any hypersurface dividing M^n_f into two parts M_1, M_2. For any subsets $W_i \subset M_i$, there exists x_1 in one of W_i, say W_1, and a subset W in another one, W_2, such that the unique minimal geodesic joint x_1 and any $x_2 \in W$ intersects S at q with*

$$d(x_1, q) \geq d(x_2, q), \tag{3-2}$$

and

$$\mathrm{vol}_f\, W_2 \leq 2\, \mathrm{vol}_f\, W. \tag{3-3}$$

Analogously to [Dai et al. 2018, Lemma 4.2], we have the following volume estimate by using the Laplacian comparison method in smooth metric measure spaces:

Lemma 3.4. *Let M^n_f, S, W and x_1 be as in Lemma 3.3. Then for any $p > \frac{1}{2}n$,*

$$\begin{aligned} \mathrm{vol}_f\, W \leq 2^{n-1}\big[&e^{aD} D\, \mathrm{vol}_f\, S' \\ &+ e^{(a+\frac{a}{2p})D} C(n,p)(\bar{\kappa}(p,f,a,D))^{\frac{1}{2}}\, \mathrm{vol}_f\, B_{x_1}(D)\big], \end{aligned} \tag{3-4}$$

where $D=\sup_{x\in W} d(x_1,x)$ *and* S' *is the set of intersection points with* S *of geodesics* γ_{xx_1} *for all* $x\in W$.

Proof. Let $\Gamma\subset S_{x_1}$ be the set of unit vectors v such that $\gamma_v=\gamma_{x_1x_2}$ for some $x_2\in W$. Using the polar coordinate $(\theta,t)\in S_{x_1}\times\mathbb{R}^+$ and $e^{-f}d\,\mathrm{vol}=\mathcal{A}_f(\theta,t)\,d\theta\wedge dt$. Recalling [Wu 2018, Theorem 3.1], we have

$$\frac{\partial}{\partial t}\frac{\mathcal{A}_f}{t^{n-1}e^{at}}=\Big(\Delta_f t-\frac{n-1}{t}-a\Big)\frac{\mathcal{A}_f}{t^{n-1}e^{at}}\le\psi\frac{\mathcal{A}_f}{t^{n-1}e^{at}} \tag{3-5}$$

with $\psi=(\Delta_f t-(n-1)/t-a)_+$. Integrating (3-5) from t to s gives

$$\begin{aligned}\mathcal{A}_f(s,\theta)&\le\Big(\frac{s}{t}\Big)^{n-1}e^{a(s-t)}\Big(\mathcal{A}_f(t,\theta)+\int_t^s\psi\mathcal{A}_f(l,\theta)\,dl\Big)\\&\le 2^{n-1}e^{\frac{as}{2}}\Big(\mathcal{A}_f(t,\theta)+\int_t^s\psi\mathcal{A}_f(l,\theta)\,dl\Big)\end{aligned}$$

for any $\frac{1}{2}s\le t\le s$. For any $\theta\in\Gamma$, let $s_1(\theta)$ and $s_2(\theta)$ be the minimum and maximum radius, respectively, such that $\exp_{x_1}(s_i\theta)\in W$, and $s(\theta)$ such that $\exp_{x_1}(s(\theta)\theta)\in S$. Then Lemma 3.3 implies that $2s(\theta)\ge s_2(\theta)\ge s_1(\theta)\ge s(\theta)$. Thus,

$$\begin{aligned}\mathrm{vol}_f W&\le\int_\Gamma\int_{s_1(\theta)}^{s_2(\theta)}\mathcal{A}_f(s,\theta)\,ds\,d\theta\\&\le 2^{n-1}e^{as(\theta)}\int_\Gamma\int_{s_1(\theta)}^{s_2(\theta)}\Big(\mathcal{A}_f(s(\theta),\theta)+\int_{s(\theta)}^s\psi\mathcal{A}_f(l,\theta)\,dl\Big)\,ds\,d\theta\\&\le 2^{n-1}e^{aD}D\Big(\int_\Gamma\mathcal{A}_f(s(\theta),\theta)d\theta+\int_0^D\int_\Gamma\psi\mathcal{A}_f(s,\theta)\,ds\,d\theta\Big).\end{aligned} \tag{3-6}$$

On the other hand,

$$\mathrm{vol}_f S'=\int_\Gamma\frac{\mathcal{A}_f(s(\theta),\theta)}{\cos\alpha(\theta)}\,d\theta\ge\int_\Gamma\mathcal{A}_f(s(\theta),\theta)\,d\theta,$$

where $\alpha(\theta)$ is the angle between S and the radical geodesic $\exp_{x_1}(s\theta)$. Applying this result, the Hölder inequality, (1-3) and the Laplacian comparison (2-3) successively in (3-6), we obtain

$$\begin{aligned}\mathrm{vol}_f W&\le 2^{n-1}e^{aD}D[\mathrm{vol}_f S'+\|\psi\|_{1,f,B_{x_1}(D)}]\\&\le 2^{n-1}e^{aD}D[\mathrm{vol}_f S'+\|\psi\|^*_{2p,f,B_{x_1}(D)}\,\mathrm{vol}_f B_{x_1}(D)]\\&\le 2^{n-1}e^{aD}D[\mathrm{vol}_f S'+e^{\frac{aD}{2p}}\|\psi\|^*_{2p,f,a}(D)\,\mathrm{vol}_f B_{x_1}(D)]\\&\le 2^{n-1}e^{aD}D\big[\mathrm{vol}_f S'+e^{\frac{aD}{2p}}C(n,p)D^{-1}(\bar{\kappa}(p,f,a,D))^{\frac{1}{2}}\,\mathrm{vol}_f B_{x_1}(D)\big],\end{aligned}$$

which completes the required estimate. □

Lemma 3.4 enables us to obtain a local Cheeger's constant estimate.

Lemma 3.5. *Let M^n_f, S, W and x_1 be the same as in Lemma 3.3. For $p > \frac{1}{2}n$, there exists $\varepsilon = \varepsilon(p, n, a)$ such that if $\bar{\kappa}(p, f, a, 1) \le \varepsilon$, then for a geodesic ball $B = B_x(r)$, $r \le \frac{1}{2}$ which is divided equally by S, we have*

$$\operatorname{vol}_f B_x(r) \le 2^{n+3} r e^a \operatorname{vol}_f(S \cap B_x(2r)).$$

Proof. To begin, we choose $W_i = B_x(2r) \cap M_i$ with M_i as in Lemma 3.3; then

$$\operatorname{vol}_f(B \cap M_1) = \operatorname{vol}_f(B \cap M_2) \le \min\{\operatorname{vol}_f W_1, \operatorname{vol}_f W_2\} \le 2 \operatorname{vol}_f W,$$

and $D \le 2r$ and $S' \subset S \cap B_x(2r)$. Thus, by Lemma 3.4, we have

$$\begin{aligned} \operatorname{vol}_f B_x(r) &\le 4 \operatorname{vol}_f W \\ &\le 2^{n+1}\big[2re^{2ar} \operatorname{vol}_f(S \cap B_x(2r)) \\ &\qquad + e^{2r(a+\frac{a}{2p})} C(n, p)(\bar{\kappa}(p, f, a, 2r))^{\frac{1}{2}} \operatorname{vol}_f B_x(2r)\big]. \end{aligned} \tag{3-7}$$

Next we aim at canceling the curvature inequality term. Since volume doubling property (2-5) implies that

$$\operatorname{vol}_f B_x(2r) \le 2\frac{V(n, a, 2r)}{V(n, a, r)} \cdot \operatorname{vol}_f B_x(r) \le 2^{n+1} e^a \operatorname{vol}_f B_x(r) \tag{3-8}$$

holds for $\bar{\kappa}(p, f, a, 2r) \le \varepsilon_0$. From the curvature inequality (2-6) and $r \le \frac{1}{2}$, the curvature condition reduces to $\bar{\kappa}(p, f, a, 1) \le 2^{-1/p} e^{-a/p} \varepsilon_0$, and

$$e^{2r(a+\frac{a}{2p})} C(n, p)(\bar{\kappa}(p, f, a, 2r))^{\frac{1}{2}} \le e^{a+\frac{a}{2p}} C(n, p)(2^{\frac{1}{p}} e^{\frac{a}{p}} \bar{\kappa}(p, f, a, 1))^{\frac{1}{2}}. \tag{3-9}$$

Inserting (3-8) and (3-9) into (3-7) gives

$$\begin{aligned} \operatorname{vol}_f B_x(r) &\le 2^{n+2} e^a r \operatorname{vol}_f(S \cap B_x(2r)) \\ &\qquad + 2^{2n+2+\frac{1}{2p}} e^{2a+\frac{a}{p}} C(n, p)(\bar{\kappa}(p, f, a, 1))^{\frac{1}{2}} \operatorname{vol}_f B_x(r). \end{aligned} \tag{3-10}$$

Here we used $r \le \frac{1}{2}$. Hence, we get the required result by choosing

$$\varepsilon(n, p, a) = \min\{2^{-\frac{1}{p}} e^{-\frac{a}{p}} \varepsilon_0, (2 \cdot 2^{2n+2+\frac{1}{2p}} e^{2a+\frac{a}{p}} C(n, p))^{-2}\}$$

and regrouping (3-10). □

Volume doubling property (2-5) indicates that the volume quotient of a concentric geodesic ball is lower bound by a function of the quotient of corresponding radius. The next theorem not only offers the other direction but also extends [Dai et al. 2018, Theorem 3.3] to the case of integral Bakry–Émery Ricci curvature. Actually, by fixing the upper bound to $\frac{1}{2}$ and the bigger radius to 1, we obtain the other radius.

Theorem 3.6. *Let M_f^n be a complete smooth metric measure space with $\partial_r f \geq -a$ along all minimal geodesic segments for some constant $a \geq 0$. For $p > \frac{1}{2}n$, there exists $\varepsilon = \varepsilon(n, p, a) > 0$ and $r_0 = r_0(n, a) > 0$ such that if $\bar{\kappa}(p, f, a, 1) \leq \varepsilon$, then*

$$\frac{\operatorname{vol}_f B_x(r_0)}{\operatorname{vol}_f B_x(1)} \leq \frac{1}{2} \quad \forall x \in M_f^n. \tag{3-11}$$

Our proof follows the idea in [Dai et al. 2018, Theorem 3.3], but using the approach directly runs into obstacles. The difficulties were conquered by repeating the process of choosing the radius $k-1$ times with k depending on a.

Proof. For any $x \in M_f^n$ and $i = 2, \dots, k$, let $r_1 = 1$, choose points $x_i \in B_x(r_{i-1})$ with $r_i < \frac{1}{3}r_{i-1}$ and $d_i = d_i(x, x_i) = \frac{1}{2}(r_{i-1} - r_i) > \frac{1}{3}r_{i-1}$, then

$$B_x(r_i) \subset B_{x_i}(d_i + r_i) \setminus B_{x_i}(d_i - r_i) \subset B_{x_i}(d_i + r_i) \subset B_x(r_{i-1}), \quad i = 2, 3, \dots, k.$$

Moreover,

$$\frac{\operatorname{vol}_f B_x(r_i)}{\operatorname{vol}_f B_x(r_{i-1})} \leq \frac{\operatorname{vol}_f(B_{x_i}(d_i + r_i) \setminus B_{x_i}(d_i - r_i))}{\operatorname{vol}_f B_{x_i}(d_i + r_i)} \leq 1 - \frac{\operatorname{vol}_f B_{x_i}(d_i - r_i)}{\operatorname{vol}_f B_{x_i}(d_i + r_i)}.$$

By (2-4), we have

$$\begin{aligned}
&\frac{\operatorname{vol}_f B_{x_i}(d_i - r_i)}{\operatorname{vol}_f B_{x_i}(d_i + r_i)} \\
&\qquad \geq \left(\frac{d_i - r_i}{d_i + r_i}\right)^n e^{-a(d_i + r_i)} [1 - C(n, p) e^{(1 + \frac{1}{2p})a} \bar{\kappa}^{\frac{1}{2}}(p, f, a, d_i + r_i)]^{2p} \\
&\qquad \geq \left(\frac{d_i - r_i}{d_i + r_i}\right)^n e^{-a} [1 - C(n, p) e^{(1 + \frac{1}{2p})a} \cdot (2e^a)^{\frac{1}{2p}} \bar{\kappa}^{\frac{1}{2}}(p, f, a, 1)]^{2p} \\
&\qquad = \left(\frac{d_i - r_i}{d_i + r_i}\right)^n e^{-a} [1 - C(n, p) e^{(1 + \frac{1}{p})a} 2^{\frac{1}{2p}} \bar{\kappa}^{\frac{1}{2}}(p, f, a, 1)]^{2p}.
\end{aligned}$$

Here we used the curvature inequality (2-6) and $d_i + r_i \leq 1$ in the second inequality. Choose a $q = q(n)$ such that

$$\frac{1-q}{1+q} = \left(\tfrac{3}{4}\right)^{\frac{1}{n}},$$

then for any $r_i \leq \frac{1}{3}q r_{i-1}$, since $d_i > \frac{1}{3}r_{i-1}$, we have

$$\left(\frac{d_i - r_i}{d_i + r_i}\right)^n \geq \tfrac{3}{4}.$$

Choose $\varepsilon \leq \varepsilon_0$ such that

$$(1 - C(n, p) e^{(1 + \frac{1}{p})a} 2^{\frac{1}{p}} \bar{\kappa}^{\frac{1}{2}}(p, f, a, 1))^{2p} \geq \tfrac{2}{3}. \tag{3-12}$$

Then

$$\frac{\operatorname{vol}_f B_{x_i}(d_i - r_i)}{\operatorname{vol}_f B_{x_i}(d_i + r_i)} \geq \tfrac{3}{4} \cdot e^{-a} \cdot \tfrac{2}{3} = \tfrac{1}{2} e^{-a},$$

and

$$\frac{\operatorname{vol}_f B_x(r_i)}{\operatorname{vol}_f B_x(r_{i-1})} \leq 1 - \tfrac{1}{2} e^{-a}.$$

Hence,

$$\frac{\operatorname{vol}_f B_x(r_k)}{\operatorname{vol}_f B_x(r_1)} = \prod_{i=2}^{k} \frac{\operatorname{vol}_f B_x(r_i)}{\operatorname{vol}_f B_x(r_{i-1})} = \left(1 - \tfrac{1}{2} e^{-a}\right)^{k-1}$$

with $r_k = \left(\tfrac{1}{3} q\right)^{k-1} r_1$. We choose the integer $k = k(a)$ such that

$$\left(1 - \tfrac{1}{2} e^{-a}\right)^{k-1} \leq \tfrac{1}{2} < \left(1 - \tfrac{1}{2} e^{-a}\right)^{k-2}.$$

Since $r_1 = 1$, the proof is completed by choosing $r_0 = r_k$ and $\varepsilon \leq \varepsilon_0$ satisfying (3-12). □

We now turn to prove the Theorem 1.1.

Proof of Theorem 1.1. Our first step is to show that the estimation (1-1) holds for some radius $r_0 = r_0(n, p, a)$ if $\bar{\kappa}(p, f, a, 1) \leq \varepsilon_1$ for some small constant $\varepsilon_1 = \varepsilon_1(n, p, a) > 0$. By Theorem 3.6, we assume that $\varepsilon_1 = \varepsilon_1(n, p, a)$ is chosen such that

$$\frac{\operatorname{vol}_f B_y(2r_0)}{\operatorname{vol}_f B_y\left(\tfrac{1}{10}\right)} \leq \tfrac{1}{2} \quad \forall\, y \in M_f^n$$

holds for some $r_0 = r_0(n, a)$. Given any $y_0 \in M_f^n$, let Ω be a smooth subdomain of $B_{y_0}(r_0)$. Assume that Ω is connected and its boundary $S = \partial\Omega$ divides M_f^n into Ω and Ω^c. For any $y \in \Omega$, let r_y be the smallest radius such that

$$\operatorname{vol}_f(B_y(r_y) \cap \Omega) = \operatorname{vol}_f(B_y(r_y) \cap \Omega^c) = \tfrac{1}{2} \operatorname{vol}_f B_y(r_y). \tag{3-13}$$

From $\Omega \subset B_y(2r_0)$ and $\operatorname{vol}_f B_y(2r_0) \leq \tfrac{1}{2} \operatorname{vol}_f B_y\left(\tfrac{1}{10}\right)$, it follows that $r_y \leq \tfrac{1}{10}$. Since Ω has a covering

$$\Omega \subset \cup_{y \in \Omega} B_y(2r_y),$$

thanks to the Vitali covering lemma (see [Lin and Yang 2002, §1.3]), there exists a countable family of disjoint balls $\{B_{y_i}(2r_i)\}$ such that $\Omega \subset \cup_i B_{y_i}(10r_i)$. On one hand, choosing ε_1 such that $\bar{\kappa}(p, a, f, r) \leq \varepsilon_0$ for all $r \leq 1$, and using the volume doubling property (2-5) leads to

$$\begin{aligned} \operatorname{vol}_f \Omega &\leq \sum_i \operatorname{vol}_f B_{y_i}(10r_i) \leq 2 \cdot 10^n \cdot \sum_i e^{10 r_i a} \operatorname{vol}_f B_{y_i}(r_i) \\ &\leq 2 \cdot 10^n \cdot e^a \sum_i \operatorname{vol}_f B_{y_i}(r_i). \end{aligned} \tag{3-14}$$

Choosing ε_1 as in Lemma 3.5 and using the disjoint of the balls $\{B_{x_i}(2r_i)\}$ gives

$$\operatorname{vol}_f \partial\Omega \geq \sum_i \operatorname{vol}_f(B_{y_i}(2r_i) \cap S) \geq 2^{-(n+3)} e^{-a} \sum_i (\operatorname{vol}_f B_{y_i}(r_i)) r_i^{-1}. \tag{3-15}$$

Combining (3-14) with (3-15) we obtain

$$\begin{aligned} \frac{\operatorname{vol}_f \partial\Omega}{(\operatorname{vol}_f \Omega)^{\frac{n-1}{n}}} &\geq 10^{-(n-1)} 2^{-(n+4-\frac{1}{n})} e^{-2a+\frac{a}{n}} \frac{\sum_i (\operatorname{vol}_f B_{y_i}(r_i)) r_i^{-1}}{\left(\sum_i (\operatorname{vol}_f B_{y_i}(r_i))\right)^{\frac{n-1}{n}}} \\ &\geq 2^{-n-1} \cdot 10^{-n} e^{-2a+\frac{a}{n}} \frac{\sum_i (\operatorname{vol}_f B_{y_i}(r_i)) r_i^{-1}}{\sum_i (\operatorname{vol}_f B_{y_i}(r_i))^{\frac{n-1}{n}}} \\ &\geq 2^{-1} \cdot 10^{-2n} e^{-2a+\frac{a}{n}} \inf_i \frac{(\operatorname{vol}_f B_{y_i}(r_i)) r_i^{-1}}{(\operatorname{vol}_f B_{y_i}(r_i))^{\frac{n-1}{n}}} \\ &\geq 2^{-1} \cdot 10^{-2n} e^{-2a+\frac{a}{n}} \inf_i [r_i^{-1} \operatorname{vol}_f^{\frac{1}{n}} B_{y_i}(r_i)]. \end{aligned} \tag{3-16}$$

On the other hand, since $d(y_i, y_0) \leq r_0$, then $B_{y_0}(r_0) \subset B_{y_i}(2r_0)$. Using the volume doubling property (2-5) with $r_i \leq \frac{1}{10}$ and (3-13) yields

$$\operatorname{vol}_f B_{y_i}(r_i) \geq \tfrac{1}{2}(10r_i)^n e^{-\frac{a}{10}} \operatorname{vol}_f B_{y_i}\left(\tfrac{1}{10}\right) \geq (10r_i)^n e^{-\frac{a}{10}} \operatorname{vol}_f B_{y_0}(r_0).$$

Inserting the above inequality into (3-16), we obtain

$$\frac{\operatorname{vol}_f \partial\Omega}{(\operatorname{vol}_f \Omega)^{\frac{n-1}{n}}} \geq 5 \cdot 10^{-2n} e^{-2a+\frac{a}{n}-\frac{a}{10n}} \operatorname{vol}_f^{\frac{1}{n}} B_{y_0}(r_0) \geq 10^{-2n} e^{-2a} \operatorname{vol}_f^{\frac{1}{n}} B_{y_0}(r_0).$$

Hence,

$$(\operatorname{vol}_f B_{y_0}(r_0))^{-\frac{1}{n}} \inf_\Omega \left\{ \frac{\operatorname{vol}_f \partial\Omega}{(\operatorname{vol}_f \Omega)^{\frac{n-1}{n}}} \right\} \geq 10^{-2n} e^{-2a}.$$

Our task now is to show (1-1) holds for any radius $R \leq 1$ and for $\bar{\kappa}(p, f, a, 1) \leq \varepsilon_2$ with $\varepsilon_2 = \varepsilon_2(n, p, a) > 0$. Let $r_1 = R/r_0 \leq 1/r_0$. After a scaling, its sufficient to check that $\bar{\kappa}(p, f, a, r_1) \leq \varepsilon_1$. Choose ε_2 satisfying $\varepsilon_2 \leq \varepsilon_0$, and then (2-5) holds for all $R \leq 1$. Now if $r_1 \leq 1$, by (2-6),

$$\bar{\kappa}(p, f, a, r_1) \leq 2^{\frac{1}{p}} e^{\frac{a}{p}} \bar{\kappa}(p, f, a, 1) \leq 2^{\frac{1}{p}} e^{\frac{a}{p}} \varepsilon_2.$$

If $1 < r_1 \leq 1/r_0$, then by (2-7) we have

$$\bar{\kappa}(p, f, a, r_1) \leq 2^{\frac{n+1}{p}} e^{\frac{ar_1}{p}} \bar{\kappa}(p, f, a, 1) \leq 2^{\frac{n+1}{p}} e^{\frac{a}{pr_0}} r_0^{-2} \varepsilon_2.$$

Using the two cases, let

$$\varepsilon_2 = \min\{2^{-\frac{1}{p}} e^{-\frac{a}{p}} \varepsilon_1, 2^{-\frac{n+1}{p}} e^{-\frac{a}{pr_0}} r_0^2 \varepsilon_1, \varepsilon_0\}.$$

The proof is completed by setting $\varepsilon = \varepsilon_2$. $\square$

Theorem A.1 implies that the normalized constants in (3-1) are equivalent, that is,

$$\mathrm{ID}^*_{n,f}\, B_x(R) = \mathrm{SD}^*_{n,f}\, B_x(R).$$

Hence, Theorem 1.1 gives the following Sobolev inequality:

Corollary 3.7. *If $\bar{\kappa}(p, f, a, 1) \le \varepsilon$ for the ε in Theorem 1.1, then for any $R \le 1$,*

$$\|\nabla h\|^*_{1,f,B_x(R)} \ge 10^{-2n} e^{-2a} R^{-1} \|h\|^*_{\frac{n}{n-1},f,B_x(R)} \quad \forall h \in C_0^\infty(B_x(R)) \tag{3-17}$$

and

$$\|\nabla h\|^*_{2,f,B_x(R)} \ge \frac{n-2}{2(n-1)} 10^{-2n} e^{-2a} R^{-1} \|h\|^*_{\frac{2n}{n-2},f,B_x(R)} \quad \forall h \in C_0^\infty(B_x(R)). \tag{3-18}$$

Applying (3-17) to $h^{\frac{2(n-1)}{n-2}}$, together with the Hölder inequality, we get (3-18).

The first eigenvalue of the f-Laplacian is defined by

$$\lambda_1(B_x(R)) = \inf_{h \in C_0^\infty(B_x(R))} \frac{\int_{B_x(R)} |\nabla h|^2 \, d\,\mathrm{vol}_f}{\int_{B_x(R)} h^2 \, d\,\mathrm{vol}_f}.$$

As in Cheeger's inequality [1970] we have:

Corollary 3.8. *With the same assumption as in Theorem 1.1, for $p > \frac{1}{2}n$, there exists $\varepsilon = \varepsilon(n, p, a) > 0$ such that $\bar{\kappa}(p, f, a, 1) \le \varepsilon$, then for any $R \le 1$, the first eigenvalue of the Dirichlet f-Laplacian has lower bound*

$$\lambda_1(B_x(R)) \ge C(n, p, a) R^{-2}, \quad C(n, p, a) = \frac{(n-2)^2}{4(n-1)^2} 10^{-4n} e^{-4a}.$$

Proof. Suppose $\Delta_f h = -\lambda h$ for some $\lambda > 0$, and normalize h such that

$$\fint_{B_x(R)} h^2 e^{-f} \, d\,\mathrm{vol} = 1$$

and $h = 0$ on $\partial B_x(R)$. Then using (3-18) we have

$$\begin{aligned} \lambda = (\|\nabla h\|^*_{2,f,B_x(R)})^2 &\ge \Big(\frac{n-2}{2(n-1)} 10^{-2n} e^{-2a} R^{-1} \|h\|^*_{\frac{2n}{n-2},f,B_x(R)} \Big)^2 \\ &\ge \frac{(n-2)^2}{4(n-1)^2} 10^{-4n} e^{-4a} R^{-2} (\|h\|^*_{2,f,B_x(R)})^2 \\ &= \frac{(n-2)^2}{4(n-1)^2} 10^{-4n} e^{-4a} R^{-2}. \end{aligned}$$

□

4. Applications

In this section, we prove the maximum principle and the gradient estimate for integral Bakry–Émery Ricci curvature with the help of the normalized local Dirichlet Sobolev constant estimate.

Let $C_s(\Omega)$ be the normalized local Sobolev constant of $\Omega \subset B_x(R) \subset M^n_f$ such that

$$\|h\|^*_{\frac{2n}{n-2},f,\Omega} \le C_s(\Omega)\|\nabla h\|^*_{2,f,\Omega} \quad \forall h \in C^\infty_0(\Omega). \tag{4-1}$$

Obviously, $C_s(\Omega)$ is the smallest constant such that (4-1) holds for all $h \in C^\infty_0(\Omega)$. Since $h \in C^\infty_0(B_x(R))$, then (3-18) gives

$$C_s(\Omega) \le C_s(B_x(R)) \le \frac{2(n-1)}{n-2} 10^{2n} e^{2a} R. \tag{4-2}$$

Theorem 4.1. *Let M^n_f be a smooth metric measure space and $\Omega \subset B_x(R) \subset M^n_f$ be a domain. For $p > \frac{1}{2}n$ and any function u with $u|_{\partial\Omega} = 0$, we have*

$$\|u\|^*_{\infty,f,\Omega} \le C^2_s(B_x(R))C(n, p)\|\Delta_f u\|^*_{p,f,B_x(R)},$$

where $C(n, p)$ is a constant depending on n and p.

The proof of this result is quite similar to the one used in [Petersen and Wei 2001, Theorem 3.1] with $s = \frac{1}{2}n$ due to the Sobolev inequality and the self-adjoint of Δ_f, and we omit it.

Corollary 4.2. *With the same assumption as in Theorem 4.1, for $p > \frac{1}{2}n$ and any function $u : \Omega \subset M^n_f \to \mathbb{R}$ with $\Delta_f u \ge -h$, where h is nonnegative on Ω, we have*

$$\sup_\Omega u \le \sup_{\partial\Omega} u + C(n, p) \cdot C^2_s(B_x(R)) \cdot \|h\|^*_{p,f,\Omega}.$$

Proof. Without loss of generality, we can assume that $\sup_{x\in\partial\Omega} u(x) = 0$. Then we have the Dirichlet problem

$$\Delta_f v = -h \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega. \tag{4-3}$$

Hence $u - v$ is f-subharmonic, and $u - v \le 0$ on $\partial\Omega$. By the maximum principle we get $u \le v$ in $\bar{\Omega}$, that is $\sup_\Omega u \le \|v\|_{\infty,f,\Omega}$. Using Theorem 4.1 we complete the proof. □

Combining Corollary 4.2 with (4-2) gives Theorem 1.4.

Following the idea in [Dai et al. 2018, Theorem 5.2], we use the standard and powerful Nash–Moser iteration and establish the gradient estimate. We give the details of the proof because there are still many differences compared to the previous arguments.

Theorem 4.3. *Let M_f^n be a complete smooth metric measure space. Assume that M_f^n satisfies $\partial_r f \geq -a$ along all minimal geodesic segments for some constant $a \geq 0$. For $p > \frac{1}{2}n$, if u is a function on $B_x(R)$ with $u = 0$ on $\partial B_x(R)$ and satisfying*

$$\Delta_f u = h,$$

then

$$\begin{aligned}\sup_{B_x(\frac{R}{2})} |\nabla u|^2 \leq C(n,p) R^{-2} \frac{\operatorname{vol}_f B_x(R)}{\operatorname{vol}_f B_x\big(\frac{3}{4}R\big)} \big((\|h\|^*_{2p,f,B_x(R)})^2 + (\|u\|^*_{2,f,B_x(R)})^2\big) \\ \times \big(R^{-2} C_s^2(B_x(R))(1 + a + R^{-2} C_s^2(B_x(R)) e^{\frac{a}{p}} \bar{\kappa}(p,f,a,R)) \\ + (R^{-2} C_s^2(B_x(R)) e^{\frac{a}{p}} \bar{\kappa}(p,f,a,R))^{\frac{2p}{2p-n}}\big)^{\frac{n}{2}}.\end{aligned}$$

Proof. By scaling we assume $R = 1$. We omit the volume form $e^{-f} d\operatorname{vol}$ for convenience. From the Bochner formula, we have

$$\begin{aligned}\Delta_f |\nabla u|^2 &= 2|\operatorname{Hess} u|^2 + 2\langle \nabla u, \nabla \Delta_f u\rangle + 2\operatorname{Ric}_f(\nabla u, \nabla u) \\ &\geq 2\langle \nabla u, \nabla h\rangle - 2|\operatorname{Ric}_{f-}||\nabla u|^2.\end{aligned} \tag{4-4}$$

Let $v = |\nabla u|^2 + \|h\|^*_{p,f}$ and $v = |\nabla u|^2$ if h is constant. For any $\eta \in C_0^\infty(B_x(1))$, $l > 1$, we have

$$\begin{aligned}\int |\nabla(\eta v^l)|^2 &= -\int (\eta v^l)[v^l \Delta_f \eta + 2\langle \nabla \eta, \nabla v^l\rangle + \eta \Delta_f v^l] \\ &= \int v^{2l}(-\eta \Delta_f \eta) - 2\int v^l \langle \nabla \eta, \eta \nabla v^l\rangle \\ &\quad - l\int \eta^2 v^{2l-1} \Delta_f v - l(l-1)\int \eta^2 v^{2l-2} |\nabla v|^2 \\ &\leq \int v^{2l}(-\eta \Delta_f \eta) + \frac{l}{l-1}\int v^{2l}|\nabla \eta|^2 + \frac{l-1}{l}\int \eta^2 |\nabla v^l|^2 \\ &\quad - l\int \eta^2 v^{2l-1}(2\langle \nabla u, \nabla h\rangle - 2|\operatorname{Ric}_{f-}|v) - \frac{l-1}{l}\int \eta^2 |\nabla v^l|^2 \\ &= \int v^{2l}(-\eta \Delta_f \eta) + \frac{l}{l-1}\int v^{2l}|\nabla \eta|^2 \\ &\quad - 2l\int \eta^2 v^{2l-1}\langle \nabla u, \nabla h\rangle + 2l\int \eta^2 v^{2l}|\operatorname{Ric}_{f-}|.\end{aligned} \tag{4-5}$$

Here we used Young's inequality $2xy \leq \varepsilon x^2 + \frac{1}{\varepsilon} y^2$ with $\varepsilon = \frac{l}{l-1}$ and (4-4) in the above inequality. Integrating by parts and using $|\nabla u| \le v^{1/2}$ gives

$$\begin{aligned}
&-2l \int \eta^2 v^{2l-1} \langle \nabla u, \nabla h\rangle \\
&\quad = 2l \int \eta^2 v^{2l-1} h^2 + 4l \int \eta h v^{2l-1} \langle \nabla \eta, \nabla u\rangle \\
&\qquad\qquad + 2l(2l-1) \int \eta^2 h v^{2l-2} \langle \nabla u, \nabla v\rangle \\
&\quad \le 2l \int \eta^2 v^{2l-1} h^2 + 2 \int (2\sqrt{l}\,\eta v^{l-\frac{1}{2}} h)(\sqrt{l}\, v^l |\nabla \eta|) \\
&\qquad\qquad + 2 \int ((4l-2)\eta v^{l-\frac{1}{2}} h)\big(\tfrac{1}{2}\eta |\nabla v^l|\big) \\
&\quad = [6l + (4l-2)^2] \int \eta^2 v^{2l-1} h^2 \\
&\qquad\qquad + l \int v^{2l} |\nabla \eta|^2 + \tfrac{1}{4} \int |\nabla(\eta v^l) - v^l \nabla \eta|^2 \\
&\quad \le (16l^2 - 10l + 4) \int \eta^2 v^{2l-1} h^2 + \big(l + \tfrac{1}{2}\big) \int v^{2l} |\nabla \eta|^2 + \tfrac{1}{2} |\nabla(\eta v^l)|^2.
\end{aligned}$$

Here we used the L^2-Hölder inequality and

$$\eta^2 |\nabla v^l|^2 = |\nabla(\eta v^l) - v^l \nabla \eta|^2 \le 2|\nabla(\eta v^l)|^2 + 2v^{2l} |\nabla \eta|^2.$$

Inserting (4-6) into (4-5) and regrouping gives

$$\begin{aligned}
\int |\nabla(\eta v^l)|^2 \le \int v^{2l}(-2\eta \Delta_f \eta) + \Big(\frac{2l}{l-1} + 2l + 1\Big) \int v^{2l} |\nabla \eta|^2 \\
+ 4(8l^2 - 5l + 2) \int \eta^2 v^{2l-1} h^2 + 4l \int \eta^2 v^{2l} |\mathrm{Ric}_{f-}|.
\end{aligned} \tag{4-6}$$

In order to control $\Delta_f \eta$, choose a cut-off function $\phi \in C_0^\infty(B_x(1))$ such that $0 \le \phi \le 1$. For $0 < r < 1$, $\phi(t) \equiv 1$; for $t \in [0, r]$, $\phi(t) \equiv 0$; for $t \ge 1$, and $\phi' \le 0$. Then define $\eta(y) = \phi(r(y))$ and let $r(y) = d(x, y)$ be a distance function from x. Thus $|\nabla \eta| = |\phi'|$, and

$$\begin{aligned}
\Delta_f \eta = \phi'' + \phi' \Delta_f r \ge \phi'' + \phi' \Big(\psi + \frac{n-1}{r} + a\Big) \\
\ge -|\phi''| - |\phi'|\psi - \frac{n-1}{r} |\phi'| - a|\phi'|,
\end{aligned} \tag{4-7}$$

where $\psi = (\Delta_f r - (n-1)/r - a)_+$. Hence, for $l \ge n/(n-2)$, (4-6) becomes

$$\begin{aligned}
\int |\nabla(\eta v^l)|^2 \le C(n) l^2 \int \big((|\phi''| + |\phi'| r^{-1} + a)\eta v^{2l} + |\phi'|\psi \eta v^{2l} \\
+ |\phi'|^2 v^{2l} + \eta^2 h^2 v^{2l-1} + \eta^2 v^{2l} |\mathrm{Ric}_{f-}|\big).
\end{aligned}$$

Notice that this formula remains valid for $l = 1$. Indeed,

$$|\nabla(\eta v^{\frac{1}{2}})|^2 = \left| v^{\frac{1}{2}}\nabla\eta + \eta\frac{|\nabla u|}{v^{1/2}}\nabla|\nabla u|\right|^2 \le 2v|\nabla\eta|^2 + 2\eta^2|\operatorname{Hess} u|^2,$$

and

$$\begin{aligned}\int \eta^2|\operatorname{Hess} u|^2 &= -\int \nabla_i u(2\eta\nabla_j\nabla_i\nabla_j u + \eta^2\nabla_i\Delta_f u + \eta^2(\mathrm{Ric}_f)_{ij}\nabla_j u)\\ &\le \tfrac{1}{2}\int \eta^2|\operatorname{Hess} u|^2 + 3\int|\nabla\eta|^2 v + 2\int\eta^2h^2 + \int\eta^2|\mathrm{Ric}_{f-}|v.\end{aligned}$$

Next we use C_s denote $C_s(B_x(1))$ for simplicity. Letting $\beta = n/(n-2)$, and applying the Sobolev inequality (4-1), then for $l \ge n/(n-2)$ and $l = 1$,

$$\begin{aligned}(4\text{-}8)\quad &\left(⨍_{B_x(1)}(\eta^2v^{2l})^\beta\right)^{\frac{1}{\beta}}\\ &\le C_s^2 ⨍_{B_x(1)}|\nabla(\eta v^l)|^2\\ &\le C_s^2C(n)l^2 ⨍_{B_x(1)}(|\phi''|+|\phi'|r^{-1}+a)\eta v^{2l}\\ &\quad + C_s^2C(n)l^2 ⨍_{B_x(1)}(|\phi'|\psi\eta v^{2l}+|\phi'|^2v^{2l}+\eta^2h^2v^{2l-1}+\eta^2v^{2l}|\mathrm{Ric}_{f-}|).\end{aligned}$$

The integration involving Bakry–Émery Ricci curvature can be estimated as follows:

$$\begin{aligned}&⨍_{B_x(1)}\eta^2v^{2l}|\mathrm{Ric}_{f-}|\\ &\le \|\mathrm{Ric}_{f-}\|^*_{p,f,B_x(1)}\left(⨍_{B_x(1)}(\eta^2v^{2l})^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\\ &\le e^{\frac{a}{p}}\|\mathrm{Ric}_{f-}\|^*_{p,a,f}(1)\left(⨍_{B_x(1)}\eta^2v^{2l}\right)^{\frac{p-1}{p}q}\left(⨍_{B_x(1)}(\eta^2v^{2l})^\beta\right)^{\frac{p-1}{p}(1-q)}\\ &\le e^{\frac{a}{p}}\bar\kappa(p,f,a,1)\left[\varepsilon\left(⨍_{B_x(1)}(\eta^2v^{2l})^\beta\right)^{\frac{1}{\beta}} + \varepsilon^{-\frac{n}{2p-n}}\left(⨍_{B_x(1)}\eta^2v^{2l}\right)\right],\end{aligned}$$

where $q = q(n,p) = (2p-n)/(2(p-1)) > 0$ is determined by $q + (1-q)\beta = p/(p-1)$, and we also used the Young's inequality

$$xy \le \varepsilon x^b + \varepsilon^{-\frac{b^*}{b}}y^{b^*} \quad \forall x,y \ge 0,\ b > 1, \quad \frac{1}{b^*}+\frac{1}{b} = 1,$$

where

$$b = \frac{p}{(1-q)(p-1)\beta}, \quad b^* = \frac{p}{(p-1)q}.$$

By choosing $\varepsilon = (4C_s^2 C(n) l^2 e^{\frac{a}{p}} \bar{\kappa}(p, f, a, 1))^{-1}$, we obtain

$$C_s^2 C(n) l^2 \fint_{B_x(1)} \eta^2 v^{2l} |\mathrm{Ric}_{f-}| \le \frac{1}{4}\left(\fint_{B_x(1)} (\eta^2 v^{2l})^\beta\right)^{\frac{1}{\beta}} + C(n,p)(C_s^2 l^2 e^{\frac{a}{p}} \bar{\kappa}(p,f,a,1))^{\frac{2p}{2p-n}} \fint_{B_x(1)} \eta^2 v^{2l}. \tag{4-9}$$

For the term $\fint_{B_x(1)} \eta^2 h^2 v^{2l-1}$, since $v \ge \|h^2\|^*_{p,f,B_x(1)}$, we have

$$\fint_{B_x(1)} \eta^2 h^2 v^{2l-1} \le \frac{1}{\|h^2\|^*_{p,f,B_x(1)}} \fint_{B_x(1)} \eta^2 h^2 v^{2l} \le \left(\fint_{B_x(1)} (\eta^2 h^2 v^{2l})^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}.$$

Now the same argument as above with $\varepsilon = (4C_s^2 C(n) l^2)^{-1}$ gives

$$C_s^2 C(n) l^2 \fint_{B_x(1)} \eta^2 h^2 v^{2l-1} \le \frac{1}{4}\left(\fint_{B_x(1)} (\eta^2 v^{2l})^\beta\right)^{\frac{1}{\beta}} + C(n,p)(C_s^2 l^2)^{\frac{2p}{2p-n}} \fint_{B_x(1)} \eta^2 v^{2l}. \tag{4-10}$$

For the term with ψ, applying the Hölder inequality, (1-3) and the Laplacian comparison (2-3) gives

(4-11)
$$\begin{aligned} C_s^2 C(n) l^2 & \fint_{B_x(1)} |\phi'| \psi \eta v^{2l} \\ & \le C_s^2 C(n) l^2 \|\psi\|^*_{2p,f,B_x(1)} \cdot \|\eta \phi' v^{2l}\|^*_{\frac{2p}{2p-1},f,B_x(1)} \\ & \le C_s^2 C(n) l^2 e^{\frac{a}{2p}} \|\psi\|^*_{2p,a,f}(1) \cdot \|\eta \phi' v^{2l}\|^*_{\frac{2p}{2p-1},f,B_x(1)} \\ & \le C_s^2 C(n) l^2 e^{\frac{a}{2p}} C(n,p)(\bar{\kappa}(p,f,a,1))^{\frac{1}{2}} \cdot \|\eta \phi' v^{2l}\|^*_{\frac{2p}{2p-1},f,B_x(1)}. \end{aligned}$$

Note that for

$$\alpha = \frac{p(n-2)}{n(2p-1)} = \frac{1}{\beta}\frac{p}{2p-1} < 1,$$

we have

$$
\begin{aligned}
\text{(4-12)}\quad &\|\eta\phi' v^{2l}\|^*_{\frac{2p}{2p-1}}\\
&= \left[\fint_{B_x(1)} (\eta^2 v^{2l})^{\alpha\beta} (|\phi'|^2 v^{2l})^{\frac{p}{2p-1}} \right]^{\frac{2p-1}{2p}}\\
&\le \left[\left(\fint_{B_x(1)} (\eta^2 v^{2l})^{\beta} \right)^{\alpha} \left(\fint_{B_x(1)} (|\phi'|^2 v^{2l})^{\frac{np}{np+2p-n}} \right)^{\frac{np+2p-n}{n(2p-1)}} \right]^{\frac{2p-1}{2p}}\\
&\le \left[\left(\fint_{B_x(1)} (\eta^2 v^{2l})^{\beta} \right)^{\alpha} \left(\fint_{B_x(1)} |\phi'|^2 v^{2l} \right)^{\frac{p}{2p-1}} \right]^{\frac{2p-1}{2p}}\\
&\le \varepsilon \left(\fint_{B_x(1)} (\eta^2 v^{2l})^{\beta} \right)^{\frac{1}{\beta}} + \frac{1}{4\varepsilon} \fint_{B_x(1)} |\phi'|^2 v^{2l},
\end{aligned}
$$

where we used the Hölder inequality; note that $p > \frac{1}{2}n$ implies $np/(np+2p-n) < 1$ in the second inequality. By setting

$$\varepsilon = (4C_s^2 C(n) l^2 e^{\frac{a}{2p}} C(n,p) (\bar{\kappa}(p,f,a,1))^{1/2})^{-1}$$

and inserting (4-12) into (4-11) we obtain

$$
\begin{aligned}
\text{(4-13)}\quad C_s^2 C(n) l^2 \fint_{B_x(1)} \psi\eta|\phi'| v^{2l} &\le \frac{1}{4} \left(\fint_{B_x(1)} (\eta^2 v^{2l})^{\beta} \right)^{\frac{1}{\beta}}\\
&\quad + (C_s^4 C^2(n) l^4 e^{\frac{a}{p}} C^2(n,p) \bar{\kappa}(p,f,a,1)) \fint_{B_x(1)} |\phi'|^2 v^{2l}.
\end{aligned}
$$

Inserting (4-9), (4-10) and (4-13) into (4-8) gives

$$
\begin{aligned}
\text{(4-14)}\quad &\left(\fint_{B_x(1)} (\eta^2 v^{2l})^{\beta} \right)^{\frac{1}{\beta}}\\
&\le 4C_s^2 C(n) l^2 \fint_{B_x(1)} \Big(|\phi''| + \frac{|\phi'|}{r} + a \Big) \eta v^{2l}\\
&\quad + 4C_s^2 C(n) l^2 (1 + C_s^2 l^2 e^{\frac{a}{p}} C^2(n,p) \bar{\kappa}(p,f,a,1)) \fint_{B_x(1)} |\phi'|^2 v^{2l}\\
&\quad + C(n,p) (C_s^2 l^2)^{\frac{2p}{2p-n}} [1 + (e^{\frac{a}{p}} \bar{\kappa}(p,f,a,1))^{\frac{2p}{2p-n}}] \Big(\fint_{B_x(1)} \eta^2 v^{2l} \Big).
\end{aligned}
$$

Define $l_i = \frac{1}{2}\beta^i$, $i \ge 0$, and $r_i = \frac{3}{4} - \sum_{j=0}^{i} 2^{-j-1}$. Choose the cut-off functions $\eta_i = \phi_i(r) \in C_0^\infty(B_x(r_i))$ such that

$$\eta_i \equiv 1 \quad \text{on } B_x(r_{i+1}); \quad |\phi_i'| \le 2^{i+1}, \quad |\phi_i''| \le 2^{2i+2}.$$

Then (4-14) becomes

$$
\begin{aligned}
\|v\|^*_{\beta^{i+1},f,B_x(r_{i+1})}
&\le C(n,p)\bigl\{C_s^2(\beta^{4i})(1+a+C_s^2e^{\frac{a}{p}}\bar\kappa(p,f,a,1))\\
&\qquad\qquad + (C_s^2\beta^{2i})^{\frac{2p}{2p-n}}[1+(e^{\frac{a}{p}}\bar\kappa(p,f,a,1))^{\frac{2p}{2p-n}}]\bigr\}\|v\|^*_{\beta^i,f,B_x(r_i)}\\
&\le C(n,p)\beta^{si}\bigl\{C_s^2(1+a+C_s^2e^{\frac{a}{p}}\bar\kappa(p,f,a,1))\\
&\qquad\qquad + C_s^{\frac{4p}{2p-n}}[1+(e^{\frac{a}{p}}\bar\kappa(p,f,a,1))^{\frac{2p}{2p-n}}]\bigr\}\|v\|^*_{\beta^i,f,B_x(r_i)},
\end{aligned}
$$

where $s=\max\{4,4p/(2p-n)\}$. Then substituting η_i into the estimate and running the iteration from $i=0$ gives

$$\|v\|^*_{\infty,B_x(\frac{1}{2})}\le C(n,p)A^{\frac{n}{2}}\|v\|^*_{1,f,B_x(\frac{3}{4})}, \tag{4-15}$$

where

$$A=C_s^2(1+a+C_s^2e^{\frac{a}{p}}\bar\kappa(p,f,a,1))+C_s^{\frac{4p}{2p-n}}[1+(e^{\frac{a}{p}}\bar\kappa(p,f,a,1))^{\frac{2p}{2p-n}}].$$

Finally, we estimate the term $\|v\|^*_{1,f,B_x(3/4)}$. For $\eta\in C_0^\infty(B_x(1))$ with $\eta\equiv 1$ in $B_x\bigl(\frac{3}{4}\bigr)$ and $|\nabla\eta|\le 5$, then

$$
\begin{aligned}
\|v\|^*_{1,f,B_x(\frac{3}{4})} &\le \frac{\operatorname{vol}_f B_x(1)}{\operatorname{vol}_f B_x\bigl(\frac{3}{4}\bigr)}\|v\|^*_{1,f,B_x(1)}\\
&= \frac{\operatorname{vol}_f B_x(1)}{\operatorname{vol}_f B_x\bigl(\frac{3}{4}\bigr)}\fint_{B_x(1)}\eta^2(|\nabla u|^2+\|h\|^*_{p,f,B_x(1)}).
\end{aligned} \tag{4-16}
$$

Using the integration by parts and Young's inequality, we have

$$
\begin{aligned}
\fint_{B_x(1)}\eta^2|\nabla u|^2 &= -2\fint_{B_x(1)}\eta\langle\nabla\eta,\nabla u\rangle u-\fint_{B_x(1)}\eta^2hu\\
&\le \frac{1}{2}\fint_{B_x(1)}\eta^2|\nabla u|^2+2\fint_{B_x(1)}|\nabla\eta|^2u^2+\frac{1}{2}\fint_{B_x(1)}(\eta^2h^2+\eta^2u^2).
\end{aligned}
$$

Regrouping the above inequality and inserting it into (4-16) gives

$$
\begin{aligned}
\|v\|^*_{1,f,B_x(\frac{3}{4})} &= \frac{\operatorname{vol}_f B_x(1)}{\operatorname{vol}_f B_x\bigl(\frac{3}{4}\bigr)}\Bigl[4\fint_{B_x(1)}|\nabla\eta|^2u^2+\fint_{B_x(1)}\eta^2u^2\\
&\qquad\qquad +\fint_{B_x(1)}\eta^2h^2+\|h^2\|^*_{p,f,B_x(1)}\Bigr]\\
&\le \frac{\operatorname{vol}_f B_x(1)}{\operatorname{vol}_f B_x\bigl(\frac{3}{4}\bigr)}[101(\|u\|^*_{2,f,B_x(1)})^2+2(\|h\|^*_{2p,f,B_x(1)})^2].
\end{aligned} \tag{4-17}
$$

Combining (4-15) with (4-17) yields

$$\sup_{B_x(\frac{1}{2})} |\nabla u|^2 \le C(n,p)\frac{\mathrm{vol}_f\, B_x(1)}{\mathrm{vol}_f\, B_x\big(\frac{3}{4}\big)}[(\|h\|^*_{2p,f,B_x(1)})^2 + (\|u\|^*_{2,f,B_x(1)})^2] \times \Big\{C_s^2(1+a+C_s^2 e^{\frac{a}{p}}\bar{\kappa}(p,f,a,1)) + C_s^{\frac{4p}{2p-n}}[1+(e^{\frac{a}{p}}\bar{\kappa}(p,f,a,1))^{\frac{2p}{2p-n}}]\Big\}^{\frac{n}{2}}.$$

Thus we get the desired result by scaling. □

Combining Theorem 4.3 and the local Sobolev constant estimate (4-2) with the volume doubling property (3-18) gives Theorem 1.5.

Appendix: Equivalence of isoperimetric and Sobolev constants for weighted measure

Here we show the equivalence between the local isoperimetric and Sobolev constants defined in Section 3 by adapting the proof of [Li 2012, Theorem 9.5]. A special case of Theorem A.1 can be found in [Cheng and Oden 1997, Proposition 1.1].

Theorem A.1. *For all $n \le \alpha \le \infty$, we have*

$$\mathrm{ID}_{\alpha,f}\, B_x(R) = \mathrm{SD}_{\alpha,f}\, B_x(R). \tag{A-1}$$

Proof. Let σ_f be an $(n-1)$-dimensional Hausdorff measure. We omit the weighted measure $e^{-f}\,d\,\mathrm{vol}$ for convenience. For $\Omega \subset B_x(R)$ with $\Omega \cap \partial B_x(R) = \varnothing$, let $\Omega_\varepsilon = \{y \in \Omega : d(y, \partial\Omega) \ge \varepsilon\}$. Construct a function by

$$h_\varepsilon(y) = \begin{cases} 0 & B_x(R)\setminus\Omega, \\ (1/\varepsilon)d(y,\partial B_x(R)) & \Omega\setminus\Omega_\varepsilon, \\ 1, & \Omega_\varepsilon. \end{cases}$$

Since the distance function d is Lipschitz, h_ε is Lipschitz with $h_\varepsilon|_{\partial\Omega} = 0$. Applying the Sobolev inequality to h_ε gives

$$\int_{B_x(R)} |\nabla h_\varepsilon| \ge \mathrm{SD}_{\alpha,f}\, B_x(R)\bigg(\int_{B_x(R)} |h_\varepsilon|^{\frac{\alpha}{\alpha-1}}\bigg)^{\frac{\alpha-1}{\alpha}} \ge \mathrm{SD}_{\alpha,f}\, B_x(R)(\mathrm{vol}_f\,\Omega_\varepsilon)^{\frac{\alpha-1}{\alpha}}. \tag{A-2}$$

The coarea formula implies

$$\lim_{\varepsilon\to 0}\int_{B_x(R)} |\nabla h_\varepsilon| e^{-f}\,d\,\mathrm{vol} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon \sigma_f(\partial\Omega_t)\,dt = \sigma_f(\partial\Omega). \tag{A-3}$$

Combining (A-2) with (A-3), together with Definition 3.1, we have

$$\mathrm{ID}_{\alpha,f}\, B_x(R) \ge \mathrm{SD}_{\alpha,f}\, B_x(R).$$

To see that $\mathrm{ID}_{\alpha,f}\, B_x(R) \le \mathrm{SD}_{\alpha,f}\, B_x(R)$, it suffices to show that

$$\int_{B_x(R)} |\nabla h| \ge \mathrm{ID}_{\alpha,f}\, B_x(R) \left(\int_{B_x(R)} h^{\frac{\alpha}{\alpha-1}} \right)^{\frac{\alpha-1}{\alpha}} \tag{A-4}$$

holds for $h|_{\partial B_x(R)} = 0$. Without loss of generality, we may assume $h \ge 0$. Let $B_t := \{y \in B_x(R) \mid h(y) > t\}$ to be the sublevel set of h. By the coarea formula,

$$\int_{B_x(R)} |\nabla h| = \int_0^\infty \sigma_f(\partial B_t)\, dt \ge \mathrm{ID}_{\alpha,f}\, B_x(R) \int_0^\infty \mathrm{vol}_f^{\frac{\alpha-1}{\alpha}} B_t\, dt. \tag{A-5}$$

Let

$$F(s) = \left(\int_0^s \mathrm{vol}_f^{\frac{\alpha-1}{\alpha}} B_t\, dt \right)^{\frac{\alpha}{\alpha-1}} - \frac{\alpha}{\alpha-1} \int_0^s t^{\frac{1}{\alpha-1}} \mathrm{vol}_f\, B_t\, dt.$$

Obviously, $F(0) = 0$ and

$$F'(s) = \frac{\alpha}{\alpha-1} \left(\int_0^s \mathrm{vol}_f^{\frac{\alpha-1}{\alpha}} B_t\, dt \right)^{\frac{1}{\alpha-1}} \mathrm{vol}_f^{\frac{\alpha-1}{\alpha}} B_s - \frac{\alpha}{\alpha-1} s^{\frac{1}{\alpha-1}} \mathrm{vol}_f\, B_s.$$

Since $B_s \subset B_t$ for $t \le s$, then $\int_0^s \mathrm{vol}_f^{\frac{\alpha-1}{\alpha}} B_t dt \ge s\, \mathrm{vol}_f^{\frac{\alpha-1}{\alpha}} B_s$ yields $F'(s) \ge 0$, hence $F(s) \ge 0$. Applying this inequality to (A-5) yields

$$\int_{B_x(R)} |\nabla h| \ge \mathrm{ID}_{\alpha,f}\, B_x(R) \left(\frac{\alpha}{\alpha-1} \int_0^\infty t^{\frac{1}{\alpha-1}} \mathrm{vol}_f\, B_t\, dt \right)^{\frac{\alpha-1}{\alpha}}. \tag{A-6}$$

Integrating by parts and using the coarea formula,

$$\begin{aligned} \frac{\alpha}{\alpha-1} \int_0^\infty t^{\frac{1}{\alpha-1}} \mathrm{vol}_f\, B_t\, dt &= \int_0^\infty \left(\frac{d(t^{\frac{\alpha}{\alpha-1}})}{dt} \int_t^\infty \int_{\partial B_s} \frac{d\sigma_f(\partial B_s)}{|\nabla h|}\, ds \right) dt \\ &= \int_0^\infty t^{\frac{\alpha}{\alpha-1}} \int_{\partial B_t} \frac{d\sigma_f(\partial B_t)}{|\nabla h|}\, dt = \int_{B_x(R)} h^{\frac{\alpha}{\alpha-1}}. \end{aligned}$$

Inserting the above equality into (A-6) gives (A-4). □

Acknowledgments

Part of this work was done while Wang was visiting UCSB. She would like to thank the China Scholarship Council (CSC) for the financial support and UCSB for the hospitality and providing a great environment for research. She also would like to thank Professor Yu Zheng for his constant support and her office mate at UCSB, Jesús Núñez-Zimbrón, for his interest and helpful conversations. We would like to thank the referee for a very careful reading of the paper and helpful suggestions. Wang is supported by the National Natural Science Foundation of China (No. 11671141). Wei is supported by National Science Foundation (Grant No. DMS-1506393).

References

[Aubry 2007] E. Aubry, "Finiteness of π_1 and geometric inequalities in almost positive Ricci curvature", *Ann. Sci. École Norm. Sup.* (4) **40**:4 (2007), 675–695. MR Zbl

[Aubry 2009] E. Aubry, "Bounds on the volume entropy and simplicial volume in Ricci curvature L^p-bounded from below", *Int. Math. Res. Not.* **2009**:10 (2009), 1933–1946. MR Zbl

[Bakry and Émery 1985] D. Bakry and M. Émery, "Diffusions hypercontractives", pp. 177–206 in *Séminaire de Probabilités, XIX*, edited by J. Azéma and M. Yor, Lecture Notes in Math. **1123**, Springer, 1985. MR Zbl

[Cao and Zhou 2010] H.-D. Cao and D. Zhou, "On complete gradient shrinking Ricci solitons", *J. Differential Geom.* **85**:2 (2010), 175–185. MR Zbl

[Chavel 1993] I. Chavel, *Riemannian geometry: a modern introduction*, Cambridge Tracts in Math. **108**, Cambridge Univ. Press, 1993. MR Zbl

[Cheeger 1970] J. Cheeger, "A lower bound for the smallest eigenvalue of the Laplacian", pp. 195–199 in *Problems in analysis* (Princeton, 1969), edited by R. C. Gunning, Princeton Univ. Press, 1970. MR Zbl

[Cheeger and Colding 1996] J. Cheeger and T. H. Colding, "Lower bounds on Ricci curvature and the almost rigidity of warped products", *Ann. of Math.* (2) **144**:1 (1996), 189–237. MR Zbl

[Cheng and Oden 1997] S.-Y. Cheng and K. Oden, "Isoperimetric inequalities and the gap between the first and second eigenvalues of an Euclidean domain", *J. Geom. Anal.* **7**:2 (1997), 217–239. MR Zbl

[Dai et al. 2000] X. Dai, P. Petersen, and G. Wei, "Integral pinching theorems", *Manuscripta Math.* **101**:2 (2000), 143–152. MR Zbl

[Dai et al. 2018] X. Dai, G. Wei, and Z. Zhang, "Local Sobolev constant estimate for integral Ricci curvature bounds", *Adv. Math.* **325** (2018), 1–33. MR Zbl

[Gromov 1979] M. Gromov, "Paul Levy's isoperimetric inequality", preprint 33, Inst. Hautes Études Sci., 1979, http://www.ihes.fr/~gromov/PDF/11[33].pdf.

[Li 2012] P. Li, *Geometric analysis*, Cambridge Studies in Advanced Math. **134**, Cambridge Univ. Press, 2012. MR Zbl

[Lin and Yang 2002] F. Lin and X. Yang, *Geometric measure theory: an introduction*, Advanced Math. **1**, Science Press, Beijing, 2002. MR Zbl

[Lott 2003] J. Lott, "Some geometric properties of the Bakry–Émery–Ricci tensor", *Comment. Math. Helv.* **78**:4 (2003), 865–883. MR Zbl

[Munteanu and Wang 2011] O. Munteanu and J. Wang, "Smooth metric measure spaces with non-negative curvature", *Comm. Anal. Geom.* **19**:3 (2011), 451–486. MR Zbl

[Petersen and Wei 1997] P. Petersen and G. Wei, "Relative volume comparison with integral curvature bounds", *Geom. Funct. Anal.* **7**:6 (1997), 1031–1045. MR Zbl

[Petersen and Wei 2001] P. Petersen and G. Wei, "Analysis and geometry on manifolds with integral Ricci curvature bounds, II", *Trans. Amer. Math. Soc.* **353**:2 (2001), 457–478. MR Zbl

[Rose 2017] C. Rose, "Heat kernel upper bound on Riemannian manifolds with locally uniform Ricci curvature integral bounds", *J. Geom. Anal.* **27**:2 (2017), 1737–1750. MR Zbl

[Rose and Stollmann 2017] C. Rose and P. Stollmann, "The Kato class on compact manifolds with integral bounds on the negative part of Ricci curvature", *Proc. Amer. Math. Soc.* **145**:5 (2017), 2199–2210. MR Zbl

[Tian and Zhang 2016] G. Tian and Z. Zhang, "Regularity of Kähler–Ricci flows on Fano manifolds", *Acta Math.* **216**:1 (2016), 127–176. MR Zbl

[Wei and Wylie 2009] G. Wei and W. Wylie, "Comparison geometry for the Bakry–Émery Ricci tensor", *J. Differential Geom.* **83**:2 (2009), 377–405. MR Zbl

[Wu 2018] J.-Y. Wu, "Comparison geometry for integral Bakry–Émery Ricci tensor bounds", *J. Geom. Anal.* (online publication March 2018).

[Zhang and Zhu 2017] Q. S. Zhang and M. Zhu, "Li–Yau gradient bound for collapsing manifolds under integral curvature condition", *Proc. Amer. Math. Soc.* **145**:7 (2017), 3117–3126. MR Zbl

[Zhang and Zhu 2018] Q. S. Zhang and M. Zhu, "Li–Yau gradient bounds on compact manifolds under nearly optimal curvature conditions", *J. Funct. Anal.* **275**:2 (2018), 478–515. MR Zbl

Received March 7, 2018. Revised June 20, 2018.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 300 No. 1 May 2019

0030-8730(201905)300:1;1-L