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BRAIDING GAPPED BOUNDARIES OF  
DIJKGRAAF–WITTEN THEORIES**

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**We study representations of the braid groups from braiding gapped boundaries of Dijkgraaf–Witten theories and their twisted generalizations, which are (twisted) quantum doubled topological orders in two spatial dimensions. We show that the braid representations associated to Lagrangian algebras are all monomial with respect to some specific bases. We give explicit formulas for the monomial matrices and the ground state degeneracy of the Kitaev models that are Hamiltonian realizations of Dijkgraaf–Witten theories. Our results imply that braiding gapped boundaries alone cannot provide universal gate sets for topological quantum computing with gapped boundaries.**

## 1. Introduction

Interesting new directions in topological quantum computing include its extension from anyons to gapped boundaries and symmetry defects, with the hope that anyonic systems with nonuniversal computational power can be enhanced to achieve universality. Enrichment of topological physics in two spatial dimensions by gapped boundaries has been investigated intensively, but their computing power has not been analyzed in detail yet. One interesting case is gapped boundaries of Dijkgraaf–Witten theories both for their experimental relevance and as theoretical exemplars (see [Cong et al. 2016; 2017a; 2017b]).

In this paper, we study representations of the braid groups from braiding gapped boundaries of Dijkgraaf–Witten theories and their twisted generalizations, which are (twisted) quantum doubled topological orders in two spatial dimensions. We show that the resulting braid (pure braid) representations are all monomial with respect to some specific bases, and their entries are roots of unity; hence all such representation

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images of the braid groups are finite groups. The finiteness of the image of the braid representation from braiding gapped boundaries of twisted Dijkgraaf–Witten theories follows directly from [Etingof et al. 2008], since they are braid representations obtained from group-theoretical braided fusion categories. Besides, we give explicit formulas for the monomial matrices and the ground state degeneracy of the Kitaev models that are Hamiltonian realizations of Dijkgraaf–Witten theories. The universal gate sets from [Cong et al. 2017c] include a nontopological measurement primitive. Our results imply that braiding gapped boundaries alone cannot provide universal gate sets for topological quantum computing with gapped boundaries.

For a topological order of the form  $\mathcal{C} = \mathcal{Z}(\mathcal{S})$ , where  $\mathcal{S}$  is some unitary fusion category, gapped boundaries are modeled by Lagrangian algebras (see [Cong et al. 2016]). For these models the ground manifolds have the form

$$\mathrm{Hom}_{\mathcal{C}}(1, A_1 \otimes \cdots \otimes A_n),$$

where the  $A_i$ 's are the Lagrangian algebras modeling the gapped boundaries; see [Cong et al. 2016, Section 3] for details. Recall that a Lagrangian algebra in any modular (tensor) category is a commutative étale algebra whose quantum dimension is maximal. A group theoretical modular category (GTMC) admitting a Lagrangian algebra is a category of the form  $\mathcal{C} = \mathcal{Z}(\mathrm{Vec}_G^\omega)$  for some finite group  $G$  and some  $\omega \in Z^3(G, \mathbb{C}^\times)$ , where  $\mathcal{Z}$  denotes the Drinfeld center and  $\mathrm{Vec}_G^\omega$  is the category of finite-dimensional  $G$ -graded vector spaces with associativity constraint twisted by  $\omega \in H^3(G, \mathbb{C}^\times)$ ; see [Davydov et al. 2013; Davydov and Simmons 2017].

Kitaev [2003] proposed Hamiltonian realizations of Dijkgraaf–Witten theories, whose topological orders are GTMCs. Moreover, extensions of these Hamiltonian realizations to surfaces with boundaries can be constructed from Lagrangian algebras [Bravyi and Kitaev 1998; Bombin and Martin-Delgado 2008; Beigi et al. 2011; Kitaev and Kong 2012].

Lagrangian algebras in GTMCs are in one-to-one correspondence with indecomposable module categories of  $\mathrm{Vec}_G^\omega$  [Davydov et al. 2013], which are in bijection with pairs  $(H, \gamma)$ , where  $H$  is a subgroup of  $G$  and  $\gamma \in C^2(H, \mathbb{C}^\times)$  such that  $\delta(\gamma) = \omega|_{H \times 3}$ , all up to conjugation [Natale 2017]. A more direct description of the relationship between Lagrangian algebras and pairs  $(H, \gamma)$  can be found in [Davydov 2010].

Recently, a quantum computing scheme to use gapped boundaries to achieve universality has been proposed [Cong et al. 2016; 2017a; 2017b; 2017c]. Braiding gapped boundaries can be either added to braiding anyons as in Kitaev's original proposal or as new computing primitives supplemented with other topological operations. Gapped boundaries lead to additional degeneracy of the topologically protected subspace, which potentially allows the implementation of more powerful gates. More precisely, the new gates come from representation matrices of the

braid groups,  $\mathcal{B}_n$ , on objects of the GTMCs that are tensor products of Lagrangian algebras. However, a characterization of the computational power of these new braid representations, mathematically a study of the representation images, was left as an important open problem [Cong et al. 2016; 2017c].

The goal of this paper is to provide such a characterization. We find a canonical monomial structure for Lagrangian algebras in  $\mathcal{Z}(\text{Vec}_G^\omega)$ , which allows us to compute things more easily. This paper is organized as follows. Section 2 develops the theory of monomial representations. Specifically, it shows how to calculate invariants for a representation of  $G$  using the monomial structure. In Section 3 we recall the notion of a monomial twisted Yetter–Drinfeld module. We use the theory developed in Section 2 to give an explicit description and a basis for  $\text{Hom}_{\mathcal{Z}(\text{Vec}_G^\omega)}(\mathbb{C}, V^{\otimes n})$  if  $V$  is a monomial object. Next, we describe the representation of  $\mathcal{B}_n$  with respect to this basis. Theorem 3.3 states the representation is monomial and Theorem 3.4 gives an explicit formula for the nonzero entries. In Section 4 we prove that every Lagrangian algebra in  $\mathcal{Z}(\text{Vec}_G^\omega)$  has a canonical monomial structure. Then the results of Section 3 are applied to Lagrangian algebras in  $\mathcal{Z}(\text{Vec}_G^\omega)$ . We finish the section developing some examples and applications.

## 2. Monomial representations

In this section, we recall some basic definitions and results on monomial representations of groups.

**Definition 2.1.** A *monomial space* is a triple  $V = (V, X, (V_x)_{x \in X})$ , where

- (i)  $V$  is a finite-dimensional complex vector space,
- (ii)  $X$  is a finite set,
- (iii)  $(V_x)_{x \in X}$  is a family of one-dimensional subspaces of  $V$ , indexed by  $X$ , such that  $V = \bigoplus_{x \in X} V_x$ .

Let  $G$  be a group. By a *monomial representation* of  $G$  on  $V$  we mean a group homomorphism

$$\Gamma : G \rightarrow \text{GL}(V)$$

such that for every  $g \in G$ ,  $\Gamma(g)$  permutes the  $V_x$ 's; hence,  $\Gamma$  induces an action by permutation of  $G$  on  $X$ . We will denote  $\Gamma(g)(v)$  by  $g \triangleright v$ .

If  $V$  is a representation of  $G$ , we denote by  $V^G$  the subspace of  $G$ -invariant vectors, i.e.,

$$V^G = \{v \in V : g \triangleright v = v, \text{ for all } g \in G\}.$$

For each  $x \in X$ , we denote by  $\text{Sta}_G(x)$  the stabilizer of  $x$  and by  $\mathcal{O}_G(x)$  the

$G$ -orbit of  $x$ . For  $G$  finite, and a representation  $V$  define

$$\text{Av}_G : V \rightarrow V, \quad v \mapsto \frac{1}{|G|} \sum_{g \in G} g \triangleright v.$$

It is easy to see that  $\text{Av}_G$  is a  $G$ -linear projection onto  $V^G$ . We define

$$\text{Av}_G(V_{\mathcal{O}}) := \text{Av}_G(V_x), \quad x \in \mathcal{O}(x),$$

since for any  $x' \in \mathcal{O}_G(x)$ ,  $\text{Av}_G(V_x) = \text{Av}_G(V_{x'})$ .

We say that an element  $x \in X$  is *regular* under the monomial action of  $G$  if  $\Gamma(g)$  is the identity map on  $V_x$ , for all  $g \in \text{Sta}_G(x)$ .

Let us write  $X/G$  for the set of orbits of the action of  $G$  on  $X$ , and  $\tilde{X}$  for the regular ones.

**Proposition 2.2** [Karpilovsky 1985, Lemma 9.1]. *Let  $V = (V, X, (V_x)_{x \in X})$  be a monomial representation of  $G$ .*

- (a)  $x \in X$  is a regular element if and only if  $\text{Av}_G(V_x) \neq 0$ .
- (b) If  $x \in X$  is a regular element under the monomial action of  $G$ , then so are all elements in the  $G$ -orbit of  $x$ .
- (c) The triple

$$\mathbf{V}^G = (V^G, \tilde{X}, (\text{Av}_G(V_{\mathcal{O}}))_{\mathcal{O} \in \tilde{X}})$$

is a monomial space.

- (d) The dimension of  $V^G$  is equal to the number of regular  $G$ -orbits under the monomial action of  $G$  on  $X$ .

Let  $V = (V, X, (V_x)_{x \in X})$  and  $V' = (V', Y, (V'_y)_{y \in Y})$  be monomial spaces. A linear isomorphism  $T : V \rightarrow V'$  is called an *isomorphism of monomial spaces* if  $T(V_x) = V'_y$  for any  $x \in X$ .

**Proposition 2.3.** *Let  $V = (V, X, (V_x)_{x \in X})$  and  $V' = (V', Y, (V'_y)_{y \in Y})$  be monomial representations of a finite group  $G$ . If  $T : V \rightarrow V'$  is a  $G$ -linear isomorphism of monomial spaces, then  $T|_{V^G} : V^G \rightarrow V'^G$  is an isomorphism of monomial spaces.*

*Proof.* Clearly,  $T|_{V^G} : V^G \rightarrow V'^G$  is a linear isomorphism. Let  $x \in X$  be a regular element. Since  $T$  is an isomorphism of monomial spaces, there is some  $y \in Y$  such that  $T(V_x) = V'_y$ . In that case,

$$\text{Av}_G(V'_y) = \text{Av}_G(T(V_x)) = T(\text{Av}_G(V_x)).$$

This implies  $y$  is regular, because  $\text{Av}_G(V_x) \neq \{0\}$  and  $T$  is an isomorphism. It also gives  $T|_{V^G}(\text{Av}_G(V_{\mathcal{O}(x)})) = \text{Av}_G(V'_{\mathcal{O}(y)})$ , which means  $T|_{V^G}$  is an isomorphism of monomial spaces.  $\square$

### 3. Monomial representation of the braid group

In this section we recall the notion of monomial twisted Yetter–Drinfeld module introduced in [Galindo and Rowell 2014, Definition 4.12] and prove that the representation of the braid groups  $\mathcal{B}_n$  over  $\text{Hom}_{\mathcal{Z}(\text{Vec}_G^\omega)}(\mathbb{C}, V^{\otimes n})$  is monomial if  $V$  is monomial.

**3A. Dijkgraaf–Witten theories.** Let  $G$  be a discrete group. A (normalized) 3-cocycle  $\omega \in Z^3(G, \mathbb{C}^\times)$  is a map  $\omega : G \times G \times G \rightarrow \mathbb{C}^\times$  such that

$$\omega(ab, c, d)\omega(a, b, cd) = \omega(a, b, c)\omega(a, bc, d)\omega(b, c, d), \quad \omega(a, 1, b) = 1,$$

for all  $a, b, c, d \in G$ .

Let us recall the description of the modular category  $\mathcal{Z}(\text{Vec}_G^\omega)$ , the Drinfeld center of the category  $\text{Vec}_G^\omega$  sometimes called the category of twisted Yetter–Drinfeld modules. The category  $\mathcal{Z}(\text{Vec}_G^\omega)$  is braided equivalent to the representations of the twisted Drinfeld double defined by Dijkgraaf, Pasquier and Roche [Dijkgraaf et al. 1991, Section 3.2].

Given  $\omega \in Z^3(G; \mathbb{C}^\times)$ , we define

$$\omega(g, g'; h) := \frac{\omega({}^g g' h, g, g')\omega(g, g', h)}{\omega(g, g' h, g')}, \quad \omega(g; f, h) := \frac{\omega({}^g f, g, h)}{\omega(g, f, h)\omega({}^g f, {}^g h, g)},$$

for  $f, g, g', h \in G$ .

The objects of  $\mathcal{Z}(\text{Vec}_G^\omega)$  are  $G$ -graded vector spaces  $V = \bigoplus_{g \in G} V_g$  with a linear map  $\triangleright : \mathbb{C}^\omega G \otimes V \rightarrow V$  such that  $1 \triangleright v = v$  for all  $v \in V$ , and

$$(gh) \triangleright v = \omega(g, h; k)(g \triangleright (h \triangleright v)), \quad g, h, k \in G, \quad v \in V_k,$$

satisfying the compatibility condition

$$g \triangleright V_h \subseteq V_{ghg^{-1}}, \quad g, h \in G.$$

Morphisms in  $\mathcal{Z}(\text{Vec}_G^\omega)$  are  $G$ -linear  $G$ -homogeneous maps. The tensor product of  $V = \bigoplus_{g \in G} V_g$  and  $W = \bigoplus_{g \in G} W_g$  is  $V \otimes W$  as vector space, with

$$(V \otimes W)_g = \bigoplus_{h \in G} V_h \otimes W_{h^{-1}g},$$

and for all  $v \in V_g, w \in W_l$ ,

$$h \triangleright (v \otimes w) = \omega(h; g, l)(h \triangleright v) \otimes (h \triangleright w).$$

For  $V, W, Z \in \mathcal{Z}(\text{Vec}_G^\omega)$ , the associativity constraint is defined by

$$\begin{aligned} a_{V, W, Z} : (V \otimes W) \otimes Z &\rightarrow V \otimes (W \otimes Z), \\ (v_g \otimes w_h) \otimes z_k &\mapsto \omega(g, h, k)^{-1} v_g \otimes (w_h \otimes z_k) \end{aligned}$$

for all  $g, h, k \in G$ ,  $v_g \in V_x$ ,  $w_h \in W_h$ ,  $z_k \in Z_k$ . The category is tensor braided, with braiding  $c_{V,W} : V \otimes W \rightarrow W \otimes V$ ,  $V, W \in \mathcal{Z}(\text{Vec}_G^\omega)$ ,

$$c_{V,W}(v \otimes w) = (g \triangleright w) \otimes v, \quad g \in G, v \in V_g, w \in W.$$

**3B. Braid group representation of twisted Yetter–Drinfeld modules.** Since the braided category  $\mathcal{Z}(\text{Vec}_G^\omega)$  is not strict, we must be careful about the way we associate terms when we consider tensor products with more than two objects. For a list of objects  $A_1, A_2, \dots, A_n \in \mathcal{Z}(\text{Vec}_G^\omega)$ , we define

$$A_1 \otimes \cdots \otimes A_n := (\cdots (A_1 \otimes A_2) \otimes \cdots \otimes A_{n-1}) \otimes A_n,$$

and an isomorphism by

$$(1) \quad \sigma'_i = (a_{A_1 \otimes \cdots \otimes A_{i-1}, A_{i+1}, A_i}^{-1} \otimes \text{id}_{A_{i+2} \otimes \cdots \otimes A_n}) \circ (\text{id}_{A_1 \otimes \cdots \otimes A_{i-1}} \otimes c_{A_i, A_{i+1}} \otimes \text{id}_{A_{i+2} \otimes \cdots \otimes A_n}) \\ \circ (a_{A_1 \otimes \cdots \otimes A_{i-1}, A_i, A_{i+1}} \otimes \text{id}_{A_{i+2} \otimes \cdots \otimes A_n}),$$

where  $a_{V,W,Z}$  denotes the associativity constraint.

If  $A = A_1 = \cdots = A_n$ , there exists a unique group homomorphism

$$\rho_n : \mathcal{B}_n \rightarrow \text{Aut}_{\mathcal{Z}(\text{Vec}_G^\omega)}(A^{\otimes n})$$

sending the generator  $\sigma_i \in \mathcal{B}_n$  to  $\sigma'_i$ .

In general, the pure braid group  $\mathcal{P}_n$  acts on  $A_1 \otimes \cdots \otimes A_n$ , in the sense that there exists a group homomorphism  $\rho_n : \mathcal{P}_n \rightarrow \text{Aut}_{\mathcal{Z}(\text{Vec}_G^\omega)}(A_1 \otimes \cdots \otimes A_n)$ .

**3C. Crossed  $G$ -sets.** Let  $G$  be a group. We will recall the definition of (left) *crossed  $G$ -set* introduced in [Freyd and Yetter 1989]. A crossed  $G$ -set is a left  $G$ -set  $X$  and a grading function  $|\cdot| : X \rightarrow G$  such that

$$|gx| = g|x|g^{-1},$$

for all  $x \in X$ ,  $g \in G$ . If  $X$  and  $Y$  are crossed  $G$ -sets, a  $G$ -equivariant map  $f : X \rightarrow Y$  is a morphism of crossed  $G$ -sets if  $|f(x)| = |x|$  for all  $x \in X$ .

If  $X$  and  $Y$  are crossed  $G$ -sets, the cartesian product  $X \times Y$  is a crossed  $G$ -set with the diagonal action and grading map  $|(x, y)| = |x| |y|$ .

The category of crossed  $G$ -sets is a braided category with braiding

$$c_{X,Y} : X \times Y \rightarrow Y \times X, \\ (x, y) \mapsto (|x| \triangleright y, x).$$

Thus, given a crossed  $G$ -set  $X$  the braid group  $\mathcal{B}_n$  acts on  $X^n$ , in the following way:

$$\sigma'_i := \text{id}_{X^{i-1}} \times c_{X,X} \times \text{id}_{X^{n-(i-1)}}.$$

**3D. Monomial objects of  $\mathcal{Z}(\text{Vec}_G^\omega)$ .** Let  $G$  be a finite group and  $\omega \in Z^3(G, \mathbb{C}^\times)$  be a 3-cocycle.

**Definition 3.1** [Galindo and Rowell 2014]. A monomial Yetter–Drinfeld module is a monomial space  $\mathbf{V} = (V, X, (V_x)_{x \in X})$  such that  $V \in \mathcal{Z}(\text{Vec}_G^\omega)$ , the twisted  $G$ -action  $\triangleright$  permutes the  $V_x$ 's, and each  $V_x$  is  $G$ -homogeneous.

**Remark 3.2.** (a) If  $\mathbf{V} = (V, X, (V_x)_{x \in X})$  is a monomial Yetter–Drinfeld module, the set  $X$  is a crossed  $G$ -set with the induced  $G$ -action and the grading map.

(b) If  $\mathbf{V} = (V, X, (V_x)_{x \in X})$  is a monomial Yetter–Drinfeld module, the action of  $G$  on  $(V_e, X_e, (V_x)_{x \in X_e})$  is monomial, where  $X_e := \{x \in X : |x| = e\}$  and  $V_e = \bigoplus_{x \in X_e} V_x$ .

**Theorem 3.3.** Let  $G$  be a finite group,  $\omega \in Z^3(G, \mathbb{C}^\times)$ . If  $\mathbf{V} = (V, X, (V_x)_{x \in X})$  is a monomial Yetter–Drinfeld module in  $\mathcal{Z}(\text{Vec}_G^\omega)$ , then

- (a) the action of  $\mathcal{B}_n$  on  $\text{Hom}_{\mathcal{Z}(\text{Vec}_G^\omega)}(\mathbb{C}, V^{\otimes n})$  is monomial,
- (b) the dimension of  $\text{Hom}_{\mathcal{Z}(\text{Vec}_G^\omega)}(\mathbb{C}, V^{\otimes n})$  is equal to the number of regular  $G$ -orbits under the monomial action of  $G$  on

$$(X^n)_e := \{(x_1, \dots, x_n) : |x_1| \cdots |x_n| = e\}.$$

*Proof.* The action of  $G$  on  $(V_e^{\otimes n}, (X^n)_e, (V_x)_{x \in X_e})$  is monomial. Hence, by Proposition 2.2, the triple

$$\mathbf{V}_e^G := \left( (V_e^{\otimes n})^G, \widetilde{(X^n)_e}, (\text{Av}_G((V_e^{\otimes n})_{\mathcal{O}}))_{\mathcal{O} \in \widetilde{(X^n)_e}} \right)$$

is a monomial space. Since  $\text{Hom}_{\mathcal{Z}(\text{Vec}_G^\omega)}(\mathbb{C}, V^{\otimes n}) = (V^{\otimes n})_e^G$ , and each of the automorphisms  $\sigma'$  are morphisms in  $\mathcal{Z}(\text{Vec}_G^\omega)$ ,

$$\sigma'|_{V_e^{\otimes n}} : (V_e^{\otimes n}, (X^n)_e, (V_x)_{x \in X_e}) \rightarrow (V_e^{\otimes n}, (X^n)_e, (V_x)_{x \in X_e})$$

is a  $G$ -linear isomorphism of monomial spaces. It follows from Proposition 2.3 that  $\sigma'|_{(V^{\otimes n})_e^G}$  is an isomorphism of monomial spaces. Thus, the linear representation

$$\begin{aligned} \rho_n : \mathcal{B}_n &\rightarrow \text{GL}((V_e^{\otimes n})^G), \\ \sigma &\mapsto \sigma' \end{aligned}$$

is a monomial representation of  $\mathcal{B}_n$ . The second part follows immediately from Proposition 2.2.  $\square$

**3E. Monomial matrices of the braid representation.** In this subsection we obtain concrete formulas for the monomial braid representations associated to a monomial Yetter–Drinfeld module.

Let  $G$  be a finite group,  $\omega \in Z^3(G, \mathbb{C}^\times)$ , and  $\mathbf{V} = (V, X, (V_x)_{x \in X})$  be a monomial Yetter–Drinfeld module. If we fix nonzero vectors  $\mathcal{S} := \{v_x \in V_x : x \in X\}$ , the twisted  $G$ -action defines a map

$$\lambda_X : G \times X \rightarrow \mathbb{C}^\times,$$

by  $g \triangleright v_x = \lambda_X(g; x)v_{gx}$ , where  $g \in G, x \in X$ .

For the monomial Yetter–Drinfeld module  $\mathbf{V}^{\otimes n} = (V^{\otimes n}, X^n, (V_x)_{x \in X^n})$  and the basis  $\mathcal{S}^{\otimes n} := \{v_{x_1} \otimes \cdots \otimes v_{x_n} : v_{x_i} \in \mathcal{S}, 1 \leq i \leq n\}$ , the action is determined by the map  $\lambda_{X^n} : G \times X^n \rightarrow \mathbb{C}^\times$ ,

$$(2) \quad \lambda_{X^n}(g; x_1, \dots, x_n) := \prod_{i=1}^n \lambda_X(g; x_i) \omega(g; |x_1| |x_2| \cdots |x_{n-1}|, |x_n|)^{-1} \\ \times \omega(g; |x_1| \cdots |x_{n-2}|, |x_{n-1}|)^{-1} \cdots \omega(g; |x_1|, |x_2|)^{-1},$$

that is,

$$g \triangleright (v_{x_1} \otimes \cdots \otimes v_{x_n}) = \lambda_{X^n}(g; x_1, \dots, x_n) (v_{gx_1} \otimes \cdots \otimes v_{gx_n}),$$

for all  $g \in G, x_1, x_2, \dots, x_n \in X$ . Hence an element  $(x_1, \dots, x_n) \in (X^n)_e$  is regular if and only if

$$(3) \quad \lambda_{X^n}(g; x_1, \dots, x_n) = 1, \quad \text{for all } g \in \bigcap_{i=1}^n \text{Sta}(x_i).$$

Let  $\mathcal{R} \subset X_e^n$  be a set of representatives of the regular orbits of  $X_e^{\times n}$ . Let  $\mathcal{S}_{\text{reg}} = \{v_{x_1} \otimes \cdots \otimes v_{x_n} : (x_1, \dots, x_n) \in \mathcal{R}\}$ . By Proposition 2.2, the set  $\{\text{Av}_G(v) : v \in \mathcal{S}_{\text{reg}}\}$  is a basis of  $(V^{\otimes n})_e^G$ .

To express the action of the generator  $\sigma_i \in \mathcal{B}_n$  in terms of  $\{\text{Av}_G(v) : v \in \mathcal{S}_{\text{reg}}\}$ , for each  $\mathbf{x} = (x_1, \dots, x_n) \in \mathcal{R}$  choose  $g_{\mathbf{x}} \in G$  such that  $g_{\mathbf{x}} \triangleright \sigma'_i(\mathbf{x}) = \mathbf{y}$ , where  $\mathbf{y} \in \mathcal{R}$  and  $\sigma'_i(\mathbf{x}) = (x_1, \dots, x_{i-1}, |x_i| x_{i+1}, x_i, x_{i+2}, \dots, x_n)$ . Hence there is  $\beta_{i,\mathbf{x}} \in \mathbb{C}^\times$  such that  $g_{\mathbf{x}} \triangleright \sigma'_i(v_{x_1} \otimes \cdots \otimes v_{x_n}) = \beta_{i,\mathbf{x}} v_{y_1} \otimes \cdots \otimes v_{y_n}$ .

Since the action of the generator  $\sigma_i \in \mathcal{B}_n$  is given by

$$(4) \quad \sigma'_i(v_{x_1} \otimes \cdots \otimes v_{x_n}) = \omega(|x_1| \cdots |x_{i-1}|, |x_i| |x_{i+1}| |x_i|^{-1}, |x_i|) \\ \times \lambda_X(|x_i|; x_{i+1}) \omega(|x_1| \cdots |x_{i-1}|, |x_i|, |x_{i+1}|)^{-1} \\ \times v_{x_1} \otimes \cdots \otimes v_{x_{i-1}} \otimes v_{|x_i| x_{i+1}} \otimes v_{x_i} \otimes \cdots \otimes v_{x_n},$$

we have

$$(5) \quad \beta_{i,\mathbf{x}} = \omega(|x_1| \cdots |x_{i-1}|, |x_i| |x_{i+1}| |x_i|^{-1}, |x_i|) \\ \times \lambda_X(|x_i|; x_{i+1}) \omega(|x_1| \cdots |x_{i-1}|, |x_i|, |x_{i+1}|)^{-1} \lambda_{X^n}(g_{\mathbf{x}}; \sigma'_i(\mathbf{x})).$$

**Theorem 3.4.** *Let  $G$  be a finite group,  $\omega \in Z^3(G, \mathbb{C}^\times)$  and  $\mathbf{V} = (V, X, (V_x)_{x \in X})$  be a monomial Yetter–Drinfeld module. Let  $Y$  be the set of all regular elements in  $X_e^n$  and let  $\mathcal{R} \subset Y$  be a set of representatives of the  $G$ -orbits of  $Y$ .*

- (a) *The projection  $\pi : Y \rightarrow \mathcal{R}$  is map of  $\mathcal{B}_n$ -sets. The image of  $\mathbf{x} \in \mathcal{R}$  by the generator  $\sigma_i \in \mathcal{B}_n$  will be denoted by  $\sigma_i \triangleright \mathbf{x}$ .*

(b) Let  $\mathcal{S}_{\text{reg}} = \{v_{x_1} \otimes \cdots \otimes v_{x_n} : (x_1, \dots, x_n) \in \mathcal{R}\}$ . The action of the generator  $\sigma_i \in \mathcal{B}_n$  in the basis  $\{\text{Av}_G(v_{\mathbf{x}}) : \mathbf{x} \in \mathcal{R}\}$  is given by

$$\sigma_i(\text{Av}_G(v_{\mathbf{x}})) = \beta_{i,\mathbf{x}} \text{Av}_G(v_{\sigma_i \triangleright \mathbf{x}}),$$

where  $\beta_{i,\mathbf{x}}$  was defined in (5).

*Proof.* The first part is a consequence of Theorem 3.3.

For the second part, recall that the number  $\beta_{i,\mathbf{x}}$  and the element  $g_{\mathbf{x}} \in G$  are such that

$$g_{\mathbf{x}} \triangleright \sigma(v_{\mathbf{x}}) = \beta_{i,\mathbf{x}} v_{\sigma_i \triangleright \mathbf{x}}.$$

Hence,

$$\begin{aligned} \sigma_i(\text{Av}_G(v_{\mathbf{x}})) &= \text{Av}_G(\sigma_i(v_{\mathbf{x}})) \\ &= g_{\mathbf{x}} \triangleright \text{Av}_G(\sigma_i(v_{\mathbf{x}})) = \text{Av}_G(g_{\mathbf{x}} \triangleright \sigma_i(v_{\mathbf{x}})) \\ &= \text{Av}_G(\beta_{i,\mathbf{x}} v_{\sigma_i \triangleright \mathbf{x}}) = \beta_{i,\mathbf{x}} \text{Av}_G(v_{\sigma_i \triangleright \mathbf{x}}). \end{aligned} \quad \square$$

**Example 3.5.** Let  $G$  be a finite group and  $X$  be a left crossed  $G$ -set. Then the linearization  $V_X := \bigoplus_{x \in X} \mathbb{C}x$  is an (untwisted) Yetter–Drinfeld module in  $\mathcal{Z}(\text{Vec}_G)$ . Clearly  $\lambda_X \equiv 1$ , thus every element in  $(X^n)_e$  is regular. Hence the canonical projection

$$(X^n)_e \rightarrow (X^n)_e // G,$$

is an epimorphism of  $\mathcal{B}_n$ -sets. In other words, the linear representation of  $\mathcal{B}_n$  on  $\text{Hom}_{\mathcal{Z}(\text{Vec}_G)}(\mathbb{C}, V_X^{\otimes n})$  is the linearization of the permutation action of  $\mathcal{B}_n$  on  $(X^n)_e // G$ .

#### 4. Braid groups representations associated to Lagrangian algebras

In this section, we prove that every Lagrangian algebra in  $\mathcal{Z}(\text{Vec}_G^\omega)$  has a canonical monomial structure. Then the results of Section 3 can be applied to Lagrangian algebras in  $\mathcal{Z}(\text{Vec}_G^\omega)$ .

**4A. Lagrangian algebras.** Following Corollary 3.17 of [Davydov and Simmons 2017], we will describe the Lagrangian algebra on  $\mathcal{Z}(G, \omega)$  associated to a pair  $(H, \gamma)$ , where  $H \subseteq G$  is a subgroup and  $\gamma : H \times H \rightarrow \mathbb{C}^\times$  is a map such that

$$\frac{\gamma(ab, c)\gamma(a, b)}{\gamma(a, bc)\gamma(b, c)} = \omega(a, b, c), \quad a, b, c \in H.$$

Let  $\mathbb{C}_\gamma[H] = \bigoplus_{h \in H} \mathbb{C}e_h$  be the group algebra of  $H$  with the multiplication

$$e_{h_1}e_{h_2} = \gamma(h_1, h_2)e_{h_1h_2}, \quad h_1, h_2 \in H.$$

The vector space  $\mathbb{C}_\gamma[H] = \bigoplus_{h \in H} \mathbb{C}e_h$ , is a commutative algebra in  $\mathcal{Z}(\text{Vec}_H^\omega)$ , where

the  $H$ -action is given by

$$h_1 \triangleright e_{h_2} = \epsilon(h_1, h_2) e_{h_1 h_2 h_1^{-1}}, \quad \epsilon(h_1, h_2) := \frac{\gamma(h_1, h_2)}{\gamma(h_1 h_2, h_1)}, \quad h_1, h_2 \in H,$$

and grading  $|e_h| = h$  for all  $h \in H$ .

Let  $\text{Map}(G, \mathbb{C}_\gamma[H])$  be the vector space of all set-theoretic maps from  $G$  to  $\mathbb{C}_\gamma[H]$ . With the grading given by

$$|a| = f \quad \Leftrightarrow \quad |a(x)| = x^{-1} f x \quad \text{for all } x \in G,$$

and twisted  $G$ -action

$$(g \triangleright a)(x) := \omega(x^{-1}, g^{-1}; |a|)^{-1} a(g^{-1} \triangleright x), \quad g, x \in G,$$

$\text{Map}(G, \mathbb{C}_\gamma[H])$  is a twisted Yetter–Drinfeld module.

The Lagrangian algebra  $L(H, \gamma)$  is the Yetter–Drinfeld submodule

$$L(H, \gamma) := \{a \in \text{Maps}(G, \mathbb{C}_\gamma[H]) \mid a(xh) = \omega(h^{-1}, x^{-1}; |a|) h^{-1} \triangleright a(x)\};$$

see [Davydov and Simmons 2017] for more details.

**4B. Monomial structure of the Lagrangian algebras  $L(H, \gamma)$ .** In this section we will prove that every Lagrangian algebra of the form  $L(H, \gamma)$  has a canonical monomial structure.

Let  $G$  be a group and  $H \subset G$  be a subgroup. We can regard  $G \times H$  as a left  $H$ -set with actions given by  $h \triangleright (g, h') = (gh^{-1}, hh'h^{-1})$ . Then we can consider the set of  $H$ -orbits that we will denote by  $G \times_H H$ . The set  $G \times_H H$  is equipped with a left  $G$ -action given by left multiplication on the first component.

**Definition 4.1.** Let  $L(H, \gamma)$  be a Lagrangian. For each  $g \in G$  and  $f \in H$ , define  $\chi_{g,f} \in L(H, \gamma)$  by

$$(6) \quad \chi_{g,f}(x) = \begin{cases} 0, & x \notin gH, \\ \omega(h^{-1}, g^{-1}; {}^g f) \epsilon(h^{-1}, f) e_{hf h^{-1}}, & x = gh, \text{ where } h \in H. \end{cases}$$

**Remark 4.2.** The function  $\chi_{g,h}$  can be characterized as the unique map in  $L(H, \gamma)$  with support  $gH$  and such that  $\chi_{g,h}(g) = e_h$ .

**Lemma 4.3.** Let  $L(H, \gamma)$  be a Lagrangian algebra in  $\mathcal{Z}(G, \omega)$ . Then

$$(7) \quad \chi_{gh,f} = \omega(h, (gh)^{-1}; {}^{gh} f) \epsilon(h, f) \chi_{g,h} f, \quad g \in G, f, h \in H.$$

$$(8) \quad l \triangleright \chi_{g,f} = \omega((lg)^{-1}, l^{-1}; {}^g f) \chi_{lg,f}, \quad g, l \in G, h \in H.$$

*Proof.* Since the supports of  $\chi_{gh,f}$  and  $\chi_{g,h} f$  are  $gH$ , and

$$\chi_{gh,f}(g) = \chi_{gh,f}(ghh^{-1}) = \omega(h, (gh)^{-1}; {}^{gh} f) \epsilon(h, f) \chi_{g,h} f(g),$$

we obtain (7).

By the definition of the action of  $G$  we have

$$l \triangleright \chi_{g,f}(lg) = \omega((lg)^{-1}, l^{-1}; {}^s f) \chi_{g,f}(g) = \omega((lg)^{-1}, l^{-1}; {}^s f) e_f.$$

Since  $l \triangleright \chi_{g,f}$  and  $\chi_{gl,f}$  are supported in  $glH$ , we get (8).  $\square$

It follows from Lemma 4.3 that  $\mathbb{C}\chi_{gh,hf} = \mathbb{C}\chi_{g,f}$ . Then for any  $(g, h) \in G \times_H H$  the space  $\mathbb{C}\chi_{g,f}$  is well defined.

**Theorem 4.4.** *Let  $L(H, \gamma)$  be a Lagrangian algebra in  $\mathcal{Z}(G, \omega)$ . Then  $L(H, \gamma)$  with the decomposition*

$$L(H, \gamma) = \bigoplus_{(g,h) \in G \times_H H} \mathbb{C}\chi_{g,h}$$

*is a monomial twisted Yetter–Drinfeld module.*

*Proof.* First we will check that in fact the sum  $\sum_{(g,h) \in G \times_H H} \mathbb{C}\chi_{g,h}$  is direct. Since  $\text{supp}(\chi_{g,f}) = gH$ , we have that  $\chi_{g,f}$  and  $\chi_{g',f}$  are linearly independent if  $gH \neq g'H$ . Hence it is sufficient to check linear independence of the collections  $\{\chi_{g,f}\}_{f \in H}$ , with  $g$  fixed. But if  $f \neq f'$ ,  $|\chi_{r,f}| \neq |\chi_{r,f'}|$ . It follows that the sum  $\sum_{(g,h) \in G \times_H H} \mathbb{C}\chi_{g,h}$  is direct.

In order to see that  $L(H, \gamma) = \sum_{(g,h) \in G \times_H H} \mathbb{C}\chi_{g,h}$ , fix  $\mathcal{R} \subset G$ , a set of representatives of the left coset of  $H$  in  $G$ . Let  $a \in L(H, \gamma)$ . For each  $r \in \mathcal{R}$ , suppose

$$(9) \quad a(r) = \sum_{f \in H} \lambda_{r,f} e_f.$$

Then we have

$$(10) \quad a = \sum_{r \in \mathcal{R}, f \in H} \lambda_{r,f} \chi_{r,f} \in \sum_{(g,h) \in G \times_H H} \mathbb{C}\chi_{g,h}.$$

By (8) and the fact that  $|\chi_{g,f}| = gf g^{-1}$ , we obtain that  $L(H, \gamma)$  is a monomial twisted Yetter–Drinfeld module.  $\square$

**Corollary 4.5.** *Let  $G$  be a finite group,  $\omega \in \mathcal{Z}^3(G, \mathbb{C}^\times)$ . If  $L(H, \gamma)$  is a Lagrangian algebra in  $\mathcal{Z}(\text{Vec}_G^\omega)$ , then*

- (a) *the action of  $\mathcal{B}_n$  on  $\text{Hom}_{\mathcal{Z}(\text{Vec}_G^\omega)}(\mathbb{C}, L(H, \gamma)^{\otimes n})$  is monomial,*
- (b) *the dimension of  $\text{Hom}_{\mathcal{Z}(\text{Vec}_G^\omega)}(\mathbb{C}, L(H, \gamma)^{\otimes n})$  is equal to the number of regular  $G$ -orbits under the monomial action of  $G$  on*

$$(G \times_H H)^{\times n}_e := \{(g_1, h_1), \dots, (g_n, h_n) : g_1 h_1 g_1^{-1} g_2 h_2 g_2^{-1} \cdots g_n h_n g_n^{-1} = e\}.$$

*Proof.* This follows from Theorem 4.4 and Theorem 3.3.  $\square$

We will fix a set of representatives of the left cosets of  $G$  in  $H$ ,  $\mathcal{R} \subset G$ . Thus every element  $g \in G$  has a unique factorization  $g = rh$ ,  $h \in H$ ,  $r \in \mathcal{R}$ . We assume  $e \in \mathcal{R}$ . The uniqueness of the factorization  $G = \mathcal{R}H$  implies that there are well defined maps

$$\triangleright : G \times \mathcal{R} \rightarrow \mathcal{R}, \quad \kappa : G \times \mathcal{R} \rightarrow H,$$

determined by the condition

$$gr = (g \triangleright r)\kappa(g, h), \quad g \in G, r \in \mathcal{R}.$$

As a crossed  $G$ -set we can identify  $G \times_H H$  with  $\mathcal{R} \times H$  with action

$$g \triangleright (r, h) := (g \triangleright r, \kappa^{(g,r)} h), \quad r \in \mathcal{R}, h \in H, g \in G,$$

and grading map

$$|-| : \mathcal{R} \times H \rightarrow G \quad (r, h) \mapsto rhr^{-1}.$$

It follows from Theorem 4.4 that  $B_{\mathcal{R}} := \{\chi_{r,h} \mid r \in \mathcal{R}, h \in H\}$  is a basis for  $L(H, \gamma)$ .

In order to apply the results of Section 3E, we only need to compute the map  $\lambda_{\mathcal{R} \times H} : G \times (\mathcal{R} \times H) \rightarrow \mathbb{C}^\times$ , such that

$$g \triangleright \chi_{r,h} = \lambda_{\mathcal{R} \times H}(g; r, h) \chi_{g \triangleright r, \kappa^{(g,r)} h}, \quad g \in G, r \in \mathcal{R}, h \in H.$$

Using Lemma 4.3 we obtain

$$(11) \quad \lambda_{\mathcal{R} \times H}(g; r, h) = \omega((gr)^{-1}, g^{-1}; rh) \omega(\kappa(g, r), (gr)^{-1}; {}^{gr}h) \epsilon(\kappa(g, r), h),$$

for all  $g \in G, r \in \mathcal{R}, h \in H$ .

By (3), we have that an element  $t = ((r_1, h_1), \dots, (r_n, h_n)) \in (\mathcal{R} \times H)_e^n$  is regular if and only if

$$(12) \quad \lambda_{(\mathcal{R} \times H)^n}(g; (r_1, h_1), \dots, (r_n, h_n)) = 1, \quad \text{for all } g \in \bigcap_{i=1}^n r_i^{-1} C_H(h_i) r_i,$$

where  $\lambda_{(\mathcal{R} \times H)^n}$  was defined in (2) as a function of  $\lambda_{\mathcal{R} \times H}$  and  $\omega$ .

**4C. Applications and examples.** In this last section we present some applications of the results of the previous section.

**4C1. Central subgroups.**

**Proposition 4.6.** *Let  $G$  be a finite group and  $L(H, \gamma)$  a Lagrangian algebra in  $\mathcal{Z}(\text{Vec}_G)$ , where  $H \subset G$  is a central subgroup. Then*

$$\dim(\text{Hom}_{\mathcal{Z}(\text{Vec}_G)}(\mathbb{C}, L(H, \gamma)^{\otimes n})) = |G|^{n-1}.$$

Moreover, the representation of  $B_n$  is actually a representation of  $S_n$ .

*Proof.* Since  $H$  is a central subgroup,  $g \triangleright (r, h) = (g \triangleright r, h)$  and

$$|\chi_{r_1, h_1} \otimes \cdots \otimes \chi_{r_k, h_k}| = h_1 \cdots h_k,$$

for any  $r_1, \dots, r_k \in R, h_1, \dots, h_n \in H$ . Hence,

$$|(\mathcal{R} \times H)_e^n| = |(\mathcal{R}^n/G)||H^{n-1}| = [G : H]^n |H|^{n-1} = |G|^{n-1}.$$

To determine the number of orbits, notice that  $\epsilon : H \times H \rightarrow \mathbb{C}^\times$  is a bicharacter such that  $\epsilon(h_1, h_2)\epsilon(h_2, h_1) = 1$ . Then, by (12), an element

$$((r_1, h_1), \dots, (r_n, h_n)) \in (\mathcal{R} \times H)_e^n$$

is regular if and only if

$$\prod_{i=1}^n \epsilon(h, h_i) = 1, \quad \text{for all } h \in H.$$

But  $\prod_{i=1}^n \epsilon(h, h_i) = \epsilon(h, h_1 \cdots h_n) = \epsilon(h', e) = 1$ . Hence every element is regular. By Corollary 4.5 the dimension of  $\text{Hom}_{\mathcal{Z}(\text{Vec}_G)}(\mathbb{C}, L(H, \gamma)^{\otimes n})$  is  $|G|^{n-1}$ .

Finally, using (4), we see that

$$\begin{aligned} \sigma'_i \circ \sigma'_i(\chi_{r_1, h_1} \otimes \cdots \otimes \chi_{r_n, h_n}) &= \epsilon(h_i, h_{i+1})\epsilon(h_{i+1}, h_i)(\chi_{r_1, h_1} \otimes \cdots \otimes \chi_{r_n, h_n}) \\ &= \chi_{r_1, h_1} \otimes \cdots \otimes \chi_{r_n, h_n}. \end{aligned}$$

Hence representation of  $\mathcal{B}_n$  factors as a representation of  $S_n$ .  $\square$

**4C2.** *Lagrangian algebra of the form  $L(H, 1)$ .* The Lagrangian algebras  $L(H, 1)$  as an object in  $\mathcal{Z}(\text{Vec}_G)$  are completely determined by the crossed  $G$ -set  $G \times_H H$ , and the monomial representation  $\text{Hom}(\mathbb{C}, L(H, 1)^{\otimes n})$  is a permutation representation; see Example 3.5. Let us see some extreme cases:

*Case  $H = \{e\}$ .* In this case the crossed  $G$ -set is  $G$  with the regular action and grading map the constant map  $e$ . It is clear that the braiding  $c_{G, G}$  is just the flip map

$$(g_1, g_2) \mapsto (g_2, g_1),$$

hence, really the symmetric group  $S_n$  acts on  $G^n$ .

The set of  $G$ -orbits is in bijection with  $G^{n-1}$ ,

$$\mathcal{O}(G^n) \rightarrow G^{n-1},$$

$$\mathcal{O}_G(g_1, g_2, \dots, g_n) \mapsto (e, g_1^{-1}g_2, \dots, g_1^{-1}g_n).$$

Using the previous map the action of  $\mathfrak{S}_n$  is given by

$$\sigma_1(g_1, \dots, g_{n-1}) = (g_1^{-1}, g_1^{-1}g_2, \dots, g_1^{-1}g_{n-1})$$

and

$$\sigma_i(g_1, \dots, g_i, g_{i+1}, \dots, g_{n-1}) = (g_1, \dots, g_{i+1}, g_i, \dots, g_{n-1}), \quad 1 < i < n.$$

It is clear that permutation action of  $S_n$  on  $G^{n-1}$  is faithful; thus the image is isomorphic to  $S_n$ .

*Case  $H = G$ .* In this case the crossed  $G$ -set is  $G$  with the action by conjugation and grading map the identity map. Hence, the braiding is given by

$$c_{G,G} : (x, y) \mapsto (y, y^{-1}xy).$$

Note  $c_{G,G}$  is symmetric if and only if  $G$  is abelian.

If  $G$  is abelian,  $G_e^n = \{(g_1, \dots, g_{n-1}, (g_1, \dots, g_{n-1})^{-1})\}$  is the set of orbits and, as in the previous example, the group  $S_n$  acts faithfully.

**4C3. Dihedral group.** Every time we take  $H$  to be a normal subgroup of  $G$ , the following proposition provides a way to simplify the situation.

**Proposition 4.7.** *Let  $G$  be a finite group,  $H \trianglelefteq G$ , and  $\mathcal{R}$  a collection of representatives for  $G/H$ . Define  $B_\gamma[H] \in \mathcal{Z}(\text{Vec}_G)$  as*

$$B(H, \gamma) := \text{span}\{b_{r,h} \mid r \in \mathcal{R}, h \in H\},$$

with grading  $|b_{r,h}| = h$  and the  $G$ -action

$$(13) \quad g \triangleright b_{r,h} = \epsilon(\kappa(g, r)^{r^{-1}}h) b_{g \triangleright r, g h}.$$

Then, the mapping

$$B(H, \gamma) \rightarrow L(H, \gamma), \quad b_{r,h} \mapsto \chi_{r, r^{-1}h}$$

is an isomorphism in  $\mathcal{Z}(\text{Vec}_G)$ .

*Proof.* We need to show the map preserves the grading and the  $G$ -representation. We have

$$|\chi_{r, r^{-1}h}| = {}^r(r^{-1}h) = h = |b_{r,h}|.$$

Now, since

$$g \cdot \chi_{r, r^{-1}h} = \epsilon(\kappa(g, r), {}^{r^{-1}}h) \chi_{g \triangleright r, \kappa(g, r)({}^{r^{-1}}h)},$$

and

$$\kappa(g, r)({}^{r^{-1}}h) = ({}^{g \triangleright r})^{-1} g h g^{-1},$$

we have that

$$g \triangleright b_{r,h} = \epsilon(\kappa(g, r), {}^{r^{-1}}h) a_{g \triangleright r, ({}^{g \triangleright r})^{-1}(g h)}.$$

Hence, by (13) the map is equivariant. □

Proposition 4.7 works particularly well when  $\gamma = 1$ , since (13) is just

$$g \triangleright b_{r,h} = b_{g \triangleright r, g h}.$$

Thus, the action of  $G$  is “decoupled”. We use this idea in the following example.

Let  $G = D_{2k}$  be the dihedral groups of order  $2k$  and  $H = \langle r \rangle$ . We take  $\mathcal{R} = \{e, s\} = \{s^i\}_{i \in \mathbb{Z}/2\mathbb{Z}}$ . Then

$$|b_{s^{i_1}, r^{j_1}} \otimes \cdots \otimes b_{s^{i_n}, r^{j_n}}| = r^{\sum_{m=1}^n j_m},$$

and

$$\dim(B(H, \gamma)_e^{\otimes n}) = 2^n \times k^{n-1}.$$

Since

$$(s^i r^j)(s^k) = s^{i+k} r^{(-1)^k j},$$

we have

$$(s^i r^j) \triangleright s^k = s^{i+k} \quad \text{and} \quad \kappa(s^i r^j, s^k) = r^{(-1)^k j}.$$

Hence, the action, on the set label is

$$s^i r^j (s^k, r^l) = (s^{i+k}, r^l).$$

It follows that the number of orbits in  $(\mathcal{R} \times H)_e^n$  is

$$2^{n-1} \times k^{n-1} = |G|^{n-1}.$$

Since  $\gamma = 1$  all orbits are regular and so  $\dim(\text{Hom}_{\mathcal{Z}(\text{Vec}_G)}(\mathbb{C}, L(H, 1)^{\otimes n})) = |G|^{n-1}$ .

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