Pacific Journal of Mathematics

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XIU JI, TONGZHU LI AND HUAFEI SUN

Volume 300 No. 1 May 2019

PACIFIC JOURNAL OF MATHEMATICS
Vol. 300, No. 1, 2019
dx.doi.org/10.2140/pjm.2019.300.17

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Let $f : M^n \to \mathbb{R}_1^{n+1}$ be an n-dimensional umbilic-free spacelike hypersurface in the $(n+1)$-dimensional Lorentzian space $\mathbb{R}_1^{n+1}$. One can define the conformal metric g on f which is invariant under the conformal transformation group of $\mathbb{R}_1^{n+1}$. We classify the n-dimensional spacelike hypersurfaces with constant sectional curvature with respect to the conformal metric g when $n \geq 3$. Such spacelike hypersurfaces are obtained by the standard construction of cylinders, cones or hypersurfaces of revolution over certain spirals in the 2-dimensional Lorentzian space forms $\mathbb{S}_1^2(1)$, $\mathbb{R}_1^2$, $\mathbb{R}_{1+}^2$, respectively.

1. Introduction

Recently the Möbius geometry of submanifolds in Riemannian space forms has been studied extensively and many special hypersurfaces were classified under the Möbius transformation group (see [Guo et al. 2012; Hu and Li 2003; Li et al. 2013; Li and Wang 2003]). As its parallel generalization, the conformal geometry of submanifolds in Lorentzian space forms is another important branch of submanifold theory; however there are fewer results (see [Li and Nie 2013; 2018; Nie 2015]). In this paper, we investigate the spacelike hypersurfaces with constant conformal sectional curvature. Since the conformal geometry of spacelike hypersurfaces in Lorentzian space forms $M_1^{n+1}(c)$ is uniform by the conformal map (2-4) (see Section 2), we only consider the hypersurfaces in $\mathbb{R}_1^{n+1}$.

Let $f : M^n \to \mathbb{R}_1^{n+1}$ be an n-dimensional umbilic-free spacelike hypersurface in the $(n+1)$-dimensional Lorentzian space $\mathbb{R}_1^{n+1}$. Given the first fundamental form $I = df \cdot df$ as well as a local orthonormal basis $\{e_i\}$ and the dual basis $\{\theta_i\}$, we denote $II = \sum_{ij} h_{ij}\theta_i \otimes \theta_j$ the second fundamental form and $H = \frac{1}{n}\sum_i h_{ii}$ the mean curvature. The conformal metric of f,

$$g = \rho^2 df \cdot df = \frac{n}{n-1}(\| II \|^2 - nH^2)I, \tag{1-1}$$

is a Riemannian metric which is invariant under the conformal transformations

MSC2010: 53A30, 53B25.

Keywords: conformal metric, conformal sectional curvature, conformal second fundamental form, curvature-spiral.

of $\mathbb{R}^{n+1}_1$. Together with another quadratic form (called the conformal second fundamental form) they form a complete system of invariants for the spacelike hypersurface ($n \geq 3$) in conformal geometry of the Lorentzian space $\mathbb{R}^{n+1}_1$ (see Section 2). In this framework, a notable class of spacelike hypersurfaces are those with constant conformal sectional curvature (i.e., constant sectional curvature with respect to the conformal metric g). Here we classify them up to a conformal transformation of the Lorentzian space $\mathbb{R}^{n+1}_1$, and our main result is stated below.

Theorem 1.1. *Let $f : M^n \to \mathbb{R}^{n+1}_1$, $n \geq 3$, be an umbilic-free spacelike hypersurface with constant conformal sectional curvature δ. Then locally f is conformally equivalent to one of the following hypersurfaces*:

(i) *A cylinder over a curvature-spiral in a Lorentzian 2-plane $\mathbb{R}^2_1$ (where $\delta \leq 0$).*

(ii) *A cone over a curvature-spiral in a de Sitter 2-sphere $\mathbb{S}^2_1 \subset \mathbb{R}^3_1$ (where $\delta < 0$).*

(iii) *A rotational hypersurface over a curvature-spiral in a Lorentzian hyperbolic 2-plane $\mathbb{R}^2_{1+} \subset \mathbb{R}^2_1$ (the constant curvature δ could be positive, negative or zero).*

(iv) *A cone over the hyperbolic torus $\mathbb{H}^q(-\sqrt{a^2-1}) \times \mathbb{S}^p(a)$, $a > 1$, (where $\delta = 0$).*

The *curvature-spiral* $\gamma(s) \in N^2_1(\epsilon)$ in a 2-dimensional Lorentzian space form $N^2_1(\epsilon)(= \mathbb{S}^2_1(1),\ \mathbb{R}^2_1,\ \mathbb{R}^2_{1+}$ for Gauss curvature $\epsilon = 1, 0, -1$, respectively) is a spacelike curve which is determined by the intrinsic equation

$$\left[\frac{d}{ds}\frac{1}{\kappa}\right]^2 + \epsilon\left[\frac{1}{\kappa}\right]^2 = -\delta, \tag{1-2}$$

where s is the arc-length parameter, and κ denotes the geodesic curvature of the spacelike curve γ, and δ is a real constant. Note that (1-2) is equivalent to the harmonic oscillator equation $(1/\kappa)'' + \epsilon/\kappa = 0$ for the function $\kappa(s)$. It is easy to see that for fixed ϵ and δ the solution curve is unique (because $N^2_1(\epsilon)$ is two-point homogeneous, since any two solutions with arbitrary initial values are congruent to each other).

The Lorentzian hyperbolic 2-plane $\mathbb{R}^2_{1+} \subset \mathbb{R}^2_1$ is defined by

$$\mathbb{R}^2_{1+} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\},$$

endowed with the Lorentzian metric $ds^2 = \frac{1}{y^2}(-dx^2 + dy^2)$. The Gauss curvature of $\mathbb{R}^2_{1+}$ is $\epsilon = -1$ with respect to the Lorentzian metric ds^2. Let $\mathbb{H}^2_1(-1)$ be a 2-dimensional anti-de Sitter sphere; there exists a standard isometric embedding

$$\phi : \mathbb{R}^2_{1+} \to \mathbb{H}^2_1(-1), \quad \phi(x, y) = \left(\frac{y^2 - x^2 + 1}{2y}, \frac{x}{y}, \frac{y^2 - x^2 - 1}{2y}\right). \tag{1-3}$$

The rest of this paper is organized as follows. In Section 2, we study the conformal geometry of spacelike hypersurfaces in Lorentzian space forms $M_1^{n+1}(c)$. In Section 3, we construct some examples of the spacelike hypersurfaces with constant conformal sectional curvature. In Section 4, we give the proof of Theorem 1.1.

2. Conformal geometry of spacelike hypersurfaces

In this section, following Wang [1998], we define some conformal invariants on a spacelike hypersurface and give a congruent theorem of the spacelike hypersurfaces under the conformal group of Lorentzian space forms $M_1^{n+1}(c)$.

Let $\mathbb{R}_s^{n+2}$ be the real vector space $\mathbb{R}^{n+2}$ with the Lorentzian product $\langle\, ,\rangle_s$ given by

$$\langle X, Y\rangle_s = -\sum_{i=1}^{s} x_i y_i + \sum_{j=s+1}^{n+2} x_j y_j.$$

For any $a > 0$, the standard sphere $\mathbb{S}^{n+1}(a)$, the hyperbolic space $\mathbb{H}^{n+1}(-a)$, the de Sitter space $\mathbb{S}_1^{n+1}(a)$ and the anti-de Sitter space $\mathbb{H}_1^{n+1}(-a)$ are defined by

$$\begin{aligned}
\mathbb{S}^{n+1}(a) &= \{x \in \mathbb{R}^{n+2} \mid x\cdot x = a^2\},\\
\mathbb{H}^{n+1}(-a) &= \{x \in \mathbb{R}_1^{n+2} \mid \langle x, x\rangle_1 = -a^2\},\\
\mathbb{S}_1^{n+1}(a) &= \{x \in \mathbb{R}_1^{n+2} \mid \langle x, x\rangle_1 = a^2\},\\
\mathbb{H}_1^{n+1}(-a) &= \{x \in \mathbb{R}_2^{n+2} \mid \langle x, x\rangle_2 = -a^2\}.
\end{aligned}$$

Let $M_1^{n+1}(c)$ be a Lorentzian space form. When $c=0$, $M_1^{n+1}(c)=\mathbb{R}_1^{n+1}$; when $c=1$, $M_1^{n+1}(c) = \mathbb{S}_1^{n+1}(1)$; when $c = -1$, $M_1^{n+1}(c) = \mathbb{H}_1^{n+1}(-1)$.

Denoting by C^{n+2} the cone in $\mathbb{R}_2^{n+3}$ and by $\mathbb{Q}_1^{n+1}$ the conformal compactification space in $\mathbb{R}P^{n+3}$,

$$C^{n+2} = \{X \in \mathbb{R}_2^{n+3} \mid \langle X, X\rangle_2 = 0,\ X \neq 0\}, \quad \mathbb{Q}_1^{n+1} = \{[X] \in \mathbb{R}P^{n+2} \mid \langle X, X\rangle_2 = 0\}.$$

Let $O(n+3, 2)$ be the Lorentzian group of $\mathbb{R}_2^{n+3}$ keeping the Lorentzian product $\langle X, Y\rangle_2$ invariant. Then $O(n+3, 2)$ is a transformation group on $\mathbb{Q}_1^{n+1}$ defined by

$$T([X]) = [XT], \quad X \in C^{n+2}, \quad T \in O(n+3, 2).$$

Topologically, $\mathbb{Q}_1^{n+1}$ is identified with the compact space $\mathbb{S}^n \times \mathbb{S}^1/\mathbb{S}^0$, which is endowed by a standard Lorentzian metric

$$h = g_{\mathbb{S}^n} \oplus (-g_{\mathbb{S}^1}),$$

where $g_{\mathbb{S}^k}$ denotes the standard metric of the k-dimensional sphere $\mathbb{S}^k$. Therefore, $\mathbb{Q}_1^{n+1}$ has conformal metric $[h] = \{e^\tau h\}$, $\tau \in C^\infty(\mathbb{Q}_1^{n+1})$, and $[O(n+3, 2)]$ is the conformal transformation group of $\mathbb{Q}_1^{n+1}$(see [Cahen and Kerbrat 1983; O'Neill 1983]).

Setting $P=\{[X]\in\mathbb{Q}_1^{n+1}\mid x_1=x_{n+3}\}$, $P_-=\{[X]\in\mathbb{Q}_1^{n+1}\mid x_{n+3}=0\}$, and $P_+=\{[X]\in\mathbb{Q}_1^{n+1}\mid x_1=0\}$, we can define the following conformal diffeomorphisms

$$\begin{aligned}\sigma_0:\mathbb{R}_1^{n+1}&\to\mathbb{Q}_1^{n+1}\backslash P, && u\mapsto\left[\left(\frac{1+\langle u,u\rangle_1}{2},u,\frac{\langle u,u\rangle_1-1}{2}\right)\right],\\ \sigma_1:\mathbb{S}_1^{n+1}(1)&\to\mathbb{Q}_1^{n+1}\backslash P_+, && u\mapsto[(1,u)],\\ \sigma_{-1}:\mathbb{H}_1^{n+1}(-1)&\to\mathbb{Q}_1^{n+1}\backslash P_-, && u\mapsto[(u,1)].\end{aligned}\tag{2-4}$$

We may regard $\mathbb{Q}_1^{n+1}$ as the common compactification of $\mathbb{R}_1^{n+1}$, $\mathbb{S}_1^{n+1}(1)$, $\mathbb{H}_1^{n+1}(-1)$.

Let $f:M^n\to M_1^{n+1}(c)$ be a spacelike hypersurface. Using σ_c, we obtain the hypersurface $\sigma_c\circ f:M^n\to\mathbb{Q}_1^{n+1}$ in $\mathbb{Q}_1^{n+1}$. From [Cahen and Kerbrat 1983], we have the following theorem:

Theorem 2.1. *Two hypersurfaces $f,\bar f:M^n\to M_1^{n+1}(c)$ are conformally equivalent if and only if there exists $T\in O(n+3,2)$ such that $\sigma_c\circ f=T(\sigma_c\circ\bar f):M^n\to\mathbb{Q}_1^{n+1}$.*

Since $f:M^n\to M_1^{n+1}(c)$ is a spacelike hypersurface, $(\sigma_c\circ f)_*(TM^n)$ is a positive definite subbundle of $T\mathbb{Q}_1^{n+1}$. For any local lift Z of the standard projection $\pi:C^{n+2}\to\mathbb{Q}_1^{n+1}$, we get a local lift $y=Z\circ\sigma_c\circ f:U\to C^{n+1}$ of $\sigma_c\circ f:M\to\mathbb{Q}_1^{n+1}$ in an open subset U of M^n. Thus $\langle \mathrm{d}y,\mathrm{d}y\rangle_2=\rho^2\langle df,df\rangle_s$ is a local metric, where $\rho\in C^\infty(U)$. We denote by Δ and κ the Laplacian operator and the normalized scalar curvature with respect to the local positive definite metric $\langle \mathrm{d}y,\mathrm{d}y\rangle_2$, respectively. Much as in the proof of Theorem 1.2 in [Wang 1998], we can get the following theorem:

Theorem 2.2. *Let $f:M^n\to M_1^{n+1}(c)$ be a spacelike hypersurface, then the 2-form $g=-(\langle\Delta y,\Delta y\rangle_2-n^2\kappa)\langle dy,dy\rangle_2$ is a globally defined conformal invariant. Moreover, g is positive definite at any nonumbilical point of M^n.*

We call g the conformal metric of the spacelike hypersurface f, and there exists a unique lift

$$Y:M^n\to C^{n+2}$$

such that $g=\langle \mathrm{d}Y,\mathrm{d}Y\rangle_2$. We call Y the conformal position vector of the spacelike hypersurface f. Theorem 2.2 implies the following:

Theorem 2.3. *Two spacelike hypersurfaces $f,\bar f:M^n\to M_1^{n+1}(c)$ are conformally equivalent if and only if there exists $T\in O(n+3,2)$ such that $\overline{Y}=YT$, where Y and $\overline{Y}$ are the conformal position vectors of f and $\bar f$, respectively.*

Let $\{E_1,\dots,E_n\}$ be a local orthonormal basis of M^n with respect to g with dual basis $\{\omega_1,\dots,\omega_n\}$. Denote $Y_i=E_i(Y)$ and define

$$N=-\frac{1}{n}\Delta Y-\frac{1}{2n^2}\langle\Delta Y,\Delta Y\rangle_2 Y,$$

where Δ is the Laplace operator of g, then we have

$$\langle N, Y\rangle_2 = 1,\ \langle N, N\rangle_2 = 0,\ \langle N, Y_k\rangle_2 = 0,\ \langle Y_i, Y_j\rangle_2 = \delta_{ij}, \quad 1 \le i, j, k \le n.$$

We may decompose $\mathbb{R}^{n+3}_2$ such that

$$\mathbb{R}^{n+3}_2 = \mathrm{span}\{Y, N\} \oplus \mathrm{span}\{Y_1, \dots, Y_n\} \oplus \mathbb{V},$$

where $\mathbb{V} \perp \mathrm{span}\{Y, N, Y_1, \dots, Y_n\}$. We call $\mathbb{V}$ the conformal normal bundle of f, which is a linear bundle. Let ξ be a local section of $\mathbb{V}$ and $\langle \xi, \xi\rangle_2 = -1$, then $\{Y, N, Y_1, \dots, Y_n, \xi\}$ forms a moving frame in $\mathbb{R}^{n+3}_2$ along M^n. We write the structure equations as follows:

$$\begin{aligned}
\mathrm{d}Y &= \sum_i \omega_i Y_i, \qquad \mathrm{d}N = \sum_{ij} A_{ij}\omega_j Y_i + \sum_i C_i \omega_i \xi,\\
\mathrm{d}Y_i &= -\sum_j A_{ij}\omega_j Y - \omega_i N + \sum_j \omega_{ij} Y_j + \sum_j B_{ij}\omega_j \xi,\\
\mathrm{d}\xi &= \sum_i C_i \omega_i Y + \sum_{ij} B_{ij}\omega_j Y_i,
\end{aligned} \tag{2-5}$$

where $\omega_{ij} (= -\omega_{ji})$ are the connection 1-forms on M^n with respect to $\{\omega_1, \dots, \omega_n\}$. It is clear that $A = \sum_{ij} A_{ij}\omega_j \otimes \omega_i$, $B = \sum_{ij} B_{ij}\omega_j \otimes \omega_i$, $C = \sum_i C_i \omega_i$ are globally defined conformal invariants. We call A, B and C the Blaschke tensor, the conformal second fundamental form, and the conformal 1-form, respectively. The covariant derivatives of these tensors are defined by

$$\begin{aligned}
\sum_j C_{i,j}\omega_j &= dC_i + \sum_k C_k \omega_{kj},\\
\sum_k A_{ij,k}\omega_k &= dA_{ij} + \sum_k A_{ik}\omega_{kj} + \sum_k A_{kj}\omega_{ki},\\
\sum_k B_{ij,k}\omega_k &= dB_{ij} + \sum_k B_{ik}\omega_{kj} + \sum_k B_{kj}\omega_{ki}.
\end{aligned}$$

By exterior differentiation of the structure equations (2-5), we can get the integrable conditions of the structure equations

$$A_{ij} = A_{ji}, \quad B_{ij} = B_{ji}, \tag{2-6}$$

$$A_{ij,k} - A_{ik,j} = B_{ij}C_k - B_{ik}C_j, \tag{2-7}$$

$$B_{ij,k} - B_{ik,j} = \delta_{ij}C_k - \delta_{ik}C_j, \tag{2-8}$$

$$C_{i,j} - C_{j,i} = \sum_k (B_{ik}A_{kj} - B_{jk}A_{ki}), \tag{2-9}$$

$$R_{ijkl} = B_{il}B_{jk} - B_{ik}B_{jl} + A_{ik}\delta_{jl} + A_{jl}\delta_{ik} - A_{il}\delta_{jk} - A_{jk}\delta_{il}. \tag{2-10}$$

Furthermore, we have

$$\begin{aligned}
&\operatorname{tr}(A)=\frac{1}{2n}(n^2\kappa-1), \qquad R_{ij}=\operatorname{tr}(A)\delta_{ij}+(n-2)A_{ij}+\sum_k B_{ik}B_{kj},\\
&(1-n)C_i=\sum_j B_{ij,j}, \qquad \sum_{ij} B_{ij}^2=\frac{n-1}{n}, \qquad \sum_i B_{ii}=0,
\end{aligned} \tag{2-11}$$

where κ is the normalized scalar curvature of g. From (2-11), we see that when $n\geq 3$, all coefficients in the structure equations are determined by the conformal metric g and the conformal second fundamental form B, thus we get the congruent theorem:

Theorem 2.4. *Two spacelike hypersurfaces $f, \bar{f}: M^n \to M_1^{n+1}(c)$, $n \geq 3$, are conformally equivalent if and only if there exists a diffeomorphism $\varphi: M^n \to M^n$ which preserves the conformal metric g and the conformal second fundamental form B.*

The second covariant derivative of the conformal second fundamental form B_{ij} is defined by

$$\sum_m B_{ij,km}\omega_m = dB_{ij,k}+\sum_m B_{mj,k}\omega_{mi}+\sum_m B_{im,k}\omega_{mj}+\sum_m B_{ij,m}\omega_{mk}. \tag{2-12}$$

Thus we have the following Ricci identities

$$B_{ij,kl}-B_{ij,lk}=\sum_m B_{mj}R_{mikl}+\sum_m B_{im}R_{mjkl}. \tag{2-13}$$

Next we give the relations between the conformal invariants and the isometric invariants of a spacelike hypersurface in $\mathbb{R}_1^{n+1}$.

Assume that $f: M^n \to \mathbb{R}_1^{n+1}$ is an umbilic-free spacelike hypersurface. Let $\{e_1, \ldots, e_n\}$ be an orthonormal local basis with respect to the induced metric $I=\langle df, df\rangle_1$ with dual basis $\{\theta_1, \ldots, \theta_n\}$. Let e_{n+1} be a normal vector field of f, $\langle e_{n+1}, e_{n+1}\rangle_1=-1$. Let $II=\sum_{ij} h_{ij}\theta_i\otimes\theta_j$ denote the second fundamental form, $H=\frac{1}{n}\sum_i h_{ii}$ the mean curvature. Denote by Δ_M the Laplacian operator and κ_M the normalized scalar curvature for I. By the structure equation of $f: M^n \to \mathbb{R}_1^{n+1}$ we get

$$\Delta_M f = nHe_{n+1}. \tag{2-14}$$

There is a local lift of f

$$y: M^n \to C^{n+2}, \qquad y=\left(\frac{\langle f, f\rangle_1+1}{2}, f, \frac{\langle f, f\rangle_1-1}{2}\right).$$

It follows from (2-14) that $\langle \Delta y, \Delta y\rangle_2 - n^2\kappa_M = \frac{n}{n-1}(-|II|^2+n|H|^2) = -e^{2\tau}$. Therefore the conformal metric g, the conformal position vector of f, and ξ are expressed as

$$\begin{aligned} g &= \frac{n}{n-1}(|II|^2 - n|H|^2)\langle \mathrm{d}f, \mathrm{d}f\rangle_1 = e^{2\tau} I, \quad Y = e^{\tau} y, \\ \xi &= -Hy + (\langle f, e_{n+1}\rangle_1, e_{n+1}, \langle f, e_{n+1}\rangle_1). \end{aligned} \tag{2-15}$$

By a direct calculation we get the following expression of the conformal invariants

$$\begin{aligned} A_{ij} &= e^{-2\tau}\big[\tau_i\tau_j - h_{ij}H - \tau_{i,j} + \tfrac{1}{2}(-|\nabla\tau|^2 + |H|^2)\delta_{ij}\big], \\ B_{ij} &= e^{-\tau}(h_{ij} - H\delta_{ij}), \quad C_i = e^{-2\tau}\Big(H\tau_i - H_i - \sum_j h_{ij}\tau_j\Big), \end{aligned} \tag{2-16}$$

where $\tau_i = e_i(\tau)$ and $|\nabla\tau|^2 = \sum_i \tau_i^2$, and $\tau_{i,j}$ is the Hessian of τ for I and $H_i = e_i(H)$.

3. Typical examples

In this section, we construct some spacelike hypersurfaces with constant conformal sectional curvature. Such spacelike hypersurfaces are obtained by the standard construction of cylinders, cones or hypersurfaces of revolution over curvature-spirals in $N_1^2(\epsilon)$. A key observation is that the conformal metric of those spacelike hypersurfaces constructed over these curvature-spirals are of the form

$$g = \kappa(s)^2(ds^2 + I^{n-1}_{-\epsilon}),$$

where $I^{n-1}_{-\epsilon}$ is the metric of the $(n-1)$-dimensional Riemannian space form of constant curvature $-\epsilon$. For such metric forms we have the following result:

Lemma 3.1. *The metric $g = \kappa(s)^2(ds^2 + I^{n-1}_{-\epsilon})$ given above has constant curvature δ if and only if the function $\kappa(s)$ satisfies* (1-2).

This lemma is easy to prove using exterior differential forms and we omit the proof. Next we give the explicit construction of the spacelike hypersurfaces.

Example 3.2. The cylinder in $\mathbb{R}^{n+1}_1$ over a curve $\gamma(s) \subset \mathbb{R}^2_1$ is defined by

$$f : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n+1}_1, \quad f(s, y) = (\gamma(s), y),$$

where $y \in \mathbb{R}^{n-1}$.

The first and the second fundamental form of the cylinder f are given by

$$I = ds^2 + I_{\mathbb{R}^{n-1}}, \quad II = \kappa ds^2,$$

where $\kappa(s)$ is the geodesic curvature of $\gamma(s) \subset \mathbb{R}^2_1$, and $I_{\mathbb{R}^{n-1}}$ denotes the standard metric of the $(n-1)$-dimensional Euclidean space $\mathbb{R}^{n-1}$. Thus the principal curvatures of the cylinder are $(\kappa, 0, \dots, 0)$, and the mean curvature $H = \frac{\kappa}{n}$. From (2-15), we see that the conformal metric of the cylinder f is $g = \kappa^2(s)(ds^2 + I_{\mathbb{R}^{n-1}})$. By Lemma 3.1 we have the following result:

Proposition 3.3. *The cylinder in $\mathbb{R}^{n+1}_1$ over $\gamma(s) \subset \mathbb{R}^2_1$ as in Example 3.2 is of constant conformal sectional curvature if and only if $\gamma(s)$ is a curvature-spiral in $\mathbb{R}^2_1$.*

Example 3.4. The cone in $\mathbb{R}^{n+1}_1$ over a curve $\gamma(s) \subset \mathbb{S}^2_1(1) \subset \mathbb{R}^3_1$ is defined by

$$f : \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^{n-2} \to \mathbb{R}^{n+1}_1, \quad f(s, t, y) = (t\gamma(s), y),$$

where $y \in \mathbb{R}^{n-2}$ and $\mathbb{R}^+ = \{t \mid t > 0\}$.

The first and the second fundamental form of the cone f are given by

$$I = t^2 ds^2 + I_{\mathbb{R}^{n-1}}, \quad II = t\kappa ds^2,$$

where $\kappa(s)$ is the geodesic curvature of $\gamma(s) \subset \mathbb{S}^2_1(1)$. Thus the principal curvatures of the cone are $(\frac{\kappa}{t}, 0, \dots, 0)$, and the mean curvature $H = \frac{\kappa}{nt}$. From (2-15), we know that the conformal position vector of the cone f is

$$Y = \kappa\left(\frac{t^2 + |y|^2 + 1}{2t},\ \gamma(s),\ \frac{y}{t},\ \frac{t^2 + |y|^2 - 1}{2t}\right).$$

Note that

$$i(t, y) = \left(\frac{t^2+|y|^2+1}{2t}, \frac{y}{t}, \frac{t^2+|y|^2-1}{2t}\right) : \mathbb{R}^+ \times \mathbb{R}^{n-2} = \mathbb{H}^{n-1} \to \mathbb{H}^{n-1} \subset \mathbb{R}^n_1 \tag{3-17}$$

is nothing but the identity map of $\mathbb{H}^{n-1}$, since $\mathbb{R}^+ \times \mathbb{R}^{n-2} = \mathbb{H}^{n-1}$ is the upper half-space endowed with the standard hyperbolic metric. From (2-15), the conformal metric of the cone f is $g = \frac{\kappa^2}{t^2}(t^2 ds^2 + I_{\mathbb{R}^{n-1}}) = \kappa^2(ds^2 + I_{\mathbb{H}^{n-1}})$, where $I_{\mathbb{H}^{n-1}}$ denotes the standard hyperbolic metric of $\mathbb{H}^{n-1}$. By Lemma 3.1 we have the following result:

Proposition 3.5. *The cone in $\mathbb{R}^{n+1}_1$ over $\gamma(s) \subset \mathbb{S}^2_1(1) \subset \mathbb{R}^3_1$ as in Example 3.4 is of constant conformal sectional curvature if and only if $\gamma(s)$ is a curvature-spiral in $\mathbb{S}^2_1$.*

Example 3.6. Let $\mathbb{R}^2_{1+} = \{(x, y) \mid y > 0\}$ be the Lorentzian hyperbolic 2-plane. The rotational hypersurface in $\mathbb{R}^{n+1}_1$ over a curve $\gamma(s) \subset \mathbb{R}^2_{1+}$ is defined by

$$f : \mathbb{R} \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1}_1, \quad f(s, \theta) = (x(s), y(s)\theta),$$

where $\theta \in \mathbb{S}^{n-1}$ is the standard round sphere, and $\gamma(s) = (x(s), y(s)) \subset \mathbb{R}^2_{1+}$.

Denote the covariant differentiation of the metric ds^2 by D in $\mathbb{R}^2_{1+}$. For $\gamma(s) = (x(s), y(s)) \subset R^2_{1+}$, let $\dot{x}$ denote the derivative $\frac{\partial x}{\partial s}$ and so on. Choose the unit tangent vector $\alpha = \frac{1}{y}(\dot{x}, \dot{y})$ and the unit normal vector $\beta = \frac{1}{y}(\dot{y}, \dot{x})$. The geodesic curvature is computed via $\kappa(s) = \langle D_\alpha \alpha, \beta\rangle = \frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{y^2} + \frac{\dot{x}}{y}$. The rotational hypersurface f has the unit normal vector $\eta = \frac{1}{y}(\dot{y}, \dot{x}\theta)$. The first and the second fundamental form of the rotational hypersurface f are given by

$$I = df \cdot df = y^2(ds^2 + I_{\mathbb{S}^{n-1}}), \quad II = -df \cdot d\eta = (y\kappa - \dot{x})ds^2 - \dot{x} I_{\mathbb{S}^{n-1}}.$$

Thus the principal curvatures of the rotational hypersurface f are $\frac{y\kappa-\dot{x}}{y^2}, \frac{-\dot{x}}{y^2}, \ldots, \frac{-\dot{x}}{y^2}$. From (2-15), we know that the conformal metric of the rotational hypersurface f is $g = \kappa^2(x)(ds^2 + I_{\mathbb{S}^{n-1}})$. By Lemma 3.1 we have the following result:

Proposition 3.7. *The rotational hypersurface in $\mathbb{R}^{n+1}_1$ over $\gamma(s) \subset \mathbb{R}^2_{1+}$ as in Example 3.6 is of constant conformal sectional curvature if and only if $\gamma(s)$ is a curvature-spiral in $\mathbb{R}^2_{1+}$.*

Example 3.8. Let p, q be any two given natural numbers with $p + q < n$ and let a be a real number $a > 1$. We define the cone over the hyperbolic torus $\mathbb{H}^q(-\sqrt{a^2-1}) \times \mathbb{S}^p(a) \subset \mathbb{S}^{p+q+1}_1(1)$, as follows:

$$f : \mathbb{H}^q(-\sqrt{a^2-1}) \times \mathbb{S}^p(a) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \to \mathbb{R}^{n+1}_1, \quad f(u', u'', t, u''') = (tu', tu'', u'''),$$

where $u' \in \mathbb{H}^q(-\sqrt{a^2-1}),\ u'' \in \mathbb{S}^p(a),\ u''' \in \mathbb{R}^{n-p-q-1}$.

Let $b = \sqrt{a^2-1}$. One of the normal vectors of f can be taken as $e_{n+1} = \left(\frac{a}{b}u', \frac{b}{a}u'', 0\right)$. The first and second fundamental form of f are given by

$$I = t^2(\langle du', du'\rangle_1 + du'' \cdot du'') + dt \cdot dt + du''' \cdot du''',$$
$$II = -\langle dx, de_{n+1}\rangle_1 = -t\left(\frac{a}{b}\langle du', du'\rangle_1 + \frac{b}{a} du'' \cdot du''\right).$$

Thus the mean curvature of f satisfies

$$H = \frac{-pb^2 - qa^2}{nabt},$$

and

$$e^{2\tau} = \frac{n}{n-1}\left[\sum_{ij} h^2_{ij} - nH^2\right] = \frac{p(n-p)b^4 - 2pqa^2b^2 + q(n-q)a^4}{(n-1)a^2b^2t^2} := \frac{\alpha^2}{t^2}.$$

Let id_k denote the k-dimensional identical mapping. From (2-16), we have

$$B = b_1 \,\mathrm{id}_q \oplus b_2 \,\mathrm{id}_p \oplus b_3 \,\mathrm{id}_{n-q-p}, \quad A = a_1 \,\mathrm{id}_q \oplus a_2 \,\mathrm{id}_p \oplus a_3 \,\mathrm{id}_{n-q-p},$$

where

$$b_1 = \frac{pb^2 - (n-q)a^2}{nab\alpha}, \qquad b_2 = \frac{qa^2 - (n-p)b^2}{nab\alpha}, \qquad b_3 = \frac{pb^2 + qa^2}{nab\alpha},$$

and

$$a_1 = \frac{(pb^2+qa^2)^2-(pb^2+qa^2)2na^2+n^2a^2b^2}{2n^2a^2b^2\alpha^2},$$

$$a_2 = \frac{(pb^2+qa^2)^2-(pb^2+qa^2)2nb^2+n^2a^2b^2}{2n^2a^2b^2\alpha^2},$$

$$a_3 = \frac{(pb^2+qa^2)^2-n^2a^2b^2}{2n^2a^2b^2\alpha^2}.$$

Using these equations and (2-10), it is easy to prove the following result:

Proposition 3.9. *Let $f : M^n \to \mathbb{R}^{n+1}_1$ be a cone over a hyperbolic torus*

$$\mathbb{H}^q(-\sqrt{a^2-1}) \times \mathbb{S}^p(a).$$

If f has constant conformal sectional curvature δ, then $\delta = 0$, $p = q = 1$ and $n = 3$.

A spacelike hypersurface is called a conformal isoparametric spacelike hypersurface if the conformal 1-form vanishes and the eigenvalues of the conformal second fundamental form are constant. In [Li and Nie 2018] and [Nie and Wu 2008], the authors characterized the cone over the hyperbolic torus as follows:

Theorem 3.10 [Li and Nie 2018]. *Let $f : M^n \to M^{n+1}_1(c)$ be a conformal isoparametric spacelike hypersurface with r distinct principal curvatures. If $r \geq 3$, then $r = 3$, and locally f is conformally equivalent to the cone over the hyperbolic torus $\mathbb{H}^q(-\sqrt{a^2-1}) \times \mathbb{S}^p(a)$.*

4. The proof of Theorem 1.1

The hypothesis of constant conformal sectional curvature implies that the spacelike hypersurface is conformally flat. A classical result says that a spacelike hypersurface $f : M^n \to M^{n+1}_1(c) (n \geq 4)$ is conformally flat if and only if there exists a principle curvature which has multiplicity at least $n-1$ everywhere. Since our classification theorem is local, we consider the following two cases:

(1) The spacelike hypersurface has only two distinct principal curvatures.

(2) The 3-dimensional spacelike hypersurface has three distinct principal curvatures.

First, we consider case (1). Let $f : M^n \to \mathbb{R}^{n+1}_1$, $n \geq 3$, be a spacelike hypersurface with two distinct principal curvatures; one of which is simple, while the other has multiplicity $n-1$.

Lemma 4.1. *Let $f : M^n \to \mathbb{R}^{n+1}_1$, $n \geq 3$, be a spacelike hypersurface with two distinct principal curvatures. If the conformal sectional curvature has constant δ, then* *we can choose an orthonormal basis* $\{E_1, \dots, E_n\}$ *with respect to the conformal metric* g *such that*

$$(B_{ij}) = \mathrm{diag}\left(\frac{n-1}{n}, \frac{-1}{n}, \dots, \frac{-1}{n}\right); \quad C_2 = \dots = C_n = 0; \tag{4-18}$$
$$\omega_{1\alpha} = -C_1\omega_\alpha; \quad \delta = C_{1,1} - (C_1)^2; \quad C_{\alpha,\alpha} = -(C_1)^2, \quad \alpha \geq 2.$$

Proof. We take a local orthonormal basis $\{E_1, \dots, E_n\}$, with respect to g, under which,

$$(B_{ij}) = \mathrm{diag}(b_1, \underbrace{b_2, \dots, b_2}_{n-1}).$$

From the fourth equation in (2-11), we assume $b_1 = \frac{n-1}{n}$ and $b_2 = -\frac{1}{n}$. Since the spacelike hypersurface has constant conformal curvature δ, by (2-11), we have

$$(A_{ij}) = \mathrm{diag}\left(\frac{\delta}{2} - \frac{2n-1}{2n^2}, \frac{\delta}{2} + \frac{1}{2n^2}, \dots, \frac{\delta}{2} + \frac{1}{2n^2}\right).$$

In this section, we make use of the following convention on the range of indices

$$1 \leq i, j, k \leq n, \quad 2 \leq \alpha, \beta, \gamma \leq n.$$

From $dB_{ij} + \sum_k B_{kj}\omega_{ki} + \sum_k B_{ik}\omega_{kj} = \sum_k B_{ij,k}\omega_k$ and (2-8), we can get

$$B_{1\alpha,\alpha} = -C_1, \quad \omega_{1\alpha} = -C_1\omega_\alpha, \quad C_\alpha = 0, \quad 2 \leq \alpha \leq n, \tag{4-19}$$
$$(B_{ij,k} = 0 \text{ otherwise}).$$

Using $dC_i + \sum_k C_k\omega_{ki} = \sum_k C_{i,k}\omega_k$ and (4-19), we get

$$C_{\alpha,\alpha} = -(C_1)^2; \quad C_{\alpha,k} = 0, \ \alpha \neq k. \tag{4-20}$$

From (4-19), we see that

$$d\omega_{1\alpha} = -dC_1 \wedge \omega_\alpha - C_1^2\omega_1 \wedge \omega_\alpha - C_1 \sum_\gamma \omega_\gamma \wedge \omega_{\gamma\alpha},$$

and

$$d\omega_{1\alpha} - \sum_j \omega_{1j} \wedge \omega_{j\alpha} = -\tfrac{1}{2}\sum_{kl} R_{1\alpha kl}\omega_k \wedge \omega_l.$$

Thus we have

$$R_{1\alpha 1\alpha} = C_{1,1} - (C_1)^2, \quad R_{1\alpha\beta\alpha} - C_{1,\beta} = 0. \tag{4-21}$$

□

From Lemma 4.1, we know that the distributions

$$D_1 = \mathrm{span}\{E_1\}, \quad D_2 = \mathrm{span}\{E_2, E_3, \dots, E_n\}$$

are integrable. Any integral submanifold of distribution D_1 is a curve γ, and any integral submanifold of distribution D_2 is an $(n-1)$-dimensional submanifold L. Thus locally, we have

$$M^n = \gamma \times L.$$

Under the orthonormal basis $\{E_1, \dots, E_n\}$ as in Lemma 4.1, $\{Y, N, Y_1, \dots, Y_n, \xi\}$ forms a moving frame in $\mathbb{R}^{n+3}_2$ along M^n. We define

$$F = -\frac{1}{n}Y - \xi, \quad X_1 = -C_1 Y - Y_1, \quad P = -a_2 Y + N + C_1 X_1 - \frac{1}{n}F.$$

Therefore we have

$$\begin{aligned} &\langle F, F\rangle = -1, \quad \langle X_1, X_1\rangle = 1, \quad \langle P, P\rangle = -C_{1,1}, \\ &\langle F, P\rangle = 0, \quad \langle F, X_1\rangle = 0, \quad \langle X_1, P\rangle = 0. \end{aligned} \tag{4-22}$$

From Lemma 4.1 and the structure equation of f we can derive

$$\begin{aligned} E_1(F) &= X_1, & E_\alpha(F) &= 0, \\ E_1(X_1) &= P + F, & E_\alpha(X_1) &= 0, \\ E_1(P) &= C_1 P + C_{1,1} X_1, & E_\alpha(P) &= 0. \end{aligned} \tag{4-23}$$

Thus the subspace $V_1 = \operatorname{span}\{F, X_1, P\}$ is fixed along M^n. From $\delta = C_{1,1} - (C_1)^2$, we get

$$E_1(C_{1,1}) = 2C_1 C_{1,1}, \quad E_\alpha(C_{1,1}) = 0. \tag{4-24}$$

We define $T = -a_2 Y - N + C_1 Y_1 - \frac{1}{n}\xi$, then we have

$$T \perp V_1, \quad \langle T, T\rangle = C_{1,1}, \quad \langle T, Y_\alpha\rangle = 0, \quad 2 \le \alpha \le n.$$

From (4-24), Lemma 4.1, and the structure equation of f we can derive

$$\begin{aligned} &E_\alpha(T) = -C_{1,1} Y_\alpha, \quad E_1(T) = C_1 T, \quad E_\beta(Y_\alpha) = \sum_\gamma \omega_{\alpha\gamma}(E_\beta) Y_\gamma, \\ &E_\alpha(Y_\alpha) = \sum_\beta \omega_{\alpha\beta}(E_\alpha) Y_\beta + T, \quad E_1(Y_\alpha) = \sum_\beta \omega_{\alpha\beta}(E_1) Y_\beta, \quad \alpha \ne \beta. \end{aligned} \tag{4-25}$$

Thus the subspace $V_2 = \operatorname{span}\{T, Y_2, Y_3, \dots, Y_n\}$ is fixed along M^n, and $V_1 \perp V_2$.

Using theory of linear first-order differential equations for $C_{1,1}$, (4-24) means that $C_{1,1} \equiv 0$ or $C_{1,1} \ne 0$ on an open subset $U \subset M^n$. Thus we need to consider the following three subcases: (1) $C_{1,1} \equiv 0$ on M^n; (2) $C_{1,1} < 0$ on M^n; (3) $C_{1,1} > 0$ on M^n. We will treat them case by case.

Proposition 4.2. *Under the assumptions in Lemma 4.1, if $C_{1,1} \equiv 0$ on M^n, then f is conformally equivalent to a cylinder over a curvature-spiral in $\mathbb{R}^2_1$.*

Proof. Since $C_{1,1} \equiv 0$, we have $\langle P, P\rangle = 0$. From (4-23), we know that P determines a fixed direction. Hence up to a conformal transformation we can write the fixed direction $P \in \mathbb{R}^{n+3}_2$ and constant subspace $V_1 \subset \mathbb{R}^{n+3}_2$ as follows:

$$P = (1, 0, \dots, 0, 1),$$
$$V_1 = \mathrm{span}\{F, X_1, P\} = \mathrm{span}\{(0, 1, 0, \dots, 0),\ (0, 0, 1, 0, \dots, 0),\ (1, 0, \dots, 0, 1)\}.$$

From (4-22), we have

$$\langle F, P\rangle = \langle F, (1,0,0,\dots,0,1)\rangle = 0, \quad \langle X_1, P\rangle = \langle X_1, (1,0,0,\dots,0,1)\rangle = 0. \tag{4-26}$$

Let $\{\kappa_1, \kappa_2, \dots, \kappa_2\}$ be the principal curvatures of the spacelike hypersurface f, then $e^\tau = |\kappa_1 - \kappa_2|$. We choose an orthonormal basis $\{e_1, \dots, e_n\}$ of TM^n with respect to the first fundamental form $I = df \cdot df$ such that $(h_{ij}) = \mathrm{diag}\{\kappa_1, \kappa_2, \dots, \kappa_2\}$; then $\{E_i = e^{-\tau} e_i, 1 \le i \le n\}$ is an orthonormal basis as in Lemma 4.1. From (2-15) and (4-26), we can deduce

$$\kappa_2 = 0, \quad E_1(\tau) = -C_1. \tag{4-27}$$

On the other hand, we have $\langle Y_\alpha, P\rangle = 0$ which implies that

$$E_\alpha(\tau) = 0. \tag{4-28}$$

Let $\{\tilde{\omega}_1, \dots, \tilde{\omega}_n\}$ be the dual basis of $\{e_1, \dots, e_n\}$, and $\{\tilde{\omega}_{ij}\}$ be the corresponding connection forms. Since $\omega_i = e^\tau \tilde{\omega}_i,\ 1 \le i \le n$, its connections have the relations

$$\omega_{ij} = \tilde{\omega}_{ij} + e_i(\tau)\tilde{\omega}_j - e_j(\tau)\tilde{\omega}_i.$$

Equations (4-27) and (4-28) imply that $\tilde{\omega}_{1\alpha} = 0$. Thus the spacelike hypersurface f is conformally equivalent to the hypersurface given by Example 3.2. By Proposition 3.3, we finish the proof of Proposition 4.2. □

Proposition 4.3. *Under the assumptions in Lemma 4.1, if $C_{1,1} < 0$ on M^n, then f is conformally equivalent to a cone over a curvature-spiral in $\mathbb{S}^2_1$.*

Proof. Since $C_{1,1} < 0$, by (4-22), the vector field P is a spacelike vector field in $\mathbb{R}^{n+3}_2$. Thus up to a conformal transformation we can write

$$V_1 = \mathrm{span}\{F, X_1, P\} = \mathrm{span}\{(0,1,0,\dots,0),\ (0,0,1,0,\dots,0),\ (0,0,0,1,\dots,0)\}.$$

Let f have principal curvatures $\{\kappa_1, \kappa_2, \dots, \kappa_2\}$. Since $e = (1, 0, \dots, 0, 1) \perp V$, we have $\langle F, e\rangle = \langle X_1, e\rangle = 0$ which implies $\kappa_2 = 0,\ E_1(\tau) = -C_1$. By (2-15), we obtain $e^{2\tau} = \kappa_1^2$. Setting

$$\bar{P} = \frac{P}{\sqrt{-C_{1,1}}}, \quad \theta = \frac{T}{\sqrt{-C_{1,1}}},$$

then $\langle \bar{P}, \bar{P}\rangle = 1$ and $\langle \theta, \theta\rangle = -1$. From (4-23), we know

$$\bar{P} : \gamma \to \mathbb{S}^2_1 \subset R^3_1 = V_1$$

is a curve, and (4-25) gives that

$$\theta : L \to \mathbb{H}^{n-1} \subset \mathbb{R}^n_1$$

is a standard embedding and the sectional curvature of $\theta(L)$ is -1. Since $\dim L = \dim \mathbb{H}^{n-1} = n-1$, we know that $\theta : L \to \mathbb{H}^{n-1}$ is a standard isometric isomorphism. By (3-17), we have the standard isometric isomorphism

$$\theta : L \to \mathbb{H}^{n-1} = \mathbb{R}^+ \times \mathbb{R}^{n-2}.$$

Since $P + T = -C_{1,1} Y$,

$$Y = \frac{1}{\sqrt{-C_{1,1}}}(\bar{P}, \theta) : M^n = \gamma \times L \to \mathbb{S}^2_1 \times \mathbb{H}^{n-1} = \mathbb{S}^2_1 \times \mathbb{R}^+ \times \mathbb{R}^{n-2} \subset \mathbb{R}^{n+3}_1.$$

Therefore

$$g = \langle dY, dY\rangle = -\frac{1}{C_{1,1}}(ds^2 + I_{\mathbb{H}^{n-1}}).$$

Thus the spacelike hypersurface f is conformally equivalent to the hypersurface given by Example 3.4. By Proposition 3.5, we finish the proof of Proposition 4.3. □

Proposition 4.4. *Under the assumptions in Lemma 4.1, if $C_{1,1} > 0$ on M^n, then f is conformally equivalent to a rotational hypersurface over a curvature-spiral in $\mathbb{R}^2_{1+}$.*

Proof. Since $C_{1,1} > 0$, we have $\langle P, P\rangle < 0$. Thus up to a conformal transformation we can write

$$V_1 = \text{span}\{F, X_1, P\} = \text{span}\{(1, 0, \dots, 0),\ (0, \dots, 0, 1),\ (0, 1, 0, \dots, 0)\}.$$

Thus $e = (1, 0, \dots, 0, 1) \in V_1$, and $\langle Y_\alpha, e\rangle = 0,\ 2 \le \alpha \le n$, which imply that $E_\alpha(\tau) = 0,\ 2 \le \alpha \le n$. Setting

$$\bar{P} = \frac{P}{\sqrt{C_{1,1}}}, \quad \theta = \frac{T}{\sqrt{C_{1,1}}},$$

then $\langle \bar{P}, \bar{P}\rangle = -1$ and $\langle \theta, \theta\rangle = 1$. From (4-23), we know

$$\bar{P} : \gamma \to \mathbb{H}^2_1 \subset \mathbb{R}^3_2 = V_1$$

is a curve. From (4-25), we see that

$$\theta : L \to \mathbb{S}^{n-1} \subset \mathbb{R}^n$$

is a standard embedding and the sectional curvature of $\theta(L)$ is 1. Since $\dim L = n-1$,

$\theta : L \to \mathbb{S}^{n-1}$ is a standard isometric isomorphism. Since $P + T = -C_{1,1}Y$,

$$Y = \frac{-1}{\sqrt{C_{1,1}}}(\bar{P}, \theta) : \gamma \times L \to \mathbb{H}^2_1 \times \mathbb{S}^{n-1}.$$

Denote $\bar{P} = (u_1, u_2, u_3) \in \mathbb{H}^2_1$, then

$$Y = \frac{u_3 - u_1}{\sqrt{C_{1,1}}}\left(\frac{u_1}{u_1 - u_3}, \frac{u_2}{u_1 - u_3}, \frac{u_3}{u_1 - u_3}, \frac{\theta}{u_1 - u_3}\right).$$

Thus the hypersurface $f : \mathbb{R} \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1}_1$ is given by $f = \left(\frac{u_2}{u_1 - u_3}, \frac{\theta}{u_1 - u_3}\right)$. Note that

$$\varphi(u_1, u_2, u_3) = \left(\frac{u_2}{u_1 - u_3}, \frac{1}{u_1 - u_3}\right)$$

is the inverse mapping of the local isometric correspondence $\phi : \mathbb{R}^2_{1+} \to \mathbb{H}^2_1$ (see (1-3)). Thus the spacelike hypersurface f is conformally equivalent to the hypersurface given by Example 3.6. By Proposition 3.7, we finish the proof of Proposition 4.4. □

Next we consider case (2). Let $f : M^3 \to \mathbb{R}^4_1$ be a spacelike hypersurface with three distinct principal curvatures.

Proposition 4.5. *Let $f : M^3 \to \mathbb{R}^4_1$ be a spacelike hypersurface with three distinct principal curvatures. If the conformal sectional curvature has constant δ, then $\delta = 0$ and f is conformally equivalent to the spacelike hypersurface defined by Example 3.8.*

To prove Proposition 4.5, we need the following two lemmas.

Lemma 4.6. *Under the same assumptions as in Proposition 4.5, there exist a local orthonormal basis $\{E_1, E_2, E_3\}$, consisting of eigenvectors of B such that*

$$\begin{aligned} &B_{11,2} = \frac{b_3 - b_1}{b_1 - b_2}C_2, \quad B_{11,3} = \frac{b_2 - b_1}{b_1 - b_3}C_3, \quad B_{22,1} = \frac{b_3 - b_2}{b_2 - b_1}C_1,\\ &B_{22,3} = \frac{b_1 - b_2}{b_2 - b_3}C_3, \quad B_{33,1} = \frac{b_2 - b_3}{b_3 - b_1}C_1, \quad B_{33,2} = \frac{b_1 - b_3}{b_3 - b_2}C_2. \end{aligned} \tag{4-29}$$

Proof. Since f is of constant conformal curvature, from (2-11), we have

$$R_{ij} = 2\delta\delta_{ij} = \sum_k B_{ik}B_{kj} + (tr A)\delta_{ij} + A_{ij}. \tag{4-30}$$

Thus we can take a local orthonormal basis $\{E_1, E_2, E_3\}$ such that

$$(B_{ij}) = \mathrm{diag}(b_1, b_2, b_3), \quad (A_{ij}) = \mathrm{diag}(a_1, a_2, a_3). \tag{4-31}$$

Using (4-30) and (4-31), we have

$$B_{ij,k}(b_i+b_j)+A_{ijk}=0, \quad 1\le i,\, j,\, k\le 3. \tag{4-32}$$

Using (2-7) and (2-8) and (4-32), we can obtain (4-29) and

$$B_{ij,k}=A_{ij,k}=0, \quad i\neq j,\ i\neq k,\ k\neq j. \tag{4-33}$$

Thus we complete the proof. □

If the conformal 1-form C is equal to 0, by Lemma 4.6, we know that the eigenvalues of the conformal second fundamental form are constant. Thus the spacelike hypersurface is a conformal isoparametric spacelike hypersurface. By Proposition 3.9 and Theorem 3.10, we can prove Proposition 4.5. Next we need to prove $C=0$.

Lemma 4.7. *Under the same assumptions as in Proposition 4.5, the conformal* 1*-form C is equal to* 0.

Proof. Let $\{\omega_1, \omega_2, \omega_3\}$ be the dual of the local orthonormal basis $\{E_1, E_2, E_3\}$ in Lemma 4.6, and $\{\omega_{ij}\}$ the connection forms. Using covariant derivatives of B_{ij},

$$\omega_{ij}=\frac{B_{ij,i}}{b_i-b_j}\omega_i+\frac{B_{ij,j}}{b_i-b_j}\omega_j, \qquad i\neq j, \quad 1\le i,\, j\le 3. \tag{4-34}$$

Using (2-8), we have $B_{ij,j}=B_{jj,i}-C_i,\ i\neq j$. Using (2-12), (4-33) and (4-29), we can obtain

$$B_{12,31}=(B_{11,3}-B_{22,3})\frac{B_{12,1}}{b_1-b_2}+(B_{12,1}-B_{32,3})\frac{B_{13,1}}{b_1-b_3},$$

$$\begin{aligned} B_{12,13}=&\frac{3(b_2B_{11,3}-b_1B_{223})}{(b_1-b_2)^2}C_2\\ &+\left(C_{2,3}-\frac{B_{32,3}}{b_3-b_2}C_3\right)\frac{3b_1}{b_2-b_1}+\frac{B_{32,3}B_{13,1}}{b_3-b_2}. \end{aligned} \tag{4-35}$$

From (2-10) and the Ricci identity (2-13), we have $C_{i,j}-C_{j,i}=(b_i-b_j)A_{ij}=0$, and $B_{12,31}=B_{12,13}$. Using (4-35), we can derive

$$b_1C_{2,3}=\frac{b_1b_2+2b_3^2}{(b_2-b_3)(b_3-b_1)}C_3C_2=-C_3C_2,$$

where we use $b_1+b_2+b_3=0$ and $b_1^2+b_2^2+b_3^2=\frac{2}{3}$. Similarly $b_2C_{1,3}=-C_3C_1$ and $b_3C_{1,2}=-C_2C_1$. Thus

$$b_kC_{i,j}=-C_iC_j, \quad i\neq j,\ i\neq k,\ k\neq j. \tag{4-36}$$

Using the covariant derivative of C_i and taking the derivative for (4-36) along E_k, we obtain

$$B_{kk,k}C_{i,j}+b_k\left[C_{i,jk}-C_{k,j}\frac{B_{ki,k}}{b_k-b_i}-C_{i,k}\frac{B_{kj,k}}{b_k-b_j}\right]=-C_i\left[C_{j,k}-C_k\frac{B_{jk,k}}{b_k-b_j}\right]-C_j\left[C_{i,k}-C_k\frac{B_{ik,k}}{b_k-b_i}\right]. \tag{4-37}$$

If $b_1b_2b_3=0$, we can assume that $b_1=0$. From (2-11), we know that $b_2=-b_3=\frac{1}{\sqrt{3}}$. Using (4-29) we have $C=0$.

We assume $b_1b_2b_3\neq 0$. From (4-29), (4-36), (4-37) and $B_{kk,k}=-B_{jj,k}-B_{ii,k}$, we conclude that

$$b_kC_{i,jk}=-\frac{4}{3}\frac{C_iC_jC_k}{b_ib_jb_k}=-\frac{4}{3}\frac{C_1C_2C_3}{b_1b_2b_3}. \tag{4-38}$$

Since $C_{i,jk}=C_{j,ik}=C_{k,ij}$ and $b_i\neq b_j,\ i\neq j$, from (4-38) we get

$$C_{1,23}=0,\qquad C_1C_2C_3=0.$$

We can assume that $C_1=0$, and (4-34) can be written as

$$\omega_{12}=\frac{B_{12,1}}{b_1-b_2}\omega_1,\qquad \omega_{13}=\frac{B_{13,1}}{b_1-b_3}\omega_1,\qquad \omega_{23}=\frac{B_{23,2}}{b_2-b_3}\omega_2+\frac{B_{23,3}}{b_2-b_3}\omega_3. \tag{4-39}$$

Using the covariant derivative of C_i and

$$d\omega_{ij}-\sum_k\omega_{ik}\wedge\omega_{kj}=-\tfrac{1}{2}\sum_{kl}R_{ijkl}\omega_k\wedge\omega_l,$$

we can derive

$$\begin{aligned}
&\frac{3C_3^2[b_2^3-b_3^3-6b_1b_2^2+6b_1^2b_2]}{(b_1-b_3)^2(b_3-b_2)}+\frac{27b_1^2b_3C_2^2}{(b_1-b_2)^2(b_3-b_2)}\\
&\qquad\qquad=3b_1C_{3,3}+(b_1-b_3)^2\delta,\\
&\frac{3C_2^2[b_3^3-b_2^3-6b_1b_3^2+6b_1^2b_3]}{(b_1-b_2)^2(b_2-b_3)}+\frac{27b_1^2b_2C_3^2}{(b_1-b_3)^2(b_2-b_3)}\\
&\qquad\qquad=3b_1C_{2,2}+(b_1-b_2)^2\delta,\\
&\frac{3C_3^2[b_1^3-b_3^3+3b_2^3+15b_1b_2^2]}{(b_3-b_2)^2(b_3-b_1)}+\frac{3C_2^2[b_1^3-b_2^3+3b_3^3+15b_1b_3^2]}{(b_3-b_2)^2(b_2-b_1)}\\
&\qquad\qquad=3b_3C_{2,2}+3b_2C_{3,3}+(b_3-b_2)^2\delta,
\end{aligned} \tag{4-40}$$

where we use $B_{ij,j}=B_{jj,i}-C_i,\ \mathrm{tr}(B)=0$ and $|B|^2=\frac{2}{3}$. We can eliminate $C_{2,2}$

and $C_{3,3}$ in (4-40) and derive

$$\frac{3[2b_1^4+2b_2^4+2b_3^4-9b_1^2b_2^2-6b_1^3b_2-12b_1b_2^3]}{(b_1-b_3)^3}C_3^2 + \frac{3[2b_1^4+2b_2^4+2b_3^4-9b_1^2b_3^2-6b_1^3b_3-12b_1b_3^3]}{(b_1-b_2)^3}C_2^2 = -b_1(b_2-b_3)^2\delta. \tag{4-41}$$

On the other hand, using the covariant derivative of C_i and $C_1=0$, we have

$$\begin{aligned} C_{1,1} &= \frac{3b_1C_2^2}{(b_2-b_1)^2}+\frac{3b_1C_3^2}{(b_3-b_1)^2},\\ C_{2,1} &= C_{1,2}=0, \quad C_{3,1}=C_{1,3}=0,\\ C_{2,2} &= E_2(C_2)+\frac{3b_2C_3^2}{(b_3-b_2)^2},\\ C_{2,3} &= E_3(C_2)-\frac{3b_3C_2C_3}{(b_3-b_2)^2},\\ C_{3,2} &= E_2(C_3)-\frac{3b_2C_2C_3}{(b_3-b_2)^2},\\ C_{3,3} &= E_3(C_3)+\frac{3b_3C_2^2}{(b_3-b_2)^2}. \end{aligned} \tag{4-42}$$

Using the second covariant derivative of the conformal 1-form C defined by

$$\sum_m C_{i,jm}\omega_m = dC_{i,j}+\sum_m C_{m,j}\omega_{mi}+\sum_m C_{i,m}\omega_{mj},$$

and combining (4-29) and (4-42), we can deduce

$$\begin{aligned} C_{3,23} = {} & E_3(E_2(C_3)) - 3\left[\frac{b_1-b_2}{(b_2-b_3)^3}-\frac{6b_2b_1(b_1-b_2)}{(b_3-b_2)^4(b_1-b_3)}\right]C_3^2C_2\\ & +3(C_{2,2}-C_{3,3})\frac{b_3C_2}{(b_3-b_2)^2}\\ & -\frac{3b_2}{(b_3-b_2)^2}\left[C_3C_{2,3}+C_2C_{3,3}+\frac{3b_3}{(b_3-b_2)^2}(C_3^2C_2-C_2^3)\right],\end{aligned}$$

$$\begin{aligned} C_{3,32} = {} & E_2(E_3(C_3)) + 3\left[\frac{b_1-b_3}{(b_3-b_2)^3}-\frac{6b_3b_1(b_1-b_3)}{(b_3-b_2)^4(b_1-b_2)}\right]C_2^3\\ & +\frac{6b_3}{(b_3-b_2)^2}C_2C_{2,2}-\frac{18b_3b_2}{(b_3-b_2)^4}C_3^2C_2.\end{aligned}$$

Using the Ricci identity $C_{3,23} - C_{3,32} = \sum_l C_l R_{l323} = \delta C_2$, we obtain

$$\text{(4-43)}\quad \frac{\delta}{3} C_2 = \frac{3b_1b_3b_2 + b_2b_1^2 + 5b_3b_1^2 - 8b_1b_3^2 - 2b_3b_2^2 - b_1b_2^2 + 2b_2b_3^2}{(b_3-b_2)^4(b_1-b_2)} C_2^3 + \frac{3b_1b_3b_2 + b_3b_1^2 + 5b_2b_1^2 - 8b_1b_2^2 - 2b_2b_3^2 - b_1b_3^2 + 2b_3b_2^2}{(b_3-b_2)^4(b_1-b_3)} C_2C_3^2 - \frac{b_3}{(b_3-b_2)^2} C_2C_{2,2} - \frac{b_2}{(b_3-b_2)^2} C_2C_{3,3} + \frac{2b_2}{b_1(b_3-b_2)^2} C_2C_3^2,$$

where we use the equation

$$\begin{aligned} E_3(E_2(C_3)) - E_2(E_3(C_3)) &= [E_3, E_2](C_3) \\ &= (\omega_{23}(E_3)E_3 - \omega_{32}(E_2)E_2)(C_3) \\ &= \frac{3b_3}{(b_3-b_2)^2} C_2C_{3,3} - \frac{9b_3^2}{(b_3-b_2)^4} C_2^3 - \frac{3b_2}{(b_3-b_2)^2} C_3C_{3,2} - \frac{9b_2^2}{(b_3-b_2)^4} C_3^2C_2. \end{aligned}$$

From the third equation in (4-40) and (4-43), noting that $b_1b_2b_3 \neq 0$, we can deduce

$$\text{(4-44)}\quad \frac{2b_2}{b_1(b_3-b_2)^2} C_2C_3^2 = 0.$$

We can assume that $C_2 = 0$. Next we prove that $C_3 = 0$. In fact, if $C_3 \neq 0$, from (4-39), we have

$$\omega_{12} = 0, \quad \omega_{13} = \frac{B_{13,1}}{b_1 - b_3}\omega_1, \quad \omega_{23} = \frac{B_{23,2}}{b_2 - b_3}\omega_2.$$

Using

$$d\omega_{12} - \sum_k \omega_{1k} \wedge \omega_{k2} = -\tfrac{1}{2} \sum_{kl} R_{12kl}\omega_k \wedge \omega_l,$$

we can derive

$$\text{(4-45)}\quad \delta = R_{1212} = \frac{-9b_1b_2C_3^2}{(b_1-b_3)^2(b_2-b_3)^2}.$$

Since $C_2 = 0$, (4-41) becomes

$$\text{(4-46)}\quad \frac{3[2b_1^4 + 2b_2^4 + 2b_3^4 - 9b_1^2b_2^2 - 6b_1^3b_2 - 12b_1b_2^3]}{(b_3-b_1)^2(b_3-b_2)^2} C_3^2 = -b_1(b_1-b_3)\delta.$$

Combining (4-45) and (4-46), we have

$$\text{(4-47)}\quad [2b_1^4 + 2b_2^4 + 2b_3^4 + 12b_1b_2(b_1b_3 - b_2^2)]C_3^2 = 0.$$

Using $b_1+b_2+b_3=0$ and $b_1^2+b_2^2+b_3^2=\frac{2}{3}$, we see that

$$b_1^4+b_2^4+b_3^4=\tfrac{2}{9} \quad \text{and} \quad b_1b_3-b_2^2=-\tfrac{1}{3}.$$

Thus (4-47) is written as

$$\left(\tfrac{4}{9}-4b_1b_2\right)C_3^2=0.$$

Since $C_3\neq 0$, $\frac{4}{9}-4b_1b_2=0$ which implies that b_1, b_2, b_3 are constant. Thus (4-29) means that $C=0$, which is a contradiction. Thus $C_3=0$ and $C=0$. This completes the proof. □

Combining Propositions 4.2, 4.3, 4.4 and 4.5, we finish the proof of Theorem 1.1.

Acknowledgements

The authors thank the referee for valuable suggestions that improved the clarity and readability of the paper. The authors are supported by the grants No. 11571037 and No. 11471021 of NSFC.

References

[Cahen and Kerbrat 1983] M. Cahen and Y. Kerbrat, "Domaines symétriques des quadriques projectives", *J. Math. Pures Appl.* (9) **62**:3 (1983), 327–348. MR Zbl

[Guo et al. 2012] Z. Guo, T. Li, L. Lin, X. Ma, and C. Wang, "Classification of hypersurfaces with constant Möbius curvature in $\mathbb{S}^{m+1}$", *Math. Z.* **271**:1-2 (2012), 193–219. MR Zbl

[Hu and Li 2003] Z. Hu and H. Li, "Submanifolds with constant Möbius scalar curvature in S^n", *Manuscripta Math.* **111**:3 (2003), 287–302. MR Zbl

[Li and Nie 2013] T. Li and C. Nie, "Conformal geometry of hypersurfaces in Lorentz space forms", *Geometry* **2013** (2013), art. id. 549602. Zbl

[Li and Nie 2018] T. Li and C. Nie, "Spacelike Dupin hypersurfaces in Lorentzian space forms", *J. Math. Soc. Japan* **70**:2 (2018), 463–480. MR Zbl

[Li and Wang 2003] H. Li and C. Wang, "Möbius geometry of hypersurfaces with constant mean curvature and scalar curvature", *Manuscripta Math.* **112**:1 (2003), 1–13. MR Zbl

[Li et al. 2013] T. Li, X. Ma, and C. Wang, "Willmore hypersurfaces with constant Möbius curvature in $\mathbb{R}^{n+1}$", *Geom. Dedicata* **166** (2013), 251–267. MR Zbl

[Nie 2015] C. X. Nie, "Blaschke isoparametric hypersurfaces in the conformal space $\mathbb{Q}_1^{n+1}$, I", *Acta Math. Sin.* (*Engl. Ser.*) **31**:11 (2015), 1751–1758. MR Zbl

[Nie and Wu 2008] C. X. Nie and C. X. Wu, "Space-like hypersurfaces with parallel conformal second fundamental forms in the conformal space", *Acta Math. Sin.* (*Chin. Ser.*) **51**:4 (2008), 685–692. In Chinese. MR

[O'Neill 1983] B. O'Neill, *Semi-Riemannian geometry*, Pure and Applied Math. **103**, Academic Press, New York, 1983. MR Zbl

[Wang 1998] C. Wang, "Moebius geometry of submanifolds in $\mathbb{S}^n$", *Manuscripta Math.* **96**:4 (1998), 517–534. MR Zbl

Received April 11, 2017. Revised May 16, 2018.

XIU JI
DEPARTMENT OF MATHEMATICS
BEIJING INSTITUTE OF TECHNOLOGY
BEIJING
CHINA
jixiu1106@163.com

TONGZHU LI
DEPARTMENT OF MATHEMATICS
BEIJING INSTITUTE OF TECHNOLOGY
BEIJING
CHINA
litz@bit.edu.cn

HUAFEI SUN
BEIJIN KEY LABORATORY ON MCAAI
BEIJING
CHINA
huafeisun@bit.edu.cn

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 300 No. 1 May 2019
