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We present a classification of the complete, simply connected, contact metric (κ, μ)-spaces as homogeneous contact metric manifolds, by studying the base space of their canonical fibration. According to the value of the Boeckx invariant, it turns out that the base is a complexification or a paracomplexification of a sphere or of a hyperbolic space. In particular, we obtain a new homogeneous representation of the contact metric (κ, μ)-spaces with Boeckx invariant less than -1.

1. Introduction

The study of the curvature tensor of associated metrics to a contact form is a central theme in contact metric geometry. Actually some important classes of contact metric manifolds can be defined using it. We recall for example that Sasakian manifolds, the odd-dimensional analogues of Kähler manifolds, can be characterized by

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

where X, Y are any vector fields and ξ denotes the characteristic vector field of the contact metric manifold. A meaningful generalization of this curvature condition is

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY),$$

where κ, μ are real numbers and $2h$ is the Lie derivative of the structure tensor φ in the direction of the characteristic vector field ξ.

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The contact metric manifolds with this property were introduced by Blair, Koufogiorgos and Papantoniou [1995], and are called *contact metric* (κ, μ)*-spaces* in the literature. These spaces have many interesting geometric properties; first of all, they are stable under $\mathcal{D}$-homothetic deformations and moreover in the non-Sasakian case, i.e., when $\kappa \neq 1$, the curvature tensor of the associated metric is completely determined. Looking at contact metric manifolds as strongly pseudoconvex (almost) CR manifolds, it was shown in [Dileo and Lotta 2009] that the (κ, μ) condition is equivalent to the local CR-symmetry with respect to the Webster metric, according to the general notion in [Kaup and Zaitsev 2000]. In this context, another characterization was given by Boeckx and Cho [2008] in terms of the parallelism of the Tanaka–Webster curvature.

Boeckx gave a crucial contribution to the problem of classifying these manifolds; after showing that every non-Sasakian contact (κ, μ)-space is locally homogeneous and strongly locally φ-symmetric [Boeckx 1999], he defined a scalar invariant I_M which completely determines a contact (κ, μ)-space M locally up to equivalence and up to $\mathcal{D}$-homothetic deformations of its contact metric structure [Boeckx 2000].

A standard example is the tangent sphere bundle T_1M of a Riemannian manifold M with constant sectional curvature $c \neq 1$. Being a hypersurface of TM, which is equipped with a natural almost-Kähler structure (J, G), where G is the Sasaki metric, T_1M inherits a standard contact metric structure (for more details, see for instance [Blair 2010]). In particular, the Webster metric g of T_1M is a scalar multiple of G. The corresponding Boeckx invariant is given by

$$I_{T_1M} = \frac{1+c}{|1-c|}.$$

Hence, as c varies in $\mathbb{R} \smallsetminus \{1\}$, I_{T_1M} assumes all real values strictly greater than -1.

The case $I \leqslant -1$ seems to lead to models of different nature. Namely, Boeckx found examples of contact metric (κ, μ)-spaces, for every value of the invariant $I \leqslant -1$, namely a two parameter family of (abstractly constructed) Lie groups with a left-invariant contact metric structure. However, he gave no geometric description of these examples; in particular, to our knowledge, nothing can be found in the literature regarding the topological structure of these manifolds.

One of the first aims of this paper is to fill this gap, showing that simply connected, complete contact metric (κ, μ)-spaces of dimension $2n+1$ (where $n > 1$) with $I < -1$ are exhausted by a one parameter family of invariant contact metric structures on the homogeneous space

$$\mathrm{SO}(n, 2)/\,\mathrm{SO}(n).$$

Actually, we provide a unified treatment of all the models with $I_M \neq \pm 1$. Our classification is accomplished intrinsically, by studying the canonical fibration of non-Sasakian contact metric (κ, μ)-spaces with Boeckx invariant $I_M \neq \pm 1$ and

Boeckx invariant	model space	base space
$I_M > 1$	$\mathrm{SO}(n+2)/\mathrm{SO}(n)$	$\mathrm{SO}(n+2)/(\mathrm{SO}(n)\times\mathrm{SO}(2))$
$-1 < I_M < 1$	$\mathrm{SO}(n+1,1)/\mathrm{SO}(n)$	$\mathrm{SO}(n+1,1)/(\mathrm{SO}(n)\times\mathrm{SO}(1,1))$
$I_M < -1$	$\mathrm{SO}(n,2)/\mathrm{SO}(n)$	$\mathrm{SO}(n,2)/(\mathrm{SO}(n)\times\mathrm{SO}(2))$

Table 1. Simply connected complete contact metric (κ, μ)-spaces with $I_M \neq \pm 1$.

endowing the base spaces of a canonical connection. Here we refer to the fibration $M \to M/\xi$ over the leaf space of the foliation determined by the Reeb vector field; as such, it depends only on the contact form of M. First, in Theorem 7, non-Sasakian contact metric (κ, μ)-spaces with Boeckx invariant not equal to ± 1 are characterized by admitting a transitive Lie group of automorphisms whose Lie algebra $\mathfrak{g}$ has a (canonical) symmetric decomposition. This decomposition yields a reductive decomposition for the base space B of the canonical fibration and the associated canonical connection makes B an affine symmetric space (Corollary 8).

Next we show that B admits a uniquely determined standard invariant complex or paracomplex structure, by which it is a complexification or a paracomplexification of the sphere S^n or of the hyperbolic space $\mathbb{H}^n$, according to the value of the Boeckx invariant of the (κ, μ)-space. After identifying the possible base spaces B, in the final section we construct explicitly our models as homogeneous contact metric manifolds fiberings onto them. In conclusion, we obtain the classification list in Table 1. This table also provides a new geometric interpretation of the Boeckx invariant.

2. Preliminaries

Let M be an odd-dimensional smooth manifold. An *almost contact structure* on M is a triple consisting of a $(1, 1)$ tensor field φ, a vector field ξ, and a 1-form η satisfying

$$\varphi^2 = -\operatorname{id} + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

An almost contact manifold always admits a *compatible metric*, namely a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields X, Y on M. If such a metric g satisfies also

$$\mathrm{d}\eta(X, Y) = g(X, \varphi Y),$$

then (φ, ξ, η, g) is called a *contact metric structure* on M. In this case η is a contact form; we shall denote by D the corresponding contact distribution $D = \ker(\eta)$ and by $\mathcal{D}$ the module of smooth sections of D.

A contact metric manifold M is said to be a *K-contact manifold* if its characteristic vector field ξ is Killing. This condition is equivalent to the vanishing of the $(1, 1)$ tensor field

$$h := \tfrac{1}{2}\mathcal{L}_\xi \varphi,$$

where $\mathcal{L}_\xi$ is Lie differentiation in the direction of ξ.

If the curvature tensor R of a contact metric manifold M satisfies the condition

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

for all vector fields X, Y on M, then M is a *Sasakian* manifold. In this case ξ is a Killing vector field and hence M is a K-contact manifold.

A *contact metric (κ, μ)-space* is a contact metric manifold $(M, \varphi, \xi, \eta, g)$ such that

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY),$$

where $X, Y \in \mathfrak{X}(M)$ are arbitrary vector fields and κ, μ are real numbers. The (κ, μ) condition is invariant under D_a-homothetic deformations. We recall that a D_a-homothetic deformation of a contact metric manifold $(M, \varphi, \xi, \eta, g)$ is given by the following changing of the structural tensors of M:

$$\bar{\eta} := a\eta, \quad \bar{\xi} := \tfrac{1}{a}\xi, \quad \bar{g} = ag + a(a-1)\eta \otimes \eta, \tag{1}$$

where a is a positive constant.

By direct computations one can check that a D_a-homothetic deformation transforms a contact metric (κ, μ) space into a contact metric $(\bar{\kappa}, \bar{\mu})$ space where

$$\bar{\kappa} = \frac{\kappa + a^2 - 1}{a^2}, \quad \bar{\mu} = \frac{\mu + 2a - 2}{a}.$$

In particular, a D_a-homothetic deformation of a contact metric manifold $(M, \varphi, \xi, \eta, g)$ satisfying $R(X, Y)\xi = 0$ yields

$$\bar{R}(X, Y)\xi = \frac{a^2 - 1}{a^2}(\bar{\eta}(Y)X - \bar{\eta}(X)Y) + \frac{2a - 2}{a}(\bar{\eta}(Y)\bar{h}X - \bar{\eta}(X)\bar{h}Y).$$

Blair, Koufogiorgos, and Papantoniou [1995] proved the following result.

Theorem 1. *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric (κ, μ) manifold. Then $\kappa \leqslant 1$. Moreover, if $\kappa = 1$ then $h = 0$ and $(M, \varphi, \xi, \eta, g)$ is Sasakian. If $\kappa < 1$, the contact metric structure is not Sasakian and M admits three mutually orthogonal integrable distributions $\mathcal{D}(0)$, $\mathcal{D}(\lambda)$, and $\mathcal{D}(-\lambda)$ corresponding to the eigenspaces of h, where $\lambda = \sqrt{1-\kappa}$.*

The explicit expression of the Riemannian curvature tensor of a non-Sasakian contact metric (κ,μ)-manifold is known (see [Boeckx 1999, Theorem 5]).

Theorem 2. *Let M be a contact metric (κ,μ)-space. If $\kappa \neq 1$, then*

$$\begin{aligned}
g(R(X,Y)Z,W) = {} & \bigl(1-\tfrac{1}{2}\mu\bigr)(g(Y,Z)g(X,W)-g(X,Z)g(Y,W)) \\
& + g(Y,Z)g(hX,W)-g(X,Z)g(hY,W) \\
& - g(Y,W)g(hX,Z)+g(X,W)g(hY,Z) \\
& + \frac{1-\mu/2}{1-\kappa}(g(hY,Z)g(hX,W)-g(hX,Z)g(hY,W)) \\
& - \tfrac{1}{2}\mu(g(\varphi Y,Z)g(\varphi X,W)-g(\varphi X,Z)g(\varphi Y,W)) \\
& + \frac{\kappa-\mu/2}{1-\kappa}(g(\varphi hY,Z)g(\varphi hX,W)-g(\varphi hY,W)g(\varphi hX,Z)) \\
& + \mu g(\varphi X,Y)g(\varphi Z,W) \\
& + \eta(X)\eta(W)\bigl(\bigl(\kappa-1+\tfrac{1}{2}\mu\bigr)g(Y,Z)+(\mu-1)g(hY,Z)\bigr) \\
& - \eta(X)\eta(Z)\bigl(\bigl(\kappa-1+\tfrac{1}{2}\mu\bigr)g(Y,W)+(\mu-1)g(hY,W)\bigr) \\
& + \eta(Y)\eta(Z)\bigl(\bigl(\kappa-1+\tfrac{1}{2}\mu\bigr)g(X,W)+(\mu-1)g(hX,W)\bigr) \\
& - \eta(Y)\eta(W)\bigl(\bigl(\kappa-1+\tfrac{1}{2}\mu\bigr)g(X,Z)+(\mu-1)g(hX,Z)\bigr).
\end{aligned}$$

The class of non-Sasakian contact metric (κ,μ)-spaces coincides with the class of contact metric manifolds with nonvanishing η-parallel tensor h, according to [Blair, Koufogiorgos, and Papantoniou 1995, Lemma 3.8] and the following result of Boeckx and Cho [2005]:

Theorem 3. *Let (M,φ,ξ,η,g) be a contact metric manifold which is not K-contact. If $g((\nabla_X h)Y,Z)=0$ for all vector fields X, Y, Z orthogonal to ξ, then M is a contact metric (κ,μ)-space.*

Finally, we recall also the following characterization in the context of CR geometry (we refer to [Blair 2010, §6.4; Dragomir and Tomassini 2006] for a general reference on this topic):

Theorem 4 [Dileo and Lotta 2009, Theorem 3.2]. *Let (M,HM,J,η) be a pseudo-Hermitian manifold. Assume that the Webster metric g_η is not Sasakian. The following conditions are equivalent*:

(1) *The Webster metric g_η is locally* CR-*symmetric.*

(2) *The underlying contact metric structure satisfies the (κ,μ) condition.*

Non-Sasakian contact metric (κ,μ)-spaces have been completely classified by Boeckx [2000]. In this case $\kappa<1$ and the real number

$$I_M := \frac{1-\mu/2}{\sqrt{1-\kappa}}$$

is an invariant of the (κ, μ)-structure, which we call *Boeckx invariant*. Indeed we have:

Theorem 5 [Boeckx 2000]. *Let $(M_i, \varphi_i, \xi_i, \eta_i, g_i)$, $i = 1, 2$, be two non-Sasakian (κ_i, μ_i)-spaces of the same dimension. Then $I_{M_1} = I_{M_2}$ if and only if, up to a D-homothetic transformation, the two spaces are locally isometric as contact metric spaces. In particular, if both spaces are simply connected and complete, they are globally isometric up to a D-homothetic deformation.*

Next we recall the notions of straight and twisted complexifications of a Lie triple system (LTS). For more details we refer the reader to [Bertram 2000; 2001]. Given a Lie triple system $(\mathfrak{m}, [\ ,\ ,\ ])$ we shall write as usual

$$R(X, Y)Z := -[X, Y, Z].$$

We shall also write $(\mathfrak{m}, R)$ instead of $(\mathfrak{m}, [\ ,\ ,\ ])$. An *invariant complex structure* on $\mathfrak{m}$ is a complex structure $J : \mathfrak{m} \to \mathfrak{m}$ such that for every $X, Y, Z \in \mathfrak{m}$,

$$[X, Y, JZ] = J[X, Y, Z].$$

An *invariant paracomplex structure* I on $\mathfrak{m}$ is a paracomplex structure on $\mathfrak{m}$ (i.e., an endomorphism of $\mathfrak{m}$ such that $I^2 = \mathrm{id}_{\mathfrak{m}}$ and the ± 1 eigenspaces of I have the same dimension) satisfying

$$[X, Y, IZ] = I[X, Y, Z]$$

for every $X, Y, Z \in \mathfrak{m}$.

For every LTS $\mathfrak{m}$ endowed with an invariant (para-)complex structure, the corresponding simply connected symmetric space G/H is canonically endowed with a G-invariant almost (para-)complex structure and vice versa (see [Bertram 2000, Proposition III.1.4]).

An invariant (para-)complex structure J on a Lie triple system $(\mathfrak{m}, [\ ,\ ,\ ])$ is called *straight* if

$$[JX, Y, Z] = [X, JY, Z]$$

or *twisted* if

$$[JX, Y, Z] = -[X, JY, Z].$$

Accordingly, a *straight* or respectively *twisted* (para-)complex symmetric space is an affine symmetric space $M = G/H$ endowed with an invariant almost (para-)complex structure $\mathcal{J}$ such that

$$R(\mathcal{J}X, Y)Z = R(X, \mathcal{J}Y)Z$$

or respectively

$$R(\mathcal{J}X, Y)Z = -R(X, \mathcal{J}Y)Z,$$

where R is the curvature of M.

A *(para-)complexification* of an LTS $\mathfrak{m}$ is an LTS $(\mathfrak{q}, [\ ,\ ,\ ])$ together with an invariant (para-)complex structure J and an automorphism τ such that $\tau J + J\tau = 0$, $\tau^2 = \mathrm{id}_{\mathfrak{q}}$, and the LTS $\mathfrak{q}^\tau$ given by the space of τ-fixed points of $\mathfrak{q}$ is isomorphic to $\mathfrak{m}$. The (para-)complexification $(\mathfrak{q}, [\ ,\ ,\ ], J, \tau)$ of $\mathfrak{m}$ is called *straight* or *twisted* respectively if J is a straight or twisted.

We recall that every LTS $(\mathfrak{m}, R)$ has a unique straight complexification given by the $\mathbb{C}$-trilinear extension $R_{\mathbb{C}} : \mathfrak{m}_{\mathbb{C}} \times \mathfrak{m}_{\mathbb{C}} \times \mathfrak{m}_{\mathbb{C}} \to \mathfrak{m}_{\mathbb{C}}$ of R [Bertram 2001, Proposition 2.1.4]. The existence of a twisted complexification or paracomplexification of $\mathfrak{m}$ is instead related to the existence of a particular $(1, 3)$-tensor, the *Jordan extension* of R.

Let $M = G/H$ be a symmetric space endowed with an invariant almost (para-)complex structure $\mathcal{J}$. The *structure tensor* of $\mathcal{J}$ is the $(1, 3)$-tensor

$$T(X, Y)Z = -\tfrac{1}{2}(R(X, Y)Z - \mathcal{J}R(X, \mathcal{J}^{-1}Y)Z).$$

This tensor satisfies the following two properties:

(JT1) $T(X, Y)Z = T(Z, Y)X$,

(JT2) $T(U, V)T(X, Y, Z)$
$$= T(T(U, V)X, Y, Z) - T(X, T(U, V)Y, Z) + T(X, Y, T(U, V)Z).$$

Now, a *Jordan triple system* is a pair (V, T), where V is a vector space and $T : V \times V \times V \to V$ is a trilinear map satisfying (JT1), (JT2), called a *Jordan triple product* on V.

Observe that if T is a JT product on V, then

$$[x, y, z] := T(x, y)z - T(y, x)z$$

is a LT product on V.

Let T be a JT product on an LTS $(\mathfrak{m}, R)$. We set

$$R_T(x, y) := -T(x, y) + T(y, x).$$

T is said to be a *Jordan extension* of R if $R = R_T$.

Theorem 6 [Bertram 2000, Theorem III.4.4]. *Let* $(\mathfrak{m}, R)$ *be an LTS. The following objects are in one-to-one correspondence*:

(1) *twisted complexifications of* R,

(2) *twisted paracomplexifications of* R,

(3) *Jordan extensions of* R.

In the next section we shall be concerned with the following basic examples, studying their interplay with the classification of contact metric (κ, μ)-manifolds.

Consider the Lie triple systems $(\mathbb{R}^n, R)$ and $(\mathbb{R}^n, -R)$, associated respectively to the sphere S^n and the hyperbolic space $\mathbb{H}^n$, where R is

$$R(x, y)z := 2(\langle y, z\rangle x - \langle x, z\rangle y).$$

On $(\mathbb{R}^n, R)$ one can consider the following JT product:

$$T(x, y)z = \langle x, z\rangle y - \langle x, y\rangle z - \langle y, z\rangle x.$$

Then, according to Bertram [2000, Proposition IV.1.5], the corresponding twisted complexification and paracomplexification of S^n are the symmetric spaces

$$\mathrm{SO}(n+2)/(\mathrm{SO}(n) \times \mathrm{SO}(2))$$

and

$$\mathrm{SO}(n+1, 1)/(\mathrm{SO}(n) \times \mathrm{SO}(1, 1)).$$

In the case of $\mathbb{H}^n$, one can consider $-T$; the corresponding twisted complexification is (see [Bertram 2000, p. 91])

$$\mathrm{SO}(n, 2)/(\mathrm{SO}(n) \times \mathrm{SO}(2)).$$

3. A characterization of contact metric (κ, μ)-spaces

Let $(M, \varphi, \xi, \eta, g)$ be a connected homogeneous contact metric manifold. Consider a Lie group G acting transitively on M as a group of automorphisms of the contact metric structure, and denote by H the isotropy subgroup of G at $x_o \in M$. The natural map $j : G/H \to M$ given by $j(aH) = ax_o$ is a diffeomorphism. Thus G/H is a homogeneous Riemannian space and in particular it is a reductive homogeneous space (see, e.g., [Tricerri and Vanhecke 1983]). Fix a reductive decomposition of the Lie algebra $\mathfrak{g}$ of G:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \tag{2}$$

where $\mathfrak{h} = \mathrm{Lie}(H)$. The identity component G^o of G acts again transitively on M, and the isotropy subgroup of G^o at x_o is $H \cap G^o$. Let

$$\pi : G^o \to G^o/H \cap G^o \simeq M$$

be the natural fibration of G^o onto the homogeneous space $G^o/H \cap G^o$. Since $\mathrm{Lie}(H) = \mathrm{Lie}(H \cap G^o)$, (2) is also a reductive decomposition for $G^o/H \cap G^o$. Then $\mathfrak{m}$ decomposes into the direct sum of two $H \cap G^o$-invariant subspaces:

$$\mathfrak{m} = \mathbb{R}J \oplus \mathfrak{b},$$

where J is the vector of $\mathfrak{m}$ corresponding to $\xi_{\underline{o}}$ and $\mathfrak{b}$ corresponds to the determination of the contact distribution $D = \ker(\eta)$ at $\underline{o} := \pi(e) \cong x_o$, where e is the neutral element of G.

Now, homogeneity ensures that the contact form η is regular (see [Boothby and Wang 1958, § II]); hence we have a canonical fibration of M, given by (see also [Musso 1991, p. 225])

$$G^o/H\cap G^o \to G^o/S^o(H\cap G^o),$$

where S^o is the identity component of the closed Lie subgroup

$$S := \{h \in G^o \mid \mathrm{Ad}(h)^*\tilde{\eta} = \tilde{\eta}\}$$

of G^o. Here $\tilde{\eta}$ denotes the one form on G^o pull back of η via π. We have that $H\cap G^o \subset S$ [Boothby and Wang 1958, Lemma II.4].

Moreover, the Lie algebra $\bar{\mathfrak{h}}$ of $\overline{H} := S^o(H\cap G^o)$ is given by

$$\bar{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{R}J,$$

and we have the following decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \bar{\mathfrak{h}} \oplus \mathfrak{b}. \tag{3}$$

Our first aim is to characterize the non-Sasakian contact metric (κ, μ)-spaces as homogeneous contact metric manifolds for which decomposition (3) is *symmetric*, i.e.,

$$[\bar{\mathfrak{h}}, \bar{\mathfrak{h}}] \subset \bar{\mathfrak{h}}, \quad [\bar{\mathfrak{h}}, \mathfrak{b}] \subset \mathfrak{b}, \quad [\mathfrak{b}, \mathfrak{b}] \subset \bar{\mathfrak{h}}.$$

Using this, in Corollary 8, we shall be able to endow B of G^o-invariant affine connections making it an affine symmetric space.

Theorem 7. *Let $(M, \varphi, \xi, \eta, g)$ be a simply connected, complete, contact metric manifold. Assume M is not K-contact. Then the following conditions are equivalent*:

(a) *M is a contact metric (κ, μ)-space.*

(b) *M admits a transitive, effective Lie group of automorphisms G whose Lie algebra $\mathfrak{g}$ is a symmetric Lie algebra with symmetric decomposition* (3).

Proof. (a) $\Rightarrow$ (b): According to [Boeckx 1999], $(M, \varphi, \xi, \eta, g)$ is a homogeneous contact metric manifold. Let $G = \mathrm{Aut}(M)$ be the Lie group of all the automorphisms of the contact metric structure of M, and H be the isotropy subgroup of G at $x_o \in M$.

We fix a reductive decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \tag{4}$$

where $\mathfrak{g}$ and $\mathfrak{h}$ are respectively the Lie algebras of G and H. Keeping the notation above we consider also the decompositions

$$\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}J \oplus \mathfrak{b} = \bar{\mathfrak{h}} \oplus \mathfrak{b}.$$

By Theorem 4, for every $x \in M$ there exists a local CR-symmetry at x. Since M is simply connected and complete, the local CR-symmetries are actually globally defined. Let σ be the CR-symmetry at $\underline{o} = eH$. We recall that σ is an isometric CR diffeomorphism of M, whose differential at $\underline{o}$ is $-\mathrm{Id}$ on $D_{\underline{o}}$. In particular, it is an automorphism of the contact metric structure and an affine automorphism of the canonical G-invariant affine connection $\widetilde{\nabla}$ associated to (4). Hence, denoting by $\widetilde{T}$ the torsion of $\widetilde{\nabla}$, we have that, for every $X, Y, Z \in \mathfrak{b} \subset \mathfrak{m}$:

$$\begin{aligned} g_{\underline{o}}(\widetilde{T}(X,Y),Z) &= g_{\underline{o}}(\sigma_\star \widetilde{T}(X,Y),\sigma_\star Z) = g_{\underline{o}}(\widetilde{T}(\sigma_\star X,\sigma_\star Y),\sigma_\star Z) \\ &= -g_{\underline{o}}(\widetilde{T}(X,Y),Z), \end{aligned}$$

which yields that $[X,Y]_{\mathfrak{m}} = -\widetilde{T}_{\underline{o}}(X,Y) \in \mathbb{R}J$, and hence $[\mathfrak{b},\mathfrak{b}] \subset \bar{\mathfrak{h}}$.

The curvature tensor $\widetilde{R}$ of $\widetilde{\nabla}$ and the Reeb vector field ξ are also preserved by σ. Hence for every $X, Y, Z \in \mathfrak{b}$:

$$\begin{aligned} g_{\underline{o}}(\widetilde{R}(J,X)Y,Z) &= g_{\underline{o}}(\sigma_\star \widetilde{R}(J,X)Y,\sigma_\star Z) = g_{\underline{o}}(\widetilde{R}(\sigma_\star J,\sigma_\star X)\sigma_\star Y,\sigma_\star Z) \\ &= -g_{\underline{o}}(\widetilde{R}(J,X)Y,Z), \end{aligned}$$

moreover, since $\widetilde{\nabla}\mathcal{D} \subset \mathcal{D}$ we have that $\widetilde{R}(J,X)Y \in \mathcal{D}_{\underline{o}}$; thus

$$[[J,X]_{\mathfrak{h}},Y] = 0$$

for every $X, Y \in \mathfrak{b}$. Since G is effective on M, the adjoint representation $\mathrm{ad} : \mathfrak{h} \to \mathrm{End}(\mathfrak{m})$ is injective; therefore, using also $[\mathfrak{h},J] = 0$, we conclude that $[J,X]_{\mathfrak{h}} = 0$.

Finally we prove that $[J,X] \in \mathfrak{b}$; indeed we have

$$\begin{aligned} g_{\underline{o}}(\widetilde{T}(J,X),J) &= g_{\underline{o}}(\sigma_\star \widetilde{T}(J,X),\sigma_\star J) = g_{\underline{o}}(\widetilde{T}(\sigma_\star J,\sigma_\star X),\sigma_\star J) \\ &= -g_{\underline{o}}(\widetilde{T}(J,X),J). \end{aligned}$$

This completes the proof of (b).

(b) $\Rightarrow$ (a): Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a reductive decomposition for the homogeneous contact metric space $M = G/H$, where H is the isotropy subgroup of G at a point $x_o \in M$.

Let ∇ and $\widetilde{\nabla}$ respectively the Levi-Civita connection of g and the canonical affine connection on M associated to the fixed reductive decomposition. If we set $A = \nabla - \widetilde{\nabla}$, then

$$(\nabla_X h)Y = (\widetilde{\nabla}_X h)Y + A(X,hY) - hA(X,Y).$$

Now, since the tensor $h = \frac{1}{2}\mathcal{L}_\xi \varphi$ is invariant under automorphisms of the contact metric structure, it is parallel with respect to the canonical connection $\widetilde{\nabla}$ [Kobayashi and Nomizu 1969, p. 193] and hence

$$(\nabla_X h)Y = A(X,hY) - hA(X,Y). \tag{5}$$

Since $\widetilde{\nabla}$ is a metric connection, for $X, Y, Z \in \mathfrak{X}(M)$ we have that

$$g(A(X, Y), Z) + g(Y, A(X, Z)) = 0. \tag{6}$$

Then for every $X, Y, Z \in \mathfrak{X}(M)$,

$$2g(A(X, Y), Z) = -g(\widetilde{T}(X, Y), Z) + g(\widetilde{T}(Y, Z), X) - g(\widetilde{T}(Z, X), Y). \tag{7}$$

Now observe that for every $X, Y \in \mathfrak{b}$,

$$\widetilde{T}_{\underline{o}}(X, Y) = -[X, Y]_{\mathfrak{m}},$$

and

$$[X, Y] \in \mathfrak{h} \oplus \mathbb{R}J,$$

since $\mathfrak{g} = \bar{\mathfrak{h}} \oplus \mathfrak{b}$ a symmetric decomposition by assumption. Thus $\widetilde{T}_{\underline{o}}(X, Y) \in \mathbb{R}J$. Hence for every $X, Y, Z \in \mathcal{D}$,

$$g(\widetilde{T}(X, Y), Z) = 0,$$

and then, by (7),

$$g(A(X, Y), Z) = 0.$$

Thus, using (5), we obtain that

$$g((\nabla_X h)Y, Z) = 0$$

for every $X, Y, Z \in \mathcal{D}$. This implies that M is a contact metric (κ, μ)-space according to Theorem 3. □

Corollary 8. *Let $M = G/H$ be a simply connected, complete, non-Sasakian contact metric (κ, μ)-manifold. Then the base space $B = G^o/\overline{H}$ of the canonical fibration of M is an affine symmetric space.*

Proof. It suffices to prove that $B = G^o/\overline{H}$ is a homogeneous reductive space with respect to decomposition (3); indeed, the associated canonical G^o-invariant connection makes B a locally symmetric affine manifold. Observe that B is simply connected since the fibers of the canonical fibration are connected (see [Boothby and Wang 1958, Theorem II.4]). Since the canonical invariant connection is always complete (see [Kobayashi and Nomizu 1969, Chapter X, Corollary 2.5]), B is actually a symmetric space.

To prove our claim, we recall that $H \cap G^o \subset S$; thus $S^o \subset S^o(H \cap G^o) \subset S$ and $\mathrm{Lie}(S^o) = \bar{\mathfrak{h}}$. Since $[\bar{\mathfrak{h}}, \mathfrak{b}] \subset \mathfrak{b}$ and S^o is connected, it follows that $\mathrm{Ad}(S^o)\mathfrak{b} \subset \mathfrak{b}$ and hence, since also $\mathrm{Ad}(H \cap G^o)(\mathfrak{b}) \subset \mathfrak{b}$, we conclude that $\mathrm{Ad}(\overline{H})\mathfrak{b} \subset \mathfrak{b}$, as claimed. □

We remark that the affine symmetric structure on B thus obtained a priori depends on the initial choice of a reductive decomposition (2) of $\mathfrak{g}$. In the next section, we shall see that actually different choices lead to the same affine symmetric space, up to isomorphism (see Corollary 10).

4. The base space of the canonical fibration

The aim of this section is to give a complete classification of the symmetric base spaces B of the canonical fibrations of simply connected, complete, non-Sasakian contact metric (κ, μ)-manifolds with Boeckx invariant $I_M \neq \pm 1$. We obtain that B is a twisted complexification or paracomplexification of the sphere S^n, or of the hyperbolic space $\mathbb{H}^n$ according to this table:

Boeckx invariant	base space	type
$I_M > 1$	$\mathrm{SO}(n+2)/(\mathrm{SO}(n)\times\mathrm{SO}(2))$	complexification of S^n
$-1 < I_M < 1$	$\mathrm{SO}(n+1,1)/(\mathrm{SO}(n)\times\mathrm{SO}(1,1))$	paracomplexification of S^n
$I_M < -1$	$\mathrm{SO}(n,2)/(\mathrm{SO}(n)\times\mathrm{SO}(2))$	complexification of $\mathbb{H}^n$

Keeping the notations above, we identify the tangent space of B at the base point with the linear subspace $\mathfrak{b} \cong \mathcal{D}_o$. Moreover we denote by $\mathfrak{b}_+$ and $\mathfrak{b}_-$ the subspaces of $\mathfrak{b}$ corresponding respectively to the eigenspaces $\mathcal{D}_o(\lambda)$ and $\mathcal{D}_o(-\lambda)$ of $h_o : \mathfrak{b} \to \mathfrak{b}$.

We start by computing the curvature of B.

Proposition 9. *Let $(M, \varphi, \xi, \eta, g)$ be a simply connected, complete, non-Sasakian contact metric (κ, μ)-manifold and B the base space of the canonical fibration of M. If $\overline{\nabla}$ is the canonical affine connection on B associated to any reductive decomposition of type* (3), *then the curvature tensor $\bar{R}$ of $\overline{\nabla}$ at the base point $o \in B$ is given by*

$$
\begin{aligned}
\bar{R}_o(X,Y)Z = {} & \big((1-\tfrac{1}{2}\mu)g(Y,Z)+g(hY,Z)\big)X \\
& - \big((1-\tfrac{1}{2}\mu)g(X,Z)+g(hX,Z)\big)Y \\
& + \Big(\frac{1-\mu/2}{1-\kappa}g(hY,Z)+g(Y,Z)\Big)hX \\
& - \Big(\frac{1-\mu/2}{1-\kappa}g(hX,Z)+g(X,Z)\Big)hY \\
& + \big((1-\tfrac{1}{2}\mu)g(\varphi Y,Z)+g(\varphi hY,Z)\big)\varphi X \\
& - \big((1-\tfrac{1}{2}\mu)g(\varphi X,Z)+g(\varphi hX,Z)\big)\varphi Y \\
& + \Big(\frac{1-\mu/2}{1-\kappa}g(\varphi hY,Z)+g(\varphi Y,Z)\Big)\varphi hX \\
& - \Big(\frac{1-\mu/2}{1-\kappa}g(\varphi hX,Z)+g(\varphi X,Z)\Big)\varphi hY \\
& + (\mu-2)g(\varphi X,Y)\varphi Z - 2g(\varphi X,Y)\varphi hZ.
\end{aligned}
\tag{8}
$$

Proof. For every $X, Y, Z \in \mathfrak{b}$ we have

$$\bar{R}_o(X,Y)Z = -[[X,Y]_J + [X,Y]_{\mathfrak{h}}, Z]$$

(see [Kobayashi and Nomizu 1969, Chapter X]), and hence

$$\bar{R}_o(X, Y)Z = \widetilde{R}(X, Y)Z - [[X, Y]_J, Z], \tag{9}$$

where $[X, Y]_J$ and $[X, Y]_{\mathfrak{h}}$ are the components of $[X, Y] \in \mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}J \oplus \mathfrak{b}$ respectively in $\mathbb{R}J$ and $\mathfrak{h}$; $\widetilde{R}$ is the curvature tensor of the canonical connection of the homogeneous reductive space M with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

Let ∇ be the Levi-Civita connection of g and R the curvature tensor of ∇. If we set $A := \widetilde{\nabla} - \nabla$, then a standard computation yields:

$$\begin{aligned}\widetilde{R}(X, Y)Z = R(X, Y)Z - A(X, A(Y, Z)) + A(Y, A(X, Z))&\\ + A(\widetilde{T}(X, Y), Z) + (\widetilde{\nabla}_X A)(Y, Z) - (\widetilde{\nabla}_Y A)(X, Z)&,\end{aligned}$$

for every $X, Y, Z \in \mathfrak{X}(M)$. Moreover, since A is a G-invariant tensor, we have that A is parallel with respect to the canonical connection $\widetilde{\nabla}$ and hence

$$\widetilde{R}(X, Y)Z = R(X, Y)Z - A(X, A(Y, Z)) + A(Y, A(X, Z)) + A(\widetilde{T}(X, Y), Z),$$

and (9) becomes

$$\begin{aligned}\bar{R}_o(X, Y)Z = R(X, Y)Z - A(X, A(Y, Z)) + A(Y, A(X, Z))&\\ + A(\widetilde{T}(X, Y), Z) - [[X, Y]_J, Z]&.\end{aligned}$$

We already observed in the proof of Theorem 7 that for every $X, Y, Z \in \mathcal{D}$,

$$g(A(X, Y), Z) = 0, \quad g(\widetilde{T}(X, Y), Z) = 0;$$

hence

$$A(X, Y) = g(A(X, Y), \xi)\xi, \tag{10}$$

$$\widetilde{T}(X, Y) = g(\widetilde{T}(X, Y), \xi)\xi = -g([X, Y], \xi)\xi = 2g(X, \varphi Y)\xi. \tag{11}$$

In (11) we are using the parallelism of the distributions $\mathcal{D}(\pm\lambda)$ with respect to $\widetilde{\nabla}$, which is a consequence of the fact that $\widetilde{\nabla}h = 0$.

Moreover, we have

$$A(X, \xi) = \widetilde{\nabla}_X \xi - \nabla_X \xi = \varphi X + \varphi h X. \tag{12}$$

Then, using (10), (11), (12), specializing at the point o we obtain

$$\begin{aligned}\bar{R}_o(X, Y)Z &= R(X, Y)Z - g(A(Y, Z), J)A(X, J) + g(A(X, Z), J)A(Y, J)\\ &\qquad + 2g(X, \varphi Y)A(J, Z) + [\widetilde{T}(X, Y), Z]\\ &= R(X, Y)Z - g(A(Y, Z), J)(\varphi X + \varphi h X)\\ &\qquad + g(A(X, Z), J)(\varphi Y + \varphi h Y) + 2g(X, \varphi Y)A(J, Z)\\ &\qquad + 2g(X, \varphi Y)[J, Z],\end{aligned} \tag{13}$$

where $X, Y, Z \in \mathfrak{b}$. The $(1, 1)$-tensor $A(X, \cdot)$ is a skew symmetric tensor, since $\widetilde{\nabla} g = 0$. In particular,

$$g(A(X, Y), \xi) = -g(Y, A(X, \xi)),$$

so that, by (12)

$$g(A(X, Y), \xi) = -g(Y, \varphi X + \varphi h X).$$

Thus, (13) becomes

$$\begin{aligned} \bar{R}_o(X, Y)Z = R(X, Y)Z &+ g(Z, \varphi Y + \varphi h Y)(\varphi X + \varphi h X) \\ &- g(\varphi X + \varphi h X, Z)(\varphi Y + \varphi h Y) + 2g(X, \varphi Y)A(J, Z) \\ &+ 2g(X, \varphi Y)[J, Z]. \end{aligned}$$

Now, using Theorem 7,

$$\widetilde{T}_o(J, Z) = -[J, Z]_{\mathfrak{m}} = -[J, Z];$$

on the other hand,

$$\begin{aligned} \widetilde{T}(\xi, W) &= \widetilde{\nabla}_\xi W - \widetilde{\nabla}_W \xi - [\xi, W] = \nabla_\xi W + A(\xi, W) - [\xi, W] \\ &= -\varphi W - \varphi h W + A(\xi, W), \end{aligned}$$

for every W vector field on M. Thus,

$$\begin{aligned} \bar{R}_o(X, Y)Z &= R(X, Y)Z + g(Z, \varphi Y + \varphi h Y)(\varphi X + \varphi h X) \\ &\qquad - g(\varphi X + \varphi h X, Z)(\varphi Y + \varphi h Y) + 2g(X, \varphi Y)A(J, Z) \\ &\qquad - 2g(X, \varphi Y)(-\varphi Z - \varphi h Z + A(J, Z)) \\ &= R(X, Y)Z + g(Z, \varphi Y + \varphi h Y)(\varphi X + \varphi h X) \\ &\qquad - g(\varphi X + \varphi h X, Z)(\varphi Y + \varphi h Y) + 2g(X, \varphi Y)(\varphi Z + \varphi h Z). \end{aligned}$$

Finally, taking into account the explicit expression of the curvature tensor R of M (see Theorem 2), we obtain (8). □

Corollary 10. *The affine base spaces* $(B, \overline{\nabla})$ *of a simply connected, complete, non-Sasakian, contact metric* (κ, μ)*-manifold are all mutually equivalent affine symmetric spaces.*

For a non-Sasakian contact metric (κ, μ)-space the restriction of the $(1, 1)$ tensor φ to the horizontal distribution does not induce a complex structure on the base space, as occurs in the homogeneous Sasakian case, because $h \neq 0$. However, we shall see in the following that B admits a *standard* complex or paracomplex structure, according to the following definition and Theorem 13.

Definition 11. Let (M,φ,ξ,η,g) be a contact metric (κ,μ)-manifold and $(B,\bar{\bar{\nabla}})$ the base space of the canonical fibration of M.

A G^o-invariant almost complex structure $\mathcal{J}$ on B will be called *standard complex structure* provided its determination at the base point o is of the form

$$\mathcal{J}_o=\begin{cases} a\varphi & \text{on } \mathfrak{b}_+,\\ \frac{1}{a}\varphi & \text{on } \mathfrak{b}_-,\end{cases}\tag{14}$$

where a is a positive constant.

A *standard paracomplex structure* on B is a G^o-invariant almost paracomplex structure on B whose determination at the base point o is of the form

$$\mathcal{I}_o=\begin{cases} a\varphi & \text{on } \mathfrak{b}_+,\\ -\frac{1}{a}\varphi & \text{on } \mathfrak{b}_-,\end{cases}\tag{15}$$

where a is a positive constant.

Remark 12. A (para-)complex structure J on the vector space $\mathfrak{b}$ defined as in (14) or (15) does not induce in general a G^o-invariant almost complex or paracomplex structure on B.

Theorem 13. *Let (M,φ,ξ,η,g) be a simply connected, complete, contact metric (κ,μ)-manifold and let $(B,\bar{\bar{\nabla}})$ be the symmetric base space of the canonical fibration of M. Then*:

(1) $|I_M|>1$ *if and only if B admits a standard complex structure.*

(2) $|I_M|<1$ *if and only if B admits a standard paracomplex structure.*

Moreover, in each case such a standard complex or paracomplex structure is uniquely determined; precisely, it corresponds to the following value of the constant a in (14), (15):

$$a=\sqrt{\frac{I_M+1}{I_M-1}}$$

when $|I_M|>1$, *and*

$$a=\sqrt{-\frac{I_M+1}{I_M-1}}$$

when $|I_M|<1$.

Proof. Let $(\mathfrak{b},[\ ,\ ,\ ])$ be the Lie triple system associated to the symmetric space $(B,\bar{\bar{\nabla}})$. The Lie triple product $[\ ,\ ,\ ]$ is given by the curvature $\bar{R}$ of $\bar{\bar{\nabla}}$ at the base point o:

$$[X,Y,Z]=-\bar{R}_o(X,Y)Z.$$

Let $J : \mathfrak{b} \to \mathfrak{b}$ be a complex structure on $\mathfrak{b}$ of the form

$$J = \begin{cases} a\varphi & \text{on } \mathfrak{b}_+, \\ \frac{1}{a}\varphi & \text{on } \mathfrak{b}_-, \end{cases} \tag{16}$$

where a is a real parameter, $a > 0$.

For every $X_+, Y_+, Z_+ \in \mathfrak{b}_+$ and $X_-, Y_-, Z_- \in \mathfrak{b}_-$, using (8) and (16), by a direct computation, one can check that

$$\begin{gathered} \bar{R}(X_+, Y_+)JZ_+ = J\bar{R}(X_+, Y_+)Z_+, \quad \bar{R}(X_+, Y_+)JZ_- = J\bar{R}(X_+, Y_+)Z_-, \\ \bar{R}(X_-, Y_-)JZ_+ = J\bar{R}(X_-, Y_-)Z_+, \quad \bar{R}(X_-, Y_-)JZ_- = J\bar{R}(X_-, Y_-)Z_-, \\ \bar{R}(X_+, Y_-)JZ_- = \tfrac{1}{a}(2\lambda - \mu + 2)g(\varphi X_+, Y_-)Z_-, \\ J\bar{R}(X_+, Y_-)Z_- = -a(\mu - 2 + 2\lambda)g(\varphi X_+, Y_-)Z_-. \end{gathered}$$

Hence, the condition

$$\bar{R}(X_+, Y_-)JZ_- = J\bar{R}(X_+, Y_-)Z_-$$

is satisfied for every $X_+ \in \mathfrak{b}_+$, $Y_-, Z_- \in \mathfrak{b}_-$ if and only if there exists $a > 0$ such that $2\lambda - \mu + 2 = -a^2(\mu - 2 + 2\lambda)$.

If $\mu - 2 + 2\lambda = 0$ then also $2\lambda - \mu + 2 = 0$. It follows that $\kappa = 1$, but by assumption M is non-Sasakian, then it must be $\mu - 2 + 2\lambda \neq 0$ and

$$-\frac{2\lambda - \mu + 2}{2\lambda + \mu - 2} > 0.$$

This condition is equivalent to requiring that $|I_M| > 1$.

Finally,

$$\begin{gathered} \bar{R}(X_+, Y_-)JZ_+ = -a(2\lambda + \mu - 2)g(\varphi X_+, Y_-)Z_+, \\ J\bar{R}(X_+, Y_-)Z_+ = \tfrac{1}{a}(2\lambda - \mu + 2)g(\varphi X_+, Y_-)Z_+. \end{gathered}$$

Thus,

$$\bar{R}(X_+, Y_-)JZ_+ = J\bar{R}(X_+, Y_-)Z_+$$

for every $X_+, Z_+ \in \mathfrak{b}_+$, $Y_- \in \mathfrak{b}_-$ if and only if there exist $a > 0$ such that $2\lambda - \mu + 2 = -a^2(2\lambda + \mu - 2)$.

We conclude that the complex structure J is invariant if and only if $|I_M| > 1$. Moreover, in this case

$$a = \sqrt{\frac{2 - \mu + 2\lambda}{2 - \mu - 2\lambda}}.$$

With analogous considerations, we obtain that the paracomplex structure defined on $\mathfrak{b}$ by

$$I = \begin{cases} a\varphi & \text{on } \mathfrak{b}_+, \\ -\frac{1}{a}\varphi & \text{on } \mathfrak{b}_-, \end{cases} \tag{17}$$

where $a > 0$, is an invariant paracomplex structure if and only if $-1 < I_M < 1$. In this case,

$$a = \sqrt{-\frac{2-\mu+2\lambda}{2-\mu-2\lambda}}. \qquad \square$$

Remark 14. Cappelletti-Montano, Carriazo, and Martín-Molina [2013] showed that every non-Sasakian contact metric (κ,μ)-manifold (M,φ,ξ,η,g) with $|I_M| > 1$ admits a Sasakian structure $(\tilde{\varphi},\xi,\eta,\tilde{g})$ obtained by deforming the $(1,1)$-tensor φ and the Riemannian metric g as

$$\tilde{\varphi} = \epsilon \frac{1}{(1-\kappa)\sqrt{(2-\mu)^2-4(1-\kappa)}} \mathcal{L}_\xi h \circ h, \quad \tilde{g} = -\mathrm{d}\eta(\,\cdot\,,\tilde{\varphi}\,\cdot\,) + \eta\otimes\eta,$$

where

$$\epsilon = \begin{cases} 1 & \text{if } I_M > 1, \\ -1 & \text{if } I_M < -1. \end{cases}$$

Moreover, for every point of M there exists a local CR-symmetry [Dileo and Lotta 2009, Theorem 3.2]. Observe that the CR-symmetries preserve the tensor field h, and hence they preserve also $\tilde{\varphi}$ and $\tilde{g}$. Thus, $(M,\tilde{\varphi},\xi,\eta,\tilde{g})$ is a Sasakian φ-symmetric space [Dileo and Lotta 2009, Proposition 3.3] and fibers over a Kähler manifold $(B,\bar{\mathcal{J}},\bar{g})$ that is a Hermitian symmetric space [Takahashi 1977]. One can check that $\bar{\mathcal{J}}$ coincides with the standard complex structure $\mathcal{J}$ on B in our sense.

Proposition 15. *The standard (para-)complex structure on the base space* $(B,\bar{\bar{\nabla}})$ *of a simply connected, complete, non-Sasakian, contact metric* (κ,μ)*-manifold* M *with* $|I_M| > 1$ ($|I_M| < 1$) *is actually a twisted (para-)complex* G^o*-invariant structure.*

Proof. This can be easily verified directly using (8). $\square$

Theorem 16. *Let* M^{2n+1} *be a simply connected, complete, non-Sasakian, contact metric* (κ,μ)*-manifold. Then:*

(a) $I_M > 1$ *if and only if its twisted complex symmetric base space* $(B,\bar{\bar{\nabla}},\mathcal{J})$ *is the complexification* $\mathrm{SO}(n+2)/(\mathrm{SO}(n)\times\mathrm{SO}(2))$ *of* S^n.

(b) $-1 < I_M < 1$ *if and only if its twisted paracomplex symmetric base space* $(B,\bar{\bar{\nabla}},\mathcal{I})$ *is the paracomplexification* $\mathrm{SO}(n+1,1)/(\mathrm{SO}(n)\times\mathrm{SO}(1,1))$ *of* S^n.

(c) $I_M < -1$ *if and only if its twisted complex symmetric base space* $(B,\bar{\bar{\nabla}},\mathcal{J})$ *is the complexification* $\mathrm{SO}(n,2)/(\mathrm{SO}(n)\times\mathrm{SO}(2))$ *of* $\mathbb{H}^n$.

Proof. Consider the Lie triple system $(\mathfrak{b}, [\ ,\ ,\ ])$ associated to the canonical symmetric base space $(B, \overline{\nabla})$. The Lie triple commutator $[\ ,\ ,\ ] : \mathfrak{b} \times \mathfrak{b} \times \mathfrak{b} \to \mathfrak{b}$ is given by

$$[X, Y, Z] = -\bar{R}_o(X, Y)Z,$$

where $\bar{R}$ is the curvature of $\overline{\nabla}$. By direct computation, using Proposition 9 we see that the linear mapping

$$\tau : X \in \mathfrak{b} \mapsto \tfrac{1}{\lambda} hX \in \mathfrak{b}$$

is an involutive automorphism of the LTS $(\mathfrak{b}, [\ ,\ ,\ ])$. Thus the space $\mathfrak{b}^\tau$ of the τ-fixed elements of $\mathfrak{b}$, together with the induced Lie triple bracket, is a Lie triple system. Actually, since

$$\mathfrak{b}^\tau = \mathfrak{b}_+,$$

and because the restriction $\bar{R}_+$ of $\bar{R}$ to $\mathfrak{b}_+$ is given by

$$\bar{R}_+(X_+, Y_+)Z_+ = (2 - \mu + 2\lambda)(g(Y_+, Z_+)X_+ - g(X_+, Z_+)Y_+),$$

we have that the LTS $(\mathfrak{b}_+, \bar{R}_+)$ is isomorphic to the LTS belonging to the sphere S^n or the hyperbolic space $\mathbb{H}^n$, according to the circumstance that the Boeckx invariant I_M is greater than -1 or less than -1 respectively; indeed we have $2 - \mu + 2\lambda = 2\lambda(I_M + 1)$.

Suppose $|I_M| > 1$. Let J be the twisted complex structure on $\mathfrak{b}$ corresponding to the standard complex structure $\mathcal{J}$ of B. Observe that $J\tau + \tau J = 0$, since $\varphi h + h\varphi = 0$. Then $(\mathfrak{b}, [\ ,\ ,\ ], J, \tau)$ is a twisted complexification of $(\mathfrak{b}_+, \bar{R}_+)$.

We recall that, by definition, the structure tensor T of $\mathcal{J}$ at the base point o is

$$T_o(X, Y)Z = -\tfrac{1}{2}(\bar{R}_o(X, Y)Z + J\bar{R}_o(X, JY)Z),$$

and that its restriction T_+ to $\mathfrak{b}_+$ yields the Jordan extension $(\mathfrak{b}_+, T_+)$ of the LTS $(\mathfrak{b}_+, \bar{R}_+)$, uniquely associated to its twisted complexification $(\mathfrak{b}, [\ ,\ ,\ ], J, \tau)$ (see Theorem 6).

Computing T_+ we obtain

$$\begin{aligned} T_+(X_+, Y_+)Z_+ &= -\tfrac{1}{2}\big(\bar{R}(X_+, Y_+)Z_+ + J\bar{R}(X_+, JY_+)Z_+\big) \\ &= \tfrac{1}{2}(\mu - 2 - 2\lambda)\big(g(Y_+, Z_+)X_+ - g(X_+, Z_+)Y_+ + g(X_+, Y_+)Z_+\big). \end{aligned}$$

Hence, taking into account the complexification diagrams of the sphere and of the hyperbolic space [Bertram 2000, Chapter IV], we obtain assertions (a) and (c).

Now suppose $|I_M| < 1$ and denote by I the twisted paracomplex structure on $\mathfrak{b}$ corresponding to the standard paracomplex structure $\mathcal{I}$ of B at the base point. We have that $I\tau + \tau I = 0$, since $\varphi h + h\varphi = 0$, and hence $(\mathfrak{b}, [\ ,\ ,\ ], I, \tau)$ is a twisted paracomplexification of $(\mathfrak{b}^\tau, \bar{R}_+)$. The structure tensor of $\mathcal{I}$ at the base point o is

$$T_o(X, Y)Z = -\tfrac{1}{2}(\bar{R}_o(X, Y)Z - I\bar{R}_o(X, IY)Z).$$

Then the Jordan extension of $\bar{R}_+$ uniquely associated to the twisted paracomplexification $(\mathfrak{b}, [\ ,\ ], I, \tau)$ of the LTS $(\mathfrak{b}_+, -\bar{R}_+)$ is

$$\begin{aligned} T(X_+, Y_+)Z_+ &= -\tfrac{1}{2}\big(\bar{R}(X_+, Y_+)Z_+ - I\bar{R}(X_+, IY_+)Z_+\big) \\ &= -\tfrac{1}{2}(2-\mu+2\lambda)\big(g(Y_+, Z_+)X_+ - g(X_+, Z_+)Y_+ + g(X_+, Y_+)Z_+\big). \end{aligned}$$

Then, comparing again with the complexification diagram of the sphere we obtain assertion (b). □

5. Homogeneous model spaces of contact metric (κ, μ)-spaces

In this section we complete our classification, showing that one can actually construct a contact metric (κ,μ)-space with prescribed Boeckx invariant starting from each of the symmetric spaces in the table on page 50. More precisely, we prove

Theorem 17. *The simply connected, complete, contact metric (κ,μ)-spaces of dimension $2n+1$ (where $n>1$) with Boeckx invariant different from ± 1 can be classified as follows*:

(a) *The homogeneous space* $\mathrm{SO}(n,2)/\mathrm{SO}(n)$ *carries a one-parameter family of invariant contact metric (κ,μ)-structures whose Boeckx invariant assumes all the values in* $]-\infty,-1[$.

(b) *The homogeneous space* $\mathrm{SO}(n+2)/\mathrm{SO}(n)$ *carries a one-parameter family of invariant contact metric (κ,μ)-structures whose Boeckx invariant assumes all the values in* $]1,+\infty[$.

(c) *The homogeneous space* $\mathrm{SO}(n+1,1)/\mathrm{SO}(n)$ *carries a one-parameter family of invariant contact metric (κ,μ)-structures whose Boeckx invariant assumes all the values in* $]-1,1[$.

Proof. Starting from a fixed Hermitian or para-Hermitian symmetric structure on each of the symmetric spaces,

$$\begin{aligned} B_1 &= \mathrm{SO}(n+2)/(\mathrm{SO}(n)\times\mathrm{SO}(2)), \\ B_2 &= \mathrm{SO}(n,2)/(\mathrm{SO}(n)\times\mathrm{SO}(2)), \\ B_3 &= \mathrm{SO}(n+1,1)/(\mathrm{SO}(n)\times\mathrm{SO}(1,1)), \end{aligned}$$

we shall construct explicitly a one-parameter family of invariant contact metric (κ,μ)-structures on the homogeneous spaces

$$\begin{aligned} M_1 &= \mathrm{SO}(n+2)/\mathrm{SO}(n), \\ M_2 &= \mathrm{SO}(n,2)/\mathrm{SO}(n), \\ M_3 &= \mathrm{SO}(n+1,1)/\mathrm{SO}(n), \end{aligned}$$

with $I_{M_1}>1$, $I_{M_2}<-1$, and $-1<I_{M_3}<1$.

We first consider the symmetric Lie algebras $\mathfrak{g}_1 := \mathfrak{so}(n+2)$ and $\mathfrak{g}_2 := \mathfrak{so}(n,2)$ with symmetric decompositions

$$\mathfrak{g}_i = \mathfrak{h}_i \oplus \mathfrak{b}_i,$$

where

$$\mathfrak{h}_1 = \mathfrak{h}_2 := \left\{ \left[\begin{array}{cc|c} 0 & -\lambda & \multirow{2}{*}{$\mathbf{0}$} \\ \lambda & 0 & \\ \hline \mathbf{0} & \mathbf{0} & a \end{array}\right] : \lambda \in \mathbb{R},\ a \in \mathfrak{so}(n) \right\} = \mathfrak{so}(2) \oplus \mathfrak{so}(n),$$

$$\mathfrak{b}_1 := \left\{ \left[\begin{array}{c|c} \multirow{2}{*}{$\mathbf{0}$} & -v^T \\ & -w^T \\ \hline v\ \ w & \mathbf{0} \end{array}\right] : v, w \in \mathbb{R}^n \right\} \simeq T_o B_1,$$

$$\mathfrak{b}_2 := \left\{ \left[\begin{array}{c|c} \multirow{2}{*}{$\mathbf{0}$} & v^T \\ & w^T \\ \hline v\ \ w & \mathbf{0} \end{array}\right] : v, w \in \mathbb{R}^n \right\} \simeq T_o B_2.$$

The $\mathrm{Ad}(\mathrm{SO}(2) \times \mathrm{SO}(n))$-invariant almost complex structure $J_i : \mathfrak{b}_i \to \mathfrak{b}_i$ defined by

$$J_i(v\ w) = (-1)^i (w\ \ -v),$$

and the $\mathrm{Ad}(\mathrm{SO}(2) \times \mathrm{SO}(n))$-invariant metric G_i on $\mathfrak{b}_i$

$$G_i((v\ w), (u\ z)) = \langle v, u \rangle + \langle w, z \rangle,$$

determine an invariant Hermitian symmetric structure $(\mathcal{J}_i, \bar{g}_i)$ on B_i; here $\langle\ \rangle$ denotes the standard inner product on $\mathbb{R}^n$ and $(v\ w)$ denotes the matrix

$$\left[\begin{array}{cc|c} 0 & 0 & -w^T \\ 0 & 0 & -v^T \\ \hline v & w & \mathbf{0} \end{array}\right]$$

in the case $i = 1$, and the matrix

$$\left[\begin{array}{cc|c} 0 & 0 & w^T \\ 0 & 0 & v^T \\ \hline v & w & \mathbf{0} \end{array}\right]$$

in the case $i = 2$. Observe that the decomposition of $\mathfrak{g}_i$,

$$\mathfrak{g}_i = \mathfrak{so}(n) \oplus \mathfrak{m}_i, \tag{18}$$

$$\mathfrak{m}_i := \mathbb{R}\xi \oplus \mathfrak{b}_i, \quad \xi := \left[\begin{array}{cc|c} 0 & -1 & \multirow{2}{*}{$\mathbf{0}$} \\ 1 & 0 & \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right],$$

is a reductive decomposition for M_i. Indeed, for every

$$a = \left[\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & a \end{array}\right] \in \mathrm{SO}(n), \qquad X = s\xi + (v\, w) \in \mathfrak{m}_i,$$

we have that $\mathrm{Ad}(a)X = s\xi + (av\, aw)$. In particular, we have $\mathrm{Ad}(a)\xi = \xi$ for every $a \in \mathrm{SO}(n)$.

We have a natural decomposition of $\mathfrak{b}_i$,

$$\mathfrak{b}_i = \mathfrak{p}_i \oplus \mathfrak{q}_i,$$

where

$$\mathfrak{p}_i := \{(v\, 0) \mid v \in \mathbb{R}^n\}, \qquad \mathfrak{q}_i := \{(0\, w) \mid w \in \mathbb{R}^n\}.$$

By using this decomposition, we define on $\mathfrak{m}_i$ a (1, 1) tensor φ_i, an inner product g_i, and a 1-form η_i as follows:

$$\varphi_i(Z) := \begin{cases} \alpha JZ & \text{if } Z \in \mathfrak{p}_i, \\ \frac{1}{\alpha} JZ & \text{if } Z \in \mathfrak{q}_i, \\ 0 & \text{if } Z \in \mathbb{R}\xi, \end{cases} \tag{19}$$

$$g_i(X, Y) := st + \tfrac{1}{2}\big(\alpha\langle v, u\rangle + \tfrac{1}{\alpha}\langle w, z\rangle\big), \qquad \eta_i(X) := s,$$

where $\alpha > 0$, and $X = s\xi + (v\, w)$, $Y = t\xi + (u\, z)$ are arbitrary elements of $\mathfrak{m}_i$. These tensors are $\mathrm{Ad}(\mathrm{SO}(n))$-invariant. Indeed for every $a \in \mathrm{SO}(n)$,

$$\begin{aligned} \mathrm{Ad}(a)\varphi_i X &= \mathrm{Ad}(a)\big((-1)^i\big(\alpha(0\, -v) + \tfrac{1}{\alpha}(w\, 0)\big)\big) \\ &= (-1)^i\big(\alpha(0\, -av) + \tfrac{1}{\alpha}(aw\, 0)\big) \\ &= \varphi_i\, \mathrm{Ad}(a)X, \end{aligned}$$

$$\begin{aligned} g_i(\mathrm{Ad}(a)X, \mathrm{Ad}(a)Y) &= g(s\xi + (av\, aw), t\xi + (au\, az)) \\ &= st + \tfrac{1}{2}\big(\alpha\langle av, au\rangle + \tfrac{1}{\alpha}\langle aw, az\rangle\big) \\ &= st + \tfrac{1}{2}\big(\alpha\langle v, u\rangle + \tfrac{1}{\alpha}\langle w, z\rangle\big) \\ &= g(X, Y). \end{aligned}$$

Finally, since $\mathrm{Ad}(a)\xi = \xi$, we also have that $\mathrm{Ad}(a)^*\eta_i = \eta_i$. Observe that the invariance of η_i implies that, for every $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{X}(M_i)$,

$$0 = (\mathcal{L}_{X^*}\eta_i)Y = X^*(\eta_i Y) - \eta_i([X^*, Y]),$$

where X^* is the fundamental vector field determined by X. Thus, for every $X, Y \in \mathfrak{m}_i$

$$\begin{aligned} 2\mathrm{d}\eta_i(X^*, Y^*) &= X^*(\eta_i Y^*) - Y^*(\eta_i X^*) - \eta_i([X^*, Y^*]) \\ &= -\eta_i([Y^*, X^*]) = -\eta_i([X, Y]^*). \end{aligned}$$

Evaluating this formula at the base point $o \in M_i$ yields

$$2(\mathrm{d}\eta_i)_o(X, Y) = -\eta_i([X, Y]_{\mathfrak{m}_i}). \tag{20}$$

By direct computations, using (19), (20), we obtain that

$$(\mathrm{d}\eta_i)_o(X, Y) = g_i(X, \varphi_i Y), \quad X, Y \in \mathfrak{m}_i.$$

This proves that the invariant tensors $(\varphi_i, \xi, \eta_i, g_i)$ make up a contact metric structure on M_i. Moreover it is a K-contact structure if and only if $\alpha = 1$. Indeed, since ξ and φ_i are invariant tensors on M_i, they are parallel with respect to the canonical connection $\widetilde{\nabla}$ associated to the decomposition (18), hence,

$$\begin{aligned}(\mathcal{L}_\xi \varphi_i)Y &= [\xi, \varphi_i Y] - \varphi_i[\xi, Y]\\ &= \widetilde{\nabla}_\xi \varphi_i Y - \widetilde{T}(\xi, \varphi_i Y) - \varphi_i(\widetilde{\nabla}_\xi Y - \widetilde{T}(\xi, Y))\\ &= -\widetilde{T}(\xi, \varphi_i Y) + \varphi_i \widetilde{T}(\xi, Y),\end{aligned}$$

then

$$\begin{aligned}2(h_i)_o(v\ w) &= (\mathcal{L}_\xi \varphi_i)_o(v\ w)\\ &= [\xi, \varphi_i(v\ w)] - \varphi_i[\xi, (v\ w)]\\ &= (-1)^i\big[\xi, \big(\tfrac{1}{\alpha}w\ \ -\alpha v\big)\big] - \varphi_i(-w\ v)\\ &= (-1)^i\big(\alpha v\ \tfrac{1}{\alpha}w\big) - (-1)^i\big(\tfrac{1}{\alpha}v\ \alpha w\big)\\ &= (-1)^i\Big(\frac{\alpha^2-1}{\alpha}v\ \ -\frac{\alpha^2-1}{\alpha}w\Big).\end{aligned}$$

Applying Theorem 7, we see that $(\varphi_i, \xi, \eta_i, g_i)$ is a contact metric (κ, μ)-structure on M_i for every $\alpha > 0$, $\alpha \neq 1$; moreover, by construction, $\mathcal{J}_i$ is a standard complex structure on the base space B_i of the canonical fibration of M_i, in the sense of Definition 11. In particular if $0 < \alpha < 1$ then, by the uniqueness result in Theorem 13, we must have

$$\sqrt{\frac{I_{M_1}+1}{I_{M_1}-1}} = \frac{1}{\alpha}, \quad \sqrt{\frac{I_{M_2}+1}{I_{M_2}-1}} = \alpha,$$

or equivalently

$$I_{M_1} = \frac{1+\alpha^2}{1-\alpha^2}, \quad I_{M_2} = -\frac{1+\alpha^2}{1-\alpha^2}.$$

Thus, as α varies in $]0, 1[$, I_{M_1} assumes all the values in $]1, +\infty[$ and I_{M_2} assumes all the values in $]-\infty, -1[$.

Now we consider the Lie algebra $\mathfrak{g} := \mathfrak{so}(n+1, 1)$ with symmetric decomposition $\mathfrak{g} = \bar{\mathfrak{h}} \oplus \mathfrak{b}$, where

$$\bar{\mathfrak{h}} := \left\{ \left[\begin{array}{cc|c} 0 & \lambda & \multirow{2}{*}{$\mathbf{0}$} \\ \lambda & 0 & \\ \hline \mathbf{0} & \mathbf{0} & a \end{array} \right] : \lambda \in \mathbb{R},\ a \in \mathfrak{so}(n) \right\} = \mathfrak{so}(1, 1) \oplus \mathfrak{so}(n),$$

$$\mathfrak{b} := \left\{ \left[\begin{array}{c|c} \multirow{2}{*}{$\mathbf{0}$} & v^T \\ & -w^T \\ \hline v\ \ w & \mathbf{0} \end{array} \right] : v, w \in \mathbb{R}^n \right\} \simeq T_o B_3.$$

Let $(\mathcal{I}, \bar{g})$ be the para-Hermitian structure on B_3 determined by the $\mathrm{Ad}(\mathrm{SO}(1, 1) \times \mathrm{SO}(n))$-invariant structure (I, G) on $\mathfrak{b}$:

$$I(v\, w) := -(w\, v), \quad G((v\, w), (u\, z)) := \langle v, u \rangle - \langle w, z \rangle,$$

where $(v\ w)$ denotes the matrix

$$\left[\begin{array}{c|c} \multirow{2}{*}{$\mathbf{0}$} & v^T \\ & -w^T \\ \hline v\ \ w & \mathbf{0} \end{array} \right] \in \mathfrak{b}.$$

The homogeneous space $\mathrm{SO}(n+1, 1)/\mathrm{SO}(n)$ is reductive with respect to the decomposition

$$\mathfrak{so}(n+1, 1) = \mathfrak{so}(n) \oplus \mathfrak{m},$$

where

$$\mathfrak{m} := \mathfrak{so}(1, 1) \oplus \mathfrak{b} = \mathbb{R}\xi \oplus \mathfrak{b},$$

$$\xi := \left[\begin{array}{cc|c} 0 & 1 & \multirow{2}{*}{$\mathbf{0}$} \\ 1 & 0 & \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right];$$

indeed

$$\mathrm{Ad}(a)(s\xi + (v\, w)) = s\xi + (av\, aw),$$

for every $a \in \mathrm{SO}(n)$, $X = s\xi + (v\ w) \in \mathfrak{m}$.

Now we consider the natural decomposition of $\mathfrak{b}$:

$$\mathfrak{b} = \mathfrak{p} \oplus \mathfrak{q},$$

where

$$\mathfrak{p} := \{(v\, 0) \mid v \in \mathbb{R}^n\} \subset \mathfrak{b},$$

$$\mathfrak{q} := \{(0\, w) \mid w \in \mathbb{R}^n\} \subset \mathfrak{b}.$$

Using this decomposition, we define on $\mathfrak{m}$ the following Ad(SO(n))-invariant tensors:

$$
\begin{aligned}
&\varphi(Z) := \begin{cases} -\alpha I Z & \text{if } Z \in \mathfrak{p}, \\ \frac{1}{\alpha} I Z & \text{if } Z \in \mathfrak{q}, \\ 0 & \text{if } Z \in \mathbb{R}\xi, \end{cases} \\
&g(X,Y) := st + \tfrac{1}{2}\big(\alpha\langle v,u\rangle + \tfrac{1}{\alpha}\langle w,z\rangle\big), \quad \eta(X) := s,
\end{aligned}
\tag{21}
$$

where $\alpha > 0$ and $X = s\xi + (v\ w)$, $Y = t\xi + (u\ z)$ are any matrices in $\mathfrak{m}$. One checks by the same method used above that (φ, ξ, η, g) is a contact metric (κ, μ)-structure. Moreover

$$
2h_o(v\ w) = \Big(-\frac{\alpha^2+1}{\alpha} v\ \frac{\alpha^2+1}{\alpha} w\Big).
$$

Then applying again Theorem 13 we get

$$
I_{M_3} = \frac{\alpha^2 - 1}{\alpha^2 + 1}
$$

and hence, as α varies in $\mathbb{R}^*_+$, I_{M_3} assumes all the values in $]-1, 1[$. $\square$

Remark 18. Of course, in the case $I > 1$ we recover, up to isomorphism, the unit tangent sphere bundle $T_1 M$ of a Riemannian manifold (M, g) with constant sectional curvature $c > 0$, $c \neq 1$.

In the case $I < -1$, we obtain a new homogeneous representation of the contact metric (κ, μ)-manifolds M with $I_M < -1$, different from the Lie group representation furnished by Boeckx. Actually these models can be geometrically interpreted also as tangent hyperquadric bundle over Lorentzian space forms, as shown in [Loiudice and Lotta 2018].

Remark 19. The homogeneous model spaces of contact metric (κ, μ)-manifolds here obtained also appear in the classification list of the simply connected sub-Riemannian symmetric spaces carried out by Bieliavsky, Falbel, and Gorodski [1999]. However, in their paper the contact metric structures are not considered.

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