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LOCAL SOBOLEV CONSTANT ESTIMATE FOR INTEGRAL BAKRY–ÉMERY RICCI CURVATURE

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We extend several geometrical results in Dai et al. (2018) for Riemannian manifolds with integral curvature to complete smooth metric measure spaces with integral Bakry–Émery Ricci curvature.

1. Introduction

Sobolev inequalities not only encode rich analytical and geometrical information about manifold, but have wide applications in differential geometry. A useful method to estimate the Sobolev constant is to estimate the isoperimetric constant since they are equivalent [Cheeger 1970; Chavel 1993; Li 2012]. A key issue for the isoperimetric constant study is the volume control, which is given by the Ricci curvature lower bound. After Petersen and Wei [1997] generalized the classical Laplacian and volume comparison to the integral Ricci curvature bound, many results for the pointwise Ricci lower bound have been extended to the integral Ricci curvature bound, see, e.g., [Petersen and Wei 2001; Aubry 2007; 2009; Dai et al. 2000; Tian and Zhang 2016; Dai et al. 2018; Zhang and Zhu 2017; 2018; Rose 2017; Rose and Stollmann 2017]. In particular, Dai, Wei and Zhang [Dai et al. 2018] obtained the local isoperimetric constant estimate for integral Ricci curvature.

An n-dimensional smooth metric measure space, denoted by $M_f^n := (M^n, g, e^{-f}\,d\,\mathrm{vol})$, is a complete n-dimensional Riemannian manifold (M^n, g) coupled with a weighted volume $e^{-f}\,d\,\mathrm{vol}$ for some $f \in C^\infty(M)$, where $d\,\mathrm{vol}$ is the usual Riemannian volume element on M. For a smooth metric measure space M_f^n, a natural generalization of the Ricci curvature is the Bakry–Émery Ricci curvature [Bakry and Émery 1985] defined by

$$\mathrm{Ric}_f := \mathrm{Ric} + \mathrm{Hess}\, f.$$

A great deal of effort has been devoted to the study of smooth metric measure spaces with Bakry–Émery Ricci curvature bounded below, and some of the earlier works are [Lott 2003; Wei and Wylie 2009; Cao and Zhou 2010; Munteanu and

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Wang 2011]. However, limited work has been done for the integral Bakry–Émery Ricci curvature.

Recently, Wu [2018] has extended the volume comparison in [Petersen and Wei 1997] to the integral Bakry–Émery Ricci curvature case. In this paper we extend the local isoperimetric constant estimate in [Dai et al. 2018] to integral Bakry–Émery Ricci curvature and give some applications.

To state the results, we fix some notations. Given $x \in M_f^n$, let $\rho_f(x)$ be the smallest eigenvalue of $\mathrm{Ric}_f : T_xM \to T_xM$, and

$$\mathrm{Ric}_{f-}^H = ((n-1)H - \rho_f(x))_+ = \max\{0, (n-1)H - \rho_f(x)\}.$$

Denote $B_x(R) \subset M$ by the ball with radius R, centered at x. Various weighted L^p norms of the function h on a smooth metric measure space M_f^n are

$$\|h\|_{p,f,B_x(R)} = \left(\int_{B_x(R)} |h|^p e^{-f} \, d\,\mathrm{vol} \right)^{\frac{1}{p}},$$

$$\|h\|_{p,f,a}(R) = \sup_{x \in M_f^n} \left[\int_0^R \int_{\mathbb{S}^{n-1}} |h|^p e^{-at} \mathcal{A}_f(t,\theta) \, d\theta \, dt \right]^{\frac{1}{p}}.$$

Here $d\theta$ is the volume element of the unit sphere $\mathbb{S}^{n-1}$ and $\mathcal{A}_f(t,\theta)$ is the volume element of weighted measure $e^{-f} d\,\mathrm{vol} = \mathcal{A}_f(t,\theta)\, d\theta \wedge dt$. Clearly, $\|\mathrm{Ric}_{f-}^H\|_{p,f,B_x(R)} \equiv 0$ if and only if $\mathrm{Ric}_f \geq (n-1)H$. For convenience, we assume $H = 0$ and write $\mathrm{Ric}_{f-}^0 = \mathrm{Ric}_{f-}$ in the whole paper. Then Ric_{f-}^0 is the negative part of ρ_f, denoted by ρ_{f-}. The following scale-invariant curvature quantity will be useful:

$$\bar{\kappa}(p, f, a, R) = R^2 \sup_{x \in M_f^n} \left[\fint_{B_x(R)} \rho_{f-}^p e^{-at} \mathcal{A}_f(t,\theta)\, d\theta \wedge dt \right]^{\frac{1}{p}},$$

where

$$\mathrm{vol}_f B_x(R) = \int_{B_x(R)} e^{-f} \, d\,\mathrm{vol} \quad \text{and} \quad \fint_{B_x(R)} = \frac{1}{\mathrm{vol}_f B_x(R)} \int_{B_x(R)}.$$

For a fixed point $x \in M$, let $r(y) := d(y,x)$ be a distance function from x to y. In geodesic polar coordinates at x, let $\nabla r = \partial_r$. Our first result gives an estimation of the local normalized Dirichlet isoperimetric constant under a lower bound of $\partial_r f$.

Theorem 1.1. *Let M_f^n be a complete smooth metric measure space. Assume that $\partial_r f \geq -a$ along all minimal geodesic segments for some constant $a \geq 0$. For $p > \frac{1}{2}n$, there exists $\varepsilon = \varepsilon(n,p,a) > 0$ such that if $\bar{\kappa}(p,f,a,1) \leq \varepsilon$, then for any $x \in M_f^n$, $\partial B_x(R) \neq \varnothing$, $R \leq 1$, we have the estimate*

$$\mathrm{ID}_{n,f}^* B_x(R) \geq 10^{-2n} e^{-2a} R^{-1}, \tag{1-1}$$

where

$$\mathrm{ID}^*_{n,f} B_x(R) = \mathrm{vol}_f B_x(R)^{-\frac{1}{n}} \inf_{\Omega} \left\{ \frac{\mathrm{vol}_f \partial\Omega}{(\mathrm{vol}_f \Omega)^{\frac{n-1}{n}}} \right\}.$$

Here the infimum runs over all subdomains $\Omega \subset B_x(R)$ with smooth boundary and $\partial\Omega \cap \partial B_x(R) = \varnothing$.

Remark 1.2. Clearly, the local normalized Dirichlet constant has explicit and accurate dependency of the growth of f; Theorem 1.1 will recover to [Dai et al. 2018, Theorem 1.1] when f is a constant. The smallness of $\bar{\kappa}(p, f, a, 1)$ is necessary; see the counterexample in [Dai et al. 2018, §6] when f is constant and $\bar{\kappa}(p, f, a, 1)$ is bounded. Also the result is not true when $p \le \frac{1}{2}n$; see details in [Aubry 2007].

It is well known that the classical Dirichlet isoperimetric and Sobolev constants are equivalent; see, e.g., [Li 2012]. In Section 3, we introduce their weighted versions for smooth metric measure spaces; see Definitions 3.1, 3.2. A similar proof shows that they are also equivalent; see Theorem A.1. Hence we have:

Proposition 1.3. *With the assumptions of Theorem 1.1, the Sobolev inequality*

$$\fint_{B_x(R)} |\nabla h| e^{-f} \, d\,\mathrm{vol} \ge 10^{-2n} e^{-2a} R^{-1} \left(\fint_{B_x(R)} h^{\frac{n}{n-1}} e^{-f} \, d\,\mathrm{vol} \right)^{\frac{n-1}{n}} \tag{1-2}$$

holds for all $h \in C_0^\infty(B_x(R))$.

Recall that the f-Laplacian of M_f^n is

$$\Delta_f = \Delta - \nabla f \cdot \nabla.$$

Given the normalized form of integral in Proposition 1.3, we denote the normalized L^p norm for function h by

$$\|h\|^*_{p,f,B_x(R)} = \|h\|_{p,f,B_x(R)} (\mathrm{vol}_f B_x(R))^{-\frac{1}{p}},$$

and

$$\|h\|^*_{p,f,a}(R) = \sup_{x \in M_f^n} \left(\fint_{B_x(R)} h e^{-at} \mathcal{A}_f(t,\theta) \, d\theta \wedge dt \right)^{\frac{1}{p}}$$

with the same $\mathcal{A}_f(t,\theta)\, d\theta \wedge dt$ as above. It is easy to observe that

$$\|h\|^*_{p,f,B_x(R)} \le e^{\frac{aR}{p}} \|h\|^*_{p,f,a}(R), \tag{1-3}$$

and the normalized L^∞ norm is independent of f satisfying $\|h\|^*_{\infty,f,B_x(R)} = \sup_{B_x(R)} h$. By employing the above Sobolev inequality (1-2), we extend the maximum principle in [Petersen and Wei 2001; Dai et al. 2018] to the integral Bakry–Émery Ricci curvature situation.

Theorem 1.4. *Let M_f^n be a complete smooth metric measure space. Assume that $\partial_r f \geq -a$ along all minimal geodesic segments for some constant $a \geq 0$. For $p > \frac{1}{2}n$, there exists an $\varepsilon = \varepsilon(n, p, a) > 0$ and $C = C(n, p, a) > 1$ such that if $\bar{\kappa}(p, f, a, 1) \leq \varepsilon$ and $R \leq 1$, then for any function $u : \Omega(\subset B_x(R)) \subset M_f^n \to \mathbb{R}$ with $\Delta_f u \geq h$, we have*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + C \cdot R^2 \cdot \|h_-\|^*_{p,f,\Omega},$$

where h_- denotes the negative part of the function h.

Also we have the gradient estimate.

Theorem 1.5. *Let M_f^n be a complete smooth metric measure space. Assume that $\partial_r f \geq -a$ along all minimal geodesics for some constant $a \geq 0$. For $p > \frac{1}{2}n$, there exists an $\varepsilon = \varepsilon(n, p, a) > 0$ and $C(n, p, a) > 1$ such that if $\bar{\kappa}(p, f, a, 1) \leq \varepsilon$ and $R \leq 1$ and u is a function on $B_x(R)$ satisfying*

$$\Delta_f u = h,$$

then

$$\sup_{B_x(\frac{R}{2})} |\nabla u|^2 \leq C(n, p, a) R^{-2} [(\|h\|^*_{2p,f,B_x(R)})^2 + (\|u\|^*_{2,f,B_x(R)})^2].$$

An outline of this paper is as follows. In Section 2, we review the Laplacian and volume comparison for the integral Bakry–Émery Ricci curvature. In Section 3, we define local Dirichlet isoperimetric and Sobloev constants, as well as their normalized form in smooth metric measure spaces. Moreover, we estimate the normalized isoperimetric constant for integral Bakry–Émery Ricci curvature; see Theorem 1.1. In Section 4, as applications, we establish the maximum principle (Theorem 1.4) and the gradient estimate (Theorem 1.5) in a complete smooth metric measure space with integral Bakry–Émery Ricci curvature. In the Appendix, we give the proof of equivalence between the two constants defined in Section 3.

2. Preliminaries

In this section, we review the Laplacian and volume comparison for smooth metric measure spaces. Let M_f^n be a complete smooth metric measure space and $r(y) = d(y, x)$ be a distance function from $x \in M_f^n$. Assume that f satisfies $\partial_r f \geq -a$ along all minimal geodesic segments for some constant $a \geq 0$. By choosing the Euclidean space with a weighted function as the model space, that is $\mathbb{R}_a^n = (\mathbb{R}^n, g_{\mathbb{R}^n}, e^{-h} d\,\mathrm{vol})$ with $h(x) = -a|x|$ for $x \in \mathbb{R}^n$, then the f-Laplacian error term is

$$\psi(y) = \left(\Delta_f r - \frac{n-1}{r} - a\right)_+.$$

Wei and Wylie [2009] have proved that $\mathrm{Ric}_f \geq 0$ yields $\Delta_f r \leq \dfrac{n-1}{r} + a$; that is,

$\|\mathrm{Ric}_{f-}\|_{p,f,a}(r) \equiv 0$ implies that $\psi \equiv 0$. In [Wu 2018], this has been extended to integral Bakry–Émery Ricci curvature bound following the work of Petersen and Wei [1997] for integral Ricci curvature.

Theorem 2.1 [Wu 2018, Theorem 1.1]. *For any $p > \frac{1}{2}n$, we have*

$$\|\psi\|_{2p,f,a}(r) \le C(n,p)[\|\mathrm{Ric}_{f-}\|_{p,f,a}(r)]^{\frac{1}{2}}, \tag{2-1}$$

with

$$C(n,p) = \Big(\frac{(n-1)(2p-1)}{2p-n}\Big)^{\frac{1}{2}}.$$

Moreover, letting $B_x(r_2)$ and $B_x(r_1)$ be geodesic balls centered at x with radius $r_2 \ge r_1 > 0$, we have

$$\Big(\frac{\mathrm{vol}_f B_x(r_2)}{V(n,a,r_2)}\Big)^{\frac{1}{2p}} - \Big(\frac{\mathrm{vol}_f B_x(r_1)}{V(n,a,r_1)}\Big)^{\frac{1}{2p}} \le C(n,p,a,r_2)(\|\mathrm{Ric}_{f-}\|_{p,f,a}(r_2))^{\frac{1}{2}}, \tag{2-2}$$

where $V(n,a,t)$ denotes the volume of geodesic ball $B_0(t)$ in the model space $\mathbb{R}^n_a$.

Remark 2.2. In fact, the proof of [Wu 2018, Theorem 1.1] gives the following normalized form of Laplacian comparison:

$$\begin{aligned}\|\psi\|^*_{2p,f,a}(r) &\le C(n,p)(\|\mathrm{Ric}_{f-}\|^*_{p,f,a}(r))^{\frac{1}{2}} \\ &= C(n,p)r^{-1}(\kappa(p,f,a,r))^{\frac{1}{2}}.\end{aligned} \tag{2-3}$$

Remark 2.3. Here we choose the power $2p$ rather than $2p-1$, so the explicit expression of $C(n,p,a,r_2)$ is similar to the one in [Petersen and Wei 1997, Lemma 2.1] rather than the one in [Wu 2018]. If we denote the volume of $(n-1)$-dimensional unit ball in $\mathbb{R}^n$ and the weighted volume of geodesic sphere $\partial B_0(t)$ by ω_n and $A(n,a,t)$, respectively, then $A(n,a,t) = \omega_n t^{n-1}e^{at}$ and

$$\begin{aligned}C(n,p,a,r_2) &= C(n,p)\int_0^{r_2} tA(n,a,t)\Big(\frac{1}{V(n,a,t)}\Big)^{1+\frac{1}{2p}}\,dt \\ &\le C(n,p)\int_0^{r_2} tA(n,a,t)\Big(\int_0^t A(n,0,s)\,ds\Big)^{-(1+\frac{1}{2p})}\,dt \\ &= C(n,p)e^{ar_2}r_2^{1-\frac{n}{2p}}.\end{aligned}$$

Hence, (2-2) implies that

$$\left(\frac{\operatorname{vol}_f B_x(r_1)}{\operatorname{vol}_f B_x(r_2)}\right)^{\frac{1}{2p}} \geq \left(\frac{V(n,a,r_1)}{V(n,a,r_2)}\right)^{\frac{1}{2p}} \big(1 - C(n,p)e^{ar_2} r_2^{1-\frac{n}{2p}} \cdot (V(n,a,r_2))^{\frac{1}{2p}} (\|\mathrm{Ric}_{f-}\|^*_{p,f,a}(r_2))^{\frac{1}{2}}\big)$$
$$\geq e^{-\frac{ar_2}{2p}} \left(\frac{r_1}{r_2}\right)^{\frac{n}{2p}} \big(1 - C(n,p)e^{(1+\frac{1}{2p})ar_2} \bar{\kappa}^{\frac{1}{2}}(p,f,a,r_2)\big), \tag{2-4}$$

where $C(n,p)$ is a constant depending on n and p. Hence, there exists a constant $\varepsilon_0 = \varepsilon_0(n,p,a,r_0) > 0$ such that if $\bar{\kappa}(p,f,a,r_0) \leq \varepsilon_0$, then

$$\frac{\operatorname{vol}_f B_x(r)}{\operatorname{vol}_f B_x(r_0)} \geq \tfrac{1}{2} e^{-ar_0} \left(\frac{r}{r_0}\right)^n \quad \forall r \leq r_0. \tag{2-5}$$

For $r_0 \leq 1$, from (2-4), it is easy to observe that there exists a $\varepsilon_0 = \varepsilon_0(n,p,a)$, independent of r, such that (2-5) holds for $\bar{\kappa}(p,f,a,r_0) \leq \varepsilon_0$.

Remark 2.4. The scale invariant $\bar{\kappa}(p,f,a,r)$ has curvature inequalities. For any $r_1 \leq r_2$, and $\bar{\kappa}(p,f,a,r_2) \leq \varepsilon_0$, on one hand,

$$\bar{\kappa}(p,f,a,r_1) \leq r_1^2 \left(\frac{\operatorname{vol}_f B_x(r_2)}{\operatorname{vol}_f B_x(r_1)} \cdot \frac{1}{\operatorname{vol}_f B_x(r_2)} \int_0^{r_2} \int_{S^{n-1}} \rho_{f-}^p e^{-at} \mathcal{A}_f(t,\theta)\, d\theta\, dt\right)^{\frac{1}{p}}$$
$$\leq 2^{\frac{1}{p}} e^{\frac{ar_2}{p}} \left(\frac{r_1}{r_2}\right)^{2-\frac{n}{p}} \bar{\kappa}(p,f,a,r_2). \tag{2-6}$$

Hence, $\bar{\kappa}(p,f,a,r_1) \leq \varepsilon_0$ holds for $r_1 \leq 2^{-1/(2p-n)} e^{-ar_2/(2p-n)} r_2$. On the other hand, if $\bar{\kappa}(p,f,a,r_1) \leq \varepsilon_0$, using the same method as in [Petersen and Wei 2001, §2.3] and the volume doubling property (2-5), we have

$$\bar{\kappa}(p,f,a,r_2) = r_2^2 \sup_{x \in M_f^n} \left(\frac{1}{\operatorname{vol}_f B_x(r_2)} \int_0^{r_2} \int_{S^{n-1}} \rho_{f-}^p e^{-at} \mathcal{A}_f(t,\theta)\, d\theta\, dt\right)^{\frac{1}{p}}$$
$$\leq \left(\frac{r_2}{r_1}\right)^2 2^{\frac{n+1}{p}} e^{\frac{ar_1}{p}} \bar{\kappa}(p,f,a,r_1). \tag{2-7}$$

Hence, it is sufficient to work with the case where $\bar{\kappa}(p,f,a,1)$ is small and then scale the metric to obtain the curvature condition of $\bar{\kappa}(p,f,a,r)$.

3. Local Dirichlet isoperimetric constant estimate

In this section, we introduce the local isoperimetric and Sobolev constants in smooth metric measure spaces motivated by the classical ones in [Cheeger 1970; Li 2012]. Furthermore, we estimate the normalized form of these constants.

Definition 3.1. Let $B_x(r)$ be a geodesic ball with $\partial B_x(r) \neq \varnothing$ in a complete smooth metric measure space M_f^n. For $n \le \alpha \le \infty$, the Dirichlet α-isoperimetric constant of $B_x(r)$ is defined by

$$\mathrm{ID}_{\alpha,f} B_x(r) = \inf_{\Omega} \frac{\mathrm{vol}_f \partial\Omega}{(\mathrm{vol}_f \Omega)^{1-\frac{1}{\alpha}}},$$

where Ω is an open submanifold of $B_x(r)$ with $\partial\Omega \cap \partial B_x(r) = \varnothing$.

Clearly, $\mathrm{ID}_{n,f} B_x(r)$ is a scale invariant and $\mathrm{ID}_{\infty,f} B_x(r)$ is a weighted Cheeger constant.

Definition 3.2. The Dirichlet α-Sobolev constant of $B_x(r) \subset M_f^n$ is defined by

$$\mathrm{SD}_{\alpha,f} B_x(r) = \inf_{h} \frac{\|\nabla h\|_{1,f,B_x(r)}}{\|h\|_{\frac{\alpha}{\alpha-1},f,B_x(r)}},$$

where the infimum is taken over all $h \in C_0^\infty(B_x(r))$.

For convenience, we normalize the two kinds of constants above thus:

$$\begin{aligned} \mathrm{ID}^*_{\alpha,f} B_x(r) &= \mathrm{ID}_{\alpha,f} B_x(r)(\mathrm{vol}_f B_x(r))^{-\frac{1}{\alpha}}, \\ \mathrm{SD}^*_{\alpha,f} B_x(r) &= \mathrm{SD}_{\alpha,f} B_x(r)(\mathrm{vol}_f B_x(r))^{-\frac{1}{\alpha}}. \end{aligned} \tag{3-1}$$

To estimate $\mathrm{ID}^*_{\alpha,f} B_x(R)$, we need several lemmas. A suitable modification of Gromov's observation [Gromov 1979; Cheeger and Colding 1996] yields the first:

Lemma 3.3. *Let M_f^n be a complete smooth metric measure space. Assume that $\partial_r f \ge -a$ along all minimal geodesic segments for some constant $a \ge 0$. Let S be any hypersurface dividing M_f^n into two parts M_1, M_2. For any subsets $W_i \subset M_i$, there exists x_1 in one of W_i, say W_1, and a subset W in another one, W_2, such that the unique minimal geodesic joint x_1 and any $x_2 \in W$ intersects S at q with*

$$d(x_1, q) \ge d(x_2, q), \tag{3-2}$$

and

$$\mathrm{vol}_f W_2 \le 2\,\mathrm{vol}_f W. \tag{3-3}$$

Analogously to [Dai et al. 2018, Lemma 4.2], we have the following volume estimate by using the Laplacian comparison method in smooth metric measure spaces:

Lemma 3.4. *Let M_f^n, S, W and x_1 be as in Lemma 3.3. Then for any $p > \frac{1}{2}n$,*

$$\begin{aligned} \mathrm{vol}_f W \le 2^{n-1}\big[&e^{aD} D\,\mathrm{vol}_f S' \\ &+ e^{(a+\frac{a}{2p})D} C(n,p)(\bar\kappa(p,f,a,D))^{\frac{1}{2}}\,\mathrm{vol}_f B_{x_1}(D)\big], \end{aligned} \tag{3-4}$$

where $D = \sup_{x\in W} d(x_1, x)$ *and* S' *is the set of intersection points with* S *of geodesics* γ_{xx_1} *for all* $x \in W$.

Proof. Let $\Gamma \subset S_{x_1}$ be the set of unit vectors v such that $\gamma_v = \gamma_{x_1x_2}$ for some $x_2 \in W$. Using the polar coordinate $(\theta, t) \in S_{x_1} \times \mathbb{R}^+$ and $e^{-f}\, d\,\mathrm{vol} = \mathcal{A}_f(\theta, t)\, d\theta \wedge dt$. Recalling [Wu 2018, Theorem 3.1], we have

$$\frac{\partial}{\partial t}\frac{\mathcal{A}_f}{t^{n-1}e^{at}} = \Big(\Delta_f t - \frac{n-1}{t} - a\Big)\frac{\mathcal{A}_f}{t^{n-1}e^{at}} \le \psi\frac{\mathcal{A}_f}{t^{n-1}e^{at}} \tag{3-5}$$

with $\psi = (\Delta_f t - (n-1)/t - a)_+$. Integrating (3-5) from t to s gives

$$\begin{aligned}\mathcal{A}_f(s,\theta) &\le \Big(\frac{s}{t}\Big)^{n-1} e^{a(s-t)}\Big(\mathcal{A}_f(t,\theta) + \int_t^s \psi\mathcal{A}_f(l,\theta)\, dl\Big)\\ &\le 2^{n-1}e^{\frac{as}{2}}\Big(\mathcal{A}_f(t,\theta) + \int_t^s \psi\mathcal{A}_f(l,\theta)\, dl\Big)\end{aligned}$$

for any $\frac{1}{2}s \le t \le s$. For any $\theta \in \Gamma$, let $s_1(\theta)$ and $s_2(\theta)$ be the minimum and maximum radius, respectively, such that $\exp_{x_1}(s_i\theta) \in W$, and $s(\theta)$ such that $\exp_{x_1}(s(\theta)\theta) \in S$. Then Lemma 3.3 implies that $2s(\theta) \ge s_2(\theta) \ge s_1(\theta) \ge s(\theta)$. Thus,

$$\begin{aligned}\mathrm{vol}_f W &\le \int_\Gamma\int_{s_1(\theta)}^{s_2(\theta)} \mathcal{A}_f(s,\theta)\, ds\, d\theta\\ &\le 2^{n-1}e^{as(\theta)}\int_\Gamma\int_{s_1(\theta)}^{s_2(\theta)}\Big(\mathcal{A}_f(s(\theta),\theta) + \int_{s(\theta)}^s \psi\mathcal{A}_f(l,\theta)\, dl\Big)\, ds\, d\theta\\ &\le 2^{n-1}e^{aD}D\Big(\int_\Gamma \mathcal{A}_f(s(\theta),\theta)d\theta + \int_0^D\int_\Gamma \psi\mathcal{A}_f(s,\theta)\, ds\, d\theta\Big).\end{aligned} \tag{3-6}$$

On the other hand,

$$\mathrm{vol}_f S' = \int_\Gamma \frac{\mathcal{A}_f(s(\theta),\theta)}{\cos\alpha(\theta)}\, d\theta \ge \int_\Gamma \mathcal{A}_f(s(\theta),\theta)\, d\theta,$$

where $\alpha(\theta)$ is the angle between S and the radical geodesic $\exp_{x_1}(s\theta)$. Applying this result, the Hölder inequality, (1-3) and the Laplacian comparison (2-3) successively in (3-6), we obtain

$$\begin{aligned}\mathrm{vol}_f W &\le 2^{n-1}e^{aD}D[\mathrm{vol}_f S' + \|\psi\|_{1,f,B_{x_1}(D)}]\\ &\le 2^{n-1}e^{aD}D[\mathrm{vol}_f S' + \|\psi\|^*_{2p,f,B_{x_1}(D)}\,\mathrm{vol}_f B_{x_1}(D)]\\ &\le 2^{n-1}e^{aD}D[\mathrm{vol}_f S' + e^{\frac{aD}{2p}}\|\psi\|^*_{2p,f,a}(D)\,\mathrm{vol}_f B_{x_1}(D)]\\ &\le 2^{n-1}e^{aD}D\big[\mathrm{vol}_f S' + e^{\frac{aD}{2p}}C(n,p)D^{-1}(\bar\kappa(p,f,a,D))^{\frac{1}{2}}\,\mathrm{vol}_f B_{x_1}(D)\big],\end{aligned}$$

which completes the required estimate. □

Lemma 3.4 enables us to obtain a local Cheeger's constant estimate.

Lemma 3.5. *Let M_f^n, S, W and x_1 be the same as in Lemma 3.3. For $p > \frac{1}{2}n$, there exists $\varepsilon = \varepsilon(p, n, a)$ such that if $\bar{\kappa}(p, f, a, 1) \le \varepsilon$, then for a geodesic ball $B = B_x(r)$, $r \le \frac{1}{2}$ which is divided equally by S, we have*

$$\operatorname{vol}_f B_x(r) \le 2^{n+3} r e^a \operatorname{vol}_f(S \cap B_x(2r)).$$

Proof. To begin, we choose $W_i = B_x(2r) \cap M_i$ with M_i as in Lemma 3.3; then

$$\operatorname{vol}_f(B \cap M_1) = \operatorname{vol}_f(B \cap M_2) \le \min\{\operatorname{vol}_f W_1, \operatorname{vol}_f W_2\} \le 2 \operatorname{vol}_f W,$$

and $D \le 2r$ and $S' \subset S \cap B_x(2r)$. Thus, by Lemma 3.4, we have

$$\begin{aligned} \operatorname{vol}_f B_x(r) &\le 4 \operatorname{vol}_f W \\ &\le 2^{n+1}\big[2re^{2ar} \operatorname{vol}_f(S \cap B_x(2r)) \\ &\qquad + e^{2r(a+\frac{a}{2p})} C(n, p)(\bar{\kappa}(p, f, a, 2r))^{\frac{1}{2}} \operatorname{vol}_f B_x(2r)\big]. \end{aligned} \tag{3-7}$$

Next we aim at canceling the curvature inequality term. Since volume doubling property (2-5) implies that

$$\operatorname{vol}_f B_x(2r) \le 2 \frac{V(n, a, 2r)}{V(n, a, r)} \cdot \operatorname{vol}_f B_x(r) \le 2^{n+1} e^a \operatorname{vol}_f B_x(r) \tag{3-8}$$

holds for $\bar{\kappa}(p, f, a, 2r) \le \varepsilon_0$. From the curvature inequality (2-6) and $r \le \frac{1}{2}$, the curvature condition reduces to $\bar{\kappa}(p, f, a, 1) \le 2^{-1/p} e^{-a/p} \varepsilon_0$, and

$$e^{2r(a+\frac{a}{2p})} C(n, p)(\bar{\kappa}(p, f, a, 2r))^{\frac{1}{2}} \le e^{a+\frac{a}{2p}} C(n, p)(2^{\frac{1}{p}} e^{\frac{a}{p}} \bar{\kappa}(p, f, a, 1))^{\frac{1}{2}}. \tag{3-9}$$

Inserting (3-8) and (3-9) into (3-7) gives

$$\begin{aligned} \operatorname{vol}_f B_x(r) &\le 2^{n+2} e^a r \operatorname{vol}_f(S \cap B_x(2r)) \\ &\qquad + 2^{2n+2+\frac{1}{2p}} e^{2a+\frac{a}{p}} C(n, p)(\bar{\kappa}(p, f, a, 1))^{\frac{1}{2}} \operatorname{vol}_f B_x(r). \end{aligned} \tag{3-10}$$

Here we used $r \le \frac{1}{2}$. Hence, we get the required result by choosing

$$\varepsilon(n, p, a) = \min\{2^{-\frac{1}{p}} e^{-\frac{a}{p}} \varepsilon_0, (2 \cdot 2^{2n+2+\frac{1}{2p}} e^{2a+\frac{a}{p}} C(n, p))^{-2}\}$$

and regrouping (3-10). □

Volume doubling property (2-5) indicates that the volume quotient of a concentric geodesic ball is lower bound by a function of the quotient of corresponding radius. The next theorem not only offers the other direction but also extends [Dai et al. 2018, Theorem 3.3] to the case of integral Bakry–Émery Ricci curvature. Actually, by fixing the upper bound to $\frac{1}{2}$ and the bigger radius to 1, we obtain the other radius.

Theorem 3.6. *Let M_f^n be a complete smooth metric measure space with $\partial_r f \geq -a$ along all minimal geodesic segments for some constant $a \geq 0$. For $p > \frac{1}{2}n$, there exists $\varepsilon = \varepsilon(n, p, a) > 0$ and $r_0 = r_0(n, a) > 0$ such that if $\bar{\kappa}(p, f, a, 1) \leq \varepsilon$, then*

$$\frac{\operatorname{vol}_f B_x(r_0)}{\operatorname{vol}_f B_x(1)} \leq \frac{1}{2} \quad \forall x \in M_f^n. \tag{3-11}$$

Our proof follows the idea in [Dai et al. 2018, Theorem 3.3], but using the approach directly runs into obstacles. The difficulties were conquered by repeating the process of choosing the radius $k-1$ times with k depending on a.

Proof. For any $x \in M_f^n$ and $i = 2, \ldots, k$, let $r_1 = 1$, choose points $x_i \in B_x(r_{i-1})$ with $r_i < \frac{1}{3}r_{i-1}$ and $d_i = d_i(x, x_i) = \frac{1}{2}(r_{i-1} - r_i) > \frac{1}{3}r_{i-1}$, then

$$B_x(r_i) \subset B_{x_i}(d_i + r_i) \setminus B_{x_i}(d_i - r_i) \subset B_{x_i}(d_i + r_i) \subset B_x(r_{i-1}), \quad i = 2, 3, \ldots, k.$$

Moreover,

$$\frac{\operatorname{vol}_f B_x(r_i)}{\operatorname{vol}_f B_x(r_{i-1})} \leq \frac{\operatorname{vol}_f(B_{x_i}(d_i + r_i) \setminus B_{x_i}(d_i - r_i))}{\operatorname{vol}_f B_{x_i}(d_i + r_i)} \leq 1 - \frac{\operatorname{vol}_f B_{x_i}(d_i - r_i)}{\operatorname{vol}_f B_{x_i}(d_i + r_i)}.$$

By (2-4), we have

$$\begin{aligned}
&\frac{\operatorname{vol}_f B_{x_i}(d_i - r_i)}{\operatorname{vol}_f B_{x_i}(d_i + r_i)} \\
&\quad \geq \left(\frac{d_i - r_i}{d_i + r_i}\right)^n e^{-a(d_i + r_i)}[1 - C(n, p)e^{(1+\frac{1}{2p})a}\bar{\kappa}^{\frac{1}{2}}(p, f, a, d_i + r_i)]^{2p} \\
&\quad \geq \left(\frac{d_i - r_i}{d_i + r_i}\right)^n e^{-a}[1 - C(n, p)e^{(1+\frac{1}{2p})a} \cdot (2e^a)^{\frac{1}{2p}}\bar{\kappa}^{\frac{1}{2}}(p, f, a, 1)]^{2p} \\
&\quad = \left(\frac{d_i - r_i}{d_i + r_i}\right)^n e^{-a}[1 - C(n, p)e^{(1+\frac{1}{p})a}2^{\frac{1}{2p}}\bar{\kappa}^{\frac{1}{2}}(p, f, a, 1)]^{2p}.
\end{aligned}$$

Here we used the curvature inequality (2-6) and $d_i + r_i \leq 1$ in the second inequality. Choose a $q = q(n)$ such that

$$\frac{1-q}{1+q} = \left(\tfrac{3}{4}\right)^{\frac{1}{n}},$$

then for any $r_i \leq \frac{1}{3}q r_{i-1}$, since $d_i > \frac{1}{3}r_{i-1}$, we have

$$\left(\frac{d_i - r_i}{d_i + r_i}\right)^n \geq \tfrac{3}{4}.$$

Choose $\varepsilon \leq \varepsilon_0$ such that

$$(1 - C(n, p)e^{(1+\frac{1}{p})a}2^{\frac{1}{p}}\bar{\kappa}^{\frac{1}{2}}(p, f, a, 1))^{2p} \geq \tfrac{2}{3}. \tag{3-12}$$

Then

$$\frac{\operatorname{vol}_f B_{x_i}(d_i - r_i)}{\operatorname{vol}_f B_{x_i}(d_i + r_i)} \geq \tfrac{3}{4} \cdot e^{-a} \cdot \tfrac{2}{3} = \tfrac{1}{2} e^{-a},$$

and

$$\frac{\operatorname{vol}_f B_x(r_i)}{\operatorname{vol}_f B_x(r_{i-1})} \leq 1 - \tfrac{1}{2} e^{-a}.$$

Hence,

$$\frac{\operatorname{vol}_f B_x(r_k)}{\operatorname{vol}_f B_x(r_1)} = \prod_{i=2}^{k} \frac{\operatorname{vol}_f B_x(r_i)}{\operatorname{vol}_f B_x(r_{i-1})} = \left(1 - \tfrac{1}{2} e^{-a}\right)^{k-1}$$

with $r_k = \left(\frac{1}{3} q\right)^{k-1} r_1$. We choose the integer $k = k(a)$ such that

$$\left(1 - \tfrac{1}{2} e^{-a}\right)^{k-1} \leq \tfrac{1}{2} < \left(1 - \tfrac{1}{2} e^{-a}\right)^{k-2}.$$

Since $r_1 = 1$, the proof is completed by choosing $r_0 = r_k$ and $\varepsilon \leq \varepsilon_0$ satisfying (3-12). □

We now turn to prove the Theorem 1.1.

Proof of Theorem 1.1. Our first step is to show that the estimation (1-1) holds for some radius $r_0 = r_0(n, p, a)$ if $\bar{\kappa}(p, f, a, 1) \leq \varepsilon_1$ for some small constant $\varepsilon_1 = \varepsilon_1(n, p, a) > 0$. By Theorem 3.6, we assume that $\varepsilon_1 = \varepsilon_1(n, p, a)$ is chosen such that

$$\frac{\operatorname{vol}_f B_y(2r_0)}{\operatorname{vol}_f B_y\left(\frac{1}{10}\right)} \leq \tfrac{1}{2} \quad \forall\, y \in M_f^n$$

holds for some $r_0 = r_0(n, a)$. Given any $y_0 \in M_f^n$, let Ω be a smooth subdomain of $B_{y_0}(r_0)$. Assume that Ω is connected and its boundary $S = \partial\Omega$ divides M_f^n into Ω and Ω^c. For any $y \in \Omega$, let r_y be the smallest radius such that

$$\operatorname{vol}_f(B_y(r_y) \cap \Omega) = \operatorname{vol}_f(B_y(r_y) \cap \Omega^c) = \tfrac{1}{2} \operatorname{vol}_f B_y(r_y). \tag{3-13}$$

From $\Omega \subset B_y(2r_0)$ and $\operatorname{vol}_f B_y(2r_0) \leq \frac{1}{2} \operatorname{vol}_f B_y\left(\frac{1}{10}\right)$, it follows that $r_y \leq \frac{1}{10}$. Since Ω has a covering

$$\Omega \subset \cup_{y \in \Omega} B_y(2r_y),$$

thanks to the Vitali covering lemma (see [Lin and Yang 2002, §1.3]), there exists a countable family of disjoint balls $\{B_{y_i}(2r_i)\}$ such that $\Omega \subset \cup_i B_{y_i}(10r_i)$. On one hand, choosing ε_1 such that $\bar{\kappa}(p, a, f, r) \leq \varepsilon_0$ for all $r \leq 1$, and using the volume doubling property (2-5) leads to

$$\begin{aligned} \operatorname{vol}_f \Omega \leq \sum_i \operatorname{vol}_f B_{y_i}(10r_i) &\leq 2 \cdot 10^n \cdot \sum_i e^{10 r_i a} \operatorname{vol}_f B_{y_i}(r_i) \\ &\leq 2 \cdot 10^n \cdot e^a \sum_i \operatorname{vol}_f B_{y_i}(r_i). \end{aligned} \tag{3-14}$$

Choosing ε_1 as in Lemma 3.5 and using the disjoint of the balls $\{B_{x_i}(2r_i)\}$ gives

$$\text{vol}_f\,\partial\Omega \geq \sum_i \text{vol}_f(B_{y_i}(2r_i)\cap S) \geq 2^{-(n+3)}e^{-a}\sum_i(\text{vol}_f\,B_{y_i}(r_i))r_i^{-1}. \tag{3-15}$$

Combining (3-14) with (3-15) we obtain

$$\begin{aligned}\frac{\text{vol}_f\,\partial\Omega}{(\text{vol}_f\,\Omega)^{\frac{n-1}{n}}} &\geq 10^{-(n-1)}2^{-(n+4-\frac{1}{n})}e^{-2a+\frac{a}{n}}\frac{\sum_i(\text{vol}_f\,B_{y_i}(r_i))r_i^{-1}}{\left(\sum_i(\text{vol}_f\,B_{y_i}(r_i))\right)^{\frac{n-1}{n}}}\\ &\geq 2^{-n-1}\cdot 10^{-n}e^{-2a+\frac{a}{n}}\frac{\sum_i(\text{vol}_f\,B_{y_i}(r_i))r_i^{-1}}{\sum_i(\text{vol}_f\,B_{y_i}(r_i))^{\frac{n-1}{n}}}\\ &\geq 2^{-1}\cdot 10^{-2n}e^{-2a+\frac{a}{n}}\inf_i\frac{(\text{vol}_f\,B_{y_i}(r_i))r_i^{-1}}{(\text{vol}_f\,B_{y_i}(r_i))^{\frac{n-1}{n}}}\\ &\geq 2^{-1}\cdot 10^{-2n}e^{-2a+\frac{a}{n}}\inf_i[r_i^{-1}\,\text{vol}_f^{\frac{1}{n}}\,B_{y_i}(r_i)].\end{aligned} \tag{3-16}$$

On the other hand, since $d(y_i, y_0)\leq r_0$, then $B_{y_0}(r_0)\subset B_{y_i}(2r_0)$. Using the volume doubling property (2-5) with $r_i\leq\frac{1}{10}$ and (3-13) yields

$$\text{vol}_f\,B_{y_i}(r_i)\geq\tfrac{1}{2}(10r_i)^n e^{-\frac{a}{10}}\,\text{vol}_f\,B_{y_i}\left(\tfrac{1}{10}\right)\geq(10r_i)^n e^{-\frac{a}{10}}\,\text{vol}_f\,B_{y_0}(r_0).$$

Inserting the above inequality into (3-16), we obtain

$$\frac{\text{vol}_f\,\partial\Omega}{(\text{vol}_f\,\Omega)^{\frac{n-1}{n}}}\geq 5\cdot 10^{-2n}e^{-2a+\frac{a}{n}-\frac{a}{10n}}\,\text{vol}_f^{\frac{1}{n}}\,B_{y_0}(r_0)\geq 10^{-2n}e^{-2a}\,\text{vol}_f^{\frac{1}{n}}\,B_{y_0}(r_0).$$

Hence,

$$(\text{vol}_f\,B_{y_0}(r_0))^{-\frac{1}{n}}\inf_\Omega\left\{\frac{\text{vol}_f\,\partial\Omega}{(\text{vol}_f\,\Omega)^{\frac{n-1}{n}}}\right\}\geq 10^{-2n}e^{-2a}.$$

Our task now is to show (1-1) holds for any radius $R\leq 1$ and for $\bar\kappa(p, f, a, 1)\leq\varepsilon_2$ with $\varepsilon_2=\varepsilon_2(n, p, a)>0$. Let $r_1=R/r_0\leq 1/r_0$. After a scaling, its sufficient to check that $\bar\kappa(p, f, a, r_1)\leq\varepsilon_1$. Choose ε_2 satisfying $\varepsilon_2\leq\varepsilon_0$, and then (2-5) holds for all $R\leq 1$. Now if $r_1\leq 1$, by (2-6),

$$\bar\kappa(p, f, a, r_1)\leq 2^{\frac{1}{p}}e^{\frac{a}{p}}\bar\kappa(p, f, a, 1)\leq 2^{\frac{1}{p}}e^{\frac{a}{p}}\varepsilon_2.$$

If $1<r_1\leq 1/r_0$, then by (2-7) we have

$$\bar\kappa(p, f, a, r_1)\leq 2^{\frac{n+1}{p}}e^{\frac{ar_1}{p}}\bar\kappa(p, f, a, 1)\leq 2^{\frac{n+1}{p}}e^{\frac{a}{pr_0}}r_0^{-2}\varepsilon_2.$$

Using the two cases, let

$$\varepsilon_2=\min\{2^{-\frac{1}{p}}e^{-\frac{a}{p}}\varepsilon_1, 2^{-\frac{n+1}{p}}e^{-\frac{a}{pr_0}}r_0^2\varepsilon_1, \varepsilon_0\}.$$

The proof is completed by setting $\varepsilon=\varepsilon_2$. □

Theorem A.1 implies that the normalized constants in (3-1) are equivalent, that is,

$$\mathrm{ID}^*_{n,f} B_x(R) = \mathrm{SD}^*_{n,f} B_x(R).$$

Hence, Theorem 1.1 gives the following Sobolev inequality:

Corollary 3.7. *If $\bar{\kappa}(p, f, a, 1) \le \varepsilon$ for the ε in Theorem 1.1, then for any $R \le 1$,*

$$\|\nabla h\|^*_{1,f,B_x(R)} \ge 10^{-2n} e^{-2a} R^{-1} \|h\|^*_{\frac{n}{n-1},f,B_x(R)} \quad \forall h \in C_0^\infty(B_x(R)) \tag{3-17}$$

and

$$\|\nabla h\|^*_{2,f,B_x(R)} \ge \frac{n-2}{2(n-1)} 10^{-2n} e^{-2a} R^{-1} \|h\|^*_{\frac{2n}{n-2},f,B_x(R)} \quad \forall h \in C_0^\infty(B_x(R)). \tag{3-18}$$

Applying (3-17) to $h^{\frac{2(n-1)}{n-2}}$, together with the Hölder inequality, we get (3-18).

The first eigenvalue of the f-Laplacian is defined by

$$\lambda_1(B_x(R)) = \inf_{h \in C_0^\infty(B_x(R))} \frac{\int_{B_x(R)} |\nabla h|^2 \, d\,\mathrm{vol}_f}{\int_{B_x(R)} h^2 \, d\,\mathrm{vol}_f}.$$

As in Cheeger's inequality [1970] we have:

Corollary 3.8. *With the same assumption as in Theorem 1.1, for $p > \frac{1}{2}n$, there exists $\varepsilon = \varepsilon(n, p, a) > 0$ such that $\bar{\kappa}(p, f, a, 1) \le \varepsilon$, then for any $R \le 1$, the first eigenvalue of the Dirichlet f-Laplacian has lower bound*

$$\lambda_1(B_x(R)) \ge C(n, p, a) R^{-2}, \quad C(n, p, a) = \frac{(n-2)^2}{4(n-1)^2} 10^{-4n} e^{-4a}.$$

Proof. Suppose $\Delta_f h = -\lambda h$ for some $\lambda > 0$, and normalize h such that

$$\fint_{B_x(R)} h^2 e^{-f} \, d\,\mathrm{vol} = 1$$

and $h = 0$ on $\partial B_x(R)$. Then using (3-18) we have

$$\begin{aligned}
\lambda = (\|\nabla h\|^*_{2,f,B_x(R)})^2 &\ge \Big(\frac{n-2}{2(n-1)} 10^{-2n} e^{-2a} R^{-1} \|h\|^*_{\frac{2n}{n-2},f,B_x(R)}\Big)^2 \\
&\ge \frac{(n-2)^2}{4(n-1)^2} 10^{-4n} e^{-4a} R^{-2} (\|h\|^*_{2,f,B_x(R)})^2 \\
&= \frac{(n-2)^2}{4(n-1)^2} 10^{-4n} e^{-4a} R^{-2}.
\end{aligned}$$

□

4. Applications

In this section, we prove the maximum principle and the gradient estimate for integral Bakry–Émery Ricci curvature with the help of the normalized local Dirichlet Sobolev constant estimate.

Let $C_s(\Omega)$ be the normalized local Sobolev constant of $\Omega \subset B_x(R) \subset M_f^n$ such that

$$\|h\|^*_{\frac{2n}{n-2},f,\Omega} \le C_s(\Omega)\|\nabla h\|^*_{2,f,\Omega} \quad \forall h \in C_0^\infty(\Omega). \tag{4-1}$$

Obviously, $C_s(\Omega)$ is the smallest constant such that (4-1) holds for all $h \in C_0^\infty(\Omega)$. Since $h \in C_0^\infty(B_x(R))$, then (3-18) gives

$$C_s(\Omega) \le C_s(B_x(R)) \le \frac{2(n-1)}{n-2} 10^{2n} e^{2a} R. \tag{4-2}$$

Theorem 4.1. *Let M_f^n be a smooth metric measure space and $\Omega \subset B_x(R) \subset M_f^n$ be a domain. For $p > \frac{1}{2}n$ and any function u with $u|_{\partial\Omega} = 0$, we have*

$$\|u\|^*_{\infty,f,\Omega} \le C_s^2(B_x(R))C(n,p)\|\Delta_f u\|^*_{p,f,B_x(R)},$$

where $C(n,p)$ is a constant depending on n and p.

The proof of this result is quite similar to the one used in [Petersen and Wei 2001, Theorem 3.1] with $s = \frac{1}{2}n$ due to the Sobolev inequality and the self-adjoint of Δ_f, and we omit it.

Corollary 4.2. *With the same assumption as in Theorem 4.1, for $p > \frac{1}{2}n$ and any function $u : \Omega \subset M_f^n \to \mathbb{R}$ with $\Delta_f u \ge -h$, where h is nonnegative on Ω, we have*

$$\sup_\Omega u \le \sup_{\partial\Omega} u + C(n,p) \cdot C_s^2(B_x(R)) \cdot \|h\|^*_{p,f,\Omega}.$$

Proof. Without loss of generality, we can assume that $\sup_{x\in\partial\Omega} u(x) = 0$. Then we have the Dirichlet problem

$$\Delta_f v = -h \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega. \tag{4-3}$$

Hence $u - v$ is f-subharmonic, and $u - v \le 0$ on $\partial\Omega$. By the maximum principle we get $u \le v$ in $\overline{\Omega}$, that is $\sup_\Omega u \le \|v\|_{\infty,f,\Omega}$. Using Theorem 4.1 we complete the proof. □

Combining Corollary 4.2 with (4-2) gives Theorem 1.4.

Following the idea in [Dai et al. 2018, Theorem 5.2], we use the standard and powerful Nash–Moser iteration and establish the gradient estimate. We give the details of the proof because there are still many differences compared to the previous arguments.

Theorem 4.3. *Let M_f^n be a complete smooth metric measure space. Assume that M_f^n satisfies $\partial_r f \geq -a$ along all minimal geodesic segments for some constant $a \geq 0$. For $p > \frac{1}{2}n$, if u is a function on $B_x(R)$ with $u = 0$ on $\partial B_x(R)$ and satisfying*

$$\Delta_f u = h,$$

then

$$\begin{aligned}\sup_{B_x(\frac{R}{2})} |\nabla u|^2 \leq C(n,p)R^{-2}\frac{\operatorname{vol}_f B_x(R)}{\operatorname{vol}_f B_x(\frac{3}{4}R)}\big((\|h\|^*_{2p,f,B_x(R)})^2 + (\|u\|^*_{2,f,B_x(R)})^2\big)\\ \times\big(R^{-2}C_s^2(B_x(R))(1+a+R^{-2}C_s^2(B_x(R))e^{\frac{a}{p}\bar{\kappa}}(p,f,a,R))\\ + (R^{-2}C_s^2(B_x(R))e^{\frac{a}{p}\bar{\kappa}}(p,f,a,R))^{\frac{2p}{2p-n}}\big)^{\frac{n}{2}}.\end{aligned}$$

Proof. By scaling we assume $R = 1$. We omit the volume form $e^{-f}d\,\mathrm{vol}$ for convenience. From the Bochner formula, we have

$$\begin{aligned}\Delta_f|\nabla u|^2 &= 2|\operatorname{Hess} u|^2 + 2\langle\nabla u, \nabla\Delta_f u\rangle + 2\mathrm{Ric}_f(\nabla u,\nabla u)\\ &\geq 2\langle\nabla u,\nabla h\rangle - 2|\mathrm{Ric}_{f-}||\nabla u|^2.\end{aligned}\tag{4-4}$$

Let $v = |\nabla u|^2 + \|h\|^*_{p,f}$ and $v = |\nabla u|^2$ if h is constant. For any $\eta \in C_0^\infty(B_x(1))$, $l > 1$, we have

$$\begin{aligned}\int|\nabla(\eta v^l)|^2 &= -\int(\eta v^l)[v^l\Delta_f\eta + 2\langle\nabla\eta,\nabla v^l\rangle + \eta\Delta_f v^l]\\ &= \int v^{2l}(-\eta\Delta_f\eta) - 2\int v^l\langle\nabla\eta,\eta\nabla v^l\rangle\\ &\quad - l\int\eta^2 v^{2l-1}\Delta_f v - l(l-1)\int\eta^2 v^{2l-2}|\nabla v|^2\\ &\leq \int v^{2l}(-\eta\Delta_f\eta) + \frac{l}{l-1}\int v^{2l}|\nabla\eta|^2 + \frac{l-1}{l}\int\eta^2|\nabla v^l|^2\\ &\quad - l\int\eta^2 v^{2l-1}(2\langle\nabla u,\nabla h\rangle - 2|\mathrm{Ric}_{f-}|v) - \frac{l-1}{l}\int\eta^2|\nabla v^l|^2\\ &= \int v^{2l}(-\eta\Delta_f\eta) + \frac{l}{l-1}\int v^{2l}|\nabla\eta|^2\\ &\quad - 2l\int\eta^2 v^{2l-1}\langle\nabla u,\nabla h\rangle + 2l\int\eta^2 v^{2l}|\mathrm{Ric}_{f-}|.\end{aligned}\tag{4-5}$$

Here we used Young's inequality $2xy \leq \varepsilon x^2 + \frac{1}{\varepsilon}y^2$ with $\varepsilon = \frac{l}{l-1}$ and (4-4) in the

above inequality. Integrating by parts and using $|\nabla u| \le v^{1/2}$ gives

$$\begin{aligned}
&-2l\int \eta^2 v^{2l-1}\langle \nabla u, \nabla h\rangle \\
&\quad = 2l\int \eta^2 v^{2l-1}h^2 + 4l\int \eta h v^{2l-1}\langle \nabla \eta, \nabla u\rangle \\
&\qquad\qquad + 2l(2l-1)\int \eta^2 h v^{2l-2}\langle \nabla u, \nabla v\rangle \\
&\quad \le 2l\int \eta^2 v^{2l-1}h^2 + 2\int (2\sqrt{l}\eta v^{l-\frac{1}{2}}h)(\sqrt{l}v^l|\nabla\eta|) \\
&\qquad\qquad + 2\int ((4l-2)\eta v^{l-\frac{1}{2}}h)\big(\tfrac{1}{2}\eta|\nabla v^l|\big) \\
&\quad = [6l+(4l-2)^2]\int \eta^2 v^{2l-1}h^2 \\
&\qquad\qquad + l\int v^{2l}|\nabla\eta|^2 + \tfrac{1}{4}\int |\nabla(\eta v^l) - v^l\nabla\eta|^2 \\
&\quad \le (16l^2-10l+4)\int \eta^2 v^{2l-1}h^2 + \big(l+\tfrac{1}{2}\big)\int v^{2l}|\nabla\eta|^2 + \tfrac{1}{2}|\nabla(\eta v^l)|^2.
\end{aligned}$$

Here we used the L^2-Hölder inequality and

$$\eta^2|\nabla v^l|^2 = |\nabla(\eta v^l) - v^l\nabla\eta|^2 \le 2|\nabla(\eta v^l)|^2 + 2v^{2l}|\nabla\eta|^2.$$

Inserting (4-6) into (4-5) and regrouping gives

$$\begin{aligned}
\int |\nabla(\eta v^l)|^2 &\le \int v^{2l}(-2\eta\Delta_f\eta) + \Big(\frac{2l}{l-1}+2l+1\Big)\int v^{2l}|\nabla\eta|^2 \\
&\quad + 4(8l^2-5l+2)\int \eta^2 v^{2l-1}h^2 + 4l\int \eta^2 v^{2l}|\mathrm{Ric}_{f-}|.
\end{aligned} \tag{4-6}$$

In order to control $\Delta_f\eta$, choose a cut-off function $\phi \in C_0^\infty(B_x(1))$ such that $0 \le \phi \le 1$. For $0 < r < 1$, $\phi(t) \equiv 1$; for $t \in [0, r]$, $\phi(t) \equiv 0$; for $t \ge 1$, and $\phi' \le 0$. Then define $\eta(y) = \phi(r(y))$ and let $r(y) = d(x, y)$ be a distance function from x. Thus $|\nabla\eta| = |\phi'|$, and

$$\begin{aligned}
\Delta_f\eta &= \phi'' + \phi'\Delta_f r \ge \phi'' + \phi'\Big(\psi + \frac{n-1}{r} + a\Big) \\
&\ge -|\phi''| - |\phi'|\psi - \frac{n-1}{r}|\phi'| - a|\phi'|,
\end{aligned} \tag{4-7}$$

where $\psi = (\Delta_f r - (n-1)/r - a)_+$. Hence, for $l \ge n/(n-2)$, (4-6) becomes

$$\begin{aligned}
\int |\nabla(\eta v^l)|^2 \le C(n)l^2\int \big((|\phi''| + |\phi'|r^{-1} + a)\eta v^{2l} + |\phi'|\psi\eta v^{2l} \\
+ |\phi'|^2 v^{2l} + \eta^2 h^2 v^{2l-1} + \eta^2 v^{2l}|\mathrm{Ric}_{f-}|\big).
\end{aligned}$$

Notice that this formula remains valid for $l = 1$. Indeed,

$$|\nabla(\eta v^{\frac{1}{2}})|^2 = \left| v^{\frac{1}{2}}\nabla\eta + \eta\frac{|\nabla u|}{v^{1/2}}\nabla|\nabla u| \right|^2 \le 2v|\nabla\eta|^2 + 2\eta^2|\operatorname{Hess} u|^2,$$

and

$$\begin{aligned}\int \eta^2|\operatorname{Hess} u|^2 &= -\int \nabla_i u(2\eta\nabla_j\nabla_i\nabla_j u + \eta^2\nabla_i\Delta_f u + \eta^2(\mathrm{Ric}_f)_{ij}\nabla_j u)\\ &\le \tfrac{1}{2}\int \eta^2|\operatorname{Hess} u|^2 + 3\int|\nabla\eta|^2 v + 2\int \eta^2 h^2 + \int \eta^2|\mathrm{Ric}_{f-}|v.\end{aligned}$$

Next we use C_s denote $C_s(B_x(1))$ for simplicity. Letting $\beta = n/(n-2)$, and applying the Sobolev inequality (4-1), then for $l \ge n/(n-2)$ and $l = 1$,

$$\begin{aligned}(4\text{-}8)\quad &\left(\fint_{B_x(1)}(\eta^2 v^{2l})^\beta\right)^{\frac{1}{\beta}}\\ &\le C_s^2\fint_{B_x(1)}|\nabla(\eta v^l)|^2\\ &\le C_s^2 C(n)l^2\fint_{B_x(1)}(|\phi''|+|\phi'|r^{-1}+a)\eta v^{2l}\\ &\quad + C_s^2 C(n)l^2\fint_{B_x(1)}(|\phi'|\psi\eta v^{2l}+|\phi'|^2 v^{2l}+\eta^2 h^2 v^{2l-1}+\eta^2 v^{2l}|\mathrm{Ric}_{f-}|).\end{aligned}$$

The integration involving Bakry–Émery Ricci curvature can be estimated as follows:

$$\begin{aligned}&\fint_{B_x(1)}\eta^2 v^{2l}|\mathrm{Ric}_{f-}|\\ &\quad\le \|\mathrm{Ric}_{f-}\|^*_{p,f,B_x(1)}\left(\fint_{B_x(1)}(\eta^2 v^{2l})^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\\ &\quad\le e^{\frac{a}{p}}\|\mathrm{Ric}_{f-}\|^*_{p,a,f}(1)\left(\fint_{B_x(1)}\eta^2 v^{2l}\right)^{\frac{p-1}{p}q}\left(\fint_{B_x(1)}(\eta^2 v^{2l})^\beta\right)^{\frac{p-1}{p}(1-q)}\\ &\quad\le e^{\frac{a}{p}}\bar{\kappa}(p,f,a,1)\left[\varepsilon\left(\fint_{B_x(1)}(\eta^2 v^{2l})^\beta\right)^{\frac{1}{\beta}} + \varepsilon^{-\frac{n}{2p-n}}\left(\fint_{B_x(1)}\eta^2 v^{2l}\right)\right],\end{aligned}$$

where $q = q(n,p) = (2p-n)/(2(p-1)) > 0$ is determined by $q + (1-q)\beta = p/(p-1)$, and we also used the Young's inequality

$$xy \le \varepsilon x^b + \varepsilon^{-\frac{b^*}{b}}y^{b^*} \quad \forall x, y \ge 0,\ b > 1, \quad \frac{1}{b^*} + \frac{1}{b} = 1,$$

where

$$b = \frac{p}{(1-q)(p-1)\beta}, \quad b^* = \frac{p}{(p-1)q}.$$

By choosing $\varepsilon = (4C_s^2C(n)l^2e^{\frac{a}{p}}\bar{\kappa}(p,f,a,1))^{-1}$, we obtain

$$
\begin{aligned}
(4\text{-}9)\quad & C_s^2C(n)l^2 \fint_{B_x(1)} \eta^2 v^{2l} |\mathrm{Ric}_{f-}| \\
&\le \frac{1}{4}\left(\fint_{B_x(1)} (\eta^2 v^{2l})^{\beta} \right)^{\frac{1}{\beta}} \\
&\qquad + C(n,p)(C_s^2 l^2 e^{\frac{a}{p}}\bar{\kappa}(p,f,a,1))^{\frac{2p}{2p-n}} \fint_{B_x(1)} \eta^2 v^{2l}.
\end{aligned}
$$

For the term $\fint_{B_x(1)} \eta^2 h^2 v^{2l-1}$, since $v \ge \|h^2\|^*_{p,f,B_x(1)}$, we have

$$
\fint_{B_x(1)} \eta^2 h^2 v^{2l-1} \le \frac{1}{\|h^2\|^*_{p,f,B_x(1)}} \fint_{B_x(1)} \eta^2 h^2 v^{2l} \le \left(\fint_{B_x(1)} (\eta^2 h^2 v^{2l})^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}}.
$$

Now the same argument as above with $\varepsilon = (4C_s^2C(n)l^2)^{-1}$ gives

$$
\begin{aligned}
(4\text{-}10)\quad & C_s^2C(n)l^2 \fint_{B_x(1)} \eta^2 h^2 v^{2l-1} \\
&\le \frac{1}{4}\left(\fint_{B_x(1)} (\eta^2 v^{2l})^{\beta} \right)^{\frac{1}{\beta}} + C(n,p)(C_s^2 l^2)^{\frac{2p}{2p-n}} \fint_{B_x(1)} \eta^2 v^{2l}.
\end{aligned}
$$

For the term with ψ, applying the Hölder inequality, (1-3) and the Laplacian comparison (2-3) gives

(4-11)

$$
\begin{aligned}
& C_s^2C(n)l^2 \fint_{B_x(1)} |\phi'|\psi\eta v^{2l} \\
&\le C_s^2C(n)l^2 \|\psi\|^*_{2p,f,B_x(1)} \cdot \|\eta\phi' v^{2l}\|^*_{\frac{2p}{2p-1},f,B_x(1)} \\
&\le C_s^2C(n)l^2 e^{\frac{a}{2p}} \|\psi\|^*_{2p,a,f}(1) \cdot \|\eta\phi' v^{2l}\|^*_{\frac{2p}{2p-1},f,B_x(1)} \\
&\le C_s^2C(n)l^2 e^{\frac{a}{2p}} C(n,p)(\bar{\kappa}(p,f,a,1))^{\frac{1}{2}} \cdot \|\eta\phi' v^{2l}\|^*_{\frac{2p}{2p-1},f,B_x(1)}.
\end{aligned}
$$

Note that for

$$
\alpha = \frac{p(n-2)}{n(2p-1)} = \frac{1}{\beta}\frac{p}{2p-1} < 1,
$$

we have

$$
\begin{aligned}
(4\text{-}12)\quad &\|\eta\phi' v^{2l}\|^*_{\frac{2p}{2p-1}} \\
&= \left[\fint_{B_x(1)} (\eta^2 v^{2l})^{\alpha\beta} (|\phi'|^2 v^{2l})^{\frac{p}{2p-1}} \right]^{\frac{2p-1}{2p}} \\
&\le \left[\left(\fint_{B_x(1)} (\eta^2 v^{2l})^{\beta} \right)^{\alpha} \left(\fint_{B_x(1)} (|\phi'|^2 v^{2l})^{\frac{np}{np+2p-n}} \right)^{\frac{np+2p-n}{n(2p-1)}} \right]^{\frac{2p-1}{2p}} \\
&\le \left[\left(\fint_{B_x(1)} (\eta^2 v^{2l})^{\beta} \right)^{\alpha} \left(\fint_{B_x(1)} |\phi'|^2 v^{2l} \right)^{\frac{p}{2p-1}} \right]^{\frac{2p-1}{2p}} \\
&\le \varepsilon \left(\fint_{B_x(1)} (\eta^2 v^{2l})^{\beta} \right)^{\frac{1}{\beta}} + \frac{1}{4\varepsilon} \fint_{B_x(1)} |\phi'|^2 v^{2l},
\end{aligned}
$$

where we used the Hölder inequality; note that $p > \frac{1}{2}n$ implies $np/(np+2p-n) < 1$ in the second inequality. By setting

$$
\varepsilon = (4C_s^2 C(n) l^2 e^{\frac{a}{2p}} C(n,p)(\bar{\kappa}(p,f,a,1))^{1/2})^{-1}
$$

and inserting (4-12) into (4-11) we obtain

$$
\begin{aligned}
(4\text{-}13)\quad C_s^2 C(n) l^2 \fint_{B_x(1)} \psi\eta |\phi'| v^{2l} &\le \frac{1}{4} \left(\fint_{B_x(1)} (\eta^2 v^{2l})^{\beta} \right)^{\frac{1}{\beta}} \\
&\quad + (C_s^4 C^2(n) l^4 e^{\frac{a}{p}} C^2(n,p) \bar{\kappa}(p,f,a,1)) \fint_{B_x(1)} |\phi'|^2 v^{2l}.
\end{aligned}
$$

Inserting (4-9), (4-10) and (4-13) into (4-8) gives

$$
\begin{aligned}
(4\text{-}14)\quad &\left(\fint_{B_x(1)} (\eta^2 v^{2l})^{\beta} \right)^{\frac{1}{\beta}} \\
&\le 4C_s^2 C(n) l^2 \fint_{B_x(1)} \left(|\phi''| + \frac{|\phi'|}{r} + a \right) \eta v^{2l} \\
&\quad + 4C_s^2 C(n) l^2 (1 + C_s^2 l^2 e^{\frac{a}{p}} C^2(n,p) \bar{\kappa}(p,f,a,1)) \fint_{B_x(1)} |\phi'|^2 v^{2l} \\
&\quad + C(n,p)(C_s^2 l^2)^{\frac{2p}{2p-n}} [1 + (e^{\frac{a}{p}} \bar{\kappa}(p,f,a,1))^{\frac{2p}{2p-n}}] \left(\fint_{B_x(1)} \eta^2 v^{2l} \right).
\end{aligned}
$$

Define $l_i = \frac{1}{2}\beta^i$, $i \ge 0$, and $r_i = \frac{3}{4} - \sum_{j=0}^{i} 2^{-j-1}$. Choose the cut-off functions $\eta_i = \phi_i(r) \in C_0^\infty(B_x(r_i))$ such that

$$
\eta_i \equiv 1 \quad \text{on } B_x(r_{i+1}); \quad |\phi_i'| \le 2^{i+1}, \quad |\phi_i''| \le 2^{2i+2}.
$$

Then (4-14) becomes

$$\begin{aligned}
&\|v\|^*_{\beta^{i+1},f,B_x(r_{i+1})}\\
&\quad\le C(n,p)\big\{C_s^2(\beta^{4i})(1+a+C_s^2e^{\frac{a}{p}}\bar\kappa(p,f,a,1))\\
&\qquad\qquad+(C_s^2\beta^{2i})^{\frac{2p}{2p-n}}[1+(e^{\frac{a}{p}}\bar\kappa(p,f,a,1))^{\frac{2p}{2p-n}}]\big\}\|v\|^*_{\beta^i,f,B_x(r_i)}\\
&\quad\le C(n,p)\beta^{si}\big\{C_s^2(1+a+C_s^2e^{\frac{a}{p}}\bar\kappa(p,f,a,1))\\
&\qquad\qquad+C_s^{\frac{4p}{2p-n}}[1+(e^{\frac{a}{p}}\bar\kappa(p,f,a,1))^{\frac{2p}{2p-n}}]\big\}\|v\|^*_{\beta^i,f,B_x(r_i)},
\end{aligned}$$

where $s=\max\{4,4p/(2p-n)\}$. Then substituting η_i into the estimate and running the iteration from $i=0$ gives

$$\|v\|^*_{\infty,B_x(\frac12)}\le C(n,p)A^{\frac{n}{2}}\|v\|^*_{1,f,B_x(\frac34)},\tag{4-15}$$

where

$$A=C_s^2(1+a+C_s^2e^{\frac{a}{p}}\bar\kappa(p,f,a,1))+C_s^{\frac{4p}{2p-n}}[1+(e^{\frac{a}{p}}\bar\kappa(p,f,a,1))^{\frac{2p}{2p-n}}].$$

Finally, we estimate the term $\|v\|^*_{1,f,B_x(3/4)}$. For $\eta\in C_0^\infty(B_x(1))$ with $\eta\equiv1$ in $B_x\big(\frac34\big)$ and $|\nabla\eta|\le5$, then

$$\begin{aligned}
\|v\|^*_{1,f,B_x(\frac34)}&\le\frac{\operatorname{vol}_f B_x(1)}{\operatorname{vol}_f B_x\big(\frac34\big)}\|v\|^*_{1,f,B_x(1)}\\
&=\frac{\operatorname{vol}_f B_x(1)}{\operatorname{vol}_f B_x\big(\frac34\big)}\fint_{B_x(1)}\eta^2(|\nabla u|^2+\|h\|^*_{p,f,B_x(1)}).
\end{aligned}\tag{4-16}$$

Using the integration by parts and Young's inequality, we have

$$\begin{aligned}
\fint_{B_x(1)}\eta^2|\nabla u|^2&=-2\fint_{B_x(1)}\eta\langle\nabla\eta,\nabla u\rangle u-\fint_{B_x(1)}\eta^2hu\\
&\le\frac12\fint_{B_x(1)}\eta^2|\nabla u|^2+2\fint_{B_x(1)}|\nabla\eta|^2u^2+\frac12\fint_{B_x(1)}(\eta^2h^2+\eta^2u^2).
\end{aligned}$$

Regrouping the above inequality and inserting it into (4-16) gives

$$\begin{aligned}
\|v\|^*_{1,f,B_x(\frac34)}&=\frac{\operatorname{vol}_f B_x(1)}{\operatorname{vol}_f B_x\big(\frac34\big)}\Big[4\fint_{B_x(1)}|\nabla\eta|^2u^2+\fint_{B_x(1)}\eta^2u^2\\
&\qquad\qquad+\fint_{B_x(1)}\eta^2h^2+\|h^2\|^*_{p,f,B_x(1)}\Big]\\
&\le\frac{\operatorname{vol}_f B_x(1)}{\operatorname{vol}_f B_x\big(\frac34\big)}[101(\|u\|^*_{2,f,B_x(1)})^2+2(\|h\|^*_{2p,f,B_x(1)})^2].
\end{aligned}\tag{4-17}$$

Combining (4-15) with (4-17) yields

$$\sup_{B_x(\frac{1}{2})} |\nabla u|^2 \le C(n,p)\frac{\mathrm{vol}_f B_x(1)}{\mathrm{vol}_f B_x(\frac{3}{4})}[(\|h\|^*_{2p,f,B_x(1)})^2 + (\|u\|^*_{2,f,B_x(1)})^2]$$
$$\times \{C_s^2(1+a+C_s^2 e^{\frac{a}{p}}\bar{\kappa}(p,f,a,1)) + C_s^{\frac{4p}{2p-n}}[1+(e^{\frac{a}{p}}\bar{\kappa}(p,f,a,1))^{\frac{2p}{2p-n}}]\}^{\frac{n}{2}}.$$

Thus we get the desired result by scaling. □

Combining Theorem 4.3 and the local Sobolev constant estimate (4-2) with the volume doubling property (3-18) gives Theorem 1.5.

Appendix: Equivalence of isoperimetric and Sobolev constants for weighted measure

Here we show the equivalence between the local isoperimetric and Sobolev constants defined in Section 3 by adapting the proof of [Li 2012, Theorem 9.5]. A special case of Theorem A.1 can be found in [Cheng and Oden 1997, Proposition 1.1].

Theorem A.1. *For all $n \le \alpha \le \infty$, we have*

$$\mathrm{ID}_{\alpha,f} B_x(R) = \mathrm{SD}_{\alpha,f} B_x(R). \tag{A-1}$$

Proof. Let σ_f be an $(n-1)$-dimensional Hausdorff measure. We omit the weighted measure $e^{-f}d\,\mathrm{vol}$ for convenience. For $\Omega \subset B_x(R)$ with $\Omega \cap \partial B_x(R) = \varnothing$, let $\Omega_\varepsilon = \{y \in \Omega : d(y, \partial\Omega) \ge \varepsilon\}$. Construct a function by

$$h_\varepsilon(y) = \begin{cases} 0 & B_x(R)\setminus\Omega, \\ (1/\varepsilon)d(y,\partial B_x(R)) & \Omega\setminus\Omega_\varepsilon, \\ 1, & \Omega_\varepsilon. \end{cases}$$

Since the distance function d is Lipschitz, h_ε is Lipschitz with $h_\varepsilon|_{\partial\Omega} = 0$. Applying the Sobolev inequality to h_ε gives

$$\int_{B_x(R)} |\nabla h_\varepsilon| \ge \mathrm{SD}_{\alpha,f} B_x(R)\left(\int_{B_x(R)} |h_\varepsilon|^{\frac{\alpha}{\alpha-1}}\right)^{\frac{\alpha-1}{\alpha}}$$
$$\ge \mathrm{SD}_{\alpha,f} B_x(R)(\mathrm{vol}_f \Omega_\varepsilon)^{\frac{\alpha-1}{\alpha}}. \tag{A-2}$$

The coarea formula implies

$$\lim_{\varepsilon\to 0}\int_{B_x(R)} |\nabla h_\varepsilon| e^{-f}\, d\,\mathrm{vol} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon \sigma_f(\partial\Omega_t)\,dt = \sigma_f(\partial\Omega). \tag{A-3}$$

Combining (A-2) with (A-3), together with Definition 3.1, we have

$$\mathrm{ID}_{\alpha,f} B_x(R) \ge \mathrm{SD}_{\alpha,f} B_x(R).$$

To see that $\mathrm{ID}_{\alpha,f} B_x(R) \le \mathrm{SD}_{\alpha,f} B_x(R)$, it suffices to show that

$$\int_{B_x(R)} |\nabla h| \ge \mathrm{ID}_{\alpha,f} B_x(R) \Big(\int_{B_x(R)} h^{\frac{\alpha}{\alpha-1}} \Big)^{\frac{\alpha-1}{\alpha}} \tag{A-4}$$

holds for $h|_{\partial B_x(R)} = 0$. Without loss of generality, we may assume $h \ge 0$. Let $B_t := \{y \in B_x(R) \mid h(y) > t\}$ to be the sublevel set of h. By the coarea formula,

$$\int_{B_x(R)} |\nabla h| = \int_0^\infty \sigma_f(\partial B_t)\, dt \ge \mathrm{ID}_{\alpha,f} B_x(R) \int_0^\infty \mathrm{vol}_f^{\frac{\alpha-1}{\alpha}} B_t \, dt. \tag{A-5}$$

Let

$$F(s) = \Big(\int_0^s \mathrm{vol}_f^{\frac{\alpha-1}{\alpha}} B_t \, dt \Big)^{\frac{\alpha}{\alpha-1}} - \frac{\alpha}{\alpha-1} \int_0^s t^{\frac{1}{\alpha-1}} \mathrm{vol}_f B_t \, dt.$$

Obviously, $F(0) = 0$ and

$$F'(s) = \frac{\alpha}{\alpha-1} \Big(\int_0^s \mathrm{vol}_f^{\frac{\alpha-1}{\alpha}} B_t \, dt \Big)^{\frac{1}{\alpha-1}} \mathrm{vol}_f^{\frac{\alpha-1}{\alpha}} B_s - \frac{\alpha}{\alpha-1} s^{\frac{1}{\alpha-1}} \mathrm{vol}_f B_s.$$

Since $B_s \subset B_t$ for $t \le s$, then $\int_0^s \mathrm{vol}_f^{\frac{\alpha-1}{\alpha}} B_t dt \ge s \,\mathrm{vol}_f^{\frac{\alpha-1}{\alpha}} B_s$ yields $F'(s) \ge 0$, hence $F(s) \ge 0$. Applying this inequality to (A-5) yields

$$\int_{B_x(R)} |\nabla h| \ge \mathrm{ID}_{\alpha,f} B_x(R) \Big(\frac{\alpha}{\alpha-1} \int_0^\infty t^{\frac{1}{\alpha-1}} \mathrm{vol}_f B_t \, dt \Big)^{\frac{\alpha-1}{\alpha}}. \tag{A-6}$$

Integrating by parts and using the coarea formula,

$$\begin{aligned} \frac{\alpha}{\alpha-1} \int_0^\infty t^{\frac{1}{\alpha-1}} \mathrm{vol}_f B_t \, dt &= \int_0^\infty \Big(\frac{d(t^{\frac{\alpha}{\alpha-1}})}{dt} \int_t^\infty \int_{\partial B_s} \frac{d\sigma_f(\partial B_s)}{|\nabla h|} \, ds \Big) dt \\ &= \int_0^\infty t^{\frac{\alpha}{\alpha-1}} \int_{\partial B_t} \frac{d\sigma_f(\partial B_t)}{|\nabla h|} \, dt = \int_{B_x(R)} h^{\frac{\alpha}{\alpha-1}}. \end{aligned}$$

Inserting the above equality into (A-6) gives (A-4). □

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