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# THE GRAPH LAPLACIAN AND MORSE INEQUALITIES

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**We provide an interpretation of the discrete version of Morse inequalities, following Witten's approach via supersymmetric quantum mechanics (*J. Differential Geom.* 17:4 (1982), 661-692), adapted by Forman to finite graphs, as a particular instance of Morse–Witten theory for cell complexes (*Topology* 37:5 (1998), 945–979). We describe the general framework of graph quantum mechanics and we produce discrete versions of the Hodge theorems and energy cut-offs within this formulation.**

## Introduction

The understanding of physical phenomena, as well as the behavior of information in networks, have been recently studied from the combinatorial and algebraic perspective.

In this paper we consider a toy version of quantum mechanics [Del Vecchio 2012; Mnëv 2016] based on a graph-theoretic analogue of the Schrödinger equation. To a finite graph we associate a *partition function*, a discretization of the Feynman path integral which can be used to count special types of paths on graphs and compute topological invariants. It relies on the discretized version of the Laplace operator

$$\Delta = \nabla^2 = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2},$$

which depends on the combinatorics of the graph. We apply this framework to Morse theory, and we are able to recover Morse inequalities, by following Witten's viewpoint of critical points of Morse functions.

The main idea can be summarized as follows: after introducing the supersymmetric version of quantum mechanics of graphs in terms of the graph Laplacian, we describe a version of Morse theory on graphs first by partially ordering the set of edges and vertices of $\Gamma$ by declaring each vertex lesser than each edge of which it is

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an endpoint. With respect to this ordering, a *discrete Morse function* is a real-valued function $f$ on the set of edges and vertices of $\Gamma$ such that, for all $\sigma \in \Gamma$,

$$\#\{\tau > \sigma \mid f(\tau) \le f(\sigma)\} \le 1,$$
$$\#\{\tau < \sigma \mid f(\tau) \le f(\sigma)\} \le 1.$$

A *critical point* $\sigma$ of $f$ is one for which the two sets above are empty.

Forman [1998] provided a combinatorial interpretation via cell complexes of Witten's approach for Morse inequalities. He shows that, given a Morse function $f$ on a CW-complex $X$, there is always an equivalent Morse function $g$ that is *flat* and *self-indexing* (see Definitions 0.7 and 1.3 in [Forman 1998]), which means:

(1) The values of $f$ are nondecreasing as the dimension increases.

(2) The values of $f$ on critical cells are given by the dimension of the corresponding cell.

It turns out that, by considering the particular case of finite graphs, the computations of the deformed Laplacian and Morse–Witten complex are explicit, regardless of whether the function is self-indexing or not. In particular we deduce that the height function on trees (Section 5) gives rise to the correct Morse complex, after the deformation procedure.

The Morse inequalities state that Betti numbers $h_0$ and $h_1$ are bounded by the number of critical vertices and critical edges, respectively. Using Theorem 1.11, we arrive at these inequalities, drawing inspiration from [Witten 1982]. The idea is as follows. Deform the supersymmetric Laplacian $\Delta$ using the Morse function $f$ with real parameter $s$ by taking boundary operator $d_s = \exp(fs)d\exp(-fs)$ and coboundary $d_s^* = \exp(-fs)d^*\exp(fs)$. The Hodge theorems still hold for the deformed Laplacian

$$\Delta_s = d_s^* d_s + d_s d_s^*,$$

and, after taking a limit $s \to \infty$, there is an energy level $a$ for which the cutoff complex $C_a^\bullet$ approaches the Morse complex as $s$ approaches infinity.

This combinatorial approach, based on the linear algebraic properties of the deformed Laplacian and incidence matrices, gives an intuitive interpretation of Witten's proof of Morse inequalities. In our description, the analytical issues of explicitly describing the spectra of deformed Laplacian operators (for which Witten requires the approximation of the operators around the critical points by using the Morse coordinates) do not exist, since the operators are finite-dimensional.

## 1. Graph quantum mechanics

In this section we introduce the combinatorial version of quantum mechanics for finite graphs.

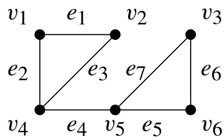


**Figure 1.** A connected unoriented graph, with two independent closed paths.

***The graph Laplacian.*** For the purpose of this paper we consider finite graphs $\Gamma = (V, E)$, i.e., a finite set $V$ of vertices and a finite set $E$ of edges $e = (v_i, v_j)$. See Figures 1 and 2, for example. We will distinguish between unoriented and oriented graphs when necessary.

**Definition 1.1.** The adjacency matrix $A_\Gamma$ (or simply $A$) of the graph $\Gamma$ is

$$A_\Gamma(i, j) = \begin{cases} 1 & \text{if } (v_i, v_j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

**Definition 1.2.** The valence matrix $\mathrm{val}_\Gamma$ of the graph $\Gamma$ is the diagonal matrix such that the entry $(i, i)$ is the number of neighbors of the vertex $v_i$.

**Definition 1.3.** The even graph Laplacian $\Delta_{+,\Gamma}$ is defined by

$$\Delta_{+,\Gamma} = \mathrm{val}_\Gamma - A_\Gamma \tag{1}$$

***Orientation on graphs.*** In order to define the incidence matrix of $\Gamma$, we choose an orientation, that is, a particular order of the pairs $(v_i, v_j)$.

**Definition 1.4.** Let $\Gamma$ be an oriented graph. The incidence matrix $I_\Gamma$ (or simply $I$) is a $|V| \times |E|$ matrix defined by

$$I_\Gamma(k, l) = \begin{cases} -1 & \text{if } e_l \text{ starts at } v_k, \\ 1 & \text{if } e_l \text{ ends at } v_k, \\ 0 & \text{otherwise.} \end{cases}$$

The following relationship between the even Laplacian and the incidence matrix follows form the combinatorial description of the even Laplacian.

**Proposition 1.5.** *The even graph Laplacian $\Delta_{+,\Gamma}$ can be written in terms of the incidence matrix as follows*: $\Delta_{+,\Gamma} = I_\Gamma I_\Gamma^*$.

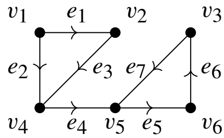


**Figure 2.** An oriented labeled graph.

Note that from Definition 1.3 it follows that the even Laplacian is independent of the orientation. Based on Proposition 1.5, we have the following definition of the odd Laplacian, which can be regarded as an operator on functions on edges.

**Definition 1.6.** The odd Laplacian $\Delta_{-,\Gamma}$ is defined by

$$\Delta_{-,\Gamma} = I_\Gamma^* I_\Gamma. \tag{2}$$

***The state evolution.***

**Definition 1.7.** An even quantum state $\Psi_+$ on a graph $\Gamma$ is a complex-valued function on the vertices $\Gamma_0$, that is, $\Psi_+ \in \mathbb{C}^{|V|}$. Similarly, an odd state $\Psi_-$ is an element of $\mathbb{C}^{|E|}$.

The quantum theory is given by the Schrödinger equation

$$\frac{\partial}{\partial t}\Psi_{+,t} = -\Delta_+ \Psi_{+,t}$$

which is solved by

$$\exp(-\Delta_+ t)\Psi_{+,0}.$$

We denote $Z(t) = \exp(-\Delta_{+,t})$, the (even) *partition function* of $\Gamma$. Indeed, if $\Gamma$ is regular, then

$$Z(t)(i, j) = \exp(tA)\exp(-\operatorname{val} t)(i, j) = \sum_{\gamma : i \to j} \frac{t^{|\gamma|}}{|\gamma|!} e^{-\operatorname{val} t},$$

which is an integral over a space of paths with measure $\frac{t^{|\gamma|}}{|\gamma|!}$ and integrand $e^{-\operatorname{val} t}$. The "action" on a path is the sum of the valences over the vertices it traverses. $Z(t)$ is therefore a discretization of the Feynman path integral. Furthermore,

$$\left.\frac{d^k Z}{dt^k}\right|_0 (i, j)$$

gives a signed count of *generalized walks* [Yu 2017] of length $k$ from vertex $i$ to $j$: a sequence

$$(v_1, e_1), (v_2, e_2), \ldots, (v_k, e_k)$$

of pairs of vertices and edges in which $v_j$ and $v_{j+1}$ are endpoints to $e_j$ (not necessarily distinct) for all $j$. In other words, a new path is a sequence of vertices which may traverse an edge while remaining stationary at a vertex. The precise combinatorial interpretation of the partition function can be found in [Del Vecchio 2012; Mnëv 2016; Yu 2017]. The sign of such a path is determined by the number of $j$ such that $v_{j+1} \neq v_j$.

***The supersymmetric version.*** In the supersymmetric theory, we extend $\Delta$ to the entire simplicial cochain complex $C^\bullet$ of $\Gamma$ with $\Delta := \Delta_+ \oplus \Delta_-$, where $\Delta_-$ and $\Delta_+$ operate on edges and vertices, respectively. The entries of $(\Delta_-)^k$ give signed counts of another type of special path: sequences of edges $e_1, e_2, \dots, e_k$ such that $e_j$ is adjacent to $e_{j+1}$, with the sign determined by the number of $j$ such that $e_j$ meets $e_{j+1}$ with opposite orientation. For further details, see, e.g., [Del Vecchio 2012; Yu 2017].

*Graph Hodge theory.* There is a close relationship between the graph Laplacian and the topology of the graph. More precisely, we have the following relationship between $\Delta$ and the cohomology groups of $\Gamma$:

**Proposition 1.8.** *The dimension of* $\ker(\Delta_+)$ *is the number of connected components of* $\Gamma$.

*Proof.* By Lemma A.1 in the Appendix, $\ker(\Delta_+) = \ker(I)$. Now, if the vertices $\{v_{\alpha_1}, v_{\alpha_2}, \dots, v_{\alpha_k}\}$ are all the elements of a connected component of $\Gamma$, the state

$$\Psi_+(v) = \begin{cases} 1 & \text{if } v = v_{\alpha_j},\ 1 \le j \le k, \\ 0 & \text{otherwise,} \end{cases}$$

is a generator of $\ker(I)$. □

**Proposition 1.9.** *The dimension of* $\ker(\Delta_-)$ *is the number of independent cycles of* $\Gamma$.

*Proof.* Once again, by Lemma A.1, $\ker(\Delta_-) = \ker(I^*)$. Now, the combinatorial interpretation of the elements of the kernel of $I^*$ is in terms of closed paths and this can be interpreted in terms of closed currents. More precisely, one may think of a state $\Psi_-$ as a current, i.e., a function taking values on edges, obeying Kirchhoff's first law: at each vertex, in- and outgoing currents balance. Thus, a current achieves balance if and only if the current is assigned to a closed path. □

These two propositions lead to the following:

**Theorem 1.10.** *The cohomology groups of* $\Gamma$ *can be calculated as*

$$H^0(\Gamma, \mathbb{C}) = \ker(I^*) = \ker(\Delta_+)$$

*and*

$$H^1(\Gamma, \mathbb{C}) = \ker(I) = \ker(\Delta_-).$$

Thus $\Delta$ satisfies a "discrete" Hodge theorem.

We can further simplify the calculation of cohomology by considering energy cut-offs. Since $\Delta$ is symmetric, it is diagonalizable and its eigenvalues are real (nonnegative, in fact), so the cochain complex decomposes as

$$C^\bullet = \bigoplus_{\lambda \ge 0} E_\lambda,$$

where $E_\lambda$ is the eigenspace of $\Delta$ corresponding to $\lambda$. Given an energy $a \geq 0$, let

$$C_a^\bullet = \bigoplus_{\lambda \leq a} E_\lambda$$

be the subcochain complex of $C^\bullet$ consisting of eigenspaces of energy lower than $a$.

**Theorem 1.11** (energy cut-off). $H^*(C_a^\bullet) = H^*(C^\bullet)$.

*Proof.* It is clear that $C_a^0$ contains $\ker \Delta_+$, for all $a$, thus $\ker I^*|_{C_a^0} = \ker \Delta_+ = H^0(\Gamma)$. For $H^1$, this follows from the fact that $\operatorname{coker}(I^*)$ is contained in $C_a^1$, for all $a$, and from Lemma A.1. $\square$

**Remark.** In other words, the cohomology of $\Gamma$ can be calculated by considering subcomplexes of lower energy!

## 2. Morse theory

In this section we describe a version of Morse theory on graphs by ordering the set of edges and vertices of $\Gamma$ by declaring each vertex lesser than each edge of which it is an endpoint. With respect to this ordering, we consider special states which will be the analogue of nondegenerate smooth Morse functions for manifolds. More precisely we have the following definition, originally due to Forman [2002]:

**Definition 2.1.** A *discrete Morse function* is a real-valued function $f$ on the set of edges and vertices of $\Gamma$ such that, for all $\sigma \in \Gamma$,

$$\#\{\tau > \sigma \mid f(\tau) \leq f(\sigma)\} \leq 1, \tag{3}$$

$$\#\{\tau < \sigma \mid f(\tau) \geq f(\sigma)\} \leq 1. \tag{4}$$

**Definition 2.2.** A *critical cell* (vertex or edge) $\sigma$ of $f$ is one for which the two sets above are empty. We denote by $c_0(f)$ the number of critical vertices and by $c_1(f)$ the number of critical edges.

As we have said before, our main goal is to prove the following theorem.

**Theorem 2.3** (graph Morse inequalities). *Let $h_0$ and $h_1$ be the Betti numbers of $\Gamma$. Then $h_0 \leq c_0(f)$ and $h_1 \leq c_1(f)$, for every Morse function $f$.*

*Proof.* The strategy is as follows. Using Theorem 1.11, we follow Witten's approach [1982] to Morse inequalities for Riemannian manifolds, via deformation of the supersymmetric Laplacian. The precise idea is as follows. We deform the supersymmetric Laplacian $\Delta$ using the Morse function $f$ with a real parameter $s$ by deforming the boundary and coboundary operators $d$ and $d^*$. The Hodge theorems still hold for the deformed Laplacian $\Delta_s$, and, taking a limit $s \to \infty$, there is an energy level $a$ for which the cutoff complex $C_a^\bullet$ approaches the Morse complex as $s$ approaches infinity.

Now, let us start by deforming the boundary operator.

**Definition 2.4.** The deformed boundary and coboundary operators $d_s$ and $d_s^*$ are given by

$$d_s = \exp(fs)d\exp(-fs), \quad d_s^* = \exp(-fs)d^*\exp(fs).$$

**Remark.** Note that the $\exp(fs)$ and $\exp(-fs)$ are represented by matrices of, a priori, different dimensions. For the deformed boundary operator, $\exp(-fs)$ is a diagonal $|E|\times|E|$ real matrix, whereas $\exp(fs)$ is a diagonal $|V|\times|V|$ real matrix. In the coboundary case, the situation is reversed.

**Definition 2.5.** The deformed Laplacian $\Delta_s$ is defined as $\Delta_s = d_s^* d_s + d_s d_s^*$.

Therefore we can define the deformed cochain complex $C_s^\bullet$ as

$$0 \to \mathbb{C}^{|V|} \xrightarrow{d_s^*} \mathbb{C}^{|E|} \to 0. \tag{5}$$

Similarly we can define the cut-off cochain complexes $C_{s,a}^\bullet$.

If we denote by $H_s^\bullet(\Gamma)$ the cohomology of the cochain $C_{s,a}^\bullet$, the following proposition follows from Lemma A.3 in the Appendix, since the matrices $\exp(-fs)$ and $\exp(fs)$ are both invertible.

**Proposition 2.6** (deformed energy cut-off). $H_{s,a}^\bullet(\Gamma) = H^\bullet(\Gamma)$.

Now, if we take the limit $s \to \infty$ the matrices $\Delta_{\pm,\infty}$ become quite simple, and their kernels become independent on $s$, only they only depend on the critical cells. Explicitly we have the following description:

**Proposition 2.7.** *Given a flat Morse function $f$, the matrices $\Delta_{+,\infty}$ and $\Delta_{-,\infty}$ have entries* 0 *and* 1, *and the number of zero columns is the number of corresponding critical cells.*

*Proof.* The general entries of the deformed boundary operator have the form $\exp(ks)$, where $k$ is the jump of the Morse function between an edge and its endpoint. Thus, the graph Laplacian will have an entry of the form $\exp(qs)$, with $q$ negative, if and only if the Morse value of a cell and its incident cell is different. Thus when $s \to \infty$, the 1 entries will occur exactly when there is a noncritical cell, for which the value of the cell and one of the incident cells is the same. □

From Proposition 2.7 we conclude that the dimension of $\ker(\Delta_{+,\infty})$ is $c_0(f)$ and that the dimension of $\ker(\Delta_{-,\infty})$ is $c_1(f)$. Therefore, if $a$ is arbitrarily small, the energy cut-off produces a cochain complex isomorphic to the Morse complex. More precisely, we observe that Morse homology $H_{\mathrm{Morse}}(\Gamma)$ for the chain complex $d : \ker(\Delta_{-,\infty}) \to \ker(\Delta_{+,\infty})$ is isomorphic to the cellular homology of $\Gamma$. This follows from the following fact. By collapsing gradient curves (i.e., by identifying all vertices along them), we construct a new graph $\Gamma'$. The 0- and 1-cells of $\Gamma'$ are precisely the critical 0-cells and 1-cells of the original graph $\Gamma$. Its chain complex

$C_\bullet(\Gamma')$ is isomorphic to the Morse complex and also, since $\Gamma'$ is homotopic to $\Gamma$, is quasi-isomorphic to $C_\bullet(\Gamma)$.

This concludes the proof of Theorem 2.3. □

## 3. Discrete gradient vector fields

In [Witten 1982], the super-symmetric interpretation of Morse inequalities is described in terms of *instantons*, i.e., solutions of the differential equation

$$\frac{du(t)}{dt} = -\nabla f(u(t)), u(0) = q, \tag{6}$$

where $q$ is a given noncritical point, and with boundary conditions

$$\lim_{t\to\infty} u(t) = p, \qquad \lim_{t\to-\infty} u(t) = r,$$

for which $p$ and $r$ are critical points of the Morse function $f$.

The CW-decomposition for a well behaved Morse function (a so-called Morse–Smale function, with suitable transversality conditions between the descending and ascending cells) comes equipped with an orientation, and a signed count of the number of solutions of (6) gives the Morse differential for the Morse complex.

In the graph setting, there is a discrete analogue of a gradient vector field [Forman 2002]. It turns out that noncritical cells always come in pairs. In order to see this, we observe that given a noncritical edge, it implies by definition that there exists an incident vertex with a nondecreasing value of the Morse function. In the same way, a noncritical vertex has an adjacent edge such that the Morse value is nonincreasing.

**Definition 3.1.** Let $f$ be a discrete Morse function on a graph $\Gamma$. The discrete gradient vector field of $f$, denoted by $\nabla f$, is the set of adjacent noncritical pairs $(v_{n_i}, e_{n_i})$.

Usually discrete gradient fields are represented graphically by arrows having noncritical vertices as tails and adjacent noncritical edge as arrowheads; see, e.g., Figure 6. The following definition of gradient curves for a Morse function is the graph version of gradient paths given in [Forman 1998]:

**Definition 3.2.** A gradient curve between two vertices $\sigma_{\text{initial}}$ and $\sigma_{\text{final}}$ is a finite sequence

$$\gamma : \sigma_{\text{initial}} = \sigma_0, \tau_0, \sigma_1, \tau_1, \dots, \tau_{k-1}, \sigma_k = \sigma_{\text{final}}$$

such that the following conditions are satisfied:

(1) $\sigma_i < \tau_i$ and $\sigma_{i+1} < \tau_i$.

(2) $\sigma_i \neq \sigma_{i+1}$.

(3) $f(\sigma_i) \geq f(\tau_i) > f(\sigma_{i+1})$.

We should interpret gradient curves as discrete solutions of (6). In [Witten 1982], each gradient curve is naturally equipped with a sign, so the Morse differential is obtained by counting the signed gradient curves among critical points of index differing by 1. In [Forman 1998], the sign (or algebraic multiplicity) of a gradient curve is defined as follows. Given an orientation on the vertices $\sigma_i$, the sign of a path $\gamma$, denoted by $m(\gamma)$, is said to be $+1$ if the orientation on $\sigma_{\text{final}}$ agrees with the induced orientation on $\sigma_{\text{initial}}$, and is $-1$ otherwise. Now, we can define the Morse differential $\tilde{\partial}$ from critical edges to critical vertices as follows. If $C^1$ and $C^0$ denote the vector spaces generated by critical edges and vertices, respectively, and an inner product $\langle \bullet, \bullet \rangle$ is chosen so the critical cells form an orthonormal basis, the linear operator

$$\langle \tilde{\partial}\tau, \sigma \rangle = \sum_{\sigma_1, \tau} \langle \partial\tau, \sigma_1 \rangle \sum_{\gamma \in \Gamma(\sigma_1, \sigma)} m(\gamma) \tag{7}$$

is clearly a differential, and furthermore it is the Morse differential [Forman 1998].

## 4. Examples

***The simplest case.*** Let us consider a very simple graph, that is, a graph with two vertices and one edge: $V = \{v_1, v_2\}$ and $E = \{e_1 = (v_1, v_2)\}$.

With respect to the orientation given in Figure 3 we get the following matrices:

$$I_\Gamma = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \Delta_{+,\Gamma} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \Delta_{-,\Gamma} = [2]. \tag{8}$$

Let us consider the Morse function $f$ on $\Gamma$ as illustrated in Figure 4.

It is easy to check that $c_0(f) = 1$ (the vertex $v_2$ is critical) and that $c_1(f) = 0$ (there are no critical edges). Now, the deformed boundary $d_s$ is

$$\begin{aligned} d_s = I_{\Gamma,s} = (\exp(sf)) I_\Gamma (\exp(-sf)) &= \begin{bmatrix} \exp(s) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} [\exp(-s)] \\ &= \begin{bmatrix} 1 \\ -\exp(-s) \end{bmatrix}. \end{aligned} \tag{9}$$

Therefore, the deformed even Laplacian $\Delta_{+,s}$ is

$$\Delta_{+,s} = d_s d_s^* = \begin{bmatrix} 1 & -\exp(-s) \\ -\exp(-s) & \exp(-2s) \end{bmatrix} \tag{10}$$


**Figure 3.** The graph $K_2$ (left) and oriented $K_2$ (right).

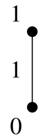


**Figure 4.** The Morse function $f$ on $K_2$.

and the odd Laplacian $\Delta_{-,s}$ is

$$\Delta_{-,s} = d_s^* d_s = [1 + \exp(-2s)]. \tag{11}$$

Therefore,

$$\Delta_{+,\infty} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; \Delta_{-,\infty} = [1]. \tag{12}$$

It is easy to check that $\dim(\ker(\Delta_{+,\infty})) = 1 = c_0(f)$ and that $\dim(\ker(\Delta_{+,\infty})) = 0 = c_1(f)$. More precisely,

$$\ker(\Delta_{+,\infty}) = \langle v_2 \rangle,$$
$$\ker(\Delta_{-,\infty}) = \langle 0 \rangle.$$

***The triangle.*** We illustrate the case in which we have two different Morse functions, one of each achieving sharpness of the Morse inequalities. Let us consider the triangle graph $K_3$ with

$$V = \{v_1, v_2, v_3\} \quad \text{and} \quad E = \{e_1 = (v_1, v_2), (v_1, v_3), (v_2, v_3)\}.$$

With respect to the orientation given in Figure 5 we get the following incidence matrix:

$$I_\Gamma = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{13}$$

Now consider the Morse function on $K_3$ illustrated in Figure 6.


**Figure 5.** The triangle $K_3$ (left) and an orientation on $K_3$ (right).

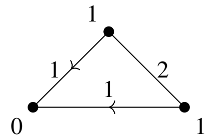


**Figure 6.** The Morse function $f$ on $K_3$ and the corresponding gradient vector field $\nabla f$.

It is easy to check that $c_0(f) = 1$ (the vertex $v_2$ is critical) and that $c_1(f) = 1$ (the edge $e_2$ is critical). Now, the deformed boundary $d_s^*$ is

(14)
$$d_s = \begin{bmatrix} \exp(s) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp(s) \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \exp(-s) & 0 & 0 \\ 0 & \exp(-2s) & 0 \\ 0 & 0 & \exp(-s) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -\exp(-s) & -1 \\ -\exp(-s) & 0 & \exp(-s) \\ 1 & \exp(-s) & 0 \end{bmatrix}.$$

Therefore, the deformed even Laplacian $\Delta_{+,s}$ is

$$\Delta_{+,s} = d_s d_s^* = \begin{bmatrix} 1+\exp(-2s) & -\exp(-s) & -\exp(-2s) \\ -\exp(-s) & 2\exp(-2s) & -\exp(-s \\ -\exp(-2s) & -\exp(-s) & 1+\exp(-2s) \end{bmatrix} \tag{15}$$

and the odd Laplacian $\Delta_{-,s}$ is

$$\Delta_{-,s} = d_s^* d_s = \begin{bmatrix} 1+\exp(-2s) & \exp(-s) & -\exp(-2s) \\ \exp(-s) & 2\exp(-2s) & -\exp(-s \\ -\exp(-2s) & -\exp(-s) & 1+\exp(-2s) \end{bmatrix}. \tag{16}$$

Therefore,

$$\Delta_{+,\infty} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \Delta_{-,\infty}. \tag{17}$$

It is easy to check that $\dim(\ker(\Delta_{+,\infty})) = 1 = c_0(f) = c_1(f) = \dim(\ker(\Delta_{-,\infty}))$ and that

$$\ker(\Delta_{+,\infty}) = \langle v_2 \rangle, \qquad \ker(\Delta_{-,\infty}) = \langle e_2 \rangle.$$

As expected, this Morse function achieves the equality for the Morse inequalities.

On the other hand, we might have considered the Morse function on $K_3$ given by Figure 7. For this function, all the vertices and edges are critical, thus $c_0(g) = c_1(g) = 3$.

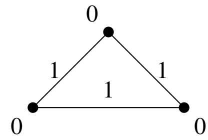


**Figure 7.** The Morse function $g$ on $K_3$.

Now, the deformed boundary $d_s^*$ is

$$d_s^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \exp(-s) & 0 & 0 \\ 0 & \exp(-s) & 0 \\ 0 & 0 & \exp(-s) \end{bmatrix} \tag{18}$$
$$= \begin{bmatrix} 0 & -\exp(-s) & -\exp(-s) \\ -\exp(-s) & 0 & \exp(-s) \\ \exp(-s) & \exp(-s) & 0 \end{bmatrix}.$$

Therefore, the deformed even Laplacian $\Delta_{+,s}$ is

$$\Delta_{+,s} = d_s^* d_s = \begin{bmatrix} 2\exp(-2s) & -\exp(-2s) & -\exp(-2s) \\ -\exp(-2s) & 2\exp(-2s) & -\exp(-2s) \\ -\exp(-2s) & -\exp(-2s) & 2\exp(-2s) \end{bmatrix} \tag{19}$$

and the odd Laplacian $\Delta_{-,s}$ is

$$\Delta_{+,s} = d_s^* d_s = \begin{bmatrix} 2\exp(-2s) & \exp(-2s) & -\exp(-2s) \\ \exp(-2s) & 2\exp(-2s) & -\exp(-2s) \\ -\exp(-2s) & -\exp(-2s) & 2\exp(-2s) \end{bmatrix}. \tag{20}$$

Therefore,

$$\Delta_{+,\infty} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \Delta_{-,\infty}, \tag{21}$$

thus

$$\ker(\Delta_{+,\infty}) = \langle v_1, v_2, v_3 \rangle, \qquad \ker(\Delta_{-,\infty}) = \langle e_1, e_2, e_3 \rangle.$$

## 5. Morse function of a tree and the boundary map

Given a vertex $v$ in $\Gamma$, let $v'$ be the critical vertex obtained by flowing along the gradient field $\nabla f$ of $f$. For an edge $e$ in $\Gamma$ with endpoints $v_0$ and $v_1$, the Morse boundary map, defined on critical edges, is given by

$$e \mapsto v_1' - v_0'. \tag{22}$$

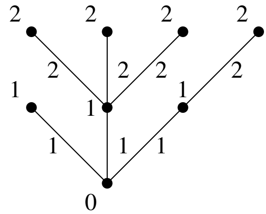


**Figure 8.** The height function for a rooted tree. Its Morse homology is trivial.

Let $T$ be a spanning tree of $\Gamma$ and $v_r$ a vertex in $T$, the *root*. Then it is an easy observation that the height function $h$ defined by

$$h(v) = \text{ edge distance from } v_r,$$
$$h(e) = \max(h(v_0), h(v_1)) \text{ if } e \text{ belongs to } T,$$
$$h(e) = \max(h(v_0), h(v_1)) + 1 \text{ otherwise,}$$

is Morse. See Figure 8 for an example.

The following proposition justifies the fact that we recover the Morse complex for such functions.

**Proposition 5.1.** *The boundary map is zero on critical edges of $h$.*

*Proof.* The critical cells of $h$ consist of $v_0$ and all the edges not contained in $T$, and the boundary map is zero on critical edges since $v_0' = v_1'$. Therefore, the boundary map coincides with the operator $\tilde{\partial}$ from (7), and this implies that the boundary maps compute graph homology; see, e.g, [Forman 1998]. □

## 6. Conclusion and perspectives

We have reproven Morse inequalities in the particular case of finite graphs, by using Witten's supersymmetric approach for quantum mechanics on Riemannian manifolds. The equality is achieved in both examples by a *height type* Morse function, which can be defined for a spanning tree in terms of the *depth* of the tree, once a root is chosen. The remaining values of the Morse function can be chosen to be larger than the maximum of the corresponding edges, so the Morse conditions (3) and (4) are satisfied. We conjecture that the sharpness of the equation is achieved by such functions in the general case of CW-complexes, for which there is a generalized notion of a spanning tree and corresponding height function. This will be part of an upcoming publication. We also intend to describe in detail how to use Witten's approach to derive Morse inequalities and define a discrete version of quantum mechanics in interesting higher dimensional examples such as real projective spaces and complexes of graphs with a monotone decreasing property [Forman 2002]. Furthermore, the locality principle of quantum mechanics should allow an extension of Witten's approach to the case of gluing of graphs, giving a topological interpretation to the gluing formulae for the even and odd Laplacians found in [Contreras et al. 2018].

## Appendix: Linear algebra and the graph Laplacian

The following are technical basic lemmas in linear algebra used throughout the paper, and they can be found in standard references for matrix linear algebra, such as [Horn and Johnson 1985].

**Lemma A.1.** *Let $A$ be a matrix and let $A^*$ be its adjoint. Then* $\ker(A) = \ker(A^*A)$.

*Proof.* It is clear that $\ker A \subseteq \ker(A^*A)$. For the other direction, if $\langle \cdot, \cdot \rangle$ is the corresponding inner product, then, for each vector $v$ in $\ker(A^*A)$,

$$\|Av\|^2 = \langle Av, Av\rangle = \langle A^*Av, v\rangle = \langle 0, v\rangle = 0,$$

thus $v \in \ker(A)$. □

**Lemma A.2.** *The matrices $AA^*$ and $A^*A$ are both nonnegative definite and their spectra coincide* (*modulo multiplicities*).

**Lemma A.3.** *Let $A$ be a $p\times q$ matrix*, *let $X$ be an invertible $p\times p$ matrix and let $Y$ be an invertible $q\times q$ matrix. Then*

$$\ker(A) \cong \ker(XAY),$$

*where the isomorphism is induced by $Y$.*

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