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SYMMETRY AND MONOTONICITY OF POSITIVE SOLUTIONS FOR AN INTEGRAL SYSTEM WITH NEGATIVE EXPONENTS

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We study the following integral system with negative exponents:

$$\begin{cases} u(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} v^{-p}(y)\,dy, \\ v(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} u^{-q}(y)\,dy, \end{cases}$$

with $\alpha > n$, $p, q > 0$ and $\frac{1}{p-1} + \frac{1}{q-1} = \frac{\alpha-n}{n}$. Such a nonlinear integral system is related to the study of the best constant of the reversed Hardy–Littlewood–Sobolev type inequality. Motivated by work of Dou, Guo and Zhu (*Adv. Math.* 312 (2017) 1–45) where they used the improved method of moving planes, we prove that each pair of positive measurable solutions is radially symmetric and monotonic increasing about some point. Our result is an extension of the work for $\alpha < n$, $p, q < 0$ of Chen, Li and Ou (*Comm. Partial Differential Equations* 30:1–3 (2005), 59–65).

1. Introduction

Let $0 < \gamma < n$, and let $s_1, s_2 > 1$ such that

$$\frac{1}{s_1} + \frac{1}{s_2} = \frac{n+\gamma}{n}.$$

The well-known Hardy–Littlewood–Sobolev inequality states that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)|x-y|^{\gamma-n} g(y)\,dx dy \le C(n,\gamma,s_1)\|f\|_{L^{s_1}(\mathbb{R}^n)}\|g\|_{L^{s_2}(\mathbb{R}^n)}, \tag{1}$$

for any $f \in L^{s_1}(\mathbb{R}^n)$ and $g \in L^{s_2}(\mathbb{R}^n)$.

It is well known that the Hardy–Littlewood–Sobolev inequality has many important applications in partial differential equations and geometry, as well as in quantum field theory. It is also known that the Hardy–Littlewood–Sobolev inequality is closely related to the Sobolev inequality in Euclidean spaces and Moser–Onofri–Beckner type inequalities on spheres (see Beckner [1993; 2008]). For more research

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results with integral equations related to the Hardy–Littlewood–Sobolev inequality, we refer the reader to [Beckner 1993; Brascamp and Lieb 1976; Chen et al. 2005a; 2016; Chen and Li 2008; Lu and Zhu 2011]. For Hardy–Littlewood–Sobolev inequalities on the Riemannian manifolds and the upper half space $\mathbb{R}^n_+$, see, e.g., [Dou 2016; Dou and Zhu 2015b; Han and Zhu 2016; Ngô and Nguyen 2017b].

To find the best constant $C(n,\gamma,s_1)$ in (1), we can maximize the functional

$$J(f,g)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)|x-y|^{\gamma-n}g(y)\,dxdy, \tag{2}$$

under the constraints

$$\|f\|_{L^{s_1}(\mathbb{R}^n)}=\|g\|_{L^{s_2}(\mathbb{R}^n)}=1. \tag{3}$$

Let (f,g) be a minimizer, or more generally a critical point of (2) under the constraints (3). Letting $u=a_1 f^{s_1-1}$, $v=a_2 g^{s_2-1}$, $p_1=\frac{1}{s_1-1}$, $p_2=\frac{1}{s_2-1}$, and by a proper choice of constants a_1 and a_2, we can see that (u,v) satisfies the following integral system in $\mathbb{R}^n$:

$$\begin{cases} u(x)=\int_{\mathbb{R}^n}|x-y|^{\gamma-n}v^{p_2}(y)\,dy,\\ v(x)=\int_{\mathbb{R}^n}|x-y|^{\gamma-n}u^{p_1}(y)\,dy,\end{cases} \tag{4}$$

with $\frac{1}{p_1+1}+\frac{1}{p_2+1}=\frac{n-\gamma}{n}$.

Under the natural integrability conditions $u\in L^{p_1}(\mathbb{R}^n)$ and $v\in L^{p_2}(\mathbb{R}^n)$, Chen, Li and Ou [Chen et al. 2005b] proved that all the solutions are radially symmetric and monotonic decreasing about some point.

Suppose that $\alpha>n$, and $0<r_1,r_2<1$ satisfying

$$\frac{1}{r_1}+\frac{1}{r_2}=\frac{n+\alpha}{n}.$$

Dou and Zhu [2015a] established the following reversed Hardy–Littlewood–Sobolev inequality (see also Ngô and Nguyen [2017a]), which can be seen as an extension of (1), which states

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)|x-y|^{\alpha-n}g(y)\,dxdy\geq C(n,\alpha,r_1)\|f\|_{L^{r_1}(\mathbb{R}^n)}\|g\|_{L^{r_2}(\mathbb{R}^n)}, \tag{5}$$

for any $f\in L^{r_1}(\mathbb{R}^n)$ and $g\in L^{r_2}(\mathbb{R}^n)$.

The corresponding Euler–Lagrange equation becomes

$$\begin{cases} u(x)=\int_{\mathbb{R}^n}|x-y|^{\alpha-n}v^{-p}(y)\,dy,\\ v(x)=\int_{\mathbb{R}^n}|x-y|^{\alpha-n}u^{-q}(y)\,dy,\end{cases} \tag{6}$$

with $\alpha > n,\ \ p, q > 0$ and,

$$\frac{1}{p-1}+\frac{1}{q-1}=\frac{\alpha-n}{n}. \tag{7}$$

The study of system (6), especially the classification result, is a crucial step in finding the best constant in the reversed Hardy–Littlewood–Sobolev inequality. Dou and Zhu [2015a] classified the positive solutions of (6) with critical exponents $p=q=\frac{n+\alpha}{\alpha-n}$ and gave its best constant. Recently, Lei [2015] considered system (6) and obtained that assumption (7) was the necessary condition for the existence of C^1 positive entire solutions. In fact, (7) is the necessary and sufficient condition for the existence of C^1 positive entire solutions, since the sufficient condition was proved by Ngô and Nguyen [2017a]. However, as far as we know, there have been few results about positive solutions of (6) with exponent condition (7) except the case of both critical exponents $p=q=\frac{n+\alpha}{\alpha-n}$. This is a problem well worth studying.

Motivated by the work of Dou, Guo and Zhu [Dou et al. 2017; Dou and Zhu 2015a], we investigate the above problem and prove the following results.

Theorem 1. *If* (u, v) *is a pair of positive measurable solutions of* (6), *then* u *and* v *are radially symmetric and monotonic increasing about some point* x_0.

In particular, if we set $u=v,\ \alpha-n=\nu$, and $p=q$, the integral system (4) will be reduced to a single equation

$$u(x)=\int_{\mathbb{R}^n}|x-y|^{\nu}u^{-p}(y)\,dy. \tag{8}$$

The negative exponent of integral equation (8) was studied by Li [2004]. Li classified the positive solutions of (8) for $p=\frac{2n+\nu}{\nu}$. It turns out that for $n=3$ and $\nu=1$, integral equation (8) is associated with some fourth order conformal covariant operator on three-dimensional compact Riemannian manifolds, arising from the study of conformal geometry. See, e.g., [Xu and Yang 2002] and [Chang and Yang 2002]. Xu [2005] proved that if $n=3$ and $\nu=1$, then $p=7$ and u must take the form $u(x)=c(1+|x|^2)^{\frac{1}{2}}$ up to dilation and translation. Furthermore, Xu [2007] proved that (8) has a C^1 positive solution if and only if $p=\frac{2n+\nu}{\nu}$. For more related results about integral systems with negative exponents, we refer the reader to [Choi and Xu 2009; Dou et al. 2016; Guo and Wei 2014; Liu et al. 2018; Ma and Wei 2008].

To prove the radial symmetry and monotonicity of the solutions, we use the improved method of moving planes introduced by Dou, Guo and Zhu [2017], which is slightly different from that introduced by Chen, Li and Ou [2006] because they rely on iteration of Hardy–Littlewood–Sobolev inequality to start the moving plane process. In this paper, by the asymptotic behavior and regularity of the solutions, we are able to establish the symmetry and monotonicity results.

2. The proof of radial symmetry and monotonicity

In this section, we use the method of moving planes to prove Theorem 1. For a given real number λ, define

$$T_\lambda = \{x \in \mathbb{R}^n \mid x_1 = \lambda\}, \quad \Sigma_\lambda = \{x \in \mathbb{R}^n \mid x_1 \le \lambda\}.$$

Let $x_\lambda = \{2\lambda - x_1, x_2, \dots, x_n\}$, $u_\lambda(x) = u(x_\lambda)$ and $v_\lambda(x) = v(x_\lambda)$.

The following lemma was proved in [Li 2004] (see also [Dou and Zhu 2015a]).

Lemma 2. *Let* (u, v) *be a pair of positive measurable solutions of* (6). *Then*:

(i) $\displaystyle\int_{\mathbb{R}^n} (1+|y|^{\alpha-n}) u^{-q}(y) dy < \infty, \quad \int_{\mathbb{R}^n} (1+|y|^{\alpha-n}) v^{-p}(y)\, dy < \infty.$

(ii) $\displaystyle 0 < a := \lim_{|x|\to\infty} |x|^{n-\alpha} u(x) = \int_{\mathbb{R}^n} v^{-p}(y)\, dx < \infty,$

$$0 < b := \lim_{|x|\to\infty} |x|^{n-\alpha} v(x) = \int_{\mathbb{R}^n} u^{-q}(y)\, dx < \infty.$$

(iii) *There exist some constants* $C_1, C_2 > 0$ *such that*

$$\frac{1+|x|^{\alpha-n}}{C_1} \le u(x) \le C_1(1+|x|^{\alpha-n}),$$

and

$$\frac{1+|x|^{\alpha-n}}{C_2} \le v(x) \le C_2(1+|x|^{\alpha-n}).$$

(iv) $u \in C^\infty(\mathbb{R}^n)$ *and* $v \in C^\infty(\mathbb{R}^n)$.

From the above lemma, we know that $p > \frac{\alpha}{\alpha-n}$ and $q > \frac{\alpha}{\alpha-n}$.

Lemma 3. *Let* (u, v) *be a pair of positive measurable solutions of* (6). *Then*

$$u(x) - u_\lambda(x) = \int_{\Sigma_\lambda} K(\lambda, x, y)(v_\lambda^{-p}(y) - v^{-p}(y))\, dy \tag{9}$$

and

$$v(x) - v_\lambda(x) = \int_{\Sigma_\lambda} K(\lambda, x, y)(u_\lambda^{-q}(y) - u^{-q}(y))\, dy, \tag{10}$$

where $K(\lambda, x, y) = |x_\lambda - y|^{\alpha-n} - |x-y|^{\alpha-n}$.

Proof. We only prove (9); by (6), we have,

$$\begin{aligned} u(x) &= \int_{\Sigma_\lambda} |x-y|^{\alpha-n} v^{-p}(y)\, dy + \int_{\mathbb{R}^n\setminus\Sigma_\lambda} |x-y|^{\alpha-n} v^{-p}(y)\, dy \\ &= \int_{\Sigma_\lambda} |x-y|^{\alpha-n} v^{-p}(y)\, dy + \int_{\Sigma_\lambda} |x-y_\lambda|^{\alpha-n} v_\lambda^{-p}(y)\, dy \end{aligned}$$

and

$$
\begin{aligned}
u_\lambda(x) &= \int_{\Sigma_\lambda} |x-y|^{\alpha-n} v^{-p}(y)\,dy + \int_{\mathbb{R}^n\setminus\Sigma_\lambda} |x-y|^{\alpha-n} v^{-p}(y)\,dy \\
&= \int_{\Sigma_\lambda} |x_\lambda-y|^{\alpha-n} v^{-p}(y)\,dy + \int_{\Sigma_\lambda} |x_\lambda-y_\lambda|^{\alpha-n} v_\lambda^{-p}(y)\,dy.
\end{aligned}
$$

Combining with $|x_\lambda - y| = |x - y_\lambda|$ and $|x_\lambda - y_\lambda| = |x - y|$, we obtain (9). □

To prove Theorem 1, we compare $u(x)$ with $u_\lambda(x)$ and $v(x)$ with $v_\lambda(x)$ on Σ_λ. The proof consists of two steps. In Step 1, we show that there exists an $N < 0$ such that for $\lambda \le N$,

$$
(11) \qquad u(x) \ge u_\lambda(x), \;\; v(x) \ge v_\lambda(x), \quad \text{for all } x \in \Sigma_\lambda.
$$

Thus we can start moving the plane continuously from $\lambda \le N$ to the right as long as (11) holds. In Step 2, we show that if the plane stops at $x_1 = \lambda_0$ for some $\lambda_0 < 0$, then $u(x)$ and $v(x)$ must be symmetric and monotonic increasing about the plane $x_1 = \lambda_0$. Otherwise, we can move the plane all the way to x_1. Since the direction of x_1 can be chosen arbitrarily, we deduce that $u(x)$ and $v(x)$ must be radially symmetric and monotonic increasing about some point.

Step 1. We first prove that there exists sufficiently negative $N < 0$ such that

$$
(12) \quad \nabla(|x|^{\frac{n-\alpha}{2}} u(x)) \cdot x = |x|^{\frac{n-\alpha}{2}} \Big(\nabla u(x) \cdot x + \frac{n-\alpha}{2} u(x) \Big) > 0, \quad \text{for all } x_1 < N.
$$

To obtain (12), it is sufficient to prove that

$$
(13) \qquad \lim_{|x|\to\infty} |x|^{n-\alpha} \Big(\nabla u(x) \cdot x + \frac{n-\alpha}{2} u(x) \Big) > 0.
$$

We only prove (13) for $i = 1$. Fix $x_2, x_3, \dots, x_n$. By Lemma 2, $u(x)$ is differentiable in x_1; then, we have

$$
\begin{aligned}
\Big| \frac{\partial u(x)}{\partial x_1} \Big| &= \Big| (\alpha-n) \int_{\mathbb{R}^n} |x-y|^{\alpha-n-2} (x_1-y_1) v^{-p}(y)\,dy \Big| \\
&\le (\alpha-n) \int_{\mathbb{R}^n} |x-y|^{\alpha-n-1} v^{-p}(y)\,dy.
\end{aligned}
$$

Now we carry out the proof in two cases.

Case (i): If $\alpha - n \ge 1$, then by Lemma 2, letting $|x_1| \ge 1$, we have

$$
\begin{aligned}
\frac{\big|\frac{\partial u(x)}{\partial x_1}\big|}{|x_1|^{\alpha-n-1}} &\le \Big| (\alpha-n) \int_{\mathbb{R}^n} (1+|y|^{\alpha-n-1}) v^{-p}(y)\,dy \Big| \\
&\le C \int_{\mathbb{R}^n} (1+|y|^{\alpha-n}) v^{-p}(y)\,dy.
\end{aligned}
$$

Thus, by using Lebesgue's dominated convergence theorem, we have

$$\lim_{|x_1|\to\infty} \frac{\left|\frac{\partial u(x)}{\partial x_1}\right|}{|x_1|^{\alpha-n-1}} = a(\alpha-n).$$

Case (ii): If $0<\alpha-n<1$, we can calculate

$$\begin{aligned}
\frac{\left|\frac{\partial u(x)}{\partial x_1}\right|}{|x|^{\alpha-n-1}} &\le (\alpha-n)\int_{\mathbb{R}^n} \frac{|x|^{n-\alpha+1}}{|x-y|^{n-\alpha+1}} v^{-p}(y)\,dy \\
&= (\alpha-n)\int_{\mathbb{R}^n\setminus B\left(x,\frac{|x|}{2}\right)} \frac{|x|^{n-\alpha+1}}{|x-y|^{n-\alpha+1}} v^{-p}(y)\,dy \\
&\qquad\qquad + (\alpha-n)\int_{B\left(x,\frac{|x|}{2}\right)} \frac{|x|^{n-\alpha+1}}{|x-y|^{n-\alpha+1}} v^{-p}(y)\,dy \\
&:= (\alpha-n)J_1(x) + (\alpha-n)J_2(x).
\end{aligned}$$

For $J_1(x)$, since $|x-y|>\frac{|x|}{2}$, we have

$$J_1(x) \le C\int_{\mathbb{R}^n\setminus B\left(x,\frac{|x|}{2}\right)} \frac{|x|^{n-\alpha+1}}{|x|^{n-\alpha+1}} v^{-p}(y)\,dy \le C\int_{\mathbb{R}^n} v^{-p}(y)\,dy. \tag{14}$$

For $J_2(x)$, by Lemma 2,

$$\begin{aligned}
J_2(x) &= C\int_{B\left(x,\frac{|x|}{2}\right)} \frac{|x|^{n-\alpha+1}}{|z|^{n-\alpha+1}} v^{-p}(x+z)\,dz \\
&\le C\int_{B\left(0,\frac{|x|}{2}\right)} \frac{|x|^{n-\alpha+1}}{|z|^{n-\alpha+1}} (1+|x+z|^{\alpha-n})^{-p}\,dz.
\end{aligned}$$

Notice that for $|x|$ large enough and $z\in B\left(0,\frac{|x|}{2}\right)$, it easy to check that $|x+z|\ge \frac{|x|}{2}\ge\frac{|z|}{2}$. Therefore,

$$\begin{aligned}
J_2(x) &\le C\int_{B\left(0,\frac{|x|}{2}\right)} \frac{|x|^{n-\alpha+1}}{|z|^{n-\alpha+1}} \cdot \frac{(1+|x+z|^{\alpha-n})^{-p+\frac{n-\alpha+1}{\alpha-n}}}{(1+|x+z|^{\alpha-n})^{\frac{n-\alpha+1}{\alpha-n}}}\,dz \\
&\le C\int_{B\left(0,\frac{|x|}{2}\right)} \frac{|x|^{n-\alpha+1}}{|z|^{n-\alpha+1}} \cdot \frac{(1+|z|^{\alpha-n})^{-p+\frac{n-\alpha+1}{\alpha-n}}}{(1+|x|^{\alpha-n})^{\frac{n-\alpha+1}{\alpha-n}}}\,dz \\
&\le C\int_{\mathbb{R}^n} \frac{(1+|z|^{\alpha-n})^{-p+\frac{n-\alpha+1}{\alpha-n}}}{|z|^{n-\alpha+1}}\,dz \\
&= C\int_{\mathbb{R}^n\setminus B(0,1)} \frac{(1+|z|^{\alpha-n})^{-p+\frac{n-\alpha+1}{\alpha-n}}}{|z|^{n-\alpha+1}}\,dz + C\int_{B(0,1)} \frac{(1+|z|^{\alpha-n})^{-p+\frac{n-\alpha+1}{\alpha-n}}}{|z|^{n-\alpha+1}}\,dz \\
&:= C J_{21}(x) + C J_{22}(x).
\end{aligned}$$

Since $0 < \alpha - n < 1$,

$$J_{22}(x) \le C \int_{B(0,1)} \frac{1}{|z|^{n-\alpha+1}}\, dz < \infty, \tag{15}$$

and

$$J_{21}(x) \le C \int_{\mathbb{R}^n \setminus B(0,1)} \frac{|z|^{-p(\alpha-n)+n-\alpha+1}}{|z|^{n-\alpha+1}}\, dz \le C \int_{\mathbb{R}^n \setminus B(0,1)} \frac{1}{|z|^{p(\alpha-n)}}\, dz < \infty, \tag{16}$$

where we use $p > \frac{\alpha}{\alpha-n}$.

Combining with (14), (15), (16) and using Lebesgue's dominated convergence theorem, we have

$$\lim_{|x_1|\to\infty} \frac{\left|\frac{\partial u(x)}{\partial x_1}\right|}{|x_1|^{\alpha-n-1}} = a(\alpha - n).$$

Recall that $a = \lim_{|x|\to\infty} |x|^{n-\alpha} u(x) = \int_{\mathbb{R}^n} v^{-p}(y)\, dx < \infty$; thus, we obtain (13).

If $|x_1| \ge -N$ and $|2\lambda - x_1| > -N$, by (12), we have

$$|x|^{\frac{n-\alpha}{2}} u(x) > |x_\lambda|^{\frac{n-\alpha}{2}} u_\lambda(x), \quad x \in \Sigma_\lambda.$$

Then we obtain

$$u(x) \ge u_\lambda(x).$$

If $|2\lambda - x_1| \le -N$, by Lemma 2, we can take λ large enough such that

$$u(x) \ge \frac{1+|x_1|^{\alpha-n}}{C_1} \ge \frac{1+|\lambda|^{\alpha-n}}{C_1} \ge C_1(1+|N|^{\alpha-n}) \ge C_1(1+|2\lambda - x_1|^{\alpha-n}) \ge u_\lambda(x).$$

Therefore (11) holds. This completes Step 1.

Step 2. For $x \in \mathbb{R}^n$, define

$$\lambda_0 = \sup\{\mu < 0 \mid u(x) \ge u_\lambda(x),\, v(x) \ge v_\lambda(x) \text{ for all } \lambda < \mu,\ x_1 \le \lambda\}.$$

It is easy to check that $|x_{\lambda_0} - y|^{\alpha-n} - |x - y|^{\alpha-n} > 0$ for any $y_1, x_1 < \lambda_0$. So by (9) and (10), for all $x \in \Sigma_{\lambda_0}$, there are the following four cases:

Case (i): $u(x) = u_{\lambda_0}(x)$, $v(x) = v_{\lambda_0}(x)$.

Case (ii): $u(x) > u_{\lambda_0}(x)$, $v(x) = v_{\lambda_0}(x)$.

Case (iii): $u(x) = u_{\lambda_0}(x)$, $v(x) > v_{\lambda_0}(x)$.

Case (iv): $u(x) > u_{\lambda_0}(x)$, $v(x) > v_{\lambda_0}(x)$.

For Case (i), we are done. Case (ii) and Case (iii) are impossible according to (9) and (10). Thus it suffices to show that Case (iv) is also impossible. We carry out the argument by contradiction. Suppose that Case (iv) is valid. We will show that there exists an $\varepsilon > 0$ such that

$$u(x) \geq u_\lambda(x), \quad v(x) \geq v_\lambda(x), \qquad \text{for all } \lambda_0 < \lambda < \lambda_0 + \varepsilon, \; x \in \Sigma_\lambda. \tag{17}$$

By (9) and (10) and Fatou's lemma, for all $x_1 \leq \lambda_0 - 1$, we can calculate

$$\begin{aligned}\lim_{|x|\to\infty} &\inf |x|^{n-\alpha}(u(x) - u_{\lambda_0}(x)) \\ &= \int_{\Sigma_{\lambda_0}} \lim_{|x|\to\infty} \inf |x|^{n-\alpha}(|x_{\lambda_0} - y|^{\alpha-n} - |x-y|^{\alpha-n})(v_{\lambda_0}^{-p}(y) - v^{-p}(y))\,dy > 0\end{aligned}$$

and

$$\begin{aligned}\lim_{|x|\to\infty} &\inf |x|^{n-\alpha}(v(x) - v_{\lambda_0}(x)) \\ &= \int_{\Sigma_{\lambda_0}} \lim_{|x|\to\infty} \inf |x|^{n-\alpha}(|x_{\lambda_0} - y|^{\alpha-n} - |x-y|^{\alpha-n})(u_{\lambda_0}^{-q}(y) - u^{-q}(y))\,dy > 0.\end{aligned}$$

Then, for any $x_1 \leq \lambda_0 - 1$, there exists $\varepsilon_1 \in (0, 1)$ such that

$$u(x) - u_{\lambda_0}(x) \geq \varepsilon_1 |x|^{\alpha-n}, \quad v(x) - v_{\lambda_0}(x) \geq \varepsilon_1 |x|^{\alpha-n}.$$

We can use continuity of u and v with respect to the variable λ to obtain that

$$\begin{aligned} u(x) - u_\lambda(x) &\geq \varepsilon_1 |x|^{\alpha-n} + (u_\lambda(x) - u_{\lambda_0}(x)) \\ &\geq \frac{\varepsilon_1}{2}|x|^{\alpha-n}, \quad \text{for all } x_1 \leq \lambda_0 - 1, \; \lambda_0 \leq \lambda \leq \lambda_0 + \varepsilon_2 \end{aligned} \tag{18}$$

and

$$\begin{aligned} v(x) - v_\lambda(x) &\geq \varepsilon_1 |x|^{\alpha-n} + (v_\lambda(x) - v_{\lambda_0}(x)) \\ &\geq \frac{\varepsilon_1}{2}|x|^{\alpha-n}, \quad \text{for all } x_1 \leq \lambda_0 - 1, \; \lambda_0 \leq \lambda \leq \lambda_0 + \varepsilon_2 \end{aligned} \tag{19}$$

for sufficiently small $\varepsilon_2 > 0$. Thus, for $\varepsilon \in (0, \varepsilon_2)$, which we choose below, it suffices to verify that for $\lambda_0 - 1 \leq x_1 \leq \lambda$,

$$u(x) \geq u_\lambda(x), \quad v(x) \geq v_\lambda(x), \quad \text{for all } \lambda_0 \leq \lambda \leq \lambda_0 + \varepsilon.$$

We only show that

$$u(x) \geq u_\lambda(x), \quad \text{for all } \lambda_0 \leq \lambda \leq \lambda_0 + \varepsilon, \; \lambda_0 - 1 \leq x_1 \leq \lambda.$$

By (13), there exists $R_1 \geq 4(|\lambda_0| + 1)$ large enough such that, for any $x \in (\Sigma_\lambda \setminus \Sigma_{\lambda_0 - 1}) \setminus B(0, R_1/2)$,

$$\nabla(|x|^{\frac{n-\alpha}{2}} \cdot x) = |x|^{\frac{n-\alpha}{2}}\left(\nabla u(x) \cdot x + \frac{n-\alpha}{2}u(x)\right) > 0.$$

We know that if $|x| \geq R_1$, then $|x_\lambda| \geq \frac{R_1}{2}$; thus

$$|x|^{\frac{n-\alpha}{2}} u(x) > |x_\lambda|^{\frac{n-\alpha}{2}} u_\lambda(x).$$

Therefore,

$$u(x) \geq u_\lambda(x), \quad \text{for } |x| \geq R_1.$$

Now we fix R_1 and consider

$$x \in (\Sigma_\lambda \setminus \Sigma_{\lambda_0 - 1}) \cap B(0, R_1).$$

By Lemma 3, we can write

$$\begin{aligned}
u(x) - u_\lambda(x) &= \int_{\Sigma_\lambda} K(\lambda, x, y)(v_\lambda^{-p}(y) - v^{-p}(y))\, dy \\
&\geq \int_{\Sigma_\lambda \setminus \Sigma_{\lambda_0 - 1}} K(\lambda, x, y)(v_\lambda^{-p}(y) - v^{-p}(y))\, dy \\
&\qquad + \int_{\Sigma_{\lambda_0 - 2} \setminus \Sigma_{\lambda_0 - 3}} K(\lambda, x, y)(v_\lambda^{-p}(y) - v^{-p}(y))\, dy \\
&\geq \int_{(\Sigma_\lambda \setminus \Sigma_{\lambda_0 - 1}) \setminus B(0, R_0)} K(\lambda, x, y)(v_\lambda^{-p}(y) - v_{\lambda_0}^{-p}(y))\, dy \\
&\qquad + \int_{(\Sigma_\lambda \setminus \Sigma_{\lambda_0 - 1}) \cap B(0, R_0)} K(\lambda, x, y)(v_\lambda^{-p}(y) - v_{\lambda_0}^{-p}(y))\, dy \\
&\qquad + \int_{\Sigma_{\lambda_0 - 2} \setminus \Sigma_{\lambda_0 - 3}} K(\lambda, x, y)(v_\lambda^{-p}(y) - v^{-p}(y))\, dy \\
&:= I_1(x) + I_2(x) + I_3(x).
\end{aligned}$$

We first estimate $I_1(x)$. Take $R_0 \geq 2R_1$ large enough such that

$$\frac{1}{R_0^{p(\alpha - n) - \alpha + 1}} \leq \varepsilon,$$

where we use $p > \frac{\alpha}{\alpha - n}$. Since

$$x \in (\Sigma_\lambda \setminus \Sigma_{\lambda_0 - 1}) \cap B(0, R_1) \quad \text{and} \quad y \in (\Sigma_\lambda \setminus \Sigma_{\lambda_0 - 1}) \setminus B(0, R_0),$$

it easy to derive that $|x - y| \sim |x_\lambda - y| \sim |y|$; then

$$\begin{aligned}
K(\lambda, x, y) &= |x - y_\lambda|^{\alpha - n} - |x - y|^{\alpha - n} \\
&\leq C \max\{|x - y_\lambda|^{\alpha - n - 1}, |x - y|^{\alpha - n - 1}\} \big| |x - y_\lambda| - |x - y| \big| \\
&\leq C |y|^{\alpha - n - 1} (\lambda - x_1).
\end{aligned}$$

Thus, by using Lemma 2,

$$\tag{20} \left|\int_{(\Sigma_\lambda\setminus\Sigma_{\lambda_0-1})\setminus B(0,R_0)} K(\lambda,x,y)(v_\lambda^{-p}(y)-v_{\lambda_0}^{-p}(y))\,dy\right|$$
$$\begin{aligned}
&\le C(\lambda-x_1)\int_{(\Sigma_\lambda\setminus\Sigma_{\lambda_0-1})\setminus B(0,R_0)} |y|^{\alpha-n-1}|y|^{-p(\alpha-n)}\,dy\\
&\le C(\lambda-x_1)\int_{\mathbb{R}^n\setminus B(0,R_0)} |y|^{\alpha-n-1-p(\alpha-n)}\,dy\\
&= C(\lambda-x_1)\int_{R_0}^{\infty} r^{\alpha-2-p(\alpha-n)}\,dr\\
&= C(\lambda-x_1)R_0^{\alpha-1-p(\alpha-n)}\\
&\le C\varepsilon(\lambda-x_1).
\end{aligned}$$

Now, we estimate $I_2(x)$. We can take $0<\varepsilon_3<\varepsilon$ small enough such that for any $\lambda_0\le\lambda\le\lambda_0+\varepsilon_3$ and $y\in(\Sigma_\lambda\setminus\Sigma_{\lambda_0-1})\cap B(0,R_0)$,

$$|v_\lambda^{-p}(y)-v_{\lambda_0}^{-p}(y)|\le C_1\varepsilon.$$

Then,

$$\tag{21}\begin{aligned}
|I_2(x)| &\le \int_{(\Sigma_\lambda\setminus\Sigma_{\lambda_0-1})\cap B(0,R_0)} |K(\lambda,x,y)(v_\lambda^{-p}(y)-v_{\lambda_0}^{-p}(y))|\,dy\\
&\le C(R_0,R_1)|x_\lambda-x|\int_{(\Sigma_\lambda\setminus\Sigma_{\lambda_0-1})\cap B(0,R_0)} |v_\lambda^{-p}(y)-v_{\lambda_0}^{-p}(y)|\,dy\\
&\le C\varepsilon(\lambda-x_1).
\end{aligned}$$

For $I_3(x)$, we can assume $\lambda_0\le\lambda\le\lambda_0+\varepsilon_3$. By (18) and (19), there exists $\delta_1>0$ which is independent of ε_3 such that

$$\tag{22} v_\lambda^{-p}(y)-v^{-p}(y)\ge\delta_1,\quad \text{for any } y\in\Sigma_{\lambda_0-2}\setminus\Sigma_{\lambda_0-3}.$$

Notice that

$$\frac{\partial K(\lambda,x,y)}{\partial x_1}\cdot x_1|_{x_1=\lambda}=2(\alpha-n)|x-y|^{\alpha-n-2}(y_1-\lambda)x_1>0,\quad y_1<\lambda.$$

Since $K(\lambda,x,y)|_{x_1=\lambda}=0$, for any

$$x\in(\Sigma_\lambda\setminus\Sigma_{\lambda_0-1})\cap B(0,R_1)\quad\text{and}\quad y\in(\Sigma_{\lambda_0-2}\setminus\Sigma_{\lambda_0-3}),$$

there exists $\delta_2>0$ which are independent of $\varepsilon_3,\varepsilon$, such that

$$\tag{23} K(\lambda,x,y)|\ge\delta_2(\lambda-x_1).$$

By (20), (21), (22), and (23), for sufficiently small $\varepsilon > 0$, we conclude that

$$u(x) - u_\lambda(x) \geq (\delta_1\delta_2 - C_1C_2\varepsilon)(\lambda - x_1) \geq 0.$$

Combining with (18) and (19), we can derive a contradiction with the definition of λ_0. This completes the proof of Step 2. Thus, we have finished the proof of the main theorem.

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