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MULTIPLICITY UPON RESTRICTION TO THE DERIVED SUBGROUP

JEFFREY D. ADLER AND DIPENDRA PRASAD

We present a conjecture on multiplicity of irreducible representations of a subgroup *H* contained in the irreducible representations of a group *G*, with *G* and *H* having the same derived groups. We point out some consequences of the conjecture, and verification of some of the consequences. We give an explicit example of multiplicity 2 upon restriction, as well as certain theorems in the context of classical groups where the multiplicity is 1.

1. Introduction

Suppose k is a local field, G is a connected reductive k-group, G' is a subgroup of G containing the derived group, and π is a smooth, irreducible, complex representation of $\mathrm{G}(k)$. In an earlier work [Adler and Prasad 2006], we showed that for many choices of G, the restriction $\mathrm{Res}_{\mathrm{G}'(k)}^{\mathrm{G}(k)}\,\pi$ decomposes without multiplicity.

A number of years ago, in the process of identifying situations where multiplicity one did not hold, one of us discovered an example of a depth-zero supercuspidal representation of $\mathrm{GU}(2d, 2d)$, a k-quasisplit group, whose restriction to $\mathrm{SU}(2d, 2d)$ decomposes with multiplicity two, and the other formulated a conjecture in the form of a reciprocity law involving enhanced Langlands parameters. In this paper, we present both the example and the conjecture, together with some consequences of the latter, and a verification of some of those consequences. Besides these, the paper proves several results by elementary means involving classical groups where multiplicity one holds.

A complete analysis of decomposition of the unitary principal series for $\mathrm{U}(n, n)$ and its restriction to $\mathrm{SU}(n, n)$ was done by Keys [1987], who also phrased his results in terms of "reciprocity" theorems for R-groups; in particular, he found cases of multiplicity greater than one.

After presenting our conjecture (Section 2), we give some of the heuristics behind it. In the formulation of the conjecture, we have considered a more general situation than that of a subgroup. We consider G_1 and G_2 to be two connected reductive groups over a local field k, and $\lambda : \mathrm{G}_1 \to \mathrm{G}_2$ to be a k-homomorphism

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that is a central isogeny when restricted to their derived subgroups, allowing us to "restrict" representations of $G_2(k)$ to $G_1(k)$. Since under such a homomorphism λ, the image of $G_1(k)$ is a normal subgroup of $G_2(k)$ with abelian quotient, all the irreducible representations of $G_1(k)$ which appear in this restriction problem for a given irreducible representation of $G_2(k)$ appear with the same multiplicity. In Section 3, we verify that for our conjectural multiplicity, this relationship does indeed hold. We show (Section 4) that if the conjecture is true for tempered representations, then via the Langlands classification it holds for all representations.

Our conjecture (for $\lambda : G_1 \to G_2$ a k-homomorphism) implies multiplicity one in situations where Langlands parameters for G_1 have abelian component groups. We list a few such situations in Section 5, and prove multiplicity one for restriction from $GU(n)$ to $U(n)$ (Section 6). Along the way, we prove multiplicity one in some other cases where it follows from elementary considerations. In Section 7, we present an example of a depth-zero supercuspidal representation of quasisplit $GU(2d, 2d)$ that decomposes with multiplicity two upon restriction to $SU(2d, 2d)$. Finally (Section 8), we give a general procedure for constructing higher multiplicities.

2. The conjecture on multiplicities

Let G_1^{qs} and G_2^{qs} be two connected quasisplit reductive groups over a local field k and let $\lambda : G_1^{qs} \to G_2^{qs}$ be a k-homomorphism that is a central isogeny when restricted to their derived subgroups. In what follows we will be twisting G_1^{qs} by a cohomology class in $H^1(\mathrm{Gal}(\bar{k}/k), G_1^{qs}(\bar{k}))$ to construct a pure inner form G_1 of G_1^{qs}. Simultaneously, by twisting G_2^{qs} by the image of this class under the map $H^1(\mathrm{Gal}(\bar{k}/k), G_1^{qs}(\bar{k})) \to H^1(\mathrm{Gal}(\bar{k}/k), G_2^{qs}(\bar{k}))$, we will have a pure inner form G_2 of G_2^{qs}, together with a map of algebraic groups that we will still call $\lambda : G_1 \to G_2$, which will appear in considerations below, all coming from an element of $H^1(\mathrm{Gal}(\bar{k}/k), G_1^{qs}(\bar{k}))$.

The map $\lambda : G_1 \to G_2$ gives rise to a "restriction" map from representations of $G_2(k)$ to those of $G_1(k)$, and from [Silberger 1979] one knows that the restriction of an irreducible representation of $G_2(k)$ is a finite direct sum of irreducible representations of $G_1(k)$. In particular, we obtain a functor $\lambda^\star : \mathcal{R}_{\mathrm{fin}}(G_2(k)) \to \mathcal{R}_{\mathrm{fin}}(G_1(k))$, where $\mathcal{R}_{\mathrm{fin}}(H)$ denotes the category of smooth, finite-length representations of a group H.

Let ${}^L G_1 = \widehat{G}_1 \rtimes W_k'$ and ${}^L G_2 = \widehat{G}_2 \rtimes W_k'$ be the L-groups associated to the quasisplit reductive groups G_1^{qs} and G_2^{qs} respectively. The map $\lambda : G_1^{qs} \to G_2^{qs}$ also gives rise to a homomorphism of L-groups,

$$ {}^L\lambda : {}^L G_2 \to {}^L G_1, $$

as well as a homomorphism of their centers,

$$ {}^L\lambda : Z(\widehat{G}_2)^{W_k} \to Z(\widehat{G}_1)^{W_k}. $$

It follows, in particular, that a character χ_1 of $\pi_0(Z(\widehat{\mathrm{G}}_1)^{W_k})$ gives rise to a character χ_2 of $\pi_0(Z(\widehat{\mathrm{G}}_2)^{W_k})$ which, by the Kottwitz isomorphism (assuming k to be nonarchimedean at this point),

$$H^1(\mathrm{Gal}(\bar{k}/k), \mathrm{G}_i^{\mathrm{qs}}(\bar{k})) \cong \mathrm{Hom}(\pi_0(Z(\widehat{\mathrm{G}}_i)^{W_k}), \mathbf{Q}/\mathrm{Z}),$$

constructs pure inner forms G_1 of $\mathrm{G}_1^{\mathrm{qs}}$ and G_2 of $\mathrm{G}_2^{\mathrm{qs}}$, together with a map $\lambda : \mathrm{G}_1 \to \mathrm{G}_2$ as before.

Let $\varphi_2 : W_k' \to {}^L\mathrm{G}_2$, and $\varphi_1 = {}^L\lambda \circ \varphi_2 : W_k' \to {}^L\mathrm{G}_1$ be associated Langlands parameters, where $W_k' = W_k \times \mathrm{SL}_2(\mathbf{C})$, with W_k the Weil group of k. Then ${}^L\lambda$ gives rise to a homomorphism of centralizers of the images of the parameters φ_1 with values in ${}^L\mathrm{G}_1$ and φ_2 with values in ${}^L\mathrm{G}_2$, and also a homomorphism of the groups of connected components of their centralizers:

$$\pi_0({}^L\lambda) : \pi_0(Z_{\widehat{\mathrm{G}}_2}(\varphi_2)) \to \pi_0(Z_{\widehat{\mathrm{G}}_1}(\varphi_1)).$$

This allows one to "restrict" representations of $\pi_0(Z_{\widehat{\mathrm{G}}_1}(\varphi_1))$ to representations of $\pi_0(Z_{\widehat{\mathrm{G}}_2}(\varphi_2))$, giving rise to the restriction functor

$$\lambda_\star : K_0(\pi_0(Z_{\widehat{\mathrm{G}}_1}(\varphi_1))) \to K_0(\pi_0(Z_{\widehat{\mathrm{G}}_2}(\varphi_2))),$$

where $K_0(H)$ denotes the Grothendieck group of finite-length representations of a group H.

The formulation of our conjecture below presumes that the local Langlands correspondence involving enhanced Langlands parameters has been achieved, giving rise to a bijection between enhanced Langlands parameters and the set of isomorphism classes of irreducible admissible representations of all pure inner forms of quasisplit groups. This will be needed for *both* of the groups G_1 and G_2; it is possible on the other hand that one could reverse this role, and use the conjectural multiplicity formula to construct an enhanced Langlands parametrization for G_2, knowing it for G_1.

Conjecture 1. (a) *Let* G_1 *and* G_2 *be two connected reductive groups over a local field k and let* $\lambda : \mathrm{G}_1 \to \mathrm{G}_2$ *be a k-homomorphism that is a central isogeny when restricted to their derived subgroups. For* $i = 1, 2$, *let* π_i *be an irreducible admissible representation of* $\mathrm{G}_i(k)$ *with Langlands parameter* φ_i. *Let*

$$\mathrm{m}(\pi_2, \pi_1) := \dim \mathrm{Hom}_{\mathrm{G}_1(k)}[\pi_1, \lambda^\star\pi_2] = \dim \mathrm{Hom}_{\mathrm{G}_1(k)}[\lambda^\star\pi_2, \pi_1].$$

Then $\mathrm{m}(\pi_2, \pi_1) = 0$ *unless* $\varphi_1 = {}^L\lambda \circ \varphi_2$.

(b) *Let* $\mathrm{G}_1^{\mathrm{qs}}$ *and* $\mathrm{G}_2^{\mathrm{qs}}$ *be two connected reductive quasisplit groups over a local field k and let* $\lambda : \mathrm{G}_1^{\mathrm{qs}} \to \mathrm{G}_2^{\mathrm{qs}}$ *be a k-homomorphism that is a central isogeny when restricted to their derived subgroups. Let* φ_1 *and* φ_2 *be Langlands parameters associated to the groups* $\mathrm{G}_1^{\mathrm{qs}}$ *and* $\mathrm{G}_2^{\mathrm{qs}}$ *with* $\varphi_1 = {}^L\lambda \circ \varphi_2$, *and let* χ_i *be characters of their component groups* $\pi_0(Z_{\widehat{\mathrm{G}}_i}(\varphi_i))$. *Then, if* $\mathrm{Hom}_{\pi_0(Z(\varphi_2))}[\chi_2, \lambda_\star\chi_1]$ *is nonzero, the* *characters χ_i define pure inner forms G_i of $\mathrm{G}_i^{\mathrm{qs}}$ together with a k-homomorphism, $\lambda : \mathrm{G}_1 \to \mathrm{G}_2$, as discussed earlier. Then if $\pi_i = \pi(\varphi_i, \chi_i)$ are the corresponding irreducible admissible representations of $\mathrm{G}_i(k)$, we have*

$$\mathrm{m}(\pi_2, \pi_1) = \dim \mathrm{Hom}_{\pi_0(Z(\varphi_2))}[\chi_2, \lambda_\star \chi_1].$$

The main heuristic for the conjectural multiplicity is the following.

(1) For any L-packet $\{\pi\}$ on any reductive group $\mathrm{G}(k)$ defined by a parameter φ (that is, $\{\pi\} = \{\pi_{(\varphi,\chi)}\}$ where one takes those characters χ of the component group which have a particular restriction to $Z(\widehat{\mathrm{G}})^{W_k}$ defining the group $\mathrm{G}(k)$ assumed to be a pure inner form of a fixed quasisplit group G^{qs}),

$$\sum_{\chi} \chi(1)\Theta(\pi_{(\varphi,\chi)})$$

is a stable distribution on $\mathrm{G}(k)$. Here, for any admissible representation π we are letting $\Theta(\pi)$ denote its character, regarded as a distribution on $\mathrm{G}(k)$.

(2) For a homomorphism $\lambda : \mathrm{G}_1 \to \mathrm{G}_2$ of reductive groups over k which is an isogeny when restricted to their derived subgroups, the pullback of a stable distribution on $\mathrm{G}_2(k)$ is a stable distribution on $\mathrm{G}_1(k)$.

(3) The restriction to $\mathrm{G}_1(k)$ of an irreducible representation π_2 of $\mathrm{G}_2(k)$ is a finite-length (completely reducible) representation of $\mathrm{G}_1(k)$, whose irreducible components are all in the same L-packet. This L-packet for $\mathrm{G}_1(k)$ depends only on the L-packet for $\mathrm{G}_2(k)$ containing π_2. If the Langlands parameter of our L-packet for $\mathrm{G}_2(k)$ is $\varphi_2 : W_k' \to {}^L\mathrm{G}_2$, then the Langlands parameter of our L-packet for $\mathrm{G}_1(k)$ is $\varphi_1 := {}^L\lambda \circ \varphi_2 : W_k' \to {}^L\mathrm{G}_1$. (This is part (a) of the conjecture.)

(4) If Conjecture 1 is true, then the pullback from $\mathrm{G}_2(k)$ to $\mathrm{G}_1(k)$ of the distribution

$$\sum_{\chi_2} \chi_2(1)\Theta(\pi_{(\varphi_2,\chi_2)}),$$

where the sum is taken over those characters χ_2 of the component group which have a particular restriction to $Z(\widehat{G}_2)^{W_k}$ defining the group $\mathrm{G}_2(k)$ assumed to be a pure inner form of a fixed quasisplit group $\mathrm{G}_2^{\mathrm{qs}}(k)$, is a stable distribution on $\mathrm{G}_1(k)$ as we check now.

By Conjecture 1, the pullback of the distribution $\Theta_{\pi_2} = \Theta(\pi_{(\varphi_2,\chi_2)})$ on $\mathrm{G}_2(k)$ to $\mathrm{G}_1(k)$ is

$$\sum_{\pi_1} m(\pi_2, \pi_1)\Theta(\pi_1) = \sum_{\chi_1} \Theta(\pi_{(\varphi_1,\chi_1)}) \dim \mathrm{Hom}_{\pi_0(Z(\varphi_2))}[\chi_2, \lambda_\star \chi_1].$$

Therefore, the pullback to $G_1(k)$ of the distribution $\sum_{\chi_2} \chi_2(1)\Theta(\pi_{(\varphi_2,\chi_2)})$ on $G_2(k)$ is (assuming Conjecture 1)

$$\sum_{\chi_1,\chi_2} \chi_2(1)\Theta(\pi_{(\varphi_1,\chi_1)}) \dim \mathrm{Hom}_{\pi_0(Z(\varphi_2))}[\chi_2, \lambda_\star\chi_1],$$

which is the same as

$$\sum_{\chi_1,\chi_2} \Theta(\pi_{(\varphi_1,\chi_1)}) \dim \mathrm{Hom}_{\pi_0(Z(\varphi_2))}[\chi_2(1)\chi_2, \lambda_\star\chi_1],$$

where the sum is taken over all pairs of characters χ_1, χ_2 with particular restrictions to $Z(\widehat{G}_1)^{W_k}$ and $Z(\widehat{G}_2)^{W_k}$. Observe that those characters χ_2 whose restrictions to $Z(\widehat{G}_2)^{W_k}$ are not compatible with the restriction of χ_1 to $Z(\widehat{G}_1)^{W_k}$ contribute 0 to the sum. Therefore, we can take the sum over all χ_2. The sum then is the same as

$$(*) \qquad \sum_{\chi_1} \Theta(\pi_{(\varphi_1,\chi_1)}) \dim \mathrm{Hom}_{\pi_0(Z(\varphi_2))}[R, \lambda_\star\chi_1],$$

where $R = \sum \chi_2(1)\chi_2$ is the regular representation of $\pi_0(Z(\varphi_2))$.

By Schur orthogonality,

$$\dim \mathrm{Hom}_{\pi_0(Z(\varphi_2))}[\chi_2, \lambda_\star\chi_1] = \frac{1}{|\pi_0(Z(\varphi_2))|} \sum_{g\in\pi_0(Z(\varphi_2))} \chi_1(\lambda^\star g)\overline{\chi}_2(g),$$

where $\lambda^\star$ denotes the map $\pi_0({}^L\lambda)\colon \pi_0(Z(\varphi_2)) \to \pi_0(Z(\varphi_1))$. So

$$\dim \mathrm{Hom}_{\pi_0(Z(\varphi_2))}[R, \lambda_\star\chi_1] = \frac{1}{|\pi_0(Z(\varphi_2))|} \sum_{g\in\pi_0(Z(\varphi_2))} \chi_1(\lambda^\star g)\chi_R(g),$$

where R is the regular representation of $\pi_0(Z(\varphi_2))$ and χ_R its character, thus

$$\chi_R(g) = \begin{cases} 0 & \text{if } g \text{ is not the identity,} \\ |\pi_0(Z(\varphi_2))| & \text{if } g \text{ is the identity.} \end{cases}$$

Therefore,

$$\dim \mathrm{Hom}_{\pi_0(Z(\varphi_2))}[R, \lambda_\star\chi_1] = \chi_1(1).$$

By $(*)$ it follows that the pullback of the distribution $\sum_{\chi_2} \chi_2(1)\Theta(\pi_{(\varphi_2,\chi_2)})$ on $G_2(k)$ to $G_1(k)$ is equal to $\sum_{\chi_1} \chi_1(1)\Theta(\pi_{(\varphi_1,\chi_1)})$, where the sum is taken over those χ_1 with a given restriction to $Z(\widehat{G}_1)^{W_k}$. Thus the pullback of the distribution $\sum_{\chi_2} \chi_2(1)\Theta(\pi_{(\varphi_2,\chi_2)})$ on $G_2(k)$ to $G_1(k)$ is a stable distribution on $G_1(k)$ which is what we set out to prove.

Remark 2. A weaker version of our conjecture says that the pullback to $G_1(k)$ of the stable character $\sum_\chi \chi(1)\Theta_\chi$ on $G_2(k)$ is $\sum_\mu \mu(1)\Theta_\mu$ on $G_1(k)$, where both of the sums are over the characters of component groups defining fixed pure inner forms that are G_2 and G_1, respectively.

3. Some remarks on the multiplicity formula

Conjecture 1 relating $\mathrm{m}(\pi_2, \pi_1)$ with $\dim \mathrm{Hom}_{\pi_0(Z(\varphi_2))}[\lambda_\star\chi_1, \chi_2]$ can be considered as a set of assertions keeping π_2 fixed and varying π_1, or keeping π_1 fixed and varying π_2, say, inside an L-packet for $\mathrm{G}_2(k)$. It is easy to see that for G_1 and G_2 two reductive groups over a local field k, and $\lambda : \mathrm{G}_1 \to \mathrm{G}_2$, a k-homomorphism that is a central isogeny when restricted to their derived subgroups, the image of $\mathrm{G}_1(k)$ inside $\mathrm{G}_2(k)$ is a normal subgroup, and therefore every irreducible representation of $\mathrm{G}_1(k)$ that appears inside a given irreducible representation π_2 of $\mathrm{G}_2(k)$ does so with the same multiplicity (depending, of course, on π_2). This section aims to prove this as a consequence of our Conjecture 1.

This section is meant to prove that $\dim \mathrm{Hom}_{\pi_0(Z(\varphi_2))}[\lambda_\star\chi_1, \chi_2]$ remains constant when χ_2 is a fixed character of $\pi_0(Z(\varphi_2))$ but χ_1 varies among characters of $\pi_0(Z(\varphi_1))$. This is achieved by combining Corollary 4 with Lemma 5. We begin with the following lemma whose straightforward proof will be omitted.

Lemma 3. *Let N be a normal subgroup of a finite group G with $A = G/N$ an abelian group. Let π be an irreducible representation of N. Then any two irreducible representations π_1 and π_2 of G containing π on restriction to N are twists of each other by characters of G/N, i.e.,*

$$\pi_2 \cong \pi_1 \otimes \chi,$$

for $\chi : G/N \to \mathbb{C}^\times$.

Corollary 4. *If N is a normal subgroup of a group G with $A = G/N$ a finite abelian group, and π an irreducible representation of N, then all irreducible G-submodules of $\mathrm{Ind}_N^G(\pi)$ appear in it with the same multiplicity.*

Lemma 5. *Let G_1 and G_2 be two connected reductive groups over a local field k and let $\lambda\colon \mathrm{G}_1 \to \mathrm{G}_2$ be a k-homomorphism that is a central isogeny when restricted to their derived subgroups, and giving rise to a homomorphism ${}^L\lambda\colon {}^L\mathrm{G}_2 \to {}^L\mathrm{G}_1$ of the L-groups. Let $\varphi_2 : W_k' \to {}^L\mathrm{G}_2$, and $\varphi_1 = {}^L\lambda \circ \varphi_2 : W_k' \to {}^L\mathrm{G}_1$ be associated Langlands parameters. Then for the associated homomorphism of finite groups $\lambda^\star\colon \pi_0(Z_{\widehat{\mathrm{G}}_2}(\varphi_2)) \to \pi_0(Z_{\widehat{\mathrm{G}}_1}(\varphi_1))$, the image is normal with abelian cokernel.*

Proof. It suffices to prove the lemma separately in the two cases:

(1) $\lambda\colon \mathrm{G}_1 \to \mathrm{G}_2$ is injective as a homomorphism of algebraic groups.

(2) $\lambda\colon \mathrm{G}_1 \to \mathrm{G}_2$ is surjective as a homomorphism of algebraic groups.

We will address only the first case, the other being very similar.

Assume then that $\lambda\colon \mathrm{G}_1 \to \mathrm{G}_2$ is injective, and thus $\widehat{\lambda}\colon \widehat{\mathrm{G}}_2 \to \widehat{\mathrm{G}}_1$ is surjective with kernel, say, $\widehat{Z}$. Use $\varphi_2\colon W_k' \to {}^L\mathrm{G}_2$ and $\varphi_1 = {}^L\lambda \circ \varphi_2 : W_k' \to {}^L\mathrm{G}_1$ to give $\widehat{\mathrm{G}}_2$

and $\widehat{\mathrm{G}}_1$, a W_k'-group structure, such that we have an exact sequence of W_k'-groups,

$$1 \to \widehat{Z} \to \widehat{\mathrm{G}}_2 \to \widehat{\mathrm{G}}_1 \to 1.$$

This gives rise to a long exact sequence of W_k'-cohomology sets:

$$1 \to \widehat{Z}^{W_k'} \to \widehat{\mathrm{G}}_2^{W_k'} \to \widehat{\mathrm{G}}_1^{W_k'} \to H^1(W_k', \widehat{Z}) \to \cdots .$$

Equivalently, we have the exact sequence of groups,

$$1 \to Z_{\widehat{\mathrm{G}}_2}(\varphi_2)/\widehat{Z}^{W_k'} \to Z_{\widehat{\mathrm{G}}_1}(\varphi_1) \to A \to 1,$$

where A is a subgroup of $H^1(W_k', \widehat{Z})$, a locally compact abelian group. Taking π_0 of the terms in the above exact sequence which all fit together in a long exact sequence of π_i's (higher homotopy groups), the assertion in the lemma follows on noting that if $E_1 \to E_2$ is a surjective map of locally compact and locally connected topological groups, then the induced map $\pi_0(E_1) \to \pi_0(E_2)$ is also surjective. □

4. Reduction of the conjecture to the case of tempered representations

As before, let G_1 and G_2 be two reductive groups over a local field k, and let $\lambda : \mathrm{G}_1 \to \mathrm{G}_2$ be a k-homomorphism that is a central isogeny when restricted to their derived subgroups, giving rise to the restriction functor

$$\lambda^\star : \mathcal{R}_{\mathrm{fin}}(\mathrm{G}_2(k)) \to \mathcal{R}_{\mathrm{fin}}(\mathrm{G}_1(k)).$$

Lemma 6. *Let V be a finite-length representation of $\mathrm{G}_2(k)$ with maximal semisimple quotient Q. Then $\lambda^\star Q$ is the maximal semisimple quotient of $\lambda^\star V$, a finite-length representation of $G_1(k)$.*

Proof. It suffices to observe that a finite-length representation of $\mathrm{G}_2(k)$ is semisimple if and only if its image under $\lambda^\star$ is a finite-length, semisimple representation of $\mathrm{G}_1(k)$. If $\mathrm{Z}(\mathrm{G}_1)(k)\cdot\mathrm{G}_1(k)$ is of finite index in $\mathrm{G}_2(k)$, such as when k is of characteristic zero, then this is easy to see. By a theorem of Silberger [1979], irreducible representations of $\mathrm{G}_2(k)$ remain finite-length semisimple representations when restricted to $\mathrm{G}_1(k)$, and the lemma follows in general. □

To set up the next result, let $P_2 = M_2N_2$ be a Levi factorization of a parabolic subgroup in G_2. If we let $P_1 = \lambda^{-1}(P_2)$, $M_1 = \lambda^{-1}(M_2)$, and $N_1 = \lambda^{-1}(N_2)$, then $P_1 = M_1N_1$ is a Levi factorization of a parabolic subgroup in G_1. Then $\lambda : M_1 \to M_2$ gives us a restriction functor $\mathcal{R}_{\mathrm{fin}}(M_2(k)) \to \mathcal{R}_{\mathrm{fin}}(M_1(k))$ that we will also denote by $\lambda^\star$. Since λ gives an isomorphism $\mathrm{G}_1(k)/P_1(k) \to \mathrm{G}_2(k)/P_2(k)$, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{R}_{\mathrm{fin}}(\mathrm{G}_2(k)) & \xrightarrow{\lambda^\star} & \mathcal{R}_{\mathrm{fin}}(\mathrm{G}_1(k)) \\ {\scriptstyle \mathrm{Ind}_{P_2(k)}^{\mathrm{G}_2(k)}}\uparrow & & {\scriptstyle \mathrm{Ind}_{P_1(k)}^{\mathrm{G}_1(k)}}\uparrow \\ \mathcal{R}_{\mathrm{fin}}(M_2(k)) & \xrightarrow{\lambda^\star} & \mathcal{R}_{\mathrm{fin}}(M_1(k)) \end{array}$$

Lemma 7. *Let σ_2 be an irreducible, essentially tempered representation of $M_2(k)$ with strictly positive exponents along the center $\mathrm{Z}(M_2)(k)$ of $M_2(k)$. Write*

$$\lambda^\star \sigma_2 = \sum_\alpha m_\alpha \sigma_{1,\alpha},$$

a sum of irreducible, essentially tempered representations of $M_1(k)$ with (finite) multiplicities m_α. Let π_2 be the Langlands quotient of the standard module $\mathrm{Ind}_{P_2(k)}^{\mathrm{G}_2(k)} \sigma_2$, and $\pi_{1,\alpha}$ the Langlands quotients of $\mathrm{Ind}_{P_1(k)}^{\mathrm{G}_1(k)} \sigma_{1,\alpha}$. Then

$$\lambda^\star \pi_2 = \sum_\alpha m_\alpha \pi_{1,\alpha}.$$

Proof. Clearly,

$$\lambda^\star \mathrm{Ind}_{P_2(k)}^{\mathrm{G}_2(k)} \sigma_2 = \mathrm{Ind}_{P_1(k)}^{\mathrm{G}_1(k)} \lambda^\star \sigma_2 = \sum_\alpha m_\alpha \mathrm{Ind}_{P_1(k)}^{\mathrm{G}_1(k)} \sigma_{1,\alpha}.$$

Since "taking maximal semisimple quotient" commutes with direct sum, our result follows from Lemma 6. □

Corollary 8. *If Conjecture 1 is true for tempered representations, then it is true in general.*

Proof. Every representation π_2 of $\mathrm{G}_2(k)$ can be realized as a Langlands quotient of a standard module $\mathrm{Ind}_{P_2(k)}^{\mathrm{G}_2(k)} \sigma_2$ for an essentially tempered representation σ_2 of $M_2(k)$. The Langlands parameter $\varphi_2\colon W_F' \to {}^L\mathrm{G}_2$ for π_2 is the same as the Langlands parameter φ_2 for σ_2 considered as a map $W_F' \xrightarrow{\varphi_2} {}^L M_2 \to {}^L\mathrm{G}_2$. The component groups of these parameters, and thus the representations of these component groups, correspond as discussed in [Prasad 2019, §5]. Therefore, our result is a consequence of Lemma 7. □

5. Consequences of the conjecture

If the group of connected components $\pi_0(Z_{\widehat{\mathrm{G}}_1}(\varphi_1))$ is known to be abelian, as is the case when G_1 is any of the groups SL_n, U_n, SO_n, and Sp_n, then our conjecture predicts that for any homomorphism $\lambda\colon \mathrm{G}_1 \to \mathrm{G}_2$ of connected reductive algebraic groups that is an isomorphism up to center (i.e., $\bar\lambda\colon \mathrm{G}_1/\mathrm{Z}_1 \to \mathrm{G}_2/\mathrm{Z}_2$ is an isomorphism of algebraic groups, where Z_i is the center of G_i), any irreducible representation of $\mathrm{G}_2(k)$ when restricted via λ to $\mathrm{G}_1(k)$ decomposes as a sum of irreducible representations of $\mathrm{G}_1(k)$ with multiplicity ≤ 1.

We note that by our earlier work [Adler and Prasad 2006], we know that multiplicity is ≤ 1 whenever the pair $(\mathrm{G}_1, \mathrm{G}_2)$ is $(\mathrm{SL}_n, \mathrm{GL}_n)$, or (when the characteristic of k is not two) either $(\mathrm{O}_n, \mathrm{GO}_n)$ or $(\mathrm{Sp}_n, \mathrm{GSp}_n)$. In the next section, we will see that multiplicity ≤ 1 also holds for $(\mathrm{U}_n, \mathrm{GU}_n)$. The paper [Gee and Taïbi 2018] shows that multiplicity ≤ 1 holds for the pair $(\mathrm{SO}_n, \mathrm{GSO}_n)$ if k has characteristic zero.

6. Generalities on restriction to unitary and special unitary groups

Let E/k denote a separable quadratic extension of nonarchimedean local fields, $N = N_{E/k}$ the norm map from $E^\times$ to $k^\times$, and E_1 the kernel of this map.

Let B denote a nondegenerate E/k-hermitian form on some E-vector space V of some dimension r. Then we can form algebraic groups $\mathrm{SU}(V, B)$, $\mathrm{U}(V, B)$, and $\mathrm{GU}(V, B)$ whose k-points consist respectively of the elements of $\mathrm{SL}(r, E)$ that preserve B; the elements of $\mathrm{GL}(r, E)$ that preserve B; and the elements of $\mathrm{GL}(r, E)$ that preserve B up to a scalar in $k^\times$. The group $\mathrm{GU}(V, B)$ comes equipped with a map $\mu\colon \mathrm{GU}(V, B) \to \mathrm{GL}_1$ called the *similitude character*. We will write our algebraic groups as $\mathrm{SU}(r)$, $\mathrm{U}(r)$, and $\mathrm{GU}(r)$ when V and B are understood.

If G is a group, H is a subgroup, and $G/Z(G)H$ is cyclic, then every irreducible representation of G restricts to H without multiplicity. How far can we exploit this fact?

Theorem 9. *Let p be the residual characteristic of k.*

(a) *All irreducible representations of* $\mathrm{GU}(r)(k)$ *decompose without multiplicity upon restriction to* $\mathrm{U}(r)(k)$. *Such a restriction is irreducible when r is odd, and has at most two components when r is even.*

(b) *All irreducible representations of* $\mathrm{U}(r)(k)$ *decompose without multiplicity upon restriction to* $\mathrm{SU}(r)(k)$ *when r is coprime to p, or $k = \mathbf{Q}_p$ (p odd).*

(c) *All irreducible representations of* $\mathrm{GU}(r)(k)$ *decompose without multiplicity upon restriction to* $\mathrm{SU}(r)(k)$ *when r is odd and coprime to p.*

Proof. (a) Let $\mu\colon \mathrm{GU}(r) \to \mathrm{GL}(1)$ denote the similitude character. Clearly the group $\mathrm{GU}(r)$ contains the scalar matrices eI_r for all $e \in E^\times$, and for such matrices the similitude is $N_{E/k}(e)$. Therefore, the image under μ of the center of $\mathrm{GU}(r)(k)$ is $N_{E/k}(E^\times)$, so μ thus gives an isomorphism

$$\frac{\mathrm{GU}(r)}{Z(\mathrm{GU}(r))\,\mathrm{U}(r)} \xrightarrow{\sim} \frac{\mathrm{Im}(\mu)}{N(E^\times)}.$$

A scalar $a \in k^\times$ is a similitude for some linear transformation g of V if and only if for all $v, w \in V$, we have that $B(gv, gw) = a \cdot B(v, w)$. That is, B and $a \cdot B$

are equivalent Hermitian forms. It is known that two Hermitian forms over a non-archimedean local field k are equivalent if and only if their discriminants, which are elements of $k^\times/N(E^\times)$, are the same. Therefore, B and aB are equivalent if and only if $\operatorname{disc} B = a^r \operatorname{disc} B$ in $k^\times/N(E^\times) \cong \mathbf{Z}/2$. Thus, if r is even, then B and aB are equivalent for a an arbitrary element of $k^\times$, but if r is odd, then a must lie in $N(E^\times)$. Thus,

$$\frac{\mathrm{GU}(r)}{\mathrm{Z}(\mathrm{GU}(r))\,\mathrm{U}(r)} \cong \mathbf{Z}/2 \quad \text{or} \quad \{1\}.$$

(b) Let R_E and P_E denote the ring of integers and prime ideal for E. The determinant character gives us an isomorphism,

$$\det\colon \frac{\mathrm{U}(r)(k)}{Z(\mathrm{U}(r))(k)\,\mathrm{SU}(r)(k)} \xrightarrow{\sim} \frac{E_1}{(E_1)^r}.$$

As an abstract group, E_1 inherits a direct product decomposition from $R_E^\times \cong k_E^\times \times (1+P_E)$. Thus, E_1 is a direct product of a cyclic group (of order coprime to p) and a pro-p-group A, implying that E_1/E_1^r is cyclic if and only A/A^r is cyclic. But this latter quotient is trivial if r is coprime to p, and is cyclic if $k = \mathbf{Q}_p$ (p odd).

(c) This follows from the previous two parts of the theorem. □

7. An example of multiplicity upon restriction

Let ϖ be a uniformizer of k, E/k an unramified quadratic extension, R_k and R_E the rings of integers in k and E, and $\mathfrak{f}$ and $\mathfrak{f}_E$ the residue fields. Let V be a $4d$-dimensional hermitian space over E, with hyperbolic basis $\{e_1, f_1, \ldots, e_{2d}, f_{2d}\}$. Thus, $\langle e_i, f_i\rangle = 1$ for all $1 \le i \le 2d$, and all the other products being 0. Let $\mathrm{U}(V)$ be the corresponding unitary group. Define the lattice $\mathcal{L}$ in E by

$$\mathcal{L} = \operatorname{span}_{R_E}\{e_1, f_1, \ldots, e_d, f_d, \varpi e_{d+1}, f_{d+1}, \ldots, \varpi e_{2d}, f_{2d}\}.$$

Clearly, $\mathcal{L}^\vee := \{v \in V \mid \langle v, \ell\rangle \in R_E \text{ for all } \ell \in \mathcal{L}\}$ is given by

$$\mathcal{L}^\vee = \operatorname{span}_{R_E}\{e_1, f_1, \ldots, e_d, f_d, e_{d+1}, \varpi^{-1} f_{d+1}, \ldots, e_{2d}, \varpi^{-1} f_{2d}\}.$$

Observe that

$$\varpi\mathcal{L}^\vee \subseteq \mathcal{L} \subseteq \mathcal{L}^\vee,$$

and $\mathcal{L}^\vee/\mathcal{L}$ and $\mathcal{L}/\varpi\mathcal{L}^\vee$ are $2d$-dimensional hermitian spaces over $\mathfrak{f}_E$ with natural hermitian structures. For example, given two elements ℓ_1 and ℓ_2 in $\mathcal{L}^\vee$ with images $\overline{\ell}_1$ and $\overline{\ell}_2$ in $\mathcal{L}^\vee/\mathcal{L}$, the hermitian structure on $\mathcal{L}^\vee/\mathcal{L}$ is defined by having $\langle\overline{\ell}_1, \overline{\ell}_2\rangle$ as the image of $\varpi\langle\ell_1, \ell_2\rangle$ (which belongs to R_E) in $\mathfrak{f}_E$.

Define $K = \mathrm{U}(\mathcal{L})$ to be the stabilizer of the lattice $\mathcal{L}$ in $\mathrm{U}(V)$, i.e., $\mathrm{U}(\mathcal{L}) = \{g \in \mathrm{U}(V) \mid g\ell \in \mathcal{L} \text{ for all } \ell \in \mathcal{L}\}$. If an element of $\mathrm{U}(V)$ preserves $\mathcal{L}$, then it clearly preserves $\mathcal{L}^\vee$ and $\varpi\mathcal{L}$, giving a map $\mathrm{U}(\mathcal{L}) \to \mathrm{U}(2d, \mathfrak{f}) \times \mathrm{U}(2d, \mathfrak{f})$. Similarly, we have a map $\mathrm{SU}(\mathcal{L}) \to \mathrm{S}(\mathrm{U}(2d) \times \mathrm{U}(2d))(\mathfrak{f})$.

Let $g_0 \in \mathrm{GU}(V)$ be defined (for $i \le d$) by

$$e_i \mapsto e_{d+i}, \quad f_i \mapsto \varpi^{-1} f_{d+i}, \quad e_{d+i} \mapsto \varpi^{-1} e_i, \quad f_{d+i} \mapsto f_i.$$

Clearly, g_0 has similitude factor ϖ^{-1}, and $g_0\mathcal{L} = \mathcal{L}^\vee$. Therefore, we have

$$g_0\,\mathrm{U}(\mathcal{L}) g_0^{-1} = \mathrm{U}(\mathcal{L}^\vee).$$

Thus conjugation by g_0 induces an isomorphism of $\mathrm{U}(\mathcal{L})$ into $\mathrm{U}(\mathcal{L}^\vee)$, making the diagram

$$\begin{tikzcd}
\mathrm{U}(\mathcal{L}) \arrow[r, "g_0"] \arrow[d] & \mathrm{U}(\mathcal{L}^\vee) \arrow[d] \\
\mathrm{U}(2d, \mathfrak{f}) \times \mathrm{U}(2d, \mathfrak{f}) \arrow[r, "j"] & \mathrm{U}(2d, \mathfrak{f}) \times \mathrm{U}(2d, \mathfrak{f})
\end{tikzcd}$$

commute, where $j(x, y) = (y, x)$.

Theorem 10. *Let ρ be any irreducible cuspidal representation of* $\mathrm{U}(2d)(\mathfrak{f})$ *such that $\rho \not\cong \rho\chi$, where χ is a quadratic character of* $\mathrm{U}(2d)(\mathfrak{f})$ *trivial on* $\mathrm{SU}(2d)(\mathfrak{f})$. *Let $\sigma := \mathrm{infl}(\rho \otimes \rho\chi)$ denote the inflation of $\rho \otimes \rho\chi$ from* $(\mathrm{U}(2d) \times \mathrm{U}(2d))(\mathfrak{f})$ *to* $\mathrm{U}(\mathcal{L})$ *and let $\pi = \text{c-Ind}_{\mathrm{U}(\mathcal{L})}^{\mathrm{U}(V)} \sigma$. Then $\pi \oplus \pi^{g_0}$ extends to an irreducible representation $\widetilde{\pi}$ of* $\mathrm{GU}(V)$ *whose restriction to* $\mathrm{SU}(V)$ *decomposes with multiplicity two.*

Proof. From [Moy and Prasad 1996, Proposition 6.6], π is an irreducible, supercuspidal representation of $\mathrm{U}(V)$. Let π also denote one of its extensions to $Z(\mathrm{GU}(V))\,\mathrm{U}(V)$. From the last sentence of [Moy and Prasad 1994, Theorem 5.2], $\pi^{g_0} \not\cong \pi$, so the sum $\pi \oplus \pi^{g_0}$ extends to an irreducible (also supercuspidal) representation $\widetilde{\pi}$ of $\mathrm{GU}(V)$. By the induction-restriction formula (observe that by the explicit description of $\mathrm{U}(\mathcal{L})$, $\det : \mathrm{U}(\mathcal{L}) \to E_1$ is surjective, and hence $\mathrm{U}(\mathcal{L})\,\mathrm{SU}(V) = \mathrm{U}(V)$),

$$\pi|_{\mathrm{SU}(V)} = \text{c-Ind}_{\mathrm{SU}(\mathcal{L})}^{\mathrm{SU}(V)} (\sigma|_{\mathrm{SU}(\mathcal{L})}),$$

$$\pi^{g_0}|_{\mathrm{SU}(V)} = \text{c-Ind}_{\mathrm{SU}(\mathcal{L})}^{\mathrm{SU}(V)} (\sigma^{g_0}|_{\mathrm{SU}(\mathcal{L})}).$$

Since $\rho \otimes \rho\chi \cong \rho\chi \otimes \rho$ as representations of $\mathrm{S}(\mathrm{U}(2d) \times \mathrm{U}(2d))(\mathfrak{f})$, we have that $\sigma \cong \sigma^{g_0}$ as representations of $\mathrm{SU}(\mathcal{L})$, so

$$\widetilde{\pi}|_{\mathrm{SU}(V)} = (\pi \oplus \pi^{g_0})|_{\mathrm{SU}(V)} = 2 \cdot \text{c-Ind}_{\mathrm{SU}(\mathcal{L})}^{\mathrm{SU}(V)} (\sigma|_{\mathrm{SU}(\mathcal{L})}). \qquad \square$$

In order to have an example of multiplicity at least two, it is thus sufficient to find a representation ρ of $\mathrm{U}(2d)(\mathfrak{f})$ such that $\rho \not\cong \rho\chi$, as in the theorem. In fact, most irreducible Deligne–Lusztig cuspidal representations of $\mathrm{U}(2d)(\mathfrak{f})$ will have this property, as they restrict irreducibly to $\mathrm{SU}(2d)(\mathfrak{f})$.

Remark 11. In a future work, we will expand upon the example in Theorem 10, whose essence is the following. Given a supercuspidal representation of $G_2(k)$ whose restriction to $G_1(k)$ has regular components (in the sense of Kaletha [2016]), then the components occur with multiplicity one. (Nevins [2015] already verified this for many cases.) If the components are not regular, then higher multiplicities can occur.

Our example begins with ρ, an irreducible cuspidal representation of $\mathrm{U}(2d)(\mathfrak{f})$ that arises via Deligne–Lusztig induction from a character θ of the group of $\mathfrak{f}$-points of an anisotropic torus $\mathsf{T} \subset \mathrm{U}(2d)$. Suppose also that the restriction of θ to $\mathsf{T}(\mathfrak{f}) \cap \mathrm{SU}(2d)(\mathfrak{f})$ remains regular so that the restriction of ρ to $\mathrm{SU}(2d)(\mathfrak{f})$ remains irreducible. The torus $\mathsf{T} \times \mathsf{T} \subset \mathrm{U}(2d) \times \mathrm{U}(2d)$ lifts to give an unramified torus $T \subset \mathrm{GU}(V)$, and the character $\theta \otimes \theta\chi$ can be inflated and extended to give a character Θ of T. The representation $\widetilde{\pi}$ of $\mathrm{GU}(V)$ that we have constructed in the theorem is a regular supercuspidal representation in the sense of Kaletha [2016], but the irreducible components of its restriction to $\mathrm{SU}(V)$ are not since our character Θ of T, when restricted to $T \cap \mathrm{SU}(V)$, is not regular because of the presence of the element $g_0 \in \mathrm{GU}(V)$.

For depth-zero supercuspidal representations of quasisplit unitary groups, the parahoric that we have used is the only one that can lead to higher multiplicities.

8. Generalities on constructing higher multiplicities

In this section, we discuss some generalities underlying the example of the previous section, which will be useful for constructing higher multiplicities in general.

Let G be a group, and N a normal subgroup of G such that

$$G/N \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2.$$

A good example to keep in mind is $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, the quaternion group of order 8, and $N = \{\pm 1\}$. Let ω_1 and ω_2 be two distinct, nontrivial characters of G that are trivial on N.

Suppose π is an irreducible representation of G such that

$$\pi \cong \pi \otimes \omega_1 \cong \pi \otimes \omega_2.$$

By [Gelbart and Knapp 1982, §2], $\pi|_N$ must be one of

(1) a sum of four inequivalent, irreducible representations, or

(2) a sum of two copies of an irreducible representation.

Deciding which of these two options we have is a subtle question, and this is what we wish to do here.

Let $N_1 = \ker\{\omega_1 \colon G \to \mathbf{Z}/2\}$, so that $G \supset N_1 \supset N$. Because $\pi \cong \pi \otimes \omega_1$, $\pi|_{N_1}$ is equal to $\pi_1 \oplus \pi_2$, a sum of inequivalent, irreducible representations. Further, since $\pi \cong \pi \otimes \omega_2$, we have

$$(\pi_1 \oplus \pi_2) \cong (\pi_1 \oplus \pi_2) \otimes \omega_{21},$$

where ω_{21} is equal to $\omega_2|_{N_1}$, a nontrivial character of N_1 of order 2. Therefore, we have the following two possibilities:

(i) $\pi_1 \cong \pi_1 \otimes \omega_{21}$.

(ii) $\pi_2 \cong \pi_1 \otimes \omega_{21}$.

In case (i), π_1, which is an irreducible representation of N_1, decomposes when restricted to N into two inequivalent irreducible representations, and therefore π has at least two inequivalent irreducible subrepresentations when restricted to N; hence, in case (i),

$$\pi|_N = \text{a sum of 4 inequivalent, irreducible representations.}$$

In case (ii), clearly $\pi|_N$ is twice an irreducible representation.

How does one then construct an example of an irreducible representation π of G for which $\pi|_N$ is twice an irreducible representation? We start with an irreducible representation π_1 of N_1 such that the following equivalent conditions hold:

(i) π_1 does not extend to a representation of G.

(ii) $\pi_1^g \ncong \pi_1$ for some $g \in G$.

Given such a representation π_1 of N_1, next we must ensure that

$$\pi_1^g \cong \pi_1 \otimes \omega_{21} \quad \text{for } g \in G \setminus N.$$

If we understand N_1, together with the action of G on the representations of N_1, then the condition

$$\pi_1^g \cong \pi_1 \otimes \omega_{21} \ncong \pi_1$$

is checkable, constructing an irreducible representation $\pi = \mathrm{Ind}_{N_1}^G \pi_1$ of G such that

$$\pi|_N = 2\pi_1|_N.$$

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UNKNOTTING NUMBER AND KHOVANOV HOMOLOGY

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We show that the order of h-torsion homology classes in the Bar-Natan deformation of Khovanov homology with $\mathbb{Z}/2\mathbb{Z}$-coefficients is a lower bound for the unknotting number. This is not a bound for the slice genus, unlike most lower bounds for the unknotting number, and only vanishes for the unknot. We give examples of knots for which this is a better lower bound than $|s(K)/2|$, where $s(K)$ is the Rasmussen s invariant defined by the Bar-Natan spectral sequence.

1. Introduction

Khovanov [2000] introduced a knot (and link) invariant which categorifies the Jones polynomial, now known as *Khovanov homology*. This invariant is constructed by applying a specific TQFT to the cube of resolutions corresponding to a projection of the knot. Using a different TQFT, Bar-Natan [2005] defined a deformation of Khovanov homology, called Bar-Natan homology. Let $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. This invariant is a bigraded $\mathbb{F}[h]$-module, and in this paper we introduce a lower bound for the unknotting number in terms of the order of *h-torsion* elements.

For a knot K, let $\mathcal{H}_{BN}(K)$ denote the Bar-Natan homology of K. A homology class $\alpha \in \mathcal{H}_{BN}(K)$ is called *torsion* if $h^n.\alpha = 0$ for a positive integer n. The smallest n with this property is called the *order* of α, denoted by $\mathrm{ord}(\alpha)$. Let $T_{BN}(K)$ denote the set of torsion classes in $\mathcal{H}_{BN}(K)$.

Definition 1.1. For a knot K in $\mathbb{R}^3$, we define

$$\mathfrak{u}(K) := \max_{\alpha \in T_{BN}(K)} \mathrm{ord}(\alpha).$$

Theorem 1.2. *For any knot K, $\mathfrak{u}(K)$ is a lower bound for the unknotting number of K.*

Let K_+ and K_- be knot diagrams that differ in a single crossing c. Orient them

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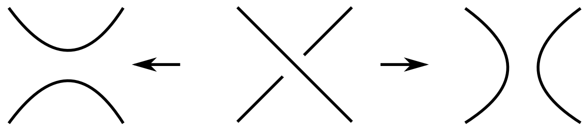

Figure 1. The two possible resolutions of a crossing.

such that c is a positive crossing in K_+ and a negative crossing in K_-. We prove Theorem 1.2 by introducing chain maps

$$f_c^+ : \mathcal{C}_{BN}(K_+) \to \mathcal{C}_{BN}(K_-) \quad \text{and} \quad f_c^- : \mathcal{C}_{BN}(K_-) \to \mathcal{C}_{BN}(K_+) \tag{1}$$

such that the induced maps by $f_c^- \circ f_c^+$ and $f_c^+ \circ f_c^-$ on $\mathcal{H}_{BN}(K_+)$ and $\mathcal{H}_{BN}(K_-)$, respectively, are equal to multiplication by h. In [Alishahi and Dowlin 2018], we introduce similar chain maps for the Lee homology and prove the knight move conjecture [Khovanov 2000; Bar-Natan 2002] for knots with unknotting number smaller than 3.

Despite the algebraic definition of the chain maps (1), we show that they can be described in terms of the cobordism maps associated to specific cobordisms from K_+ to $K_-\#H$, and K_- to $K_+\#mH$, where H is the right-handed Hopf link and mH is its mirror. In [Alishahi and Eftekhary 2016], we use corresponding cobordism maps for the knot Floer homology to deduce a lower bound for the unknotting number, in terms of the order of torsion classes in variants of knot Floer homology.

This paper is organized as follows. Section 2 reviews the Bar-Natan chain complex and collects some results we will need later. Section 3 proves Theorem 1.2. Section 4 gives a geometric description, using cobordism maps, for the chain maps, defined algebraically, in the process of proving Theorem 1.2 in Section 3. Finally, Section 5 gives examples of knots for which our invariant (Definition 1.1) is a better lower bound compared to the s-invariant, i.e., $\mathfrak{u}(K) > |s(K)|/2$.

2. Background

In this section, we review the Bar-Natan chain complex, describe its module structure and discuss some of its basic properties.

Bar-Natan's deformation of Khovanov homology. Let K be an oriented knot or link diagram in $\mathbb{R}^2$ with n crossings. Denote the set of crossings in K by $\mathfrak{C} = \{c_1, \dots, c_n\}$. Each crossing can be resolved in two different ways, the 0-resolution and the 1-resolution; see Figure 1.

For any vertex v of $\{0, 1\}^n$, let K_v denote the complete resolution obtained by replacing the crossing c_i by its v_i-resolution. Let k_v denote the number of connected components of K_v.

There is a partial order on $\{0, 1\}^n$ by setting $u \leq v$ if $u_i \leq v_i$ for all $1 \leq i \leq n$. Denote $u \lessdot v$ if $u < v$ and $|v| - |u| = 1$, where $|v|$ denotes $\sum_i v_i$. Corresponding to each edge of the cube, i.e., a pair $u \lessdot v$, there is an embedded cobordism in $\mathbb{R}^2 \times [0, 1]$ from K_u to K_v, constructed by attaching an embedded one-handle near the crossing c_i where $u_i < v_i$. If $k_u > k_v$, the cobordism *merges* two circles, and otherwise *splits* two circles.

Set $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. Let $\boldsymbol{A}$ denote the 2-dimensional Frobenius algebra over $\mathbb{F}[h]$ with basis $\{x_+, x_-\}$ and multiplication and comultiplication defined as:

$$\begin{aligned}
&x_+ \otimes x_+ \overset{m}{\mapsto} x_+, & & \\
&x_- \otimes x_+ \overset{m}{\mapsto} x_-, & \qquad & x_+ \overset{\Delta}{\mapsto} x_+ \otimes x_- + x_- \otimes x_+ + h x_+ \otimes x_+, \\
&x_+ \otimes x_- \overset{m}{\mapsto} x_-, & & x_- \overset{\Delta}{\mapsto} x_- \otimes x_-. \\
&x_- \otimes x_- \overset{m}{\mapsto} h x_-, & &
\end{aligned}$$

The Bar-Natan chain complex is obtained by applying the (1+1)-dimensional TQFT corresponding to $\boldsymbol{A}$ to the above cube of cobordisms for K. More precisely, corresponding to a vertex $v \in \{0, 1\}^n$, a Khovanov generator is a labeling of the circles in K_v by x_+ or x_-. The module $\mathcal{C}_{BN}(K_v)$ is defined as the free $\mathbb{F}[h]$-module generated by the Khovanov generators corresponding to v and

$$\mathcal{C}_{BN}(K) := \bigoplus_{v \in \{0,1\}^n} \mathcal{C}_{BN}(K_v).$$

The differential δ_{BN} decomposes along the edges; for any $u \lessdot v$ the component

$$\delta_{BN}^{u,v} : \mathcal{C}_{BN}(K_u) \to \mathcal{C}_{BN}(K_v)$$

is defined by the multiplication if K_v is obtained from K_u by merging two circles; otherwise, it is defined by the comultiplication. The Bar-Natan chain complex, $(\mathcal{C}_{BN}(K), \delta_{BN})$, was studied by Bar-Natan [2005]; the homology is denoted by $\mathcal{H}_{BN}(K)$. For simplicity, we denote the differential by δ.

The chain complex is bigraded by the homological grading gr_h, and an internal grading gr_q, called *quantum* grading. The homological grading for each summand $\mathcal{C}_{BN}(K_v)$ of $\mathcal{C}_{BN}(K)$ is given by $|v| - n_-$, where $n_\bullet$ denotes the number of $\bullet$-crossings in K for $\bullet \in \{+, -\}$. The quantum grading for each Khovanov generator x at a vertex v is given by

$$\mathrm{gr}_q(x) = n_+ - 2n_- + |v| + k_v^+ - k_v^-,$$

where $k_v^\bullet$ denotes the number of circles labeled by $x_\bullet$ in K_v, for $\bullet = +, -$. Furthermore, the formal variable h has homological grading 0 and quantum grading -2. Note that if K is a knot, the bigraded module $\mathcal{H}_{BN}(K)$ is independent of the choice of orientation on K.

Module structure on Bar-Natan homology and basepoint action. Let K be a knot diagram and p be a point on K away from the crossings. The choice of p induces a module structure on the Khovanov homology of K described in [Khovanov 2003]. We recall this structure for Bar-Natan homology. Choose a small unknot U near p and disjoint from K such that merging the unknot with K gives a knot or link diagram isotopic to K. Then, attaching the corresponding embedded one-handle to $K \amalg U$ gives an embedded cobordism in $\mathbb{R}^3 \times I$ from $K \amalg U$ to K, and its associated cobordism map, denoted by m_p,

$$m_p : \mathcal{C}_{BN}(K \amalg U) = \mathcal{C}_{BN}(K) \otimes_{\mathbb{F}[h]} \boldsymbol{A} \to \mathcal{C}_{BN}(K),$$

is given by the multiplication map m of $\boldsymbol{A}$. More precisely, for a Khovanov generator $x \in \mathcal{C}_{BN}(K_v)$, $m_p(x \otimes x_\bullet)$ is the Khovanov generator obtained from x by multiplying the label of the circle containing p with $x_\bullet$.

Similarly, let

$$\Delta_p : \mathcal{C}_{BN}(K) \to \mathcal{C}_{BN}(K) \otimes_{\mathbb{F}[h]} \boldsymbol{A}$$

denote the cobordism map associated to the inverse cobordism from K to $K \amalg U$.

If K is related to another knot diagram K' by a Reidemeister move away from the basepoint p, it is straightforward that the chain homotopy equivalence between $\mathcal{C}_{BN}(K)$ and $\mathcal{C}_{BN}(K')$, defined in [Bar-Natan 2005], commutes with m_p. On the other hand, any Reidemeister move which crosses p is equivalent to a sequence of Reidemeister moves away from p. So it induces an $\boldsymbol{A}$-module structure on the Bar-Natan homology of the underlying knot K.

For a point p on K, let

$$\boldsymbol{x}_p : \mathcal{C}_{BN}(K) \to \mathcal{C}_{BN}(K)$$

be the chain map $\boldsymbol{x}_p(a) = m_p(a \otimes x_-)$, defined as in [Hedden and Ni 2013]. Therefore, for a Khovanov generator $x \in \mathcal{C}_{BN}(K_v)$ if the circle containing p is labeled by x_+ then $\boldsymbol{x}_p(x)$ is the Khovanov generator obtained from x by changing the label of this circle to x_-, otherwise $\boldsymbol{x}_p(x) = hx$. Thus, $\boldsymbol{x}_p \circ \boldsymbol{x}_p = h\boldsymbol{x}_p$ and $\boldsymbol{x}_p$ reduces the quantum grading by 2.

Lemma 2.1. *For any point $p \in K$ away from the crossings and $a \in \mathcal{C}_{BN}(K)$,*

$$\Delta_p(a) = a \otimes x_- + (\boldsymbol{x}_p(a) + ha) \otimes x_+.$$

Proof. It is enough to check this relation for each Khovanov generator $x \in \mathcal{C}_{BN}(K_v)$. If the circle containing p is labeled by x_+, then by definition

$$\Delta_p(x) = x \otimes x_- + \boldsymbol{x}_p(x) \otimes x_+ + hx \otimes x_+ = x \otimes x_- + (\boldsymbol{x}_p(x) + hx) \otimes x_+.$$

If the circle containing p is labeled by x_-, then $\boldsymbol{x}_p(x) + hx = 0$, and by definition

$$\Delta_p(x) = x \otimes x_-. \qquad \square$$

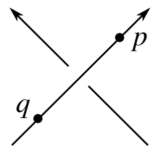

Figure 2. Two points lying on opposite sides of a single crossing.

In contrast to the Khovanov homology, the chain homotopy type of the module multiplication map $\boldsymbol{x}_p$ is not independent of the marked point p. In fact, [Hedden and Ni 2013, Lemma 2.3] may be generalized to describe the difference of $\boldsymbol{x}_p$ and $\boldsymbol{x}_q$ when p and q lie on the opposite sides of a crossing as follows.

Lemma 2.2. *Let $p, q \in K$ be points away from the crossings that lie on the opposite sides of a single crossing as in Figure 2. Then, $\boldsymbol{x}_p + \boldsymbol{x}_q$ is homotopy equivalent to multiplication by h.*

Before proving this lemma, we review some of the properties of the cobordism maps associated with oriented *saddle* moves. Assume K and L are oriented link diagrams such that L is obtained from K by an oriented saddle move as in Figure 3. This saddle move represents an oriented, embedded saddle cobordism in $\mathbb{R}^3 \times [0, 1]$ from K to L. Let

$$\mathfrak{f} : \mathcal{C}_{BN}(K) \to \mathcal{C}_{BN}(L)$$

denote the chain map on the Bar-Natan chain complex associated with this cobordism.

Lemma 2.3. *For any point p on K away from the crossings,*

$$\mathfrak{f} \circ \boldsymbol{x}_p = \boldsymbol{x}_p \circ \mathfrak{f}.$$

Proof. To prove this, it suffices to check two elementary cases:

(1) K is one circle and L is a disjoint union of two circles, and so $\mathfrak{f} = \Delta$. Then,

$$\Delta \circ \boldsymbol{x}_p(x_+) = x_- \otimes x_- = \boldsymbol{x}_q(\Delta x_+) = \boldsymbol{x}_{q'}(\Delta x_+)$$

and

$$\Delta \circ \boldsymbol{x}_p(x_-) = hx_- \otimes x_- = \boldsymbol{x}_q(x_- \otimes x_-) = \boldsymbol{x}_{q'}(x_- \otimes x_-).$$

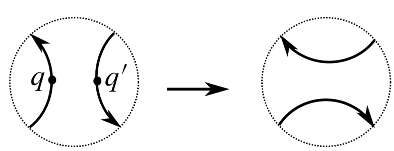

Figure 3. Saddle move: q and q' are the attaching points of the corresponding one-handle.

(2) K is a disjoint union of two circles and L is one circle, and so $\mathfrak{f} = m$. Then,

$$\begin{aligned}
\boldsymbol{x}_p \circ m(x_+ \otimes x_+) &= \ x_- &&= m \circ \boldsymbol{x}_p(x_+ \otimes x_+),\\
\boldsymbol{x}_p \circ m(x_+ \otimes x_-) &= \ hx_- &&= m \circ \boldsymbol{x}_p(x_+ \otimes x_-),\\
\boldsymbol{x}_p \circ m(x_- \otimes x_+) &= \ hx_- &&= m \circ \boldsymbol{x}_p(x_- \otimes x_+),\\
\boldsymbol{x}_p \circ m(x_- \otimes x_-) &= \ h^2x_- &&= m \circ \boldsymbol{x}_p(x_- \otimes x_-). \qquad \square
\end{aligned}$$

Let $\bar{\mathfrak{f}} : \mathcal{C}_{BN}(L) \to \mathcal{C}_{BN}(K)$ be the chain map associated with the inverse saddle move.

Lemma 2.4. *With the above notation fixed, for any $a \in \mathcal{C}_{BN}(K)$ we have*

$$\bar{\mathfrak{f}} \circ \mathfrak{f}(a) = ha + \boldsymbol{x}_q(a) + \boldsymbol{x}_{q'}(a). \tag{2}$$

Proof. Given a vertex $v \in \{0, 1\}^n$, if q and q' lie on the same connected component of the complete resolution K_v, then $\boldsymbol{x}_q|_{\mathcal{C}_{BN}(K_v)} = \boldsymbol{x}_{q'}|_{\mathcal{C}_{BN}(K_v)}$. Thus, for any $a \in \mathcal{C}_{BN}(K_v)$, (2) follows from

$$m\Delta(x_+) = hx_+ \quad \text{and} \quad m\Delta(x_-) = hx_-.$$

Otherwise, if q and q' belong to distinct connected components of K_v, for any Khovanov generator $x \in \mathcal{C}_{BN}(K_v)$ the statement follows from one of the following relations, depending on the labels of the circles containing q and q':

$$\begin{aligned}
&\Delta m(x_+ \otimes x_+) = hx_+ \otimes x_+ + x_- \otimes x_+ + x_+ \otimes x_-,\\
&\Delta m(x_- \otimes x_+) = x_- \otimes x_- = hx_- \otimes x_+ + m(x_- \otimes x_-) \otimes x_+ + x_- \otimes x_-,\\
&\Delta m(x_+ \otimes x_-) = x_- \otimes x_- = hx_+ \otimes x_- + x_- \otimes x_- + x_+ \otimes m(x_- \otimes x_-),\\
&\Delta m(x_- \otimes x_-) = hx_- \otimes x_- = hx_- \otimes x_- + m(x_- \otimes x_-) \otimes x_- + x_- \otimes m(x_- \otimes x_-). \qquad \square
\end{aligned}$$

Proof of Lemma 2.2. The proof is similar to the proof of [Hedden and Ni 2013, Lemma 2.3]. Denote the crossing between p and q by c. Let $K_\bullet$ be the diagram obtained from K by applying the $\bullet$-resolution at c. One may orient K_0 and K_1 such that K_1 is obtained from K_0 by an oriented saddle move, and up to appropriate grading shifts, $\mathcal{C}_{BN}(K)$ is given by the mapping cone

$$\mathfrak{f} : \mathcal{C}_{BN}(K_0) \to \mathcal{C}_{BN}(K_1),$$

where $\mathfrak{f}$ is the corresponding cobordism map. Let $\bar{\mathfrak{f}}$ denote the cobordism map associated with the inverse cobordism from K_1 to K_0. Under this decomposition we define

$$H(a_0, a_1) := (\bar{\mathfrak{f}}(a_1), 0).$$

Then,

$$\begin{aligned}\delta H(a_0,a_1)+H\delta(a_0,a_1)&=\delta(\bar{\mathfrak{f}}(a_1),0)+H(\delta a_0,\mathfrak{f}(a_0)+\delta a_1)\\&=(\delta\bar{\mathfrak{f}}(a_1),\mathfrak{f}\bar{\mathfrak{f}}(a_1))+(\bar{\mathfrak{f}}(\mathfrak{f}(a_0)+\delta a_1),0)\\&=(\bar{\mathfrak{f}}\mathfrak{f}(a_0),\mathfrak{f}\bar{\mathfrak{f}}(a_1)),\end{aligned}$$

and thus it follows from Lemma 2.4 that H is a chain homotopy between $\boldsymbol{x}_p+\boldsymbol{x}_q$ and multiplication by h. □

Corollary 2.5. *Assume K and L are oriented link diagrams so that L is obtained from K by an oriented saddle move. If the attaching points of the saddle lie on the same connected component of K, then $\bar{\mathfrak{f}}\circ\mathfrak{f}$ is chain homotopic to multiplication by h. As before, $\mathfrak{f}$ and $\bar{\mathfrak{f}}$ denote the chain maps associated with the corresponding saddle cobordism and its inverse, respectively.*

Proof. Let p and q be the attaching points of the saddle. Consider an arc $\alpha\subset K$ connecting p to q. Moving the point p along α, it would cross an even number of crossings until it gets to q; thus Lemma 2.2 implies that $\boldsymbol{x}_p$ is homotopy equivalent to $\boldsymbol{x}_q$. Then, by Lemma 2.4 we have that $\bar{\mathfrak{f}}\mathfrak{f}$ is homotopy equivalent to multiplication by h. □

3. Lower bound for unknotting number

The goal of this section is to prove Theorem 1.2.

Let C be a chain complex of $\mathbb{F}[h]$-modules. Recall that a homology class $\alpha\in\mathrm{H}_\star(C)$ is called *torsion* if $h^n\alpha=0$ for some positive n, and the smallest such n is called the *order* of α, denoted by $\mathrm{ord}(\alpha)$. Let $T(C)$ be the set of torsion homology classes in $\mathrm{H}_\star(C)$ and define

$$\mathfrak{u}(C):=\max_{\alpha\in T(C)}\mathrm{ord}(\alpha).$$

Lemma 3.1. *Given chain complexes C and C' of $\mathbb{F}[h]$-modules, together with chain maps*

$$f:C\to C'\quad\text{and}\quad g:C'\to C$$

so that both $f_\star\circ g_\star$ and $g_\star\circ f_\star$ are equal to multiplication by h^n for some $n>0$, then

$$|\mathfrak{u}(C)-\mathfrak{u}(C')|\le n.$$

Proof. For any homology class $\alpha\in T(C)$ we have $f_\star(\alpha)\in T(C')$, and it follows from $g_\star\circ f_\star(\alpha)=h^n\alpha$ that

$$\mathrm{ord}(h^n\alpha)\le\mathrm{ord}(f_\star(\alpha))\le\mathrm{ord}(\alpha).$$

Thus, $\mathrm{ord}(\alpha)\le\mathrm{ord}(f_\star(\alpha))+n$, and so $\mathfrak{u}(C)\le\mathfrak{u}(C')+n$. Similarly, $\mathfrak{u}(C')\le\mathfrak{u}(C)+n$, which proves the claim. □

Suppose K_+ and K_- are oriented knot diagrams so that K_- is obtained from K_+ by changing one positive crossing, denoted by c, to a negative crossing. For $i = 0, 1$, denote the i-resolution of K_+ at the crossing c by K_i. We orient K_0 and K_1 such that they are related by an oriented saddle move. Let $\mathfrak{f}$ and $\bar{\mathfrak{f}}$ denote the cobordism maps corresponding to the saddle cobordism from K_0 to K_1 and its inverse from K_1 to K_0, respectively. The Bar-Natan chain complexes $\mathcal{C}_{BN}(K_+)$ and $\mathcal{C}_{BN}(K_-)$, up to grading shifts, are given by the mapping cones of $\mathfrak{f}$ and $\bar{\mathfrak{f}}$, respectively.

Choose the points p and q on the opposite sides of the crossing c, as in Figure 2 and define

$$\begin{aligned} &f_c^+ : \mathcal{C}_{BN}(K_+) \to \mathcal{C}_{BN}(K_-), && f_c^- : \mathcal{C}_{BN}(K_-) \to \mathcal{C}_{BN}(K_+), \\ &f_c^+(a_0, a_1) = ((\boldsymbol{x}_p + \boldsymbol{x}_q)(a_1), a_0), && f_c^-(a_1, a_0) = ((\boldsymbol{x}_p + \boldsymbol{x}_q)(a_0), a_1), \end{aligned} \tag{3}$$

where $a_i \in \mathcal{C}_{BN}(K_i)$.

Lemma 3.2. *Both f_c^+ and f_c^- are chain maps.*

Proof. Let $\delta_\bullet$ denote the differential of $\mathcal{C}_{BN}(K_\bullet)$ for $\bullet \in \{0, 1, +, -\}$. Then,

$$f_c^+\delta_+(a_0, a_1) = f_c^+(\delta_0 a_0, \delta_1 a_1 + \mathfrak{f}(a_0)) = ((\boldsymbol{x}_p + \boldsymbol{x}_q)(\delta_1 a_1 + \mathfrak{f}(a_0)), \delta_0 a_0).$$

Both $\mathfrak{f}$ and $\bar{\mathfrak{f}}$ commute with $\boldsymbol{x}_p + \boldsymbol{x}_q$ by Lemma 2.3. On the other hand, for any $v \in \{0, 1\}^{n-1}$, the points p and q lie on the same component of either K_{0v} or K_{1v}, the complete resolutions of K_0 and K_1 by v, respectively. Thus, $\boldsymbol{x}_p + \boldsymbol{x}_q$ vanishes on one of $\mathcal{C}_{BN}(K_{0v})$ or $\mathcal{C}_{BN}(K_{1v})$, and so

$$\mathfrak{f} \circ (\boldsymbol{x}_p + \boldsymbol{x}_q) = (\boldsymbol{x}_p + \boldsymbol{x}_q) \circ \mathfrak{f} = \bar{\mathfrak{f}} \circ (\boldsymbol{x}_p + \boldsymbol{x}_q) = (\boldsymbol{x}_p + \boldsymbol{x}_q) \circ \bar{\mathfrak{f}} = 0.$$

As a result, we get

$$f_c^+\delta_+(a_0, a_1) = ((\boldsymbol{x}_p + \boldsymbol{x}_q)\delta_1(a_1), \delta_0(a_0))$$

and

$$\delta_- f_c^+(a_0, a_1) = \delta_-((\boldsymbol{x}_p + \boldsymbol{x}_q)a_1, a_0) = ((\boldsymbol{x}_p + \boldsymbol{x}_q)\delta_1(a_1), \delta_0(a_0)).$$

Therefore, f_c^+ is a chain map. The proof for f_c^- is similar. □

Corollary 3.3. *With the above notation fixed*, $|\mathfrak{u}(K_+) - \mathfrak{u}(K_-)| \leq 1$.

Proof. By Lemma 2.2 the induced maps on homology by both $f_c^+ \circ f_c^-$ and $f_c^- \circ f_c^+$ are equal to multiplication by h. Thus the claim follows from Lemma 3.1. □

Proof of Theorem 1.2. Consider a diagram for K such that we get a diagram for the unknot after switching N crossings $\{c_1, \ldots, c_N\}$, where N is the unknotting number of K. Abusing the notation we denote the diagram by K. For any $i = 1, \ldots, N$, let K_i be the diagram obtained from K after switching the crossings $c_1, \ldots, c_i$. The diagrams K_{i-1} and K_i differ in a single crossing for each i, so it follows from Corollary 3.3 that $|\mathfrak{u}(K_{i-1}) - \mathfrak{u}(K_i)| \leq 1$. Thus, $|\mathfrak{u}(K) - \mathfrak{u}(\text{Unknot})| = \mathfrak{u}(K) \leq N$. □

Setting $h = 1$, one may think of $(\mathcal{C}_{BN}(K), \delta)$ as a filtered chain complex of $\mathbb{F}$-modules, where the differential increases homological grading by 1 and does not decrease quantum grading. This gives a spectral sequence from $\mathrm{Kh}(K) = \mathrm{Kh}(K; \mathbb{F})$ to $\mathbb{F} \oplus \mathbb{F}$, called the Bar-Natan spectral sequence [Turner 2006].

Lemma 3.4. *For any knot K, if the Bar-Natan spectral sequence collapses in the n-th page, then $\mathfrak{u}(K) = n - 1$.*

Proof. The power of h induces a filtration on the chain complex $(\mathcal{C}_{BN}(K), \delta)$. This gives a spectral sequence, closely related to the Bar-Natan spectral sequence, from $\mathrm{Kh}(K) \otimes_{\mathbb{F}} \mathbb{F}[h]$ to $\mathcal{H}_{BN}(K)$. For each k, the k-th page of this spectral sequence has a free part F^k and a torsion part T^k. Inductively, it is straightforward to prove that $F^k = E^k_{BN} \otimes_{\mathbb{F}} \mathbb{F}[h]$, where E^k_{BN} denotes the k-th page of the Bar-Natan spectral sequence, and its differential on the k-th page is obtained by multiplying the differential on E^k_{BN} by h^k. So, the maximum order of elements in T^k is $k - 1$. If this spectral collapses in the n-th page, $T^n = T_{BN}(K)$ implies that $\mathfrak{u}(K) = n - 1$. Since, this spectral sequence collapses in the same page as the Bar-Natan spectral sequence, we are done. $\square$

Lemma 3.5. *The Bar-Natan spectral sequence for a knot K collapses in the first page if and only if K is the unknot.*

Proof. It follows from [Kronheimer and Mrowka 2011] that K is the unknot if and only if the reduced Khovanov homology of K, $\mathrm{Khr}(K) = \mathrm{Khr}(K; \mathbb{F})$, is isomorphic to $\mathbb{F}$. Since, $\mathrm{Kh}(K) \cong \mathrm{Khr}(K) \oplus \mathrm{Khr}(K)$ we conclude that $\mathrm{Kh}(K) \cong \mathbb{F} \oplus \mathbb{F}$, and so $E^1 = E^\infty$ if and only if K is the unknot. $\square$

Corollary 3.6. *If K is not the unknot, then $\mathfrak{u}(K) > 0$.*

Proof. If K is nontrivial, by Lemma 3.5, the Bar-Natan spectral sequence for K collapses in n-th page for some $n > 1$. Then, Lemma 3.4 implies $\mathfrak{u}(K) = n - 1 > 0$. $\square$

4. A geometric interpretation of the chain maps

Suppose K and K' are oriented pointed knots, i.e., oriented knots with marked points on them, such that K' is obtained from K by a sequence of crossing changes. To any such sequence, Eftekhary and the author associate a decorated cobordism from K to a connected sum of K' with some right- or left-handed Hopf links. Then, by the corresponding cobordism maps for knot Floer homology [Alishahi and Eftekhary 2016, Section 8.2], we define chain maps between the knot Floer chain complexes of K and K' satisfying the assumptions of Lemma 3.1 [Alishahi and Eftekhary 2016, Section 8.3]. As a result, one gets a lower bound for the unknotting number in term of the *u-torsion* in knot Floer homology.

Following the approach in [Alishahi and Eftekhary 2016], one may use cobordism maps for Bar-Natan homology to define chain maps between $\mathcal{C}_{BN}(K)$ and $\mathcal{C}_{BN}(K')$

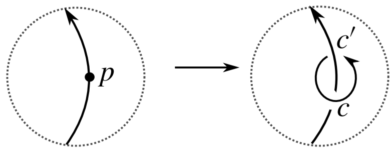

Figure 4. Connected sum with right-handed Hopf link at p.

which satisfy the assumptions of Lemma 3.1. The goal of this section is to show that for any crossing change process, these chain maps are equal to the ones defined in Section 3.

Connected sum with a Hopf link. A connected sum formula for Khovanov homology has been studied in [Khovanov 2000]. In this section, we recall a special case of this formula, taking a connected sum with the Hopf link, for the Bar-Natan homology.

Let H be the right-handed Hopf link. Choose an arbitrary point $p \in K$. We obtain an oriented diagram L for the link $K\#H$ by changing K locally in a neighborhood of p, as in Figure 4.

Denote the new crossings by c and c'. For $i, j \in \{0, 1\}$, let L_{ij} denote the diagram obtained from L by applying i- and j-resolutions at the crossings c and c', respectively. We orient these diagrams such that their orientations coincide with the orientation of K outside the above neighborhood of p. The oriented diagrams L_{01} and L_{10} are isotopic to K, while L_{00} and L_{11} are isotopic to a disjoint union of K with an unknot near the point p. The chain complex $\mathcal{C}_{BN}(L)$ is given by the mapping cone

$$\begin{tikzcd}
\mathcal{C}_{BN}(L_{10}) = \mathcal{C}_{BN}(K) \arrow[r, "\Delta_p"] & \mathcal{C}_{BN}(L_{11}) = \mathcal{C}_{BN}(K) \otimes_{\mathbb{F}[h]} \boldsymbol{A} \\
\mathcal{C}_{BN}(L_{00}) = \mathcal{C}_{BN}(K) \otimes_{\mathbb{F}[h]} \boldsymbol{A} \arrow[u, "m_p"] \arrow[r, "m_p"] & \mathcal{C}_{BN}(L_{01}) = \mathcal{C}_{BN}(K) \arrow[u, "\Delta_p"']
\end{tikzcd}$$

From now on, we use this decomposition to write any $a \in \mathcal{C}_{BN}(L)$ as $a = (a_{00}, a_{01}, a_{10}, a_{11})$ where $a_{ij} \in \mathcal{C}_{BN}(L_{ij})$ and so $a_{00}, a_{11} \in \mathcal{C}_{BN}(K) \otimes \boldsymbol{A}$ while $a_{10}, a_{01} \in \mathcal{C}_{BN}(K)$.

We define chain maps $i : \mathcal{C}_{BN}(K) \to \mathcal{C}_{BN}(L)$ and $p : \mathcal{C}_{BN}(L) \to \mathcal{C}_{BN}(K)$ as

$$i(a) = (0, 0, 0, a \otimes x_+) \quad \text{and} \quad p(a_{00}, a_{01}, a_{10}, a_{11}) = a_{00}^-, \tag{4}$$

where

$$a_{00} = a_{00}^+ \otimes x_+ + a_{00}^- \otimes x_-.$$

It is clear that both i and p are chain maps.

Lemma 4.1. *The sequence*

$$0 \to \mathcal{H}_{BN}(K) \xrightarrow{i_\star} \mathcal{H}_{BN}(L) \xrightarrow{p_\star} \mathcal{H}_{BN}(K) \to 0$$

is a split exact sequence.

Proof. First, we prove that $\operatorname{im}(i_\star) = \ker(p_\star)$. Consider $\alpha \in \ker(p_\star)$ and let $a \in \mathcal{C}_{BN}(L)$ be a cycle representing α. Since, $p(a) = a_{00}^- = \delta b$ for some $b \in \mathcal{C}_{BN}(K)$, after adding $\delta(b \otimes x_-, 0, 0, 0)$ to a, we may assume that $a_{00}^- = 0$. Then, $\delta a = 0$ implies that

$$0 = m_p(a_{00}) + \delta a_{01} = a_{00}^+ + \delta a_{01},$$

and thus

$$a + \delta(a_{01} \otimes x_+, 0, 0, 0) = (0, 0, a_{01} + a_{10}, a_{11}).$$

Again, it follows from $\delta a = 0$ and Lemma 2.1 that

$$\delta a_{11} = \Delta_p(a_{10} + a_{01}) = (a_{10} + a_{01}) \otimes x_- + (\boldsymbol{x}_p(a_{10} + a_{01}) + h(a_{10} + a_{01})) \otimes x_+.$$

Therefore, $\delta a_{11}^- = a_{10} + a_{01}$ where $a_{11} = a_{11}^+ \otimes x_+ + a_{11}^- \otimes x_-$ and so

$$(0, 0, a_{10}+a_{01}, a_{11}) + \delta(0, 0, a_{11}^-, 0) = (0, 0, 0, a_{11} + \Delta_p a_{11}^-) = i(a_{11}^+ + \boldsymbol{x}_p(a_{11}^-) + h a_{11}^-).$$

Note that the last equality follows from Lemma 2.1.

Next, let

$$r : \mathcal{C}_{BN}(L) \to \mathcal{C}_{BN}(K) \quad \text{and} \quad s : \mathcal{C}_{BN}(K) \to \mathcal{C}_{BN}(L)$$

be the chain maps defined as

$$r(a_{00}, a_{01}, a_{10}, a_{11}) = a_{11}^+ + \boldsymbol{x}_p(a_{11}^-) + h a_{11}^- \quad \text{and} \quad s(a) = (\boldsymbol{x}_p(a) \otimes x_+ + a \otimes x_-, 0, 0, 0),$$

where $a_{11} = a_{11}^+ \otimes x_+ + a_{11}^- \otimes x_-$. It is clear that r and s are chain maps such that $r \circ i = \mathrm{id}$ and $p \circ s = \mathrm{id}$. So $i_\star$ and $p_\star$ are injective and surjective, respectively, and the sequence splits. □

The homomorphism $p_\star$ preserves the homological grading and decreases the quantum grading by 1, while $i_\star$ increases the homological grading by 2 and the quantum grading by 5.

Combining the chain maps i, p, r and s, we define chain maps

$$\mathsf{i} : \mathcal{C}_{BN}(K) \to \mathcal{C}_{BN}(L) \quad \text{and} \quad \mathsf{p} : \mathcal{C}_{BN}(L) \to \mathcal{C}_{BN}(K)$$

as

$$\mathsf{i} = i + s \quad \text{and} \quad \mathsf{p} = p + r. \tag{5}$$

Similarly, there exist chain maps i and p for the connected sum of K with the left-handed Hopf link, $K \# mH$, so that the induced homomorphisms on homology give a split exact sequence. The only difference is that $p_\star$ increases the homological

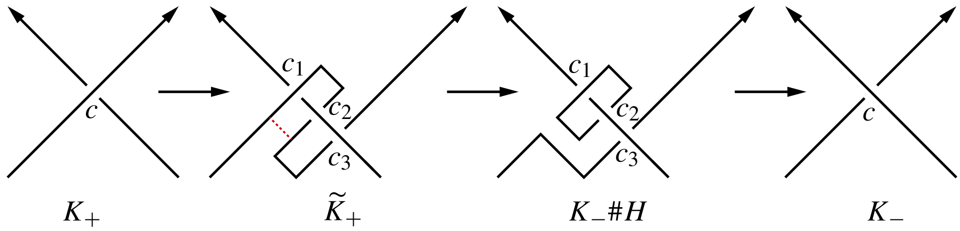

Figure 5. Changing a positive crossing to a negative crossing with an oriented saddle move followed by the removal of a Hopf link summand.

grading by 2 and quantum grading by 5, while $i_\star$ preserves the homological grading and decreases the quantum grading by 1. Also, we define analogous chain maps

$$\mathsf{i}: \mathcal{C}_{BN}(K) \to \mathcal{C}_{BN}(K\#mH) \quad \text{and} \quad \mathsf{p}: \mathcal{C}_{BN}(K\#mH) \to \mathcal{C}_{BN}(K).$$

Crossing change and cobordism maps. As before, let K_+ be an oriented knot diagram with a specific positive crossing c, and K_- be the oriented knot diagram obtained from K_+ by changing c into a negative crossing. As in Figure 5, after a Reidemeister II move near the crossing c on K_+, followed by an oriented saddle move, one gets a diagram for $K_-\#H$. Here, H is the right-handed Hopf link.

As in Figure 5, we denote the knot diagram obtained from K_+ by the specified Reidemeister II move by $\widetilde{K}_+$. Further, let $\mathsf{h} : \mathcal{C}_{BN}(K_+) \to \mathcal{C}_{BN}(\widetilde{K}_+)$ and $\widetilde{\mathsf{h}} : \mathcal{C}_{BN}(\widetilde{K}_+) \to \mathcal{C}_{BN}(K_+)$ be the chain homotopy equivalences corresponding to this move as defined in [Bar-Natan 2005, Section 4.3]. For the reader's convenience, we recall the definition of h and $\widetilde{\mathsf{h}}$. For $i, j = 0, 1$, let $\mathcal{C}^{ij}_{BN}(\widetilde{K}_+)$ denote the direct sum of the summands of $\mathcal{C}_{BN}(\widetilde{K}_+)$ corresponding to vertices v of the cube so that $v(c_2) = i$ and $v(c_3) = j$. Also, let $\mathsf{h}^{ij} = \pi_{ij} \circ \mathsf{h}$ and $\widetilde{\mathsf{h}}^{ij} = \widetilde{\mathsf{h}} \circ \iota_{ij}$ where π_{ij} is the projection of $\mathcal{C}_{BN}(\widetilde{K}_+)$ on $\mathcal{C}^{ij}_{BN}(\widetilde{K}_+)$ and ι_{ij} is the inclusion of $\mathcal{C}^{ij}_{BN}(\widetilde{K}_+)$ in $\mathcal{C}_{BN}(\widetilde{K}_+)$. Then, $\mathsf{h}^{00} = \mathsf{h}^{11} = \widetilde{\mathsf{h}}^{00} = \widetilde{\mathsf{h}}^{11} = 0$, $\mathsf{h}^{10} = \widetilde{\mathsf{h}}^{10} = \mathrm{id}$ and $\mathsf{h}^{01} = g \otimes x_+$, where g is the cobordism map corresponding to the saddle move along the dashed curve in Figure 6. In other words, h^{01} is a chain map for the cobordism which is the union of a saddle and a cup. Finally, $\widetilde{\mathsf{h}}^{01}$ is the cobordism map for the inverse of the aforementioned saddle move union a cap; see Figure 6.

Let $\mathfrak{f} : \mathcal{C}_{BN}(\widetilde{K}_+) \to \mathcal{C}_{BN}(K_-\#H)$ and $\bar{\mathfrak{f}} : \mathcal{C}_{BN}(K_-\#H) \to \mathcal{C}_{BN}(\widetilde{K}_+)$ denote the cobordism maps for the saddle move in Figure 5.

Theorem 4.2. *With the above notation fixed,* $f_c^+ = \mathsf{p} \circ \mathfrak{f} \circ \mathsf{h}$ *and* $f_c^- = \widetilde{\mathsf{h}} \circ \bar{\mathfrak{f}} \circ \mathsf{i}$, *where* i *and* p *are the chain maps defined in* (5).

Proof. We prove both equalities by looking at the cube of resolutions for the three crossings c_1, c_2 and c_3. For $i = 0, 1$, let $\mathcal{C}^i_{BN}(K_+)$ and $\mathcal{C}^i_{BN}(K_-)$ denote the summands of $\mathcal{C}_{BN}(K_+)$ and $\mathcal{C}_{BN}(K_-)$, respectively, corresponding to the i-resolution at c.

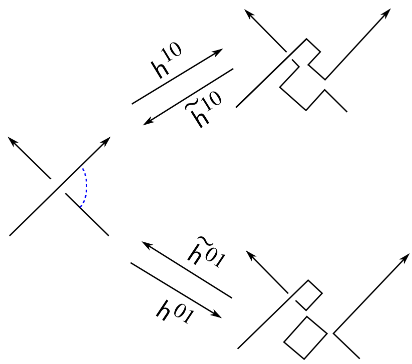

Figure 6. $\mathsf{h}^{10} = \widetilde{\mathsf{h}}^{10} = \mathrm{id}$, while h^{01} and $\widetilde{\mathsf{h}}^{01}$ are the maps associated with the cobordism corresponding to the saddle move on the dashed curve union a cup and its inverse, respectively.

Assume $a = (a_0, a_1)$ is an element in $\mathcal{C}_{BN}(K)$ such that $a_i \in \mathcal{C}^i_{BN}(K_+)$. It follows from the definition of h that $\mathsf{h}^{ii}(a_j) = 0$ for any $i, j = 0, 1$. Further, considering the definition of p, it is enough to compute $\mathfrak{f}(\mathsf{h}^{01}(a_0))$ and $\mathfrak{f}(\mathsf{h}^{10}(a_1))$. As in Figure 7, $\mathsf{h}^{01}(a_0) = \Delta_p(a_0) \otimes x_+$ and so

$$\mathfrak{f}(\mathsf{h}^{01}(a_0)) = \Delta_p(a_0) = a_0 \otimes x_- + (\boldsymbol{x}_p(a_0) + ha_0) \otimes x_+ \tag{6}$$

by Lemma 2.1.

On the other hand, $\mathsf{h}^{10}(a_1) = a_1$ and, as Figure 8 shows,

$$\mathfrak{f}(\mathsf{h}^{10}(a_1)) = \Delta_q(a_1) = a_1 \otimes x_- + (\boldsymbol{x}_q(a_1) + ha_1) \otimes x_+. \tag{7}$$

Thus, by (6) and (7) and the definition of p, we have

$$\mathsf{p}(\mathfrak{f} \circ \mathsf{h}(a)) = (\boldsymbol{x}_p(a_1) + \boldsymbol{x}_q(a_1), a_0).$$

Let $a = (a_0, a_1)$ be an element in $\mathcal{C}_{BN}(K_-)$ where $a_i \in \mathcal{C}^i_{BN}(K_-)$ for $i = 0, 1$. By definition,

$$\mathsf{i}(a_0) = (\boldsymbol{x}_p(a_0) \otimes x_+ + a_0 \otimes x_-, 0, 0, a_0 \otimes x_+),$$
$$\mathsf{i}(a_1) = (\boldsymbol{x}_q(a_1) \otimes x_+ + a_1 \otimes x_-, 0, 0, a_1 \otimes x_+).$$

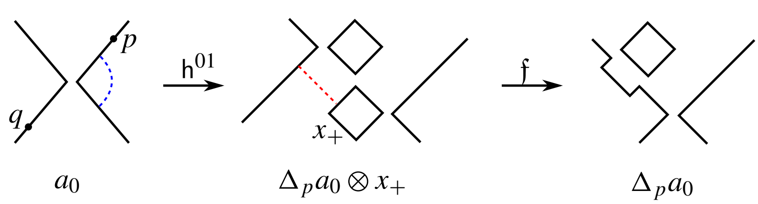

Figure 7. Evaluation of $\mathfrak{f} \circ \mathsf{h}^{01}$ over an element $a_0 \in \mathcal{C}^0_{BN}(K_+)$.

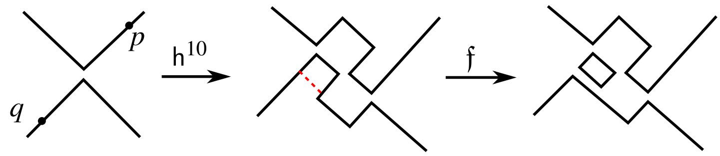

Figure 8. Evaluation of $\mathfrak{f} \circ \mathsf{h}^{10}$ over an element $a_1 \in \mathcal{C}^1_{BN}(K_+)$.

The only nonzero components of $\widetilde{\mathsf{h}}$ are $\widetilde{\mathsf{h}}^{01}$ and $\widetilde{\mathsf{h}}^{10}$, so

$$\widetilde{\mathsf{h}} \circ \bar{\mathfrak{f}} \circ \mathsf{i}(a) = (\widetilde{\mathsf{h}}^{01} \circ \bar{\mathfrak{f}}(\boldsymbol{x}_q(a_1) \otimes x_+ + a_1 \otimes x_-), \widetilde{\mathsf{h}}^{10} \circ \bar{\mathfrak{f}}(a_0 \otimes x_+)).$$

By Figure 8, we have

$$\widetilde{\mathsf{h}}^{10} \circ \bar{\mathfrak{f}}(a_0 \otimes x_+) = \widetilde{\mathsf{h}}^{10}(a_0) = a_0,$$

and by Figure 7,

$$\begin{aligned}
&\widetilde{\mathsf{h}}^{01} \circ \bar{\mathfrak{f}}(\boldsymbol{x}_q(a_1) \otimes x_+ + a_1 \otimes x_-) \\
&\quad = \widetilde{\mathsf{h}}^{01}(\Delta_q(\boldsymbol{x}_q(a_1) \otimes x_+ + a_1 \otimes x_-)) \\
&\quad = \widetilde{\mathsf{h}}^{01}((\boldsymbol{x}_q(a_1) \otimes x_-) \otimes x_+ + (a_1 \otimes x_- + (\boldsymbol{x}_q(a_1) + ha_1) \otimes x_+) \otimes x_-) \\
&\quad = \boldsymbol{x}_q(a_1) + \boldsymbol{x}_p(a_1).
\end{aligned}$$

□

Remark 4.3. Similarly, one may change K_- by a Reidemeister II move near c to get a diagram $\widetilde{K}_-$, so that $K_+ \# mH$ is obtained from $\widetilde{K}_-$ by a saddle move. Let $\mathsf{h} : \mathcal{C}_{BN}(K_-) \to \mathcal{C}_{BN}(\widetilde{K}_-)$ and $\widetilde{\mathsf{h}} : \mathcal{C}_{BN}(\widetilde{K}_-) \to \mathcal{C}_{BN}(K_-)$ be the corresponding chain homotopy equivalences and

$$\mathfrak{f} : \mathcal{C}_{BN}(\widetilde{K}_-) \to \mathcal{C}_{BN}(K_+ \# mH)$$

and

$$\bar{\mathfrak{f}} : \mathcal{C}_{BN}(K_+ \# mH) \to \mathcal{C}_{BN}(\widetilde{K}_-)$$

be the cobordism maps for the saddle move. Then, by the same argument, one can show that $f_c^- = \mathsf{p} \circ \mathfrak{f} \circ \mathsf{h}$ and $f_c^+ = \widetilde{\mathsf{h}} \circ \bar{\mathfrak{f}} \circ \mathsf{i}$.

5. Examples

The Rasmussen's s invariant gives a lower bound for the slice genus, $|s(K)|/2$, and thus the unknotting number [Rasmussen 2010]. We used Cotton Seed's package, Knotkit [Seed 2011], to compute $\mathfrak{u}$ and s (defined using the Bar-Natan spectral sequence) for some knots with more than 12 crossings. We found some examples where $\mathfrak{u}$ is a better lower bound compared to $|s|/2$; for instance, $|s|/2$ for the knots $13n689$, $13n1166$, $13n2504$ and $13n2807$ is equal to 1, while $\mathfrak{u}$ is equal to 2.

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LIGHT GROUPS OF ISOMORPHISMS OF BANACH SPACES AND INVARIANT LUR RENORMINGS

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SOPHIE GRIVAUX AND CHRISTIAN ROSENDAL

Megrelishvili (2001) defines *light groups* of isomorphisms of a Banach space as the groups on which the weak and strong operator topologies coincide, and proves that every bounded group of isomorphisms of Banach spaces with the point of continuity property (PCP) is light. We investigate this concept for isomorphism groups G of classical Banach spaces X without the PCP, specially isometry groups, and relate it to the existence of G-invariant LUR or strictly convex renormings of X.

1. Introduction

The general objective of this note is to determine conditions on a bounded group of isomorphisms of Banach spaces that ensure the existence of a locally uniformly rotund (LUR) renorming invariant under the action of this group. In particular, we are interested in this context in the notion of lightness for such groups.

Light groups. A frequent problem in functional analysis is to determine under which conditions weak convergence and norm convergence coincide. For example, it is well known that conditions of convexity of the norm of a Banach space ensure that weak and strong convergence are equivalent on its unit sphere. The corresponding problem for isomorphisms of Banach spaces (or more generally of locally convex spaces) was studied by Megrelishvili [2001] in the context of group representations, using the concept of fragmentability.

Given a (real) Banach space X, we denote by $\mathrm{L}(X)$ the set of bounded linear operators on X, and by $\mathrm{GL}(X)$ the group of bounded isomorphisms of X. We

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also denote by Isom(X) the group of surjective linear isometries of X. If G is a subgroup of $\mathrm{GL}(X)$, we write $G \leqslant \mathrm{GL}(X)$. Recall that given a Banach space X, the strong operator topology on $\mathrm{L}(X)$ is the topology of pointwise convergence, i.e., the initial topology generated by the family of functions $f_x : \mathrm{L}(X) \to X,\ x \in X$, given by $f_x(T) = Tx,\ T \in \mathrm{L}(X)$, and the weak operator topology on $\mathrm{L}(X)$ is generated by the family of functions $f_{x,x^*} : \mathrm{L}(X) \to \mathbb{R},\ x \in X, x^* \in X^*$, given by $f_{x,x^*}(T) = x^*(Tx),\ T \in \mathrm{L}(X)$.

Definition 1.1 [Megrelishvili 2001]. A group $G \leqslant \mathrm{GL}(X)$ of isomorphisms on a Banach space X is *light* if the weak operator topology (WOT) and the strong operator topology (SOT) coincide on G.

Observe that since the two operator topologies are independent of the specific choice of norm on X, the same holds for lightness of G.

Well-known examples of light groups are the groups $\mathrm{U}(H)$ of unitary operators on Hilbert spaces H. However, the main result of [Megrelishvili 2001] concerning light groups indicates that we have a similar phenomenon in a more general context. Recall that a Banach space X has the point of continuity property (PCP) if for every norm-closed nonempty bounded subset C of X, the identity on C has a point of continuity from the weak to the norm topology:

Theorem 1.2 [Megrelishvili 2001]. *If X is a Banach space with the point of continuity property* (*PCP*) *and if $G \leqslant \mathrm{GL}(X)$ is bounded in norm, then G is light.*

In particular, if X has the Radon–Nikodym property (RNP) (e.g., if X is reflexive or is a separable dual space), then every bounded subgroup of $\mathrm{GL}(X)$ is light. For example, the isometry group of ℓ_1, $\mathrm{Isom}(\ell_1)$, is light.

We note here that in the literature (and indeed in [Megrelishvili 2001]) PCP sometimes appears as the formally weaker condition *"every weakly-closed nonempty bounded subset has a weak-to-norm point of continuity for the identity"*. However, as was pointed out to us by G. Godefroy, if X satisfies this definition and F is norm-closed and bounded, then any point of continuity of the weak closure $\overline{F}^w$ belongs to F, so the two definitions are equivalent. In fact, if $x \in \overline{F}^w$ is a weak-to-norm point of continuity for the identity, there exists a net $(x_\alpha)_{\alpha \in I} \subset F$ such that $x_\alpha \xrightarrow{w} x$. Hence, $x_\alpha \xrightarrow{\|\cdot\|} x$ and, since F is norm-closed, $x \in F$.

Bounded nonlight groups. A natural question that arises from Megrelishvili's result is to investigate in which respect his result is optimal, and whether "smallness" assumptions on G or weaker assumptions than the PCP on X could imply that G is light. We show (Theorem 4.6) that any separable space containing an isomorphic copy of c_0 admits a bounded cyclic group of isomorphisms which is not light. This shows that we cannot really expect further general results in this direction.

Megrelishvili gives the group $\mathrm{Isom}(C([-1,1]^2))$ as an example of a nonlight group. His proof uses a construction of Helmer [1980] of a separately continuous group action on $[-1,1]^2$ that is not jointly continuous, and the equivalence of pointwise compactness and weak compactness of bounded subsets of $C(K)$. This leads to the following question:

Question 1.3. For which compact sets K is the isometry group $\mathrm{Isom}(C(K))$ not light?

We first prove (Proposition 4.3) that the isometry group of c, the space of real convergent sequences, is not light. Neither is the isometry group of $C(\{0,1\}^{\mathbb{N}})$ (Corollary 5.6). On the other hand, as a consequence of Theorem 5.9, we show that the isometry group of $C[0,1]$ is light, while those of the spaces $C([0,1]^n)$, $n \geq 2$, are not light. These constructions simplify the initial example of Megrelishvili.

Light groups and LUR renormings. In another direction, we study the relation between light groups and the existence of LUR renormings invariant under the action of the group. Recall that a norm $\|\cdot\|$ on X is *rotund* or *strictly convex* if whenever the vectors x, y belong to the unit sphere S_X of X and $\|x+y\| = 2$, $x = y$. It is *locally uniformly convex* (LUC) or *locally uniformly rotund* (LUR) at a vector $x_0 \in X$ if whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence of vectors of X such that $\lim\|x_n\| = \|x_0\|$ and $\lim\|x_0 + x_n\| = 2\|x_0\|$, $\lim\|x_n - x_0\| = 0$. Another equivalent definition (in fact, Lovaglia's original definition) is the following: the norm is LUR at a vector $x_0 \in S_X$ if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, x_0) > 0$ such that

$$\frac{\|x+y\|}{2} \leq 1 - \delta \quad \text{whenever } \|x-y\| \geq \varepsilon \text{ and } \|y\| = 1.$$

The norm is said to be LUR in X if it is LUR at every point $x_0 \neq 0$ of X (or, equivalently, of S_X). The property of the dual norm $\|\cdot\|_*$ on X^* being strictly convex or LUR is closely related to the differentiability of the norm $\|\cdot\|$ on X, in the senses of Fréchet and Gateaux, respectively. All (real) separable Banach spaces admit an equivalent LUR renorming. For this and much more on renormings of Banach spaces, see [Deville et al. 1993].

A fundamental result in the study of LUR renorming is the following theorem:

Theorem 1.4 [Lancien 1993]. *If X is a separable Banach space with the RNP*, X *admits an isometry invariant LUR renorming.*

If $G \leqslant \mathrm{GL}(X, \|\cdot\|)$ is a bounded group of isomorphisms on X, the norm on X defined by

$$|||x||| = \sup_{g\in G}\|gx\|, \; x \in X,$$

is a G-invariant renorming of X. In other words, $G \leqslant \mathrm{Isom}(X, |||\cdot|||)$. So a consequence of Theorem 1.4 is that whenever X is a separable space with the RNP and G is a bounded group of isomorphisms on X, there exists a G-invariant LUR renorming of X. The existence of G-invariant LUR renormings for general groups of isomorphisms G was first investigated by Ferenczi and Rosendal [2013]. They studied problems of maximal symmetry in Banach spaces, analyzing the structure of subgroups of $\mathrm{GL}(X)$ when X is a separable reflexive Banach space. An example of a super-reflexive space with no maximal bounded group of isomorphisms was also exhibited in [Ferenczi and Rosendal 2013].

The relation between light groups and G-invariant LUR renormings is given by Theorem 2.3. We observe that if a Banach space X admits a G-invariant LUR renorming, then G is light. In fact, this is true even if the norm is LUR only on a dense subset of S_X. We also show that the converse assertion is false: although the isometry group of $C[0, 1]$ is light, $C[0, 1]$ admits no strictly convex isometry invariant renorming (Corollary 5.12). This link between the existence of a G-invariant LUR renorming and the lightness of G is a natural one: if X is a Banach space with an LUR norm $\|\cdot\|$, the weak topology and the norm topology coincide on the unit sphere of $(X, \|\cdot\|)$.

Light groups and distinguished families. Ferenczi and Galego [2010] investigated groups that may be seen as the group of isometries of a Banach space under some renorming. Among other results, they proved that if X is a separable Banach space and G is a finite group of isomorphisms of X such that $-\mathrm{Id} \in G$, X admits an equivalent norm $|||\cdot|||$ such that $G = \mathrm{Isom}(X, |||\cdot|||)$. They also proved that if X is a separable Banach space with LUR norm $\|\cdot\|$ and if G is an infinite countable bounded isometry group of X such that $-\mathrm{Id} \in G$ and such that G admits a point $x \in X$ with $\inf_{g\neq\mathrm{Id}} \|gx - x\| > 0$, then $G = \mathrm{Isom}(X, |||\cdot|||)$ for some equivalent norm $|||\cdot|||$ on X. In [Ferenczi and Rosendal 2011], a point x satisfying the condition

$$\inf_{g\neq\mathrm{Id}} \|gx - x\| > 0$$

is called a *distinguished point* of X for the group G.

Ferenczi and Rosendal [2011] generalized results of [Ferenczi and Galego 2010] to certain uncountable Polish groups, and also defined the concept of a *distinguished family* for the action of a group G on a Banach space X. It is clear that if G is an isometry group with a distinguished point, G is SOT-discrete. However, the following question remained open: if G is an isomorphism group of X which is SOT-discrete, should X have a distinguished point for G? In Proposition 6.1 we see that the answer to this question is negative, and we give an example of an infinite countable group of isomorphisms G of c_0 which is SOT-discrete but does not admit a distinguished point for G. In addition, this group is also not light.

Light groups on quasinormed spaces. Although Megrelishvili has defined the concept of light group only for groups of isomorphisms on locally convex spaces, we can extend the definition to quasinormed spaces, even if they are not locally convex. We finish this article by investigating whether the isometry groups of the quasinormed spaces ℓ_p and $L_p[0, 1]$, $0 < p < 1$, are light.

2. LUR renormings and light groups

Let G be a bounded group of isomorphisms on a Banach space $(X, \|\cdot\|)$. In this section we are interested in the existence of a G-invariant LUR renorming of X, i.e., in the existence of an equivalent norm $|||\cdot|||$ on X which is both invariant under the action of G and is LUR; or in the existence of a G-invariant *dense LUR renorming*, meaning a renorming which is invariant under the action of G and is LUR on a dense subset of S_X. When $G = \mathrm{Isom}(X, \|\cdot\|)$ we shall talk about *isometry invariant* renormings. Our first result is the following:

Proposition 2.1. *Let X be a Banach space and let $G \leqslant \mathrm{GL}(X)$. If G is SOT-compact and if X admits an LUR renorming, X admits a G-invariant LUR renorming.*

Proof. Suppose that $\|\cdot\|$ is an equivalent LUR norm on X. The formula

$$|||x||| = \sup_{T\in G} \|Tx\|, \qquad x \in X,$$

defines a G-invariant LUR renorming of X. Indeed, suppose that x_n and x are vectors of X such that $|||x_n||| = |||x||| = 1$ for every $n \in \mathbb{N}$ and $\lim |||x_n + x||| = 2$. Then we can find elements T_n, $n \in \mathbb{N}$, of G such that $\lim \|T_n x_n + T_n x\| = 2$. By SOT-compactness of G we can assume without loss of generality that T_n tends SOT to some element T of G, from which it follows that $\|T_n x_n + Tx\|$ converges to 2. Since $\|T_n x_n\| \leq |||x_n||| = 1$ for every $n \in \mathbb{N}$ and $\|Tx\| \leq |||x||| = 1$, we deduce that $\|Tx\| = 1$ and that $\|T_n x_n\|$ converges to 1. In particular, if we set, for every $n \in \mathbb{N}$, $y_n = T_n x_n / \|T_n x_n\|$, then y_n belongs to the unit sphere of $(X, \|\cdot\|)$ and

$$\|y_n + Tx\| = \frac{1}{\|T_n x_n\|} \big\| T_n x_n + \|T_n x_n\| Tx \big\| \to 2.$$

By the LUR property of $\|\cdot\|$ at the point Tx, we deduce that y_n converges to Tx. So $T_n x_n$ converges to Tx. So $|||x_n - x||| = |||T_n x_n - T_n x|||$ converges to 0 since both $T_n x_n$ and $T_n x$ converge to Tx. This shows that $|||\cdot|||$ is LUR. □

It is also worth mentioning that every SOT-compact group of isomorphisms is light:

Proposition 2.2. *Let $G \leqslant \mathrm{GL}(X)$ be a group of isomorphisms of a Banach space X. If G is SOT-compact, then G is light.*

Proof. The assumption implies that G is also WOT compact, since the WOT is weaker than the SOT. However, the WOT is also Hausdorff, and so the two topologies must agree on G. In other words, G is light. □

Ferenczi and Rosendal [2011] investigated LUR renormings in the context of transitivity of norms. Recall that a norm $\|\cdot\|$ on X is called *transitive* if the orbit $\mathscr{O}(x)$ of every point $x \in S_X$ under the action of the isometry group $\mathrm{Isom}(X)$ is the whole sphere S_X. If for every $x \in X$ the orbit $\mathscr{O}(x)$ is dense in S_X, we say that $\|\cdot\|$ is *almost transitive*, and if the closed convex hull of $\mathscr{O}(x)$ is the unit ball B_X, we say that $\|\cdot\|$ is *convex transitive*. Ferenczi and Rosendal proved that if an almost transitive norm on a Banach space is LUR at some point of the unit sphere, it is uniformly convex. They also proved that if a convex transitive norm on a Banach space is LUR on a dense subset of the unit sphere, it is almost transitive and uniformly convex.

In the next theorem, we exhibit a relation between the existence of LUR renormings and light groups.

Theorem 2.3. *Let $G \leqslant \mathrm{GL}(X)$ be a group of isomorphisms of a Banach space X. If X admits a G-invariant renorming which is LUR on a dense subset of S_X, then G is light.*

Proof. Let $\|\cdot\|$ be a G-invariant renorming of X which is LUR on a dense subset of S_X. Let (T_α) be a net of elements of G which converges WOT to the identity operator Id on X, and assume that T_α does not converge SOT to Id. Let $x \in S_X$ be such that $T_\alpha x$ does not converge to x. Without loss of generality, we can suppose that the norm is LUR at x, and that there exists $\delta > 0$ such that, for every α, $\|T_\alpha x - x\| \geq \delta$. Since $\|T_\alpha x\| = \|x\| = 1$ for every α, the LUR property forbids having $\lim \|T_\alpha x + x\| = 2$. So we can assume that there exists $\varepsilon > 0$ such that $\|T_\alpha x + x\| \leq 2 - \varepsilon$ for every α.

Let $\phi \in X^*$ with $\|\phi\| = 1$ be such that $\phi(x) = 1$. Since T_α converges WOT to Id, $\phi(T_\alpha x) \to 1$. On the other hand,

$$\begin{aligned} 2 - \delta &\geq \|T_\alpha x + x\| \\ &= \max_{\psi \in X^*,\ \|\psi\|=1} |\psi(T_\alpha x + x)| \geq |\phi(T_\alpha x) + 1| \quad \text{for every } \alpha, \end{aligned}$$

which contradicts the WOT convergence of T_α to Id. □

Remark 2.4. In fact, the proof of Theorem 2.3 gives us a formally stronger result: if X admits a G-invariant renorming which is LUR on a dense subset of S_X then G is *orbitwise light*. Megrelishvili [2003] defines a group $G \leqslant GL(X)$ as orbitwise light (or orbitwise Kadec) if for every $x \in X$ the orbit $\mathscr{O}(x) = \{Tx\ ;\ T \in G\}$ is a set on which the weak and the strong topologies coincide. It is readily seen that if G is orbitwise light, then it is light, but whether the converse holds is still an open question.

3. Light groups and distinguished points

As recalled in the introduction, Lancien [1993] proved that if X is separable with the RNP, X admits an isometry invariant LUR renorming. Although separable spaces always admit LUR renormings, the generalization of Lancien's result to all separable spaces is false. For example, if $X = C([-1, 1]^2)$ and $G = \mathrm{Isom}(X)$ then, since G is not light [Megrelishvili 2001], by Theorem 2.3 there is no equivalent G-invariant (not even dense) LUR renorming. Another example mentioned in [Ferenczi and Rosendal 2011] is the case where $X = L_1[0, 1]$ and $G = \mathrm{Isom}(L_1[0, 1])$. In this case there is not even a strictly convex G-invariant renorming.

Here we discuss conditions which clarify the relations between the two properties of a group $G \leqslant \mathrm{GL}(X)$ being light and X having a G-invariant LUR renorming, in the case when G is SOT-discrete. The following notion was defined in [Ferenczi and Rosendal 2011].

Definition 3.1. Let X be a Banach space, let $G \leqslant \mathrm{GL}(X)$ be a bounded group of isomorphisms of X, and let $x \in X$. We say that x is *distinguished* for G (or for the action of G on X) if

$$\inf_{T \neq \mathrm{Id}} \|Tx - x\| > 0.$$

If $\{x_1, \dots, x_n\}$ is a finite family of vectors of X, then it is distinguished for G if

$$\inf_{T \neq \mathrm{Id}} \max_{1 \leq i \leq n} \|Tx_i - x_i\| > 0,$$

or, equivalently, if the n-tuple $(x_1, \dots, x_n)$ is distinguished for the canonical action of G on X^n.

This notion does not depend on the choice of an equivalent norm on X. Note also that G is SOT-discrete exactly when it admits a distinguished finite family of vectors. We also have, considering the adjoint action of G on X^*:

Lemma 3.2. *Assume that $G \leqslant \mathrm{GL}(X)$ is light. If G acts as an SOT-discrete group on X, then G acts as an SOT-discrete group on X^*.*

Proof. Define $\psi : G \to \mathrm{GL}(X^*)$ by setting $\psi(T)(x^*) = x^* \circ T^{-1}$ for every $T \in G$ and $x^* \in X^*$. We want to show that $\psi(G)$ is an SOT-discrete subgroup of $\mathrm{GL}(X^*)$. It suffices to show the existence of $\varepsilon > 0$ and $x_1^*, \dots, x_n^* \in S_{X^*}$ such that the only element T of G such that $\|\psi(T)(x_i^*) - x_i^*\| < \varepsilon$ for every $1 \leq i \leq n$ is the identity operator Id_X on X. Since G is light and acts as an SOT-discrete group on X, it is WOT-discrete. So there exist $\varepsilon > 0$, $x_1, \dots, x_m \in S_X$ and $x_1^*, \dots, x_n^* \in S_{X^*}$ such that the only element T of G such that $|x_i^*(T^{-1}x_j - x_j)| < \varepsilon$ for every $1 \leq i \leq n$ and $1 \leq j \leq m$ is $T = \mathrm{Id}_X$. The conclusion follows immediately. □

Lemma 3.3. *Let X be a Banach space, let G be a bounded subgroup of* $\mathrm{GL}(X)$, *and let $\{x_1, \dots, x_n\}$ be a distinguished family of vectors for the action of G on X.*

Let $\|\cdot\|$ be a G-invariant norm on X which is LUR at x_i for every $1 \le i \le n$. For any functional $x_i^* \in S_{X^*,\|\cdot\|^*}$ such that $x_i^*(x_i) = \|x_i\|$ for every $1 \le i \le n$, the family $\{x_1^*, \dots, x_n^*\}$ is distinguished for the action of G on X^*.

Proof. Assume $\|x_i\|=1$ for every $1\le i\le n$. Let $\alpha = \inf_{T\neq \mathrm{Id}_X} \max_{1\le i\le n} \|Tx_i - x_i\| > 0$. For every $T \neq \mathrm{Id}_X$, choose $1 \le i \le n$ such that $\|T^{-1}x_i - x_i\| \ge \alpha$. By the LUR property of the norm at x_i, there exists $\varepsilon > 0$ depending on α but not on i such that $\|T^{-1}x_i + x_i\| \leqslant 2 - \varepsilon$. So $x_i^*(T^{-1}x_i) \leqslant 1 - \varepsilon$. From this it follows, using the notation introduced in the proof of Lemma 3.2, that $\psi(T)(x_i^*)(x_i) - x_i^*(x_i) \leqslant -\varepsilon$, so that $\|\psi(T)(x_i^*) - x_i^*\| \ge \varepsilon$. This being true for every $T \neq \mathrm{Id}_X$, $\{x_1^*, \dots, x_n^*\}$ is distinguished for the action of G on X^*. ☐

As a direct corollary, we obtain:

Corollary 3.4. *Let X be a Banach space, let $G \leqslant \mathrm{GL}(X)$ be SOT-discrete, and assume that X admits a G-invariant dense LUR renorming. If there exists a distinguished family of cardinality n for the action of G on X, there also exists a distinguished family of cardinality n for the action of G on X^*.*

4. Bounded groups which are not light

Isometry groups are especially relevant to our study. We introduce the following definition:

Definition 4.1. A Banach space X is *light* if $\mathrm{Isom}(X)$ is a light subgroup of $\mathrm{GL}(X)$.

Observe that since the isometry group of a Banach space $(X, \|\cdot\|)$ is not invariant by equivalent renorming, the notion of lightness for a Banach space depends very much on the choice of the norm. In our terminology, Megrelishvili [2001] proved that all spaces with the PCP are light but that $C([0, 1]^2)$ is not light. Also, we have the following example:

Example 4.2. The space c_0 is light.

In fact, every isometry T of c_0 (endowed with the usual supremum norm) has the form

$$T((x_k)_{k\in\mathbb{N}}) = (\varepsilon_k x_{\sigma(k)})_{k\in\mathbb{N}}, \quad (x_k)_{k\in\mathbb{N}} \in c_0,$$

where $(\varepsilon_k)_{k\in\mathbb{N}} \in \{-1, 1\}^{\mathbb{N}}$ and σ is a permutation of $\mathbb{N}$. For $i \in \mathbb{N}$, denote by φ_i the i-th coordinate functional on c_0. Let (T_α) be a net in $\mathrm{Isom}(c_0)$, such that $T_\alpha \xrightarrow{\mathrm{WOT}} \mathrm{Id}$. Write each T_α as

$$T_\alpha((x_k)_{k\in\mathbb{N}}) = (\varepsilon_{\alpha,k} x_{\sigma_\alpha(k)})_{k\in\mathbb{N}}, \quad (x_k)_{k\in\mathbb{N}} \in c_0,$$

with $(\varepsilon_{\alpha,k})_{k\in\mathbb{N}} \in \{-1, 1\}^{\mathbb{N}}$ and σ_α is a permutation of $\mathbb{N}$. We have for every $x \in c_0$ and every $i \in \mathbb{N}$,

$$|\varphi_i(T_\alpha x) - \varphi_i(x)| = |\varepsilon_{\alpha,i} x_{\sigma_\alpha(i)} - x_i| \to 0.$$

Since this holds for every x belonging to the space c_{00} of finitely supported sequences, we must eventually have $\sigma_\alpha(i) = i$ and $\varepsilon_{\alpha,i} = 1$. Hence $\|T_\alpha x - x\| \to 0$ for every $x \in c_{00}$, and by density of c_{00} in c_0, $\|T_\alpha x - x\| \to 0$ for every $x \in c_0$.

Another proof of Example 4.2 is based on the observation that c_0 admits a particular LUR renorming, namely the Day's renorming given by

$$\|x\|_D = \sup\left\{\left(\sum_{k=1}^{n} \frac{x_{\sigma(k)}^2}{4^k}\right)^{\frac{1}{2}}\right\}, \quad x \in c_0,$$

where the supremum is taken over all $n \in \mathbb{N}$ and all permutations σ of $\mathbb{N}$ (see [Deville et al. 1993, p. 69]). Since this renorming is isometry invariant, it follows from Theorem 2.3 that c_0 is light.

Note that Day's renorming is actually defined on ℓ_∞, and therefore on the space c of convergent real sequences. In view of Proposition 4.3, it may be amusing to observe that Day's renorming is not strictly convex on c (not even on a dense subset of S_c). In fact, it is not strictly convex at the point $(1, 1, \ldots)$ since for every $x = (x_k)_{k\in\mathbb{N}} \in c$ such that $\|x\|_\infty = 1$ and $|x_k| = 1$ for infinitely many indices k, we have $\|x\|_D = \|(1, 1, \ldots)\|_D$.

We now provide an elementary example of a space which is not light.

Proposition 4.3. *There exists a subgroup $G \leqslant \mathrm{Isom}(c)$ which has a distinguished point, but whose dual action on ℓ_1 is not SOT-discrete. In particular the space c is not light.*

Proof. Define G as the subgroup of isometries T of c of the form

$$T((x_k)_{k\in\mathbb{N}}) = (\varepsilon_k x_k)_{k\in\mathbb{N}}, \quad (x_k)_{k\in\mathbb{N}} \in c,$$

where the sequence $(\varepsilon_k)_k \in \{-1, 1\}^{\mathbb{N}}$ is eventually constant. We easily see that $(1, 1, \ldots)$ is a distinguished point for G. On the other hand, the dual space of c identifies isomorphically with ℓ_1, where $\varphi = (y_k)_{k\in\mathbb{N}} \in \ell_1$ acts on an element $x = (x_k)_{k\in\mathbb{N}} \in c$ by the formula

$$\varphi(x) = y_1 \lim_{k\to\infty} x_k + \sum_{k=2}^{\infty} y_k\, x_{k-1}.$$

For every $n \in \mathbb{N}$, define the operator $T_n \in G$ by setting, for every $(x_k)_{k\in\mathbb{N}} \in c$,

$$T_n(x_1, x_2, \ldots, x_{n-1}, x_n, x_{n+1}, \ldots) = (x_1, x_2, \ldots, x_{n-1}, -x_n, x_{n+1}, \ldots).$$

Obviously $T_n \overset{\mathrm{SOT}}{\not\to} \mathrm{Id}$, but for every $n \in \mathbb{N}$ and every $x \in c$ we have

$$\varphi(T_n(x)) = y_1 \lim_{k\to\infty} (T_n(x))_k + \sum_{k=2}^{\infty} y_k (T_n(x))_{k-1} = \left(y_1 \lim_{k\to\infty} x_k + \sum_{k=2}^{\infty} y_k\, x_{k-1}\right) - 2y_n x_n$$

which tends to $\varphi(x)$ as n tends to infinity. Hence $T_n \xrightarrow{\text{WOT}} \text{Id}$ and G is not light, which implies that Isom(c) itself is not light. Actually the inequality $|(T_n^*\varphi - \varphi)(x)| = 2|y_n x_n| \leq 2|y_n|\|x\|$, $x \in c$, $\varphi \in \ell_1$, implies that T_n^* tends SOT to Id, so the dual action of G on ℓ_1 is not SOT-discrete. □

Remark 4.4. Note that the nonlight subgroup G of Isom(c) constructed in the proof of Proposition 4.3 has the property that all its elements belong to the group Isom$_f(c)$ of isometries which are finite rank perturbations of the identity.

We observe the following relation between groups acting on a space and on a complemented subspace.

Lemma 4.5. *Assume Y embeds complementably in X. If every bounded group of isomorphisms on X is light, then every bounded group of isomorphisms on Y is light.*

Proof. Let Z be a closed subspace of X such that $X \simeq Y \oplus Z$. Let $G \leqslant GL(Y)$ be a bounded subgroup and for each $T \in G$, consider the operator $\tilde{T} = T \oplus \text{Id}_Y \in GL(X)$. These operators form a bounded subgroup $\widetilde{G}$ of $GL(X)$ which is therefore light.

Let $(T_\alpha)_{\alpha \in I}$ be a net in G such that $T_\alpha \xrightarrow{\text{WOT}} \text{Id}_Y$. Then $\widetilde{T}_\alpha \xrightarrow{\text{WOT}} \text{Id}_X$, and since $\widetilde{G}$ is light, $\widetilde{T}_\alpha \xrightarrow{\text{SOT}} \text{Id}_X$. Since for every $y \in Y$,

$$\|T_\alpha(y) - y\|_Y = \|\widetilde{T}_\alpha(y, 0) - (y, 0)\|_X \to 0,$$

it follows that $T_\alpha \xrightarrow{\text{SOT}} \text{Id}_Y$. □

Assume that X is separable and that $G \leqslant \text{GL}(X)$ is a bounded group of isomorphisms on X. As we have seen, if X has the RNP or if G is SOT-compact, then X admits a G-invariant LUR-renorming. It is natural to wonder whether the assumption on G may be weakened somewhat and, in particular, whether a similar result holds true for cyclic groups G. We show that it is not the case.

Theorem 4.6. *Let X be a separable Banach space containing an isomorphic copy of c_0. Then* GL(X) *contains a WOT-indiscrete bounded cyclic subgroup G with a distinguished point in X. In particular, G is not light.*

Proof. Consider the space $c(\mathbb{R}^2)$ of convergent sequences in the euclidean space $\mathbb{R}^2$ with the supremum norm. We define an isometry T of $c(\mathbb{R}^2)$ by setting

$$T((x_n)_{n\in\mathbb{N}}) = (R_n x_n)_{n\in\mathbb{N}} \quad \text{for every } x = (x_n)_{n\in\mathbb{N}} \in c(\mathbb{R}^2),$$

where, for every $n \in \mathbb{N}$,

$$R_n = \begin{pmatrix} \cos\left(\frac{2\pi}{n}\right) & -\sin\left(\frac{2\pi}{n}\right) \\ \cos\left(\frac{2\pi}{n}\right) & \sin\left(\frac{2\pi}{n}\right) \end{pmatrix}$$

is the rotation of $\mathbb{R}^2$ of angle $2\pi/n$. Observe that, since $\lim_n 2\pi/n = 0$, we have

$$\lim_n T((x_n)_{n\in\mathbb{N}}) = \lim_n (R_n x_n)_{n\in\mathbb{N}} = \lim_n (x_n)_{n\in\mathbb{N}} \quad \text{for every } x = (x_n)_{n\in\mathbb{N}} \in c(\mathbb{R}^2).$$

As also $R_n^{k!} = \mathrm{Id}_{\mathbb{R}^2}$ whenever $k \geq n$, we deduce that $T^{k!} \xrightarrow{\mathrm{WOT}} \mathrm{Id}$. So the cyclic subgroup $\langle T \rangle$ of $\mathrm{GL}(c(\mathbb{R}^2))$ generated by T is indiscrete in the WOT.

On the other hand, if we define $x = (x_n)_{n\in\mathbb{N}} \in c(\mathbb{R}^2)$ by setting $x_n = (1, 0)$ for every $n \in \mathbb{N}$, we find that, for every $k \in \mathbb{N}$,

$$\|T^k x - x\|_{c(\mathbb{R}^2)} \geq \|R_{2k}^k x_{2k} - x_{2k}\|_2 = \|(-1, 0) - (1, 0)\|_2 = 2.$$

So x is a distinguished point for $\langle T \rangle$.

Observe that $c(\mathbb{R}^2) \simeq c \oplus c \simeq c_0 \oplus c_0 \simeq c_0$, so T can be seen as an automorphism of c_0. Also, if X is a separable Banach space containing c_0, then c_0 is complemented in X by Sobczyk's theorem, i.e., X can be written as $X = c_0 \oplus Y$ for some subspace Y of X. Then Lemma 4.5 applies. Actually the group G generated by $S = T \oplus \mathrm{I}$ on X is not light, since $S^{k!} \xrightarrow{\mathrm{WOT}} \mathrm{Id}$, while G has a distinguished point in X. □

Remark 4.7. It follows from Theorem 4.6 that any separable Banach space X containing an isomorphic copy of c_0 admits a renorming $|||\cdot|||$ such that $(X, |||\cdot|||)$ is not light.

We finish this section with the following observation:

Lemma 4.8. *Suppose G is an abelian group acting by isometries on a metric space (X, d) without isolated points, and inducing a dense orbit $G \cdot x$ for some element $x \in X$. Then, for every $\varepsilon > 0$, there exists $g \in G \setminus \{1\}$ such that* $\sup_{z\in X} d(gz, z) < \varepsilon$.

Proof. Indeed, since X has no isolated points and the orbit $G \cdot x$ is dense, we may pick $g \in G$ so that $0 < d(gx, x) < \varepsilon$. For any y in $G \cdot x$, written $y = hx$ for $h \in G$, we have

$$d(gy, y) = d(ghx, hx) = d(hgx, hx) = d(gx, x) < \varepsilon.$$

The result follows by density. □

As a particular instance, note that if G is an SOT-discrete group of isometries of a Banach space X of dimension > 1 with a dense orbit on S_X, then G cannot be abelian.

5. LUR and strictly convex isometry invariant renormings

Theorem 2.3 leads to the following question:

Question 5.1. Does there exist a light Banach space X which admits no isometry invariant LUR renorming?

It was observed in [Ferenczi and Rosendal 2011] that $X = L_1[0,1]$ does not admit any isometry invariant dense LUR renorming. In fact, since the norm of $L_1[0,1]$ is almost transitive and is not strictly convex, any equivalent renorming is just a multiple of the original norm, so it is not strictly convex either, and hence is not LUR. Thus $L_1[0,1]$ could be a natural example of a light Banach space which admits no isometry invariant LUR renorming. However, this is not the case:

Proposition 5.2. *The space $L_1[0,1]$ is not light.*

Proof. For every $n \in \mathbb{N}$, define $\varphi_n : [0,1] \to [0,1]$ by setting

$$\varphi_n(t) = t + \frac{1-\cos(2^n \pi t)}{2^n \pi}, \quad t \in [0,1],$$

and $T_n : L_1[0,1] \to L_1[0,1]$ by

$$T_n(f)(t) = \varphi_n'(t) f(\varphi_n(t)), \quad f \in L_1[0,1],\ t \in [0,1].$$

Note that φ_n is a differentiable bijection from $[0,1]$ into itself. So T_n is a surjective linear isometry of $L_1[0,1]$. Moreover, $T_n \overset{\mathrm{SOT}}{\not\to} \mathrm{Id}$, since for $f \equiv 1$ we have

$$\|T_n(1) - 1\|_1 = \|\sin(2^n \pi x)\|_1 = \frac{2}{\pi} \text{ for every } n \in \mathbb{N}.$$

On the other hand, $T_n \xrightarrow{\mathrm{WOT}} \mathrm{Id}$. To prove this, we need to check that

$$\int_0^1 T_n(f)(t) g(t)\,dt \to \int_0^1 f(t) g(t)\,dt \quad \text{for every } f \in L_1[0,1] \text{ and } g \in L_\infty[0,1].$$

By the linearity of T_n and the density of step functions in $L_1[0,1]$, it is sufficient to consider the case where f is the indicator function of a segment $I_{m,k} = \left[\frac{2k}{2^m}, \frac{2(k+1)}{2^m}\right]$, where $m \geq 1$ and $0 \leq k \leq 2^{m-1} - 1$. In this case the function φ_n is a bijection from $I_{m,k}$ into itself for every $n \geq m$. Thus $f \circ \varphi_n = \varphi_n$, and

$$\begin{aligned}\int_0^1 T_n(f)(t) g(t)\,dt &= \int_0^1 \varphi_n'(t) f(\varphi_n(t)) g(t)\,dt = \int_0^1 \varphi_n'(t) f(t) g(t)\,dt \\ &= \int_0^1 f(t) g(t)\,dt + \int_0^1 \sin(2^n \pi t) f(t) g(t)\,dt.\end{aligned}$$

The result then follows from the Riemann–Lebesgue lemma. □

Remark 5.3. Another space of which it is well known that it does not admit any LUR renorming is ℓ_∞. Indeed ℓ_∞ does not admit any equivalent norm with the Kadec–Klee property ([Deville et al. 1993, Chapter 2, Theorem 7.10]), while every LUR norm satisfies the Kadec–Klee property ([Deville et al. 1993, Chapter 2, Proposition 1.4]). However, ℓ_∞ does admit a strictly convex renorming (see [Diestel 1975, p. 120]). We note here that it does not admit any isometry invariant strictly convex renorming. To see this, consider the points $x = (1, 1, 0, 1, 0, 1, 0, \ldots)$ and

$y = (-1, 1, 0, 1, 0, 1, 0, \ldots)$. Setting $z = (x + y)/2 = (0, 1, 0, 1, 0, 1, 0, \ldots)$, it is readily seen that there exist two isometries T and S of ℓ_∞ such that $Tx = y$ and $Sx = z$. So, for any isometry invariant renorming $|||\cdot|||$ of ℓ_∞ we have $|||x||| = |||y||| = |||z|||$, and therefore $|||\cdot|||$ cannot be strictly convex.

Proposition 5.4. *The space ℓ_∞ is not light.*

Proof. Consider the sequence of isometries $T_n : \ell_\infty \to \ell_\infty,\ n \in \mathbb{N}$, defined by

$$T_n(x_1, \ldots, x_{n-1}, x_n, x_{n+1}, \ldots) = (x_1, \ldots, x_{n-1}, -x_n, x_{n+1}, \ldots), \quad x = (x_k)_{k\in\mathbb{N}} \in \ell_\infty.$$

Notice that $T_n \overset{SOT}{\not\to} \mathrm{Id}$, since $\|T_n(1, 1, \ldots) - (1, 1, \ldots)\|_\infty = 2$ for every $n \in \mathbb{N}$. On the other hand, observe that $T_n \xrightarrow{WOT} \mathrm{Id}$. Indeed, if $(e_j)_{j\in\mathbb{N}}$ denotes the canonical basis of ℓ_∞, the sequence $(\beta(e_j))_{j\in\mathbb{N}}$ belongs to ℓ_1 for every $\beta \in \ell_\infty^*$. In particular, $\beta(e_j) \to 0$. Thus $\beta(T_n x - x) = -2x_n\beta(e_n) \to 0$ for every $x \in \ell_\infty$ and $\beta \in \ell_\infty^*$, showing that $T_n \xrightarrow{WOT} \mathrm{Id}$. □

A similar proof allows us to construct many examples of $C(K)$-spaces which are not light.

Theorem 5.5. *Let K be a compact space with infinitely many connected components. Then $C(K)$ is not light.*

Proof. We claim that, under the assumption of Theorem 5.5, there exists a sequence $(N_n)_{n\in\mathbb{N}}$ of disjoint clopen subsets of K. Indeed, choose two points x_1 and y_1 of K which belong to two different connected components of K. Since the connected component of a point x of K is the intersection of all the clopen subsets of K containing x, there exists a clopen subset K_1 of K such that $x_1 \in K_1$ and $y_1 \in L_1 := K \setminus K_1$. The two sets K_1 and L_1 are compact, and one of them, say K_1, has infinitely many connected components. We set then $N_1 = L_1$, and repeat the argument starting from the compact set K_1. This yields a sequence $(N_n)_{n\in\mathbb{N}}$ of disjoint clopen subsets of K.

For each integer $n \in \mathbb{N}$, define $T_n \in \mathrm{Isom}(C(K))$ by setting, for every $f \in C(K)$ and every $x \in K$,

$$T_n(f)(x) = \begin{cases} -f(x) & \text{if } x \in N_n, \\ f(x) & \text{otherwise.} \end{cases}$$

If χ_n denotes the indicator function of the set N_n, we have $T_n(f) = f(1 - 2\chi_n)$ for every $f \in C(K)$. Applying this to the constant function $f \equiv 1$, we have $\|T_n(f) - f\|_\infty = 2$ for every $n \in \mathbb{N}$, so that $T_n \overset{SOT}{\not\to} \mathrm{Id}$. On the other hand, the same kind of argument as in Proposition 5.4 shows that $T_n \xrightarrow{WOT} \mathrm{Id}$. Indeed, we have $\Phi(T_n f - f) = -2\Phi(f\chi_n)$ for every functional $\Phi \in C(K)^*$ and every $f \in C(K)$. For every sequence $(\alpha_n)_{n\in\mathbb{N}} \in c_0$, the series $\sum_{n\in\mathbb{N}} \alpha_n f\chi_n$ converges in $C(K)$, so that the series $\sum_{n\in\mathbb{N}} \alpha_n \Phi(f\chi_n)$ converges. It follows that $\Phi(f\chi_n) \to 0$ as $n \to +\infty$, which proves our claim. □

As a direct consequence of Theorem 5.5 we retrieve the result, proved in Proposition 4.3 above, that the space c of convergent sequences is not light. Also, we immediately deduce that the space of continuous functions on the Cantor space is not light.

Corollary 5.6. *The space $C(\{0,1\}^{\mathbb{N}})$ is not light.*

In view of the results above, combined with the known fact that the space $C([0,1]^2)$ is not light, it may seem reasonable to conjecture that none of the spaces $C(K)$, where K is any infinite compact space, is light. However, our next result shows that this is not the case.

Theorem 5.7. *Let K be an infinite compact connected space. Then $C(K)$ is light if and only if the topologies of pointwise and uniform convergence coincide on the group* Homeo(K) *of homeomorphisms of K.*

Proof. Suppose first that the topologies of pointwise and uniform convergence coincide on Homeo(K). Let $(T_\alpha)_{\alpha\in I}$ be a net of isometries of $C(K)$ such that $T_\alpha \xrightarrow{\text{WOT}} \text{Id}$. By the Banach–Stone theorem and the connectedness of K, each isometry T_α of $C(K)$ has the form

$$T_\alpha(f) = \varepsilon_\alpha\, f\circ\varphi_\alpha \quad \text{for every } f\in C(K),$$

where $\varepsilon_\alpha \in \{-1,1\}$ and $\varphi_\alpha \in \text{Homeo}(K)$. Since $T_\alpha \xrightarrow{\text{WOT}} \text{Id}$, $\varepsilon_\alpha \to 1$, so we can suppose without loss of generality that $\varepsilon_\alpha = 1$ for every $\alpha\in I$. Moreover, the fact that $T_\alpha \xrightarrow{\text{WOT}} \text{Id}$ also implies that the net $(\varphi_\alpha)_{\alpha\in I}$ converges pointwise to the identity function id_K on K. Our assumption then implies that $(\varphi_\alpha)_{\alpha\in I}$ converges uniformly to id_K on K, from which it easily follows that $T_\alpha \xrightarrow{\text{SOT}} \text{Id}$. Thus $C(K)$ is light.

Conversely, suppose that $C(K)$ is light. Let $(\varphi_\alpha)_{\alpha\in I}$ be a net of elements of Homeo(K) which converges pointwise to $\varphi\in\text{Homeo}(K)$. Consider the isometries T_α and T of $C(K)$ defined by

$$T_\alpha(f) = f\circ\varphi_\alpha \quad\text{and}\quad T(f) = f\circ\varphi \quad \text{for every } f\in C(K).$$

Then $T_\alpha \xrightarrow{\text{WOT}} \text{Id}$. Since $C(K)$ is light, $T_\alpha \xrightarrow{\text{SOT}} \text{Id}$ and thus $(\varphi_\alpha)_{\alpha\in I}$ converges to φ uniformly on K. □

Remark 5.8. Theorem 5.7 characterizes the lightness of $C(K)$ for infinite connected compact spaces K. We may naturally wonder whether the connectedness assumption is really necessary. It is indeed the case: there exist compact spaces K with infinitely many connected components which are *rigid* in the sense that their homeomorphism group Homeo(K) is trivial (it consists uniquely of the identity map on K). The existence of such compacta is proved in [de Groot and Wille 1958] (see the remark on p. 443 at the end of Section 2). By Theorem 5.5, $C(K)$ is not light for such a compact K, but the topologies of pointwise and uniform convergence obviously coincide on Homeo(K).

Birkhoff [1934] studied various topologies on so-called "transformation spaces", in particular on the groups of homeomorphisms of topological spaces. He introduced the notions of A-, B- and C-convergence of sequences of homeomorphisms on a given space X corresponding respectively to pointwise convergence, continuous convergence, and continuous convergence in both directions. Since on compact spaces continuous convergence and uniform convergence coincide, Theorem 5.7 can be rephrased, using Birkhoff's language, as saying that for compact connected spaces K, $C(K)$ is light if and only if A- and B-convergence coincide on $\mathrm{Homeo}(K)$. Now, it is observed in [Birkhoff 1934, Theorem 18] that A-convergence implies B- and C-convergence for homeomorphisms of (disjoint or not) finite unions of segments of the real line (this is essentially the content of Dini's second convergence theorem), while if K contains an n-dimensional region with $n \geq 2$ (i.e., an open set homeomorphic to an open subset of $\mathbb{R}^n$), A-convergence implies neither B- nor C-convergence for homeomorphisms of K ([Birkhoff 1934, Theorem 19]). In more modern language, under this assumption there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of homeomorphisms of K such that φ_n converges pointwise but not uniformly on K to the identity function on K. Combined with Theorem 5.7 above, this yields:

Theorem 5.9. *Let K be an infinite compact connected space.*

(a) *If K is homeomorphic to a finite union of segments of $\mathbb{R}$, $C(K)$ is light.*

(b) *If K contains an n-dimensional region for some $n \geq 2$, $C(K)$ is not light.*

For instance, the space $C[0, 1]$ is light, while spaces $C([0, 1]^n)$, $n \geq 2$, are not light. We thus retrieve in a natural way the original example of Megrelishvili of a nonlight space.

Theorem 5.9 allows us to answer Question 5.1 in the negative. Although $C[0, 1]$ is light, it does not admit any isometry invariant LUR renorming. In fact, $C[0, 1]$ does not admit any isometry invariant strictly convex renorming. In order to prove this, we need the following lemma:

Lemma 5.10. *Let $f \in C[0, 1]$ be such that there exists an interval $[a, b] \subset [0, 1]$, $a < b$, on which f is strictly monotone. Then there exists $g \in C[0, 1]$ with the following three properties*:

(a) $\|f\|_\infty = \|g\|_\infty = \|(f + g)/2\|_\infty$;

(b) $\|f - g\|_\infty > 0$;

(c) *There exist two homeomorphisms φ and ψ of $[0, 1]$ such that $g = f \circ \varphi$ and $(f + g)/2 = f \circ \psi$.*

Proof. Let $0 \leq a < b \leq 1$ be such that f is strictly monotone on $[a, b]$. Without loss of generality, suppose f is strictly increasing on $[a, b]$. Let $\xi : [a, b] \to [f(a), f(b)]$

be an increasing homeomorphism such that $\xi \not\equiv f|_{[a,b]}$. Define $g \in C[0, 1]$ and a homeomorphism $\varphi : [0, 1] \to [0, 1]$ by

$$g(x) = \begin{cases} \xi(x) & \text{if } x \in [a, b], \\ f(x) & \text{otherwise,} \end{cases} \qquad \varphi(x) = \begin{cases} f^{-1}(\xi(x)) & \text{if } x \in [a, b], \\ x & \text{otherwise.} \end{cases}$$

Then, $g = f \circ \varphi$, $\|g\|_\infty = \|f\|_\infty = \|(f + g)/2\|_\infty$ and $\|f - g\|_\infty > 0$. Moreover, $f \circ \psi = (f + g)/2$, where $\psi : [0, 1] \to [0, 1]$ is the homeomorphism defined by

$$\psi(x) = \begin{cases} f^{-1}\left(\dfrac{\xi(x) + f(x)}{2}\right) & \text{if } x \in [a, b], \\ x & \text{otherwise.} \end{cases}$$

□

Proposition 5.11. *Let* $|||\cdot|||$ *be an isometry invariant renorming of* $C[0, 1]$. *Then there exists a dense subset of* $C[0, 1]$ *where* $|||\cdot|||$ *is not strictly convex.*

Proof. Let $f \in C[0, 1]$ be a nonconstant and affine function, and take g, φ and ψ as in Lemma 5.10. Since $f \mapsto f \circ\varphi$ and $f \mapsto f \circ \psi$ define surjective linear isometries of $C[0, 1]$,

$$|||g||| = |||f \circ \varphi||| = |||f||| = |||f \circ \psi||| = \left|\left|\left|\frac{f+g}{2}\right|\right|\right|.$$

So $|||\cdot|||$ is not strictly convex at the point f. The result then follows from the fact that the set of piecewise linear functions is dense in $C[0, 1]$. □

Combining Theorem 5.9 and Proposition 5.11, we obtain:

Corollary 5.12. *The space* $C[0, 1]$ *is light, but does not admit any isometry invariant LUR renorming.*

Remark 5.13. Using the same arguments as in the proofs of Proposition 5.11, Theorem 5.7, and Theorem 5.9 one can prove that $C_0(\mathbb{R})$ is light, but does not admit a strictly convex isometry invariant renorming either.

Remark 5.14. The examples presented in this section show that there is no general relation between closed subspaces and their respective isometry groups, in terms of being light, apart from Lemma 4.5. In fact:

(1) c_0 is a closed subspace of c, c_0 is light, but c is not;

(2) c is isometrically isomorphic to a closed subspace of $C[0, 1]$, c is not light but $C[0, 1]$ is light.

Corollary 5.12 gives us a positive answer to Question 5.1. On the other hand, Remark 2.4 suggests the following new question:

Question 5.15. Does there exist a Banach space X and an orbitwise light group $G \leqslant GL(X)$ such that X admits no G-invariant LUR renorming?

The next proposition shows that the isometry group of $C[0, 1]$ also gives a positive answer to Question 5.15:

Proposition 5.16. *The group* $\mathrm{Isom}(C[0, 1])$ *is orbitwise light.*

Proof. Let $f \in C[0, 1]$ and let $(g_\alpha)_{\alpha\in I}$ be a net in the orbit $\mathcal{O}(f)$ of f under the action of the group $\mathrm{Isom}(C[0, 1])$ such that g_α converges weakly to $g \in \mathcal{O}(f)$. By the Banach–Stone theorem, there exist homeomorphisms $\varphi, \varphi_\alpha \in \mathrm{Hom}([0, 1])$ and $\varepsilon, \varepsilon_\alpha \in \{-1, 1\}$ such that $g = \varepsilon \cdot f \circ \varphi$ and $g_\alpha = \varepsilon_\alpha \cdot f \circ \varphi_\alpha$. Since g_α converges weakly to g (hence, pointwise), we can assume that the φ_α are increasing homeomorphisms, $\varepsilon = \varepsilon_\alpha = 1$ for every $\alpha \in I$ and $g = f$.

Suppose by contradiction that $f \circ \varphi_\alpha$ does not converge uniformly to f. Then we can assume that there exists $\varepsilon > 0$ and for every $\alpha \in I$ there exists $x_\alpha \in [0, 1]$ such that $|f(\varphi_\alpha(x_\alpha)) - f(x_\alpha)| > 2\varepsilon$. We also can assume that $x_\alpha \to x \in [0, 1]$ and $x_\alpha \le x$ for every α. Then by the continuity of f at the point x,

$$|f(\varphi_\alpha(x_\alpha)) - f(x)| > \varepsilon.$$

Let $\delta > 0$ be such that $|x-y| < \delta \Longrightarrow |f(x)-f(y)| < \frac{\varepsilon}{8}$. Then $\varphi_\alpha(x_\alpha) \notin (x-\delta, x+\delta)$ for every α, and $\varphi_\alpha(x_\alpha) < x-\delta$ for infinitely many indices $\alpha \in I$, or $\varphi_\alpha(x_\alpha) > x+\delta$ for infinitely many indices $\alpha \in I$. Without loss of generality, we may assume that

$$\varphi_\alpha(x_\alpha) < x - \delta, \qquad \text{for every } \alpha \in I.$$

We also may assume that

$$x - \delta < x_\alpha \le x, \qquad \text{for every } \alpha \in I$$

(the cases $\varphi_\alpha(x_\alpha) > x + \delta$ and/or $x < x_\alpha < x + \delta$ for every $\alpha \in I$ are similar).

Let $\alpha_1 \in I$ and let

$$y_{1,1} = \varphi_{\alpha_1}(x_{\alpha_1}).$$

We claim that for every $n \ge 2$, there exists a finite sequence in $[0, 1]$,

$$y_{n,1} < y_{n,2} < \cdots < y_{n,2n-1} < x - \delta,$$

such that

$$|f(y_{n,2k+1}) - f(x)| > \varepsilon - \frac{\varepsilon}{8} - \frac{\varepsilon}{2^{k+4}}\left(\sum_{j=0}^{n-k-2} \frac{1}{2^j}\right) > \frac{3\varepsilon}{4} \quad \text{for } k = 0, 1, \ldots, n-1$$

and

$$|f(y_{n,2k}) - f(x)| < \frac{\varepsilon}{8} + \frac{\varepsilon}{2^{k+3}}\left(\sum_{j=0}^{n-k-1} \frac{1}{2^j}\right) < \frac{\varepsilon}{4} \quad \text{for } k = 1, 2, \ldots, n-1.$$

Notice that the existence of such a sequence for every $n \ge 2$ contradicts the uniform continuity of f on $[0, 1]$. Hence it suffices to prove this claim in order to complete the proof of Proposition 5.16.

We prove this claim by induction. Since $f \circ \varphi_\alpha$ converges pointwise to f and $x_\alpha \to x$, we can take $\alpha_2 \succcurlyeq \alpha_1$ such that $x_{\alpha_1} < x_{\alpha_2} < x$, $|f(\varphi_\beta(y_{1,1})) - f(y_{1,1})| < \varepsilon/16$ and $|f(\varphi_\beta(x_{\alpha_1})) - f(x_{\alpha_1})| < \varepsilon/16$ for every $\beta \succcurlyeq \alpha_2$. Let

$$y_{2,1} = \varphi_{\alpha_2}(y_{1,1}), \quad y_{2,2} = \varphi_{\alpha_2}(x_{\alpha_1}) \quad \text{and} \quad y_{2,3} = \varphi_{\alpha_2}(x_{\alpha_2}).$$

Since φ_{α_2} is an increasing homeomorphism and $y_{1,1} < x - \delta < x_{\alpha_1} < x_{\alpha_2}$, we have $y_{2,1} < y_{2,2} < y_{2,3}$ and $y_{2,3} = \varphi_{\alpha_2}(x_{\alpha_2}) < x - \delta$. Moreover,

$$|f(y_{2,1}) - f(x)| > \varepsilon - \frac{\varepsilon}{8} - \frac{\varepsilon}{16}, \quad |f(y_{2,2}) - f(x)| < \frac{\varepsilon}{8} + \frac{\varepsilon}{16}, \quad |f(y_{2,3}) - f(x)| > \varepsilon - \frac{\varepsilon}{8},$$

which proves the inequalities for $n = 2$.

Suppose the inequalities hold for n. Let $\alpha_{n+1} \succcurlyeq \alpha_n$ such that $x_{\alpha_n} < x_{\alpha_{n+1}} < x$, $|f(\varphi_\beta(y_{n,r})) - f(y_{n,r})| < \varepsilon/2^{n+3}$ and $|f(\varphi_\beta(x_{\alpha_n})) - f(x_{\alpha_n})| < \varepsilon/2^{n+3}$ for every $r = 1, 2, \ldots, 2n-1$ and every $\beta \succcurlyeq \alpha_2$. Let

$$y_{n+1,r} = \varphi_{\alpha_{n+1}}(y_{n,r}) \quad \text{for } r = 1, \ldots, 2n-1,$$

and let

$$y_{n+1,2n} = \varphi_{\alpha_{n+1}}(x_{\alpha_n}) \quad \text{and} \quad y_{n+1,2n+1} = \varphi_{\alpha_{n+1}}(x_{\alpha_{n+1}}).$$

It follows that

$$|f(y_{n+1,2k+1}) - f(x)| > \varepsilon - \frac{\varepsilon}{8} - \frac{\varepsilon}{2^{k+4}}\left(\sum_{j=0}^{n-k-1} \frac{1}{2^j}\right) \quad \text{for } k = 0, 1, \ldots, n,$$

and

$$|f(y_{n+1,2k}) - f(x)| < \frac{\varepsilon}{8} + \frac{\varepsilon}{2^{k+3}}\left(\sum_{j=0}^{n-k} \frac{1}{2^j}\right) \quad \text{for } k = 1, 2, \ldots, n.$$

Since $\varphi_{\alpha_{n+1}}$ is an increasing homeomorphism, and $y_{n,1} < y_{n,2} < \cdots < y_{n,2n-1} < x - \delta < x_{\alpha_n} < x_{\alpha_{n+1}}$, we have $y_{n+1,1} < y_{n+1,2} < \cdots < y_{n+1,2n+1} = \varphi_{\alpha_{n+1}}(x_{\alpha_{n+1}}) < x - \delta$, which proves the claim. $\square$

6. An example of a group with a discrete orbit but no distinguished point

In this section we solve a problem of [Ferenczi and Rosendal 2011], mentioned in the introduction, by exhibiting an SOT-discrete group of isomorphisms of c_0 which admits no distinguished point. More generally, we show the following:

Proposition 6.1. *For any integer $r \geq 2$, there exists a bounded infinite SOT-discrete group of isomorphisms of c_0 of the form $\mathrm{Id} + F$, $F \in L(c_0)$ of finite rank, admitting a distinguished family of cardinality r, but none of cardinality $r - 1$.*

Proof. Since $c_0 \simeq \ell_1^r \oplus_\infty c_0$, it is enough to define the group G as an infinite bounded SOT-discrete group of isomorphisms on $\ell_1^r \oplus_\infty c_0$.

Let $(e_n)_{n\in\mathbb{N}}$ be the canonical basis of c_0, and let $(U_n)_{n\in\mathbb{N}}$ be the sequence of isometries of c_0 defined by setting, for every $n, m \in \mathbb{N}$, $U_n(e_n) = -e_n$ and $U_n(e_m) = e_m$ whenever $m \neq n$. Let $(\phi_n)_{n\in\mathbb{N}}$ be dense in the unit sphere of ℓ_∞^r, and define the rank-one operator $R_n : \ell_1^r \to c_0$ by $R_n(x) = \phi_n(x)e_n$, $x \in \ell_1$. We then define an operator T_n on $\ell_1^r \oplus_\infty c_0$ in matrix form as

$$T_n = \begin{pmatrix} \mathrm{Id} & 0 \\ R_n & U_n \end{pmatrix}.$$

It is readily checked that $T_n^2 = \mathrm{Id}$ for every $n \in \mathbb{N}$ and that for all distinct integers $n_1, \dots, n_k$,

$$T_{n_1} \dots T_{n_k} = \begin{pmatrix} \mathrm{Id} & 0 \\ R_{n_1} + \dots + R_{n_k} & U_{n_1} \dots U_{n_k} \end{pmatrix}.$$

Therefore the group G generated by the operators T_n is abelian. Furthermore, since for every $x \in \ell_1^r$

$$\|(R_{n_1} + \dots + R_{n_k})x\| = \|\phi_{n_1}(x)e_{n_1} + \dots + \phi_{n_k}(x)e_{n_k}\| \le \max_i \|\phi_{n_i}(x)\| \cdot \|x\|,$$

it follows that $\|T_{n_1} + \dots + T_{n_k}\| \le 2$; thus G is a bounded subgroup of $\mathrm{GL}(\ell_1^r \oplus_\infty c_0)$.

We claim that no family $\{x_1, \dots, x_{r-1}\}$ of $\ell_1^r \oplus c_0$ is distinguished for G. Indeed, writing each vector x_i as (y_i, z_i) with $y_i \in \ell_1^r$ and $z_i \in c_0$, we note that $U_n z_i \to z_i$ for every $1 \le i \le r-1$. Since the vectors $y_1, \dots, y_{r-1}$ generate a subspace of dimension strictly less than r of ℓ_1^r, there exists a norm 1 functional $\phi \in \ell_\infty^r$ such that $\phi(y_i) = 0$ for every $1 \le i \le r-1$. Let $D \subset \mathbb{N}$ be such that $\phi_n \to \phi$ in ℓ_∞^r as n tends to infinity along D. Then $R_n(y_i) \to 0$ as n tends to infinity along D, and therefore $T_n(x_i) \to x_i$ as n tends to infinity along D for every $1 \le i \le r-1$. So the family $\{x_1, \dots, x_{r-1}\}$ is not distinguished for G.

On the other hand, if we denote by $(f_1, \dots, f_r)$ the canonical basis of ℓ_1^r, then the family $\{f_1 \oplus 0, \dots, f_r \oplus 0\}$ is distinguished for G. To check this, note that for any operator $T \in \mathrm{GL}(\ell_1^r \oplus_\infty c_0)$ of the form

$$T = \begin{pmatrix} \mathrm{Id} & 0 \\ \sum_{k\in F} R_k & U \end{pmatrix},$$

where F in a nonempty subset of $\mathbb{N}$, and U is an isometry of c_0, we have

$$\|T(f_s \oplus 0) - f_s \oplus 0\| = \max_{k\in F} |\phi_k(f_s)| \qquad \text{for every } 1 \le s \le r.$$

Since, for each $k \in F$, ϕ_k is normalized in ℓ_∞^r, $|\phi_k(f_s)| \ge 1$ for at least one index s. It follows that

$$\max_{1\le s\le r} \|T(f_s \oplus 0) - f_s \oplus 0\| \ge 1,$$

and so

$$\inf_{T\in G,\, T\neq \mathrm{Id}} \{\max_{1\le s\le r} \|T(f_s\oplus 0)-f_s\oplus 0\|\}\ge 1.$$

Hence $\{f_1, \dots, f_r\}$ is a distinguished family for G. □

We immediately deduce the following:

Corollary 6.2. *The group of isomorphisms of c_0 which is constructed in the proof of Proposition 6.1 is not light.*

Proof. For every $x\in\ell_1^r$, the sequence $(R_n(x))_{n\in\mathbb{N}}$ tends weakly to 0 in c_0. We also know that the sequence $(U_n)_{n\in\mathbb{N}}$ tends WOT to Id. Therefore $(T_n)_{n\in\mathbb{N}}$ also tends WOT to Id. On the other hand, we have for every $x\in\ell_1^r$ and every $n\in\mathbb{N}$ that

$$\|T_n(x\oplus 0)-x\oplus 0\| = \|R_n(x)\| = \|\phi_n(x)\|.$$

By the density of the sequence $(\phi_n)_{n\in\mathbb{N}}$ in the unit sphere of ℓ_∞^r, this implies that the sequence $(T_n(x\oplus 0))_{n\in\mathbb{N}}$ does not tend to x in norm, and thus $(T_n)_{n\in\mathbb{N}}$ does not tend SOT to Id. □

We have thus proved:

Corollary 6.3. *There exists a bounded group G of isomorphisms of c_0 which is infinite, not light, SOT-discrete, and does not admit a distinguished point.*

Proof. Take $r=2$ in Proposition 6.1. □

7. Quasinormed spaces

Although Megrelishvili has defined the concept of light group of isomorphisms only for locally convex spaces, we can extend the definition to quasinormed spaces, even if these spaces are not locally convex. One could ask if there is a general answer for the isometry groups of nonlocally convex spaces, in terms of being light. The spaces ℓ_p and $L_p[0,1]$, $0<p<1$, are examples that give a negative answer to this question.

Recall that for $0<p<1$, $(L_p[0,1])^*=\{0\}$, i.e., the only linear continuous functional $f: L_p[0,1]\to\mathbb{R}$ is the constant function $f\equiv 0$ (see [Kalton et al. 1984, p. 18]). Considering the sequence $(T_n)_{n\in\mathbb{N}}$ constantly equal to $-\mathrm{Id}$, we observe that $T_n \overset{\mathrm{SOT}}{\not\to} \mathrm{Id}$ while $T_n \xrightarrow{\mathrm{WOT}} \mathrm{Id}$. So $L_p[0,1]$ is trivially nonlight for every $0<p<1$. On the other hand, we have:

Proposition 7.1. *For $0<p<1$, the space ℓ_p is light.*

Proof. Let $0<p<1$, and let $(T_\alpha)_{\alpha\in I}$ be a net in $\mathrm{Isom}(\ell_p)$ such that $T_\alpha \xrightarrow{\mathrm{WOT}} \mathrm{Id}$. Each T_α acts on vectors $(x_n)_{n\in\mathbb{N}}\in\ell_p$ as $T_\alpha((x_n)_{n\in\mathbb{N}}) = (\varepsilon_n^{(\alpha)} x_{\sigma_\alpha(n)})_{n\in\mathbb{N}}$, where σ_α is a permutation of $\mathbb{N}$ and $(\varepsilon_n^{(\alpha)})_{n\in\mathbb{N}}$ is a sequence of elements of $\{-1,1\}$ (the proof of this statement is similar to the case where $p>1$ and $p\neq 2$, which can be found in [Banach 1932, p. 178]). Assume, by contradiction, that $T_\alpha \overset{\mathrm{SOT}}{\not\to} \mathrm{Id}$.

Then there exist $x \in \ell_p$, $\varepsilon > 0$ and an infinite sequence $(\alpha_i)_{i\in\mathbb{N}}$ of indices in I such that $\|T_{\alpha_i}x - x\|_p^p > \varepsilon$ for every $i \in \mathbb{N}$. Since $x \in \ell_p$, there exists $N \in \mathbb{N}$ such that $\sum_{k=N+1}^{\infty} |x_k|^p < \varepsilon/2$. The dual space of ℓ_p identifies isomorphically with ℓ_∞, where $\Phi = (y_k)_{k\in\mathbb{N}}$ acts on an element $x = (x_k)_{k\in\mathbb{N}} \in \ell_p$ by the formula $\Phi(x) = \sum_{k\in\mathbb{N}} y_k x_k$ (see [Kalton et al. 1984, p. 21]). Considering for $1 \le j \le N$ the functionals Φ_j identified with the vectors of the canonical basis $e_j \in \ell_\infty$, as well as the vectors $e_k \in \ell_p$ for $1 \le k \le N$, we obtain by the WOT convergence of T_α to Id that

$$\Phi_j(T_\alpha(e_k)) - \Phi_j(e_k) = \varepsilon_j^{(\alpha)} \delta_{\sigma_\alpha(k),j} - \delta_{k,j} \to 0,$$

where $\delta_{k,j} = 1$ if $k = j$ and $\delta_{k,j} = 0$ if $k \neq j$. In particular, $\varepsilon_k^{(\alpha)} \delta_{\sigma_\alpha(k),k} \to 1$ for every $1 \le k \le N$. So we may assume that the permutations σ_α fix the first N integers and that $\varepsilon_k^{(\alpha)} = 1$ for every $1 \le k \le N$. Hence we have for every $i \in \mathbb{N}$

$$\sum_{k=N+1}^{\infty} |(T_{\alpha_i}(x))_k|^p = \sum_{k=N+1}^{\infty} |x_k|^p < \frac{\varepsilon}{2}.$$

However, taking $z = (z_k)_{k\in\mathbb{N}} \in \ell_p$ defined by $z_k = 0$ if $1 \le k \le N$ and $z_k = 1$ if $k > N$, we have $\|z\|_p^p = \sum_{k=N+1}^{\infty} |x_k|^p < \varepsilon/2$ and

$$\|T_{\alpha_i}x - x\|_p^p = \|T_{\alpha_i}z - z\|_p^p \le \|T_{\alpha_i}z\|_p^p + \|z\|_p^p = \varepsilon$$

for every $i \in \mathbb{N}$, which is a contradiction. □

We finish the paper with a few related questions and comments.

8. Questions and comments

Our first question concerns renormings of the space c. Since c is not light, as proved in Proposition 4.3, it does not admit any isometry invariant LUR renorming. But it may still admit an isometry invariant strictly convex renorming.

Question 8.1. Does c admit an isometry invariant strictly convex renorming?

We have observed in Section 4 that if the isometry group $\mathrm{Isom}(X)$ of a Banach space X of dimension > 1 acts almost transitively on S_X and is SOT-discrete, it is not abelian.

Question 8.2. Suppose X is a separable Banach space of dimension > 1 and $G \leqslant \mathrm{Isom}(X)$ is an SOT-discrete amenable subgroup. Can G have a dense orbit on S_X?

We have seen in Corollary 6.3 that there exists a bounded group G of isomorphisms of c_0 which is infinite, not light, SOT-discrete, and does not admit a distinguished point. We may wonder about the role of the space c_0 in this construction. For example, we can ask:

Question 8.3. Does there exist a reflexive space X with an SOT-discrete bounded group $G \leqslant GL(X)$ that does not admit a distinguished point?

Of course such a group G, if it exists, must be light, as all reflexive spaces are light. Noting that the example of Proposition 4.3 is a group of finite rank perturbations of the identity on the space c_0, a question in the same vein is:

Question 8.4. Does there exist a reflexive space X with an SOT-discrete infinite bounded group $G \leqslant GL(X)$ such that all elements of G are finite rank perturbations of the identity?

This question is relevant to [Ferenczi and Rosendal 2013], where isometry groups on complex, reflexive, separable, hereditarily indecomposable spaces are studied. A negative answer would imply that all isometry groups on such spaces act almost trivially, i.e., there would exist an isometry invariant decomposition $F \oplus H$ of the space where F is finite dimensional and all elements of the group act as multiples of the identity on H, [Ferenczi and Rosendal 2013, Theorem 6.9].

Another natural space which could be investigated in this context is the universal space of Gurariĭ, whose isometry group possesses a very rich structure (see [Gurariĭ 1966] for its definition and [Garbulińska and Kubiś 2011] for a recent survey).

Question 8.5. Is the isometry group of the Gurariĭ space light?

Finally, whether the converse to Megrelishvili's result holds remains an open question:

Question 8.6. Does a Banach space X have the PCP if and only if all bounded subgroups of $GL(X)$ are light?

The answer is positive when X has an unconditional basis; this follows from Theorem 4.6, the fact that an unconditional basis whose span does not contain c_0 must be boundedly complete, and the fact that separable dual spaces have the RNP and therefore the PCP.

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SOME UNIFORM ESTIMATES FOR SCALAR CURVATURE TYPE EQUATIONS

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We consider the prescribed scalar curvature equation on an open set Ω of $\mathbb{R}^n$, $-\Delta u = V u^{(n+2)/(n-2)} + u^{n/(n-2)}$ with $V \in C^{1,\alpha}$ $(0 < \alpha \le 1)$, and we prove the inequality $\sup_K u \times \inf_\Omega u \le c$ where K is a compact set of Ω.

In dimension 4, we have an idea on the supremum of the solution of the prescribed scalar curvature if we control the infimum. For this case we suppose the scalar curvature $C^{1,\alpha}$ $(0 < \alpha \le 1)$.

1. Introduction and main result

In our work, we denote by $\Delta = \nabla^i \nabla_i$ the Laplace–Beltrami operator in dimension $n \ge 2$.

Without loss of generality, we suppose $\Omega = B = B_2(0)$ the ball of radius 2 centered at 0 of $\mathbb{R}^n$.

Here, we study some a priori estimates of type $\sup \times \inf$ for a perturbed prescribed scalar curvature equation in all dimensions $n \ge 4$.

We have a counterexample to the sharp $\sup \times \inf$ inequality for the prescribed scalar curvature [Chen and Lin 1997, Proposition 4.3]. In our work the perturbation by a subcritical term is a sufficient condition to obtain such an inequality.

The $\sup \times \inf$ inequality is characteristic of those equations as the usual Harnack inequalities are for harmonic functions.

Note that the prescribed scalar curvature equation was studied a lot. We can find — see, for example, [Aubin 1998; Bahoura 2004; Brezis and Merle 1991; Brezis et al. 1993; Chen and Lin 1997; 1998; Li 1993; 1995; 1996; 1999; Li and Shafrir 1994; Li and Zhang 2004; Li and Zhu 1999; Shafrir 1992] — many results about uniform estimates in dimensions $n = 2$ and $n \ge 3$.

In dimension 2, the corresponding equation is

$$(E_0) \qquad -\Delta u = V e^u.$$

Note that Shafrir [1992] obtained an inequality of type $\sup u + C \inf u < c$ with only an L^∞ assumption on V.

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To obtain exactly the estimate $\sup u + \inf u < c$, Brezis, Li and Shafrir [Brezis et al. 1993] assumed that the prescribed scalar curvature V is Lipschitz continuous. Later, Chen and Lin [1998] proved that, if V is uniformly Hölder continuous, we can obtain a $\sup + \inf$ inequality.

In dimension $n \geq 3$, the prescribed curvature equation on general manifold M is

$$(E_0') \qquad -\Delta u + R_g u = V u^{(n+2)/(n-2)}.$$

When $M = \mathbb{S}_n$, Li [1993; 1995; 1996] proved a priori estimates for the solutions of the previous equation. He used the notion of simple isolated points and some flatness conditions on V.

If we suppose $n = 3, 4$, we can find in [Li and Zhang 2004; Li and Zhu 1999] a uniform bound for the energy and a $\sup \times \inf$ inequality. Note that Li and Zhu [1999] proved the compactness of the solutions to the Yamabe problem using the positive mass theorem.

In [Bahoura 2004], we can see (on a bounded domain of $\mathbb{R}^4$) that we have a uniform estimate for the solutions of (E_0') ($n = 4$ and Euclidean case) by assuming that those solutions are bounded below by a positive constant; in this case we have assumed that the prescribed scalar curvature V is only Lipschitz.

Here we extend some result of [Bahoura 2004] to equations with nonlinear terms or with minimal condition on the prescribed scalar curvature.

For the Euclidean case, Chen and Lin [1997] got some a priori estimates for general equations

$$(E_0'') \qquad -\Delta u = V u^{(n+2)/(n-2)} + g(u)$$

with some assumption on g and the Li-flatness conditions on V.

Here, we give some a priori estimates with some minimal conditions on the prescribed curvature, for perturbed scalar curvature equation, in all dimensions $n \geq 4$.

In our work, we use the *blow-up* analysis, the *moving-plane* method and a flatness condition (of order 1) for the prescribed scalar curvature. Note that the flatness condition which we use is also obtained by a *moving-plane* argument of Chen and Lin [1997]. The method of moving plane was developed in particular by Gidas, Ni and Nirenberg [Gidas et al. 1979] and Serrin [1971].

First, consider the equation

$$(E_1) \qquad -\Delta u = V u^{(n+2)/(n-2)} + u^{n/(n-2)}$$

with $0 < a \leq V(x) \leq b$ and $\|V\|_{C^{1,\alpha}} \leq A$, $0 < \alpha \leq 1$.

We have:

Theorem 1. *For all $a, b, A, \alpha > 0$ $(0 < \alpha \leq 1)$, and all compact sets K of Ω of dimension $n \geq 4$, there is a positive constant $c = c(a, b, A, \alpha, K, \Omega, n)$ such that*

$$\sup_K u \times \inf_\Omega u \leq c$$

for all solutions u of (E_1) relative to V.

Now, if we suppose $V \in C^1(\Omega)$ and $V \geq a > 0$, we have:

Theorem 2. *For all $a > 0$, V and all compact K of Ω of dimension $n \geq 4$, there is a positive constant $c = c(a, V, K, \Omega, n)$ such that*

$$\sup_K u \times \inf_\Omega u \leq c$$

for all solutions u of (E_1) relative to V.

Now, we suppose $n = 4$, and we consider the equation (prescribed scalar curvature equation)

$$(E_2) \qquad -\Delta u = V u^3 \quad \text{on } \Omega \subset \mathbb{R}^4$$

with $0 < a \leq V(x) \leq b$ and $\|V\|_{C^{1,\alpha}} \leq A$, $0 < \alpha \leq 1$. We have:

Theorem 3. *For all $a, b, m, A, \alpha > 0$ $(0 < \alpha \leq 1)$ and all compact K of Ω, there is a positive constant $c = c(a, b, m, A, \alpha, K, \Omega)$ such that*

$$\sup_K u \leq c \quad \textit{if} \quad \min_\Omega u \geq m$$

for all solutions u of (E_2) relative to V.

If we suppose $n = 4$ and $V \in C^1(\Omega)$ and $V \geq a > 0$ on Ω, we have:

Theorem 4. *For all $a, m > 0$, $V \in C^1(\Omega)$ and all compact $K \in \Omega$, there is a positive constant $c = c(a, m, V, K, \Omega)$ such that*

$$\sup_K u \leq c \quad \textit{if} \quad \min_\Omega u \geq m$$

for all solutions u of (E_2) relative to V.

2. Proofs of the theorems

Proof of Theorems 1 and 2.

Proof of Theorem 1. Without loss of generality, we suppose $\Omega = B_1$ the unit ball of $\mathbb{R}^n$. We want to prove an a priori estimate around 0.

Let (u_i) and (V_i) be sequences of functions on Ω such that

$$-\Delta u_i = V_i {u_i}^{(n+2)/(n-2)} + u_i^{n/(n-2)}, \quad u_i > 0,$$

with $0 < a \leq V_i(x) \leq b$ and $\|V_i\|_{C^{1,\alpha}} \leq A$.

We argue by contradiction, and we suppose that the $\sup \times \inf$ is not bounded.

We have that for all c, $R > 0$ there exists $u_{c,R}$ a solution of (E_1) such that

$$(H) \qquad R^{n-2} \sup_{B(0,R)} u_{c,R} \times \inf_M u_{c,R} \geq c.$$

Proposition (blow-up analysis). *There is a sequence of points $(y_i)_i$, $y_i \to 0$, and two sequences of positive real numbers $(l_i)_i$ and $(L_i)_i$ (see below), $l_i \to 0$ and $L_i \to +\infty$, such that, if we set $v_i(y) = u_i(y + y_i)/u_i(y_i)$, we have*

$$0 < v_i(y) \leq \beta_i \leq 2^{(n-2)/2},$$

$$\beta_i \to 1,$$

$$v_i(y) \to \left(\frac{1}{1+|y|^2}\right)^{(n-2)/2} \text{ uniformly on all compact sets of } \mathbb{R}^n,$$

$$l_i^{(n-2)/2} u_i(y_i) \times \inf_{B_1} u_i \to +\infty,$$

Proof. We use the hypothesis (H); we take two sequences $R_i > 0$, $R_i \to 0$, and $c_i \to +\infty$ such that

$$R_i{}^{(n-2)} \sup_{B(0,R_i)} u_i \times \inf_{B_1} u_i \geq c_i \to +\infty.$$

Let $x_i \in B(x_0, R_i)$ be a point such that $\sup_{B(0,R_i)} u_i = u_i(x_i)$ and $s_i(x) = (R_i - |x - x_i|)^{(n-2)/2} u_i(x)$, $x \in B(x_i, R_i)$. Then $x_i \to 0$.

We have

$$\max_{B(x_i,R_i)} s_i(x) = s_i(y_i) \geq s_i(x_i) = R_i{}^{(n-2)/2} u_i(x_i) \geq \sqrt{c_i} \to +\infty.$$

We set

$$l_i = R_i - |y_i - x_i|, \qquad \bar{u}_i(y) = u_i(y_i + y), \qquad v_i(z) = \frac{u_i[y_i + (z/[u_i(y_i)]^{2/(n-2)})]}{u_i(y_i)}.$$

Clearly, we have $y_i \to x_0$.

We take

$$L_i = \frac{l_i}{(c_i)^{1/2(n-2)}} [u_i(y_i)]^{2/(n-2)} = \frac{[s_i(y_i)]^{2/(n-2)}}{c_i^{1/2(n-2)}} \geq \frac{c_i^{1/(n-2)}}{c_i^{1/2(n-2)}} = c_i^{1/2(n-2)} \to +\infty.$$

If $|z| \leq L_i$, then $y = [y_i + z/[u_i(y_i)]^{2/(n-2)}] \in B(y_i, \delta_i l_i)$ with $\delta_i = 1/(c_i)^{1/2(n-2)}$ and $|y - y_i| < R_i - |y_i - x_i|$; thus, $|y - x_i| < R_i$ and $s_i(y) \leq s_i(y_i)$. We can write

$$u_i(y)(R_i - |y - y_i|)^{(n-2)/2} \leq u_i(y_i)(l_i)^{(n-2)/2}.$$

But $|y-y_i| \leq \delta_i l_i$, $R_i > l_i$ and $R_i - |y-x_i| \geq R_i - |x_i - y_i| - \delta_i l_i > l_i - \delta_i l_i = l_i(1-\delta_i)$. We obtain

$$0 < v_i(z) = \frac{u_i(y)}{u_i(y_i)} \leq \left[\frac{l_i}{l_i(1-\delta_i)}\right]^{(n-2)/2} \leq 2^{(n-2)/2}.$$

We set $\beta_i = (1/(1-\delta_i))^{(n-2)/2}$; clearly, we have $\beta_i \to 1$.

The function v_i satisfies

$$-\Delta v_i = \widetilde{V}_i {v_i}^{(n+2)/(n-2)} + \frac{v_i^{n/(n-2)}}{[u_i(y_i)]^{2/(n-2)}},$$

where $\widetilde{V}_i(y) = V_i[y + y/[u_i(y_i)]^{2/(n-2)}]$. Without loss of generality, we can suppose that $\widetilde{V}_i \to V(0) = n(n-2)$.

We use the elliptic estimates of the Ascoli and Ladyzhenskaya theorems to have the uniform convergence of (v_i) to v on a compact set of $\mathbb{R}^n$. The function v satisfies

$$-\Delta v = n(n-2)v^{N-1}, \qquad v(0) = 1, \qquad 0 \leq v \leq 1 \leq 2^{(n-2)/2}, \qquad N = \frac{2n}{n-2}.$$

By the maximum principle, we have $v > 0$ on $\mathbb{R}^n$. If we use the result of Caffarelli, Gidas and Spruck [Caffarelli et al. 1989], we obtain $v(y) = (1/(1+|y|^2))^{(n-2)/2}$. We have the same properties as in [Bahoura 2004]. □

Remark. When we use the convergence on compact sets of the sequence (v_i), we can take an increasing sequence of compact sets and we see that we can obtain a sequence (ϵ_i) such that $\epsilon_i \to 0$ and after we choose $(\widetilde{R}_i)$ such that $\widetilde{R}_i \to +\infty$ and finally

$$\widetilde{R}_i^{n-2} \|v_i - v\|_{B(0,\widetilde{R}_i)} \leq \epsilon_i.$$

We can say that we are in the case of [Chen and Lin 1997, step 1 of the proof of Theorem 1.2].

Fundamental point (a consequence of the blow-up). According to the work of Chen and Lin [1997, step 2 of the proof of Theorem 1.3], in the blow-up point, the prescribed scalar curvature V is such that

$$(P_0) \qquad \lim_{i\to+\infty} |\nabla V_i(y_i)| = 0.$$

Polar coordinates (moving-plane method). Now we must use the same method as in [Bahoura 2004, Theorem 1]. We will use the moving-plane method.

We must prove [Bahoura 2004, Lemma 2].

We set $t \in]-\infty, -\log 2]$ and $\theta \in \mathbb{S}_{n-1}$:

$$w_i(t,\theta) = e^{(n-2)t/2} u_i(y_i + e^t\theta) \quad \text{and} \quad \overline{V}_i(t,\theta) = V_i(y_i + e^t\theta).$$

We consider the operator $L=\partial_{tt}+\Delta_\sigma-(n-2)^2/4$, with Δ_σ the Laplace–Beltrami operator on $\mathbb{S}_{n-1}$.

The function w_i satisfies

$$-Lw_i=\overline{V}_i {w_i}^{N-1}+e^t\times {w_i}^{n/(n-2)},\qquad N=\frac{2n}{n-2}.$$

Remark. Here w_i is a solution to the previous equation with a perturbed term which contains e^t. The term e^t is fundamental in the computations; it corrects the variation of V_i.

For $\lambda\le 0$, we set

$$t^\lambda=2\lambda-t,\qquad w_i^\lambda(t,\theta)=w_i(t^\lambda,\theta),\qquad \overline{V}_i^\lambda(t,\theta)=\overline{V}_i(t^\lambda,\theta).$$

First, like in [Bahoura 2004], we have the following lemma.

Lemma 5. *Let A_λ be the property*

$$A_\lambda=\{\lambda\le 0 \mid \textit{there exists } (t_\lambda,\theta_\lambda)\in\,]\lambda,t_i]\times\mathbb{S}_{n-1},\ {w_i}^\lambda(t_\lambda,\theta_\lambda)-w_i(t_\lambda,\theta_\lambda)\ge 0\}.$$

Then there is $\nu\le 0$ such that, for $\lambda\le\nu$, A_λ is not true.

Remark. Here we choose $t_i=\log\sqrt{l_i}$, where l_i is chosen as in the proposition.

Like in proof of the Theorem 1 of [Bahoura 2004], we want to prove the following lemma.

Lemma 6. *For $\lambda\le 0$ we have*

$${w_i}^\lambda-w_i\le 0\quad\Longrightarrow\quad -L({w_i}^\lambda-w_i)\le 0$$

on $]\lambda,t_i]\times\mathbb{S}_{n-1}$.

Like in [Bahoura 2004], we have:

A useful point. Let $\xi_i=\sup\{\lambda\le\bar\lambda_i=2+\log\eta_i \mid {w_i}^\lambda-w_i<0 \text{ on }]\lambda,t_i]\times\mathbb{S}_{n-1}\}$. The real ξ_i exists.

First,

$$w_i(2\xi_i-t,\theta)=w_i[(\xi_i-t+\xi_i-\log\eta_i-2)+(\log\eta_i+2)].$$

Proof of Lemma 6. In fact, for each i we have $\lambda=\xi_i\le\log\eta_i+2$, where $\eta_i=[u_i(y_i)]^{(-2)/(n-2)}$.

Note that

$$w_i(2\xi_i-t,\theta)=w_i[(\xi_i-t+\xi_i-\log\eta_i-2)+(\log\eta_i+2)];$$

if we use the definition of w_i, then for $\xi_i \le t$,

$$w_i(2\xi_i - t,\theta) = e^{[(n-2)(\xi_i - t+\xi_i - \log\eta_i - 2)]/2} e^{n-2} v_i[\theta e^2 e^{(\xi_i - t)+(\xi_i-\log\eta_i-2)}] \le 2^{(n-2)/2} e^{n-2} = \bar{c}.$$

We know that

$$-L(w_i^{\xi_i} - w_i) = [\overline{V}_i^{\xi_i}(w_i^{\xi_i})^{(n+2)/(n-2)} - \overline{V}_i {w_i}^{(n+2)/(n-2)}] + [e^{t^{\xi_i}}(w_i^{\xi_i})^{n/(n-2)} - e^t {w_i}^{n/(n-2)}].$$

We denote by Z_1 and Z_2 the terms

$$Z_1 = (\overline{V}_i^{\xi_i} - \overline{V}_i)(w_i^{\xi_i})^{(n+2)/(n-2)} + \overline{V}_i[(w_i^{\xi_i})^{(n+2)/(n-2)} - {w_i}^{(n+2)/(n-2)}]$$

and

$$Z_2 = e^{t^{\xi_i}}[(w_i^{\xi_i})^{n/(n-2)} - {w_i}^{n/(n-2)}] + {w_i}^{n/(n-2)}(e^{t^{\xi_i}} - e^t).$$

Like in the proof of Theorem 1 of [Bahoura 2004], we have

$$ {w_i}^{\xi_i} \le w_i \quad \text{and} \quad w_i^{\xi_i}(t,\theta) \le \bar{c} \quad \text{for all } (t,\theta) \in [\xi_i, -\log 2] \times \mathbb{S}_{n-1},$$

where $\bar{c}$ is a positive constant independent of i and $w_i^{\xi_i}$ for $\xi_i \le \log\eta_i + 2$.

The (P_0) *hypothesis*. Now we use (P_0) (this hypothesis is the same hypothesis as in the first part of the paper: $|\nabla V_i(y_i)| \to 0$). We write

$$|\nabla V_i(y_i + e^t\theta) - \nabla V_i(y_i)| \le A e^{\alpha t},$$

Thus,

$$\left| V_i(y_i + e^{t^{\xi_i}}\theta) - V_i(y_i + e^t\theta) - \langle \nabla V_i(y_i) \mid \theta\rangle (e^{t^{\xi_i}} - e^t)\right| \le \frac{A}{1+\alpha}[e^{(1+\alpha)t^{\xi_i}} - e^{(1+\alpha)t}].$$

Then

$$|V_i^{\xi_i} - V_i| \le |o(1)|(e^t - e^{t^{\xi_i}}).$$

Thus, $Z_1 \le |o(1)|(w_i^{\xi_i})^{(n+2)/(n-2)}(e^t - e^{t^{\xi_i}})$ and $Z_2 \le (w_i^{\xi_i})^{n/(n-2)} \times (e^{t^{\xi_i}} - e^t)$. Then

$$-L(w_i^{\xi_i} - w_i) \le (w_i^{\xi_i})^{n/(n-2)}[(|o(1)|{w_i^{\xi_i}}^{2/(n-2)} - 1)(e^t - e^{t^{\xi_i}})] \le 0.$$

The lemma is proved. □

We set

$$\xi_i = \sup\{\mu \le \log\eta_i + 2 \mid w_i^{\mu}(t,\theta) - w_i(t,\theta) \le 0 \text{ for all } (t,\theta) \in [\mu, t_i] \times \mathbb{S}_{n-1}\},$$

with t_0 small enough.

Like in the proof of Theorem 1 of [Bahoura 2004], the maximum principle implies

$$\min_{\theta\in\mathbb{S}_{n-1}} w_i(t_i,\theta) \le \max_{\theta\in\mathbb{S}_{n-1}} w_i(2\xi_i - t_i).$$

But

$$w_i(t_i,\theta) = e^{t_i} u_i(y_i + e^{t_i}\theta) \ge e^{t_i} \min u_i \quad \text{and} \quad w_i(2\xi_i - t_i) \le \frac{c_0}{u_i(y_i)};$$

thus,

$$l_i^{(n-2)/2} u_i(y_i) \times \min u_i \le c.$$

The proposition is contradicted. □

Proof of Theorem 2. The proof of Theorem 2 is similar to the proof of Theorem 1. Only the "fundamental point" changes.

According to the work of Chen and Lin [1997, step 2 of the proof of Theorem 1.1], in the blow-up point, the prescribed scalar curvature V is such that

$$\nabla V(0) = 0.$$

The function ∇V is continuous on $B_r(0)$ (with r small enough), so it is uniformly continuous and we write (because $y_i \to 0$)

$$|\nabla V(y_i + y) - \nabla V(y_i)| \le \epsilon \quad \text{for } |y| \le \delta \ll r \text{ for all } i.$$

Thus,

$$|V^{\xi_i} - V| \le o(1)(e^t - e^{t^{\xi_i}}).$$

We see that we have the same computations as in the "polar coordinates" in the proof of Theorem 1. □

Proof of Theorems 3 and 4. Here, only the "polar coordinates" change; the proposition of the first theorem stays true. First, we have:

Fundamental point (a consequence of the blow-up). According to the work of Chen and Lin [1997, step 2 of the proof of Theorem 1.3], in the blow-up point, the prescribed scalar curvature V is such that:

Case 1 (Theorem 3). $\lim_{i\to+\infty} |\nabla V_i(y_i)| = 0.$

We write

$$|\nabla V_i(y_i + e^t\theta) - \nabla V_i(y_i)| \le A e^{\alpha t}.$$

Thus,

$$|V_i^{\xi_i} - V_i| \le |o(1)|(e^t - e^{t^{\xi_i}}).$$

Case 2 (Theorem 4). $\nabla V(0) = 0.$

The function ∇V is continuous on $B_r(0)$ (for r small enough), so it is uniformly continuous and we write (because $y_i \to 0$)

$$|\nabla V(y_i + y) - \nabla V(y_i)| \le \epsilon \quad \text{for } |y| \le \delta \ll r \text{ for all } i.$$

Thus,

$$|V^{\xi_i} - V| \le o(1)(e^t - e^{t^{\xi_i}}),$$

Conclusion of the proofs of Theorems 3 and 4. Finally, we can note that we are in the case of Theorem 2 of [Bahoura 2004]. We have the same computations if we consider the function

$$\overline{w}_i(t,\theta) = w_i(t,\theta) - \frac{m}{2} e^t.$$

We set $L = \partial_{tt} + \Delta_\sigma - 1$, where Δ_σ is the Laplace–Beltrami operator on $\mathbb{S}_3$, and $\overline{V}_i(t,\theta) = V_i(y_i + e^t\theta)$.

Like in [Bahoura 2004], we want to prove the following lemma.

Lemma 7. $$\overline{w}_i^{\xi_i} - \overline{w}_i \le 0 \quad \Longrightarrow \quad -L(\overline{w}_i^{\xi_i} - \overline{w}_i) \le 0.$$

Proof of Lemma 7. We have

$$-L(\overline{w}_i^{\xi_i} - \overline{w}_i) = \overline{V}_i^{\xi_i} (w_i^{\xi_i})^3 - \overline{V}_i w_i^3.$$

Then

$$-L(\overline{w}_i^{\xi_i} - \overline{w}_i) = (\overline{V}_i^{\xi_i} - \overline{V}_i)(w_i^{\xi_i})^3 + [(w_i^{\xi_i})^3 - w_i^3]\overline{V}_i.$$

For $t \in [\xi_i, t_i]$ and $\theta \in \mathbb{S}_3$,

$$|\overline{V}_i^{\xi_i}(t,\theta) - \overline{V}_i(t,\theta)| = |V_i(y_i + e^{2\xi_i - t}\theta) - V_i(y_i + e^t\theta)| \le |o(1)|(e^t - e^{2\xi_i - t}).$$

The real $t_i = \log\sqrt{l_i} \to -\infty$, where l_i is chosen as in the proposition of Theorem 1.

But if $\overline{w}_i^{\xi_i} - \overline{w}_i \le 0$, we obtain

$$w_i^{\xi_i} - w_i \le \frac{m}{2}(e^{2\xi_i - t} - e^t) < 0.$$

Using the fact that $0 < w_i^{\xi_i} < w_i$, we have

$$(w_i^{\xi_i})^3 - w_i^3 = (w_i^{\xi_i} - w_i)[(w_i^{\xi_i})^2 + w_i^{\xi_i} w_i + (w_i)^2] \le 3(w_i^{\xi_i} - w_i) \times (w_i^{\xi_i})^2.$$

Thus, we have for $t \in [\xi_i, t_i]$ and $\theta \in \mathbb{S}_3$

$$(w_i^{\xi_i})^3 - {w_i}^3 \le 3\frac{m}{2}(w_i^{\xi_i})^2(e^{2\xi_i - t} - e^t).$$

We can write

$$(**) \qquad -L(\overline{w}_i^{\xi_i} - \overline{w}_i) \le (w_i^{\xi_i})^2\big(\tfrac{3}{2}m\overline{V}_i - |o(1)|{w_i}^{\xi_i}\big)(e^{2\xi_i - t} - e^t).$$

We know that, for $t \leq \log(l_i) - \log 2 + \log \eta_i$, we have

$$w_i(t,\theta) = e^t \times \frac{u_i(y_i + e^t\theta/u_i(y_i))}{u_i(y_i)} \leq 2e^t.$$

We find

$${w_i}^{\xi_i}(t,\theta) \leq 2e^2\sqrt{\frac{8}{a}},$$

because $\xi_i - \log \eta_i \leq 2 + \frac{1}{2}\log(8/V(0))$ and $\xi_i \leq t \leq t_i$.

Finally, $(**)$ is negative and the lemma is proved. □

Now, if we use the Hopf maximum principle, we obtain

$$\min_{\theta\in\mathbb{S}^3} \overline{w}_i(t_i,\theta) \leq \max_{\theta\in\mathbb{S}^3} \overline{w}_i(2\xi_i - t_i,\theta),$$

which implies that

$$l_i u_i(y_i) \leq c.$$

It is a contradiction. □

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COMPLEMENTED COPIES OF $c_0(\tau)$ IN TENSOR PRODUCTS OF $L_p[0,1]$

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Let X be a Banach space and τ an infinite cardinal. We show that if τ has uncountable cofinality, $p \in [1,\infty)$, and either the Lebesgue–Bochner space $L_p([0,1],X)$ or the injective tensor product $L_p[0,1]\widehat{\otimes}_\varepsilon X$ contains a complemented copy of $c_0(\tau)$, then so does X. We show also that if $p \in (1,\infty)$ and the projective tensor product $L_p[0,1]\widehat{\otimes}_\pi X$ contains a complemented copy of $c_0(\tau)$, then so does X.

1. Introduction and preliminaries

We use standard set-theoretical and Banach space theory terminology as may be found, e.g., in [Jech 2003] and [Johnson and Lindenstrauss 2001]. We denote by B_X the closed unit ball of the Banach space X. If X and Y are Banach spaces, we denote by $\mathcal{L}(X,Y)$ the space of all bounded linear operators from X to Y and by $\mathcal{K}(X,Y)$ the subspace of all compact linear operators. We say that Y contains a copy (resp. a complemented copy) of X, and write $X \hookrightarrow Y$ (resp. $X \overset{c}{\hookrightarrow} Y$), if X is isomorphic to a subspace (resp. complemented subspace) of Y. The *density character* of X, denoted by $\mathrm{dens}(X)$, is the smallest cardinality of a dense subset of X.

A Banach space X has the *bounded approximation property* if there exists $\lambda > 0$ such that, for every compact subset K of X and every $\varepsilon > 0$, there exists a finite rank operator $T : X \to X$ such that $\|T\| \le \lambda$ and $\|x - T(x)\| < \varepsilon$ for every $x \in K$.

We shall denote the projective and injective tensor norms by $\|\cdot\|_\pi$ and $\|\cdot\|_\varepsilon$, respectively. The projective (resp. injective) tensor product of X and Y is the completion of $X \otimes Y$ with respect to $\|\cdot\|_\pi$ (resp. $\|\cdot\|_\varepsilon$) and will be denoted by $X\widehat{\otimes}_\pi Y$ (resp. $X\widehat{\otimes}_\varepsilon Y$).

For a nonempty set Γ, $c_0(\Gamma)$ denotes the Banach space of all real-valued maps f on Γ with the property that for each $\varepsilon > 0$, the set $\{\gamma \in \Gamma : |f(\gamma)| \ge \varepsilon\}$ is finite, equipped with the supremum norm. We will refer to $c_0(\Gamma)$ as $c_0(\tau)$ when

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the cardinality of Γ (denoted by $|\Gamma|$) is equal to τ. This space will be denoted by c_0 when $\tau = \aleph_0$. By $\ell_\infty(\Gamma)$ we will denote the Banach space of all bounded real-valued maps on Γ, with the supremum norm. This space will be denoted by ℓ_∞ when $\Gamma = \mathbb{N}$.

Given X a Banach space and $p \in [1, \infty)$, we denote by $L_p([0, 1], X)$ the Lebesgue–Bochner space of all (classes of equivalence of) measurable functions $f : [0, 1] \to X$ such that the scalar function $\|f\|^p$ is integrable, equipped with the complete norm

$$\|f\|_p = \left[\int_0^1 \|f(t)\|^p \, dt\right]^{\frac{1}{p}}.$$

These spaces will be denoted by $L_p[0, 1]$ when $X = \mathbb{R}$.

A measurable function $f : [0, 1] \to X$ is *essentially bounded* if there exists $\varepsilon > 0$ such that the set $\{t \in [0, 1] : \|f(t)\| \geq \varepsilon\}$ has Lebesgue measure zero, and we denote by $\|f\|_\infty$ the infimum of all such numbers $\varepsilon > 0$. By $L_\infty([0, 1], X)$ we will denote the space of all (classes of equivalence of) essentially bounded functions $f : [0, 1] \to X$, equipped with the complete norm $\|\cdot\|_\infty$.

Recall that if τ is an infinite cardinal then the *cofinality* of τ, denoted by $\mathrm{cf}(\tau)$, is the least cardinal α such that there exists a family of ordinals $\{\beta_j : j \in \alpha\}$ satisfying $\beta_j < \tau$ for all $j \in \alpha$, and $\sup\{\beta_j : j \in \alpha\} = \tau$. A cardinal τ is said to be *regular* when $\mathrm{cf}(\tau) = \tau$; otherwise, it is said to be *singular*.

Many papers in the history of the geometry of Banach spaces have been devoted to establishing results about when certain Banach spaces contain complemented copies of c_0 or $c_0(\tau)$ for uncountable cardinals τ; see, for example, [Amir and Lindenstrauss 1968; Argyros et al. 2002; Cembranos 1984; Cembranos and Mendoza 1997; Emmanuele 1988; Sobczyk 1941; Zippin 1977]. The starting points of our research are three of these results related to the space c_0, i.e., Theorems 1, 2 and 3 below.

We begin by recalling the following immediate consequence of the classical Cembranos–Freniche theorem [Cembranos 1984, Main theorem; Freniche 1984, Corollary 2.5].

Theorem 1. *For each $p \in [1, \infty)$,*

$$c_0 \overset{c}{\hookrightarrow} L_p[0, 1]\widehat{\otimes}_\varepsilon \ell_\infty.$$

However, $c_0 \overset{c}{\not\hookrightarrow} \ell_\infty$ (see, e.g., [Diestel and Uhl 1977, Corollary 11, p. 156]).

On the other hand, Oja proved the following stability property.

Theorem 2 [Oja 1991, Theorem 3b]. *If X is a Banach space and $p \in (1, \infty)$, then*

$$c_0 \overset{c}{\hookrightarrow} L_p[0, 1]\widehat{\otimes}_\pi X \Longrightarrow c_0 \overset{c}{\hookrightarrow} X.$$

Observe that Theorem 2 does not hold for $p = 1$. Indeed, $L_1([0, 1], X)$ is linearly isometric to $L_1[0,1]\hat{\otimes}_\pi X$ [Ryan 2002, Example 2.19, p. 29] and Emmanuele obtained the following result.

Theorem 3 [Emmanuele 1988, Main theorem]. *If X is a Banach space and $p\in[1,\infty)$, then*

$$c_0 \hookrightarrow X \Longrightarrow c_0 \overset{c}{\hookrightarrow} L_p([0,1],X).$$

So, in particular, $L_p([0,1],\ell_\infty)$ contains a complemented copy of c_0, but once again $c_0 \overset{c}{\not\hookrightarrow} \ell_\infty$.

We recall also that, denoting by $\|\cdot\|_{\Delta_p}$ the natural tensor norm induced on $L_p[0,1]\otimes X$ by $L_p([0,1],X)$ and by $L_p[0,1]\widehat{\otimes}_{\Delta_p}X$ the completion of $L_p[0,1]\otimes X$ with this norm, the space $L_p([0,1],X)$ is linearly isometric to $L_p[0,1]\widehat{\otimes}_{\Delta_p}X$ [Defant and Floret 1993, Chapters 7.1 and 7.2].

Thus, we are naturally led to the following problem.

Problem 4. For X a Banach space, $p\in[1,\infty)$, and τ an infinite cardinal, we want to know under which conditions

$$c_0(\tau) \overset{c}{\hookrightarrow} L_p[0,1]\widehat{\otimes}_\alpha X \Longrightarrow c_0(\tau) \overset{c}{\hookrightarrow} X$$

holds, where α denotes either the projective, injective or natural norm.

This problem becomes more interesting if we keep in mind that, in general, it is not so simple to determine whether the tensor products of E and X contain complemented copies of a certain space F, even when E contains no complemented copies of F. Indeed, there are a number of elementary questions about this topic that remain unanswered. For instance, it is not known whether $l_\infty\widehat{\otimes}_\pi l_\infty$ contains a complemented copy of c_0 or not [Cabello Sánchez et al. 2006, Remark 3].

In the present paper, we will prove that for every Banach space X, $p\in(1,\infty)$ and an infinite cardinal τ,

$$c_0(\tau) \overset{c}{\hookrightarrow} L_p[0,1]\widehat{\otimes}_\pi X \Longrightarrow c_0(\tau) \overset{c}{\hookrightarrow} X.$$

Additionally, if τ has uncountable cofinality, then for every $p\in[1,\infty)$,

$$c_0(\tau) \overset{c}{\hookrightarrow} L_p[0,1]\widehat{\otimes}_\varepsilon X \Longrightarrow c_0(\tau) \overset{c}{\hookrightarrow} X$$

and

$$c_0(\tau) \overset{c}{\hookrightarrow} L_p([0,1],X) \Longrightarrow c_0(\tau) \overset{c}{\hookrightarrow} X.$$

This paper is organized as follows. We study complemented copies of $c_0(\tau)$ in the injective (Section 2), projective (Section 3) and natural (Section 4) tensor products with $L_p[0,1]$.

2. Complemented copies of $c_0(\tau)$ in $X\hat{\otimes}_\varepsilon Y$ spaces

The goal of this section is to prove Theorem 7. We recall that given Banach spaces X and Y, the operator $S : X\hat{\otimes}_\varepsilon Y \to \mathcal{K}(X^*, Y)$ satisfying

$$S(v)(x^*) = \sum_{i=1}^{j} x^*(a_i)b_i$$

for every $x^* \in X^*$ and $v = \sum_{i=1}^{j} a_i \otimes b_i \in X \otimes Y$, is a linear isometry onto its image.

We will need the following key lemma.

Lemma 5. *Let X and Y be Banach spaces. Suppose that X has the bounded approximation property. Then there exist sets $A \subset X$ and $B \subset X^*$ such that* $\max(|A|, |B|) \leq \operatorname{dens}(X)$ *and for every $u \in X\hat{\otimes}_\varepsilon Y$ and $\delta > 0$ there exist $x_1, \ldots, x_m \in A$ and $\varphi_1, \ldots, \varphi_m \in B$ satisfying*

$$\left\| u - \sum_{n=1}^{m} x_n \otimes S(u)(\varphi_n) \right\|_\varepsilon < \delta.$$

Proof. By hypothesis, there exists $\lambda \geq 1$ such that for every finite-dimensional subspace Z of X there exists a finite rank operator T on X such that $\|T\| \leq \lambda$ and $T(x) = x$ for all $x \in Z$ [Casazza 2001, Theorem 3.3.(3), p. 288].

Let D be a dense subset of X with $|D| = \operatorname{dens}(X)$ and let $\mathcal{F}$ be the family of all finite, nonempty subsets of D. For each $F \in \mathcal{F}$, fix a finite rank operator T_F on X such that $\|T_F\| \leq \lambda$ and $T_F(d) = d$ for all $d \in F$. Let m_F be the dimension of $T_F(X)$, $\{x_1^F, \ldots, x_{m_F}^F\}$ be a basis of $T_F(X)$ and $\varphi_1^F, \ldots, \varphi_{m_F}^F \in X^*$ such that

$$T_F(x) = \sum_{n=1}^{m_F} \varphi_n^F(x) x_n^F,$$

for every $x \in X$. Define

$$A = \bigcup_{F \in \mathcal{F}} \{x_1^F, \ldots, x_{m_F}^F\} \quad \text{and} \quad B = \bigcup_{F \in \mathcal{F}} \{\varphi_1^F, \ldots, \varphi_{m_F}^F\}.$$

We claim that A and B have the desired properties. Indeed, notice that

$$|A| \leq |\mathcal{F}| \sup_{F \in \mathcal{F}} |\{x_1^F, \ldots, x_{m_F}^F\}| \leq \max(|D|, \aleph_0) = |D|$$

and similarly $|B| \leq |D|$.

Next, let $u \in X\hat{\otimes}_\varepsilon Y$ and $\delta > 0$ be given. There exists $v = \sum_{j=1}^{k} d_j \otimes y_j \in X \otimes Y$ such that $d_1, \ldots, d_k \in D$, $d_i \neq d_j$ if $i \neq j$, and

$$\|u - v\|_\varepsilon < \frac{\delta}{\lambda + 1}.$$

Writing $G = \{d_1, \dots, d_k\}$, we see that

$$\sum_{n=1}^{m_G} x_n^G \otimes S(v)(\varphi_n^G) = \sum_{j=1}^{k}\biggl(\sum_{n=1}^{m_G} \varphi_n^G(d_j) x_n^G\biggr) \otimes y_j = \sum_{j=1}^{k} T_G(d_j) \otimes y_j = v.$$

Furthermore, since

$$\biggl\| \sum_{n=1}^{m_G} x_n^G \otimes \varphi_n^G \biggr\|_\varepsilon = \sup_{x \in B_X} \biggl\| \sum_{n=1}^{m_G} \varphi_n^G(x) x_n^G \biggr\| = \|T_G\| \le \lambda,$$

we obtain

$$\begin{aligned}\biggl\| \sum_{n=1}^{m_G} x_n^G \otimes S(u-v)(\varphi_n^G) \biggr\|_\varepsilon &= \sup_{x^* \in B_{X^*}} \biggl\| \sum_{n=1}^{m_G} x^*(x_n^G) S(u-v)(\varphi_n^G) \biggr\| \\ &\le \|u-v\|_\varepsilon \sup_{x^* \in B_{X^*}} \biggl\| \sum_{n=1}^{m_G} x^*(x_n^G)(\varphi_n^G) \biggr\| \\ &< \frac{\delta}{\lambda+1} \biggl\| \sum_{n=1}^{m_G} x_n^G \otimes \varphi_n^G \biggr\|_\varepsilon \le \frac{\lambda\delta}{\lambda+1}.\end{aligned}$$

Thus,

$$\biggl\| u - \sum_{n=1}^{m_G} x_n^G \otimes S(u)(\varphi_n^G) \biggr\|_\varepsilon < \delta$$

and we are done. □

The following result [Galego and Cortes 2017] will also be used frequently throughout this work.

Theorem 6 [Galego and Cortes 2017, Theorem 2.4]. *Let X be a Banach space and τ be an infinite cardinal. The following are equivalent*:

(1) *X contains a complemented copy of $c_0(\tau)$.*

(2) *There exist a family $(x_j)_{j\in\tau}$ equivalent to the unit-vector basis of $c_0(\tau)$ in X and a weak*-null family $(x_j^*)_{j\in\tau}$ in X^* such that*, *for each $j, k \in \tau$*,

$$x_j^*(x_k) = \delta_{jk}.$$

(3) *There exist a family $(x_j)_{j\in\tau}$ equivalent to the unit-vector basis of $c_0(\tau)$ in X and a weak*-null family $(x_j^*)_{j\in\tau}$ in X^* such that*

$$\inf_{j\in\tau} |x_j^*(x_j)| > 0.$$

Theorem 7. *Let X and Y be Banach spaces and τ be an infinite cardinal. If X has the bounded approximation property and* $\mathrm{cf}(\tau) > \mathrm{dens}(X)$, *then*

$$c_0(\tau) \overset{c}{\hookrightarrow} X\hat{\otimes}_\varepsilon Y \Longrightarrow c_0(\tau) \overset{c}{\hookrightarrow} Y.$$

Proof. Let $A \subset X$ and $B \subset X^*$ be the sets provided by Lemma 5. By Theorem 6, there exist families $(u_i)_{i\in\tau}$ in $X\hat{\otimes}_\varepsilon Y$ and $(\psi_i)_{i\in\tau}$ in $(X\hat{\otimes}_\varepsilon Y)^*$ such that $(u_i)_{i\in\tau}$ is equivalent to the usual unit-vector basis of $c_0(\tau)$, $(\psi_i)_{i\in\tau}$ is weak*-null and $\psi_i(u_j) = \delta_{ij}$ for each $i, j \in \tau$. Let $s = \sup_{i\in\tau} \|\psi_i\| < \infty$.

For each $i \in \tau$ there exist $x_1^i, \dots, x_{m_i}^i \in A$ and $\varphi_1^i, \dots, \varphi_{m_i}^i \in B$ such that

$$\Bigl\| u_i - \sum_{n=1}^{m_i} x_n^i \otimes S(u_i)(\varphi_n^i)\Bigr\|_\varepsilon < \frac{1}{2s}$$

and hence

$$\frac{1}{2} < \sum_{n=1}^{m_i} |\psi_i(x_n^i \otimes S(u_i)(\varphi_n^i))|.$$

Put $\mathcal{M} = \{m_i : i \in \tau\}$ and for each $m \in \mathcal{M}$ define $\alpha_m = \{i \in \tau : m_i = m\}$. Since $\mathcal{M}$ is countable and τ has uncountable cofinality, there exists $M \in \mathcal{M}$ such that $|\alpha_M| = \tau$. Setting $\tau_1 = \alpha_M$, we have

$$\frac{1}{2} < \sum_{n=1}^{M} |\psi_i(x_n^i \otimes S(u_i)(\varphi_n^i))| \quad \text{for all } i \in \tau_1.$$

Next, for each $i \in \tau_1$ there exists $1 \le n_i \le M$ satisfying

$$\frac{1}{2M} < |\psi_i(x_{n_i}^i \otimes S(u_i)(\varphi_{n_i}^i))|.$$

Let $\mathcal{N} = \{n_i : i \in \tau_1\}$ and for each $n \in \mathcal{N}$ consider $\beta_n = \{i \in \tau_1 : n_i = n\}$. Since $\mathcal{N}$ is finite, there exists $N \in \mathcal{N}$ such that $|\beta_N| = \tau$. Setting $\tau_2 = \beta_N$, we obtain

$$\frac{1}{2M} < |\psi_i(x_N^i \otimes S(u_i)(\varphi_N^i))| \quad \text{for all } i \in \tau_2.$$

Now let $\mathcal{A} = \{x_N^i : i \in \tau_2\}$ and for each $a \in \mathcal{A}$ put $\gamma_a = \{i \in \tau_2 : x_N^i = a\}$. Since $\mathrm{cf}(\tau) > \mathrm{dens}(X) \ge |\mathcal{A}|$, there exists $x_0 \in \mathcal{A}$ such that $|\gamma_{x_0}| = \tau$. Setting $\tau_3 = \gamma_{x_0}$, we get

$$\frac{1}{2M} < |\psi_i(x_0 \otimes S(u_i)(\varphi_N^i))| \quad \text{for all } i \in \tau_3.$$

Finally, let $\mathcal{B} = \{\varphi_N^i : i \in \tau_3\}$, and for each $\varphi \in \mathcal{B}$ put $\lambda_\varphi = \{i \in \tau_3 : \varphi_N^i = \varphi\}$. Since $\mathrm{cf}(\tau) > \mathrm{dens}(X) \ge |\mathcal{B}|$, there exists $\varphi_0 \in \mathcal{B}$ such that $|\lambda_{\varphi_0}| = \tau$. Setting $\tau_4 = \lambda_{\varphi_0}$, we obtain

$$\frac{1}{2M} < |\psi_i(x_0 \otimes S(u_i)(\varphi_0))| \quad \text{for all } i \in \tau_4. \tag{2-1}$$

For each $i \in \tau_4$, write $y_i = S(u_i)(\varphi_0) \in Y$ and consider the linear functional $y_i^* \in Y^*$ defined by $y_i^*(y) = \psi_i(x_0 \otimes y)$, for every $y \in Y$. By (2-1), we have

$$\frac{1}{2M} < |y_i^*(y_i)| \leq \|\psi_i\|\|x_0\|\|y_i\| \leq s\|x_0\|\|y_i\| \quad \text{for all } i \in \tau_4,$$

and therefore

$$\frac{1}{2Ms\|x_0\|} < \|y_i\| \quad \text{for all } i \in \tau_4. \tag{2-2}$$

Denote by $(e_i)_{i\in\tau}$ the unit-vector basis of $c_0(\tau)$ and let $T : c_0(\tau) \to X\hat{\otimes}_\varepsilon Y$ be an isomorphism from $c_0(\tau)$ onto its image such that $T(e_i) = u_i$ for each $i \in \tau$. Consider $P : X\hat{\otimes}_\varepsilon Y \to Y$ the linear operator defined by $P(u) = S(u)(\varphi_0)$ for every $u \in X\hat{\otimes}_\varepsilon Y$. The inequality (2-2) then yields

$$\|(P \circ T)(e_i)\| = \|y_i\| \geq \frac{1}{2Ms\|x_0\|} > 0 \quad \text{for all } i \in \tau_4$$

and thus, by [Rosenthal 1970, remark following Theorem 3.4], there exists $\tau_5 \subset \tau_4$ such that $|\tau_5| = \tau$ and $P \circ T_{|c_0(\tau_5)}$ is an isomophism onto its image. This shows that $(y_i)_{i\in\tau_5} = (P(T(e_i))_{i\in\tau_5}$ is equivalent to the unit-vector basis of $c_0(\tau_5)$ in Y. Notice also that

$$(y_i^*(y))_{i\in\tau_5} = (\psi_i(x_0 \otimes y))_{i\in\tau_5} \in c_0(\tau_5) \quad \text{for all } y \in Y,$$

since $(\psi_i)_{i\in\tau}$ is weak*-null by hypothesis. Thus, $(y_i^*)_{i\in\tau_5}$ is weak*-null in Y^*. Combining these facts with (2-1), an appeal to Theorem 6 yields a complemented copy of $c_0(\tau)$ in Y. □

Note that according to Theorem 1, the above result is optimal. Moreover, Theorem 7 does not hold for cardinals with uncountable cofinality equal to the density of X. Indeed, by [Galego and Hagler 2012, Theorem 4.5] it follows that $c_0(\tau) \overset{c}{\hookrightarrow} \ell_1(\tau)\hat{\otimes}_\varepsilon\ell_\infty(\tau)$, however according to [Diestel and Uhl 1977, Corollary 11, p. 156], $c_0(\tau) \overset{c}{\not\hookrightarrow} \ell_\infty(\tau)$.

As a direct application of Theorem 7, we have:

Corollary 8. *Let X be a Banach space, $p \in [1,\infty)$ and τ an infinite cardinal with* $\operatorname{cf}(\tau) > \aleph_0$. *Then*

$$c_0(\tau) \overset{c}{\hookrightarrow} L_p[0,1]\hat{\otimes}_\varepsilon X \Longrightarrow c_0(\tau) \overset{c}{\hookrightarrow} X.$$

3. Complemented copies of $c_0(\tau)$ in $L_p[0,1]\hat{\otimes}_\pi X$ spaces

We will use a convenient characterization of $L_p[0,1]\hat{\otimes}_\pi X$ as a sequence space.

3.1. ***The spaces $L_p^{\mathrm{weak}}(X)$ and $L_p\langle X\rangle$.*** We will denote by $(\chi_n)_{n\geq 1}$ the Haar system, that is, the sequence of functions defined on [0, 1] by $\chi_1(t)=1$, for every $t\in[0,1]$, and

$$\chi_{2^k+j}(t)=\begin{cases} 1 & \text{if } t\in\left[\frac{2j-2}{2^{k+1}},\frac{2j-1}{2^{k+1}}\right),\\ -1 & \text{if } t\in\left[\frac{2j-1}{2^{k+1}},\frac{2j}{2^{k+1}}\right),\\ 0 & \text{otherwise,}\end{cases}$$

for each $k\geq 0$ and $1\leq j\leq 2^k$. It is well known (see [Lindenstrauss and Tzafriri 1977, p. 19; 1979, p. 155]) that the Haar system is an unconditional basis of $L_p[0,1]$, $p\in(1,\infty)$, and we will denote its unconditional basis constant by K_p. Following [Bu 2002; Dowling 2004], we renorm $L_p[0,1]$ by

$$\|f\|_p^{\mathrm{new}}=\sup\left\{\left\|\sum_{n=1}^{\infty}\theta_n\alpha_n\chi_n\right\|_p:\theta_n=\pm 1,n\geq 1\right\}$$

for each $f=\sum_{n=1}^{\infty}\alpha_n\chi_n\in L_p[0,1]$. Then

$$\|\cdot\|_p\leq\|\cdot\|_p^{\mathrm{new}}\leq K_p\|\cdot\|_p$$

and $(\chi_n)_{n\geq 1}$ is a monotone, unconditional basis with respect to $\|\cdot\|_p^{\mathrm{new}}$. We will use $L_p^{\mathrm{new}}[0,1]$ to denote $L_p[0,1]$ equipped with the norm $\|\cdot\|_p^{\mathrm{new}}$.

Now, for each $n\geq 1$ let

$$e_n^p=\frac{\chi_n}{\|\chi_n\|_p^{\mathrm{new}}}.$$

The sequence $(e_n^p)_{n\geq 1}$ is a normalized, unconditional basis of $L_p^{\mathrm{new}}[0,1]$ whose unconditional basis constant is 1. Further, by [Lindenstrauss and Tzafriri 1977, p. 19], $(e_n^p)_{n\geq 1}$ is also a boundedly complete basis.

Given X a Banach space and $p,q\in(1,\infty)$ satisfying $1/p+1/q=1$, we denote by $L_p^{\mathrm{weak}}(X)$ the space

$$\left\{(x_n)_{n\geq 1}\in X^{\mathbb{N}}:\sum_{n=1}^{\infty}x^*(x_n)e_n^p \text{ converges in } L_p^{\mathrm{new}}[0,1] \text{ for each } x^*\in X^*\right\}$$

equipped with the norm

$$\|\bar{x}\|_p^{\mathrm{weak}}=\sup\left\{\left\|\sum_{n=1}^{\infty}x^*(x_n)e_n^p\right\|_p^{\mathrm{new}}:x^*\in B_{X^*}\right\},$$

and by $L_p\langle X\rangle$ the space

$$\left\{(x_n)_{n\geq 1}\in X^{\mathbb{N}}:\sum_{n=1}^{\infty}|x_n^*(x_n)|<\infty \text{ for each } (x_n^*)_{n\geq 1}\in L_q^{\mathrm{weak}}(X^*)\right\}$$

with the norm

$$\|\bar{x}\|_{L_p\langle X\rangle} = \sup\left\{\sum_{n=1}^{\infty}|x_n^*(x_n)| : (x_n^*)_{n\geq 1} \in B_{L_q^{\text{weak}}(X^*)}\right\},$$

where $\bar{x} = (x_n)_{n\geq 1}$. With their own respective norms, $L_p^{\text{weak}}(X)$ and $L_p\langle X\rangle$ are Banach spaces [Bu 2002].

For each $n \geq 1$, we will denote by

$$I_n : X \to X^{\mathbb{N}}$$

the natural inclusion

$$I_n(x) = (\delta_{mn}x)_{m\geq 1} \quad \text{for all } x \in X.$$

It is easy to see that $\|I_n(x)\|_p^{\text{weak}} = \|x\|$ and furthermore, by [Lindenstrauss and Tzafriri 1977, Proposition 1.c.7], we know $\|I_n(x)\|_{L_p\langle X\rangle} \leq 2\|x\|$, for every $x \in X$.

We shall consider also the following closed subspace of $L_p^{\text{weak}}(X)$:

$$F_p(X) = \left\{\bar{x} = (x_n)_{n\geq 1} \in L_p^{\text{weak}}(X) : \left\|\bar{x} - \sum_{n=1}^{m} I_n(x_n)\right\|_p^{\text{weak}} \to 0\right\}.$$

Next, we recall some results obtained in [Bu 2002].

Theorem 9 [Bu 2002, Theorem 2.4]. *Given X a Banach space, $p \in (1,\infty)$ and $\bar{x} = (x_n)_{n\geq 1} \in L_p\langle X\rangle$, the series $\sum_{n=1}^{\infty} I_n(x_n)$ converges to $\bar{x}$ in $L_p\langle X\rangle$.*

The next result gives a sequential representation of $L_p[0,1]\hat{\otimes}_\pi X$.

Theorem 10 [Bu 2002, Theorem 3.4]. *Let X be a Banach space and $p \in (1,\infty)$. The function $\Psi : L_p\langle X\rangle \to L_p[0,1]\hat{\otimes}_\pi X$ defined by*

$$\Psi(\bar{x}) = \sum_{n=1}^{\infty} e_n^p \otimes x_n$$

for each $\bar{x} = (x_n)_{n\geq 1} \in L_p\langle X\rangle$ is an isomorphism onto $L_p[0,1]\hat{\otimes}_\pi X$.

Theorem 11. *Let X be a Banach space and $p, q \in (1,\infty)$ such that $1/p + 1/q = 1$. Then $L_q^{\text{weak}}(X)$ is isomorphic to $\mathcal{L}(L_p[0,1], X)$ and its subspace $F_q(X)$ is isomorphic to $\mathcal{K}(L_p[0,1], X)$.*

Proof. Let $(e_n^*)_{n\geq 1}$ be the sequence of coordinate functionals in $L_p[0,1]^*$ with respect to the basis $(e_n^p)_{n\geq 1}$. It is easy to check that the usual isometry from $L_p[0,1]^*$ onto $L_q[0,1]$ associates the functional e_n^* to e_n^q.

Fix $\bar{x} = (x_n)_{n\geq 1} \in L_q^{\text{weak}}(X)$ and $f = \sum_{n=1}^{\infty} \alpha_n e_n^p \in L_p[0,1]$. We claim that the series $\sum_{n=1}^{\infty} \alpha_n x_n$ converges in X. Indeed, given $k \geq j \geq 1$ we have

$$\begin{aligned}
\left\| \sum_{n=j}^{k} \alpha_n x_n \right\| &= \left\| \sum_{n=j}^{k} e_n^*(f) x_n \right\| = \sup_{x^* \in B_{X^*}} \left| \sum_{n=j}^{k} e_n^*(f) x^*(x_n) \right| \\
&= \sup_{x^* \in B_{X^*}} \left| \left(\sum_{n=j}^{k} x^*(x_n) e_n^* \right) \left(\sum_{m=j}^{k} e_m^*(f) e_m^p \right) \right| \\
&\leq \sup_{x^* \in B_{X^*}} \left\| \sum_{n=j}^{k} x^*(x_n) e_n^* \right\| \left\| \sum_{m=j}^{k} e_m^*(f) e_m^p \right\| \\
&= \sup_{x^* \in B_{X^*}} \left\| \sum_{n=j}^{k} x^*(x_n) e_n^q \right\|_q \left\| \sum_{m=j}^{k} e_m^*(f) e_m^p \right\| \leq \|\bar{x}\|_q^{\text{weak}} \left\| \sum_{m=j}^{k} e_m^*(f) e_m^p \right\|
\end{aligned}$$

and therefore the partial sums of the series $\sum_{n=1}^{\infty} \alpha_n x_n$ form a Cauchy sequence in X, which establishes our claim.

This proves that $\mathcal{I} : L_q^{\text{weak}}(X) \to \mathcal{L}(L_p[0,1], X)$ given by

$$\mathcal{I}(\bar{x})(f) = \sum_{n=1}^{\infty} \alpha_n x_n$$

for each $\bar{x} = (x_n)_{n \geq 1} \in L_q^{\text{weak}}(X)$ and $f = \sum_{n=1}^{\infty} \alpha_n e_n^p \in L_p[0,1]$ is a well-defined linear operator satisfying $\|\mathcal{I}(\bar{x})\| \leq \|\bar{x}\|_q^{\text{weak}}$.

Let us show now that $\mathcal{I}$ is an isomorphism onto $\mathcal{L}(L_p[0,1], X)$. Fix $S \in \mathcal{L}(L_p[0,1], X)$ and consider $\bar{y} = (S(e_n^p))_{n \geq 1}$. We claim that $\bar{y} \in L_q^{\text{weak}}$. Indeed, for each $m \geq 1$ and $x^* \in B_{X^*}$ we have

$$\begin{aligned}
&\left\| \sum_{n=1}^{m} x^*(S(e_n^p)) e_n^q \right\|_q^{\text{new}} \\
&\quad = \sup_{\theta_n = \pm 1} \left\| \sum_{n=1}^{m} \theta_n x^*(S(e_n^p)) e_n^q \right\|_q = \sup_{\theta_n = \pm 1} \sup_{g \in B_{L_p[0,1]}} \left| x^* \left(\sum_{n=1}^{m} \theta_n e_n^*(g) S(e_n^p) \right) \right| \\
&\quad \leq \sup_{\theta_n = \pm 1} \sup_{g \in B_{L_p[0,1]}} \left\| S \left(\sum_{n=1}^{m} \theta_n e_n^*(g) e_n^p \right) \right\| \leq \|S\| \sup_{\theta_n = \pm 1} \sup_{g \in B_{L_p[0,1]}} \left\| \sum_{n=1}^{m} \theta_n e_n^*(g) e_n^p \right\|_p \\
&\quad = \|S\| \sup_{g \in B_{L_p[0,1]}} \left\| \sum_{n=1}^{m} e_n^*(g) e_n^p \right\|_p^{\text{new}} \leq \|S\| \sup_{g \in B_{L_p[0,1]}} \left\| \sum_{n=1}^{\infty} e_n^*(g) e_n^p \right\|_p^{\text{new}} \\
&\quad \leq K_p \|S\| \sup_{g \in B_{L_p[0,1]}} \left\| \sum_{n=1}^{\infty} e_n^*(g) e_n^p \right\|_p = K_p \|S\|.
\end{aligned}$$

Since $(e_n^q)_{n\geq 1}$ is a boundedly complete basis, the claim is established. This shows that $\mathcal{I}' : \mathcal{L}(L_p[0, 1], X) \to L_q^{\text{weak}}(X)$ defined by $\mathcal{I}'(S) = (S(e_n^p))_{n\geq 1}$ is a bounded linear operator with $\|\mathcal{I}'\| \leq K_p$. Furthermore, it is easy to see that $\mathcal{I}'$ is the inverse of $\mathcal{I}$. Thus, $\mathcal{I}$ is an isomorphism onto $\mathcal{L}(L_p[0, 1], X)$.

Next we will show that $\mathcal{I}$ maps $F_q(X)$ onto $\mathcal{K}(L_p[0, 1], X)$. It is clear that $\mathcal{I}(F_q(X))$ is subset of $\mathcal{K}(L_p[0, 1], X)$. Next, fix $T \in \mathcal{K}(L_p[0, 1], X)$. Since $\mathcal{I}$ is onto $\mathcal{L}(L_p[0, 1], X)$, there exists a unique $\bar{y} = (y_n)_{n\geq 1} \in L_q^{\text{weak}}(X)$ such that $\mathcal{I}(\bar{y}) = T$. We will show that $\bar{y} \in F_q(X)$. Fix $\varepsilon > 0$ and denote by $(P_n)_{n\geq 1}$ the sequence of projections associated to the basis $(e_n^p)_n$. Since $(e_n^*)_{n\geq 1}$ is a Schauder basis of $L_p[0, 1]^*$ and T is compact, the sequence $(P_n^*)_{n\geq 1}$ converges uniformly to the identity operator on the compact set $T^*(B_{X^*})$. Hence, there exists $N \geq 1$ such that $\| P_m^*(T^*(x^*)) - T^*(x^*)\| < \varepsilon/K_p$ for every $x^* \in B_{X^*}$ and $m \geq N$, and thus $\|T \circ P_m - T\| \leq \varepsilon/K_p$ for every $m \geq N$. It is easy to see that

$$\mathcal{I}\left(\sum_{n=1}^{m} I_n(y_n)\right) = T \circ P_m$$

for every $m \geq 1$. Therefore we have

$$\left\|\bar{y} - \sum_{n=1}^{m} I_n(y_n)\right\|_q^{\text{weak}} \leq \|\mathcal{I}^{-1}\| \|T - T \circ P_m\| < \varepsilon$$

for every $m \geq N$, and thus $\bar{y} \in F_q(X)$. The proof is complete. □

3.2. ***The duals of the spaces $L_p\langle X\rangle$ and $F_q(X)$.*** It is well known that $L_p\langle X\rangle^*$ is linearly isomorphic to $\mathcal{L}(L_p[0, 1], X^*)$ [Ryan 2002, Theorem 2.9] and that $F_q(X)^*$ is linearly isomorphic to $L_p[0, 1]\widehat{\otimes}_\pi X^*$ [Ryan 2002, Theorem 5.33].

This subsection will be devoted to obtaining convenient characterizations of the duals of the spaces $F_q(X)$ and $L_p\langle X\rangle$.

Proposition 12. *Given X a Banach space, $p \in (1, \infty)$, $\bar{x} = (x_n)_{n\geq 1} \in L_p\langle X\rangle$ and $\varphi \in L_p\langle X\rangle^*$, the series $\sum_{n=1}^{\infty}(\varphi \circ I_n)(x_n)$ converges absolutely.*

Proof. For each $n \geq 1$, let $\theta_n = \text{sign}(\varphi \circ I_n)(x_n)$. Then $\bar{y} = (\theta_n x_n)_{n\geq 1} \in L_p\langle X\rangle$ and by Theorem 9 we have

$$\sum_{n=1}^{\infty}|(\varphi \circ I_n)(x_n)| = \sum_{n=1}^{\infty}(\varphi \circ I_n)(\theta_n x_n) = \varphi(\bar{y}),$$

as desired. □

Similarly to the previous proposition, we have:

Proposition 13. *Given X a Banach space, $p \in (1, \infty)$, $\bar{x} = (x_n)_{n\geq 1} \in F_p(X)$ and $\varphi \in F_p(X)^*$, the series $\sum_{n=1}^{\infty}(\varphi \circ I_n)(x_n)$ converges absolutely.*

Proof. For each $n \geq 1$, let $\theta_n = \operatorname{sign}(\varphi \circ I_n)(x_n)$. Since $(e_n^p)_{n\geq 1}$ is an unconditional basis with unconditional constant equal to 1, it follows that the series $\sum_{n=1}^{\infty} \theta_n x^*(x_n) e_n^p$ converges in $L_p^{\text{new}}[0,1]$ for every $x^* \in X^*$. Moreover, for every $k \geq 1$ and $x^* \in X^*$ we have

$$\left\| \sum_{n=k}^{\infty} \theta_n x^*(x_n) e_n^p \right\|_p^{\text{new}} = \left\| \sum_{n=k}^{\infty} x^*(x_n) e_n^p \right\|_p^{\text{new}}$$

and so $(\theta_n x_n)_{n\geq 1} \in F_p(X)$. Thus, $\sum_{n=1}^{\infty} \theta_n (\varphi \circ I_n)(x_n)$ converges. □

Proposition 14. *Let X be a Banach space and $p, q \in (1, \infty)$ such that $1/p + 1/q = 1$. A sequence $\bar{x}^* = (x_n^*)_{n\geq 1}$ of elements of X^* belongs to $L_p\langle X^*\rangle$ if, and only if, the series $\sum_{n=1}^{\infty} x_n^*(x_n)$ converges absolutely for each $\bar{x} = (x_n)_{n\geq 1} \in F_q(X)$. Furthermore, in this case one has*

$$\|\bar{x}^*\|_{L_p\langle X\rangle} \leq \sup\left\{ \sum_{n=1}^{\infty} |x_n^*(x_n)| : \bar{x} = (x_n)_{n\geq 1} \in B_{F_q(X)} \right\} < \infty.$$

Proof. Let us show the nontrivial implication. Let $\bar{x}^* = (x_n^*)_{n\geq 1}$ be a sequence of elements of X^* such that the series $\sum_{n=1}^{\infty} x_n^*(x_n)$ converges absolutely for each $\bar{x} = (x_n)_{n\geq 1} \in F_q(X)$. We claim that

$$\mathcal{S}(\bar{x}^*) = \sup\left\{ \sum_{n=1}^{\infty} |x_n^*(x_n)| : \bar{x} = (x_n)_{n\geq 1} \in B_{F_q(X)} \right\} < \infty.$$

Indeed, for each $m \geq 1$, consider the set

$$U_m = \left\{ \bar{x} = (x_n)_{n\geq 1} \in B_{F_q(X)} : \sum_{n\geq 1} |x_n^*(x_n)| \leq m \right\}.$$

It is easy to check that U_m is a closed, absolutely convex subset of $B_{F_q(X)}$. Since $B_{F_q(X)} = \bigcup_{m\geq 1} U_m$ has nonempty interior, by Baire's theorem there exists $M \geq 1$ such that U_M has nonempty interior. The absolute convexity of U_M implies that 0 is an interior point of U_M, that is, there exists $r > 0$ satisfying

$$\{\bar{x} = (x_n)_{n\geq 1} \in B_{F_q(X)} : \|\bar{x}\|_q^{\text{weak}} \leq r\} \subset U_M.$$

This proves that $\mathcal{S}(\bar{x}^*) \leq M/r$ and our claim is established.

Next, let us show that $\bar{x}^* = (x_n^*)_{n\geq 1} \in L_p\langle X^*\rangle$. Fix $\bar{x}^{**} = (x_n^{**})_{n\geq 1} \in L_q^{\text{weak}}(X^{**})$, $m \geq 1$ and $\varepsilon > 0$. Put $Y = \operatorname{span}\{x_1^{**}, \ldots, x_m^{**}\}$. By the principle of local reflexivity [Martínez-Abejón 1999, Theorem 2], there exists a linear operator $T : Y \to X$ satisfying $\|T\| \leq 1 + \varepsilon$ and $x_n^*(T(x_n^{**})) = x_n^{**}(x_n^*)$ for each $1 \leq n \leq m$. Put $\bar{y} = (y_n)_{n\geq 1} \in F_q(X)$, where $y_n = T(x_n^{**})$, if $1 \leq n \leq m$, and $y_n = 0$ otherwise.

Since $(e_n^q)_{n\geq 1}$ is an unconditional basis, by [Lindenstrauss and Tzafriri 1977, p. 18] we have

$$\begin{aligned}\|\bar{y}\|_q^{\text{weak}} &= \sup_{x^*\in B_{X^*}}\Big\|\sum_{n=1}^{m}(x^*\circ T)(x_n^{**})e_n^q\Big\|_q^{\text{new}}\\ &\leq (1+\varepsilon)\sup_{\varphi\in B_{X^{***}}}\Big\|\sum_{n=1}^{m}\varphi(x_n^{**})e_n^q\Big\|_q^{\text{new}}\\ &\leq (1+\varepsilon)\sup_{\varphi\in B_{X^{***}}}\Big\|\sum_{n=1}^{\infty}\varphi(x_n^{**})e_n^q\Big\|_q^{\text{new}} = (1+\varepsilon)\|\bar{x}^{**}\|_q^{\text{weak}}\end{aligned}$$

and hence

$$\sum_{n=1}^{m}|x_n^{**}(x_n^*)| \leq \mathcal{S}(\bar{x}^*)\|\bar{y}\|_q^{\text{weak}} \leq (1+\varepsilon)\mathcal{S}(\bar{x}^*)\|\bar{x}^{**}\|_q^{\text{weak}}.$$

Since $\varepsilon>0$ was arbitrary, we obtain

$$\sum_{n=1}^{m}|x_n^{**}(x_n^*)| \leq \mathcal{S}(\bar{x}^*)\|\bar{x}^{**}\|_q^{\text{weak}}$$

for each $m\geq 1$, which in turn implies

$$\sum_{n=1}^{\infty}|x_n^{**}(x_n^*)| \leq \mathcal{S}(\bar{x}^*)\|\bar{x}^{**}\|_q^{\text{weak}}.$$

Thus, $\bar{x}^*\in L_p\langle X^*\rangle$ and $\|\bar{x}^*\|_{L_p\langle X\rangle}\leq \mathcal{S}(\bar{x}^*)$, as desired. □

Theorem 15. *Let X be a Banach space and $p,q\in(1,\infty)$ such that $1/p+1/q=1$. The function $\mathcal{H}: F_q(X)^*\to L_p\langle X^*\rangle$ defined by*

$$\mathcal{H}(\varphi) = (\varphi\circ I_n)_{n\geq 1}$$

for each $\varphi\in F_q(X)^*$ is a linear isometry onto $L_p\langle X^*\rangle$.

Proof. Given $\varphi\in F_q(X)^*$, Propositions 13 and 14 imply that $(\varphi\circ I_n)_{n\geq 1}\in L_p\langle X^*\rangle$. Thus, $\mathcal{H}$ is well defined. It is clear that $\mathcal{H}$ is linear.

By Proposition 13, we have

$$\begin{aligned}\|\mathcal{H}(\varphi)\|_{L_p\langle X^*\rangle} &\leq \sup\Big\{\sum_{n=1}^{\infty}|(\varphi\circ I_n)(x_n)| : \bar{x}=(x_n)_{n\geq 1}\in B_{F_q(X)}\Big\}\\ &= \sup\Big\{\Big|\sum_{n=1}^{\infty}(\varphi\circ I_n)(x_n)\Big| : \bar{x}=(x_n)_{n\geq 1}\in B_{F_q(X)}\Big\}\\ &= \sup\{|\varphi(\bar{x})| : \bar{x}=(x_n)_{n\geq 1}\in B_{F_q(X)}\} = \|\varphi\|,\end{aligned}$$

where the first equality follows immediately from the proof of Proposition 13. On the other hand,

$$\begin{aligned}\|\mathcal{H}(\varphi)\|_{L_p\langle X^*\rangle} &= \sup\left\{\sum_{n=1}^{\infty}|x_n^{**}(\varphi\circ I_n)| : \bar{x}^{**}=(x_n^{**})_{n\geq 1}\in B_{L_q^{\text{weak}}(X^{**})}\right\}\\ &\geq \sup\left\{\sum_{n=1}^{\infty}|(\varphi\circ I_n)(x_n)| : \bar{x}=(x_n)_{n\geq 1}\in B_{F_q(X)}\right\} = \|\varphi\|.\end{aligned}$$

This shows that $\mathcal{H}$ is an isometry onto its image.

Finally, given $\bar{x}^*=(x_n^*)_{n\geq 1}\in L_p\langle X^*\rangle$, the function $\psi : F_q(X)\to\mathbb{R}$ defined by $\psi(\bar{x})=\sum_{n=1}^{\infty}x_n^*(x_n)$ for each $\bar{x}=(x_n)_{n\geq 1}\in F_q(X)$ is a linear functional on $F_q(X)$ and it is clear that $\mathcal{H}(\psi)=\bar{x}^*$. This completes the proof. □

Next, we establish an isomorphism from $L_p\langle X\rangle^*$ onto $L_q^{\text{weak}}(X^*)$.

Theorem 16. *Let X be a Banach space and $p, q\in(1,\infty)$ such that $1/p+1/q=1$. The function $\mathcal{J} : L_q^{\text{weak}}(X^*)\to L_p\langle X\rangle^*$ given by*

$$\mathcal{J}(\bar{x}^*)(\bar{x})=\sum_{n=1}^{\infty}x_n^*(x_n)$$

for each $\bar{x}^*=(x_n)_{n\geq 1}\in L_q^{\text{weak}}(X^*)$ and $\bar{x}=(x_n)_{n\geq 1}\in L_p\langle X\rangle$ is an isomorphism onto $L_p\langle X\rangle^*$.

Proof. Let $\Psi : L_p\langle X\rangle\to L_p[0,1]\hat{\otimes}_\pi X$ be the isomorphism defined in Theorem 10, $\mathcal{I} : L_q^{\text{weak}}(X^*)\to\mathcal{L}(L_p[0,1],X^*)$ be the isomorphism defined in Theorem 11, and consider $\Phi : \mathcal{L}(L_p[0,1],X^*)\to(L_p[0,1]\hat{\otimes}_\pi X)^*$ the canonical linear isometry [Ryan 2002, p. 24]. Given $\bar{x}^*=(x_n)_{n\geq 1}\in L_q^{\text{weak}}(X^*)$ and $\bar{x}=(x_n)_{n\geq 1}\in L_p\langle X\rangle$, we have

$$\begin{aligned}(\Psi^*\circ\Phi\circ\mathcal{I})(\bar{x}^*)(\bar{x}) &= (\Phi\circ\mathcal{I})(\bar{x}^*)(\Psi(\bar{x})) = \sum_{n=1}^{\infty}(\Phi\circ\mathcal{I})(\bar{x}^*)(e_n^p\otimes x_n)\\ &= \sum_{n=1}^{\infty}\mathcal{I}(\bar{x}^*)(e_n^p)(x_n) = \mathcal{J}(\bar{x}^*)(\bar{x}) = \sum_{n=1}^{\infty}x_n^*(x_n)\end{aligned}$$

and therefore $\mathcal{J}=\Psi^*\circ\Phi\circ\mathcal{I}$. The proof is complete. □

3.3. ***Complemented copies of $c_0(\tau)$ in $L_p\langle X\rangle$ spaces.*** The next lemma will play a crucial role in the proof of Theorem 18.

Lemma 17. *Let X be a Banach space, τ be an infinite cardinal and $p, q\in(1,\infty)$ such that $1/p+1/q=1$. Suppose that $(\bar{x}_i)_{i\in\tau}=((x_n^i)_{n\geq 1})_{i\in\tau}$ is a family equivalent to the canonical basis of $c_0(\tau)$ in $L_p\langle X\rangle$ and let $(\varphi_i)_{i\in\tau}$ be a bounded family in*

$L_p\langle X\rangle^*$. *Then for each* $\varepsilon > 0$ *there exists* $M \geq 0$ *such that*

$$\left|\sum_{n=M+1}^{\infty}(\varphi_i \circ I_n)(x_n^i)\right| < \varepsilon, \quad \textit{for all } i \in \tau.$$

Proof. We recall that the series $\sum_{n=1}^{\infty}(\varphi_i \circ I_n)(x_n^i)$ converges absolutely for each $i \in \tau$, by Proposition 12. Let $s = \sup_{i\in\tau} \|\psi_i\| < \infty$.

Suppose the thesis does not hold. Then there exists $\varepsilon > 0$ such that, for each $m \geq 0$, there exists $i \in \tau$ satisfying

$$\left|\sum_{n=m+1}^{\infty}(\varphi_i \circ I_n)(x_n^i)\right| \geq \varepsilon.$$

We proceed by induction. For $M_0 = 0$, there exists $i_1 \in \tau$ such that

$$\left|\sum_{n=1}^{\infty}(\varphi_{i_1} \circ I_n)(x_n^{i_1})\right| \geq \varepsilon.$$

The absolute convergence of $\sum_{n=1}^{\infty}(\varphi_{i_1} \circ I_n)(x_n^{i_1})$ yields $M_1 \geq 1$ such that

$$\sum_{n=M_1+1}^{\infty}|(\varphi_{i_1} \circ I_n)(x_n^{i_1})| < \frac{\varepsilon}{2}.$$

Thus we have

$$\left|\sum_{n=1}^{M_1}(\varphi_{i_1} \circ I_n)(x_n^{i_1})\right| > \frac{\varepsilon}{2}.$$

Suppose we have obtained, for some $k \geq 1$, strictly increasing natural numbers $0 = M_0 < M_1 < \cdots < M_k$ and distinct $i_1, \ldots, i_k \in \tau$ satisfying

$$\left|\sum_{n=N_j}^{M_j}(\varphi_{i_j} \circ I_n)(x_n^{i_j})\right| > \frac{\varepsilon}{2} > \sum_{n=M_j+1}^{\infty}|(\varphi_{i_j} \circ I_n)(x_n^{i_j})|, \tag{3-1}$$

where $N_j = M_{j-1} + 1$, for each $1 \leq j \leq k$. By hypothesis, there exists $i_{k+1} \in \tau$ such that

$$\left|\sum_{n=M_k+1}^{\infty}(\varphi_{i_{k+1}} \circ I_n)(x_n^{i_{k+1}})\right| \geq \varepsilon.$$

The absolute convergence of $\sum_{n=1}^{\infty}(\varphi_{i_{k+1}} \circ I_n)(x_n^{i_{k+1}})$ yields $M_{k+1} \geq M_k + 1$ such that

$$\sum_{n=M_{k+1}}^{\infty}|(\varphi_{i_{k+1}} \circ I_n)(x_n^{i_{k+1}})| < \frac{\varepsilon}{2}.$$

Thus we have

$$\left|\sum_{n=M_k+1}^{M_{k+1}} (\varphi_{i_{k+1}} \circ I_n)(x_n^{i_{k+1}})\right| > \frac{\varepsilon}{2}.$$

The above inequality and (3-1) imply that $i_{k+1} \notin \{i_1, \ldots, i_k\}$.

For each $j \geq 1$, consider $\bar{x}_j^* = (x_{j,n}^*)_{n\geq 1} \in F_q(X^*)$, where

$$x_{j,n}^* = \begin{cases} \varphi_{i_j} \circ I_n, & \text{if } N_j \leq n \leq M_j, \\ 0, & \text{otherwise.} \end{cases}$$

We claim that $(\bar{x}_j^*)_{j\geq 1}$ is weakly-null in $F_q(X^*)$. Indeed, fix $\psi \in F_q(X)^*$ and $\delta > 0$. Let $\mathcal{J}$ be the isomorphism defined in Theorem 16. By Theorem 15, the sequence $(\psi \circ J_n)_{n\geq 1}$ belongs to $L_p\langle X^*\rangle$, where $J_n : X^* \to (X^*)^{\mathbb{N}}$ is the usual inclusion. By Theorem 9, there exists $N \geq 1$ such that

$$\left\|\sum_{n=m}^{\infty} K_n(\psi \circ J_n)\right\|_{L_p\langle X^*\rangle} < \frac{\delta}{s\|\mathcal{J}^{-1}\|}$$

for each $m \geq N$, where $K_n : X^{**} \to (X^{**})^{\mathbb{N}}$ is the usual inclusion. Since the sequence $(N_j)_{j\geq 1}$ is strictly increasing, there exists $J \geq 1$ such that $N_j \geq N$, for all $j \geq J$. Thus we have

$$\begin{aligned} |\psi(\bar{x}_j^*)| &= \left|\sum_{n=N_j}^{M_j} (\psi \circ J_n)(x_{j,n}^*)\right| \leq \|\bar{x}_j^*\|_q^{\text{weak}} \left\|\sum_{n=N_j}^{M_j} K_n(\psi \circ J_n)\right\|_{L_p\langle X^*\rangle} \\ &\leq \|(\varphi_{i_j} \circ I_n)_{n\geq 1}\|_q^{\text{weak}} \frac{\delta}{s\|\mathcal{J}^{-1}\|} = \|\mathcal{J}^{-1}(\varphi_{i_j})\|_q^{\text{weak}} \frac{\delta}{s\|\mathcal{J}^{-1}\|} \leq \delta \end{aligned}$$

for all $j \geq J$. This establishes the claim.

Now, let $\theta_j = \mathcal{J}(\bar{x}_j^*) \in L_p\langle X\rangle^*$ for each $j \geq 1$. By our claim, $(\theta_j)_{j\geq 1}$ is weakly-null. On the other hand, by (3-1) we have

$$|\theta_j(\overline{x_{i_j}})| = \left|\sum_{n=N_j}^{M_j} (\varphi_{i_j} \circ I_n)(x_n^{i_j})\right| > \frac{\varepsilon}{2} \quad \text{for all } j \geq 1.$$

This contradicts the Dunford–Pettis property of c_0 [Fabian et al. 2010, p. 596], and we are done. □

Theorem 18. *Given X a Banach space, $p \in (1, \infty)$ and τ an infinite cardinal, we have*

$$c_0(\tau) \xhookrightarrow{c} L_p[0,1]\hat{\otimes}_\pi X \Longrightarrow c_0(\tau) \xhookrightarrow{c} X.$$

Proof. By Theorems 6 and 10, there exist families $(\overline{x_i})_{i\in\tau} = ((x_n^i)_{n\geq 1})_{i\in\tau}$ in $L_p\langle X\rangle$ and $(\psi_i)_{i\in\tau}$ in $L_p\langle X\rangle^*$ such that $(\overline{x_i})_{i\in\tau}$ is equivalent to the usual unit-vector basis of $c_0(\tau)$, $(\psi_i)_{i\in\tau}$ is weak*-null and $\psi_i(\overline{x_j}) = \delta_{ij}$ for each $i, j \in \tau$. Let $s = \sup_{i\in\tau} \|\psi_i\| < \infty$.

An appeal to Lemma 17 yields $M \geq 0$ such that

$$\Bigl|\sum_{n=M+1}^{\infty} (\varphi_i \circ I_n)(x_n^i)\Bigr| < \frac{1}{2} \quad \text{for all } i \in \tau.$$

Since $1 = \psi_i(\overline{x_i}) = \sum_{n=1}^{\infty}(\varphi_i \circ I_n)(x_n^i)$, we have $M \geq 1$ and

$$\frac{1}{2} < \sum_{n=1}^{M} |(\psi_i \circ I_n)(x_n^i)| \quad \text{for all } i \in \tau.$$

Next, for each $i \in \tau$ there exists $1 \leq n_i \leq M$ satisfying

$$\frac{1}{2M} < |(\psi_i \circ I_{n_i})(x_{n_i}^i)|.$$

Let $\mathcal{N} = \{n_i : i \in \tau\}$ and for each $n \in \mathcal{N}$ consider $\alpha_n = \{i \in \tau : n_i = n\}$. Since $\mathcal{N}$ is finite, there exists $N \in \mathcal{N}$ such that $|\alpha_N| = \tau$. Setting $\tau_1 = \alpha_N$, we obtain

$$\frac{1}{2M} < |(\psi_i \circ I_N)(x_N^i)| \quad \text{for all } i \in \tau_1. \tag{3-2}$$

For each $i \in \tau_1$, define $x_i = x_N^i \in X$ and $x_i^* = \psi_i \circ I_N \in X^*$. By (3-2), we have

$$\frac{1}{2M} < |x_i^*(x_i)| \leq \|\psi_i\|\|I_N\|\|x_i\| \leq s\|I_N\|\|y_i\| \quad \text{for all } i \in \tau_1,$$

and therefore

$$\frac{1}{2Ms\|I_N\|} < \|x_i\| \quad \text{for all } i \in \tau_1. \tag{3-3}$$

Next, let $(e_i)_{i\in\tau}$ denote the unit-vector basis of $c_0(\tau)$. By hypothesis, there exists $T : c_0(\tau) \to L_p\langle X\rangle$ an isomorphism from $c_0(\tau)$ onto its image such that $T(e_i) = \overline{x_i}$ for each $i \in \tau$. By (3-3), we have

$$\|(P_N \circ T)(e_i)\| = \|x_i\| \geq \frac{1}{2Ms\|I_N\|} > 0 \quad \text{for all } i \in \tau_1.$$

Therefore, by [Rosenthal 1970, remark following Theorem 3.4], there exists $\tau_2 \subset \tau_1$ such that $|\tau_2| = \tau$ and $P_N \circ T_{|c_0(\tau_2)}$ is an isomophism onto its image; hence, $(x_i)_{i\in\tau_2} = (P_N(T(e_i))_{i\in\tau_2}$ is equivalent to the unit-vector basis of $c_0(\tau_2)$.

Finally, given $x \in X$, observe that

$$(x_i^*(x))_{i\in\tau_2} = (\psi_i(I_N(x)))_{i\in\tau_2} \in c_0(\tau_2),$$

since $(\psi_i)_{i\in\tau}$ is weak*-null by hypothesis. This shows that $(x_i^*)_{i\in\tau_2}$ is weak*-null in X^*.

Combining these facts with (3-2), an appeal to Theorem 6 yields a complemented copy of $c_0(\tau)$ in X. $\square$

4. Complemented copies of $c_0(\tau)$ in $L_p([0,1],X)$ spaces

Let $\rho : L_p[0,1]\widehat{\otimes}_{\Delta_p} X \to L_p([0,1],X)$ be the unique linear extension of the natural mapping $g\otimes x \mapsto g(\,\cdot\,)x$, where $g \in L_p[0,1]$ and $x \in X$. By [Defant and Floret 1993, Chapters 7.1 and 7.2], ρ is a linear isometry from $L_p[0,1]\widehat{\otimes}_{\Delta_p} X$ onto $L_p([0,1],X)$.

For every integer m and $u \in L_p[0,1]$, we define

$$\sigma_m(u) = \sum_{n=1}^{m} c_n \chi_n(\cdot) \int_0^1 \chi_n(s)u(s)\,ds,$$

where $c_1 = 1$ and $c_{2^k+j} = 2^k$ for each $k \geq 0$ and $1 \leq j \leq 2^k$.

We define also the function H_m on $[0,1]\times[0,1]$ by

$$H_m(t,s) = \sum_{n=1}^{m} c_n \chi_n(t)\chi_n(s).$$

For every integer $k \geq 1$ we denote

$$I_{k,l} = \begin{cases} \left[\frac{l-1}{2^k}, \frac{l}{2^k}\right) & \text{if } 1 \leq l \leq 2^k - 1, \\ \left[1-\frac{1}{2^k}, 1\right] & \text{if } l = 2^k. \end{cases}$$

We also write $I_{0,1} = [0,1]$ and $C_{k,l} = I_{k,l} \times I_{k,l}$.

It is easy to check by induction that for each $k \geq 0$, $1 \leq l \leq 2^k$ and $m = 2^k + l$ we have

$$H_m = 2^{k+1} \sum_{i=1}^{2l} 1_{C_{k+1,i}} + 2^k \sum_{i=l+1}^{2^k} 1_{C_{k,i}},$$

[Novikov and Semenov 1997, p. 17], where 1_A denotes the characteristic function of $A \subset [0,1]$, and thus H_m is a positive function on $[0,1]\times[0,1]$. Since one has

$$\sigma_m(g) = \int_0^1 H_m(\,\cdot\,,s)g(s)\,ds$$

for each $g \in L_p[0,1]$, we conclude that σ_m is a positive operator on $L_p[0,1]$. Furthermore, $\|\sigma_m\| = 1$ and

$$\lim_{m\to\infty} \|\sigma_m(g) - g\|_p = 0 \tag{4-1}$$

for each $f \in L_p[0,1]$, by [Lindenstrauss and Tzafriri 1977, p. 3] or [Singer 1970, Example 2.3, p. 13].

Lemma 19. *Given X a Banach space, $p \in [1, \infty)$ and $f \in L_p([0, 1], X)$, the series*

$$\sum_{n=1}^{\infty} c_n \chi_n(\cdot) \int_0^1 \chi_n(s) f(s)\, ds$$

converges to f in $L_p([0, 1], X)$, where $c_1 = 1$ and $c_{2^k+j} = 2^k$ for each $k \geq 0$ and $1 \leq j \leq 2^k$.

Proof. The natural tensor norm $\|\cdot\|_{\Delta_p}$ is not an uniform cross norm, nevertheless the operator $s_m = \sigma_m \otimes I_X$ is bounded and $\|s_m\| = 1$ by [Defant and Floret 1993, Chapter 7.2]. By (4-1), we have

$$\lim_{m\to\infty} \|s_m(g \otimes x) - g \otimes x\|_{\Delta_p} = 0$$

and hence

$$\lim_{m\to\infty} \|s_m(u) - u\|_{\Delta_p} = 0$$

for every $u \in L_p[0, 1]\widehat{\otimes}_{\Delta_p} X$. The result then follows from the fact that ρ is a linear isometry onto $L_p([0, 1], X)$. □

We are now ready to prove the main result of this section.

Theorem 20. *Let X be a Banach space, τ be an infinite cardinal and $p \in [1, \infty)$. If* $\mathrm{cf}(\tau) > \aleph_0$, *then*

$$c_0(\tau) \overset{c}{\hookrightarrow} L_p([0, 1], X) \Longrightarrow c_0(\tau) \overset{c}{\hookrightarrow} X.$$

Proof. By Theorem 6, there exist families $(f_i)_{i\in\tau}$ in $L_p([0, 1], X)$ and $(\psi_i)_{i\in\tau}$ in $L_p([0, 1], X)^*$ such that $(f_i)_{i\in\tau}$ is equivalent to the usual unit-vector basis of $c_0(\tau)$, $(\psi_i)_{i\in\tau}$ is weak*-null and $\psi(f_j) = \delta_{ij}$, for each $i, j \in \tau$. Let $s = \sup_{i\in\tau} \|\psi_i\| < \infty$.

By Lemma 19, for each $i \in \tau$ we have

$$1 = |\psi_i(f_i)| \leq \sum_{n=1}^{\infty} c_n |\psi_i(\chi_n(\cdot) x_n^i)|,$$

where $x_n^i = \int_0^1 \chi_n(t) f_i(t)\, dt$, and thus there exists $m_i \geq 1$ such that

$$\frac{1}{2} < \sum_{n=1}^{m_i} c_n |\psi_i(\chi_n(\cdot) x_n^i)|.$$

Put $\mathcal{M} = \{m_i : i \in \tau\}$ and for each $m \in \mathcal{M}$ define $\alpha_m = \{i \in \tau : m_i = m\}$. Since $\mathcal{M}$ is countable and τ has uncountable cofinality, there exists $M \in \mathcal{M}$ such that $|\alpha_M| = \tau$. Setting $\tau_1 = \alpha_M$, we have

$$\frac{1}{2} < \sum_{n=1}^{M} c_n |\psi_i(\chi_n(\cdot) x_n^i)| \quad \text{for all } i \in \tau_1.$$

Next, for each $i \in \tau_1$ there exists $1 \leq n_i \leq M$ satisfying

$$\frac{1}{2M} < c_{n_i} |\psi_i(\chi_{n_i}(\,\cdot\,)x^i_{n_i})|.$$

Let $\mathcal{N} = \{n_i : i \in \tau_1\}$ and for each $n \in \mathcal{N}$ consider $\beta_n = \{i \in \tau_1 : n_i = n\}$. Since $\mathcal{N}$ is finite, there exists $N \in \mathcal{N}$ such that $|\beta_N| = \tau$. Setting $\tau_2 = \beta_N$, we obtain

$$\frac{1}{2Mc_N} < |\psi_i(\chi_N(\,\cdot\,)x^i_N)| \quad \text{for all } i \in \tau_2. \tag{4-2}$$

For each $i \in \tau_2$, write $x_i = x^i_N$ and consider the linear functional $x^*_i \in X^*$ defined by

$$x^*_i(x) = \psi_i(\chi_N(\,\cdot\,)(x)) \quad \text{for all } x \in X.$$

By (4-2), we obtain

$$\frac{1}{2Mc_N} < |x^*_i(x_i)| \leq \|\psi_i\| \|\chi_N(\,\cdot\,)x_i\|_p \leq \delta \|\chi_N\|_p \|x_i\| \quad \text{for all } i \in \tau_2,$$

and therefore

$$\Delta < \|x_i\| \quad \text{for all } i \in \tau_2, \tag{4-3}$$

where $\Delta = (2Msc_N\|\chi_N\|_p)^{-1}$.

Next, let $(e_i)_{i\in\tau}$ be the unit-vector basis of $c_0(\tau)$ and $T : c_0(\tau) \to L_p([0, 1], X)$ be an isomorphism from $c_0(\tau)$ onto its image such that $T(e_i) = f_i$ for each $i \in \tau$. Consider $P : L_p([0, 1], X) \to X$ the linear operator defined by

$$P(f) = \int_0^1 \chi_N(t) f(t)\, dt \quad \text{for all } f \in L_p([0, 1], X).$$

By (4-3), we have

$$\|(P \circ T)(e_i)\| = \|x_i\| \geq \Delta > 0 \quad \text{for all } i \in \tau_2.$$

Therefore, by [Rosenthal 1970, remark following Theorem 3.4], there exists $\tau_3 \subset \tau_2$ such that $|\tau_3| = \tau$ and $P \circ T_{|c_0(\tau_3)}$ is an isomorphism onto its image; hence,

$$(x_i)_{i\in\tau_3} = (P(T(e_i))_{i\in\tau_3}$$

is equivalent to the unit-vector basis of $c_0(\tau_3)$.

Finally, given $x \in X$, observe that

$$(x^*_i(x))_{i\in\tau_3} = (\psi_i(\chi_N(\,\cdot\,)(x)))_{i\in\tau_3} \in c_0(\tau_3),$$

since $(\psi_i)_{i\in\tau}$ is weak*-null by hypothesis. This proves that $(x^*_i)_{i\in\tau_3}$ is weak*-null in X^*.

Combining these facts with (2-1), an appeal to Theorem 6 yields a complemented copy of $c_0(\tau)$ in X. □

We do not know if the statement of Theorem 20 remains true in the case $p = \infty$.

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ON THE VOLUME BOUND IN THE DVORETZKY–ROGERS LEMMA

FERENC FODOR, MÁRTON NASZÓDI AND TAMÁS ZARNÓCZ

The classical Dvoretzky–Rogers lemma provides a deterministic algorithm by which, from any set of isotropic vectors in Euclidean d-space, one can select a subset of d vectors whose determinant is not too small. Pełczyński and Szarek improved this lower bound by a factor depending on the dimension and the number of vectors.

Pivovarov, on the other hand, determined the expectation of the square of the volume of parallelotopes spanned by d independent random vectors in $\mathbb{R}^d$, each one chosen according to an isotropic measure. We extend Pivovarov's result to a class of more general probability measures, which yields that the volume bound in the Dvoretzky–Rogers lemma is, in fact, equal to the expectation of the squared volume of random parallelotopes spanned by isotropic vectors. This allows us to give a probabilistic proof of the improvement of Pełczyński and Szarek, and provide a lower bound for the probability that the volume of such a random parallelotope is large.

1. Introduction

Given a set of isotropic vectors in Euclidean d-space $\mathbb{R}^d$ (see definition below), the Dvoretzky–Rogers lemma states that one may select a subset of d "well spread out" vectors. As a consequence, the determinant of these d vectors is at least $\sqrt{d!/d^d}$. This selection is deterministic: we start with an arbitrary element of the set, and then select more vectors one-by-one in a certain greedy manner.

Pivovarov [2010, Lemma 3, page 49], on the other hand, chooses d vectors randomly and then computes the expectation of the square of the resulting determinant. In this note, we extend Pivovarov's result to a wider class of measures, and apply this extension to obtain the improved lower bound of Pełczyński and Szarek [1991, Proposition 2.1], on the maximum of the volume of parallelotopes spanned by d vectors from the support of the measure. Thus, we give a probabilistic interpretation of the volume bound in the Dvoretzky–Rogers lemma.

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We denote the Euclidean scalar product by $\langle\,\cdot\,,\,\cdot\,\rangle$ and the induced norm by $|\cdot|$. We use the usual notation B^d for the unit ball of $\mathbb{R}^d$ centered at the origin o, and S^{d-1} for its boundary $\mathrm{bd}B^d$. We call a compact convex set $K \subset \mathbb{R}^d$ with nonempty interior a *convex body*. For detailed information on the properties of convex bodies, we refer to the books by Gruber [2007] and Schneider [2014].

Let Id_d be the identity map on $\mathbb{R}^d$. For $u, v \in \mathbb{R}^d$, let $u \otimes v : \mathbb{R}^d \to \mathbb{R}^d$ denote the *tensor product* of u and v, that is, $(u \otimes v)(x) = \langle v, x\rangle u$ for any $x \in \mathbb{R}^d$. Note that when $u \in S^{d-1}$ is a unit vector, $u \otimes u$ is the orthogonal projection to the linear subspace spanned by u.

For two functions $f(n)$, $g(n)$, we use the notation $f(n) \sim g(n)$ (as $n \to \infty$) if $\lim_{n\to\infty} f(n)/g(n) = 1$.

An *isotropic measure* is a probability measure μ on $\mathbb{R}^d$ with the following two properties.

$$\int_{\mathbb{R}^d} x \otimes x \,\mathrm{d}\mu(x) = \mathrm{Id}_d, \tag{1}$$

and the center of mass of μ is at the origin, that is,

$$\int_{\mathbb{R}^d} x \,\mathrm{d}\mu(x) = 0. \tag{2}$$

Pivovarov [2010] proved the following statement about the volume of random parallelotopes spanned by d independent, isotropic vectors.

Lemma 1 [Pivovarov 2010, Lemma 3]. *Let $x_1, \dots, x_d$ be independent random vectors distributed according to the isotropic measures $\mu_1, \dots, \mu_d$ in $\mathbb{R}^d$. Assume that $x_1, \dots, x_d$ are linearly independent with probability* 1. *Then*

$$\mathbb{E}([\det(x_1, \dots, x_d)]^2) = d!. \tag{3}$$

We note that Lutwak, Yang and Zhang [Lutwak et al. 2004, §2] established similar results for the case of discrete isotropic measures, which could also be used to prove the volumetric bounds in Theorem 5; see, for example, [Lutwak et al. 2004, formula (2.5) on page 167].

We extend Lemma 1 to a more general class of measures in the following way.

Lemma 2. *Let $x_1, \dots, x_d$ be independent random vectors distributed according to the probability measures $\mu_1, \dots, \mu_d$ in $\mathbb{R}^d$ satisfying* (1). *Assume $\mu_i(\{0\}) = 0$ for $i = 1, \dots, d$. Then* (3) *holds.*

We provide a simple and direct proof of Lemma 2 in Section 2.

Lemmas 1 and 2 yield the value of the second moment of the volume of random parallelotopes with isotropic generating vectors. On the other hand, Milman and Pajor [1989, §3.7] gave a lower bound for the p-th moment (with $0 < p < 2$) of this volume in the case when the generating vectors are selected according to the uniform distribution from an isotropic and origin-symmetric convex body; for more general results, see [Brazitikos et al. 2014, §3.5.1]. All of the previously mentioned results hold in *expectation*.

As a different approach, we mention Pivovarov's work [2010], where lower bounds on the volume of a random parallelotope are shown to hold *with high probability* under the assumption that the measures are log-concave.

For more information on properties of random parallelotopes, and random polytopes in general, we refer to the book [Schneider and Weil 2008] and the survey [Schneider 2018].

In this paper, our primary, geometric motivation in studying isotropic measures is the following celebrated theorem of John [1948], which we state in the refined form obtained by Ball [1992] (see also [Ball 1997]).

Theorem 3. *Let K be a convex body in $\mathbb{R}^d$. Then there exists a unique ellipsoid of maximal volume contained in K. Moreover, this maximal volume ellipsoid is the d-dimensional unit ball B^d if and only if there exist vectors $u_1, \dots, u_m \in \operatorname{bd} K \cap S^{d-1}$ and (positive) real numbers $c_1, \dots, c_m > 0$ such that*

$$\sum_{i=1}^{m} c_i u_i \otimes u_i = \mathrm{Id}_d, \tag{4}$$

and

$$\sum_{i=1}^{m} c_i u_i = 0. \tag{5}$$

Note that taking the trace in (4) yields $\sum_{i=1}^{m} c_i = d$. Thus, the Borel measure μ_K on $\sqrt{d}S^{d-1}$ with $\operatorname{supp}\mu_K = \{\sqrt{d}u_1, \dots, \sqrt{d}u_m\}$ and $\mu_K(\{\sqrt{d}u_i\}) = c_i/d$ $(i = 1, \dots, m)$ is a discrete isotropic measure.

If a finite system of unit vectors $u_1, \dots, u_m$ in $\mathbb{R}^d$, together with a set of positive weights $c_1, \dots, c_m$, satisfies (4) and (5), then we say that it forms a *John decomposition of the identity*. For each convex body K, there exists an affine image K' of K for which the maximal volume ellipsoid contained in K' is B^d, and K' is unique up to orthogonal transformations of $\mathbb{R}^d$.

The classical lemma of Dvoretzky and Rogers [1950] states that in a John decomposition of the identity, one can always find d vectors such that the selected vectors are not too far from an orthonormal system.

Lemma 4 (Dvoretzky–Rogers lemma [1950]). *Let $u_1, \dots, u_m \in S^{d-1}$ and let $c_1, \dots, c_m > 0$ such that* (4) *holds. Then there exists an orthonormal basis $b_1, \dots, b_d$ of $\mathbb{R}^d$ and a subset $\{x_1, \dots, x_d\} \subset \{u_1, \dots, u_m\}$ with $x_j \in \operatorname{lin}\{b_1, \dots, b_j\}$ and*

$$\sqrt{(d-j-1)/d} \le \langle x_j, b_j \rangle \le 1 \tag{6}$$

for $j = 1, \dots, d$.

Consider the parallelotope P spanned by the selected d vectors $x_1, \dots, x_d$. The volume of P is bounded from below by

$$(\mathrm{Vol}(P))^2 = [\det(x_1, \dots, x_d)]^2 \geq \frac{d!}{d^d}. \tag{7}$$

Our study of (7) is motivated in part by the recent proof [Naszódi 2016] of a conjecture of Bárány, Katchalski and Pach, where this bound is heavily relied on.

The main results of this paper are the following two theorems. Theorem 5 is essentially the same as Proposition 2.1 of [Pełczyński and Szarek 1991], however, here we give a probabilistic proof and interpretation. In Theorem 5 (ii) and (iii), we also note that when m is small the improvement on the original Dvoretzky–Rogers bound is larger.

Theorem 5. *Let $u_1, \dots, u_m \in S^{d-1}$ be unit vectors satisfying equation* (4) *with some $c_1, \dots, c_m > 0$. Then there is a subset $\{x_1, \dots, x_d\} \subset \{u_1, \dots, u_m\}$ with*

$$[\det(x_1, \dots, x_d)]^2 \geq \gamma(d, \overline{m}) \cdot \frac{d!}{d^d},$$

where $\gamma(d, \overline{m}) = \overline{m}^d / d! \binom{\overline{m}}{d}^{-1}$, and $\overline{m} = \min\{m, d(d+1)/2\}$.

Moreover, for $\gamma(d, \overline{m})$, we have:

(i) *$\gamma(d, \overline{m}) \geq \gamma(d, d(d+1)/2) \geq \frac{3}{2}$ for any $d \geq 2$ and $m \geq d$. And $\gamma(d, d(d+1)/2)$ is monotonically increasing, and $\lim_{d\to\infty} \gamma(d, d(d+1)/2) = e$.*

(ii) *Fix a $c > 1$, and consider the case when $m \leq cd$ with $c \geq 1 + 1/d$. Then*

$$\gamma(d, m) \geq \gamma(d, \lceil cd \rceil) \sim \sqrt{\frac{c-1}{c}} \left(\frac{c-1}{c}\right)^{(c-1)d} e^d, \quad \text{as } d \to \infty.$$

(iii) *Fix an integer $k \geq 1$, and consider the case when $m \leq d + k$. Then*

$$\gamma(d, m) \geq \gamma(d, d+k) \sim \frac{k! e^k}{\sqrt{2\pi}} \frac{e^d}{(d+k)^{k+1/2}}, \quad \text{as } d \to \infty.$$

We note that in (ii) and (iii), the improvements are exponentially large in d as d tends to infinity.

The following statement provides a lower bound on the probability that d independent, identically distributed random vectors selected from $\{u_1, \dots, u_m\}$ according to the distribution determined by the weights $\{c_1, \dots, c_m\}$ span a parallelotype of large volume.

Proposition 6. *Let $\lambda \in (0, 1)$. With the notation and assumptions of Theorem 5, if we choose the vectors $x_1, \dots, x_d$ independently according to the distribution $\mathbb{P}(x_\ell = u_i) = c_i/d$ for each $\ell = 1, \dots, d$ and $i = 1, \dots, m$, then with probability at least $(1-\lambda)e^{-d}$, we have*

$$[\det(x_1, \dots, x_d)]^2 \geq \lambda \gamma(d, \overline{m}) \cdot \frac{d!}{d^d}.$$

The geometric interpretation of Theorem 5 is as follows. If K is a convex polytope with n facets, and B^d is the maximal volume ellipsoid in K, then the number of contact points $u_1, \dots, u_m$ in John's theorem is at most $m \le n$. Thus, Theorem 5 yields a simplex in K of not too small volume, with one vertex at the origin.

In particular, consider $k = 1$ in Theorem 5 (iii), that is, when K is the regular simplex whose inscribed ball is B^d. Then the John decomposition of the identity determined by K consists of $d+1$ unit vectors that determine the vertices of a regular d-simplex inscribed in B^d, which we denote by Δ_d, and note that $\mathrm{Vol}(\Delta_d) = (d+1)^{(d+1)/2}/(d^{d/2}d!)$. Clearly, in this John decomposition of the identity, the volume of the simplex determined by any d of the vectors $u_1, \dots, u_{d+1}$ is

$$\mathrm{Vol}(\Delta_d)/(d+1) = \frac{(d+1)^{\frac{d-1}{2}}}{d^{d/2}d!}. \tag{8}$$

By Theorem 5, we obtain

$$\max[\det(u_{i_1}, \dots, u_{i_d})]^2 \ge \frac{(d+1)^{d-1}}{d!} \cdot \frac{d!}{d^d} = \frac{(d+1)^{d-1}}{d^d},$$

which yields the same bound for the largest volume simplex as the right-hand side of (8). Thus, Theorem 5 is sharp in this case.

We will use the following theorem in our argument.

Theorem 7 [John 1948; Pełczyński 1990; Ball 1992; Gruber and Schuster 2005]. *If a set of unit vectors satisfies* (4) (*resp.*, *both* (4) *and* (5)) *with some positive scalars* c_i', *then a subset of* m *elements also satisfies* (4) (*resp.*, *both* (4) *and* (5)) *with some positive scalars* c_i, *where*

$$d+1 \le m \le d(d+1)/2 \tag{9}$$

(*resp.*, $d+1 \le m \le d(d+3)/2$).

In Section 4, we outline a proof of Theorem 7 for two reasons. First, we will use the part when only (4) is assumed, which is only implicitly present in [Gruber and Schuster 2005]. Second, the result is described therein in terms of the contact points of a convex body with its maximal volume ellipsoid, that is, in the context of John's theorem. We, on the other hand, would like to give a presentation where the linear algebraic fact and its use in convex geometry are separated. Nevertheless, our proof is very close to the one given in [Gruber and Schuster 2005].

2. Proof of Lemma 2

The idea of the proof is to slightly rotate each distribution so that the probability that the d vectors are linearly independent is 1. Then we may apply Pivovarov's lemma, and use a limit argument as the d rotations each tend to the identity.

Let $A_1, \dots, A_d$ be matrices in $\mathrm{SO}(d)$ chosen independently of each other and of the x_i according to the unique Haar probability measure on $\mathrm{SO}(d)$. Fix an arbitrary nonzero unit vector e in $\mathbb{R}^d$. Note that $A_i x_i/|x_i|$ and $A_i e$ have the same distribution: both are uniformly chosen points of the unit sphere according to the uniform probability distribution on S^{d-1}. Further, the joint distribution of $A_1x_1/|x_1|, \dots, A_dx_d/|x_d|$ and the joint distribution of $A_1e, \dots, A_de$ are the same: they are independently chosen, uniformly distributed points on the unit sphere. It follows that

$$\mathbb{P}(A_1x_1, \dots, A_dx_d \text{ are linearly independent}) \\ = \mathbb{P}(A_1e, \dots, A_de \text{ are linearly independent}) = 1.$$

Denote the Haar measure on $Z := \mathrm{SO}(d)^d$ by ν. Thus, we have

$$\begin{aligned} 1 &= \mathbb{P}(A_1x_1, \dots, A_dx_d \text{ are linearly independent}) \\ &= \int_Z \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbb{1}_{\{A_1x_1, \dots, A_dx_d \text{ are linearly independent}\}}(x_1, \dots, x_d, A_1, \dots, A_d) \\ &\qquad\qquad \mathrm{d}\mu_1(x_1) \cdots \mathrm{d}\mu_d(x_d)\, \mathrm{d}\nu(A_1, \dots, A_d) \\ &= \int_Z \mathbb{P}(A_1x_1, \dots, A_dx_d \text{ are linearly independent} \mid A_1, \dots, A_d)\, \mathrm{d}\nu(A_1, \dots, A_d), \end{aligned}$$

where $\mathbb{1}$ denotes the indicator function.

Thus,

$$1 = \mathbb{P}[\mathbb{P}(A_1x_1, \dots, A_dx_d \text{ are linearly independent} \mid A_1, \dots, A_d) = 1]. \tag{10}$$

We call a d-tuple $(A_1, \dots, A_d) \in Z$ "good" if $A_1x_1, \dots, A_dx_d$ are linearly independent with probability 1. In (10), we obtained that the set of not good elements of Z is of measure zero.

Thus, we may choose a sequence $(A_1^{(j)}, A_2^{(j)}, \dots, A_d^{(j)})$, $j = 1, 2, \dots$ in Z, such that $\|A_i^{(j)} - \mathrm{Id}_d\| < 1/j$ for all i and j, and $(A_1^{(j)}, \dots, A_d^{(j)})$ is good for each j.

Note that for any j,

$$[\det(A_1^{(j)}x_1, \dots, A_d^{(j)}x_d)]^2 \le |A_1^{(j)}x_1|^2 |A_2^{(j)}x_2|^2 \cdots |A_d^{(j)}x_d|^2, \tag{11}$$

and

$$\mathbb{E}\big[|A_1^{(j)}x_1|^2 |A_2^{(j)}x_2|^2 \cdots |A_d^{(j)}x_d|^2\big] = d^d. \tag{12}$$

We conclude that

$$\begin{aligned} \mathbb{E}\big([\det(x_1, \dots, x_d)]^2\big) &= \mathbb{E}\big([\det \lim_{j\to\infty} (A_1^{(j)}x_1, \dots, A_d^{(j)}x_d)]^2\big) \\ &\overset{(a)}{=} \mathbb{E}\big([\lim_{j\to\infty} \det(A_1^{(j)}x_1, \dots, A_d^{(j)}x_d)]^2\big) \\ &\overset{(b)}{=} \lim_{j\to\infty} \mathbb{E}\big([\det(A_1^{(j)}x_1, \dots, A_d^{(j)}x_d)]^2\big), \end{aligned}$$

where, in (a), we use that the determinant is continuous. In (b), Lebesgue's dominated convergence theorem may be applied by (11) and (12).

Fix j and let $y_1 = A_1^{(j)} x_1, \dots, y_d = A_d^{(j)} x_d$. In order to emphasize that the assumption (2) is not needed, and also for completeness, we repeat Pivovarov's argument. For $k = 1, \dots, d-1$, let P_k denote the orthogonal projection of $\mathbb{R}^d$ onto the linear subspace $\operatorname{span}\{y_1, \dots, y_k\}^\perp$. Thus,

$$|\det(y_1, \dots, y_d)| = |y_1|\,|P_1 y_2| \cdots |P_{d-1} y_d|. \tag{13}$$

Note that with probability 1, $\operatorname{rank} P_k = d - k$. It follows from (1) that $\mathbb{E}|P_k y_{k+1}|^2 = d - k$. Fubini's theorem applied to (13) completes the proof of Lemma 2.

3. Proofs of Theorem 5 and Proposition 6

Let $u_1, \dots u_m \in S^{d-1}$ be a set of vectors satisfying (4) with some positive weights $c_1, \dots, c_m$. We set the probability of each vector u_i, $i = 1, \dots m$, as $p_i = c_i/d$, and obtain a discrete probability distribution.

Let $u_{i_1}, \dots, u_{i_d}$ be independent random vectors from the set $u_1, \dots, u_m$ chosen (with possible repetitions) according to the above probability distribution.

By Lemma 2, we have

$$\mathbb{E}\big([\det(u_{i_1}, \dots, u_{i_d})]^2\big) = \frac{d!}{d^d}.$$

Since the probability that the random vectors $u_{i_1}, \dots, u_{i_d}$ are linearly dependent is positive,

$$\max[\det(u_{i_1}, \dots, u_{i_d})]^2 > \frac{d!}{d^d}.$$

Our goal is to quantify this inequality by bounding from below the probability that the determinant is 0. Let

$$M^2 := \max[\det(u_{i_1}, \dots, u_{i_d})]^2.$$

If an element of $\{u_1, \dots, u_m\}$ is selected at least twice, then $\det(u_{i_1}, \dots, u_{i_d}) = 0$. Thus,

$$\mathbb{E}\big([\det(u_{i_1}, \dots, u_{i_d})]^2\big) \le M^2 P_1,$$

where P_1 denotes the probability that all indices are pairwise distinct. Therefore,

$$M^2 \ge \frac{d!}{d^d} \cdot \frac{1}{P_1}.$$

Note that P_1 is a degree d elementary symmetric function of the variables $p_1, \dots, p_m$. Furthermore, $p_1 + \dots + p_m = 1$ and $p_i \ge 0$ for all $i = 1, \dots, m$. It can easily be seen (using Lagrange multipliers, or by induction on m) that for fixed m and d, the maximum of P_1 is attained when $p_1 = \cdots = p_m = 1/m$. Thus,

$$P_1 \le d!\binom{m}{d}\frac{1}{m^d}.$$

In summary,

$$M^2 \ge \frac{d!}{d^d} \cdot \frac{m^d}{d!}\binom{m}{d}^{-1}.$$

First, we note that $\gamma(d,m) := \frac{m^d}{d!}\binom{m}{d}^{-1}$ is decreasing in m. Thus, by (9), we may assume that m is as large as possible, that is, $m = \frac{d(d+1)}{2}$ proving the first part of Theorem 5.

3.1. ***Proof of Theorem 5(i).*** Let $\gamma(d) := \gamma(d, d(d+1)/2)$. We show that $\gamma(d)$ is increasing in d.

With the notation $m := d(d+1)/2$, we note that $(d+1)(d+2)/2 = m+d+1$. Thus,

$$\frac{\gamma(d+1)}{\gamma(d)} = \frac{(m+d+1)^{d+1} m \cdots (m-d+1)}{m^d (m+d+1) \cdots (m+1)} = \frac{(m+d+1)^d}{m^d} \cdot \frac{m \cdots (m-d+1)}{(m+d) \cdots (m+1)}$$

Thus, we need to show that

$$1 + \frac{d+1}{m} > \sqrt[d]{\left(1+\frac{d}{m}\right)\left(1+\frac{d}{m-1}\right)\cdots\left(1+\frac{d}{m-d+1}\right)},$$

which, by the arithmetic mean/geometric mean inequality follows, if

$$1 + \frac{d+1}{m} \ge 1 + d\,\frac{\frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{m-d+1}}{d},$$

which is equivalent to

$$\frac{d}{m} \ge \frac{1}{m-1} + \frac{1}{m-2} + \cdots + \frac{1}{m-d+1}.$$

For this to hold, it is sufficient to show that for every integer or half of an integer $1 \le i \le d/2$, we have

$$\frac{2d}{(d-1)m} \ge \frac{1}{m-i} + \frac{1}{m-d+i}. \tag{14}$$

After substituting $m = d(d+1)/2$, it is easy to see that (14) holds.

Finally, $\lim_{d\to\infty} \gamma(d) = e$ follows from Stirling's formula.

3.2. ***Proof of Theorem 5 (ii) and (iii).*** Stirling's formula yields both claims.

3.3. ***Proof of Proposition 6.*** Denote the random variable $X := [\det(x_1, \dots, x_d)]^2$, and denote $E := \mathbb{E}(X) = d!/d^d$ and $q := \mathbb{P}\big(X \geq \frac{\lambda E}{P_1}\big)$, where, as in the proof of Theorem 5, $P_1 := \mathbb{P}(x_1, \dots, x_d$ are pairwise distinct).

In the proof of Theorem 5, we established

$$P_1 \leq (\gamma(d,\overline{m}))^{-1}, \text{ and thus, } q \leq \mathbb{P}\Big([\det(x_1, \dots, x_d)]^2 \geq \lambda\gamma(d,\overline{m}) \cdot \frac{d!}{d^d}\Big). \tag{15}$$

Using the fact that X is at most 1, we have

$$E \leq \frac{\lambda E}{P_1}\mathbb{P}\Big(X < \frac{\lambda E}{P_1} \text{ and } x_1, \dots, x_d \text{ are pairwise distinct}\Big) + \mathbb{P}\Big(X \geq \frac{\lambda E}{P_1}\Big).$$

That is, $E \leq \frac{\lambda E}{P_1}(P_1 - q) + q$, and thus, by (15),

$$q \geq \frac{(1-\lambda)E}{1 - \frac{\lambda E}{P_1}} \geq \frac{(1-\lambda)d!}{d^d - \lambda\gamma(d,\overline{m})d!} \geq (1-\lambda)e^{-d},$$

completing the proof of Proposition 6.

4. Proof of Theorem 7

First, observe that (4) holds with some positive scalars c_i, if and only if, the matrix Id_d/d is in the convex hull of the set $\mathcal{A} = \{v_i \otimes v_i : i = 1, \dots, m\}$ in the real vector space of $d \times d$ matrices. The set $\mathcal{A}$ is contained in the subspace of symmetric matrices with trace 1, which is of dimension $d(d+1)/2-1$. Carathéodory's theorem [Schneider 2014, Theorem 1.1.4] now yields the desired upper bound on m.

In the case when both (4) and (5) are assumed, we lift our vectors into $\mathbb{R}^{d+1}$ as follows. Let $\hat{v}_i = \sqrt{d/(d+1)}(v_i, 1/\sqrt{d}) \in \mathbb{R}^{d+1}$. It is easy to check that $|\hat{v}_i| = 1$, and that (4) holds for the vectors $\hat{v}_i$ with some positive scalars $\hat{c}_i$ if, and only if, (4) and (5) hold for the vectors v_i with scalars $c_i = \frac{d}{d+1}\hat{c}_i$. Now, $\hat{v}_i \otimes \hat{v}_i$, $i = 1, \dots, m$ are symmetric $(d+1) \times (d+1)$ matrices of trace 1, and their $(d+1, d+1)$-th entry is $1/(d+1)$. The dimension of this subspace of $\mathbb{R}^{(d+1)\times(d+1)}$ is $d(d+3)/2 - 1$, thus, again, by Carathéodory's theorem, the proof is complete.

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LIFTING OF ELLIPTIC CURVES

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Suppose that for all but finitely many primes p we are given an elliptic curve E_p defined over a finite field $\mathbb{F}_p$ of p elements. We derive a criterion for there to exist an elliptic curve E defined over $\mathbb{Q}$ for which the reduction of E modulo p is isogenous to E_p for almost all p.

1. Introduction

Throughout the article, p, q, ℓ will denote rational primes and $c_1, c_2, \ldots, c_{10}, c_{11}$ will denote positive constants which are absolute and effective unless otherwise specified. Suppose that for each prime p, we are given an elliptic curve E_p defined over a finite field $\mathbb{F}_p$ of p elements. We are interested here in the question of whether there exists an elliptic curve E over $\mathbb{Q}$ for which the reduction of $E(\text{mod } p)$ is isogenous to E_p for all but a finite set of primes p.

At the outset, it is clear that some conditions have to be imposed on the set of curves $\{E_p\}$. For example, if we choose all of them to be supersingular, E cannot exist. As we know from [Serre 1981], the set of primes of supersingular reduction has density $\frac{1}{2}$ or 0, depending on whether E has complex multiplication (CM).

On the other hand, if we consider any finite set T of primes, it is clear from the Chinese remainder theorem that we can find an elliptic curve E_T (say) over $\mathbb{Q}$ for which the reduction $E_T(\text{mod } p)$ is isogenous (even isomorphic) to E_p for $p \in T$. Moreover, we can choose E_T so that its discriminant $d(E_T)$ satisfies

$$d(E_T) \le c_1 \biggl(\prod_{p \in T} p \biggr)^3,$$

where $c_1 > 0$ is an absolute constant. In particular, if we choose T to be the set of primes $p \le N$, then

$$d(E_T) \le c_1 \exp(6N) \tag{1}$$

by Chebyshev's estimate.

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If we can find a single curve E whose reduction modulo p is isogenous to E_p for all but a finite set of primes S, then we can choose $E_T = E$ for any T disjoint from S. In this case,

$$d(E_T) \le c_2 \quad \text{as} \quad |T| \to \infty. \tag{2}$$

Here $c_2 > 0$ can be taken to be d_E, the discriminant of E. Our main result shows that even if we can find a family of curves $\{E_T\}$ for which the sequence $\{d(E_T)\}$ satisfies a bound significantly weaker than (2) but stronger than (1), then E does exist. To formulate all of this correctly, we work with conductors rather than discriminants. We get an unconditional result and a stronger conditional result assuming the generalized Riemann hypothesis (GRH), i.e., the Riemann hypothesis for Dedekind zeta functions of number fields.

2. Statement of the result

Let S be a fixed finite set of rational primes and for each prime $p \notin S$, let E_p be an elliptic curve over $\mathbb{F}_p$. For each N, consider the set

$$\Sigma_{S,N} = \{E \text{ an elliptic curve over } \mathbb{Q} \mid \text{for each rational prime } p \le N,\ p \notin S,\ E \pmod p \text{ is isogenous to } E_p\}.$$

Thus for $E \in \Sigma_{S,N}$ the primes of bad reduction are either in S or larger than N. Set

$$f_{S,N} = \min_{E \in \Sigma_{S,N}} f(E),$$

where $f(E)$ denotes the conductor of E. Denote by $E_N = E_{S,N}$ a curve in $\Sigma_{S,N}$ with conductor $f_{S,N}$.

Theorem 2.1. *Let S, N and $f_{S,N}$ be as in the previous paragraph. Assume the GRH. Suppose that*

$$f_{S,N} \le c_3 \exp(N^{c_4}), \tag{3}$$

where $c_3, c_4 > 0$ are absolute constants and $c_4 < \frac{1}{2}$. There exists an elliptic curve E over $\mathbb{Q}$ with good reduction outside S for which the reduction $E(\text{mod } p)$ is isogenous to E_p for all $p \notin S$.

The same techniques will also prove the following unconditional theorem.

Theorem 2.2. *Let S, N and $f_{S,N}$ be as in Theorem 2.1. Suppose that*

$$f_{S,N} \le c_5 N^{c_6}, \tag{4}$$

where $c_5, c_6 > 0$ are absolute constants and $c_6 < 1$. There exists an elliptic curve E over $\mathbb{Q}$ with good reduction outside S for which the reduction $E(\text{mod } p)$ is isogenous to E_p for all $p \notin S$.

Remark. We note that if we consider the subset $\Sigma^*_{S,N}$ of curves with good reduction outside S, then by Tate's conjecture (Faltings' theorem), it is contained in a finite number of isogeny classes. In particular, for infinitely many N, it must intersect a single isogeny class. This implies that there is a global lift of all the E_p, $p \notin S$. Thus, if we show that the subset $\Sigma^*_{S,N}$ is nonempty, we are done.

Remark. The Deuring lifting theorem enables us to lift a single elliptic curve over a finite field to an elliptic curve over a number field having CM. In fact, the Deuring theorem also lifts Frobenius to an endomorphism of the CM curve. We are currently investigating extensions of this result to the context studied in this note.

3. Application of the Chebotarev density theorem

For any elliptic curve E over $\mathbb{Q}$ and any prime q of good reduction, let us set (as usual),

$$a_q(E) = q + 1 - |E(\mathbb{F}_q)|.$$

Tate's conjecture (Faltings' theorem) implies that if for two curves E_1, E_2, we have $a_q(E_1) = a_q(E_2)$ for all but finitely many primes q, then E_1 is isogenous to E_2. This can be made effective using the Chebotarev density theorem.

Proposition 3.1. *Assume the GRH and let E_1 and E_2 be two elliptic curves defined over $\mathbb{Q}$ that are not isogenous. There exists a prime q of good reduction for both E_1 and E_2 with*

$$a_q(E_1) \neq a_q(E_2)$$

and

$$q \leq c_7 \left(\log[2 \cdot \max(f(E_1),\, f(E_2))]\right)^2,$$

where $c_7 > 0$ is an absolute constant. Here $f(E_1)$ and $f(E_2)$ denote the conductors of E_1 and E_2 respectively.

Proof. Serre [1981, théorème 21] proves the above bound weaker by a factor

$$(\log\log[\max(f(E_1),\, f(E_2))])^2$$

and notes in [Serre 1986, note 632.6, p. 715] that it can be improved using an argument of Faltings. Indeed, fix a prime ℓ and consider the two continuous semisimple representations

$$\rho_{1,\ell},\ \rho_{2,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Z}_\ell)$$

which are unramified outside $\ell\lambda$, where $\lambda := f(E_1)f(E_2)$. Let M be the image of $\mathbb{Z}_\ell[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$ under the map

$$\rho_{1,\ell} \times \rho_{2,\ell} : \mathbb{Z}_\ell[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})] \to M_2(\mathbb{Z}_\ell) \times M_2(\mathbb{Z}_\ell).$$

Consider the submodule N generated by the images of Frob_p for $p \leq B_\ell$, a bound to be specified. We want to find an explicit B_ℓ so that $M = N$. For this purpose, consider the $\mathbb{Z}_\ell[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module $R = M/N$. We want to show that $R = 0$.

As $\mathbb{Z}_\ell$ is Noetherian and $M_2(\mathbb{Z}_\ell) \times M_2(\mathbb{Z}_\ell)$ is finitely generated, so is any submodule. In particular, M is finitely generated and so is R.

Thus to show that $R = 0$, it suffices by Nakayama's lemma to show that $R/\ell R = 0$. In other words, it suffices to show that $M/\ell M$ is generated by the images of Frob_p for $p \leq B_\ell$.

Let K_ℓ be the fixed field of the kernel of

$$\overline{\rho_{1,\ell}} \times \overline{\rho_{2,\ell}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z}).$$

This representation is unramified outside $\ell\lambda$. By Hensel's inequality [Serre 1981, p. 129],

$$\log |d_{K_\ell}| \leq c_8 \ell^8 \log(\ell\lambda),$$

where d_{K_ℓ} is the discriminant of K_ℓ over $\mathbb{Q}$ and $c_8 > 0$ is an absolute constant. Let $f := \max(f(E_1), f(E_2))$. Then

$$\log |d_{K_\ell}| \leq 2c_8 \ell^8 \log(\ell f).$$

Since K_ℓ is Galois, using the effective Chebotarev density theorem for $K_\ell/\mathbb{Q}$ (see [Lagarias and Odlyzko 1977, pages 413 and 461]), assuming GRH, we get that every conjugacy class of $\mathrm{Gal}(K_\ell/\mathbb{Q})$ is Frob_p for some prime p satisfying

$$p \leq c_9 (\log |d_{K_\ell}|)^2.$$

Here $c_9 > 0$ is an absolute and effective constant. Hence we can take

$$B_\ell = c_9 (2c_8 \ell^8 \log(\ell f))^2 = 4c_{10} \ell^{16} (\log(\ell f))^2,$$

where $c_{10} > 0$ is an absolute constant. In particular, taking $\ell = 2$, this gives

$$B_2 = 2^{18} c_{10} (\log(2f))^2.$$

Now choose $c_7 := 2^{18} c_{10}$ to get the desired result. □

Proposition 3.2. *Let E_1 and E_2 be two elliptic curves defined over $\mathbb{Q}$ that are not isogenous. Then for any $\epsilon > 0$, there exists a prime q of good reduction for both E_1 and E_2 with*

$$q \leq c(\epsilon)(\max f(E_1), f(E_2))^{1+\epsilon},$$

such that

$$a_q(E_1) \neq a_q(E_2).$$

Here $c(\epsilon) > 0$ depends only on ϵ, and $f(E_1)$, $f(E_2)$ denote the conductors of E_1 and E_2 respectively.

Proof. Since E_1 and E_2 are defined over $\mathbb{Q}$, we know by the extensive work of Wiles and Breuil, Conrad, Diamond and Taylor [Breuil et al. 2001] that there are cuspidal eigenforms f_1 and f_2 of weight 2 and level $f(E_1)$, $f(E_2)$ respectively, with Fourier expansions

$$f_i(z) = \sum_{n\geq 1} a_n(f_i)\exp(2\pi\sqrt{-1}nz), \quad i = 1, 2,$$

where $a_n(f_i) \in \mathbb{Z}$ and $a_1(f_i) = 1$, and such that for all primes q, we have that $a_q(f_i) = a_q(E_i)$. Using the Rankin–Selberg method, Lau and Wu [2008] proved that for any $\epsilon > 0$, there exists $c(\epsilon) > 0$ and a prime q not dividing the level of f_1 or f_2 for which $a_q(f_1) \neq a_q(f_2)$ and $q \leq c(\epsilon)(\max(f(E_1), f(E_2))^{1+\epsilon}$. This means that q is a prime of good reduction for both E_1 and E_2 such that $a_q(E_1) \neq a_q(E_2)$. □

Remark. We could have used the argument of the previous proposition and in this case, we would have obtained a bound $\ll (\max(f(E_1), f(E_2)))^A$ for some absolute and effective constant $A > 0$. This bound is weaker than the above result but has the advantage that it would hold more generally.

4. Proof of Theorem 2.1

Let M and N be sufficiently large such that $M < N \leq 2M$ and consider the curves E_N and E_M. If E_N is not isogenous to E_M, then by Proposition 3.1, there exists a prime q with

$$q \leq c_7(\log[2 \cdot \max(f(E_N), f(E_M))])^2 \tag{5}$$

such that $a_q(E_N) \neq a_q(E_M)$. By the given assumption, we have

$$\max(f(E_N), f(E_M)) \leq c_3 \exp(N^{c_4}) \tag{6}$$

for some $c_4 < \frac{1}{2}$. Moreover, as $\Sigma_{S,N} \subset \Sigma_{S,M}$, we must have

$$q \geq N/2. \tag{7}$$

Hence putting (5), (6) and (7) together, we deduce that $N \leq c_{11}N^{2c_4}$, where $c_{11} > 0$ is an absolute constant. This is a contradiction as $c_4 < \frac{1}{2}$.

5. Proof of Theorem 2.2

Let M and N be sufficiently large such that $M < N \leq 2M$ and consider the curves E_N and E_M. If E_N is not isogenous to E_M, then by Proposition 3.2, for any $\epsilon > 0$, there exists a prime q with

$$q \leq c(\epsilon)(\max(f(E_N), f(E_M)))^{1+\epsilon} \tag{8}$$

such that $a_q(E_N) \neq a_q(E_M)$. By the given assumption, we have

$$\max(f(E_N),\, f(E_M)) \leq c_5 N^{c_6} \tag{9}$$

for some $c_6 < 1$. Moreover, as $\Sigma_{S,N} \subset \Sigma_{S,M}$, we must have

$$q \geq N/2. \tag{10}$$

Hence putting (8), (9) and (10) together, we deduce that

$$N \leq c_2(\epsilon) N^{c_6(1+\epsilon)},$$

where $c_2(\epsilon) > 0$ is a constant which depends on ϵ. Now choosing $\epsilon < 1/c_6 - 1$, we get a contradiction.

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LOXODROMICS FOR THE CYCLIC SPLITTING COMPLEX AND THEIR CENTRALIZERS

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We show that an outer automorphism acts loxodromically on the cyclic splitting complex if and only if it has a filling lamination and no generic leaf of the lamination is carried by a vertex group of a cyclic splitting. This is the analog for the cyclic splitting complex of Handel–Mosher's theorem on loxodromics for the free splitting complex. We also show that such outer automorphisms have virtually cyclic centralizers.

1. Introduction

The study of the mapping class group of a closed orientable surface S has benefited greatly from its action on the curve complex, $\mathcal{C}(S)$, which was shown to be hyperbolic in [Masur and Minsky 1999]. Curve complexes have been used for bounded cohomology of subgroups of mapping class groups, rigidity results, and myriad other applications.

The outer automorphism group of a finite rank free group $\mathbb{F}$, denoted by $\mathrm{Out}(\mathbb{F})$, is defined as the quotient of $\mathrm{Aut}(\mathbb{F})$ by the inner automorphisms, those which arise from conjugation by a fixed element. Much of the study of $\mathrm{Out}(\mathbb{F})$ draws parallels with the study of mapping class groups. This analogy, however, is far from perfect; there are several $\mathrm{Out}(\mathbb{F})$-complexes that act as analogs for the curve complex. Among them are the free splitting complex $\mathcal{FS}$, the cyclic splitting complex $\mathcal{FZ}$, and the free factor complex $\mathcal{FF}$, all of which have been shown to be hyperbolic [Handel and Mosher 2013a; Mann 2014; Bestvina and Feighn 2014]. Just as curve complexes have yielded useful information about mapping class groups, so too have these complexes furthered our understanding of $\mathrm{Out}(\mathbb{F})$.

The three hyperbolic $\mathrm{Out}(\mathbb{F})$-complexes mentioned above are related via coarse Lipschitz maps, $\mathcal{FS} \to \mathcal{FZ} \to \mathcal{FF}$. The loxodromics for $\mathcal{FF}$ have been identified

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with the set of fully irreducible outer automorphisms [Bestvina and Feighn 2014]. Handel and Mosher [2014] proved that an outer automorphism, ϕ, acts loxodromically on $\mathcal{FS}$ precisely when ϕ has a *filling lamination*, that is, some element of the finite set of laminations associated to ϕ (see [Bestvina et al. 2000]) is not carried by a vertex group of any free splitting. In this paper, we focus our attention on the isometry type of outer automorphisms, considered as elements of $\mathrm{Isom}(\mathcal{FZ})$.

A $\mathcal{Z}$-splitting of $\mathbb{F}$ is a splitting in which edge stabilizers are either trivial or cyclic. The cyclic splitting complex $\mathcal{FZ}$, introduced in [Mann 2014], is defined as follows (see Section 2L): vertices are one-edge $\mathcal{Z}$-splittings of $\mathbb{F}$ and k-simplices correspond to collections of $k+1$ vertices which are compatible with a common $k+1$-edge $\mathcal{Z}$-splitting. In this paper, we determine precisely which outer automorphisms act loxodromically on $\mathcal{FZ}$. Closely related to $\mathcal{Z}$-splittings are the maximally-cyclic splittings, called $\mathcal{Z}^{\max}$-splittings, in which the edge groups are required to be trivial or maximal cyclic (i.e., not contained in a larger cyclic subgroup). The results of this paper also apply to the maximally-cyclic splitting complex $\mathcal{FZ}^{\max}$ which is defined exactly as $\mathcal{FZ}$ except that splittings are required to be in the class $\mathcal{Z}^{\max}$. We will use the notation $\mathcal{FZ}^{(\max)}$ to mean either $\mathcal{FZ}$ or $\mathcal{FZ}^{\max}$.

In [Bestvina et al. 2000], the authors associate to each $\phi \in \mathrm{Out}(\mathbb{F})$ a finite set of attracting laminations, denoted by $\mathcal{L}(\phi)$. We say that a lamination $\Lambda \in \mathcal{L}(\phi)$ is $\mathcal{Z}^{(\max)}$*-filling* if no generic leaf (see Section 2N for definitions) of Λ is carried by a vertex group of a one-edge $\mathcal{Z}^{(\max)}$-splitting; we say that ϕ has a $\mathcal{Z}^{(\max)}$-filling lamination if some element of $\mathcal{L}(\phi)$ is $\mathcal{Z}^{(\max)}$-filling. We prove

Theorem 1.1. *For a free group of rank at least* 3, *an outer automorphism ϕ acts loxodromically on $\mathcal{FZ}^{(\max)}$ if and only if it has a $\mathcal{Z}^{(\max)}$-filling lamination. Furthermore*, *if ϕ has a filling lamination which is not $\mathcal{Z}^{(\max)}$-filling*, *then a power of ϕ fixes a point in $\mathcal{FZ}^{(\max)}$.*

Horbez and Wade [2015] showed that every isometry of $\mathcal{FZ}^{(\max)}$ is induced by an outer automorphism. Combining their result with [Handel and Mosher 2014, Theorem 1.1] and Theorem 1.1, this amounts to a classification of the isometries of $\mathcal{FZ}^{(\max)}$.

Corollary 1.2 (classification of isometries). *For all $\phi \in \mathrm{Isom}(\mathcal{FZ}^{(\max)})$ we have that*:

(1) *The action of ϕ on $\mathcal{FZ}^{(\max)}$ is loxodromic if and only if some element of $\mathcal{L}(\phi)$ is $\mathcal{Z}^{(\max)}$-filling.*

(2) *If the action of ϕ on $\mathcal{FZ}^{(\max)}$ is not loxodromic*, *then it has bounded orbits* (*there are no parabolic isometries*).

The proof of Theorem 1.1 relies on the description of the boundary of $\mathcal{FZ}^{(\max)}$ due to Horbez [2016]; points in the boundary of $\mathcal{FZ}^{(\max)}$ are equivalence classes of

$\mathcal{Z}^{(\max)}$-averse trees. The proof is carried out as follows. In Section 3, we extend the theory of folding paths to the boundary of Culler and Vogtmann's outer space, $\mathbb{P}\mathcal{O}$, defining a folding path guided by ϕ which is entirely contained in $\partial\mathbb{P}\mathcal{O}$. In Section 4, we show that the limit of the folding path thus constructed is $\mathcal{Z}^{(\max)}$-averse. In Section 5, we show that an outer automorphism with a filling but not a $\mathcal{Z}^{(\max)}$-filling lamination fixes (up to taking a power) a point in $\mathcal{FZ}^{(\max)}$ and conclude with a proof of Theorem 1.1.

The remainder of the paper is devoted to a study of the centralizers of automorphisms with filling laminations. We prove the following result:

Theorem 1.3. *If an outer automorphism ϕ has a $\mathcal{Z}$-filling lamination, then its centralizer in* $\mathrm{Out}(\mathbb{F})$ *is virtually cyclic. Conversely, if ϕ has a filling but not a $\mathcal{Z}$-filling lamination, then the centralizer of some power of ϕ in* $\mathrm{Out}(\mathbb{F})$ *is not virtually cyclic.*

The key tools used to prove Theorem 1.3 are the completely split train tracks introduced in [Feighn and Handel 2011] and the disintegration theory for outer automorphisms developed in [Feighn and Handel 2009]. We first show (Proposition 7.3) that the disintegration of any outer automorphism ϕ, that has a $\mathcal{Z}$-filling lamination, is virtually cyclic. Then we show that Proposition 7.3 implies the centralizer of ϕ is also virtually cyclic. Conversely, in Proposition 7.11, we show that if ϕ has a filling lamination that is not $\mathcal{Z}$-filling, then ϕ commutes with an appropriately chosen partial conjugation.

The method used to prove Theorem 1.3 provides an alternate (and simple) proof of the well-known fact, due to Bestvina, Feighn and Handel, that centralizers of fully irreducible outer automorphisms are virtually cyclic. In [Bestvina et al. 2000], the stretch factor homomorphism is used to show that the stabilizer of the lamination of a fully irreducible outer automorphism is virtually cyclic, which implies that the centralizer is also virtually cyclic. In general, little is known about the centralizers of outer automorphisms. Rodenhausen and Wade [2015] described an algorithm to find a presentation of the centralizer of a Dehn Twist automorphism. Feighn and Handel [2009] showed that the disintegration of an outer automorphism $\mathcal{D}(\phi)$ is contained in the weak center of the centralizer of ϕ. Recently, Algom-Kfir and Pfaff [2017] showed that centralizers of fully irreducible outer automorphisms with lone axes are isomorphic to $\mathbb{Z}$. We also mention a result of Kapovich and Lustig [2011]: automorphisms whose limiting trees are free have virtually cyclic centralizers.

The main motivation for examining the centralizers of loxodromic elements of $\mathcal{FZ}$ (and $\mathcal{FS}$) is to understand which automorphisms have the potential to be WPD elements for the action of $\mathrm{Out}(\mathbb{F})$ on these complexes.

Corollary 1.4. *Any outer automorphism that is loxodromic for the action of* $\mathrm{Out}(\mathbb{F})$ *on $\mathcal{FS}$ but elliptic for the action on $\mathcal{FZ}$ is not a WPD element for the action on $\mathcal{FS}$.*

The result that centralizers of loxodromic elements of $\mathcal{FZ}$ are virtually cyclic is a promising sign for the following conjecture:

Conjecture 1.5. The action of $\mathrm{Out}(\mathbb{F})$ on $\mathcal{FZ}$ is a WPD action. That is, every loxodromic element for the action satisfies WPD.

2. Preliminaries

Before proceeding, we fix a free group $\mathbb{F}$ of rank ≥ 3.

2A. ***Isometries of metric spaces.*** Let X be a Gromov hyperbolic metric space. We say that an infinite order isometry g of X is *loxodromic* if it acts with positive translation length on X: $\lim_{N\to\infty}(d(x, g^N(x))/N) > 0$ for some $x \in X$. Every loxodromic element has exactly two limit points in the Gromov boundary of X.

Given a group G acting by isometries on the hyperbolic space X, we denote by $\Lambda_X G$ the limit set of G in $\partial_\infty X$, which is defined as the intersection of $\partial_\infty X$ with the closure of the orbit of any point in X under the G-action. The following theorem, essentially due to Gromov, and formulated here for the case that G is cyclic, gives a classification of isometry groups of (possibly nonproper) Gromov hyperbolic spaces. A sketch of a proof can be found in [Caprace et al. 2015, Proposition 3.1].

Theorem 2.1 [Gromov 1987, Section 8.2]. *Let X be a hyperbolic geodesic metric space, and let G be a cyclic group acting by isometries on X. Then G is either*

- *bounded, i.e., all G-orbits in X are bounded; in this case $\Lambda_X G = \varnothing$, or*
- *horocyclic, i.e., G is not bounded and contains no loxodromic element; in this case $\Lambda_X G$ is reduced to one point, or*
- *lineal, i.e., G contains a loxodromic element, and any two loxodromic elements have the same fixed points in $\partial_\infty X$; in this case $\Lambda_X G$ consists of these two points.*

2B. ***Outer space and its compactification.*** Culler–Vogtmann *outer space*, $\mathbb{P}\mathcal{O}$, is defined in [Culler and Vogtmann 1986] as the space of simplicial, free, and minimal isometric actions of $\mathbb{F}$ on simplicial metric trees up to $\mathbb{F}$-equivariant homothety. We denote by $\mathcal{O}$ the *unprojectivized outer space*, in which the trees are considered up to isometry, rather than homothety. Each of these spaces is equipped with a natural (right) action of $\mathrm{Out}(\mathbb{F})$.

An $\mathbb{F}$-tree is an $\mathbb{R}$-tree with an isometric action of $\mathbb{F}$. An $\mathbb{F}$-tree is called *very small* if the action is minimal, arc stabilizers are either trivial or maximal cyclic, and tripod stabilizers are trivial. Outer space can be mapped into $\mathbb{R}^{\mathbb{F}}$ by the map $T \mapsto (\|g\|_T)_{g\in\mathbb{F}}$, where $\|g\|_T$ denotes the translation length of g in T. This was shown in [Culler and Morgan 1987] to be a continuous injection. The closure of the image of $\mathbb{P}\mathcal{O}$ under this embedding is compact and was identified in [Bestvina and

Feighn 1992] and [Cohen and Lustig 1995] with the space of very small $\mathbb{F}$-trees. We denote by $\overline{\mathbb{P}\mathcal{O}}$ the closure of outer space in $\mathbb{P}\mathbb{R}^{\mathbb{F}}$ and by $\partial\mathbb{P}\mathcal{O}$ its boundary. We will denote the preimage of $\overline{\mathbb{P}\mathcal{O}}$ in $\mathbb{R}^{\mathbb{F}}$ by $\overline{\mathcal{O}}$.

2C. ***Free factor system.*** A free factor system of $\mathbb{F}$ is a finite collection of conjugacy classes of proper free factors of $\mathbb{F}$ of the form $\mathcal{A} = \{[A_1], \dots, [A_k]\}$, where $k \geq 0$ and $[\cdot]$ denotes the conjugacy class of a subgroup, such that there exists a free factorization $\mathbb{F} = A_1 * \cdots * A_k * F_N$. We refer to the free factor F_N as the *cofactor* of $\mathcal{A}$, keeping in mind that it is not unique, even up to conjugacy.

The main geometric example of a free factor system is as follows: suppose G is a marked graph and K is a subgraph whose noncontractible connected components are denoted $C_1, \dots, C_k$. Let $[A_i]$ be the conjugacy class of a free factor of $\mathbb{F}$ determined by $\pi_1(C_i)$. Then $\mathcal{A} = \{[A_1], \dots, [A_k]\}$ is a free factor system. We say $\mathcal{A}$ is *realized by* K and we denote it by $\mathcal{F}(K)$.

2D. ***Marked graphs.*** We recall some basic definitions from [Bestvina and Handel 1992]. Identify $\mathbb{F}$ with $\pi_1(\mathcal{R}, *)$ where $\mathcal{R}$ is a rose with $\mathfrak{n}$ petals, $\mathfrak{n}$ being the rank of $\mathbb{F}$. A *marked graph* G is a graph of rank $\mathfrak{n}$, all of whose vertices have valence at least three, equipped with a homotopy equivalence $m : \mathcal{R} \to G$ called a *marking*. The marking determines an identification of $\mathbb{F}$ with $\pi_1(G, m(*))$. A homotopy equivalence $f : G \to G$ induces an outer automorphism of $\pi_1(G)$ and hence an element ϕ of $\mathrm{Out}(\mathbb{F})$. If f sends vertices to vertices and the restriction of f to edges is an immersion then we say that f is a *topological representative* of ϕ.

2E. ***Paths, circuits, and tightening.*** Let Γ be either a marked graph or an $\mathbb{F}$-tree. A *path* in Γ is either an isometric immersion of a (possibly infinite) closed interval $\sigma : I \to \Gamma$ or a constant map $\sigma : I \to \Gamma$. If σ is a constant map, the path will be called *trivial*. If I is finite, then any map $\sigma : I \to \Gamma$ is homotopic rel endpoints to a unique path $[\sigma]$. We say that $[\sigma]$ is obtained by *tightening* σ. If $f : \Gamma \to \Gamma$ is continuous and σ is a path in Γ, we define $f_\#(\sigma)$ as $[f(\sigma)]$. If the domain of σ is finite and Γ is either a graph or a simplicial tree, then the image has a natural decomposition into edges $E_1 E_2 \cdots E_k$ called the *edge path associated to* σ. If Γ is a tree, we may use $[x, x']$ to denote the unique geodesic path connecting x and x'.

A *circuit* is an immersion $\sigma : S^1 \to \Gamma$. For any path or circuit, let $\overline{\sigma}$ be σ with its orientation reversed. A decomposition of a path or circuit into subpaths is a *splitting* for $f : \Gamma \to \Gamma$ and is denoted $\sigma = \cdots \sigma_1 \cdot \sigma_2 \cdots$ if $f^k_\#(\sigma) = \cdots f^k_\#(\sigma_1) f^k_\#(\sigma_2) \cdots$ for all $k \geq 1$.

2F. ***Turns, directions and train track structure.*** Let Γ be an $\mathbb{F}$-tree. A direction d based at $p \in \Gamma$ is a component of $\Gamma - \{p\}$. A *turn* is an unordered pair of directions based at the same point. In the case that Γ is a simplicial tree, and p is a vertex, we identify directions at p with edges emanating from p. An *illegal turn structure* on Γ is an equivalence relation on the set of directions at each point $p \in \Gamma$. The classes of this relation are called *gates*. A turn (d, d') is *legal* if d and d' do not belong to the same gate. If in addition there are at least two gates at every vertex of Γ, then the illegal turn structure is called a *train track structure*. A path is legal if it only crosses legal turns.

2G. *Optimal morphism.* Given two $\mathbb{F}$-trees Γ and Γ', an $\mathbb{F}$-equivariant map $f : \Gamma \to \Gamma'$ is called a *morphism* if every segment of Γ can be subdivided into finitely many subintervals onto which f restricts to an isometric embedding. A morphism between $\mathbb{F}$-trees induces an illegal turn structure on the domain Γ as follows: for every $x \in \Gamma$, the map f determines a map $Df_x : D_x \to D_{f(x)}$, on the set of directions D_x at x. For $d, d' \in D_x$, we then declare $d \sim d'$ if $D(f^k)(d) = D(f^k)(d')$ for some $k \geq 0$. A morphism is called *optimal* if there are at least two gates at each point of Γ. A morphism f that induces a train track structure is an optimal morphism.

The map f is called *alignment preserving* (or a *collapse map*) if the f-image of every segment in Γ is a segment in Γ'.

2H. *Train track maps.* An optimal morphism is called a *train track map* if $f : \Gamma \to \Gamma'$ is an embedding on each edge and maps legal turns to legal turns. In particular, legal paths map to legal paths. Note that usually the term *train track map* is used for self maps, but Bestvina and Feighn [2014] defined it for a map between different $\mathbb{F}$-trees, each equipped with its own abstract train track structure.

The terminology can also be extended to graphs by passing to their universal covers. For more details on train track maps, the reader is referred to [Bestvina and Feighn 2014; Bestvina and Handel 1992].

2I. *Relative train track maps and CTs.* A *filtration* for a topological representative $f : G \to G$ of an outer automorphism ϕ, where G is a marked graph, is an increasing sequence of f-invariant subgraphs $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_M = G$. We let $H_i = \overline{G_i \setminus G_{i-1}}$ and call H_i the *i-th stratum*. A turn with one edge in H_i and the other in G_{i-1} is called *mixed* while a turn with both edges in H_i is called a *turn in* H_i. If $\sigma \subset G_i$ does not contain any illegal turns in H_i, then we say σ is *i-legal*.

We denote by M_i the submatrix of the transition matrix for f obtained by deleting all rows and columns except those labeled by edges in H_i. For the topological representatives that will be of interest to us, the transition matrices M_i will come in three flavors: M_i may be a zero matrix, it may be the 1×1 identity matrix, or it may be an irreducible matrix with Perron–Frobenius eigenvalue $\lambda_i > 1$. We will call H_i a *zero* (Z), *nonexponentially growing* (NEG), or *exponentially growing* (EG) stratum, respectively. Any stratum which is not a zero stratum is called an *irreducible stratum*.

Definition 2.2 [Bestvina and Handel 1992]. We say that $f : G \to G$ is a *relative train track map* representing $\phi \in \mathrm{Out}(F_n)$ if for every exponentially growing stratum H_r, the following hold:

(RTT i) Df maps the set of oriented edges in H_r to itself; in particular all mixed turns are legal.

(RTT ii) If $\sigma \subset G_{r-1}$ is a nontrivial path with endpoints in $H_r \cap G_{r-1}$, then so is $f_\#(\sigma)$.

(RTT iii) If $\sigma \subset G_r$ is r-legal, then $f_\#(\sigma)$ is r-legal.

Suppose that $u < r$, that H_u is irreducible, H_r is EG and each component of G_r is noncontractible, and that for each $u < i < r$, H_i is a zero stratum which is a component of G_{r-1} and each vertex of H_i has valence at least two in G_r. Then we say that H_i is *enveloped by* H_r and we define $H_r^z = \bigcup_{k=u+1}^{r} H_k$.

A path or circuit σ in a representative $f : G \to G$ is called a *periodic Nielsen path* if $f_\#^k(\sigma) = \sigma$ for some $k \geq 1$. If $k = 1$, then σ is a *Nielsen path.* A Nielsen path is *indivisible*, denoted INP, if it cannot be written as a concatenation of nontrivial Nielsen paths. If w is a closed root-free Nielsen path and E_i is an edge such that $f(E_i) = E_i w^{d_i}$, then we say E_i *is a linear edge* and we call w the *axis* of E. If E_i, E_j are distinct linear edges with the same axis such that $d_i \neq d_j$ and $d_i, d_j > 0$, then we call a path of the form $E_i w^* \bar{E}_j$ an *exceptional path.* We say that x and y are *Nielsen equivalent* if there is a Nielsen path σ in G whose endpoints are x and y. We say that a periodic point $x \in G$ is *principal* if neither of the following conditions hold:

- x is an endpoint of a nontrivial periodic Nielsen path and there are exactly two periodic directions at x, both of which are contained in the same EG stratum.
- x is contained in a component C of periodic points that is topologically a circle and each point in C has exactly two periodic directions.

A relative train track map f is called *rotationless* if each principal periodic vertex is fixed and if each periodic direction based at a principal vertex is fixed.

For an EG stratum, H_r, we call a nontrivial path $\sigma \subset G_{r-1}$ with endpoints in $H_r \cap G_{r-1}$ a *connecting path for* H_r. Let E be an edge in an irreducible stratum, H_r, and let σ be a maximal subpath of $f_\#^k(E)$ in a zero stratum for some $k \geq 1$. Then we say that σ is *taken.* A nontrivial path or circuit σ is called *completely split* if it has a splitting $\sigma = \tau_1 \cdot \tau_2 \cdots \tau_k$ where each of the τ_i's is a single edge in an irreducible stratum, an indivisible Nielsen path, an exceptional path, or a connecting path in a zero stratum which is both maximal and taken. We say that a relative train track map is *completely split* if $f(E)$ is completely split for every edge E in an irreducible stratum *and* if for every taken connecting path σ in a zero stratum, $f_\#(\sigma)$ is completely split.

The following theorem/definition is the main existence result for CTs:

Theorem 2.3 [Feighn and Handel 2011, Theorem 4.28; 2009, Corollary 3.5]. *There exists $k > 0$ depending only on $\mathtt{n}$, so that given any $\phi \in \mathrm{Out}(\mathbb{F})$ and any nested sequence of ϕ^k-invariant free factor systems, there is a **completely split improved relative train track map** (CT for short) $f : G \to G$ representing ϕ^k such that each free factor system is realized by some filtration element. The map f satisfies the following properties*:

• (*rotationless*) *$f : G \to G$ is rotationless.*

• (*completely split*) *$f : G \to G$ is completely split.*

• (*filtration*) *$\mathcal{F}$ is reduced. The core of each filtration element is a filtration element.*

• (*vertices*) *The endpoints of all indivisible periodic* (*necessarily fixed*) *Nielsen paths are* (*necessarily principal*) *vertices. The terminal endpoint of each nonfixed NEG edge is principal* (*and hence fixed*).

• (*periodic edges*) *Each periodic edge is fixed and each endpoint of a fixed edge is principal. If the unique edge E_r in a fixed stratum H_r is not a loop then G_{r-1} is a core graph and both ends of E_r are contained in G_{r-1}.*

• (*zero strata*) *If H_i is a zero stratum, then H_i is enveloped by an EG stratum H_r, each edge in H_i is r-taken and each vertex in H_i is contained in H_r and has link contained in $H_i \cup H_r$.*

• (*linear edges*) *For each linear E_i there is a closed root-free Nielsen path w_i such that $f(E_i) = E_i w_i^{d_i}$ for some $d_i \neq 0$. If E_i and E_j are distinct linear edges with the same axes then $w_i = w_j$ and $d_i \neq d_j$.*

• (*NEG Nielsen paths*) *If the highest edges in an indivisible Nielsen path σ belong to an NEG stratum then there is a linear edge E_i with w_i as in* (*linear edges*) *and there exists $k \neq 0$ such that $\sigma = E_i w_i^k \overline{E}_i$. Moreover, if ϕ is rotationless in the sense of* [*Feighn and Handel 2011*], *then we may take $k = 1$.*

It follows directly from the definitions that, for completely split paths and circuits, all cancellation under iteration of $f_\#$ is confined to the individual terms of the splitting. Moreover, $f_\#(\sigma)$ has a complete splitting which refines that of σ. Finally, just as with improved relative train track maps introduced in [Bestvina et al. 2000], every circuit or path with endpoints at vertices eventually is completely split [Feighn and Handel 2011, Lemma 4.25]. The reader is directed to [Feighn and Handel 2011, §4] for many useful properties of CTs that we will use frequently in the sequel, often without a specific reference.

2J. ***Bounded backtracking*** **(*BBT*).** Let $f : T \to T'$ be a continuous map between two $\mathbb{R}$-trees T and T'. We say that f has bounded backtracking if the f image of any path $[p, q]$ is contained in a C-neighborhood of $[f(p), f(q)]$. The smallest such C is called the *bounded backtracking constant* of f, denoted $\mathrm{BBT}(f)$.

2K. ***Folding paths.*** Given simplicial $\mathbb{F}$-trees T and T' and an optimal morphism $f : T \to T'$ Guirardel and Levitt [2007b, Section 3] construct a *canonical optimal folding path* $(T_t)_{t\in\mathbb{R}^+}$ guided by f. The tree T_t is constructed as follows. Given $a, b \in T$ with $f(a) = f(b)$, the *identification time* of a and b is defined as $\tau(a, b) = \sup_{x\in[a,b]} d_{T'}(f(x), f(a))$. Define $L := \frac{1}{2}\,\mathrm{BBT}(f)$. For each $t \in [0, L]$, one defines an equivalence relation $\sim_t$ by $a \sim_t b$ if $f(a) = f(b)$ and $\tau(a, b) < t$. The tree T_t is then a quotient of T by the equivalence relation $\sim_t$. Guirardel and Levitt prove that for each $t \in [0, L]$, T_t is an $\mathbb{R}$-tree. The collection of trees $(T_t)_{t\in[0,L]}$ comes equipped with $\mathbb{F}$-equivariant morphisms $f_{s,t} : T_t \to T_s$ for all $t < s$ and these maps satisfy the semiflow property: for all $r < s < t$, we have $f_{t,s} \circ f_{s,r} = f_{t,r}$. Moreover $T_L = T'$ and $f_{L,0} = f$. The trees $(T_t)_{t\in[0,L]}$ and the maps $(f_{s,t} : T_t \to T_s)_{t<s\in[0,L]}$ are called the *connection data* for the folding path.

2L. ***The $\mathcal{Z}$-splitting complex.*** Let $\mathcal{Z}$ be the collection of subgroups of $\mathbb{F}$ that are either trivial or cyclic. We denote by $\mathcal{Z}^{\max}$ the collection of elements of $\mathcal{Z}$ which are either trivial or closed under taking roots. We use the notation $\mathcal{Z}^{(\max)}$ to mean either $\mathcal{Z}$ or $\mathcal{Z}^{\max}$. A $\mathcal{Z}^{(\max)}$*-splitting* is a minimal, simplicial $\mathbb{F}$-tree whose edge stabilizers belong to the set $\mathcal{Z}^{(\max)}$; it is a *one-edge splitting* if there is one $\mathbb{F}$ orbit of edges. A *cyclic splitting* (resp. maximally-cyclic splitting) is a one-edge $\mathcal{Z}$-splitting (resp. $\mathcal{Z}^{\max}$-splitting) whose edge stabilizer is infinite cyclic. Two $\mathcal{Z}^{(\max)}$-splittings are *equivalent* if the corresponding Bass–Serre trees are $\mathbb{F}$-equivariantly homeomorphic. We will often blur the distinction between a splitting and its Bass–Serre tree.

If S is a one-edge free splitting (resp. $\mathcal{Z}^{(\max)}$-splitting) and v is a vertex in the Bass–Serre tree, then $\mathrm{Stab}(v)$ will be called a *vertex group* of S. Vertex groups of free splittings are free factors.

Given two $\mathcal{Z}^{(\max)}$-splittings $\overline{T}$ and T, we say that $\overline{T}$ is a *refinement* of T if there is a collapse map from $\overline{T}$ to T. Two $\mathcal{Z}^{(\max)}$-splittings T and T' are *compatible* if they have a common refinement, i.e., if there exists a tree that collapses onto both T and T'. A tree T is $\mathcal{Z}^{(\max)}$*-incompatible* if the set of $\mathcal{Z}^{(\max)}$-splittings compatible with T is empty. The (maximally-) cyclic splitting complex $\mathcal{FZ}^{(\max)}$ is the simplicial complex whose vertices are equivalence classes of one-edge $\mathcal{Z}^{(\max)}$-splittings and whose k-simplices are collections of $k+1$ pairwise compatible one-edge $\mathcal{Z}^{(\max)}$-splittings. Mann [2014] showed that $\mathcal{FZ}$ is δ-hyperbolic. More recently, Horbez [2016] used the same argument to prove that $\mathcal{FZ}^{\max}$ is δ-hyperbolic.

The results of Shenitzer [1955], Stallings [1991] and Swarup [1986] imply that every one-edge cyclic splitting of $\mathbb{F}$ is obtained from a one-edge free splitting of $\mathbb{F}$ by the "edge folding" process described as follows. Let T be a free splitting of $\mathbb{F}$, let v be a vertex of T and let G_v be its stabilizer. Consider $w \in G_v$ and $\langle w\rangle$, the cyclic group generated by w. Construct a new $\mathbb{F}$-tree T' by first choosing an edge e incident at v, then, for every $\gamma \in \mathbb{F}$, identifying γe with its orbit under $\langle \gamma w \gamma^{-1} \rangle \subseteq G_{\gamma v}$. The tree T' has an edge with stabilizer equal to $\langle w \rangle$. We say T' is obtained from T by an equivariant *edge fold*, or to be more specific, we sometimes say that T' is obtained from T by performing the *edge fold corresponding to* $\langle w \rangle$.

2M. ***$\mathcal{Z}$-averse trees and boundary of $\mathcal{FZ}$.*** A tree T in $\overline{\mathcal{O}}$ is called $\mathcal{Z}^{(\max)}$-averse [Horbez 2016, Definition 4.2] if there is no finite chain of compatibility between T and a $\mathcal{Z}^{(\max)}$-splitting: i.e., if there is no finite set of trees $(T = T_0, T_1, \ldots, T_k = T')$ in $\overline{\mathcal{O}}$ such that T' is a $\mathcal{Z}^{(\max)}$-splitting and for each $i \in \{0, \ldots, k-1\}$, the trees T_i and T_{i+1} are compatible. Two $\mathcal{Z}^{(\max)}$-averse trees, T, T', are called *equivalent* if there is a finite chain of compatible trees in $\overline{\mathcal{O}}$ relating T to T' as above. The reader will note that the notions of $\mathcal{Z}^{(\max)}$-compatibility and $\mathcal{Z}^{(\max)}$-aversity are independent of the homothety class of T; in particular, it makes sense to say that a tree in $\overline{\mathbb{P}\mathcal{O}}$ is $\mathcal{Z}$-averse, or that two trees in $\overline{\mathbb{P}\mathcal{O}}$ are equivalent. We denote by $\mathcal{X}^{(\max)}$ (resp. $\mathbb{P}\mathcal{X}^{(\max)}$) the subspace of $\overline{\mathcal{O}}$ (resp. $\overline{\mathbb{P}\mathcal{O}}$) consisting of $\mathcal{Z}^{(\max)}$-averse trees.

There is a natural map from a subset of $\partial\mathbb{P}\mathcal{O}$ to the Gromov boundary of $\mathcal{FZ}^{(\max)}$ relating the geometries at infinity of these two spaces, which we now describe. There is a map $\psi^{(\max)} : \mathbb{P}\mathcal{O} \to \mathcal{FZ}^{(\max)}$, which extends to the set of simplicial trees in $\overline{\mathcal{O}}$ with trivial edge stabilizers, defined by choosing a one-edge collapse of every simplicial tree in $\mathbb{P}\mathcal{O}$. This map is not quite $\mathrm{Out}(\mathbb{F})$-equivariant because we must make choices, however differing choices change distances by at most 2. The following theorem due to Horbez describes the boundary of the free splitting complex.

Theorem 2.4 [Horbez 2016, Theorem 0.1]. *There is a unique* $\mathrm{Out}(\mathbb{F})$*-equivariant homeomorphism*

$$\partial\psi^{(\max)} : \mathcal{X}^{(\max)}/\sim \quad \longrightarrow \quad \partial_\infty \mathcal{FZ}^{(\max)}$$

so that for all $T \in \mathcal{X}^{(\max)}$ and all sequences $(T_n) \in \mathcal{O}^{\mathbb{N}}$ converging to T, the sequence $(\psi^{(\max)}(T_n))_{n\in\mathbb{N}}$ converges to $\psi(T)$.

Given a tree $T \in \overline{\mathcal{O}}$, a $\mathcal{Z}^{(\max)}$-splitting S is called a *reducing splitting* for T, if S is compatible with some $T' \in \overline{\mathcal{O}}$, which is itself compatible with T.

2N. ***Lines and laminations.*** We briefly recall some definitions, but the reader is directed to [Bestvina et al. 2000] for details. The *space of abstract lines*, $\widetilde{\mathcal{B}} = (\partial\mathbb{F} \times \partial\mathbb{F} - \Delta)/\mathbb{Z}_2$, is the set of unordered distinct pairs of points in the boundary of $\mathbb{F}$ and is equipped with the natural (subspace/product/quotient) topology. The quotient of $\widetilde{\mathcal{B}}$ by the natural $\mathbb{F}$ action is *the space of lines in* $\mathcal{R}$ and is called $\mathcal{B}$. It is endowed with the quotient topology, which satisfies none of the separation axioms. Points in $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ will be called lines.

A closed subset Λ of $\mathcal{B}$ is an *attracting lamination for* ϕ if it is the closure of a single line β that is *bireccurrent* (every finite subpath σ of β occurs infinitely many times as an unoriented subpath of each end of β), has an *attracting neighborhood* (there is some open $U \ni \beta$ so that $\phi^k(\gamma) \to \beta$ for all $\gamma \in U$), and is not carried by a rank one ϕ-periodic free factor. The lines in Λ satisfying the above properties are called the *generic leaves* of Λ.

A subgroup A of $\mathbb{F}$ determines a subset of the boundary of $\mathbb{F}$ called $\partial A \subset \partial\mathbb{F}$. We say that A *carries* a line β if there is some lift $\widetilde{\beta}$ whose endpoints are in ∂A. We then say that the A *carries* the lamination Λ if A carries some (any) generic leaf of Λ. A lamination Λ is said to be *filling* (resp. $\mathcal{Z}^{(\max)}$*-filling*) if Λ is not carried by any vertex group of any free splitting (resp. $\mathcal{Z}^{(\max)}$-splitting).

Let $\pi_A : G_A \to \mathcal{R}$ be the immersion from the core of the cover of $\mathcal{R}$ corresponding to the subgroup A and let β be a line. Then clearly β is carried by A if and only if there exist immersions $\rho_A : \mathbb{R} \to G_A$ and $\rho : \mathbb{R} \to \mathcal{R}$ such that $\rho = \pi_A \rho_A$. If we further assume that A is finitely generated, it's easy to see that β is carried by A if and only if every finite subsegment of β can be immersed into G_A.

3. Folding in the boundary of outer space

Throughout this section, ϕ will be an outer automorphism with a $\mathcal{Z}^{(\max)}$-filling lamination Λ^+_ϕ. Our first goal is to extract from ϕ a folding path converging to a tree in $\partial\mathbb{P}\mathcal{O}$ which "witnesses" the lamination Λ^+_ϕ. The automorphism ϕ is fully irreducible relative to some maximal ϕ-invariant free factor system $\mathcal{A}$. Since ϕ has a filling lamination, $\mathcal{A}$ is not an exceptional free factor system, that is, it is not of the form $\{A\}$ or $\{A_1, A_2\}$, where $\mathbb{F} = A * \mathbb{Z}$ or $\mathbb{F} = A_1 * A_2$. Let $f : T \to T$ be the universal cover of a relative train track representative of ϕ realizing the invariant free factor system $\mathcal{A}$. Let $G = T/\mathbb{F}$ be the quotient graph, which comes with a filtration

$$\varnothing = G_0 \subset G_1 \subset \cdots \subset G_r = G$$

such that $\mathcal{F}(G_{r-1}) = \mathcal{A}$ and H_r is an EG stratum with Perron–Frobenius eigenvalue λ_ϕ. Let T_r (resp. T_{r-1}) denote the full preimage of H_r (resp. G_{r-1}) under the quotient map $T \to G$. We endow G (and hence T) with a metric by declaring all edges to have length 1. We will henceforth consider T as a point in unprojectivized outer space $\mathcal{O}$, whereby f may be thought of as an $\mathbb{F}$-equivariant map $T \to T \cdot \phi$.

Let T_0' be the tree obtained from T by equivariantly collapsing the $\mathcal{A}$-minimal subtree. Our present aim is to construct a folding path ending at $T^+_\phi := \lim_{n\to\infty} T_0'\phi^n/\lambda^n_\phi$. To accomplish this, we will construct simplicial trees T_0, T_1 and define an optimal morphism $f_0 : T_0 \to T_1$. From this we will obtain a periodic canonical optimal folding path $(f_t)_{t\in[0,L]}$ which will end at T^+_ϕ. It is worth noting that the natural map $f_0' : T_0' \to T_0'\phi$ induced by f is neither optimal nor a morphism as there may be nondegenerate intervals which are mapped to points.

We remark that existence of an optimal morphism which is a train track map representing a relative fully irreducible outer automorphism is a special case of the results of [Francaviglia and Martino 2015] and [Meinert 2015], for free products and deformation spaces, respectively. Francaviglia and Martino [2015] developed metric theory for relative outer space for free products which is used to show the existence of optimal maps. This requires a considerable amount of work due to lack of applicability of the Arzela–Ascoli theorem in this setting. In what follows, we provide a shorter proof of existence of a train track map representing ϕ in the context of free groups.

Constructing T_0. The following is based on the construction in the proof of [Bestvina and Handel 1992, Lemma 5.10]. Define a measure μ on T with support contained in the set $\{x \in T_r : f^k(x) \in T_r \text{ for all } k \geq 0\}$ as follows: choose a Perron–Frobenius eigenvector $\vec{v}$ corresponding to the PF eigenvalue λ_ϕ. For an edge e in T_r, let $\mu(e) = v_e$, where v_e is the component of $\vec{v}$ corresponding to e. Define $\mu(e) = 0$ for all edges $e \in T_{r-1}$. Let V be the set of vertices of T and let $V_m := \{x \in T : f^m(x) \in V\}$. Subdividing T at V_m divides each edge into segments that map to edge paths under f^m. If a is such a segment then define $\mu(a) = \mu(f^m(a))/\lambda_\phi^m$. The definition of μ together with the fact that relative train track maps take r-legal paths to r-legal paths implies:

Lemma 3.1. *If $[x, y]$ is an r-legal path in T, then $\mu(f_\#([x, y])) = \lambda_\phi \mu([x, y])$. If $[x, y]$ contains an initial or terminal segment of some edge in T_r, then $\mu([x, y]) > 0$.*

The measure μ defines a pseudometric d_μ on T. Collapsing the sets of μ-measure zero to make d_μ into a metric, we obtain a tree T_0. Let $p : T \to T_0$ be the collapse map.

Lemma 3.2. *T_0 is simplicial.*

Proof. We will show that the $\mathbb{F}$-orbit of any point in T_0 must be discrete. Let $x \in T_0$ and choose a point $\tilde{x} \in p^{-1}(x)$. The $\mathbb{F}$-orbit of $\tilde{x}$ in T is discrete, and to understand the orbit of x, we need only understand $\mu([\tilde{x}, g\tilde{x}])$ for $g \in \mathbb{F}$. If $[\tilde{x}, g\tilde{x}]$ contains no edges in T_r, then $\mu([\tilde{x}, g\tilde{x}]) = 0$, in which case $g \in \mathrm{Stab}(x)$. Otherwise, the segment contains an edge in T_r, and hence has positive μ-measure. Since there are only finitely many $\mathbb{F}$-orbits of edges in T_r, there is a lower bound on the μ-measure of $[\tilde{x}, g\tilde{x}]$. Hence, there is a lower bound on $d_{T_0}(x, gx)$. This concludes the proof. □

The trees T_0 and T_0' are $\mathbb{F}$-equivariantly homeomorphic. The problem with T_0' is that the "obvious" map $f_0' : T_0' \to T_0' \phi$ sends nondegenerate segments to points and, because of that, is not useful for making a folding path. The map f_0 defined in the sequel is an improvement because it can be used to construct a folding path.

Defining $f_0 : T_0 \to T_1$. Let T_1 be the tree $\lambda_\phi^{-1} T_0 \cdot \phi$: the leading coefficient indicates that the metric has been scaled by λ_ϕ^{-1}. The relative train track map $f : T \to T \cdot \phi$

naturally induces a map $f_0 : T_0 \to T_1$. For each $x \in T_0$, its pre-image $p^{-1}(x)$ is a connected subtree of T with μ-measure zero. The definition of μ guarantees that the f-image of this set is also connected and has μ-measure zero. Therefore $p \circ f \circ p^{-1}(x)$ is a single point in $T_0 \cdot \phi$, which is identified with T_1 and we define $f_0 := p \circ f \circ p^{-1}$.

Lemma 3.3. *The map f_0 is an optimal morphism.*

Proof. We first show that f_0 is a morphism, which will follow from the definition of μ and properties of relative train track maps. Given a nondegenerate segment $[x, x']$ in T_0, choose $\tilde{x} \in p^{-1}(x)$ and $\tilde{x}' \in p^{-1}(x')$. The intersection of $[\tilde{x}, \tilde{x}']$ with the vertices of T is a finite set $\{\tilde{x}_1, \dots, \tilde{x}_{k-1}\}$. Let $\tilde{x}_0 := \tilde{x}$ and $\tilde{x}_k := \tilde{x}'$. Taking the p-image of $\tilde{x}_i$ for $i \in \{0, \dots, k\}$ yields a subdivision of $[x, x']$ into finitely many subsegments $[x_i, x_{i+1}]$, some of which may be degenerate. We will ignore the degenerate subdivisions: they occur as the projections of edges in T_{r-1} (all of which have μ-measure zero).

We claim that f_0 is an isometry in restriction to each of these subsegments. Indeed, let $e = [\tilde{x}_i, \tilde{x}_{i+1}]$ be an edge in T. Assume without loss of generality that $x_i \neq x_{i+1}$ so that $\mu(e) \neq 0$ and e is therefore an edge in T_r. It is an immediate consequence of Lemma 3.1 that for each $y \in e$, we have $\mu([f(\tilde{x}_i), f(y)]) = \lambda_\phi \mu([\tilde{x}_i, y])$ and hence f_0 is an isometry in restriction to $[x_i, x_{i+1}]$.

We now address the optimality of f_0. There are three types of points to consider: points in the interior of an edge, vertices with trivial stabilizer, and vertices with nontrivial stabilizer. We have already established that f_0 is an isometry in restriction to edges, so there are two gates at each $x \in T_0$ contained in the interior of an edge. If $x \in T_0$ is a vertex with trivial stabilizer, then $p^{-1}(x)$ is a vertex (Lemma 3.1) contained in $T_r \setminus T_{r-1}$. As f is a relative train track map, there are at least two gates at $p^{-1}(x)$ and each is necessarily contained in T_r. A short path in T containing $p^{-1}(x)$ entering through the first gate and leaving through the second will be legal. Lemma 3.1 gives that f_0 is an isometry in restriction to such a path, so there are at least two gates at x.

Now let $x \in T_0$ be a vertex with nontrivial stabilizer. Then $p^{-1}(x)$ is a subtree which is the inverse image of a component of G_{r-1} under the quotient map $T \to G$. Let $\tilde{x}, \tilde{x}' \in p^{-1}(x)$ be distinct vertices in $T_r \cap T_{r-1}$ and let d (resp. d') be a direction based at $\tilde{x}$ (resp. $\tilde{x}'$) corresponding to an edge e (resp. e') in T_r. Lemma 3.1 provides that d and d' determine distinct directions at x. As mixed turns are legal, the path $\overline{e} \cup [\tilde{x}, \tilde{x}'] \cup e'$ in T is r-legal. A final application of Lemma 3.1 gives that the restriction of f_0 to the p-image of this path is an isometry, and hence that there are at least two directions at x. $\square$

The reader will note that we have proved the following:

Lemma 3.4. *The map f_0 is a train track map.*

As T_0 and T_0' are $\mathbb{F}$-equivariantly homeomorphic, there is a bijection between ($\mathbb{F}$-orbits of) edges of each. It is easily verified that the transition matrix of f_0 and that of f are equal. In particular, we will speak of edges, transition matrices, PF eigenvalues, and related notions for $f_0 : T_0 \to T_1$, without reference to this bijection.

Next, we use f_0 to construct a folding path starting at $S_0 := T_0$. This folding path will converge in $\partial\mathbb{P}\mathcal{O}$ to a tree S_L. We then prove that S_L is in fact the tree T_ϕ^+ as defined above.

Folding T_0. Applying the canonical folding path construction, we obtain a folding path $(S_t)_{t\in[0,L_1]}$ guided by $f_0 : T_0 \to T_1$ which begins at $T_0 = S_0$ and ends at $T_1 = S_{L_1}$, where $L_1 = \frac{1}{2}\,\mathrm{BBT}(f_0)$. Adapting a construction of Handel and Mosher [2011, Section 7.1], we now extend this to a *periodic fold path guided by* f_0. For each $i \in \mathbb{N}$, let $T_i = \lambda_\phi^{-i} T_0 \cdot \phi^i$, whence we have optimal morphisms $f_i : T_i \to T_{i+1}$ satisfying $\mathrm{BBT}(f_i) = \lambda_\phi^{-i}\,\mathrm{BBT}(f_0)$. For each i, inductively define $L_i := L_{i-1} + \frac{1}{2}\,\mathrm{BBT}(f_{i-1})$ and extend the folding path (which has so far been defined on $[0, L_{i-1}]$) using f_{i-1} to a folding path $(S_t)_{t\in[0,L_i]}$. Define $L := \lim_{i\to\infty} L_i$, which is finite as $\mathrm{BBT}(f_i)$ is a geometric sequence. We have thus defined the trees $(S_t)_{t\in[0,L)}$.

The notation here is less than ideal. In the above, $(T_i)_{i\in\mathbb{N}}$ is used for the trees $\lambda_\phi^{-i} T_0 \cdot \phi^i$, while $(S_t)_{t\in[0,L)}$ denotes a continuous folding path which is folded at constant speed. The reason for the differing names (S and T) is simply that the parameterizations differ; in particular $S_{L_i} = T_i$.

We now describe the maps $f_{t,s}$ for $s, t \in [0, L)$ with $s < t$. Indeed, given s, t, there is a natural choice of a map $f_{t,s} : S_s \to S_t$. Suppose $s \in [L_i, L_{i+1})$ and $t \in [L_j, L_{j+1})$. Then

$$f_{t,s} := f_{t,L_j} \circ f_{j-1} \circ f_{j-2} \circ \cdots \circ f_{i+1} \circ f_{L_{i+1},s}.$$

The semiflow property for the connection data follows from the definitions. Though our setting differs slightly from that of [Bestvina and Feighn 2014], Proposition 2.2 (5) therein can still be applied to give that each tree S_t has a well defined train track structure.

Along with the connection data, the fold path $(S_t)_{t\in[0,L)}$ forms a directed system in the category of $\mathbb{F}$-equivariant metric spaces and distance nonincreasing maps. As direct limits exist in this category, let $S_L := \varinjlim S_t$ and let $f_{L,t}$ be the direct limit maps. The proof of the following proposition is contained in Section 7.3 of [Handel and Mosher 2011], though it is not stated in this way. While Handel and Mosher deal with trees in $\mathcal{O}$ rather than $\partial\mathbb{P}\mathcal{O}$, the reader will easily verify that their proof goes through directly in our setting.

Proposition 3.5 [Handel and Mosher 2011]. *S_L is a non-trivial, minimal, $\mathbb{R}$-tree. Moreover S_t converges to S_L in the length function topology.*

We have described two trees in the boundary of outer space: $T_\phi^+ = \lim_{n\to\infty} T_0'\phi^n$ and S_L. We observe that both S_0 and T_0' are points in the relative outer space $\mathcal{O}(\mathbb{F}, \mathcal{A})$, which inherits the subspace topology from $\overline{\mathbb{P}\mathcal{O}}$. Moreover, ϕ is fully irreducible relative to $\mathcal{A}$, and as such, it acts with north-south dynamics on $\overline{\mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})}$ [Gupta 2018]. Recall that for each $i \in \mathbb{N}$, $S_{L_i} = \lambda_\phi^{-i} S_0 \cdot \phi^i$, and that $L_i \to L$. As S_L is the limit of the fold path $(S_t)_{t \in [0,L)}$, we conclude:

Lemma 3.6. $S_L = T_\phi^+$.

We conclude this section with a lemma.

Lemma 3.7. *For all $t \in [0, L)$, the tree S_t is simplicial.*

Proof. Let $t \in [0, L)$. If $t = 0$, Lemma 3.2 provides that S_0 is simplicial. Since $S_{L_i} = \lambda_\phi^{-i} S_0 \cdot \phi^i$, the lemma holds when $t = L_i$ for some $i \in \mathbb{N}$. The other possibility is that $t \in (L_i, L_{i+1})$ for some i. Since both S_{L_i} and $S_{L_{i+1}}$ have trivial edge stabilizers, Proposition 1.1 of [Horbez 2016] applies to the folding path guided by f_i and allows one to conclude that all trees S_t, $t \in [L_i, L_{i+1}]$ are simplicial, as desired. □

4. The stable tree is $\mathcal{Z}^{(\mathrm{max})}$-averse

Our present aim is to understand T_ϕ^+; we would like to show that it is $\mathcal{Z}^{(\mathrm{max})}$-averse. In this section, we will use the leaves of the topmost lamination Λ_ϕ^+ to construct a transverse covering of T_ϕ^+, and then use the transverse covering to achieve our goal.

Definition 4.1. Let G be a group and T be an $\mathbb{R}$-tree equipped with an action of G by isometries; and let $K \subseteq T$ be a subtree. We say that the action $G \curvearrowright T$ is *supported on K* if for any finite arc $J \subseteq T$, there are $g_1, \dots, g_r \in G$ such that $I \subseteq g_1 K \cup \dots \cup g_r K$.

Let I_0 be a segment of a leaf of the lamination Λ_ϕ^+ in S_0. Define the arc I_t in S_t by $I_t := f_{t,0}(I_0)$. We will denote I_L simply by I and we will call any segment in T_ϕ^+ obtained in this way *a segment of a leaf of* Λ_ϕ^+.

Lemma 4.2. *The action $\mathbb{F} \curvearrowright T_\phi^+$ is supported on I.*

Proof. Let $I = [x, y]$ and let $J = [x', y']$ be a nondegenerate arc in T_ϕ^+. The construction in Section 3 provides an optimal folding path $(S_t)_{t \in [0,L]}$, and optimal morphisms $f_{s,t} : S_t \to S_s$ for all $s, t \in [0, L]$ with $s > t$ which satisfy the semiflow property. It follows easily from the definitions that for a folding path (S_t) and any z in $S_L = T_\phi^+$, the set $f_{L,0}^{-1}(z)$ is a discrete set of points in S_0. Let $x'_0 \in f_{L,0}^{-1}(x')$ and $y'_0 \in f_{L,0}^{-1}(y')$ be points in S_0 chosen so that (x'_0, y'_0) contains no points in $f_{L,0}^{-1}(x') \cup f_{L,0}^{-1}(y')$ and define $J_0 = [x'_0, y'_0]$. Since I_0 is legal, it is never folded under the maps $f_{t,0}$, so the corresponding property already holds for I_0. Define the arc J_t in T_t by $J_t := [f_{t,0}(J_0)]$. The definitions of I_0 and J_0 guarantee that $[f_{L,0}(I_0)] = I$ and similarly for J_0. The semiflow property of the maps $f_{s,t}$ gives that for all $s, t \in [0, L]$ with $s > t$, we have $[f_{s,t}(I_t)] = I_s$ (resp. $[f_{s,t}(J_t)] = J_s$).

Since I_0 is a leaf segment and therefore legal with respect to the train track structure on S_0, it is never folded under the maps $f_{t,0}$. In particular, the length of I_t is constant in t. The maximum length of any edge in S_t tends to 0 as $t \to L$ because edge lengths can only decrease along the fold path and the metric in S_{L_i} has been scaled by λ_ϕ^{-i}. Thus, for sufficiently large t, I_t crosses an entire edge of S_t. Irreducibility of the transition matrix for f_0 implies that by further enlarging t, we may assume that I_t crosses an edge from every $\mathbb{F}$-orbit of edges in S_t.

We are now ready to complete the proof. Indeed, write J_t as an edge path $J_t = e_0 e_1 \cdots e_k$ in S_t (the first and last edges may be partial edges). Since I_t crosses every $\mathbb{F}$-orbit of edges in S_t, there exist $g_0, \dots, g_k \in \mathbb{F}$ so that for all j, $g_j I_t$ crosses the edge e_j. Now we simply use $\mathbb{F}$-equivariance of the maps $f_{L,t}$ to conclude that

$$f_{L,t}(J_t) \subseteq g_0 f_{L,t}(I_t) \cup g_1 f_{L,t}(I_t) \cup \cdots \cup g_k f_{L,t}(I_t)$$

As I_t is legal, $f_{L,t}(I_t) = I$. While J_t is not necessarily legal, it's still true that $J = [f_{L,t}(J_t)] \subseteq f_{L,t}(J_t)$, completing the proof. □

4A. *Mixing and indecomposable trees.* A tree $T \in \overline{\mathbb{P}\mathcal{O}}$ is *mixing* if for all finite subarcs $I, J \subset T$, there exist $g_0, \dots, g_k \in \mathbb{F}$ such that $J \subseteq g_0 I \cup g_1 I \cup \cdots \cup g_k I$ and $g_j I \cap g_{j+1} I \neq \varnothing$ for all $j \in \{0, \dots, k-1\}$. A tree $T \in \overline{\mathbb{P}\mathcal{O}}$ is called *indecomposable* [Guirardel 2008] if it is mixing and the g_j's can be chosen so that $g_j I \cap g_{j+1} I$ is a nondegenerate arc for each $j \in \{0, \dots, k-1\}$.

Lemma 4.3. *T_ϕ^+ is mixing.*

Proof. The proof is similar to that of Lemma 4.2, so we will retain our notation from that proof. Indeed, it's clearly enough to show that every arc J can be covered by finitely many translates with nonempty overlap of the fixed arc I and conversely that I can be covered similarly by translates of J. Recall the cover of J by translates of I constructed in proof of Lemma 4.2. Since consecutive edges in the edge path of $J_t = e_0 \cdots e_k$ intersect in a point, it follows that $g_j I_t \cap g_{j+1} I_t \neq \varnothing$ for all $j \in \{0, \dots, k-1\}$. Again, this behavior persists in the limit.

Conversely, to see that I can be covered by translates of J we use essentially the same argument as before, only now there is a slight difficulty in producing an edge in some J_t that isometrically embeds in the limit. Now J_t may have illegal turns, so we write J_t as a concatenation of maximal legal subpaths, $J_t = J_t^0 J_t^1 \cdots J_t^k$. Now $f_{L,t}(J_t)$ is a concatenation of the $f_{L,t}$-images of J_t^i, which are themselves segments in S_L. Thus, the tightened image $J = [f_{L,t}(J_t)]$ is contained in the union $f_{L,t}(J_t^0) \cup \cdots \cup f_{L,t}(J_t^k)$. Now choose an $i \in \{0, \dots, k\}$ so that $J \cap f_{L,t}(J_t^i)$ is a nondegenerate subsegment of J and replace J by the subsegment $J' = J \cap f_{L,t}(J_t^i)$. The proof of Lemma 4.2 can now be applied to J', allowing us to conclude that I can be covered by finitely many translates J' with nonempty overlaps. As J' is a subsegment of J, the same finite set of group elements witnesses the fact that I can be covered by finitely many translates J with nonempty overlaps. □

4B. *Transverse families and transverse coverings.* A subtree Y of a tree T is called *closed* [Guirardel 2004, Definition 2.4] if $Y \cap \sigma$ is either empty or a path in T for all paths $\sigma \subset T$; recall that paths are defined on closed intervals. A *transverse family* [Guirardel 2004, Definition 4.6] of an $\mathbb{R}$-tree T is a family $\mathcal{Y}$ of nondegenerate closed subtrees of T such that any two distinct subtrees in $\mathcal{Y}$ intersect in at most one point. If every path in T is covered by finitely many subtrees in $\mathcal{Y}$, then the transverse family is called a *transverse covering*.

The idea of the following definition is to start with an interval and "fill it out" into an entire subtree by translating it around, always requiring that overlaps are nondegenerate.

Definition 4.4 (the transverse family generated by Λ^+_ϕ). Let $I = [x, y]$ be a segment of a leaf of Λ^+_ϕ in T^+_ϕ. Define Y_I as the union of all arcs J such that there exists $g_0, \dots, g_k \in \mathbb{F}$ satisfying:

- $J \subseteq g_0 I \cup \cdots \cup g_k I$.
- $g_j I \cap g_{j+1} I$ is a nondegenerate segment for each $i \in \{0, \dots, k-1\}$.
- $g_0 I \cap I$ is a nondegenerate segment.

It's immediate that the collection $\mathcal{Y} = \{gY_I\}_{g\in\mathbb{F}}$ is a transverse family in T^+_ϕ since, by definition, distinct $\mathbb{F}$-translates of Y_I intersect in a point or not at all. This construction is essentially due to Guirardel–Levitt.

Lemma 4.5. *With notation as above, Y_I is indecomposable with respect to the* $\mathrm{Stab}(Y_I)$ *action. Moreover, $\mathcal{Y} = \{gY_I\}_{g\in\mathbb{F}}$ is a transverse covering of T^+_ϕ.*

Proof. We first show that Y_I is indecomposable. The proof is similar to that of Lemmas 4.2 and 4.3, so we will retain our notation from those proofs. As before, it is enough to show that every arc $J \subseteq Y_I$ can be covered by finitely many translates with nondegenerate overlap of the fixed arc I, and conversely that I can be covered by finitely many translates of J with nondegenerate overlap. The definition of Y_I guarantees that J can be covered by finitely many translates of I, so we are left to show the converse.

First, replace J by an appropriately chosen subinterval exactly as in the proof of Lemma 4.3. Now we run the proof of Lemma 4.2 with a minor modification. For $t \in [0, L)$, let J_t and I_t be as in that proof. This time, choose t large enough so that I_t crosses every $\mathbb{F}$-orbit of turns taken by a leaf of Λ^+_ϕ. By further enlarging t if necessary, we may arrange that J_t also crosses every turn taken by a leaf. Write I_t as an edge path $I_t = e_0 e_1 \cdots e_k$ in S_t, where the first and last edges may be partial edges. Since J_t crosses every $\mathbb{F}$-orbit of turns taken by a leaf in S_t, there exist

$g_0, \ldots, g_k \in \mathbb{F}$ so that for all $j \in \{0, \ldots, k-1\}$, $g_j J_t$ crosses the edge path $e_j e_{j+1}$. Now we conclude exactly as before, using $\mathbb{F}$-equivariance of the maps $f_{L,t}$ to see that

$$f_{L,t}(I_t) \subseteq g_0 f_{L,t}(J_t) \cup g_1 f_{L,t}(J_t) \cup \cdots \cup g_k f_{L,t}(J_t)$$

Since both I_t and J_t are legal, this set containment (and nondegeneracy of the overlaps) is unaffected by tightening and the proof is complete.

To see that $\mathcal{Y}$ is a transverse covering we again reference the proof of Lemma 4.2, which shows that every path in T_ϕ^+ can be covered by finitely many trees in $\mathcal{Y}$. $\square$

Lemma 4.6. *Let β be a generic leaf of Λ_ϕ^+ and let J be a finite subsegment of a realization of β in T_ϕ^+. Then there exists $g \in \mathbb{F}$ which is contained in a conjugate of* $\mathrm{Stab}(Y_I)$ *and whose axis, A_g, in T_ϕ^+ contains the segment J.*

Proof. We retain our notation from above, so that J_t is a segment in S_t which maps to J under $f_{L,t}$. We will denote the realization of β in S_t by β_t. Choose t large enough so that J_t crosses every turn taken by β_t, then lengthen J_t by following the leaf to arrange that both endpoints of J_t are vertices in the same $\mathbb{F}$-orbit. Write J_t as an edge path $J_t = e_0 e_1 \cdots e_k$. If necessary, further lengthen J_t (again following β_t) to arrange that the turn $\{e_0, \overline{e_k}\}$ is taken by a leaf. Let x_t (resp. y_t) be the initial (resp. terminal) endpoint of J_t.

Now let $g \in \mathbb{F}$ be a group element taking x_t to y_t. After postcomposing with an element of $\mathrm{Stab}(y_t)$ if necessary, we may assume that the turn $\{\overline{e_k}, g(e_0)\}$ is taken by a generic leaf of Λ_ϕ^+. We claim that the axis of g in S_t crosses J_t. Indeed, to get from x_t to y_t, one traverses the edge path $e_0 e_1 \cdots e_k$. Thus, to get from $y_t = g \cdot x_t$ to $g \cdot y_t = g^2 \cdot x_t$, one traverses the same (up to $\mathbb{F}$-orbit) edge path. As $e_0 \neq \overline{e_k}$ and S_t is a tree, we have that $d(x_t, g^2 \cdot x_t) = 2d(x_t, g \cdot x_t)$. It is an elementary exercise to show that this is equivalent to x being on the axis of g. Both β_t and the axis of g are legal, so the restriction of $f_{L,t}$ to each is an immersion. Thus, we can push this picture forward to the limit using $f_{L,t}$ to reach the desired conclusion.

We've seen that any realization of β in T_ϕ^+ is contained in a single $\mathbb{F}$-translate of Y_I. As we have arranged that every turn taken by the axis of g in S_t is also taken by a leaf, the argument given in the proof of Lemma 4.5 allows us to conclude that A_g is contained in a single $\mathbb{F}$-translate of Y_I. Thus g is contained in a conjugate of $\mathrm{Stab}(Y_I)$, as desired. $\square$

For convenience of the reader, we recall two essential facts:

Proposition 4.7 [Horbez 2016, Propositions 4.27, 4.3]. *If $T \in \overline{\mathcal{O}}$ is mixing, then T is $\mathcal{Z}^{(\max)}$-averse if and only if T is $\mathcal{Z}^{(\max)}$-incompatible.*

Lemma 4.8 [Guirardel 2008, Lemma 1.18]. *Let $T \in \overline{\mathcal{O}}$ be compatible with a $\mathcal{Z}^{(\max)}$-splitting, S. Let $H \subseteq \mathbb{F}$ be a subgroup, such that the H-minimal subtree T_H of T is indecomposable. Then H is elliptic in S.*

Proposition 4.9. *T_ϕ^+ is $\mathcal{Z}^{(\max)}$-averse.*

Proof. We assume that T_ϕ^+ is not $\mathcal{Z}^{(\max)}$-averse and argue towards a contradiction. Indeed, as T_ϕ^+ is mixing, Proposition 4.7 implies that it is compatible with a $\mathcal{Z}^{(\max)}$-splitting S. Now let $H = \mathrm{Stab}(Y_I)$. If $Y_I = T_\phi^+$, then $H = \mathbb{F}$ and Lemma 4.8 gives that $\mathbb{F}$ is elliptic in S, a contradiction as S is a nontrivial minimal splitting.

The other possibility is that Y_I is a proper subtree in T_ϕ^+, and in this situation we argue that Λ_ϕ^+ is carried by a vertex group of S. As above, we apply Lemma 4.8 to conclude that $H = \mathrm{Stab}(Y_I)$ is carried by a vertex group A of the splitting S. We have a tower of covers corresponding to subgroups as follows (we temporarily blur the distinction between $\mathbb{F}$ and the universal cover of $\mathcal{R}$):

$$\mathbb{F} \xrightarrow{\pi_{H,\mathbb{F}}} X_H \xrightarrow{\pi_{A,H}} X_A \xrightarrow{\pi_{\mathcal{R},A}} \mathcal{R}$$

We denote by G_A and G_H the core of the corresponding covers.

Let β be a generic leaf of Λ_ϕ^+. Even though H may not be finitely generated, we claim it is enough to show that every finite subsegment of β can be immersed into G_H. Indeed, by postcomposing these immersions with $\pi_{A,H}$ (also an immersion), we see that every finite subpath of β can then be immersed into G_A. Since A is finitely generated, we conclude that β can be immersed into G_A, and therefore that Λ_ϕ^+ is carried by a vertex group of the cyclic splitting S.

Let $h : \mathbb{F} \to T_\phi^+$ be an $\mathbb{F}$-equivariant map which is linear on edges and Lipschitz (it's easy to see that such maps exist). Lemma 3.1 of [Bestvina et al. 1997] gives that $\mathrm{BBT}(h)$ is finite. Color the line β_L in T_ϕ^+ red and let $\beta_{\mathbb{F}}$ be the realization of β in $\mathbb{F}$. Pull back the coloring via h to $\beta_{\mathbb{F}}$ as follows (keeping in mind the bounded cancellation): if $x \in \beta_{\mathbb{F}}$ is such that $h(x)$ is red, then color x red, otherwise do not color x. It's clear that both ends of $\beta_{\mathbb{F}}$ have red segments.

Let $J_{\mathbb{F}}$ be a subsegment of $\beta_{\mathbb{F}}$. Extend $J_{\mathbb{F}}$ along $\beta_{\mathbb{F}}$ if necessary to ensure that both endpoints of $J_{\mathbb{F}}$ are red. Define $J = h_\#(J_{\mathbb{F}})$. The fact that the endpoints of $J_{\mathbb{F}}$ are red ensures that J is a subsegment of β_L. Apply Lemma 4.6 to obtain an element $g \in H$ whose axis contains J. Color the axis of g in T_ϕ^+ blue. Pull back this coloring to the axis of g in $\mathbb{F}$ exactly as above. Equivariance of h, coupled with the fact that g is not elliptic in $\mathbb{F}$ or T_ϕ^+, implies that every subray of the axis of g in $\mathbb{F}$ contains blue points. In particular, there are blue points on either side of $J_{\mathbb{F}}$. Thus the axis of g in $\mathbb{F}$ contains the prescribed segment $J_{\mathbb{F}}$. It's now evident that $J_{\mathbb{F}}$ is contained in the H-minimal subtree of $\mathbb{F}$. This implies that $\pi_{H,\mathbb{F}}(J_{\mathbb{F}})$ is contained in the core G_H of the cover, completing the proof. □

5. Filling but not $\mathcal{Z}^{(\max)}$-filling laminations

In this section, we study filling laminations which are not $\mathcal{Z}^{(\max)}$-filling. We then use this understanding to establish the following proposition, which is a restatement of the second claim in Theorem 1.1. This section concludes with a proof of the first statement in Theorem 1.1.

Proposition 5.1. *Let ϕ be an automorphism with a filling lamination Λ^+_ϕ that is not $\mathcal{Z}^{(\max)}$-filling, so that Λ^+_ϕ is carried by a vertex group of a (maximally-) cyclic splitting S. Then there is a (maximally-) cyclic splitting S' that is fixed by a power of ϕ.*

The splitting S' is canonical in the sense that the vertex group which carries Λ^+_ϕ is as small as possible. The proof of Proposition 5.1 will require an excursion into the theory of JSJ-decompositions; the reader is referred to [Fujiwara and Papasoglu 2006] for details about JSJ theory.

We say a lamination is *elliptic* in an $\mathbb{F}$-tree T if it is is carried by a vertex stabilizer of T. Let $\mathfrak{S}$ be the set of all one-edge $\mathcal{Z}^{(\max)}$-splittings in which the lamination Λ^+_ϕ is elliptic. Since Λ^+_ϕ is filling, the set $\mathfrak{S}$ does not contain any free splittings.

Definition 5.2 (types of pairs of splittings [Rips and Sela 1997]). Let $S = A *_C B$ (or $A*_C$) and $S' = A' *_{C'} B'$ (or $A'*_{C'}$) be one-edge cyclic splittings with corresponding Bass–Serre trees T and T'. We say S is *hyperbolic* with respect to S' if there is an element $c \in C$ that acts hyperbolically on T'. We say S is *elliptic* with respect to S' if C fixes a point of T'. We say this pair is *hyperbolic-hyperbolic* if each splitting is hyperbolic with respect to the other. We define elliptic-elliptic, hyperbolic-elliptic and elliptic-hyperbolic splittings similarly.

Lemma 5.3. *With notation as above, suppose that $S, S' \in \mathfrak{S}$, and assume without loss that Λ^+_ϕ is carried by the vertex groups A and A'. Then Λ^+_ϕ is elliptic in the minimal subtree of A in T', denoted T'_A, and in the minimal subtree of A' in T, denoted $T_{A'}$.*

Proof. Since A and A' both carry Λ^+_ϕ, their intersection $A \cap A'$ also carries Λ^+_ϕ. The vertex stabilizers of $T_{A'}$ are precisely the intersection of vertex stabilizers of T with A', namely the conjugates of $A \cap A'$. Thus Λ^+_ϕ is carried by a vertex group of $T_{A'}$. □

Lemma 5.4. *With notation as above, suppose that S, S' are one-edge $\mathcal{Z}^{(\max)}$-splittings in $\mathfrak{S}$. Then S and S' are either hyperbolic-hyperbolic or elliptic-elliptic.*

Proof. The following is based on the proof of [Fujiwara and Papasoglu 2006, Proposition 2.2]. We will address the case that both the splittings are free products with amalgamations; when one or both are HNN extensions, the proof is similar. Toward a contradiction, suppose some element of C acts hyperbolically in T' and that C' is elliptic in T. Without loss of generality, we may assume that C' fixes the vertex stabilized by A in T. Suppose first that both A' and B' fix vertices in T. The two subgroups cannot fix the same vertex because they generate $\mathbb{F}$. On the other hand, if the vertices are distinct, then C' fixes an edge in T. Hence C' must be a finite index subgroup of C, in contradiction to the assumption that C is hyperbolic in T'. Thus, one of A' or B' does not fix a vertex in T.

Assume without loss that A' does not fix a vertex of T. The minimal subtree of A' in T, denoted $T_{A'}$, gives a minimal splitting of A' over an infinite index subgroup of C (i.e., a free splitting). Indeed, were A' to split over a finite index subgroup C_1 of C, then C_1 would be elliptic in T' contradicting our assumption that C is hyperbolic in T'. As C' is elliptic in T, it is also elliptic in $T_{A'}$. Now blow up the vertex stabilized by A' in T' to the free splitting of A' just obtained, and then collapse the edge stabilized by C' to get a free splitting T_0 of $\mathbb{F}$. Then B' is still elliptic in T_0. If Λ^+_ϕ is carried by B', then Λ^+_ϕ is elliptic in the free splitting T_0, which is a contradiction. If Λ^+_ϕ is carried by A', then Lemma 5.3 implies that Λ^+_ϕ is elliptic in $T_{A'}$. Thus Λ^+_ϕ is also elliptic in the free splitting T_0, again a contradiction. □

In [Fujiwara and Papasoglu 2006], the existence of JSJ decompositions for splittings with slender edge groups ([loc. cit., Theorem 5.13]) is established via an iterative process: one starts with a pair of splittings, and produces a new splitting which is a common refinement (in the case of an elliptic-elliptic pair) [loc. cit., Proposition 5.10], or an enclosing subgroup [loc. cit., Definition 4.5] (in the case of a hyperbolic-hyperbolic pair) [loc. cit., Proposition 5.8]. One then repeats this process for all the splittings under consideration, and uses an accessibility result due to Bestvina and Feighn [1991] to conclude that the process stops after finitely many iterations. In order to use techniques of Fujiwara and Papasoglu, we need only ensure that if two splittings belong to the set $\mathfrak{S}$, then the splittings created in this process also belong to $\mathfrak{S}$. By examining the construction of an enclosing subgroup for a pair of hyperbolic-hyperbolic splittings [Fujiwara and Papasoglu 2006, Proposition 4.7] and using Lemma 5.3, we see that the enclosing graph decomposition of $\mathbb{F}$ for this pair of splittings indeed belongs to $\mathfrak{S}$. Similarly, Lemma 5.3 implies that the refinement of two elliptic-elliptic splittings that are contained in $\mathfrak{S}$ is itself contained in $\mathfrak{S}$. This discussion implies that JSJ decompositions exist for cyclic splittings of $\mathbb{F}$ in which Λ^+_ϕ is elliptic.

We conclude our foray into JSJ decompositions by using the theory of deformation spaces [Forester 2002; Guirardel and Levitt 2007a] to show that the set of JSJ splittings of $\mathbb{F}$ in which Λ^+_ϕ is elliptic is finite. By passing to a power, we will then obtain a ϕ-invariant splitting in $\mathfrak{S}$.

Definition 5.5 (slide moves [Guirardel and Levitt 2007a, Section 7]). Let $e = vw$ and $f = vu$ be adjacent edges in an $\mathbb{F}$-tree T such that the edge stabilizer of f, denoted G_f, is contained in G_e. Assume that e and f are not in the same orbit as nonoriented edges. Define a new tree T' with the same vertex set as T and replace f by an edge $f' = wu$ equivariantly. Then we say f *slides* across e. Often, a slide move is described on the quotient of T by $\mathbb{F}$.

Definition 5.6 [Guirardel and Levitt 2007a; Forester 2002]. The *deformation space* $\mathcal{D}$ containing a tree T is the set of all trees T' such that there are equivariant maps from T to T' and from T' to T, up to equivariant isometry.

Definition 5.7 [Forester 2002]. A tree T is reduced if no inclusion of an edge group into either of its vertex groups is an isomorphism.

Theorem 5.8 [Guirardel and Levitt 2007a, Theorem 7.2]. *If $\mathcal{D}$ is a nonascending deformation space, then any two reduced simplicial trees $T, T' \in \mathcal{D}$ may be connected by a finite sequence of slides.*

Deformation spaces consisting of trees such that no edge stabilizer properly contains a conjugate of itself are examples of nonascending deformation spaces [Guirardel and Levitt 2007a, Section 7]. We are only interested in such deformation spaces here.

Lemma 5.9. *Given a reduced cyclic splitting S, there are only finitely many slide moves that can be performed on S. Moreover, any sequence of slide moves starting at S has bounded length.*

Proof. The first statement follows from the fact that S has finitely many orbits of edges. For the second statement, first suppose that the splitting $S/\mathbb{F}$ does not have any loops or circuits. Then it is clear that only finitely many slide moves can be performed on S. If S has a loop, then we can slide an edge f along the loop e only once. Indeed, we have $G_f \subseteq G_e$ and after sliding we have $G_{f'} \subseteq tG_e t^{-1}$, where t is the stable letter corresponding to the loop. Since $G_e \cong \mathbb{Z}$ and $G_e \cap tG_e t^{-1} = 1$, $G_{f'} \not\subseteq G_e$ which prevents sliding of f' over e. The proof in the case of a circuit is similar. □

Proof of Proposition 5.1. By assumption, there exists a one-edge cyclic splitting S such that Λ^+_ϕ is elliptic in S. The existence of JSJ decomposition for splittings in $\mathfrak{S}$ implies that the deformation space $\mathcal{D}$ for cyclic splittings in $\mathfrak{S}$ is nonempty. Since the edge stabilizer of the trees in $\mathcal{D}$ is $\mathbb{Z}$, the space $\mathcal{D}$ is nonascending. Theorem 5.8 and Lemma 5.9 together imply that the set of reduced trees in $\mathcal{D}$ is finite. As the set of reduced trees in $\mathcal{D}$ is ϕ-invariant, passing to a power yields a reduced cyclic splitting S' in $\mathcal{D}$ which is fixed by ϕ^k. The same argument works if S is a maximally-cyclic splitting. □

Proof of Theorem 1.1 (loxodromic). Suppose that ϕ has a $\mathcal{Z}^{(\max)}$-filling lamination, whereby ϕ^{-1} does as well. Applying Proposition 4.9 we conclude that both T^+_ϕ and T^-_ϕ are $\mathcal{Z}^{(\max)}$-averse. We now argue that these trees determine distinct points in $\mathcal{X}^{(\max)}$. We denote the dual lamination of a tree T by $L(T)$ [Coulbois et al. 2008]. Since the attracting lamination Λ^+_ϕ and the repelling lamination Λ^-_ϕ are different, and $\Lambda^\mp_\phi \subseteq L(T^\pm_\phi)$ and $\Lambda^\pm_\phi \not\subseteq L(T^\pm_\phi)$, we have that T^+_ϕ and T^-_ϕ are distinct points in $\overline{\mathcal{O}}$. Both trees are mixing (Lemma 4.3), but [Horbez 2016, Proposition 4.3] provides that if two mixing trees in $\overline{\mathcal{O}}$ are equivalent (i.e., determine the same point in $\mathcal{X}^{(\max)}$), then each must collapse onto the other. If there a collapse map from $T \to T'$, then $L(T) \subseteq L(T')$. So if T_ϕ^+ and T_ϕ^- were equivalent, then their dual laminations would be equal, a contradiction.

We now argue that the limit set of $\langle\phi\rangle$ acting on $\mathcal{FZ}^{(\max)}$ consists of two points. There is a minor complication arising from the fact that the folding path constructed in Section 3 consisted entirely of trees in the boundary of outer space, but Theorem 2.4 applies only to sequences in the interior. Indeed, recall from Section 3 that T denotes the universal cover of a relative train track map representing ϕ and that T_0 was obtained from T by first collapsing the $\mathbb{F}$-translates of the $\mathcal{A}$-minimal subtree in T, then further collapsing according to a measure μ. Finally, recall (Proposition 4.9) that the sequence $T_i = \lambda_\phi^{-i} T_0 \phi^i$ where $i \in \mathbb{N}$ converges to T_ϕ^+, which is $\mathcal{Z}^{(\max)}$-averse. Let $R_i = T\phi^i$ and let $R_\infty = \lim_{i\to\infty} R_i$. For all $i \in \mathbb{N}$, R_i collapses onto T_i, so R_i and T_i are compatible. That compatibility passes to the limit follows from [Guirardel and Levitt 2017, Corollary A.12], so R_∞ is compatible with T_ϕ^+ and is therefore $\mathcal{Z}$-averse. Applying Theorem 2.4 to the sequence $\{R_i\}_{i\in\mathbb{N}}$, we conclude that the image sequence $\psi(R_i)$ converges to $[T_\phi^+] \in \mathcal{X}^{(\max)}$. Finally, since the set of reducing splittings for a free simplicial $\mathbb{F}$-tree is bounded, if S is any $\mathcal{Z}^{(\max)}$-splitting we have that $S\phi^i$ converges to $[T_\phi^+]$, with a similar statement holding for iterates of ϕ^{-1}. Thus, $\Lambda_{\mathcal{FZ}}\langle\phi\rangle$ consists of exactly two points and ϕ therefore acts loxodromically on $\mathcal{FZ}^{(\max)}$.

We now prove the converse: if ϕ acts loxodromically on $\mathcal{FZ}^{(\max)}$, then ϕ has a $\mathcal{Z}^{(\max)}$-filling lamination. Indeed, if ϕ acts loxodromically on $\mathcal{FZ}^{(\max)}$, then ϕ necessarily acts loxodromically on $\mathcal{FS}$, and thus has a filling lamination Λ_ϕ^+. If the lamination is not $\mathcal{Z}^{(\max)}$-filling, then Proposition 5.1 implies that a power of ϕ fixes a point in $\mathcal{FZ}^{(\max)}$, contradicting our assumption on ϕ. Thus, Λ_ϕ^+ is $\mathcal{Z}^{(\max)}$-filling. □

6. Examples

This section will provide several examples exhibiting the range of behaviors of outer automorphisms acting on $\mathcal{FZ}$. We begin with an automorphism that acts loxodromically on $\mathcal{FZ}$.

Example 6.1 (loxodromic element). Let ϕ be a rotationless automorphism with a CT representative $f : G \to G$ satisfying the following properties:

- f has exactly two strata, each of which is EG and nongeometric.
- The lamination corresponding to the top stratum of f is filling.

An explicit example satisfying these properties can be constructed using the sage-train-tracks package written by T. Coulbois [2015]. The fact that the top lamination is filling guarantees that ϕ acts loxodromically on $\mathcal{FS}$. As both strata are nongeometric,

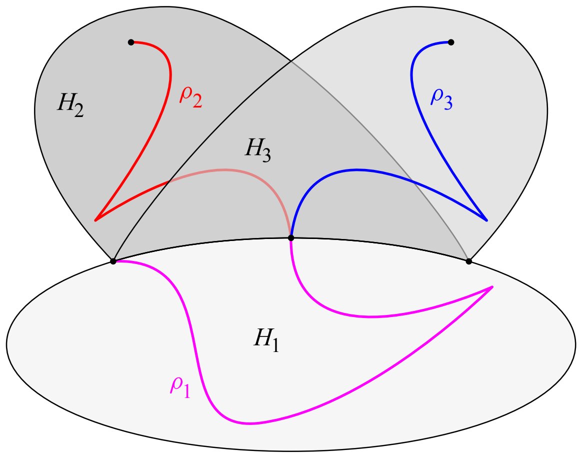

Figure 1. A CT representative for the automorphism constructed in Example 6.2, which acts with bounded orbits but no fixed point.

[Handel and Mosher 2013b, Fact 1.42(1a)] guarantees that ϕ does not fix the conjugacy class of any element of $\mathbb{F}$, and therefore cannot possibly fix a cyclic splitting. Corollary 1.2 implies that ϕ acts loxodromically.

Example 6.2 (bounded orbit without fixed point). By building on Example 6.1 and [Handel and Mosher 2014, Example 4.2], we can construct an automorphism ψ which acts on $\mathcal{FZ}$ with bounded orbits but without a fixed point. Let ψ be a three stratum automorphism obtained from f by creating a duplicate of H_2. Explicitly, ψ has a CT representative $f' : G' \to G'$ defined as follows. The graph G' is obtained by taking two copies of G and identifying them along G_1. Each edge E of G' is naturally identified with an edge of G, and $f'(E)$ is defined via this identification. Moreover, the marking of G naturally gives a marking of G' (by a larger free group). That f' is a CT is evident from the fact that f is a CT.

There are three laminations in $\mathcal{L}(\psi)$, and it's evident that none are filling. Since the top lamination in $\mathcal{L}(\phi)$ (where ϕ is as in Example 6.1) is filling, we know that $\mathcal{L}(\psi)$ must fill. Thus, ψ acts on $\mathcal{FS}$ with bounded orbits. As before, [Handel and Mosher 2013b, Fact 1.42 (1a)] implies that ψ doesn't fix the conjugacy class of any element of $\mathbb{F}$: while each stratum may have an INP, ρ_i, none of these INPs can be closed loops, nor can they be concatenated to form a closed loop. Thus, ψ does not fix any one-edge cyclic splitting and therefore must act on $\mathcal{FZ}$ with bounded orbits, but no fixed point. See Figure 1 for a pictorial representation of ψ. The INPs ρ_2 and ρ_3 must each have at least one endpoint which is not in H_1.

Example 6.3 (loxodromic element). Consider the outer automorphism $\phi : F_4 \to F_4$ given by

$$\phi(a) = ab, \quad \phi(b) = bcab, \quad \phi(c) = d, \quad \phi(d) = cd.$$

In [Reynolds 2012], it is shown that the stable tree for ϕ is indecomposable and hence $\mathcal{Z}$-averse. Therefore ϕ acts loxodromically on $\mathcal{FZ}$.

Example 6.4 (fixed point). Let $\Sigma_{2,1}$ be the surface of genus two with one puncture. Consider the free homotopy class of a simple separating curve which divides $\Sigma_{2,1}$ into two subsurfaces: a once punctured torus and a twice punctured torus. Placing a pseudo-Anosov on each of these subsurfaces and taking the outer automorphism induced by this mapping class yields an element of $\mathrm{Out}(\mathbb{F})$ that acts loxodromically on $\mathcal{FS}$, but fixes a point in $\mathcal{FZ}$. A similar example using nonseparating simple curve can be found in the proof of [Mann 2014, Proposition 3].

7. Virtually cyclic centralizers

In this section, we investigate centralizers of automorphisms acting loxodromically on $\mathcal{FZ}$. To do this, we use the machinery of completely split train tracks, and the "disintegration" procedure of [Feighn and Handel 2009], which takes a rotationless outer automorphism and returns an abelian subgroup of $\mathrm{Out}(\mathbb{F})$. The main result is:

Theorem 1.3. *An outer automorphism with a filling lamination has a virtually cyclic centralizer in* $\mathrm{Out}(\mathbb{F})$ *if and only if the lamination is* $\mathcal{Z}$*-filling.*

We begin with a terse review of disintegration for outer automorphisms.

7A. ***Disintegration and rotationless abelian subgroups in* Out($\mathbb{F}$).** Given a mapping class f in Thurston normal form, there is a straightforward way of making a subgroup of the mapping class group, called *the disintegration of* f, by "doing one piece at a time." The subgroup is easily seen to be abelian as each pair of generators can be realized as homeomorphisms with disjoint supports. The process of disintegration in $\mathrm{Out}(\mathbb{F})$ is analogous, but more difficult.

The reader is warned that we will only review those ingredients from [Feighn and Handel 2009] that will be used directly; the reader is directed there, specifically to Section 6, for complete details. Given a rotationless outer automorphism ϕ, one can form an abelian subgroup called $\mathcal{D}(\phi)$. The process of disintegrating ϕ begins by creating a finite graph, B, which records the interactions between different strata in a CT representing ϕ. As a first approximation, the components of B correspond to generators of $\mathcal{D}(\phi)$. However, there may be additional relations between strata that are unseen by B, so the number of components of B only gives an upper bound to the rank of $\mathcal{D}(\phi)$.

Let $f : G \to G$ be a CT representing the rotationless outer automorphism ϕ. While the construction of $\mathcal{D}(\phi)$ does depend on f, using different representatives will produce subgroups that are commensurable.

Let E_i, E_j be distinct linear edges in G with the same axis w so that $f(E_i) = E_i w^{d_i}$ and $f(E_j) = E_j w^{d_j}$ for integers $d_i \neq d_j$. Recall that if $d_i, d_j > 0$, then any path of the form $E_i w^* \bar{E}_j$ called an *exceptional path*. In the same scenario, if d_i and d_j have different signs, we call such a path a *quasi-exceptional path*. It would be instructive for the reader to compute the f-image of some exceptional and quasi-exceptional paths. We will need to consider a weakening of the complete splitting of paths and circuits in f. The *quasi-exceptional splitting* of a completely split path or circuit σ is the coarsening of the complete splitting obtained by considering each quasi-exceptional subpath to be a single element.

Definition 7.1. Define a finite directed graph B as follows. There is one vertex v_i^B for each nonfixed irreducible stratum H_i. If H_i is NEG, then a v_i^B-path is defined as the unique edge in H_i; if H_i is EG, then a v_i^B-path is either an edge in H_i or a taken connecting path in a zero stratum contained in H_i^z. There is a directed edge from v_i^B to v_j^B if there exists a v_i^B-path κ_i such that some term in the QE-splitting of $f_\#(\kappa_i)$ is an edge in H_j. The components of B are labeled $B_1, \dots, B_K$. For each B_s, define X_s to be the minimal subgraph of G that contains H_i for each NEG stratum with $v_i^B \in B_s$ and contains H_i^z for each EG stratum with $v_i^B \in B_s$. We say that $X_1, \dots, X_K$ are the *almost invariant subgraphs* associated to $f : G \to G$.

The reader should note that the number of components of B is left unchanged if an iterate of $f_\#$ is used in the definition, rather than $f_\#$ itself. In the sequel, we will frequently make statements about B using an iterate of $f_\#$.

For each K-tuple $\vec{a} = (a_1, \dots, a_K)$ of nonnegative integers, define

$$f_{\vec{a}}(E) = \begin{cases} f_\#^{a_i}(E) & \text{if } E \in X_i, \\ E & \text{if } E \text{ is fixed by } f. \end{cases}$$

It turns out that $f_{\vec{a}}$ is always a homotopy equivalence of G [Feighn and Handel 2009, Lemma 6.7], but in general $\langle f_{\vec{a}} \mid \vec{a} \text{ is a nonnegative tuple} \rangle$ is not abelian. To obtain an abelian subgroup, one has to pass to a certain subset of tuples which take into account interactions between the almost invariant subgraphs that are unseen by B. The reader is referred to [loc. cit., Example 6.9] for an example.

Definition 7.2. A K-tuple $(a_1, \dots, a_K)$ is called *admissible* if, for all axes μ, if

- X_s contains a linear edge E_i with axis μ and exponent d_i,
- X_t contains a linear edge E_j with axis μ and exponent d_j,
- there is a vertex v^B of B and a v^B-path $\kappa \subseteq X_r$ such that some element in the quasi-exceptional family $E_i \bar{E}_j$ is a term in the QE-splitting of $f_\#(\kappa)$,

then $a_r(d_i - d_j) = a_s d_i - a_t d_j$.

The disintegration of ϕ is then defined as

$$\mathcal{D}(\phi) = \langle f_{\vec{a}} \mid \vec{a} \text{ is admissible} \rangle,$$

which is abelian by [loc. cit., Corollary 6.16].

We now recall some useful facts concerning abelian subgroups of $\mathrm{Out}(\mathbb{F})$, which were studied in [Feighn and Handel 2009].

If an abelian subgroup H is generated by rotationless automorphisms, then all elements of H are rotationless [loc. cit., Corollary 3.13]. In this case, H is said to be rotationless. Rotationless abelian subgroups of $\mathrm{Out}(\mathbb{F})$ have finitely many attracting laminations ([loc. cit., Lemma 4.4]), i.e., if H is abelian and $\mathcal{L}(H) := \bigcup_{\phi \in H} \mathcal{L}(\phi)$, then $|\mathcal{L}(H)| < \infty$.

Feighn and Handel [2009] associated to each rotationless abelian subgroup of $\mathrm{Out}(\mathbb{F})$ a finite collection of (nontrivial) homomorphisms to $\mathbb{Z}$. Combining these, one obtains a homomorphism $\Omega : H \to \mathbb{Z}^N$ that is injective [Feighn and Handel 2009, Lemma 4.6]. An element $\psi \in H$ is said to be *generic* if all coordinates of $\Omega(\psi)$ are nonzero. For the purposes of this section, we require only two facts concerning Ω. First, some of the coordinates of Ω correspond to elements in the finite set $\mathcal{L}(H)$ (there are other coordinates, which we will not need). Second is the fact that the coordinate of $\Omega(\psi)$ corresponding to $\Lambda \in \mathcal{L}(H)$ is positive if and only if $\Lambda \in \mathcal{L}(\psi)$.

7B. ***From disintegrations to centralizers.*** In this subsection, we explain how to deduce Theorem 1.3 from the following proposition concerning the disintegration of elements acting loxodromically on $\mathcal{FZ}$. The proof of Proposition 7.3 is postponed until the next subsection.

Proposition 7.3. *If ϕ is rotationless and has a $\mathcal{Z}$-filling lamination, then $\mathcal{D}(\phi)$ is virtually cyclic.*

Proof of Theorem 1.3. Suppose $\psi \in C(\phi)$ has infinite order and assume that $\langle \phi, \psi \rangle \simeq \mathbb{Z}^2$. If no such element exists, then $C(\phi)$ is virtually cyclic, as there is a bound on the order of a finite subgroup of $\mathrm{Out}(\mathbb{F})$ [Culler 1984]. Now let H_R be the finite index subgroup of $\langle \phi, \psi \rangle$ consisting of rotationless elements [Feighn and Handel 2009, Corollary 3.14] and let ψ' be a generic element of this subgroup. If the coordinate of $\Omega(\psi')$ corresponding to the $\mathcal{Z}$-filling lamination Λ^+_ϕ is negative, then replace ψ' by $(\psi')^{-1}$, which is also generic. Since $\Lambda^+_\phi \in \mathcal{L}(\psi')$ is $\mathcal{Z}$-filling, Theorem 1.1 implies that ψ' acts loxodromically on $\mathcal{FZ}$. Since ψ' is generic in H_R, [Feighn and Handel 2009, Theorem 7.2] says that $\mathcal{D}(\psi') \cap \langle \phi, \psi \rangle$ has finite index in $\langle \phi, \psi \rangle$. This contradicts Proposition 7.3, which says that the disintegration of ψ' is virtually cyclic. $\square$

7C. ***The proof of Proposition 7.3.*** The idea of the proof is as follows. We noted above that the number of components in B only gives an upper bound to the rank of $\mathcal{D}(\phi)$; it may happen that there are interactions between the strata of f that are unseen by B (Definition 7.2). We will obtain precise information about the structure of B; it consists of one main component (B_1), and several components consisting of a single point ($B_2, \dots, B_K$). We will then show that the admissibility condition provides sufficiently many constraints so that choosing a_1 determines $a_2, \dots, a_K$. Thus, the set of admissible tuples consists of a line in $\mathbb{Z}^K$.

Let $f : G \to G$ be a CT representing ϕ with filtration $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_M = G$. Let $\Lambda^+_\phi \in \mathcal{L}(\phi)$ be $\mathcal{Z}$-filling and let $\ell \in \Lambda^+_\phi$ be a generic leaf. As Λ^+_ϕ is filling, the corresponding EG stratum is necessarily the top stratum, H_M. We will understand the graph B by studying the realization of ℓ in G. The results of [Bestvina et al. 2000, §3.1], together with Lemma 4.25 of [Feighn and Handel 2011] give that the realization of ℓ in G is completely split, and this splitting is unique. Thus, we may consider the QE-splitting of ℓ.

We begin with a lemma that allows the structure of INPs and quasi-exceptional paths to be understood inductively.

Lemma 7.4. *Let H_r be a nonfixed irreducible stratum and let ρ be a path of height $s \geq r$ which is either an INP or a quasi-exceptional path. Assume further that ρ intersects H_r nontrivially. Then one of the following holds*:

- *H_r and H_s are NEG linear strata with the same axis, each consisting of a single edge E_r (resp. E_s), and $\rho = E_s w^k \overline{E}_r$, for some $k \in \mathbb{Z}$, where w is a closed, root-free Nielsen path of height $< s$.*
- *ρ can be written as a concatenation $\rho = \beta_0\rho_1\beta_1\rho_2\beta_2 \cdots \rho_j\beta_j$, where each ρ_i is an INP of height r and each β_i is a path contained in $G - \operatorname{int}(H_r)$ (some of the β_i's may be trivial).*

Proof. The proof proceeds by strong induction on the height s of the path ρ. In the base case, $s = r$, and ρ is either an INP of height r or a quasi-exceptional path of the form described. The inductive step breaks into cases according whether H_s is an EG stratum, or an NEG stratum.

If H_s is an EG stratum, then ρ must be an INP, as there are no exceptional paths of EG height. In this case, [Feighn and Handel 2011, Lemma 4.24 (2)] provides a decomposition of ρ into subpaths of height s and maximal subpaths of height $< s$, and each of the subpaths of height $< s$ is a Nielsen path. The inductive hypothesis then guarantees that each of these Nielsen paths has the desired form. By breaking apart and combining these terms appropriately, we conclude that ρ does as well.

Suppose now that H_s is an NEG stratum and let E_s be the unique edge in H_s. Using (NEG Nielsen paths), we see that E_s must be a linear edge, and therefore that ρ is either $E_s w^k \overline{E_s}$ or $E_s w^k \overline{E'}$, where E' is another linear edge with the same axis and w is a closed root free Nielsen path of height $< s$. If H_r is NEG linear, and $E' = E_r$, then the first conclusion holds. Otherwise, we may apply the inductive hypothesis to w to obtain a decomposition as desired. This completes the proof. □

We now begin our study of the graph B. We call the component of B containing v^B_M, the vertex corresponding to the topmost stratum of f, the *main component*.

Lemma 7.5. *All nonlinear NEG strata are in the main component of B.*

Proof. Let H_r be a nonlinear NEG stratum, with single edge E_r. It is enough to show that the single edge E_r occurs as a term in the QE-splitting of ℓ (henceforth, we will say that E_r is a *QE-splitting unit* in ℓ), as this implies that there is an edge in B connecting v_M^B to v_r^B. As ℓ is filling, we know that its realization in G must cross E_r. If the corresponding QE-splitting unit of ℓ is the single edge E, then we are done. The only other possibility is that the QE-splitting unit is an INP or a quasi-exceptional path of some height $s \geq r$. An application of Lemma 7.4 shows that this is impossible, as it would imply the existence of an INP of height r or a quasi-exceptional path of the form $E_r w^* \bar{E}'$, contradicting (NEG Nielsen paths). □

Lemma 7.6. *All EG strata are in the main component of B.*

Proof. Let H_r be an EG stratum. As before, it is enough to show that some (every) edge of H_r occurs as a QE-splitting unit of ℓ. There are three types of QE-splitting units that can cross H_r: a single edge in H_r, an INP of height $\geq r$, or a quasi-exceptional path. In the first case, we are done, so suppose that every time ℓ crosses H_r, the corresponding QE-splitting unit is an INP or a quasi-exceptional path. We now argue that this situation leads to a contradiction.

We may write ℓ as a concatenation $\ell = \cdots \gamma_1 \sigma_1 \gamma_2 \sigma_2 \cdots$, where each σ_i is a QE-splitting unit of ℓ which intersects $\mathrm{int}(H_r)$, and each γ_i is a maximal concatenation of QE-splitting units of ℓ which do not intersect $\mathrm{int}(H_r)$ (some γ_i's may be trivial). By assumption, each σ_i is an INP or a QEP. Applying Lemma 7.4 to each of the σ_i's, then combining and breaking apart the terms appropriately, we see that ℓ can be written as a concatenation $\ell = \cdots \gamma_1 \rho_1 \gamma_2 \rho_2 \cdots$ where each ρ_i is the unique INP of height r or its inverse. Call this INP ρ.

We will now use the information we have about ℓ to find a $\mathcal{Z}$-splitting in which ℓ is carried by a vertex group. The existence of such a splitting will contradict our assumption that ℓ is a generic leaf of the $\mathcal{Z}$-filling lamination Λ_ϕ^+.

We now modify G to produce a 2-complex, G'', whose fundamental group is identified with $\mathbb{F}$. First assume H_r is nongeometric, so that ρ has distinct endpoints, v_0 and v_1. Let G' be the graph obtained from G by replacing each vertex v_i for $i \in \{0, 1\}$ with two vertices, v_i^u and v_i^d (u and d stand for “up” and “down”), which are to be connected by an edge E_i. For each edge E of G incident to v_i, connect it in G' to the new vertices as follows: if $E \in H_r$, then E is connected to v_i^d, and if $E \notin H_r$, then E is connected to v_i^u. G' deformation retracts onto G by collapsing the new edges, and this retraction identifies $\pi_1(G')$ with $\mathbb{F}$ via the marking of G. Let $R = [0, 1] \times [0, 1]$ be a rectangle and define G'' by gluing $\{i\} \times [0, 1]$ homeomorphically onto E_i for $i \in \{0, 1\}$, then gluing $[0, 1] \times \{0\}$ homeomorphically to the INP ρ. As only three sides of the rectangle have been glued, G'' deformation retracts onto G', and its fundamental group is again identified with $\mathbb{F}$.

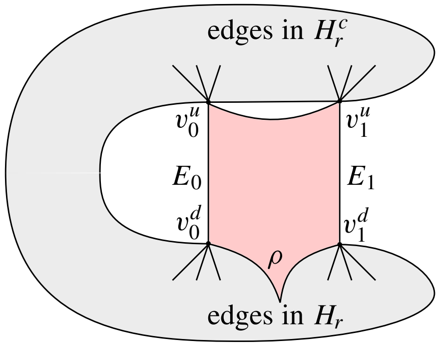

Figure 2. G'' when H_r is a nongeometric EG stratum.

The construction of G'' differs only slightly if H_r is geometric. In this case, ρ is a closed loop based at v_0 and we blow up v_0 to two vertices, v_0^u and v_0^d, that are connected by an edge E_0. Instead of gluing in a rectangle, we glue in a cylinder $R = S^1 \times [0, 1]$; $\{p\} \times [0, 1]$ is glued homeomorphically to E_0, where p is a point in S^1, and $S^1 \times \{0\}$ is glued homeomorphically to ρ.

Recall that in G, the leaf ℓ can be written as a concatenation $\ell = \cdots \gamma_1 \rho_1 \gamma_2 \rho_2 \cdots$, where each ρ_i is either ρ or $\bar{\rho}$. Thus we can realize ℓ in G' as $\ell = \cdots \gamma_1 \rho_1' \gamma_2 \rho_2' \cdots$, where each ρ_i' is either $E_0 \rho \bar{E}_1$ or $E_1 \bar{\rho} \bar{E}_0$. In G'', each ρ_i' is homotopic rel endpoints to a path that travels along the top of R, rather than down-across-and-up. Thus, after performing a (proper!) homotopy to the image of ℓ, we can arrange that it never intersects the interior of R, nor the vertical sides of R. Cutting R along its centerline yields a $\mathcal{Z}$-splitting S of $\mathbb{F}$, and ℓ is carried by a vertex group of this splitting. If H_r is nongeometric, then S is a free splitting and if H_r is geometric, then S is a cyclic splitting. In either case, so long as S is nontrivial, we have contradicted our assumption that the lamination is $\mathcal{Z}$-filling. □

Claim 7.7. *The splitting S is nontrivial.*

Proof of Claim 7.7. We first handle the case that H_r is geometric. We have described a one-edge cyclic splitting S which was obtained as follows: cut G' along the edge E_0, that is, collapse $G' - E_0$ to get a free splitting of $\mathbb{F}$, then perform the edge fold corresponding to $\langle w \rangle$ (see Section 2L for definition), where w is the conjugacy class of the INP ρ. If $G' - E_0$ is connected, then the free splitting is an HNN extension, and there is no danger of S being trivial as $\mathrm{rk}(\mathbb{F}) \geq 3$. On the other hand, if $G' - E_0$ is disconnected, then let $G^{d'}$ and $G^{u'}$ be the components of $G' - E_0$ containing v_0^d and v_0^u respectively. The free splitting which is folded to get S is precisely $\pi_1(G^{d'}) * \pi_1(G^{u'})$. In this case, $G^{d'}$ is necessarily a component of G_r and [Feighn and Handel 2011, Proposition 2.20 (2)] together with (filtration) imply that this component is a core graph. As H_r is EG, the rank of $\pi_1(G^{d'})$ is at least two and the splitting S is therefore nontrivial. To see that $\mathrm{rk}(\pi_1(G^{u'})) \geq 1$, we need only recall that ℓ is not periodic and is carried by $\pi_1(G^{c'}) * \langle w \rangle$.

In the case that H_r is nongeometric, the splitting obtained above is a free splitting. If $G' - \{E_0, E_1\}$ is connected, then the free splitting is an HNN extension, and as before S is nontrivial. If $G' - \{E_0, E_1\}$ is disconnected, then the component containing v_0^d (and by necessity v_1^d), denoted $G^{d'}$, corresponds to a vertex group of S. By the same reasoning as in the previous case, we get that $\pi_1(G^{d'})$ is nontrivial. As before, the other vertex group of S carries the leaf ℓ and hence S is a nontrivial free splitting. □

Remark 7.8. We would like the reader to note that the above proof actually gives restrictions on the way two EG strata in a CT can interact. For example, suppose that ϕ is represented by a CT, $f : G \to G$, with exactly two strata, both of which are EG. Assume further that H_1 is nongeometric and has an INP. A priori, there are three ways that H_2 can interact with H_1: (1) there is some edge E in H_2 such that $f_\#(E)$ contains an edge from H_1 as a splitting unit, (2) the $f_\#$ image of each edge in H_2 is entirely contained in H_2, or (3) whenever E is an edge from H_2 and $f_\#(E)$ crosses H_1, the corresponding splitting unit is the INP of height 1. In the first case, $\Lambda_2 \supset \Lambda_1$. In the second case, we may think of the strata as being side-by-side, rather than H_2 being stacked on top of H_1. The proof of Lemma 7.6 implies that the third possibility never happens. Indeed, the proof provides a free splitting which is ϕ-invariant and the vertex groups of this splitting form a free factor system which lies strictly between the free factor systems $\pi_1(G_1)$ and $\pi_1(G_2)$. This contradicts (filtration) in the definition of a CT, which states that the filtration $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_M = G$ must be reduced.

Before we address the NEG linear strata and conclude the proof of Proposition 7.3, we present a final lemma concerning the structure of B.

Lemma 7.9. *Assume H_r is a linear NEG stratum consisting of an edge E_r. If v_r^B is not in the main component of B, then the component of B containing v_r^B is a single point.*

Proof. This follows directly from the definition of B, together with Lemmas 7.5 and 7.6. If H_r is a linear NEG stratum, then the definition of B implies that v_r^B has no outgoing edges. For any edge in B whose terminal vertex is v_r^B, its initial vertex necessarily corresponds to a nonlinear NEG stratum or an EG stratum, and hence is in the main component of B. □

When dealing with an NEG linear stratum, we would like to carry out a similar strategy to the EG case: blow up the terminal vertex, v_0, to an edge and glue in a cylinder, thereby producing a cyclic splitting in which ℓ is carried by a vertex group. The main difficulty in implementing this comes from other linear edges with the same axis; for each such edge, one has to decide whether to glue it in G' to v_0^d or v_0^u.

Let μ be an axis with corresponding unoriented root-free conjugacy class w. Let $\mathcal{E}_\mu$ be the set of linear edges in G with axis μ. Define a relation on $\mathcal{E}_\mu$ by declaring $E \sim_R E'$ if the quasi-exceptional path $Ew^*\bar{E}'$ is a QE-splitting unit in ℓ or if both E and E' are QE-splitting units in ℓ. Then let $\sim$ be the equivalence relation generated by $\sim_R$. Note that all edges in $\mathcal{E}_\mu$ which occur as QE-splitting units in ℓ are in the same equivalence class.

As mentioned above, the difficulty in adapting the strategy used for EG stratum to the present situation lies in deciding where to glue edges (top or bottom) in G'. The existence of multiple classes in the equivalence relation $\sim$ will provide instructions for how to glue edges from $\mathcal{E}_\mu$ in G' so that the leaf never crosses the cylinder in G''.

Lemma 7.10. *There is only one equivalence class of $\sim$. Moreover, at least one edge in $\mathcal{E}_\mu$ occurs as a term in the QE-splitting of ℓ.*

Proof. Suppose for a contradiction that there is more than one equivalence class of $\sim$ and let $[E]$ be an equivalence class for which no edge in $[E]$ is a QE-splitting unit in ℓ. Now build G' as in the proof of Lemma 7.6. Let v_0 be the terminal vertex of the edges in $\mathcal{E}_\mu$ (they all have the same terminal vertex), and define G' by blowing up v_0 into two vertices, v_0^u and v_0^d, which are connected by an edge E_0. The terminal vertex of each edge of $[E]$ is to be glued in G' to v_0^u, while all other edges in G that are incident to v_0 are glued to v_0^d. Define G'' as before, gluing the bottom of a cylinder R along the closed loop w, and gluing the vertical interval above v_0 homeomorphically to the edge E_0.

The definition of $\sim$ guarantees that ℓ is carried by a vertex group of the cyclic splitting determined by cutting along the centerline of R. Indeed, whenever ℓ crosses an edge from $[E]$, the corresponding QE-splitting unit is either an INP or a quasi-exceptional path $E'w^*\bar{E}''$, where $E', E'' \in [E]$. Repeatedly applying Lemma 7.4 to each of these terms, then rearranging and combining terms appropriately, we see that ℓ can be written in G as a concatenation $\ell = \cdots \gamma_1\rho_1\gamma_2\rho_2 \cdots$ where each ρ_i is either $E'w^*\overline{E'}$ or $E'w^*\overline{E''}$ with $E', E'' \in [E]$. Thus we can realize ℓ in G' as $\ell = \cdots \gamma_1\rho_1'\gamma_2\rho_2' \cdots$, where each ρ_i' is $E'E_0w^*\bar{E}_0\overline{E'}$ or $E'E_0w^*\bar{E}_0\overline{E''}$. In G'', each ρ_i' is homotopic rel endpoints to a path that travels along the top of R, rather than down-across-and-up. Thus, we have again produced a cyclic splitting in which ℓ is carried by a vertex group.

We now argue that the splitting is nontrivial. There is a free splitting S which comes from cutting the edge E_0 in G', which cannot be a self loop. The cyclic splitting of interest S' is obtained from S by performing the edge fold corresponding to w. If $G' - E_0$ is connected, then S' is an HNN extension with edge group $\langle [w] \rangle$. As $\mathrm{rk}(\mathbb{F}) \geq 3$, the vertex group has rank at least two and we are done. Now suppose E_0 is separating so that $G' - E_0$ consists of two components. Let G'^u be the component containing the vertex v_0^u and let G'^d be the other component. The vertex groups of the splitting S' are $\pi_1(G^d)$ and $\pi_1(G^u) * \langle [w] \rangle$. The fact that v is a principal vertex guarantees that $\pi_1(G^d) \ncong \mathbb{Z}$, and the fact that G is a finite graph without valence one vertices ensures that $\pi_1(G^u)$ is nontrivial.

The proof of the second statement is exactly the same as that of the first. $\square$

Finally, we finish the proof of Proposition 7.3. As before, B_1 is the main component of B, with corresponding almost invariant subgraph X_1. All other components $B_2, \ldots, B_K$ are single points, and each almost invariant subgraph X_i consist of a single linear edge. Let $(a_1, \ldots, a_K)$ be a K-tuple and suppose that a_1 has been chosen. We claim that imposing the admissibility condition determines all other a_i's.

Suppose first that E_i, E_j are linear edges with the same axis, μ, such that $E_i \in X_1$, $E_j \in X_k$, and $E_i \sim_R E_j$. Let d_i and d_j be the exponents of E_i and E_j respectively. Applying the definition of admissibility with $s = r = 1$, $t = k$, and κ a v^B path such that $f_\#(\kappa)$ contains a quasi-exceptional path of the form $E_i w^* \overline{E}_j$ in its QE-splitting (such a κ must exist as a quasi-exceptional path of this type occurs in the QE-splitting of ℓ), we obtain the relation $a_1(d_i - d_j) = a_1 d_i - a_k d_j$. Thus a_k is determined by a_1.

Now suppose E_i and E_j are as above, but rather than being related by $\sim_R$, we only have that $E_i \sim E_j$. There is a finite chain of $\sim_R$-relations to get from E_i to E_j. At each stage in this chain, the definition of admissibility (applied with $r = 1$ and κ chosen appropriately) will impose a relation that determines the next coordinate from the previous ones. Ultimately, this determines a_k.

We have thus shown that an admissible tuple is completely determined by choosing a_1, and therefore that the set of admissible tuples forms a line in $\mathbb{Z}^K$. Therefore $\mathcal{D}(\phi)$ is virtually cyclic.

7D. *A converse to Proposition 7.3.*

Proposition 7.11. *If ϕ has a filling lamination which is not $\mathcal{Z}$-filling, then the centralizer of some power of ϕ in* $\mathrm{Out}(\mathbb{F})$ *is not virtually cyclic.*

Proof. Since ϕ has a filling lamination which is not $\mathcal{Z}$-filling, it follows by Proposition 5.1 that for some k, ϕ^k fixes a one-edge cyclic splitting S.

Suppose $S/\mathbb{F}$ is a free product with amalgamation with vertex stabilizers $\langle A, w \rangle$ and B and edge group $\langle w \rangle \subset B$. Consider the Dehn twist D_w given by S as follows: D_w acts as identity on B and conjugation by w on A. The automorphism D_w has infinite order. We claim that D_w and ϕ^k commute. Indeed, consider a generating set $\{a_1, \ldots, a_k, b_1, \ldots b_m\}$ for $\mathbb{F}$ such that the a_i's generate A and the b_i's generate B. Choose a representative Φ of ϕ such that $\Phi^k(B) = B$ and $\Phi^k(\langle A, w \rangle) = \langle A, w \rangle^b$ for some element $b \in B$. Since D_w is identity on B and $\Phi^k(B) = B$, we have $\Phi^k(D_w(b_i)) = D_w(\Phi^k(b_i))$ for all generators b_i. Since $D_w(a_i) = w a_i \overline{w}$,

$\Phi^k(w) = w$ and $\Phi^k(\langle A, w\rangle) = \langle A, w\rangle^b$, we have $D_w(\Phi^k(a_i)) = \Phi^k(D_w(a_i))$ for all generators a_i. Thus D_w and ϕ^k commute.

We now address the case that $S/\mathbb{F}$ is an HNN extension. Assume $S/\mathbb{F}$ has stable letter t, edge group $\langle w\rangle$ and vertex group $\langle A, \bar{t}wt\rangle$. Since the cyclic splitting S is obtained from a free HNN extension, with vertex group A and stable letter t, by an edge fold, we have that a basis of $\mathbb{F}$ is given by $\{a_1, a_2, \dots, a_k, t\}$, where the a_i's generate A. Consider the Dehn twist D_w determined by S such that D_w is identity on A and sends t to wt. The automorphism D_w has infinite order. Choose a representative Φ of ϕ such that $\langle A, \bar{t}wt\rangle$ is Φ^k-invariant. Then for every generator a_i, $\Phi^k(a_i)$ is a word in the a_i's and $\bar{t}wt$. Since D_w is identity on A and fixes $\bar{t}wt$, we get $\Phi^k(D_w(a_i)) = D_w(\Phi^k(a_i))$. Again, since $\langle A, \bar{t}wt\rangle$ is Φ^k-invariant, $\Phi^k(t)$ is equal to $w^m t\alpha$, where α is some word in $\langle A, \bar{t}wt\rangle$ and $m \in \mathbb{Z}$. On one hand, $\Phi^k(D_w(t)) = \Phi^k(wt) = \Phi^k(w)\Phi^k(t) = ww^m t\alpha$ and on the other hand, $D_w(\Phi^k(t)) = D_w(w^m t\alpha) = w^m D_w(t) D_w(\alpha) = w^m wt\alpha$. Thus D_w and ϕ^k commute.

Thus when ϕ^k fixes a cyclic splitting, then an infinite order element other than a power of ϕ^k exists in the centralizer of ϕ^k. $\square$

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LIE 2-ALGEBROIDS AND MATCHED PAIRS OF 2-REPRESENTATIONS: A GEOMETRIC APPROACH

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Li-Bland's correspondence between linear Courant algebroids and Lie 2-algebroids is explained at the level of linear and core sections versus graded functions, and shown to be an equivalence of categories. More precisely, decomposed VB-Courant algebroids are shown to be equivalent to split Lie 2-algebroids in the same manner as decomposed VB-algebroids are equivalent to 2-term representations up to homotopy (Gracia-Saz and Mehta). Several special cases are discussed, yielding new examples of split Lie 2-algebroids.

We prove that the bicrossproduct of a matched pair of 2-representations is a split Lie 2-algebroid and we explain this result geometrically, as a consequence of the equivalence of VB-Courant algebroids and Lie 2-algebroids. This explains in particular how the two notions of the "double" of a matched pair of representations are geometrically related. In the same manner, we explain the geometric link between the two notions of the double of a Lie bialgebroid.

1. Introduction

Lie bialgebroids and matched pairs of Lie algebroids. A matched pair of Lie algebroids is a pair of Lie algebroids A and B over a smooth manifold M, together

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with a representation of A on B and a representation of B on A,[1] satisfying some compatibility conditions, which can be interpreted in two manners: first the direct sum $A \oplus B$ carries a Lie algebroid structure over M, such that A and B are Lie subalgebroids and such that the representations give "mixed" brackets

$$[(a,0),(0,b)] = (-\nabla_b a, \nabla_a b)$$

for all $a \in \Gamma(A)$ and $b \in \Gamma(B)$. The direct sum $A \oplus B$ with this Lie algebroid structure is called here the *bicrossproduct of the matched pair*. Note that conversely, any Lie algebroid with two transverse and complementary subalgebroids defines a matched pair of Lie algebroids [Mokri 1997].

Alternatively, the fibre product $A \times_M B$, which has a double vector bundle structure with sides A and B and with trivial core, is as follows a double Lie algebroid: for $a \in \Gamma(A)$, we write $a^l : B \to A \times_M B$, $b_m \mapsto (a(m), b_m)$ for the linear section of $A \times_M B \to B$, and similarly, a section $b \in \Gamma(B)$ defines a linear section $b^l \in \Gamma_A(A \times_M B)$. The Lie algebroid structure on $A \times_M B \to B$ is defined by

$$[a_1^l, a_2^l] = [a_1, a_2]^l \quad \text{and} \quad \rho(a^l) = \widehat{\nabla}_a \in \mathfrak{X}^l(B)$$

for $a, a_1, a_2 \in \Gamma(A)$, where we denote by $\widehat{D} \in \mathfrak{X}(B)$ the linear vector field defined by a derivation D on B. The Lie algebroid structure on $A \times_M B \to A$ is defined accordingly by the Lie bracket on sections of B and the B-connection on A. The double Lie algebroid $A \times_M B$ is then called the *double of the matched pair*. Note that conversely, any double Lie algebroid with trivial core is the fibre product of two vector bundles and defines a matched pair of Lie algebroids [Mackenzie 2011].

These two constructions encoding the compatibility conditions for a matched pair of representations seem at first sight only related by the fact that they both encode matched pairs. A similar phenomenon can be observed with the notion of Lie bialgebroid: A Lie bialgebroid is a pair of Lie algebroids $A, A^* \to M$ in duality, satisfying some compatibility conditions, which can be described in two manners. First, the direct sum $A \oplus A^* \to M$ inherits a Courant algebroid structure with the two Lie algebroids A and A^* as transverse Dirac structures, and mixed brackets given by

$$[\![(a,0),(0,\alpha)]\!] = (-\boldsymbol{i}_\alpha \boldsymbol{d}_{A^*} a, \pounds_a \alpha)$$

for all $a \in \Gamma(A)$ and $\alpha \in \Gamma(A^*)$. Alternatively, the cotangent bundle T^*A, a double vector bundle with sides A and A^* and core T^*M, which is isomorphic as a double vector bundle to T^*A^*, carries two linear Lie algebroid structures. The first, on $T^*A \to A$, is the cotangent Lie algebroid induced by the linear Poisson

[1] For the sake of simplicity, we write $\nabla : \Gamma(A) \times \Gamma(B) \to \Gamma(B)$ and $\nabla : \Gamma(B) \times \Gamma(A) \to \Gamma(A)$ for the two flat connections. It is clear from the indexes which connection is meant.

structure defined on A by the Lie algebroid structure on A^*. The second, on $T^*A \simeq T^*A^* \to A^*$, is defined in the same manner by the Lie algebroid structure on A. The compatibility conditions for A and A^* to build a Lie bialgebroid are equivalent to the double Lie algebroid condition for (T^*A, A, A^*, M) [Mackenzie 2011; Gracia-Saz et al. 2018]. Again, the cotangent double of the Lie algebroid and the bicrossproduct Courant algebroid seem only related by the fact that they are two elegant ways of encoding the Lie bialgebroid conditions.

One feature of this paper is the explanation of the deeper, more intrinsic relation between the bicrossproduct of a matched pair of Lie algebroids and its double on the one hand, and between the bicrossproduct of a Lie bialgebroid and its cotangent double on the other hand. In both cases, the bicrossproduct can be understood as a purely algebraic construction, which is *geometrised* by the corresponding double Lie algebroid. More generally, we explain how the matched pair of two 2-term representations up to homotopy [Gracia-Saz et al. 2018] defines a bicrossproduct split Lie 2-algebroid, and we relate the latter to the decomposed double Lie algebroid found in [Gracia-Saz et al. 2018] to be equivalent to the matched pair of 2-representations.

These three classes of examples of bicrossproduct constructions versus double Lie algebroid constructions are described here as three special cases of the equivalence between the category of VB-Courant algebroid, and the category of Lie 2-algebroids [Li-Bland 2012].

The equivalence of VB-Courant algebroids with Lie 2-algebroids. Let us be a little more precise. Supermanifolds were introduced in the 1970s by physicists, as a formalism to describe supersymmetric field theories, and have been extensively studied since then (see, e.g., [Sardanashvily 2009; Varadarajan 2004]). A supermanifold is a smooth manifold the algebra of functions of which is enriched by anticommuting coordinates. Supermanifolds with an additional $\mathbb{Z}$-grading have been used since the late 1990s among others in relation with Poisson geometry and Lie and Courant algebroids [Ševera 2005; Roytenberg 2002; Voronov 2002].

An equivalence between Courant algebroids and $\mathbb{N}$-manifolds of degree 2 endowed with a symplectic structure and a compatible homological vector field [Roytenberg 2002] is at the heart of the current interest in $\mathbb{N}$-graded manifolds in Poisson geometry, as this algebraic description of Courant algebroids leads to possible paths to their integration [Ševera 2005; Li-Bland and Ševera 2012; Mehta and Tang 2011]. In [Jotz Lean 2018b] we showed how the category of $\mathbb{N}$-manifolds of degree 2 is equivalent to a category of double vector bundles endowed with a linear involution. The latter involutive double vector bundles are dual to double vector bundles endowed with a linear metric. In this paper we extend this correspondence to an equivalence between the category of $\mathbb{N}$-manifolds of degree 2 endowed with a homological vector field and a category of VB-Courant algebroids, i.e., metric double vector bundles endowed with a linear Courant algebroid structure. We recover in this manner Li-Bland's one-to-one correspondence between Lie 2-algebroids and VB-Courant algebroids [2012], which we better formulate as an equivalence of categories.

Li-Bland's construction of a VB-Courant algebroid from a given Lie 2-algebroid relies on the equivalence of symplectic Lie 2-algebroids with Courant algebroids [Roytenberg 2002]: given a Lie 2-algebroid, its cotangent space is a symplectic Lie 2-algebroid, which corresponds hence to a Courant algebroid. The linear property of the cotangent space induces an additional vector bundle structure on the obtained Courant algebroid, a linear structure which turns out to be compatible with the pairing, the anchor and the bracket. While this method is nice and very simple, it is not constructive in the sense that the sheaf of graded functions on the Lie 2-algebroid are not described as a sheaf of special sections of the corresponding VB-Courant algebroid. Further, the exact correspondences of the degree 2 structure with the linear pairing (that we describe in [Jotz Lean 2018b]) and of the homological vector field with the linear anchor and bracket cannot be read directly from Li-Bland's proof.

We remedy this and provide a new formulation of Li-Bland's equivalence that *does not* use Roytenberg's description [2002] of Courant algebroids via symplectic Lie 2-algebroids. Since we explain precisely how functions of degree 0, 1 and 2 on the Lie 2-algebroid side correspond to special functions and sections of the corresponding VB-Courant algebroid, the result presented here is in our opinion more convenient to work with when looking at concrete examples.

Original motivation. Let us explain in more detail our methodology and our original motivation. A VB-Lie algebroid is a double vector bundle $(D; A, B; M)$ with one side $D \to B$ endowed with a Lie algebroid bracket and an anchor that are *linear* over a Lie algebroid structure on $A \to M$. Gracia-Saz and Mehta [2010] prove that linear decompositions of VB-algebroids are equivalent to super-representations, or in other words, to 2-representations.

The definition of a VB-Courant algebroid is very similar to the one of a VB-algebroid. The Courant bracket, the anchor and the nondegenerate pairing all have to be linear. In [Jotz Lean 2018a] we prove that the standard Courant algebroid over a vector bundle can be decomposed into a connection, a Dorfman connection, a curvature term and a vector bundle map, in a manner that resembles very much the main result in [Gracia-Saz and Mehta 2010]. In other words, as linear splittings of the tangent space TE of a vector bundle E are equivalent to linear connections on the vector bundle, linear splittings of the Pontryagin bundle $TE \oplus T^*E$ over E are equivalent to a certain class of Dorfman connections [Jotz Lean 2018a]. Further, as the Lie algebroid structure on $TE \to E$ can be described in a splitting in terms of the corresponding connection, the Courant algebroid structure on $TE \oplus T^*E \to E$

is completely encoded in a splitting by the corresponding Dorfman connection [Jotz Lean 2018a].

Our original goal in this project was to show that the work done in [Jotz Lean 2018a] is in fact a very special case of a general result on linear splittings of VB-Courant algebroids, in the spirit of Gracia-Saz and Mehta's work [2010]. Along the way, we proved the equivalence of [2]-manifolds with metric double vector bundles [Jotz Lean 2018b]. This paper builds upon that equivalence and proves that a linear Lagrangian splitting of a VB-Courant algebroid decomposes the VB-Courant algebroid structure in the components of a split Lie 2-algebroid.

Note that our correspondence of decomposed VB-Courant algebroids with split Lie 2-algebroids is also described (with slightly different conventions) in the independent work of del Carpio-Marek [2015].

While the methods used in [Jotz Lean 2018b; Li-Bland 2012] do not use splittings of the [2]-manifolds and metric double vector bundles, it appears here more natural to us to work with split objects. First, the equivalence of the underlying [2]-manifolds with metric double vector bundles was already established and it is now much more convenient to work in splittings versus Lagrangian double vector bundle charts — the definition of the homological vector field that corresponds to a linear Courant algebroid structure is easily done in splittings (see Section 3B), but we did not find a good coordinate free definition of it using the techniques given by [Jotz Lean 2018b]. Second, working with splittings is necessary in order to exhibit the similarity with Gracia-Saz and Mehta's techniques [2010], which is one of our main goals. Finally, as explained below, the construction of the bicrossproduct of a matched pair of 2-representations is an algebraic description of the construction of a *decomposed* VB-Courant algebroid from a *decomposed* double Lie algebroid, just as 2-representations are equivalent to *decomposed* VB-Lie algebroids.

Application: the bicrossproduct of a matched pair of 2-representations. The equivalence of matched pairs of 2-representations with a certain class of split Lie 2-algebroids appears as a natural class of examples of our correspondence of decomposed VB-Courant algebroids with split Lie 2-algebroids. A double vector bundle $(D; A, B; M)$ with core C and two linear Lie algebroid structures on $D \to A$ and $D \to B$ is a double Lie algebroid if and only if the pair of duals $(D^*_A; D^*_B)$ is a VB-Lie bialgebroid over C^*. Equivalently, $D^*_A \oplus_{C^*} D^*_B$ is a VB-Courant algebroid over C^*, with side $A \oplus B$ and core $B^* \oplus A^*$, and with two transverse Dirac structures D^*_A and D^*_B. A decomposition of D defines on the one hand a matched pair of 2-representations [Gracia-Saz et al. 2018], and on the other hand a Lagrangian decomposition of $D^*_A \oplus_{C^*} D^*_B$, hence a split Lie 2-algebroid. Once this geometric correspondence has been found, it is straightforward to construct algebraically the split Lie 2-algebroid from the matched pair, and vice versa.

Outline, main results and applications. This paper is organised as follows.

Section 2: We describe the main result in [Jotz Lean 2018b] — the equivalence of [2]-manifolds with metric double vector bundles — and we recall the background on double Lie algebroids and matched pairs of representations up to homotopy that will be necessary for our main application on the bicrossproduct of a matched pair of 2-representations.

Section 3: We start by recalling necessary background on Courant algebroids, Dirac structures and Dorfman connections. Then we formulate in our own manner Sheng and Zhu's definition [2017] of split Lie 2-algebroids. We write in coordinates the homological vector field corresponding to a split Lie 2-algebroid, showing where the components of the split Lie 2-algebroid appear. In Section 3D, we give several classes of examples of split Lie 2-algebroids, introducing in particular the standard split Lie 2-algebroids defined by a vector bundle. Finally we describe morphisms of split Lie 2-algebroids.

Section 4: We give the definition of VB-Courant algebroids [Li-Bland 2012] and we relate split Lie 2-algebroids with Lagrangian splittings of VB-Courant algebroids, in the spirit of Gracia-Saz and Mehta's description of split VB-algebroids via 2-term representations up to homotopy [2010]. Then we describe the VB-Courant algebroids corresponding to the examples of split Lie 2-algebroids found in the preceding section, and we prove that the equivalence of categories established in [Jotz Lean 2018b] induces an equivalence of the category of VB-Courant algebroids with the category of Lie 2-algebroids.

Section 5: We construct the bicrossproduct of a matched pair of 2-representations and prove that it is a split Lie 2-algebroid. We then explain geometrically this result by studying VB-bialgebroids and double Lie algebroids.

Appendix: We give the proof of our main theorem, describing decomposed VB-Courant algebroids via split Lie 2-algebroids.

Prerequisites, notation and conventions. We write $p_M : TM \to M$, $q_E : E \to M$ for vector bundle maps. For a vector bundle $Q \to M$ we often identify without further mention the vector bundle $(Q^*)^*$ with Q via the canonical isomorphism. We write $\langle \cdot\,, \cdot \rangle$ for the canonical pairing of a vector bundle with its dual; i.e., $\langle a_m, \alpha_m \rangle = \alpha_m(a_m)$ for $a_m \in A$ and $\alpha_m \in A^*$. We use several different pairings; in general, which pairing is used is clear from its arguments. Given a section ε of E^*, we always write $\ell_\varepsilon : E \to \mathbb{R}$ for the linear function associated to it, i.e., the function defined by $e_m \mapsto \langle \varepsilon(m), e_m \rangle$ for all $e_m \in E$.

Let M be a smooth manifold. We denote by $\mathfrak{X}(M)$ and $\Omega^1(M)$ the sheaves of local smooth sections of the tangent and the cotangent bundle, respectively. For an arbitrary vector bundle $E \to M$, the sheaf of local sections of E will be written $\Gamma(E)$. Let $f : M \to N$ be a smooth map between two smooth manifolds M

and N. Then two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are said to be *f-related* if $Tf \circ X = Y \circ f$ on $\mathrm{Dom}(X) \cap f^{-1}(\mathrm{Dom}(Y))$. We write then $X \sim_f Y$. In the same manner, if $\phi : A \to B$ is a vector bundle morphism over $\phi_0 : M \to N$, then a section $a \in \Gamma_M(A)$ is ϕ-related to $b \in \Gamma_N(B)$ if $\phi(a(m)) = b(\phi_0(m))$ for all $m \in M$. We write then $a \sim_\phi b$. The dual of the morphism ϕ is in general not a morphism of vector bundles, but a relation $R_{\phi^*} \subseteq A^* \times B^*$ defined as

$$R_{\phi^*} = \{(\phi_m^* \beta_{\phi_0(m)}, \beta_{\phi_0(m)}) \mid m \in M, \beta_{\phi_0(m)} \in B^*_{\phi_0(m)}\},$$

where $\phi_m : A_m \to B_{\phi_0(m)}$ is the morphism of vector spaces.

We will say 2-representations for 2-term representations up to homotopy. We write "$[n]$-manifold" for "$\mathbb{N}$-manifolds of degree n". We refer the reader to [Jotz Lean 2018b; Bonavolontà and Poncin 2013] for a quick review of split $\mathbb{N}$-manifolds, and for our notation convention. Let E_1 and E_2 be smooth vector bundles of finite ranks r_1, r_2 over M. The [2]-manifold $E_1[-1] \oplus E_2[-2]$ has local basis sections of ${E_i}^*$ as local generators of degree i, for $i = 1, 2$, and so dimension $(p; r_1, r_2)$. A [2]-manifold $\mathcal{M} = E_1[-1] \oplus E_2[-2]$ defined in this manner by a graded vector bundle is called a *split* [2]-*manifold*. In other words, we have

$$C^\infty(\mathcal{M})^0 = C^\infty(M), \quad C^\infty(\mathcal{M})^1 = \Gamma(E_1^*) \quad \text{and} \quad C^\infty(\mathcal{M})^2 = \Gamma(E_2^* \oplus \wedge^2 E_1^*).$$

Let $\mathcal{N} := F_1[-1] \oplus F_2[-2]$ be a second [2]-manifold over a base N. A morphism $\mu : F_1[-1] \oplus F_2[-2] \to E_1[-1] \oplus E_2[-2]$ of split [2]-manifolds over the bases N and M, respectively, consists of a smooth map $\mu_0 : N \to M$, three vector bundle morphisms $\mu_1 : F_1 \to E_1$, $\mu_2 : F_2 \to E_2$ and $\mu_{12} : \wedge^2 F_1 \to E_2$ over μ_0. The morphism $\mu^\star : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{N})$ sends a degree 1 function $\xi \in \Gamma(E_1^*)$ to ${\mu_1}^\star \xi \in \Gamma(F_1^*)$, defined by $\langle \mu_1^\star \xi, f_m \rangle = \langle \xi(\mu_0(m)), \mu_1(f_m) \rangle$ for all $f_m \in F_1(m)$. The morphism $\mu^\star$ sends a degree 2 function $\xi \in \Gamma(E_2^*)$ to ${\mu_2}^\star \xi + \mu_{12}^\star \xi \in \Gamma({F_2}^* \oplus \wedge^2 {F_1}^*)$.

2. Preliminaries

We refer to Section 2.2 of [Jotz Lean 2018b] for the definition of a double vector bundle, and for the necessary background on their linear and core sections, and on their linear splittings and dualisations. Sections 2.3–2.5 of [Jotz Lean 2018b] recall the definition of a VB-algebroid, and also the equivalence of 2-term representations up to homotopy (called here "2-representations" for short) with linear decompositions of VB-algebroids [Gracia-Saz and Mehta 2010]. The notation that we use here is the same as in [Jotz Lean 2018b].

In this section we recall the correspondence of decompositions of double Lie algebroids with matched pairs of 2-representations. Then we summarise the correspondence found in [Jotz Lean 2018b] between double vector bundles endowed with a linear metric, and $\mathbb{N}$-manifolds of degree 2.

2A. ***Double Lie algebroids and matched pairs of 2-representations.*** If $(D, A; B, M)$ is a VB-algebroid with Lie algebroid structures on $D \to B$ and $A \to M$, then the dual vector bundle $D^*_B \to B$ has a *Lie–Poisson structure* (a linear Poisson structure), and the structure on D^*_B is also Lie–Poisson with respect to $D^*_B \to C^*$ [Mackenzie 2011, 3.4]. Dualising this bundle gives a Lie algebroid structure on $(D^*_B)^*_{C^*} \to C^*$. This equips the double vector bundle $((D^*_B)^*_{C^*}; C^*, A; M)$ with a VB-algebroid structure. Using the isomorphism defined by $-\langle\cdot\,,\cdot\rangle$, (see [Mackenzie 2005] and [Jotz Lean 2018b, §2.2.4] for a summary and our sign convention), the double vector bundle $(D^*_A \to C^*; A \to M)$ is also a VB-algebroid. In the same manner, if $(D \to A, B \to M)$ is a VB-algebroid then we use $\langle\cdot\,,\cdot\rangle$ to get a VB-algebroid structure on $(D^*_B \to C^*; B \to M)$.

Let $\Sigma : A \times_M B \to D$ be a linear splitting of D and denote by $(\nabla^B, \nabla^C, R_A)$ the induced 2-representation of the Lie algebroid A on $\partial_B : C \to B$ (see [Gracia-Saz and Mehta 2010]; this is also recalled in Section 2.5 of [Jotz Lean 2018b]). The linear splitting Σ induces a linear splitting $\Sigma^\star : A \times_M C^* \to D^*_A$ of D^*_A. The 2-representation of A that is associated to this splitting is then $(\nabla^{C^*}, \nabla^{B^*}, -R^*_A)$ on the complex $\partial^*_B : B^* \to C^*$. This is proved in the appendix of [Drummond et al. 2015].

A *double Lie algebroid* [Mackenzie 2011] is a double vector bundle $(D; A, B; M)$ with core C, and with Lie algebroid structures on each of $A \to M$, $B \to M$, $D \to A$ and $D \to B$ such that each pair of parallel Lie algebroids gives D the structure of a VB-algebroid, and such that the pair (D^*_A, D^*_B) with the induced Lie algebroid structures on base C^* and the pairing $\langle\cdot\,,\cdot\rangle$, is a Lie bialgebroid.

Consider a double vector bundle $(D; A, B; M)$ with core C and a VB-Lie algebroid structure on each of its sides. After a choice of splitting $\Sigma : A \times_M B \to D$, the Lie algebroid structures on the two sides of D are described by two 2-representations [Gracia-Saz and Mehta 2010]. We prove in [Gracia-Saz et al. 2018] that (D^*_A, D^*_B) is a Lie bialgebroid over C^* if and only if, for any splitting of D, the two induced 2-representations form a matched pair as in the following definition [Gracia-Saz et al. 2018].

Definition 2.1. Let $(A \to M, \rho_A, [\cdot\,,\cdot])$ and $(B \to M, \rho_B, [\cdot\,,\cdot])$ be two Lie algebroids and assume that A acts on $\partial_B : C \to B$ up to homotopy via $(\nabla^B, \nabla^C, R_A)$ and B acts on $\partial_A : C \to A$ up to homotopy via $(\nabla^A, \nabla^C, R_B)$.[2] Then we say that the two representations up to homotopy form a matched pair if

(M1) $\nabla_{\partial_A c_1} c_2 - \nabla_{\partial_B c_2} c_1 = -(\nabla_{\partial_A c_2} c_1 - \nabla_{\partial_B c_1} c_2)$,

(M2) $[a, \partial_A c] = \partial_A(\nabla_a c) - \nabla_{\partial_B c} a$,

[2] For the sake of simplicity, we write in this definition ∇ for all the four connections. It will always be clear from the indexes which connection is meant. We write ∇^A for the A-connection induced by ∇^{AB} and ∇^{AC} on $\wedge^2 B^* \otimes C$ and ∇^B for the B-connection induced on $\wedge^2 A^* \otimes C$.

(M3) $[b, \partial_B c] = \partial_B(\nabla_b c) - \nabla_{\partial_A c} b$,

(M4) $\nabla_b \nabla_a c - \nabla_a \nabla_b c - \nabla_{\nabla_b a} c + \nabla_{\nabla_a b} c = R_B(b, \partial_B c)a - R_A(a, \partial_A c)b$,

(M5) $\partial_A(R_A(a_1, a_2)b)$
$$= -\nabla_b[a_1, a_2] + [\nabla_b a_1, a_2] + [a_1, \nabla_b a_2] + \nabla_{\nabla_{a_2} b} a_1 - \nabla_{\nabla_{a_1} b} a_2,$$

(M6) $\partial_B(R_B(b_1, b_2)a)$
$$= -\nabla_a[b_1, b_2] + [\nabla_a b_1, b_2] + [b_1, \nabla_a b_2] + \nabla_{\nabla_{b_2} a} b_1 - \nabla_{\nabla_{b_1} a} b_2,$$

for all $a, a_1, a_2 \in \Gamma(A)$, $b, b_1, b_2 \in \Gamma(B)$ and $c, c_1, c_2 \in \Gamma(C)$, and

(M7) $\boldsymbol{d}_{\nabla^A} R_B = \boldsymbol{d}_{\nabla^B} R_A \in \Omega^2(A, \wedge^2 B^* \otimes C) = \Omega^2(B, \wedge^2 A^* \otimes C)$, where R_B is seen as an element of $\Omega^1(A, \wedge^2 B^* \otimes C)$ and R_A as an element of $\Omega^1(B, \wedge^2 A^* \otimes C)$.

2B. ***The equivalence of* [2]*-manifolds with metric double vector bundles.*** We quickly recall in this section the main result in [Jotz Lean 2018b].

A *metric double vector bundle* is a double vector bundle $(\mathbb{E}, Q; B, M)$ with core Q^*, equipped with a *linear symmetric nondegenerate pairing* $\mathbb{E} \times_B \mathbb{E} \to \mathbb{R}$, i.e., such that

(1) $\langle \tau_1^\dagger, \tau_2^\dagger \rangle = 0$ for $\tau_1, \tau_2 \in \Gamma(Q^*)$,

(2) $\langle \chi, \tau^\dagger \rangle = q_B^* \langle q, \tau \rangle$ for $\chi \in \Gamma_B^l(\mathbb{E})$ linear over $q \in \Gamma(Q)$, and $\tau \in \Gamma(Q^*)$ and

(3) $\langle \chi_1, \chi_2 \rangle$ is a linear function on B for $\chi_1, \chi_2 \in \Gamma_B^l(\mathbb{E})$.

Note that the *opposite* $(\overline{\mathbb{E}}; Q; B, M)$ of a metric double vector bundle $(\mathbb{E}; B; Q, M)$ is the metric double vector bundle with $\langle \cdot\,, \cdot \rangle_{\overline{\mathbb{E}}} = -\langle \cdot\,, \cdot \rangle_{\mathbb{E}}$.

A linear splitting $\Sigma : Q \times_M B \to \mathbb{E}$ is said to be *Lagrangian* if its image is maximal isotropic in $\mathbb{E} \to B$. The corresponding horizontal lifts $\sigma_Q : \Gamma(Q) \to \Gamma_B^l(\mathbb{E})$ and $\sigma_B : \Gamma(B) \to \Gamma_Q^l(\mathbb{E})$ are then also said to be *Lagrangian*. By definition, a horizontal lift $\sigma_Q : \Gamma(Q) \to \Gamma_B^l(\mathbb{E})$ is Lagrangian if and only if $\langle \sigma_Q(q_1), \sigma_Q(q_2) \rangle = 0$ for all $q_1, q_2 \in \Gamma(Q)$. Showing the existence of a Lagrangian splitting of $\mathbb{E}$ is relatively easy [Jotz Lean 2018b]. Further, if Σ^1 and $\Sigma^2 : Q \times_M B \to \mathbb{E}$ are Lagrangian, then the change of splitting $\phi_{12} \in \Gamma(Q^* \otimes Q^* \otimes B^*)$ defined by $\Sigma^2(q, b) = \Sigma^1(q, b) + \widetilde{\phi(q, b)}$ for all $(q, b) \in Q \times_M B$, is a section of $Q^* \wedge Q^* \otimes B^*$.

Example 2.2. Let $E \to M$ be a vector bundle endowed with a symmetric nondegenerate pairing $\langle \cdot\,, \cdot \rangle : E \times_M E \to \mathbb{R}$ (a *metric vector bundle*). Then $E \simeq E^*$ and the tangent double is a metric double vector bundle $(TE, E; TM, M)$ with pairing $TE \times_{TM} TE \to \mathbb{R}$ the tangent of the pairing $E \times_M E \to \mathbb{R}$. In particular, we have $\langle Te_1, Te_2 \rangle_{TE} = \ell_{\boldsymbol{d}\langle e_1, e_2 \rangle}$, $\langle Te_1, e_2^\dagger \rangle_{TE} = p_M^* \langle e_1, e_2 \rangle$ and $\langle e_1^\dagger, e_2^\dagger \rangle_{TE} = 0$ for $e_1, e_2 \in \Gamma(E)$.

Recall from [Jotz Lean 2018b, Example 3.11] that linear splittings of TE are equivalent to linear connections $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$. The Lagrangian splittings of TE are exactly the linear splittings that correspond to *metric* connections, i.e., linear connections $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ that preserve the metric: $\langle \nabla_{\cdot} e_1, e_2 \rangle + \langle e_1, \nabla_{\cdot} e_2 \rangle = \boldsymbol{d} \langle e_1, e_2 \rangle$ for $e_1, e_2 \in \Gamma(E)$.

Let $(\mathbb{E}, B; Q, M)$ be a metric double vector bundle. Define $\mathcal{C}(\mathbb{E}) \subseteq \Gamma^l_Q(\mathbb{E})$ as the $C^\infty(M)$-submodule of linear sections with isotropic image in $\mathbb{E}$. After the choice of a Lagrangian splitting $\Sigma : Q \times_M B \to \mathbb{E}$, $\mathcal{C}(\mathbb{E})$ can be written $\mathcal{C}(\mathbb{E}) := \sigma_B(\Gamma(B)) + \{\tilde{\omega} \mid \omega \in \Gamma(Q^* \wedge Q^*)\}$. This shows that $\mathcal{C}(\mathbb{E})$ together with $\Gamma^c_Q(\mathbb{E}) \simeq \Gamma(Q^*)$ span $\mathbb{E}$ as a vector bundle over Q.

An *involutive double vector bundle* is a double vector bundle (D, Q, Q, M) with core B^* equipped with a morphism $\mathcal{I} : D \to D$ of double vector bundles satisfying $\mathcal{I}^2 = \mathrm{Id}_D$ and $\pi_1 \circ \mathcal{I} = \pi_2$, $\pi_2 \circ \mathcal{I} = \pi_1$, where $\pi_1, \pi_2 : D \to Q$ are the two side projections, and with core morphism $-\mathrm{Id}_{B^*} : B^* \to B^*$. A *morphism* $\Omega : D_1 \to D_2$ of *involutive double vector bundles* is a morphism of double vector bundles such that $\Omega \circ \mathcal{I}_1 = \mathcal{I}_2 \circ \Omega$. [Jotz Lean 2018b, Proposition 3.15] proves a duality of involutive double vector bundles with metric double vector bundles: the dual $(D^*_{\pi_1}; Q, B; M)$ with core Q^* carries an induced linear metric. Conversely, given a metric double vector bundle $(\mathbb{E}; Q, B; M)$ with core Q^*, the dual $(\mathbb{E}^*_Q; Q, Q; M)$ with core B^* carries an induced involution as above. We define morphisms of metric double vector bundles as the dual morphisms to morphisms of involutive double vector bundles. A *morphism* $\Omega : \mathbb{F} \to \mathbb{E}$ of *metric double vector bundles* is hence a relation $\Omega \subseteq \overline{\mathbb{F}} \times \mathbb{E}$ that is the dual of a morphism of involutive double vector bundles $\omega : \mathbb{F}^*_P \to \mathbb{E}^*_Q$.

$$\begin{tikzcd}
\mathbb{F}^*_P \arrow[rr, "\omega"] \arrow[dr] \arrow[dd] & & \mathbb{E}^*_Q \arrow[dr] \arrow[dd] & \\
& P \arrow[rr, "\omega_P"] \arrow[dd] & & Q \arrow[dd] \\
P^{**} \arrow[rr] \arrow[dr] & & Q^{**} \arrow[dr] & \\
& N \arrow[rr, "\omega_0"] & & M
\end{tikzcd}$$

Note that the dual of Ω is compatible with the involutions if and only if Ω is an isotropic subspace of $\overline{\mathbb{F}} \times \mathbb{E}$. Equivalently [Jotz Lean 2018b], one can define a morphism $\Omega : \mathbb{F} \to \mathbb{E}$ of metric double vector bundles as a pair of maps $\omega^\star : \mathcal{C}(\mathbb{E}) \to \mathcal{C}(\mathbb{F})$ and $\omega^\star_P : \Gamma(Q^*) \to \Gamma(P^*)$ together with a smooth map $\omega_0 : N \to M$ such that

(1) $\omega^\star(\widetilde{\tau_1 \wedge \tau_2}) = \widetilde{\omega^\star_P \tau_1 \wedge \omega^\star_P \tau_2}$,

(2) $\omega^\star(q^*_Q f \cdot \chi) = q^*_P(\omega^*_0 f) \cdot \omega^\star(\chi)$ and

(3) $\omega^\star_P(f \cdot \tau) = \omega^*_0 f \cdot \omega^\star_P \tau$

for all $\tau, \tau_1, \tau_2 \in \Gamma(Q^*)$, $f \in C^\infty(M)$ and $\chi \in \mathcal{C}(\mathbb{E})$. We write MDVB for the obtained category of metric double vector bundles. The following theorem is proved in [Jotz Lean 2018b] and independently in [del Carpio-Marek 2015].

Theorem 2.3 [Jotz Lean 2018b]. *There is a* (*covariant*) *equivalence between the category of* [2]*-manifolds and the category of involutive double vector bundles.*

Combining the obtained equivalence with the (contravariant) dualisation equivalence of IDVB with MDVB yields a (contravariant) equivalence between the category of metric double vector bundles with the morphisms defined above and the category of [2]-manifolds. This equivalence establishes in particular an equivalence between split [2]-manifold $\mathcal{M} = Q[-1] \oplus B^*[-2]$ and the decomposed metric double vector bundle $(Q \times_M B \times_M Q^*, B, Q, M)$ with the obvious linear metric over B. More precisely, the obtained functor from [2]-manifolds to metric double vector bundles sends by construction a split [2]-manifold to a decomposed metric double vector bundle. Conversely, the functor from metric double vector bundles to [2]-manifolds sends decomposed metric double vector bundles to split [2]-manifolds.

We quickly describe the functors between the two categories. To construct the geometrisation functor $\mathcal{G} : [2]\text{−Man} \to \text{MDVB}$, take a [2]-manifold and consider its local trivialisations. Changes of local trivialisation define a set of cocycle conditions, that correspond exactly to cocycle conditions for a double vector bundle atlas. The local trivialisations can hence be collated to a double vector bundle, which naturally inherits a linear pairing. See [Jotz Lean 2018b] for more details, and remark that this construction is as simple as the construction of a vector bundle from a locally free and finitely generated sheaf of $C^\infty(M)$-modules.

Conversely, the algebraisation functor $\mathcal{A}$ sends a metric double vector bundle $\mathbb{E}$ to the [2]-manifold defined as follows: the functions of degree 1 are the sections of $\Gamma_Q^c(\mathbb{E}) \simeq \Gamma(Q^*)$, and the functions of degree 2 are the elements of $\mathcal{C}(\mathbb{E})$. The multiplication of two core sections $\tau_1, \tau_2 \in \Gamma(Q^*)$ is the core-linear section $\widetilde{\tau_1 \wedge \tau_2} \in \mathcal{C}(\mathbb{E})$.

Note that while that equivalence can be seen as the special case of trivial homological vector field versus trivial bracket and anchor of Li-Bland's bijection of Lie 2-algebroids with VB-Courant algebroids [Li-Bland 2012], this corollary is not given there and only a very careful study of Li-Bland's proof, which would amount to the work done in [Jotz Lean 2018b] would yield it.

3. Split Lie 2-algebroids

In this section we recall the notions of Courant algebroids, Dirac structures, dull algebroids, Dorfman connections and (split) Lie 2-algebroids.

3A. ***Courant algebroids and Dorfman connections.*** We introduce in this section a generalisation of the notion of Courant algebroid, namely the one of *degenerate Courant algebroid with pairing in a vector bundle*. Later we will see that the fat bundle associated to a VB-Courant algebroid carries a natural Courant algebroid structure with pairing in the dual of the base.

An anchored vector bundle is a vector bundle $Q \to M$ endowed with a vector bundle morphism $\rho_Q : Q \to TM$ over the identity. Consider an anchored vector bundle $(\mathsf{E} \to M, \rho)$ and a vector bundle V over the same base M together with a morphism $\tilde{\rho} : \mathsf{E} \to \mathrm{Der}(V)$, such that the symbol of $\tilde{\rho}(e)$ is $\rho(e) \in \mathfrak{X}(M)$ for all $e \in \Gamma(\mathsf{E})$. Assume that E is paired with itself via a nondegenerate pairing $\langle \cdot\,, \cdot \rangle : \mathsf{E} \times_M \mathsf{E} \to V$ with values in V. Define $\mathcal{D} : \Gamma(V) \to \Gamma(\mathsf{E})$ by $\langle \mathcal{D}v, e\rangle = \tilde{\rho}(e)(v)$ for all $v \in \Gamma(V)$. Then $\mathsf{E} \to M$ is a *Courant algebroid with pairing in V* if E is in addition equipped with an $\mathbb{R}$-bilinear bracket $[\![\cdot\,,\cdot]\!]$ on the smooth sections $\Gamma(\mathsf{E})$ such that

(CA1) $[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!]$,

(CA2) $\tilde{\rho}(e_1)\langle e_2, e_3\rangle = \langle [\![e_1, e_2]\!], e_3\rangle + \langle e_2, [\![e_1, e_3]\!]\rangle$,

(CA3) $[\![e_1, e_2]\!] + [\![e_2, e_1]\!] = \mathcal{D}\langle e_1, e_2\rangle$,

(CA4) $\tilde{\rho}[\![e_1, e_2]\!] = [\tilde{\rho}(e_1), \tilde{\rho}(e_2)]$

for all $e_1, e_2, e_3 \in \Gamma(\mathsf{E})$ and $f \in C^\infty(M)$. Equation (CA2) implies $[\![e_1, fe_2]\!] = f[\![e_1, e_2]\!] + (\rho(e_1)f)e_2$ for all $e_1, e_2 \in \Gamma(\mathsf{E})$ and $f \in C^\infty(M)$. If $V = \mathbb{R} \times M \to M$ is in addition the trivial bundle, then $\mathcal{D} = \rho^* \circ \boldsymbol{d} : C^\infty(M) \to \Gamma(\mathsf{E})$, where E is identified with E^* via the pairing. The quadruple $(\mathsf{E} \to M, \rho, \langle \cdot\,,\cdot\rangle, [\![\cdot\,,\cdot]\!])$ is then a *Courant algebroid* [Liu et al. 1997; Roytenberg 1999] and (CA4) follows then from (CA1), (CA2) and (CA3) (see [Uchino 2002] and also [Jotz Lean 2018a] for a quicker proof).

Note that Courant algebroids with a pairing in a vector bundle E were defined in [Chen et al. 2010] and called *E-Courant algebroids*. It is easy to check that Li-Bland's *AV-Courant algebroids* [2011] yield a special class of degenerate Courant algebroids with pairing in V. The examples of Courant algebroids with pairing in a vector bundle that we will get in Theorem 4.2 are *not* AV-Courant algebroids, so the two notions are distinct.

In our study of VB-Courant algebroids, we will need the following two lemmas.

Lemma 3.1 [Roytenberg 2002]. *Let $(\mathsf{E} \to M, \rho, \langle \cdot\,,\cdot\rangle, [\![\cdot\,,\cdot]\!])$ be a Courant algebroid. For all $\theta \in \Omega^1(M)$ and $e \in \Gamma(\mathsf{E})$, we have*

$$[\![e, \rho^*\theta]\!] = \rho^*(\pounds_{\rho(e)}\theta), \qquad [\![\rho^*\theta, e]\!] = -\rho^*(\boldsymbol{i}_{\rho(e)}\boldsymbol{d}\theta)$$

and so $\rho(\rho^*\theta) = 0$, which implies $\rho \circ \mathcal{D} = 0$.

Lemma 3.2 [Li-Bland 2012]. *Let $\mathsf{E} \to M$ be a vector bundle, $\rho : \mathsf{E} \to TM$ be a bundle map, $\langle \cdot, \cdot \rangle$ be a nondegenerate pairing on E, and let $\mathcal{S} \subseteq \Gamma(\mathsf{E})$ be a subspace of sections which generates $\Gamma(\mathsf{E})$ as a $C^\infty(M)$-module. Suppose that $[\![\cdot\,,\cdot]\!] : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ is a bracket satisfying*

(1) $[\![s_1, [\![s_2, s_3]\!]]\!] = [\![[\![s_1, s_2]\!], s_3]\!] + [\![s_2, [\![s_1, s_3]\!]]\!]$,

(2) $\rho(s_1)\langle s_2, s_3\rangle = \langle [\![s_1, s_2]\!], s_3\rangle + \langle s_2, [\![s_1, s_3]\!]\rangle$,

(3) $[\![s_1, s_2]\!] + [\![s_2, s_1]\!] = \rho^* \boldsymbol{d}\,\langle s_1, s_2\rangle$,

(4) $\rho[\![s_1, s_2]\!] = [\rho(s_1), \rho(s_2)]$

for any $s_i \in \mathcal{S}$, *and that* $\rho \circ \rho^* = 0$. *Then there is a unique extension of* $[\![\cdot\,,\cdot]\!]$ *to a bracket on all of* $\Gamma(\mathsf{E})$ *such that* $(\mathsf{E}, \rho, \langle\cdot\,,\cdot\rangle, [\![\cdot\,,\cdot]\!])$ *is a Courant algebroid.*

A *Dirac structure with support* [Alekseev and Xu 2001] in a Courant algebroid $\mathsf{E} \to M$ is a subbundle $D \to S$ over a submanifold S of M, such that $D(s)$ is maximal isotropic in $\mathsf{E}(s)$ for all $s \in S$ and

$$e_1|_S \in \Gamma_S(D),\ e_2|_S \in \Gamma_S(D) \quad \Rightarrow \quad [\![e_1, e_2]\!]|_S \in \Gamma_S(D)$$

for all $e_1, e_2 \in \Gamma(\mathsf{E})$. We leave to the reader the proof of the following lemma.

Lemma 3.3. *Let* $\mathsf{E} \to M$ *be a Courant algebroid and* $D \to S$ *a subbundle, with* S *a submanifold of* M. *Assume that* $D \to S$ *is spanned by the restrictions to* S *of a family* $\mathcal{S} \subseteq \Gamma(\mathsf{E})$ *of sections of* E. *Then* D *is a Dirac structure with support* S *if and only if*

(1) $\rho_{\mathsf{E}}(e)(s) \in T_sS$ *for all* $e \in \mathcal{S}$ *and* $s \in S$,

(2) D_s *is Lagrangian in* $\mathbb{E}_s$ *for all* $s \in S$ *and*

(3) $[\![e_1, e_2]\!]|_S \in \Gamma_S(D)$ *for all* $e_1, e_2 \in \mathcal{S}$.

Next we recall the notion of Dorfman connection [Jotz Lean 2018a]. Let $(Q \to M, \rho_Q)$ be an anchored vector bundle and let B be a vector bundle over M with a fibrewise pairing $\langle\cdot\,,\cdot\rangle : Q \times_M B \to \mathbb{R}$ and an $\mathbb{R}$-linear map $\delta : C^\infty(M) \to \Gamma(B)$ with $\delta(f \cdot g) = f \cdot \delta g + g \cdot \delta f$ for all $f, g \in C^\infty(M)$. A *Dorfman (Q-)connection on* B is an $\mathbb{R}$-linear map $\Delta : \Gamma(Q) \to \Gamma(\mathrm{Der}(B))$ such that

(1) Δ_q is a derivation over $\rho_Q(q) \in \mathfrak{X}(M)$,

(2) $\Delta_{fq} b = f \Delta_q b + \langle q, b\rangle \cdot \delta f$ and

(3) $\Delta_q \delta f = \delta(\rho_Q(q) f)$

for all $f \in C^\infty(M)$, $q, q' \in \Gamma(Q)$, $b \in \Gamma(B)$. The equality $\langle q, \delta f\rangle = \rho_Q(q)(f)$ follows from (2) and (3) for $q \in \Gamma(Q)$ and $f \in C^\infty(M)$.

For instance, if $B = Q^*$, the pairing is the canonical one and $\delta = \rho_Q^* \boldsymbol{d}$, we get a Q-Dorfman connection on Q^*. The map $[\![\cdot\,,\cdot]\!]_\Delta = \Delta^* : \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q)$ that is dual to Δ in the sense of dual derivations, i.e.,

$$\langle \Delta^*_{q_1} q_2, \tau\rangle = \rho_Q(q_1)\langle q_2, \tau\rangle - \langle q_2, \Delta_{q_1}\tau\rangle$$

for all $q_1, q_2 \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$, is then a *dull bracket* on $\Gamma(Q)$ in the following sense. A *dull algebroid* is an anchored vector bundle $(Q \to M, \rho_Q)$ with a bracket $[\![\cdot\,,\cdot]\!]$ on $\Gamma(Q)$ such that

$$\rho_Q[\![q_1, q_2]\!] = [\rho_Q(q_1), \rho_Q(q_2)] \tag{1}$$

and (the Leibniz identity)

$$[\![f_1q_1, f_2q_2]\!] = f_1f_2[\![q_1, q_2]\!] + f_1\rho_Q(q_1)(f_2)q_2 - f_2\rho_Q(q_2)(f_1)q_1$$

for all $f_1, f_2 \in C^\infty(M)$, $q_1, q_2 \in \Gamma(Q)$. In other words, a dull algebroid is a *Lie algebroid* if its bracket is in addition skew-symmetric and satisfies the Jacobi identity. Note that a dull bracket can easily be skew-symmetrised.

If $Q \to M$ is endowed with a dull algebroid structure, the *curvature* of a Dorfman connection $\Delta : \Gamma(Q)\times\Gamma(B) \to \Gamma(B)$ is the map $R_\Delta : \Gamma(Q)\times\Gamma(Q) \to \Gamma(\mathrm{End}(B))$ defined on $q, q' \in \Gamma(Q)$ by $R_\Delta(q, q') := \Delta_q\Delta_{q'} - \Delta_{q'}\Delta_q - \Delta_{[\![q,q']\!]}$. As always, Δ is said to be *flat* if R_Δ vanishes.

If the dull bracket on Q is skew-symmetric, $B = Q^*$ and Δ is the Dorfman connection that is dual to the bracket, then $R_\Delta \in \Omega^2(Q, \mathrm{End}(Q^*))$. The curvature satisfies then also

$$\langle \tau, \mathrm{Jac}_{[\![\cdot,\cdot]\!]}(q_1, q_2, q_3)\rangle = \langle R_\Delta(q_1, q_2)\tau, q_3\rangle \tag{2}$$

for $q_1, q_2, q_3 \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$, where

$$\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(q_1, q_2, q_3) = [\![[\![q_1, q_2]\!]_\Delta, q_3]\!] + [\![q_2, [\![q_1, q_3]\!]]\!] - [\![q_1, [\![q_2, q_3]\!]]\!]$$

is the Jacobiator of $[\![\cdot\,,\cdot]\!]$. Hence, the Dorfman connection is flat if and only if the corresponding dull bracket satisfies the Jacobi identity in Leibniz form.

3B. *Split Lie 2-algebroids.* A *homological* vector field χ on an $[n]$-manifold $\mathcal{M}$ is a derivation of degree 1 of $C^\infty(\mathcal{M})$ such that $\mathcal{Q}^2 = \frac{1}{2}[\mathcal{Q}, \mathcal{Q}]$ vanishes. A homological vector field on a [1]-manifold $\mathcal{M} = E[-1]$ is the de Rham differential $\boldsymbol{d}_E$ associated to a Lie algebroid structure on E [Vaintrob 1997]. A *Lie n-algebroid* is an $[n]$-manifold endowed with a homological vector field (an *$\mathbb{N}Q$-manifold* of degree n).

A *split Lie n-algebroid* is a split $[n]$-manifold endowed with a homological vector field. Split Lie n-algebroids were studied by Sheng and Zhu [2017] and described as vector bundles endowed with a bracket that satisfies the Jacobi identity up to some correction terms; see also [Bonavolontà and Poncin 2013]. Our definition of a split Lie 2-algebroid turns out to be a Lie algebroid version of Baez and Crans' definition of a Lie 2-algebra [2004].

Definition 3.4. A split Lie 2-algebroid $B^* \to Q$ is the pair of an anchored vector bundle[3] $(Q \to M, \rho_Q)$ and a vector bundle $B \to M$, together with a vector bundle

[3] The names that we choose for the vector bundles will become natural in a moment.

map $l : B^* \to Q$, a skew-symmetric dull bracket[4] $[\![\cdot\,,\cdot]\!] : \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q)$, a linear connection $\nabla : \Gamma(Q) \times \Gamma(B) \to \Gamma(B)$ and a vector bundle valued 3-form $\omega \in \Omega^3(Q, B^*)$, such that

(i) $\nabla^*_{l(\beta_1)}\beta_2 + \nabla^*_{l(\beta_2)}\beta_1 = 0$ for all $\beta_1, \beta_2 \in \Gamma(B^*)$,

(ii) $[\![q, l(\beta)]\!] = l(\nabla^*_q \beta)$ for $q \in \Gamma(Q)$ and $\beta \in \Gamma(B^*)$,

(iii) $\mathrm{Jac}_{[\![\cdot,\cdot]\!]} = l \circ \omega \in \Omega^3(Q, Q)$,

(iv) $R_\nabla(q_1, q_2)b = l^*\langle \boldsymbol{i}_{q_2}\boldsymbol{i}_{q_1}\omega, b\rangle$ for $q_1, q_2 \in \Gamma(Q)$ and $b \in \Gamma(B)$, and

(v) $\boldsymbol{d}_{\nabla^*}\omega = 0$.

From (iii) follows the identity $\rho_Q \circ l = 0$. In the following, we will also work with $\partial_B := l^* : Q^* \to B$, with the Dorfman connection $\Delta : \Gamma(Q) \times \Gamma(Q^*) \to \Gamma(Q^*)$ that is dual to $[\![\cdot\,,\cdot]\!]$, and with $R_\omega \in \Omega^2(Q, \mathrm{Hom}(B, Q^*))$ which is defined by $R_\omega(q_1, q_2)b = \langle \boldsymbol{i}_{q_2}\boldsymbol{i}_{q_1}\omega, b\rangle$. Then (ii) is equivalent to $\partial_B \circ \Delta_q = \nabla_q \circ \partial_B$, (iii) is $R_\omega(q_1, q_2) \circ \partial_B = R_\Delta(q_1, q_2)$ for $q, q_1, q_2 \in \Gamma(Q)$, and (iv) is $R_\nabla(q_1, q_2) = \partial_B \circ R_\omega(q_1, q_2)$ for all $q_1, q_2 \in \Gamma(Q)$.

3C. ***Split Lie-2-algebroids as split* [2]*Q-manifolds.*** Before we go on with the study of examples, we briefly describe how to construct from the objects in Definition 3.4 the corresponding homological vector fields on split [2]-manifolds. Note that local descriptions of homological vector fields are also given in [Sheng and Zhu 2017] and [Bonavolontà and Poncin 2013].

Consider a split [2]-manifold $\mathcal{M} = Q[-1] \oplus B^*[-2]$. Assume that Q is endowed with an anchor ρ_Q and a skew-symmetric dull bracket $[\![\cdot\,,\cdot]\!]$, that it acts on B via a linear connection $\nabla : \Gamma(Q) \times \Gamma(B) \to \Gamma(B)$, that ω is an element of $\Omega^3(Q, B^*)$ and that $\partial_B : Q^* \to B$ is a vector bundle morphism. Define a vector field $\mathcal{Q}$ of degree 1 on $\mathcal{M}$ by the formulas

$$\mathcal{Q}(f) = \rho_Q^* \boldsymbol{d} f \in \Gamma(Q^*)$$

for $f \in C^\infty(M)$,

$$\mathcal{Q}(\tau) = \boldsymbol{d}_Q \tau + \partial_B \tau \in \Omega^2(Q) \oplus \Gamma(B)$$

for $\tau \in \Gamma(Q^*)$ and

$$\mathcal{Q}(b) = \boldsymbol{d}_\nabla b - \langle \omega, b\rangle \in \Omega^1(Q, B) \oplus \Omega^3(Q)$$

for $b \in \Gamma(B)$. Conversely, a relatively easy degree count and study of the graded Leibniz identity for an arbitrary vector field of degree 1 on $\mathcal{M} = Q[-1] \oplus B^*[-2]$

[4]To get the definition in [Sheng and Zhu 2017], set $l_1 := -l$, $l_3 := \omega$ and consider the skew symmetric bracket $l_2 : \Gamma(Q \oplus B^*) \times \Gamma(Q \oplus B^*) \to \Gamma(Q \oplus B^*)$, $l_2((q_1, \beta_1), (q_2, \beta_2)) = ([\![q_1, q_2]\!], \nabla^*_{q_1}\beta_2 - \nabla^*_{q_2}\beta_1)$ for $q_1, q_2 \in \Gamma(Q)$ and $\beta_1, \beta_2 \in \Gamma(B^*)$. Note that this bracket satisfies a Leibniz identity with anchor $\rho_Q \circ \mathrm{pr}_Q : Q \oplus B^* \to TM$ and that the Jacobiator of this bracket is given by $\mathrm{Jac}_{l_2}((q_1, \beta_1), (q_2, \beta_2), (q_3, \beta_3)) = (-l(\omega(q_1, q_2, q_3)), \omega(q_1, q_2, l(\beta_3)) + \text{c.p.}$

shows that it must be given as above, defining therefore an anchor ρ_Q, and the structure objects $[\![\cdot\,,\cdot]\!]$, ∇, ω and ∂_B.

We show that $\mathcal{Q}^2=0$ if and only if $(\partial_B^*: B^*\to Q, [\![\cdot\,,\cdot]\!], \nabla, \omega)$ is a split Lie 2-algebroid anchored by ρ_Q. For $f\in C^\infty(M)$ we have

$$\mathcal{Q}^2(f)=\boldsymbol{d}_Q(\rho_Q^*\boldsymbol{d}f)+\partial_B(\rho_Q^*\boldsymbol{d}f)\in\Omega^2(Q)\oplus\Gamma(B).$$

Hence $\mathcal{Q}^2(f)=0$ for all $f\in C^\infty(M)$ if and only if $\partial_B\circ\rho_Q^*=0$ and $\rho_Q[\![q_1,q_2]\!]_\Delta=[\rho_Q(q_1),\rho_Q(q_2)]$ for all $q_1,q_2\in\Gamma(Q)$. Now we assume that these two conditions are satisfied. For $\tau\in\Gamma(Q^*)$ we have

$$\mathcal{Q}^2(\tau)=(\boldsymbol{d}_Q^2\tau-\langle\omega,\partial_B\tau\rangle)+(\partial_B\boldsymbol{d}_Q\tau+\boldsymbol{d}_\nabla(\partial_B\tau))\in\Omega^3(Q)\oplus\Omega^1(Q,B),$$

where $\partial_B:\Omega^k(Q)\to\Omega^{k-1}(Q,B)$ is the vector bundle morphism defined by

$$\partial_B(\tau_1\wedge\cdots\wedge\tau_k)=\sum_{i=1}^k(-1)^{i+1}\tau_1\wedge\cdots\wedge\hat{i}\wedge\cdots\tau_k\wedge\partial_B\tau_i$$

for all $\tau_1,\tau_2\in\Gamma(Q^*)$. We find $\boldsymbol{d}_Q^2\tau(q_1,q_2,q_3)=\langle\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(q_1,q_2,q_3),\tau\rangle$ and $(\partial_B\boldsymbol{d}_Q\tau)(q,\beta)=-\langle\partial_B\Delta_q\tau,\beta\rangle$, and so $\mathcal{Q}^2(\tau)=0$ for all $\tau\in\Gamma(Q^*)$ if and only if $\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(q_1,q_2,q_3)=\partial_B^*\omega(q_1,q_2,q_3)$ for all $q_1,q_2,q_3\in\Gamma(Q)$ and $\partial_B\Delta_q\tau=\nabla_q(\partial_B\tau)$ for all $q\in\Gamma(Q)$ and $\tau\in\Gamma(Q^*)$.

Finally, we find for $b\in\Gamma(B)$:

$$\mathcal{Q}^2(b)=\mathcal{Q}(\boldsymbol{d}_\nabla b)-\boldsymbol{d}_Q\langle\omega,b\rangle-\partial_B\langle\omega,b\rangle.$$

The term $\partial_B\langle\omega,b\rangle$ is an element of $\Omega^2(Q,B)$ and the term $\boldsymbol{d}_Q\langle\omega,b\rangle$ is an element of $\Omega^4(Q)$. A computation yields that the $\Omega^4(Q)$-term of $\mathcal{Q}(\boldsymbol{d}_\nabla b)$ is $-\langle\omega,\boldsymbol{d}_\nabla b\rangle$, which is defined by

$$\langle\omega,\boldsymbol{d}_\nabla b\rangle(q_1,q_2,q_3,q_4)=\sum_{\sigma\in Z_4}(-1)^\sigma\langle\omega(q_{\sigma(1)},q_{\sigma(2)},q_{\sigma(3)}),\nabla_{q_{\sigma(4)}}b\rangle,$$

where Z_4 is the group of cyclic permutations of $\{1,2,3,4\}$. The $\Omega^2(Q,B)$-term is $R_\nabla(\cdot\,,\cdot)b$ and the $\Gamma(S^2B)$-term is $\nabla_{\partial_B^*}b$ defined by $(\nabla_{\partial_B^*}b)(\beta_1,\beta_2)=\langle\nabla_{\partial_B^*\beta_1}b,\beta_2\rangle+\langle\nabla_{\partial_B^*\beta_2}b,\beta_1\rangle$ for all $\beta_1,\beta_2\in\Gamma(B^*)$. Hence $\mathcal{Q}^2(b)=0$ if and only if $\boldsymbol{d}_Q\langle\omega,b\rangle+\langle\omega,\boldsymbol{d}_\nabla b\rangle=0$, which is equivalent to $\boldsymbol{d}_{\nabla^*}\omega=0$; $\nabla_{\partial_B^*}b=0$, which is equivalent to

$$\nabla^*_{\partial_B^*\beta_1}\beta_2+\nabla^*_{\partial_B^*\beta_2}\beta_1=0$$

for all $\beta_1,\beta_2\in\Gamma(B^*)$; and

$$R_\nabla(\cdot\,,\cdot)b=\partial_B\langle\omega,b\rangle,$$

which is equivalent to $R_{\nabla^*}(q_1,q_2)\beta=\omega(q_1,q_2,\partial_B^*\beta)$ for all $q_1,q_2\in\Gamma(Q)$ and $\beta\in\Gamma(B^*)$.

3D. ***Examples of split Lie 2-algebroids.*** We describe here four classes of examples of split Lie 2-algebroids. Later we will discuss their geometric meanings. We do not verify in detail the axioms of split Lie 2-algebroids. The computations in order to do this for Examples 3D2 and 3D3 are long, but straightforward. Note that, alternatively, the next section will provide a geometric proof of the fact that the following objects are split Lie 2-algebroids, since we will find them to be equivalent to special classes of VB-Courant algebroids. Note finally that a fifth important class of examples is discussed in Section 5.

3D1. *Lie algebroid representations.* Let $(Q \to M, \rho, [\cdot\,,\cdot])$ be a Lie algebroid and $\nabla : \Gamma(Q) \times \Gamma(B) \to \Gamma(B)$ a representation of Q on a vector bundle B. Then $(0 : B^* \to Q, [\cdot\,,\cdot], \nabla, 0)$ is a split Lie 2-algebroid. It is a semidirect extension of the Lie algebroid Q (and a special case of the bicrossproduct Lie 2-algebroids defined in Section 5A): the corresponding bracket l_2 is given by $l_2(q_1+\beta_1, q_2+\beta_2) = [q_1, q_2] + (\nabla^*_{q_1}\beta_2 - \nabla^*_{q_2}\beta_1)$ for $q_1, q_2 \in \Gamma(Q)$ and $\beta_1, \beta_2 \in \Gamma(B^*)$. Hence $(Q \oplus B^* \to M,\ \rho = \rho_Q \circ \mathrm{pr}_Q, l_2)$ is simply a Lie algebroid.

3D2. *Standard split Lie 2-algebroids.* Let $E \to M$ be a vector bundle, set

$$\partial_E = \mathrm{pr}_E : E \oplus T^*M \to E,$$

consider a skew-symmetric dull bracket $[\![\cdot\,,\cdot]\!]$ on $\Gamma(TM \oplus E^*)$, with $TM \oplus E^*$ anchored by pr_{TM}, and let

$$\Delta : \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \to \Gamma(E \oplus T^*M)$$

be the dual Dorfman connection. This defines as follows a split Lie 2-algebroid structure on the vector bundles $(TM \oplus E^*, \mathrm{pr}_{TM})$ and E^*.

Let $\nabla : \Gamma(TM \oplus E^*) \times \Gamma(E) \to \Gamma(E)$ be the ordinary linear connection[5] defined by $\nabla = \mathrm{pr}_E \circ \Delta \circ \iota_E$. The vector bundle map $l = \mathrm{pr}_E^* : E^* \to TM \oplus E^*$ is just the canonical inclusion. Define ω by $\omega(v_1, v_2, v_3) = \mathrm{Jac}_{[\![\cdot,\cdot]\!]}(v_1, v_2, v_3)$. Note that since $TM \oplus E^*$ is anchored by pr_{TM}, the tangent part of the dull bracket must just be the Lie bracket of vector fields. The Jacobiator $\mathrm{Jac}_{[\![\cdot,\cdot]\!]}$ can hence be seen as an element of $\Omega^3(TM \oplus E^*, E^*)$.

A straightforward verification of the axioms shows that l, $[\![\cdot\,,\cdot]\!]$, ∇^*, ω define a split Lie 2-algebroid. For reasons that will become clearer in Section 4D1, we call *standard* this type of split Lie 2-algebroid.

3D3. *Adjoint split Lie 2-algebroids.* The *adjoint* split Lie 2-algebroids can be described as follows. Let $\mathsf{E} \to M$ be a Courant algebroid with anchor ρ_{E} and bracket $[\![\cdot\,,\cdot]\!]$ and choose a metric linear connection $\nabla : \mathfrak{X}(M) \times \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$,

[5]To see that $\nabla = \mathrm{pr}_E \circ \Delta \circ \iota_E$ is an ordinary connection, recall that since $TM \oplus E^*$ is anchored by pr_{TM}, the map $\boldsymbol{d}_{E\oplus T^*M} = \mathrm{pr}_{TM}^* \boldsymbol{d} : C^\infty(M) \to \Gamma(E \oplus T^*M)$ sends $f \to (0, \boldsymbol{d} f)$.

i.e., a linear connection that preserves the pairing. Set $\partial_{TM} = \rho_{\mathsf{E}} : \mathsf{E} \to TM$ and identify E with its dual via the pairing. The map $\Delta : \Gamma(\mathsf{E}) \times \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$,

$$\Delta_e e' = [\![e, e']\!] + \nabla_{\rho(e')} e$$

is a Dorfman connection, which we call the *basic Dorfman connection associated to* ∇. The dual skew-symmetric(!) dull bracket is given by

$$[\![e, e']\!]_\Delta = [\![e, e']\!] - \rho^* \langle \nabla_\cdot e, e' \rangle$$

for all $e, e' \in \Gamma(\mathsf{E})$. The map

$$\nabla^{\mathrm{bas}} : \Gamma(\mathsf{E}) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \quad \nabla^{\mathrm{bas}}_e X = [\rho(e), X] + \rho(\nabla_X e)$$

is a linear connection, the *basic connection associated to* ∇.

We now define the *basic curvature* $R^{\mathrm{bas}}_\Delta \in \Omega^2(\mathsf{E}, \mathrm{Hom}(TM, \mathsf{E}))$ by[6]

$$\begin{aligned} (3) \quad R^{\mathrm{bas}}_\Delta(e_1, e_2)X = -\nabla_X [\![e_1, e_2]\!] + [\![\nabla_X e_1, e_2]\!] + [\![e_1, \nabla_X e_2]\!] \\ + \nabla_{\nabla^{\mathrm{bas}}_{e_2} X} e_1 - \nabla_{\nabla^{\mathrm{bas}}_{e_1} X} e_2 - \beta^{-1} \langle \nabla_{\nabla^{\mathrm{bas}}_\cdot X} e_1, e_2 \rangle \end{aligned}$$

for all $e_1, e_2 \in \Gamma(\mathsf{E})$ and $X \in \mathfrak{X}(M)$. Note the similarity of these constructions with the one of the adjoint representation up to homotopy (see [Gracia-Saz and Mehta 2010]). The meaning of this similarity will become clear in Section 4D3. The map l is $\rho^*_{\mathsf{E}} : T^*M \to \mathsf{E}$ and the form $\omega \in \Omega^3(\mathsf{E}, T^*M)$ is given by $\omega(e_1, e_2, e_3) = \langle R^{\mathrm{bas}}_\Delta(e_1, e_2), e_3 \rangle$. Note that it corresponds to the tensor Ψ defined in [Li-Bland 2012, Definition 4.1.2] (the right-hand side of (3)). The adjoint split Lie 2-algebroids are exactly the *split symplectic Lie 2-algebroids*, and correspond hence to splittings of the tangent doubles of Courant algebroids [Jotz Lean 2018b].

3D4. *Split Lie 2-algebroid defined by a 2-representation.* Let $(\partial_B : C \to B, \nabla, \nabla, R)$ be a representation up to homotopy of a Lie algebroid A on $B \oplus C$. We anchor $A \oplus C^*$ by $\rho_A \circ \mathrm{pr}_A$ and define $\Delta : \Gamma(A \oplus C^*) \times \Gamma(C \oplus A^*) \to \Gamma(C \oplus A^*)$ by

$$\Delta_{(a,\gamma)}(c, \alpha) = (\nabla_a c, \pounds_a \alpha + \langle \nabla^*_\cdot \gamma, c \rangle),$$

and $\nabla : \Gamma(A \oplus C^*) \times \Gamma(B) \to \Gamma(B)$ by $\nabla_{(a,\gamma)} b = \nabla_a b$. The vector bundle map l is here $l = \iota_{C^*} \circ \partial_B^*$, where $\iota_{C^*} : C^* \to A \oplus C^*$ is the canonical inclusion, and the dull bracket that is dual to Δ is given by

$$[\![(a_1, \gamma_1), (a_2, \gamma_2)]\!] = ([a_1, a_2], \nabla^*_{a_1} \gamma_2 - \nabla^*_{a_2} \gamma_1)$$

[6] We have then $R^{\mathrm{bas}}_\Delta(e_1, e_2)X = -\nabla_X [\![e_1, e_2]\!]_\Delta + [\![\nabla_X e_1, e_2]\!]_\Delta + [\![e_1, \nabla_X e_2]\!]_\Delta + \nabla_{\nabla^{\mathrm{bas}}_{e_2} X} e_1 - \nabla_{\nabla^{\mathrm{bas}}_{e_1} X} e_2 - \beta^{-1} \rho^* \langle R_\nabla(X, \cdot) e_1, e_2 \rangle$. Using $-R^*_\nabla = R_{\nabla^*} = R_\nabla$ (where we identify E with its dual using $\langle \cdot\,, \cdot \rangle$), the identity $R^{\mathrm{bas}}_\Delta(e_1, e_2) = -R^{\mathrm{bas}}_\Delta(e_2, e_1)$ is then immediate.

for $a_1, a_2 \in \Gamma(A)$, $\gamma_1, \gamma_2 \in \Gamma(C^*)$. The tensor ω is given by

$$\omega((a_1,\gamma_1),(a_2,\gamma_2),(a_3,\gamma_3)) = \langle R(a_1,a_2),\gamma_3\rangle + \text{c.p.}$$

Note that if we work with the dual A-representation up to homotopy $(\partial_B^* : B^* \to C^*, \nabla^*, \nabla^*, -R^*)$, then we get the Lie 2-algebroid defined in [Sheng and Zhu 2017, Proposition 3.5] as the semidirect product of a 2-representation and a Lie algebroid. This is then also a special case of the bicrossproduct of a matched pair of 2-representations (see Section 5A). Later we will explain why the choice that we make here is more natural.

3E. ***Morphisms of (split) Lie 2-algebroids.*** In this section we quickly discuss morphisms of split Lie 2-algebroids; see also [Bonavolontà and Poncin 2013].

A morphism $\mu : (\mathcal{M}_1, \mathcal{Q}_1) \to (\mathcal{M}_2, \mathcal{Q}_2)$ of Lie 2-algebroids is a morphism $\mu : \mathcal{M}_1 \to \mathcal{M}_2$ of the underlying [2]-manifolds, such that

$$\mu^\star \circ \mathcal{Q}_2 = \mathcal{Q}_1 \circ \mu^\star : C^\infty(\mathcal{M}_2) \to C^\infty(\mathcal{M}_1). \tag{4}$$

Assume that the two [2]-manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ are split [2]-manifolds $\mathcal{M}_1 = Q_1[-1] \oplus B_1^*[-2]$ and $\mathcal{M}_2 = Q_2[-1] \oplus B_2^*[-2]$. Then the homological vector fields $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are defined as in Section 3C with two split Lie 2-algebroids; $(\rho_1 : Q_1 \to TM_1, \partial_1 : Q_1^* \to B_1, [\![\cdot,\cdot]\!]_1, \nabla^1, \omega_1)$ and $(\rho_2 : Q_2 \to TM_2, \partial_2 : Q_2^* \to B_2, [\![\cdot,\cdot]\!]_2, \nabla^2, \omega_2)$. Further, the morphism $\mu^\star : C^\infty(\mathcal{M}_2) \to C^\infty(\mathcal{M}_1)$ over $\mu_0^* : C^\infty(M_2) \to C^\infty(M_1)$ decomposes as $\mu_Q : Q_1 \to Q_2$, $\mu_B : B_1^* \to B_2^*$ and $\mu_{12} : \wedge^2 Q_1 \to B_2^*$, all morphisms over $\mu_0 : M_1 \to M_2$. We study (4) in these decompositions.

(1) The condition $\mu^\star(\mathcal{Q}_2(f)) = \mathcal{Q}_1(\mu^\star(f))$ for all $f \in C^\infty(M_2)$ is $\mu_Q^\star(\rho_2^* \boldsymbol{d} f) = \rho_1^* \boldsymbol{d}(\mu_0^* f)$ for all $f \in C^\infty(M_2)$, which is equivalent to

$$T_m\mu_0(\rho_1(q_m)) = \rho_2(\mu_Q(q_m))$$

for all $q_m \in Q_1$. In other words $\mu_Q : Q_1 \to Q_2$ over $\mu_0 : M_1 \to M_2$ is compatible with the anchors $\rho_1 : Q_1 \to TM_1$ and $\rho_2 : Q_2 \to TM_2$.

(2) The condition $\mu^\star(\mathcal{Q}_2(\tau)) = \mathcal{Q}_1(\mu^\star(\tau))$ for all $\tau \in \Gamma(Q_2^*)$ reads

$$\mu^\star(\boldsymbol{d}_2\tau + \partial_2\tau) = \partial_1(\mu_Q^\star\tau) + \boldsymbol{d}_1(\mu_Q^\star\tau)$$

for all $\tau \in \Gamma(Q_2^*)$. The left-hand side is

$$\underbrace{\mu_Q^\star(\boldsymbol{d}_2\tau) + \mu_{12}^\star(\partial_2\tau)}_{\in\Omega^2(Q_1)} + \underbrace{\mu_B^\star(\partial_2\tau)}_{\in\Gamma(B_1)}$$

and the right-hand side is

$$\partial_1(\mu_Q^\star\tau) + \boldsymbol{d}_1(\mu_Q^\star\tau) \in \Gamma(B_1) \oplus \Omega^2(Q_1).$$

Hence, $\mu^\star \circ \mathcal{Q}_2 = \mathcal{Q}_1 \circ \mu^\star$ on degree 1 functions if and only if $\mu_Q \circ \partial_1^* = \partial_2^* \circ \mu_B$ and $\mu_Q^\star(\boldsymbol{d}_2\tau) + \mu_{12}^\star(\partial_2\tau) = \boldsymbol{d}_1(\mu_Q^\star\tau)$ for all $\tau \in \Gamma(Q_2^*)$.

(3) Finally we find that $\mu^\star(\mathcal{Q}_2(b)) = \mathcal{Q}_1(\mu^\star(b))$ for all $b \in \Gamma(B_2)$ if and only if

$$\mu^\star(\boldsymbol{d}_{\nabla^2}b) = \boldsymbol{d}_{\nabla^1}(\mu_B^\star(b)) + \partial_1(\mu_{12}^\star(b)) \in \Omega^1(Q_1, B_1)$$

for all $b \in \Gamma(B_2)$ and

$$\mu_Q^\star\omega_2 = \mu_B \circ \omega_1 - \boldsymbol{d}_{\mu_0^\star\nabla^2}\mu_{12} \in \Omega^3(Q_1, \mu_0^* B_2^*).$$

In the equalities above we have used the following constructions. The form $\mu^\star(\boldsymbol{d}_{\nabla^2}b) \in \Omega^1(Q_1, B_1)$ is defined by

$$(\mu^\star(\boldsymbol{d}_{\nabla^2}b))(q_m) = \mu_{B_m}^*(\nabla^2_{\mu_Q(q_m)}b) \in B_1(m)$$

for all $q_m \in Q_1$. Recall that μ_{12} can be seen as an element of $\Omega^2(Q_1, \mu_0^* B_2^*)$. The tensors $\mu_Q^\star\omega_2 \in \Omega^2(Q_1, \mu_0^* B_2^*)$ and $\mu_B \circ \omega_1 \in \Omega^2(Q_1, \mu_0^* B_2^*)$ can be defined as follows:

$$(\mu_Q^\star\omega_2)(q_1(m), q_2(m), q_3(m)) = \omega_2(\mu_Q(q_1(m)), \mu_Q(q_2(m)), \mu_Q(q_3(m)))$$

in $B_2^*(\mu_0(m))$, and

$$(\mu_B \circ \omega_1)(q_1(m), q_2(m), q_3(m)) = \mu_B(\omega_1)(q_1(m), q_2(m), q_3(m))) \in B_2^*(\mu_0(m))$$

for all $q_1, q_2, q_3 \in \Gamma(Q_1)$. The linear connection

$$\mu_Q^\star\nabla^2 : \Gamma(Q_1) \times \Gamma(\mu_0^* B_2^*) \to \Gamma(\mu_0^* B_2^*)$$

is defined by

$$(\mu_Q^\star\nabla^2)_q(\mu_0^! \beta)(m) = {\nabla^2}^*_{\mu_Q(q(m))}\beta \in B_2^*(\mu_0(m))$$

for all $q \in \Gamma(Q_1)$ and $\beta \in \Gamma(B_2^*)$.

We call a triple (μ_Q, μ_B, μ_{12}) over μ_0 as above a *morphism of split Lie 2-algebroids*. In particular, if $\mathcal{M}_1 = \mathcal{M}_2$, $\mu_0 = \mathrm{Id}_M : M \to M$, $\mu_Q = \mathrm{Id}_Q : Q \to Q$ and $\mu_B = \mathrm{Id}_{B^*} : B^* \to B^*$, then $\mu_{12} \in \Omega^2(Q, B^*)$ is just a change of splitting. The five conditions above simplify to the following:

(1) The dull brackets are related by $[\![q, q']\!]_2 = [\![q, q']\!]_1 + \partial_B^*\mu_{12}(q, q')$.

(2) The connections are related by $\nabla_q^2 b = \nabla_q^1 b - \partial_B\langle\mu_{12}(q, \cdot), b\rangle$.

(3) The curvature terms are related by $\omega_1 - \omega_2 = \boldsymbol{d}_{1,\nabla^2}\mu_{12}$.

The operator $\boldsymbol{d}_{1,\nabla^2} : \Omega^\bullet(Q, B^*) \to \Omega^{\bullet+1}(Q, B^*)$ is defined by the dull bracket $[\![\cdot\,,\cdot]\!]_1$ and the connection ${\nabla^2}^*$.

4. VB-Courant algebroids and Lie 2-algebroids

In this section we describe and prove in detail the equivalence between VB-Courant algebroids and Lie 2-algebroids. In short, a homological vector field on a [2]-manifold defines an anchor and a Courant bracket on the corresponding metric double vector bundle. This Courant bracket and this anchor are automatically compatible with the metric and define so a linear Courant algebroid structure on the double vector bundle. Note that a correspondence of Lie 2-algebroids and VB-Courant algebroids has already been discussed by Li-Bland [2012]. Our goal is to make this result constructive by deducing it from the results in [Jotz Lean 2018b] and presenting it as the counterpart of the main result in [Gracia-Saz and Mehta 2010], and to illustrate it with several (partly new) examples.

4A. ***Definition and observations.*** We will work with the following definition of a VB-Courant algebroid, which is due to Li-Bland [2012].

Definition 4.1. A VB-Courant algebroid is a metric double vector bundle

$$\begin{tikzcd} \mathbb{E}_{\pi_Q} \arrow[r, "\pi_B"] \arrow[d] & B \arrow[d, "q_B"] \\ Q_{q_Q} \arrow[r] & M \end{tikzcd}$$

with core Q^* such that $\mathbb{E} \to B$ is a Courant algebroid and the following conditions are satisfied.

(1) The anchor map $\Theta : \mathbb{E} \to TB$ is linear. That is,

$$(5)\qquad \begin{tikzcd} \mathbb{E}_{\pi_Q} \arrow[rr, "\pi_B"] \arrow[dd] & & B \arrow[dd, "q_B"] \\ & C \arrow[dr] & \\ Q_{q_Q} \arrow[rr] & & M \end{tikzcd} \quad \xrightarrow{\Theta} \quad \begin{tikzcd} TB_{Tq_B} \arrow[rr, "p_B"] \arrow[dd] & & B \arrow[dd, "q_B"] \\ & B \arrow[dr] & \\ TM_{p_M} \arrow[rr] & & M \end{tikzcd}$$

is a morphism of double vector bundles.

(2) The Courant bracket is linear. That is,

$$[\![\Gamma^l_B(\mathbb{E}), \Gamma^l_B(\mathbb{E})]\!] \subseteq \Gamma^l_B(\mathbb{E}), \quad [\![\Gamma^l_B(\mathbb{E}), \Gamma^c_B(\mathbb{E})]\!] \subseteq \Gamma^c_B(\mathbb{E}), \quad [\![\Gamma^c_B(\mathbb{E}), \Gamma^c_B(\mathbb{E})]\!] = 0.$$

We make the following observations. Let $\rho_Q : Q \to TM$ be the side map of the anchor, i.e., if $\pi_Q(\chi) = q$ for $\chi \in \mathbb{E}$, then $Tq_B(\Theta(\chi)) = \rho_Q(q)$. In other words, if $\chi \in \Gamma^l_B(\mathbb{E})$ is linear over $q \in \Gamma(Q)$ then $\Theta(\chi)$ is linear over $\rho_Q(q)$. Let $\partial_B : Q^* \to B$ be the core map defined by the anchor Θ as

$$\Theta(\sigma^\dagger) = (\partial_B \sigma)^\uparrow \tag{6}$$

for all $\sigma \in \Gamma(Q^*)$. (∂_B is a morphism of vector bundles.) In the following, we call ρ_Q the *side-anchor* and ∂_B the *core-anchor*. The operator $\mathcal{D} = \Theta^* \boldsymbol{d} : C^\infty(B) \to \Gamma_B(\mathbb{E})$ satisfies $\mathcal{D}(q_B^* f) = (\rho_Q^* \boldsymbol{d} f)^\dagger$ for all $f \in C^\infty(M)$ and Lemma 3.1 yields immediately

$$\partial_B \circ \rho_Q^* = 0, \quad \text{which is equivalent to } \rho_Q \circ \partial_B^* = 0. \tag{7}$$

Recall that if $\chi \in \Gamma^l_B(\mathbb{E})$ is linear over $q \in \Gamma(Q)$, then $\langle \chi, \tau^\dagger \rangle = q_B^* \langle q, \tau \rangle$ for all $\tau \in \Gamma(Q^*)$.

4B. ***The fat Courant algebroid.*** Here we denote by $\widehat{\mathbb{E}} \to M$ the fat bundle, that is the vector bundle whose sheaf of sections is the sheaf of $C^\infty(M)$-modules $\Gamma^l_B(\mathbb{E})$, the linear sections of $\mathbb{E}$ over B. Gracia-Saz and Mehta [2010] showed that if $\mathbb{E}$ is endowed with a linear Lie algebroid structure over B, then $\widehat{\mathbb{E}} \to M$ inherits a Lie algebroid structure, which is called the "fat Lie algebroid". For completeness, we describe here quickly the counterpart of this in the case of a linear Courant algebroid structure on $\mathbb{E} \to B$.

Note that the restriction of the pairing on $\mathbb{E}$ to linear sections of $\mathbb{E}$ defines a nondegenerate pairing on $\widehat{\mathbb{E}}$ with values in B^*. Since the Courant bracket of linear sections is again linear, we get the following theorem.

Theorem 4.2. *Let* $(\mathbb{E}, B, Q, M)$ *be a VB-Courant algebroid. Then* $\widehat{\mathbb{E}}$ *is a Courant algebroid with pairing in* B^*.

Note that in [Jotz Lean and Kirchhoff-Lukat 2018] we explain how the Courant algebroid with pairing in E^* that is obtained from the VB-Courant algebroid $TE \oplus T^*E$, for a vector bundle E, is equivalent to the omni-Lie algebroids described in [Chen and Liu 2010; Chen et al. 2011].

We will come back in Corollary 4.8 to the structure found in Theorem 4.2. Recall that for $\phi \in \Gamma(\operatorname{Hom}(B, Q^*))$, the core-linear section $\widetilde{\phi}$ of $\mathbb{E} \to B$ is defined by $\widetilde{\phi}(b_m) = 0_{b_m} +_B \overline{\phi(b_m)}$. Note that $\widehat{\mathbb{E}}$ is also naturally paired with Q^* via $\langle \chi(m), \sigma(m) \rangle = \langle \pi_Q(\chi(m)), \sigma(m) \rangle$ for all $\chi \in \Gamma^l_B(\mathbb{E}) = \Gamma(\widehat{\mathbb{E}})$ and $\sigma \in \Gamma(Q^*)$. This pairing is degenerate since it restricts to 0 on $\operatorname{Hom}(B, Q^*) \times_M Q^*$. The following proposition can easily be proved.

Proposition 4.3. (1) *The map* $\Delta : \Gamma(\widehat{\mathbb{E}}) \times \Gamma(Q^*) \to \Gamma(Q^*)$ *defined by* $(\Delta_\chi \tau)^\dagger = [\![\chi, \tau^\dagger]\!]$ *is a flat Dorfman connection, where* $\widehat{\mathbb{E}}$ *is endowed with the anchor* $\rho_Q \circ \pi_Q$ *and paired with* Q^* *as above. The map* $\delta : C^\infty(M) \to \Gamma(Q^*)$ *sends* f *to* $\rho^* \boldsymbol{d} f$.

(2) *The map* $\nabla : \Gamma(\widehat{\mathbb{E}}) \times \Gamma(B) \to \Gamma(B)$ *defined by* $\Theta(\chi) = \widehat{\nabla}_\chi \in \mathfrak{X}^l(B)$ *is a flat connection.*

The maps Δ *and* ∇ *satisfy*

$$\partial_B \circ \Delta = \nabla \circ \partial_B \quad \textit{and} \quad [\![\chi, \widetilde{\phi}]\!]_{\widehat{\mathbb{E}}} = \widetilde{\Delta_\chi \circ \phi - \phi \circ \nabla_\chi}$$

for $\chi \in \Gamma(\widehat{\mathbb{E}})$ *and* $\phi \in \Gamma(\mathrm{Hom}(B, Q^*))$.

Proof. (1) and (2) are easy to prove. For the first equation, choose $\chi \in \Gamma^l_B(\mathbb{E})$ and $\tau \in \Gamma(Q^*)$. Then

$$(\partial_B \circ \Delta_\chi \tau)^\uparrow = \Theta(\Delta_\chi \tau^\dagger) = \Theta([\![\chi, \tau^\dagger]\!]) = [\Theta(\chi), (\partial_B \tau)^\uparrow] = (\nabla_\chi(\partial_B \tau))^\uparrow.$$

The second equation is easy to check by writing $\widetilde{\phi} = \sum_{i=1}^n \ell_{\beta_i} \cdot \tau_i^\dagger$ with $\beta_i \in \Gamma(B^*)$ and $\tau_i \in \Gamma(Q^*)$. □

Lemma 4.4. *For* $\phi, \psi \in \Gamma(\mathrm{Hom}(B, Q^*))$ *and* $\tau \in \Gamma(Q^*)$, *we have*

(1) $[\![\tau^\dagger, \widetilde{\phi}]\!] = (\phi(\partial_B \tau))^\dagger = -[\![\widetilde{\phi}, \tau^\dagger]\!]$ *and*

(2) $[\![\widetilde{\phi}, \widetilde{\psi}]\!] = \widetilde{\psi \circ \partial_B \circ \phi - \phi \circ \partial_B \circ \psi}$.

Remark 4.5. Note that (2) is the bracket of the induced Lie algebra bundle structure induced on $\mathrm{Hom}(B, Q^*)$ by ∂_B.

Proof of Lemma 4.4. We write $\phi = \sum_{i=1}^n \beta_i \otimes \tau_i$ and $\psi = \sum_{j=1}^n \beta'_j \otimes \tau_j$ with $\beta_1, \ldots, \beta_n, \beta'_1, \ldots, \beta'_n \in \Gamma(B^*)$ and $\tau_1, \ldots, \tau_n \in \Gamma(Q^*)$. Hence, we have $\widetilde{\phi} = \sum_{i=1}^n \ell_{\beta_i} \tau_i^\dagger$ and $\widetilde{\psi} = \sum_{j=1}^n \ell_{\beta'_j} \tau_j^\dagger$. First we compute

$$\left[\!\!\left[\tau^\dagger, \sum_{i=1}^n \ell_{\beta_i} \tau_i^\dagger\right]\!\!\right] = \sum_{i=1}^n (\partial_B \tau)^\uparrow(\ell_{\beta_i}) \tau_i^\dagger = \sum_{i=1}^n q_B^* \langle \partial_B \tau, \beta_i \rangle \tau_i^\dagger = \left(\sum_{i=1}^n \langle \partial_B \tau, \beta_i \rangle \tau_i\right)^\dagger$$

and we get (1). Since $\langle \tau^\dagger, \widetilde{\phi} \rangle = 0$, the second equality follows. Then we have

$$\begin{aligned}\left[\!\!\left[\sum_{i=1}^n \ell_{\beta_i} \tau_i^\dagger, \sum_{j=1}^n \ell_{\beta'_j} \tau_j^\dagger\right]\!\!\right] &= \sum_{i=1}^n \sum_{j=1}^n \ell_{\beta_i} (\partial_B \tau_i)^\uparrow(\ell_{\beta'_j}) \tau_j^\dagger - \ell_{\beta'_j} (\partial_B \tau_j)^\uparrow(\ell_{\beta_i}) \tau_i^\dagger \\ &= \left(\sum_{i=1}^n \sum_{j=1}^n \langle \partial_B \tau_i, \beta'_j \rangle \cdot \beta_i \cdot \tau_j - \langle \partial_B \tau_j, \beta_i \rangle \cdot \beta'_j \cdot \tau_i\right)^\dagger,\end{aligned}$$

which leads to (2). □

4C. ***Lagrangian decompositions of VB-Courant algebroids.*** In this section, we study in detail the structure of VB-Courant algebroids, using Lagrangian decompositions of the underlying metric double vector bundle. Our goal is the following theorem. Note the similarity of this result with Gracia-Saz and Mehta's theorem [2010] in the VB-algebroid case.

Theorem 4.6. *Let* $(\mathbb{E}; Q, B; M)$ *be a VB-Courant algebroid and choose a Lagrangian splitting* $\Sigma : Q \times_M B \to \mathbb{E}$. *Then there is a split Lie 2-algebroid structure* $(\rho_Q, l = \partial_B^*, [\![\cdot\,,\cdot]\!], \nabla, \omega)$ *on* $Q \oplus B^*$ *such that*

$$\begin{aligned} &\Theta(\sigma_Q(q)) = \widehat{\nabla}_q \in \mathfrak{X}(B), \quad [\![\sigma_Q(q), \tau^\dagger]\!] = (\Delta_q \tau)^\dagger \\ &[\![\sigma_Q(q_1), \sigma_Q(q_2)]\!] = \sigma_Q[\![q_1, q_2]\!] - \widetilde{R_\omega(q_1, q_2)}, \end{aligned} \tag{8}$$

for all $q, q_1, q_2 \in \Gamma(Q)$ *and* $\tau \in \Gamma(Q^*)$, *where* $\Delta : \Gamma(Q) \times \Gamma(Q^*) \to \Gamma(Q^*)$ *is the Dorfman connection that is dual to the dull bracket.*

Conversely, a Lagrangian splitting $\Sigma : Q \times B^* \to \mathbb{E}$ *of the metric double vector bundle* $\mathbb{E}$ *together with a split Lie 2-algebroid on* $Q \oplus B^*$ *define by* (8) *a linear Courant algebroid structure on* $\mathbb{E}$.

First we will construct the objects $[\![\cdot\,,\cdot]\!]_\Delta, \Delta, \nabla, R$ as in the theorem, and then we will prove in the Appendix that they satisfy the axioms of a split Lie 2-algebroid.

4C1. *Construction of the split Lie 2-algebroid.* First recall that, by definition, the Courant bracket of two linear sections of $\mathbb{E} \to B$ is again linear. Hence, we can denote by $[\![q_1, q_2]\!]$ the section of Q such that

$$\pi_Q \circ [\![\sigma_Q(q_1), \sigma_Q(q_2)]\!] = [\![q_1, q_2]\!] \circ q_B. \tag{9}$$

Since for each $q \in \Gamma(Q)$, the anchor $\Theta(\sigma_Q(q))$ is a linear vector field on B over $\rho_Q(q) \in \mathfrak{X}(M)$, there exists a derivation $\nabla_q : \Gamma(B) \to \Gamma(B)$ over $\rho_Q(q)$ such that $\Theta(\sigma_Q(q)) = \widehat{\nabla}_q \in \mathfrak{X}^l(B)$. This defines a linear Q-connection $\nabla : \Gamma(Q) \times \Gamma(B) \to \Gamma(B)$. For $q \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$, the bracket $[\![\sigma_Q(q), \tau^\dagger]\!]$ is a core section. It is easy to check that the map $\Delta : \Gamma(Q) \times \Gamma(Q^*) \to \Gamma(Q^*)$ defined by

$$[\![\sigma_Q(q), \tau^\dagger]\!] = (\Delta_q \tau)^\dagger$$

is a Dorfman connection.[7]

The difference of the two linear sections $[\![\sigma_Q(q_1), \sigma_Q(q_2)]\!] - \sigma_Q([\![q_1, q_2]\!]_\sigma)$ is again a linear section, which projects to 0 under π_Q. Hence, there exists a vector bundle morphism $R(q_1, q_2) : B \to Q^*$ such that $\sigma_Q([\![q_1, q_2]\!]_\sigma) - [\![\sigma_Q(q_1), \sigma_Q(q_2)]\!] = \widetilde{R(q_1, q_2)}$. This defines a map $R : \Gamma(Q) \times \Gamma(Q) \to \Gamma(\mathrm{Hom}(B, Q^*))$. We show in the Appendix that R defines a 3-form $\omega \in \Omega^3(Q, B^*)$ by $R = R_\omega$, that $(l = \partial_B^*, [\![\cdot\,,\cdot]\!], \nabla, \omega)$ is a split Lie 2-algebroid, and that $[\![\cdot\,,\cdot]\!]$ is dual to Δ.

Conversely, choose a Lagrangian splitting $\Sigma : Q \times_M B$ of a metric double vector bundle $(\mathbb{E}, Q; B, M)$ with core Q^* and let $\mathcal{S} \subseteq \Gamma_B(\mathbb{E})$ be the subset

$$\{\tau^\dagger \mid \tau \in \Gamma(Q^*)\} \cup \{\sigma_Q(q) \mid q \in \Gamma(Q)\} \subseteq \Gamma(\mathbb{E}).$$

[7] Note that condition $(C3)$ then implies that $[\![\tau^\dagger, \sigma_Q(q)]\!] = (-\Delta_q \tau + \rho_Q^* \boldsymbol{d} \langle \tau, q\rangle)^\dagger$.

Choose a split Lie 2-algebroid $(l, [\![\cdot\,,\cdot]\!], \nabla, \omega)$ on $Q \oplus B^*$ with an anchor ρ_Q on Q. Consider the Dorfman connection Δ that is dual to the dull bracket. Define then a vector bundle map $\Theta : \mathbb{E} \to TB$ over the identity on B by $\Theta(\sigma_Q(q)) = \widehat{\nabla}_q$ and $\Theta(\tau^\dagger) = (l^*\tau)^\dagger$ and a bracket $[\![\cdot\,,\cdot]\!]$ on $\mathcal{S}$ by

$$[\![\sigma_Q(q_1), \sigma_Q(q_2)]\!] = \sigma_Q[\![q_1, q_2]\!] - \widetilde{R_\omega(q_1, q_2)}, \qquad [\![\sigma_Q(q), \tau^\dagger]\!] = (\Delta_q\tau)^\dagger,$$
$$[\![\tau^\dagger, \sigma_Q(q)]\!] = (-\Delta_q\tau + \rho_Q^*\boldsymbol{d}\langle\tau, q\rangle)^\dagger, \qquad [\![\tau_1^\dagger, \tau_2^\dagger]\!] = 0.$$

We show in the Appendix that this bracket, the pairing and the anchor satisfy the conditions of Lemma 3.2, and so $(\mathbb{E}, B; Q, M)$ with this structure is a VB-Courant algebroid.

4C2. *Change of Lagrangian decomposition.* Next we study how the split Lie 2-algebroid $(\partial_B^* : B^* \to Q, \nabla, [\![\cdot\,,\cdot]\!], \omega)$ associated to a Lagrangian decomposition of a VB-Courant algebroid changes when the Lagrangian decomposition changes.

The proof of the following proposition is straightforward and left to the reader. Compare this result with the equations at the end of Section 3E, that describe a change of splittings of a Lie 2-algebroid.

Proposition 4.7. *Let $\Sigma^1, \Sigma^2 : B \times_M Q \to \mathbb{E}$ be two Lagrangian splittings and let $\phi \in \Gamma(Q^* \otimes Q^* \otimes B^*)$ be the change of lift.*

(1) *The Dorfman connections are related by $\Delta_q^2\tau = \Delta_q^1\tau - \phi(q)(\partial_B\tau)$.*

(2) *The dull brackets are consequently related by $[\![q, q']\!]_2 = [\![q, q']\!]_1 + \partial_B^*\phi(q)^*(q')$.*

(3) *The connections are related by $\nabla_q^2 = \nabla_q^1 - \partial_B \circ \phi(q)$.*

(4) *The curvature terms are related by $\omega_1 - \omega_2 = \boldsymbol{d}_{\nabla^{2*}}\phi$, where the operator $\boldsymbol{d}_{\nabla^{2*}}$ is defined with the dull bracket $[\![\cdot\,,\cdot]\!]_1$ on $\Gamma(Q)$.*

As an application, we get the following corollary of Theorems 4.2 and 4.6. Given $\Delta : \Gamma(Q) \times \Gamma(Q^*) \to \Gamma(Q^*)$ and $\nabla : \Gamma(Q) \times \Gamma(B) \to \Gamma(B)$, we define the derivations $\Diamond : \Gamma(Q) \times \Gamma(\mathrm{Hom}(B, Q^*)) \to \Gamma(\mathrm{Hom}(B, Q^*))$ by $(\Diamond_q\phi)(b) = \Delta_q(\phi(b)) - \phi(\nabla_q b)$.

Corollary 4.8. *Let $(Q \oplus B^* \to M, \rho_Q, \partial_B^*, [\![\cdot\,,\cdot]\!], \nabla, \omega)$ be a split Lie 2-algebroid. Then the vector bundle $\mathsf{E} := Q \oplus \mathrm{Hom}(B, Q^*)$ is a Courant algebroid with pairing in B^* given by $\langle (q_1, \phi_1), (q_2, \phi_2)\rangle = \phi_1^*(q_2) + \phi_2^*(q_1)$, with the anchor $\tilde{\rho} : \mathsf{E} \to \widehat{\mathrm{Der}(B)}$, $\tilde{\rho}(q, \phi)^* = \nabla_q^* + \phi^* \circ \partial_B^*$ over $\rho(q)$ and the bracket given by*

$$[\![(q_1, \phi_1), (q_2, \phi_2)]\!] = \big([\![q_1, q_2]\!]_\Delta + \partial_B(\phi_1^*(q_2)), \Diamond_{q_1}\phi_2 - \Diamond_{q_2}\phi_1 + \nabla_\cdot^*(\phi_1^*(q_2))$$
$$+ \phi_2 \circ \partial_B \circ \phi_1 - \phi_1 \circ \partial_B \circ \phi_2 + R_\omega(q_1, q_2)\big).$$

The map $\mathcal{D} : \Gamma(B^*) \to \Gamma(\mathsf{E})$ sends q to $(\partial_B^* q, \nabla_\cdot^* q)$. The bracket does not depend on the choice of splitting.

4D. ***Examples of VB-Courant algebroids and of the corresponding split Lie 2-algebroids.*** We give here some examples of VB-Courant algebroids, and we compute the corresponding classes of split Lie 2-algebroids. We find the split Lie 2-algebroids described in Section 3D. In each of the examples below, it is easy to check that the Courant algebroid structure is linear. Hence, it is easy to check geometrically that the objects described in 3D are indeed split Lie 2-algebroids.

4D1. *The standard Courant algebroid over a vector bundle.* We have discussed this example in great detail in [Jotz Lean 2018a], but not in the language of split Lie 2-algebroids. Note further that, in [Jotz Lean 2018a], we worked with general, not necessarily Lagrangian, linear splittings.

Let $q_E : E \to M$ be a vector bundle and consider the VB-Courant algebroid

$$\begin{tikzcd} TE \oplus T^*E_{\pi_E} \arrow[r, "\Phi_E:=(q_{E*}, r_E)"] \arrow[d] & TM \oplus E^* \arrow[d] \\ E_{q_E} \arrow[r] & M \end{tikzcd}$$

with base E and side $TM \oplus E^* \to M$, and with core $E \oplus T^*M \to M$, or in other words the standard (VB-)Courant algebroid over a vector bundle $q_E : E \to M$. Recall that $TE \oplus T^*E$ has a natural linear metric (see [Jotz Lean 2018a]). Linear splittings of $TE \oplus T^*E$ are in bijection with dull brackets on sections of $TM \oplus E^*$ [Jotz Lean 2018a], and so also with Dorfman connections $\Delta : \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \to \Gamma(E \oplus T^*M)$, and Lagrangian splittings of $TE \oplus T^*E$ are in bijection with skew-symmetric dull brackets on sections of $TM \oplus E^*$ [Jotz Lean 2018b].

The anchor $\Theta = \mathrm{pr}_{TE} : TE \oplus T^*E \to TE$ restricts to the map $\partial_E = \mathrm{pr}_E : E \oplus T^*M \to E$ on the cores, and defines an anchor

$$\rho_{TM\oplus E^*} = \mathrm{pr}_{TM} : TM \oplus E^* \to TM$$

on the side. In other words, the anchor of $(e, \theta)^\dagger$ is $e^\uparrow \in \mathfrak{X}^c(E)$ and if $\widetilde{(X, \varepsilon)}$ is a linear section of $TE \oplus T^*E \to E$ over $(X, \varepsilon) \in \Gamma(TM \oplus E^*)$, the anchor $\Theta(\widetilde{(X, \varepsilon)}) \in \mathfrak{X}^l(E)$ is linear over X. Let $\iota_E : E \to E \oplus T^*M$ be the canonical inclusion. In [Jotz Lean 2018a] we proved that for $q, q_1, q_2 \in \Gamma(TM \oplus E^*)$ and $\tau, \tau_1, \tau_2 \in \Gamma(E \oplus T^*M)$, the Courant–Dorfman bracket on sections of $TE \oplus T^*E \to E$ is given by

(1) $[\![\sigma(q), \tau^\dagger]\!] = (\Delta_q \tau)^\dagger$,

(2) $[\![\sigma(q_1), \sigma(q_2)]\!] = \sigma([\![q_1, q_2]\!]_\Delta) - \widetilde{R_\Delta(q_1, q_2) \circ \iota_E}$,

and that the anchor ρ is described by $\Theta(\sigma(q)) = \widehat{\nabla}_q^* \in \mathfrak{X}(E)$, where

$$\nabla : \Gamma(TM \oplus E^*) \times \Gamma(E) \to \Gamma(E)$$

is defined by $\nabla_q = \mathrm{pr}_E \circ \Delta_q \circ \iota_E$ for all $q \in \Gamma(TM \oplus E^*)$.

Hence, if we choose a Lagrangian splitting of $TE \oplus T^*E$, we find the split Lie 2-algebroid of Section 3D2.

4D2. *The VB-Courant algebroid defined by a VB-Lie algebroid.* More generally, let

$$\begin{array}{ccc} D_{\pi_A} & \xrightarrow{\pi_B} & B \\ \downarrow & & \downarrow{\scriptstyle q_B} \\ A_{q_A} & \longrightarrow & M \end{array}$$

(with core C) be endowed with a VB-Lie algebroid structure $(D \to B, A \to M)$. Then the pair (D, D^*_B) of vector bundles over B is a Lie bialgebroid, with D^*_B endowed with the trivial Lie algebroid structure. We get a linear Courant algebroid $D \oplus_B (D^*_B)$ over B with side $A \oplus C^*$,

$$\begin{array}{ccc} D \oplus_B (D^*_B) & \longrightarrow & B \\ \downarrow & & \downarrow \\ A \oplus C^* & \longrightarrow & M \end{array}$$

and core $C \oplus A^*$. We check that the Courant algebroid structure is linear. Let $\Sigma : A \times_M B \to D$ be a linear splitting of D. Recall that we can define a linear splitting of D^*_B by $\Sigma^\star : B \times_M C^* \to D^*_B$, $\langle \Sigma^\star(b_m, \gamma_m), \Sigma(a_m, b_m)\rangle = 0$ and $\langle \Sigma^\star(b_m, \gamma_m), c^\dagger(b_m)\rangle = \langle \gamma_m, c(m)\rangle$ for all $b_m \in B$, $a_m \in A$, $\gamma_m \in C^*$ and $c \in \Gamma(C)$. The linear splitting $\widetilde{\Sigma} : B \times_M (A \oplus C^*) \to D \oplus_B (D^*_B)$, $\widetilde{\Sigma}(b_m, (a_m, \gamma_m)) = (\Sigma(a_m, b_m), \Sigma^\star(b_m, \gamma_m))$ is then a Lagrangian splitting. A computation shows that the Courant bracket on $\Gamma_B(D \oplus_B (D^*_B))$ is given by

$$\begin{aligned}
&[\![\tilde\sigma_{A\oplus C^*}(a_1, \gamma_1), \tilde\sigma_{A\oplus C^*}(a_2, \gamma_2)]\!] \\
&\qquad = ([\sigma_A(a_1), \sigma_A(a_2)], \pounds_{\sigma_A(a_1)}\sigma^\star_{C^*}(\gamma_2) - \boldsymbol{i}_{\sigma_A(a_2)}\boldsymbol{d}\,\sigma^\star_{C^*}(\gamma_1)) \\
&\qquad = \big(\sigma_A[a_1, a_2] - \widetilde{R(a_1, a_2)}, \sigma^\star_{C^*}(\nabla^*_{a_1}\gamma_2 - \nabla^*_{a_2}\gamma_1) \\
&\qquad\qquad\qquad - \widetilde{\langle \gamma_2, R(a_1, \cdot)\rangle} + \widetilde{\langle \gamma_1, R(a_2, \cdot)\rangle}\big),
\end{aligned}$$

$$[\![\tilde\sigma_{A\oplus C^*}(a, \gamma), (c, \alpha)^\dagger]\!] = (\nabla_a c^\dagger, (\pounds_a \alpha + \langle \nabla^*_\cdot \gamma, c\rangle)^\dagger),$$

$$[\![(c_1, \alpha_1)^\dagger, (c_2, \alpha_2)^\dagger]\!] = 0,$$

and the anchor of $D \oplus_B (D^*_B)$ is defined by

$$\Theta(\tilde\sigma_{A\oplus C^*}(a, \gamma)) = \Theta(\sigma_A(a)) = \widehat{\nabla}_a \in \mathfrak{X}^l(B), \quad \Theta((c, \alpha)^\dagger) = (\partial_B c)^\uparrow \in \mathfrak{X}^c(B),$$

where $(\partial_B : C \to B, \nabla : \Gamma(A) \times \Gamma(B) \to \Gamma(B), \nabla : \Gamma(A) \times \Gamma(C) \to \Gamma(C), R)$ is the 2-representation of A associated to the splitting $\Sigma : A \times_M B \to D$ of the VB-algebroid $(D \to B, A \to M)$. Hence, we have found the split Lie 2-algebroid described in Section 3D4.

4D3. *The tangent Courant algebroid.* We consider here a Courant algebroid

$$(\mathsf{E}, \rho_{\mathsf{E}}, [\![\cdot\,,\cdot]\!], \langle\cdot\,,\cdot\rangle).$$

In this example, E will always be anchored by the Courant algebroid anchor map ρ_{E} and paired with itself by $\langle\cdot\,,\cdot\rangle$ and $\mathcal{D} = \boldsymbol{\beta}^{-1} \circ \rho_{\mathsf{E}}^* \circ \boldsymbol{d} : C^\infty(M) \to \Gamma(\mathsf{E})$. Note that $[\![\cdot\,,\cdot]\!]$ is not a dull bracket.

We show that, after the choice of a metric connection on E and so of a Lagrangian splitting $\Sigma^\nabla : TM \times_M \mathsf{E} \to T\mathsf{E}$ (see Example 2.2), the VB-Courant algebroid structure on $(T\mathsf{E} \to TM, \mathsf{E} \to M)$ is equivalent to the split Lie 2-algebroid defined by ∇ as in Section 3D3.

Theorem 4.9. *Choose a linear connection $\nabla : \mathfrak{X}(M) \times \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ that preserves the pairing on E. The Courant algebroid structure on $T\mathsf{E} \to TM$ can be described as follows, for all $e, e_1, e_2 \in \Gamma(\mathsf{E})$:*

(1) *The pairing is given by*

$$\langle e_1^\dagger, e_2^\dagger\rangle = 0, \quad \langle \sigma_{\mathsf{E}}^\nabla(e_1), e_2^\dagger\rangle = p_M^*\langle e_1, e_2\rangle, \quad \textit{and} \quad \langle \sigma_{\mathsf{E}}^\nabla(e_1), \sigma_{\mathsf{E}}^\nabla(e_2)\rangle = 0.$$

(2) *The anchor is given by $\Theta(\sigma_{\mathsf{E}}^\nabla(e)) = \widehat{\nabla}_e^{\mathrm{bas}}$ and $\Theta(e^\dagger) = (\rho_{\mathsf{E}}(e))^\uparrow$.*

(3) *The bracket is given by*

$$[\![e_1^\dagger, e_2^\dagger]\!] = 0, \qquad [\![\sigma_{\mathsf{E}}^\nabla(e_1), e_2^\dagger]\!] = (\Delta_{e_1} e_2)^\dagger$$

and

$$[\![\sigma_{\mathsf{E}}^\nabla(e_1), \sigma_{\mathsf{E}}^\nabla(e_2)]\!] = \sigma_{\mathsf{E}}^\nabla([\![e_1, e_2]\!]_\Delta) - \widetilde{R_\Delta^{\mathrm{bas}}(e_1, e_2)}$$

Proof. We use the characterisation of the tangent Courant algebroid in [Boumaiza and Zaalani 2009] (see also [Li-Bland 2012]): the pairing has already been discussed in Example 2.2. It is given by $\langle Te_1, Te_2\rangle = \ell_{\boldsymbol{d}\langle e_1, e_2\rangle}$ and $\langle Te_1, e_2^\dagger\rangle = p_M^*\langle e_1, e_2\rangle$. The anchor is given by $\Theta(Te) = \widehat{\pounds_{\rho_{\mathsf{E}}(e)}} \in \mathfrak{X}(TM)$ and $\Theta(e^\dagger) = (\rho_{\mathsf{E}}(e))^\uparrow \in \mathfrak{X}(TM)$. The bracket is given by $[\![Te_1, Te_2]\!] = T[\![e_1, e_2]\!]$ and $[\![Te_1, e_2^\dagger]\!] = [\![e_1, e_2]\!]^\dagger$ for all $e, e_1, e_2 \in \Gamma(\mathsf{E})$.

(1) is easy to check (see Example 2.2 and [Jotz Lean 2018b]). We here check (2), i.e., that the anchor satisfies $\Theta(\sigma_{\mathsf{E}}^\nabla(e)) = \widehat{\nabla}_e^{\mathrm{bas}}$: For $\theta \in \Omega^1(M)$ and $v_m \in TM$, we have $\Theta(\sigma_{\mathsf{E}}^\nabla(e)(v_m))(\ell_\theta) = \ell_{\pounds_{\rho_{\mathsf{E}}(e)}\theta}(v_m) - \langle \theta_m, \rho_{\mathsf{E}}(\nabla_{v_m} e)\rangle = \ell_{\nabla_e^{\mathrm{bas}*}\theta}(v_m)$ and for $f \in C^\infty(M)$, we have

$$\Theta(\sigma_{\mathsf{E}}^\nabla(e))(p_M^* f) = p_M^*(\rho_{\mathsf{E}}(e) f).$$

This proves the equality.

Then we compute the brackets of our linear and core sections. Choose sections ϕ, ϕ' of $\mathrm{Hom}(TM, \mathsf{E})$. Then $[\![Te, \widetilde{\phi}]\!] = \widetilde{\pounds_e \phi}$, with $\pounds_e \phi \in \Gamma(\mathrm{Hom}(TM, \mathsf{E}))$ defined by $(\pounds_e \phi)(X) = [\![e, \phi(X)]\!] - \phi([\rho_{\mathsf{E}}(e), X])$ for all $X \in \mathfrak{X}(M)$. The equality

$[\![\widetilde{\phi}, Te]\!] = -\widetilde{\pounds_e\phi} + \mathcal{D}\ell_{\langle\phi(\cdot),e\rangle}$ follows. For $\theta \in \Omega^1(M)$, we compute $\langle \mathcal{D}\ell_\theta, e^\dagger\rangle = \Theta(e^\dagger)(\ell_\theta) = p_M^*\langle\rho_{\mathsf{E}}(e), \theta\rangle$. Thus, $\mathcal{D}\ell_\theta = T(\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\theta) + \widetilde{\psi}$ for a section $\psi \in \Gamma(\mathrm{Hom}(TM, \mathsf{E}))$ to be determined. Since $\langle \mathcal{D}\ell_\theta, Te\rangle = \Theta(Te)(\ell_\theta) = \ell_{\pounds_{\rho_{\mathsf{E}}(e)}\theta}$, the bracket $\langle T(\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\theta) + \widetilde{\psi}, Te\rangle = \ell_{\boldsymbol{d}\langle\theta,\rho_{\mathsf{E}}(e)\rangle + \langle\psi(\cdot),e\rangle}$ must equal $\ell_{\pounds_{\rho_{\mathsf{E}}(e)}\theta}$, and we find $\langle\psi(\cdot), e\rangle = \boldsymbol{i}_{\rho_{\mathsf{E}}(e)}\boldsymbol{d}\theta$. Because $e \in \Gamma(\mathsf{E})$ was arbitrary we find $\psi(X) = -\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\boldsymbol{i}_X\boldsymbol{d}\theta$ for $X \in \mathfrak{X}(M)$. We get in particular

$$[\![\widetilde{\phi}, Te]\!] = -\widetilde{\pounds_e\phi} + T(\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\langle\phi(\cdot), e\rangle) - \widetilde{\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\boldsymbol{i}_X\boldsymbol{d}\langle\phi(\cdot), e\rangle}.$$

The formula $[\![\widetilde{\phi}, \widetilde{\phi'}]\!] = \widetilde{\phi'\circ\rho_{\mathsf{E}}\circ\phi - \phi\circ\rho_{\mathsf{E}}\circ\phi'}$ can easily be checked, as well as $[\![\widetilde{\phi}, e^\dagger]\!] = -[\![e^\dagger, \widetilde{\phi}]\!] = -(\phi(\rho_{\mathsf{E}}(e)))^\dagger$. Using this, we find now easily that

$$\begin{aligned}
[\![\sigma_{\mathsf{E}}^\nabla(e_1), \sigma_{\mathsf{E}}^\nabla(e_2)]\!] &= [\![Te_1 - \widetilde{\nabla_\cdot e_1}, Te_2 - \widetilde{\nabla_\cdot e_2}]\!]\\
&= T[\![e_1, e_2]\!] - \widetilde{\pounds_{e_1}\nabla_\cdot e_2} + \widetilde{\pounds_{e_2}\nabla_\cdot e_1} - T(\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\langle\nabla_\cdot e_1, e_2\rangle)\\
&\qquad + \widetilde{\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\boldsymbol{d}\langle\nabla_\cdot e_1, e_2\rangle} + \widetilde{\nabla_{\rho_{\mathsf{E}}(\nabla_\cdot e_1)}e_2} - \widetilde{\nabla_{\rho_{\mathsf{E}}(\nabla_\cdot e_2)}e_1}\\
&= T[\![e_1, e_2]\!]_\Delta - \widetilde{\pounds_{e_1}\nabla_\cdot e_2} + \widetilde{\pounds_{e_2}\nabla_\cdot e_1} + \widetilde{\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\boldsymbol{d}\langle\nabla_\cdot e_1, e_2\rangle}\\
&\qquad + \widetilde{\nabla_{\rho_{\mathsf{E}}(\nabla_\cdot e_1)}e_2} - \widetilde{\nabla_{\rho_{\mathsf{E}}(\nabla_\cdot e_2)}e_1}.
\end{aligned}$$

Since for all $X \in \mathfrak{X}(M)$, we have

$$\begin{aligned}
&-(\pounds_{e_1}\nabla_\cdot e_2)(X) + (\pounds_{e_2}\nabla_\cdot e_1)(X) + \beta^{-1}\rho_{\mathsf{E}}^*\boldsymbol{i}_X\boldsymbol{d}\langle\nabla_\cdot e_1, e_2\rangle\\
&= -[\![e_1, \nabla_X e_2]\!] + \nabla_{[\rho_{\mathsf{E}}(e_1), X]}e_2 + [\![e_2, \nabla_X e_1]\!] - \nabla_{[\rho_{\mathsf{E}}(e_2), X]}e_1 + \beta^{-1}\rho_{\mathsf{E}}^*\boldsymbol{i}_X\boldsymbol{d}\langle\nabla_\cdot e_1, e_2\rangle\\
&= -[\![e_1, \nabla_X e_2]\!] + \nabla_{[\rho_{\mathsf{E}}(e_1), X]}e_2 - [\![\nabla_X e_1, e_2]\!] - \nabla_{[\rho_{\mathsf{E}}(e_2), X]}e_1 + \beta^{-1}\rho_{\mathsf{E}}^*\pounds_X\langle\nabla_\cdot e_1, e_2\rangle,
\end{aligned}$$

we find that $[\![\sigma_{\mathsf{E}}^\nabla(e_1), \sigma_{\mathsf{E}}^\nabla(e_2)]\!] = T[\![e_1, e_2]\!]_\Delta - \widetilde{R_\Delta^{\mathrm{bas}}(e_1, e_2)}$. Finally we compute $[\![\sigma_{\mathsf{E}}^\nabla(e_1), e_2^\dagger]\!] = [\![Te_1 - \widetilde{\nabla_\cdot e_1}, e_2^\dagger]\!] = [\![e_1, e_2]\!]^\dagger + \nabla_{\rho_{\mathsf{E}}(e_2)}e_1^\dagger = \Delta_{e_1}e_2^\dagger$. □

4E. ***Categorical equivalence of Lie 2-algebroids and VB-Courant algebroids.*** In this section we quickly describe morphisms of VB-Courant algebroids. Then we find an equivalence between the category of VB-Courant algebroids and the category of Lie 2-algebroids. Note that a bijection between VB-Courant algebroids and Lie 2-algebroids was already described in [Li-Bland 2012].

4E1. *Morphisms of VB-Courant algebroids.* Recall from Section 2B that a morphism $\Omega : \mathbb{E}_1 \to \mathbb{E}_2$ of metric double vector bundles is an isotropic relation $\Omega \subseteq \overline{\mathbb{E}}_1 \times \mathbb{E}_2$ that is the dual of a morphism $(E_1)^*_{Q_1} \to (E_2)^*_{Q_2}$. Assume that $\mathbb{E}_1$ and $\mathbb{E}_2$ have linear Courant algebroid structures. Then Ω is a morphism of VB-Courant algebroids if it is a Dirac structure (with support) in $\overline{\mathbb{E}}_1 \times \mathbb{E}_2$.

Choose two Lagrangian splittings $\Sigma^1 : Q_1 \times B_1 \to \mathbb{E}_1$ and $\Sigma^2 : Q_2 \times B_2 \to \mathbb{E}_2$.

Then there exist four structure maps

$$\omega_0 : M_1 \to M_2, \quad \omega_Q : Q_1 \to Q_2, \quad \omega_B : B_1^* \to B_2^*, \quad \omega_{12} \in \Omega^2(Q_1, \omega_0^* B_2^*)$$

that define completely Ω. More precisely, Ω is spanned over $\mathrm{Graph}(\omega_Q : Q_1 \to Q_2)$ by sections $\tilde{b} : \mathrm{Graph}(\omega_Q) \to \Omega$,

$$\tilde{b}(q_m, \omega_Q(q_m)) = (\sigma_{B_1}(\omega_B^\star b)(q_m) + \widetilde{\omega_{12}^\star(b)}(q_m), \sigma_{B_2}(b)(\omega_Q(q_m)))$$

for all $b \in \Gamma_{M_2}(B_2)$, and $\tau^\times : \mathrm{Graph}(\omega_Q) \to \Omega$,

$$\tau^\times(q_m, \omega_Q(q_m)) = ((\omega_Q^\star \tau)^\dagger(q_m), \tau^\dagger(\omega_Q(q_m)))$$

for all $\tau \in \Gamma_{M_2}(Q_2^*)$. Note that Ω projects under $\pi_{B_1} \times \pi_{B_2}$ to $R_{\omega_B^*} \subseteq B_1 \times B_2$. If $q \in \Gamma(Q_1)$ then $\omega_Q^! q \in \Gamma_{M_1}(\omega_0^* Q_2)$ can be written as $\sum_i f_i \omega_0^! q_i$ with $f_i \in C^\infty(M_1)$ and $q_i \in \Gamma_{M_2}(Q_2)$. The pair

$$(\sigma_{B_1}(\omega_B^\star b)(q_m) + \widetilde{\omega_{12}^\star(b)}(q_m),\ \sigma_{B_2}(b)(\omega_Q(q_m)))$$

can be written as

$$\left((\sigma_{Q_1}(q) + \langle \omega_{12}(q, \cdot), b(\omega_0(m))\rangle^\dagger)(\omega_B^\star b(m)),\ \sum_i f_i(m) \sigma_{Q_2}(q_i)(b(\omega_0(m)))\right).$$

Hence, Ω is spanned by the restrictions to $R_{\omega_B^*}$ of sections

$$\left(\sigma_{Q_1}(q) \circ \mathrm{pr}_1 + \langle \omega_{12}(q, \cdot), \mathrm{pr}_2\rangle^\dagger \circ \mathrm{pr}_1, \sum_i (f_i \circ q_{B_1} \circ \mathrm{pr}_1) \cdot (\sigma_{Q_2}(q_i) \circ \mathrm{pr}_2)\right) \tag{10}$$

for all $q \in \Gamma_{M_1}(Q_1)$ and

$$((\omega_Q^\star \tau)^\dagger \circ \mathrm{pr}_1, \tau^\dagger \circ \mathrm{pr}_2) \tag{11}$$

for all $\tau \in \Gamma(Q_2^*)$.

Checking all the conditions in Lemma 3.3 on the two types of sections (10) and (11) yields that $\Omega \to R_{\omega_B^*}$ is a Dirac structure with support if and only if

(1) $\omega_Q : Q_1 \to Q_2$ over $\omega_0 : M_1 \to M_2$ is compatible with the anchors $\rho_1 : Q_1 \to TM_1$ and $\rho_2 : Q_2 \to TM_2$:

$$T_m \omega_0(\rho_1(q_m)) = \rho_2(\omega_Q(q_m))$$

for all $q_m \in Q_1$,

(2) $\partial_1 \circ \omega_Q^\star = \omega_B^\star \circ \partial_2$ as maps from $\Gamma(Q_2^*)$ to $\Gamma(B_1)$, or equivalently $\omega_Q \circ \partial_1^* = \partial_2^* \circ \omega_B$,

(3) ω_Q preserves the dull brackets up to $\partial_2^* \omega_{12}$: i.e., $\omega_Q^\star(\boldsymbol{d}_2 \tau) + \omega_{12}^\star(\partial_2 \tau) = \boldsymbol{d}_1(\omega_Q^\star \tau)$ for all $\tau \in \Gamma(Q_2^*)$.

(4) ω_B and ω_Q intertwine the connections ∇^1 and ∇^2 up to $\partial_1 \circ \omega_{12}$:

$$\omega_B^\star((\omega_Q^\star \nabla^2)_q b) = \nabla^1_q(\omega_B^\star(b)) - \partial_1 \circ \langle \omega_{12}(q,\cdot), b\rangle \in \Gamma(B_1)$$

for all $q_m \in Q_1$ and $b \in \Gamma(B^2)$, and

(5) $\omega_Q^\star \omega_{R_2} - \omega_B \circ \omega_{R_1} = -\boldsymbol{d}_{(\omega_Q^\star \nabla^2)} \omega_{12} \in \Omega^3(Q_1, \omega_0^* B_2^*)$.

We thus find that Ω is a morphism of VB-Courant algebroids if and only if it induces a morphism of split Lie 2-algebroids after any choice of Lagrangian decompositions of $\mathbb{E}_1$ and $\mathbb{E}_2$.

4E2. *Equivalence of categories.* The functors Section 2B between the category of metric double vector bundles and the category of [2]-manifolds refine to functors between the category of VB-Courant algebroids and the category of Lie [2]-algebroids.

Theorem 4.10. *The category of Lie* 2*-algebroids is equivalent to the category of VB-Courant algebroids.*

Proof. Let $(\mathcal{M}, \mathcal{Q})$ be a Lie 2-algebroid and consider the double vector bundle $\mathbb{E}_{\mathcal{M}}$ corresponding to $\mathcal{M}$. Choose a splitting $\mathcal{M} \simeq Q[-1] \oplus B^*[-2]$ of $\mathcal{M}$ and consider the corresponding Lagrangian splitting Σ of $\mathbb{E}_{\mathcal{M}}$.

By Theorem 4.6, the split Lie 2-algebroid $(Q[-1] \oplus B^*[-2], \mathcal{Q})$ defines a VB-Courant algebroid structure on the decomposition of $\mathbb{E}_{\mathcal{M}}$ and so by isomorphism on $\mathbb{E}_{\mathcal{M}}$. Further, by Proposition 4.7, the Courant algebroid structure on $\mathbb{E}_{\mathcal{M}}$ does not depend on the choice of splitting of $\mathcal{M}$, since a different choice of splitting will induce a change of Lagrangian splitting of $\mathbb{E}_{\mathcal{M}}$. This shows that the functor $\mathcal{G}$ lifts to a functor $\mathcal{G}_Q$ from the category of Lie 2-algebroids to the category of VB-Courant algebroids.

Sections 3E and 4E1 show that morphisms of split Lie 2-algebroids are sent by $\mathcal{G}$ to morphisms of decomposed VB-Courant algebroids.

The functor $\mathcal{F}$ lifts in a similar manner to a functor $\mathcal{F}_{\mathrm{VBC}}$ from the category of VB-Courant algebroids to the category of Lie 2-algebroids. The natural transformations found in the proof of Theorem 2.3 refine to natural transformations $\mathcal{F}_{\mathrm{VBC}}\mathcal{G}_Q \simeq \mathrm{Id}$ and $\mathcal{G}_Q\mathcal{F}_{\mathrm{VBC}} \simeq \mathrm{Id}$. □

Remark 4.11. Note that we use splittings and decompositions in order to obtain this equivalence of categories, which does not involve splittings and decompositions.

First, while the linear metric of the linear VB-Courant algebroid is at the heart of the equivalence of the underlying (metric) double vector bundle $(\mathbb{E}; B, Q; M)$ with the underlying [2]-manifold of the corresponding Lie 2-algebroid, the linear Courant bracket and the linear anchor do not translate to very elegant structures on the linear isotropic sections of $\mathbb{E} \to Q$ and on its core sections. Only in a decomposition, the ingredients of the linear bracket and anchor are recognised in a straightforward manner as the ingredients of a split Lie 2-algebroid.

Since our main goal was to show that, as decomposed VB-algebroids are the same as 2-representations [Gracia-Saz and Mehta 2010], decomposed VB-Courant algebroids are the same as split Lie 2-algebroids, it is natural for us to establish here our equivalence in decompositions and splittings. The main work for the "splitting free" version of the equivalence was done in [Jotz Lean 2018b]. Another approach can of course be found in [Li-Bland 2012], but the equivalence there is not really constructive, in the sense that it is difficult to even recognise the graded functions on the underlying [2]-manifold as sections of the metric double vector bundle. To our understanding, the equivalence of [2]-manifolds with metric double vector bundles is not easy to recognise in the proof of [Li-Bland 2012].

Further, our main application in Section 5 is a statement about a certain class of *decomposed* VB-Courant algebroids versus *split* Lie 2-algebroids. Similarly, in a sequel of this paper [Jotz Lean 2018c], we work exclusively with decomposed or split objects to express Li-Bland's definition of an LA-Courant algebroid [Li-Bland 2012] in a decomposition. This yields a new definition that involves the "matched pair" of a split Lie 2-algebroid with a self-dual 2-representation. This new approach is far more useful for concrete computations, since there is no need anymore to consider the tangent triple vector bundle of $\mathbb{E}$ (see [Li-Bland 2012]).

5. VB-bialgebroids and bicrossproducts of matched pairs of 2-representations

In this section we show that the bicrossproduct of a matched pair of 2-representations is a split Lie 2-algebroid and we geometrically explain this result.

5A. ***The bicrossproduct of a matched pair of 2-representations.*** We construct a split Lie 2-algebroid $(A\oplus B)\oplus C$ induced by a matched pair of 2-representations as in Definition 2.1. The vector bundle $A\oplus B\to M$ is anchored by $\rho_A\circ \mathrm{pr}_A+\rho_B\circ\mathrm{pr}_B$ and paired with $A^*\oplus B^*$ as follows:

$$\langle (a,b),(\alpha,\beta)\rangle=\alpha(a)-\beta(b)$$

for all $a\in\Gamma(A)$, $b\in\Gamma(B)$, $\alpha\in\Gamma(A^*)$ and $\beta\in\Gamma(B^*)$. The morphism $A^*\oplus B^*\to C^*$ is $\partial_A^*\circ\mathrm{pr}_{A^*}+\partial_B^*\circ\mathrm{pr}_{B^*}$. The $A\oplus B$-Dorfman connection on $A^*\oplus B^*$ is defined by

$$\Delta_{(a,b)}(\alpha,\beta)=(\nabla_b^*\alpha+\pounds_a\alpha-\langle\nabla_{\cdot}b,\beta\rangle,\nabla_a^*\beta+\pounds_b\beta-\langle\nabla_{\cdot}a,\alpha\rangle).$$

The dual dull bracket on $\Gamma(A\oplus B)$ is

$$[\![(a,b),(a',b')]\!]=([a,a']+\nabla_ba'-\nabla_{b'}a,[b,b']+\nabla_ab'-\nabla_{a'}b). \tag{12}$$

The $A\oplus B$-connection on C^* is simply given by $\nabla^*_{(a,b)}\gamma=\nabla_a^*\gamma+\nabla_b^*\gamma$ and the dual connection is $\nabla:\Gamma(A\oplus B)\times\Gamma(C)\to\Gamma(C)$,

$$\nabla_{(a,b)}c=\nabla_ac+\nabla_bc. \tag{13}$$

Finally, the form $\omega \in \Omega^3(A \oplus B, C)$ is given by

$$\omega((a_1,b_1),(a_2,b_2),(a_3,b_3)) = R(a_1,a_2)b_3 + R(a_2,a_3)b_1 + R(a_3,a_1)b_2 \\ - R(b_1,b_2)a_3 - R(b_2,b_3)a_1 - R(b_3,b_1)a_2. \tag{14}$$

The vector bundle $(A \oplus B) \oplus C \to M$ with the anchor $\rho_A \circ \mathrm{pr}_A + \rho_B \circ \mathrm{pr}_B : A \oplus B \to TM$, $l = (-\partial_A; \partial_B) : C \to A \oplus B$, ω_R and the skew-symmetric dull bracket (12) define a split Lie 2-algebroid. Moreover, we prove the following theorem:

Theorem 5.1. *The bicrossproduct of a matched pair of* 2*-representations is a split Lie* 2*-algebroid with the structure given above. Conversely if* $(A \oplus B) \oplus C$ *has a split Lie* 2*-algebroid structure such that*

(1) $[\![(a_1,0),(a_2,0)]\!] = ([a_1,a_2],0)$ *with a section* $[a_1,a_2] \in \Gamma(A)$ *for all* $a_1, a_2 \in \Gamma(A)$ *and in the same manner* $[\![(0,b_1),(0,b_2)]\!] = (0,[b_1,b_2])$ *with a section* $[b_1,b_2] \in \Gamma(B)$ *for all* $b_1, b_2 \in \Gamma(B)$*, and*

(2) $\omega((a_1,0),(a_2,0),(a_3,0)) = 0$ *and* $\omega((0,b_1),(0,b_2),(0,b_3)) = 0$ *for all* a_1, a_2 *and* a_3 *in* $\Gamma(A)$ *and* b_1, b_2 *and* b_3 *in* $\Gamma(B)$,

then A *and* B *are Lie subalgebroids of* $(A \oplus B) \oplus C$ *and* $(A \oplus B) \oplus C$ *is the bicrossproduct of a matched pair of* 2*-representations of* A *on* $B \oplus C$ *and of* B *on* $A \oplus C$*. The* 2*-representation of* A *is given by*

$$\begin{aligned} \partial_B(c) &= \mathrm{pr}_B(l(c)), & \nabla_a b &= \mathrm{pr}_B[\![(a,0),(0,b)]\!], \\ \nabla_a c &= \nabla_{(a,0)} c, & R_{AB}(a_1,a_2)b &= \omega(a_1,a_2,b) \end{aligned} \tag{15}$$

and the B*-representation is given by*

$$\begin{aligned} \partial_A(c) &= -\mathrm{pr}_A(l(c)), & \nabla_b a &= \mathrm{pr}_A[\![(0,b),(a,0)]\!], \\ \nabla_b c &= \nabla_{(0,b)} c, & R_{BA}(b_1,b_2)a &= -\omega(b_1,b_2,a). \end{aligned} \tag{16}$$

Proof. Assume first that $(A \oplus B) \oplus C$ is a split Lie 2-algebroid with (1) and (2). The bracket $[\cdot,\cdot] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ defined by $[\![(a_1,0),(a_2,0)]\!] = ([a_1,a_2],0)$ is obviously skew-symmetric and $\mathbb{R}$-bilinear. Define an anchor ρ_A on A by $\rho_A(a) = \rho_{A\oplus B}(a,0)$. Then we get immediately

$$([a_1, f a_2], 0) = [\![(a_1,0), f(a_2,0)]\!] = f([a_1,a_2],0) + \rho_{A\oplus B}(a_1,0)(f)(a_2,0),$$

which shows that $[a_1, f a_2] = f[a_1,a_2] + \rho_A(a_1)(f)a_2$ for all $a_1, a_2 \in \Gamma(A)$. Further, we find

$$\begin{aligned} \mathrm{Jac}_{[\cdot,\cdot]}(a_1,a_2,a_3) &= \mathrm{pr}_A(\mathrm{Jac}_{[\![\cdot,\cdot]\!]}((a_1,0),(a_2,0),(a_3,0))) \\ &= -(\mathrm{pr}_A \circ l \circ \omega)((a_1,0),(a_2,0),(a_3,0))) = 0 \end{aligned}$$

since ω vanishes on sections of A. Hence A is a wide subalgebroid of the split Lie

2-algebroid. In a similar manner, we find a Lie algebroid structure on B. Next we prove that (15) defines a 2-representation of A. Using (ii) in Definition 3.4 we find for $a \in \Gamma(A)$ and $c \in \Gamma(C)$ that

$$\begin{aligned}\partial_B(\nabla_a c) &= (\mathrm{pr}_B \circ l)(\nabla_{(a,0)} c)\\ &\overset{\text{(ii)}}{=} \mathrm{pr}_B[\![(a,0), l(c)]\!] = \mathrm{pr}_B[\![(a,0),(0,\mathrm{pr}_B(l(c)))]\!] = \nabla_a(\partial_B c).\end{aligned}$$

In the third equation we have used condition (1) and in the last equation the definitions of ∂_B and $\nabla_a : \Gamma(B) \to \Gamma(B)$. In the following, we will write for simplicity a for $(a,0) \in \Gamma(A \oplus B)$, etc. We easily get

$$R_{AB}(a_1,a_2)\partial_B c = \omega(a_1,a_2,\mathrm{pr}_B(l(c))) = \omega(a_1,a_2,l(c)) \overset{\text{(iv)}}{=} R_\nabla(a_1,a_2)c$$

and

$$\partial_B R_{AB}(a_1,a_2)b = (\mathrm{pr}_B \circ l \circ \omega)(a_1,a_2,b) \overset{\text{(iii)}}{=} -\mathrm{pr}_B(\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(a_1,a_2,b))$$

for all $a_1,a_2 \in \Gamma(A)$, $b \in \Gamma(B)$ and $c \in \Gamma(C)$. By condition (1) and the definition of $\nabla_a : \Gamma(B) \to \Gamma(B)$, we find

$$\begin{aligned}R_\nabla(a_1,a_2)b &= \mathrm{pr}_B[\![a_1,[\![a_2,b]\!]]\!] - \mathrm{pr}_B[\![a_2[\![a_1,b]\!]]\!] - \mathrm{pr}_B[\![[\![a_1,a_2]\!],b]\!]\\ &= -\mathrm{pr}_B(\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(a_1,a_2,b)).\end{aligned}$$

Hence, $\partial_B R_{AB}(a_1,a_2)b = R_\nabla(a_1,a_2)b$. Finally, an easy computation along the same lines shows that

$$\langle(\boldsymbol{d}_{\nabla^{\mathrm{Hom}}} R_{AB})(a_1,a_2,a_3),b\rangle = (\boldsymbol{d}_\nabla\omega)(a_1,a_2,a_3,b) \tag{17}$$

for $a_1,a_2,a_3 \in \Gamma(A)$ and $b \in \Gamma(B)$. Since $\boldsymbol{d}_\nabla\omega = 0$, we find $\boldsymbol{d}_{\nabla^{\mathrm{Hom}}} R_{AB} = 0$. In a similar manner, we prove that (16) defines a 2-representation of B. Further, by construction of the 2-representations, the split Lie 2-algebroid structure on $(A \oplus B)\oplus C)$ must be defined as in (12), (13) and (14), with the anchor $\rho_A \circ \mathrm{pr}_A + \rho_B \circ \mathrm{pr}_B$ and $l = (-\partial_A, \partial_B)$. Hence, to conclude the proof, it only remains to check that the split Lie 2-algebroid conditions for these objects are equivalent to the seven conditions in Definition 2.1 for the two 2-representations.

First, we find immediately that (M1) is equivalent to (i). Then we find by construction

$$[a,\partial_A c] + \nabla_{\partial_B c} a = -[a,\mathrm{pr}_A(l(c))] + \nabla_{\mathrm{pr}_B(l(c))} a = \mathrm{pr}_A[\![l(c),a]\!] = -\mathrm{pr}_A[\![a,l(c)]\!].$$

Hence, we find that (M2) holds if and only if $\mathrm{pr}_A[\![a,l(c)]\!] = \mathrm{pr}_A \circ l(\nabla_a c)$. But since

$$\begin{aligned}[\![a,lc]\!] &= (\mathrm{pr}_A[\![a,l(c)]\!], \nabla_a \mathrm{pr}_B\, l(c)) = (\mathrm{pr}_A[\![a,l(c)]\!], \nabla_a\partial_B(c))\\ &= (\mathrm{pr}_A[\![a,l(c)]\!], \partial_B\nabla_a c) = (\mathrm{pr}_A[\![a,l(c)]\!], \mathrm{pr}_B(l(\nabla_a c))),\end{aligned}$$

we have $\mathrm{pr}_A[\![a,l(c)]\!] = \mathrm{pr}_A \circ l(\nabla_a c)$ if and only if $[\![a,lc]\!] = l(\nabla_a c)$. Hence (M2) is satisfied if and only if $[\![a, l(c)]\!] = l(\nabla_a c)$ for all $a \in \Gamma(A)$ and $c \in \Gamma(C)$. In a similar manner, we find that (M3) is equivalent to $[\![b, lc]\!] = l(\nabla_b c)$ for all $b \in \Gamma(B)$ and $c \in \Gamma(C)$. This shows that (M2) and (M3) together are equivalent to (ii).

Next, a simple computation shows that (M4) is equivalent to $R_\nabla(b,a)c = \omega(b,a,l(c))$. Since

$$R_\nabla(a,a')c = R_{AB}(a,a')\partial_B c = \omega(a,a',\mathrm{pr}_B(l(c))) = \omega(a,a',l(c))$$

and $R_\nabla(b,b')c = \omega(b,b',l(c))$, we get that (M4) is equivalent to (iv).

Two straightforward computations show that (M5) is equivalent to

$$\mathrm{pr}_A(\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(a_1,a_2,b)) = -\,\mathrm{pr}_A(l\omega(a_1,a_2,b))$$

and that (M6) is equivalent to

$$\mathrm{pr}_B(\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(b_1,b_2,a)) = -\,\mathrm{pr}_B(l\omega(b_1,b_2,a)).$$

But since $\mathrm{pr}_B(\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(a_1,a_2,b)) = -R_\nabla(a_1,a_2)b$ by construction and

$$R_\nabla(a_1,a_2)b = \partial_B R_{AB}(a_1,a_2)b = \mathrm{pr}_B(l\omega(a_1,a_2,b)),$$

we find

$$\mathrm{pr}_B(\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(a_1,a_2,b)) = -\,\mathrm{pr}_B(l\omega(a_1,a_2,b)),$$

and in a similar manner

$$\mathrm{pr}_A(\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(b_1,b_2,a)) = -\,\mathrm{pr}_A(l\omega(b_1,b_2,a)).$$

Since $\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(a_1,a_2,a_3) = 0$, $\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(b_1,b_2,b_3) = 0$, and ω vanishes on sections of A, and respectively on sections of B, we conclude that (M5) and (M6) together are equivalent to (iii).

Finally, a slightly longer, but still straightforward computation shows that

$$(\boldsymbol{d}_{\nabla^B} R_{AB})(b_1,b_2)(a_1,a_2) - (\boldsymbol{d}_{\nabla^A} R_{BA})(a_1,a_2)(b_1,b_2) = (\boldsymbol{d}_\nabla \omega)(a_1,a_2,b_1,b_2)$$

for all $a_1, a_2 \in \Gamma(A)$ and $b_1, b_2 \in \Gamma(B)$. This, (17), the corresponding identity for R_{BA}, and the vanishing of ω on sections of A, and, respectively, on sections of B, show that (M7) is equivalent to (v). $\square$

If $C = 0$, then $R_{AB} = 0$, $R_{BA} = 0$, $\partial_A = 0$ and $\partial_B = 0$ and the matched pair of 2-representations is just a matched pair of Lie algebroids. The double is then concentrated in degree 0, with $\omega = 0$, and l_2 is the bicrossproduct Lie algebroid structure on $A \oplus B$ with anchor $\rho_A + \rho_B$ [Lu 1997; Mokri 1997]. Hence, in that case the split Lie 2-algebroid is just the bicrossproduct of a matched pair of representations and the dual (flat) Dorfman connection is the corresponding Lie derivative. The Lie 2-algebroid is in that case a genuine Lie 1-algebroid.

In the case where B has a trivial Lie algebroid structure and acts trivially up to homotopy on $\partial_A = 0 : C \to A$, the double is the semidirect product Lie 2-algebroid found in [Sheng and Zhu 2017, Proposition 3.5] (see Section 3D4).

5B. ***VB-bialgebroids and double Lie algebroids.*** Consider a double vector bundle $(D; A, B; M)$ with core C and a VB-Lie algebroid structure on each of its sides. Recall from Section 2A that $(D; A, B, M)$ is a double Lie algebroid if and only if, for any linear splitting of D, the two induced 2-representations (denoted as in Section 2A) form a matched pair [Gracia-Saz et al. 2018]. By definition of a double Lie algebroid, (D_A^*, D_B^*) is then a Lie bialgebroid over C^* [Mackenzie 2011], and so the double vector bundle

$$\begin{tikzcd} D_A^* \oplus D_B^* \arrow[r] \arrow[d] & C^* \arrow[d] \\ A \oplus B \arrow[r] & M \end{tikzcd}$$

with core $B^* \oplus A^*$ has the structure of a VB-Courant algebroid with base C^* and side $A \oplus B$. Note that we call the pair (D_A^*, D_B^*) a *VB-bialgebroid over* C^*. Conversely, a VB-Courant algebroid $(\mathbb{E}; Q, B; M)$ with two transverse VB-Dirac structures $(D_1; Q_1, B; M)$ and $(D_2; Q_2, B; M)$ defines a VB-bialgebroid (D_1, D_2) over B. It is not difficult to see that a VB-bialgebroid[8] $(D_A \to X, A \to M)$, $(D_B \to X, B \to M)$ is equivalent to a double Lie algebroid structure on

$$((D_A)_A^*; B, A; M) \simeq ((D_B)_B^*; B, A; M)$$

with core X^*.

Consider again a double Lie algebroid $(D; A, B; M)$, together with a linear splitting $\Sigma : A \times_M B \to D$. Then the "dual splittings" $\sigma_A^\star : \Gamma(A) \to \Gamma_{C^*}^l(D_A^*)$ and $\sigma_B^\star : \Gamma(B) \to \Gamma_{C^*}^l(D_B^*)$ are defined as in Section 2.2.3 in [Jotz Lean 2018b], and satisfy the equations

$$\langle \sigma_A^\star(a), \sigma_B^\star(b) \rangle = 0, \quad \langle \sigma_A^\star(a), \alpha^\dagger \rangle = -q_{C^*}^* \langle \alpha, a \rangle, \quad \langle \beta^\dagger, \sigma_B^\star(b) \rangle = q_{C^*}^* \langle \beta, b \rangle, \tag{18}$$

for all $a \in \Gamma(A)$, $b \in \Gamma(B)$, $\alpha \in \Gamma(A^*)$ and $\beta \in \Gamma(B^*)$.

Then

$$\tilde{\Sigma} : (A \oplus B) \times_M C^* \to D_A^* \oplus D_B^*,$$

defined by $\tilde{\Sigma}((a(m), b(m)), \gamma_m) = (\sigma_A^\star(a)(\gamma_m), \sigma_B^\star(b)(\gamma_m))$, is a linear Lagrangian splitting of $D_A^* \oplus D_B^*$.

Recall from Section 2A that the splitting

$$\Sigma^\star : A \times_M C^* \to D_A^*$$

[8] D_A has necessarily core B^* and D_B has core A^*.

of the VB-algebroid $(D_A^* \to C^*, A \to M)$ corresponds to the 2-representation $(\nabla^{C^*}, \nabla^{B^*}, -R^*)$ of A on the complex $\partial_B^* : B^* \to C^*$. In the same manner, the splitting $\Sigma^\star : B \times_M C^* \to D_B^*$ of the VB-algebroid $(D_B^* \to C^*, B \to M)$ corresponds to the 2-representation $(\nabla^{C^*}, \nabla^{A^*}, -R^*)$ of B on the complex $\partial_A^* : A^* \to C^*$.

We check that the split Lie 2-algebroid corresponding to the linear splitting $\tilde{\Sigma}$ of $D_A^* \oplus D_B^*$ is the bicrossproduct of the matched pair of 2-representations. The equalities in (18) imply that we have to consider $A \oplus B$ as paired with $A^* \oplus B^*$ in the nonstandard way:

$$\langle (a, b), (\alpha, \beta) \rangle = \alpha(a) - \beta(b)$$

for all $a \in \Gamma(A),\ b \in \Gamma(B),\ \alpha \in \Gamma(A^*)$ and $\beta \in \Gamma(B^*)$. The anchor of $\tilde{\sigma}(a, b) = (\sigma^\star(a), \sigma^\star(b))$ is

$$\widehat{\nabla}_a^* + \widehat{\nabla}_b^* \in \mathfrak{X}^l(C^*),$$

and the anchor of $(\alpha, \beta)^\dagger = (\beta^\dagger, \alpha^\dagger) \in \Gamma_{C^*}^c(D_A^* \oplus D_B^*)$ is

$$(\partial_B^* \beta + \partial_A^* \alpha)^\uparrow \in \mathfrak{X}^c(C^*).$$

The Courant bracket $[\![(\sigma_A^\star(a), \sigma_B^\star(b)), (\beta^\dagger, \alpha^\dagger)]\!]$ is

$$([\sigma_A^\star(a), \beta^\dagger] + \pounds_{\sigma_B^\star(b)}\beta^\dagger - \boldsymbol{i}_{\alpha^\dagger}\boldsymbol{d}_{D_B^*}\sigma_A^\star(a), [\sigma_B^\star(b), \alpha^\dagger] + \pounds_{\sigma_A^\star(a)}\alpha^\dagger - \boldsymbol{i}_{\beta^\dagger}\boldsymbol{d}_{D_A^*}\sigma_B^\star(b)),$$

where $\boldsymbol{d}_{D_A^*} : \Gamma_{C^*}\big(\bigwedge^\bullet D_B^*\big) \to \Gamma_{C^*}\big(\bigwedge^{\bullet+1} D_B^*\big)$ is defined as usual by the Lie algebroid D_A^*, and similarly for D_B^* (bear in mind that some nonstandard signs arise from the signs in (18)). The derivation $\pounds : \Gamma(D_A^*) \times \Gamma(D_B^*) \to \Gamma(D_B^*)$ is described by

$$\begin{aligned} \pounds_{\beta^\dagger}\alpha^\dagger &= 0, & \pounds_{\beta^\dagger}\sigma_B^\star(b) &= -\langle b, \nabla_\cdot^* \beta \rangle^\dagger, \\ \pounds_{\sigma_A^\star(a)}\alpha^\dagger &= \pounds_a \alpha^\dagger, & \pounds_{\sigma_A^\star(a)}\sigma_B^\star(b) &= \sigma_B^\star(\nabla_a b) + \widetilde{R(a, \cdot)b} \end{aligned}$$

in [Gracia-Saz et al. 2018, Lemma 4.8]. Similar formulae hold for

$$\pounds : \Gamma(D_B^*) \times \Gamma(D_A^*) \to \Gamma(D_A^*).$$

We get

$$[\![(\sigma_A^\star(a), \sigma_B^\star(b)), (\beta^\dagger, \alpha^\dagger)]\!] = ((\nabla_a^* \beta + \pounds_b \beta - \langle \nabla_\cdot a, \alpha \rangle)^\dagger, (\nabla_b^* \alpha + \pounds_a \alpha - \langle \nabla_\cdot b, \beta \rangle)^\dagger).$$

In the same manner, we get

$$\begin{aligned} &[\![(\sigma_A^\star(a_1), \sigma_B^\star(b_1)), (\sigma_A^\star(a_2), \sigma_B^\star(b_2))]\!] \\ &\quad = (\sigma_A^\star([a, a'] + \nabla_b a' - \nabla_{b'} a), \sigma_B^\star([b, b'] + \nabla_a b' - \nabla_{a'} b)) \\ &\qquad + \big(-\widetilde{R(a_1, a_2)} + \widetilde{R(b_1, \cdot)a_2} - \widetilde{R(b_2, \cdot)a_1}, -\widetilde{R(b_1, b_2)} + \widetilde{R(a_1, \cdot)b_2} - \widetilde{R(a_2, \cdot)b_1}\big). \end{aligned}$$

Hence we have the following result. Recall that we have found above that double Lie algebroids are equivalent to VB-Courant algebroids with two transverse VB-Dirac structures.

Theorem 5.2. *The correspondence established in Theorem 4.6, between decomposed VB-Courant algebroids and split Lie 2-algebroids, restricts to a correspondence between decomposed double Lie algebroids and split Lie 2-algebroids that are the bicrossproducts of matched pairs of 2-representations.*

In other words, decomposed VB-bialgebroids are equivalent to matched pairs of 2-representations.

Recall that if the vector bundle C is trivial, the matched pair of 2-representations is just a matched pair of the Lie algebroids A and B. The corresponding double Lie algebroid is the decomposed double Lie algebroid $(A \times_M B, A, B, M)$ found in [Mackenzie 2011]. The corresponding VB-Courant algebroid is

$$\begin{array}{ccc} A \times_M B^* \oplus A^* \times_M B & \longrightarrow & 0 \times M \\ \downarrow & & \downarrow \\ A \oplus B & \longrightarrow & M \end{array}$$

with core $B^* \oplus A^*$. In that case there is a natural Lagrangian splitting and the corresponding Lie 2-algebroid is just the bicrossproduct Lie algebroid structure defined on $A \oplus B$ by the matched pair; see also the end of Section 5. This shows that the two notions of the double of a matched pair of Lie algebroids — the bicrossproduct Lie algebroid in [Mokri 1997] and the double Lie algebroid in [Mackenzie 2011] are just the $\mathbb{N}$-geometric and the classical descriptions of the same object, and special cases of Theorem 5.2.

5C. ***Example: the two "doubles" of a Lie bialgebroid.*** Recall that a Lie bialgebroid (A, A^*) is a pair of Lie algebroids $(A \to M, \rho, [\cdot\,,\cdot])$ and $(A^* \to M, \rho_\star, [\cdot\,,\cdot]_\star)$ in duality such that $A \oplus A^* \to M$ with the anchor $\rho + \rho_\star$, the pairing

$$\langle (a_1, \alpha_1), (a_2, \alpha_2)\rangle = \alpha_1(a_2) + \alpha_2(a_1),$$

and the bracket

$$[\![(a_1, \alpha_1), (a_2, \alpha_2)]\!] = ([a_1, a_2] + \pounds_{\alpha_1} a_2 - \boldsymbol{i}_{\alpha_2} \boldsymbol{d}_{A^*} a_1, [\alpha_1, \alpha_2]_\star + \pounds_{a_1}\alpha_2 - \boldsymbol{i}_{a_2}\boldsymbol{d}_A \alpha_1)$$

is a Courant algebroid. Lie bialgebroids were originally defined in a different manner [Mackenzie and Xu 1994], and the definition above is at the origin of the abstract definition of Courant algebroids [Liu et al. 1997]. This Courant algebroid is sometimes called the bicrossproduct of the Lie bialgebroid, or the double of the Lie bialgebroid.

Mackenzie [2011] came up with an alternative notion of the double of a Lie bialgebroid. Given a Lie bialgebroid as above, the double vector bundle

$$\begin{tikzcd} T^*A \simeq T^*A^* \arrow[r] \arrow[d] & A^* \arrow[d] \\ A \arrow[r] & M \end{tikzcd}$$

is a double Lie algebroid with the following structures. The Lie algebroid structure on A defines a linear Poisson structure on A^*, and so a linear Lie algebroid structure on $T^*A^* \to A^*$. In the same manner, the Lie algebroid structure on A^* defines a linear Poisson structure on A, and so a linear Lie algebroid structure on $T^*A \to A$ (see [Gracia-Saz et al. 2018] for more details and for the matched pairs of 2-representations associated to a choice of linear splitting). The VB-Courant algebroid defined by this double Lie algebroid is $(T^*A)^*_A \oplus (T^*A)^*_{A^*}$ which is isomorphic to

$$\begin{tikzcd} TA \oplus TA^* \arrow[r] \arrow[d] & TM \arrow[d] \\ A \oplus A^* \arrow[r] & M \end{tikzcd}$$

Computations reveal that the Courant algebroid structure is just the tangent of the Courant algebroid structure on $A \oplus A^*$, and so that the two notions of the double of a Lie bialgebroid can be understood as an algebraic and a geometric interpretation of the same object.

Appendix: Proof of Theorem 4.6

Let $(\mathbb{E}; Q, B; M)$ be a VB-Courant algebroid and choose a Lagrangian splitting $\Sigma : Q \times_M B$. We prove here that the obtained split linear Courant algebroid is equivalent to a split Lie 2-algebroid. Recall the construction of the objects $\partial_B, \Delta, \nabla, [\![\cdot\,,\cdot]\!]_\sigma, R$ in Section 4C1, and recall that $\mathcal{S} \subseteq \Gamma_B(\mathbb{E})$ is the subset

$$\{\tau^\dagger \mid \tau \in \Gamma(Q^*)\} \cup \{\sigma_Q(q) \mid q \in \Gamma(Q)\} \subseteq \Gamma_B(\mathbb{E}).$$

Recall also that the tangent double $(TB \to B; TM \to M)$ has a VB-Lie algebroid structure, which is described in [Jotz Lean 2018b, Section 2.2.2]. We begin by giving two useful lemmas.

Lemma A.1. *For $\beta \in \Gamma(B^*)$, we have*

$$\mathcal{D}(\ell_\beta) = \sigma_Q(\partial_B^*\beta) + \widetilde{\nabla^*_\cdot \beta},$$

where $\nabla^*_\cdot\beta$ is seen as follows as a section of $\Gamma(\mathrm{Hom}(B, Q^*))$: $(\nabla^*_\cdot\beta)(b) = \langle \nabla^*_\cdot\beta, b\rangle \in \Gamma(Q^*)$ *for all $b \in \Gamma(B)$.*

Proof. For $\beta \in \Gamma(B^*)$, the section $\boldsymbol{d}\ell_\beta$ is a linear section of $T^*B \to B$. Since the anchor Θ is linear, the section $\mathcal{D}\ell_\beta = \Theta^*\boldsymbol{d}\ell_\beta$ is linear. Since for any $\tau \in \Gamma(Q^*)$,

$$\langle \mathcal{D}(\ell_\beta), \tau^\dagger \rangle = \Theta(\tau^\dagger)(\ell_\beta) = q_B^*\langle \partial_B \tau, \beta \rangle,$$

we find that $\mathcal{D}(\ell_\beta) - \sigma_Q(\partial_B^*\beta) \in \Gamma(\ker \pi_Q)$. Hence, $\mathcal{D}(\ell_\beta) - \sigma_Q(\partial_B^*\beta)$ is a core-linear section of $\mathbb{E} \to B$ and there exists a section ϕ of $\operatorname{Hom}(B, Q^*)$ such that $\mathcal{D}(\ell_\beta) - \sigma_Q(\partial_B^*\beta) = \widetilde{\phi}$. We have

$$\ell_{\langle \phi, q \rangle} = \langle \widetilde{\phi}, \sigma_Q(q) \rangle = \langle \mathcal{D}(\ell_\beta) - \sigma_Q(\partial_B^*\beta), \sigma_Q(q) \rangle = \Theta(\sigma_Q(q))(\ell_\beta) = \ell_{\nabla_q^*\beta}$$

and so $\phi(b) = \langle \nabla_\cdot^*\beta, b \rangle \in \Gamma(Q^*)$ for all $b \in \Gamma(B)$. □

For each $q \in \Gamma(Q)$, ∇_q, and Δ_q define a derivation $\Diamond_q$ of $\Gamma(\operatorname{Hom}(B, Q^*))$ as follows: for $\phi \in \Gamma(\operatorname{Hom}(B, Q^*))$ and $b \in \Gamma(B)$,

$$(\Diamond_q \phi)(b) = \Delta_q(\phi(b)) - \phi(\nabla_q b).$$

Lemma A.2. *For $q \in \Gamma(Q)$ and $\phi \in \Gamma(\operatorname{Hom}(B, Q^*))$, we have $[\![\sigma_Q(q), \widetilde{\phi}]\!] = \widetilde{\Diamond_q \phi}$.*

Proof. The proof is an easy computation as in the proof of Lemma 4.4. □

Now we can express all the conditions of Lemma 3.2 in terms of the objects $\partial_B, \Delta, \nabla, [\![\cdot\,,\cdot]\!]_\sigma, R$ found in Section 4C1.

Proposition A.3. *The anchor satisfies $\Theta \circ \Theta^* = 0$ if and only if $\rho_Q \circ \partial_B^* = 0$ and $\nabla^*_{\partial_B^*\beta_1}\beta_2 + \nabla^*_{\partial_B^*\beta_2}\beta_1 = 0$ for all $\beta_1, \beta_2 \in \Gamma(B^*)$.*

Proof. The composition $\Theta \circ \Theta^*$ vanishes if and only if $(\Theta \circ \Theta^*)\boldsymbol{d}F = 0$ for all linear and pullback functions $F \in C^\infty(B)$. For $f \in C^\infty(M)$,

$$\Theta(\Theta^*\boldsymbol{d}(q_B^* f)) = ((\partial_B \circ \rho_Q^*)\boldsymbol{d} f)^\uparrow.$$

For $\beta \in \Gamma(B^*)$, we find, using Lemma A.1,

$$\Theta(\Theta^*\boldsymbol{d}\ell_\beta) = \Theta(\mathcal{D}\ell_\beta) = \Theta(\sigma_Q(\partial_B^*\beta) + \widetilde{\nabla_\cdot^*\beta}) = \widehat{\nabla}_{\partial_B^*\beta} + \widetilde{\partial_B \circ \langle \nabla_\cdot^*\beta, \cdot \rangle}.$$

Here, $\partial_B \circ \langle \nabla_\cdot^*\beta, \cdot \rangle$ is as follows a morphism $B \to B$; $b \mapsto \partial_B(\langle \nabla_\cdot^*\beta, b \rangle)$. On a linear function $\ell_{\beta'}$, $\beta' \in \Gamma(B^*)$, we have $\Theta(\Theta^*\boldsymbol{d}\ell_\beta)(\ell_{\beta'}) = \ell_{\nabla^*_{\partial_B^*\beta}\beta'} + \ell_{\nabla^*_{\partial_B^*\beta'}\beta}$. On a pullback $q_B^* f$, $f \in C^\infty(M)$, this is $q_B^*(\pounds_{(\rho_Q \circ \partial_B^*)(\beta)} f)$. □

Proposition A.4. *The compatibility of Θ with the Courant algebroid bracket $[\![\cdot\,,\cdot]\!]$ is equivalent to*

(1) $\partial_B \circ R(q_1, q_2) = R_\nabla(q_1, q_2)$,

(2) $\rho_Q \circ [\![\cdot\,,\cdot]\!]_\sigma = [\cdot\,,\cdot] \circ (\rho_Q, \rho_Q)$, *or* $\Delta_q(\rho_Q^*\boldsymbol{d} f) = \rho_Q^*\boldsymbol{d}(\rho_Q(q)(f))$ *for all* $q \in \Gamma(Q)$ *and* $f \in C^\infty(M)$, *and*

(3) $\partial_B \circ \Delta = \nabla \circ \partial_B$.

Proof. We have $\Theta[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!] = [\Theta(\sigma_Q(q_1)),\Theta(\sigma_Q(q_2))] = [\widehat{\nabla}_{q_1},\widehat{\nabla}_{q_2}]$ and

$$\Theta(\sigma_Q([\![q_1,q_2]\!]_\sigma) - \widetilde{R(q_1,q_2)}) = \widehat{\nabla}_{[\![q_1,q_2]\!]_\sigma} - \widetilde{\partial_B \circ R(q_1,q_2)}.$$

Applying both derivations to a pullback function $q_B^* f$ for $f \in C^\infty(M)$ yields

$$[\widehat{\nabla}_{q_1},\widehat{\nabla}_{q_2}](q_B^* f) = q_B^*([\rho_Q(q_1),\rho_Q(q_2)]f).$$

and

$$(\widehat{\nabla}_{[\![q_1,q_2]\!]_\sigma} - \widetilde{\partial_B \circ R(q_1,q_2)})(q_B^* f) = q_B^*(\rho_Q[\![q_1,q_2]\!]_\sigma(f)).$$

Applying both vector fields to a linear function $\ell_\beta \in C^\infty(B)$, $\beta \in \Gamma(B^*)$, we get

$$[\widehat{\nabla}_{q_1},\widehat{\nabla}_{q_2}](\ell_\beta) = \ell_{\nabla^*_{q_1}\nabla^*_{q_2}\beta - \nabla^*_{q_2}\nabla^*_{q_1}\beta}$$

and

$$(\widehat{\nabla}_{[\![q_1,q_2]\!]_\sigma} - \widetilde{\partial_B \circ R(q_1,q_2)})(\ell_\beta) = \ell_{\nabla^*_{[\![q_1,q_2]\!]_\sigma}\beta - R(q_1,q_2)^*\partial_B^*\beta}.$$

Since $R_{\nabla^*}(q_1,q_2) = -(R_\nabla(q_1,q_2))^*$, we find that

$$\Theta[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!] = [\Theta(\sigma_Q(q_1)),\Theta(\sigma_Q(q_2))]$$

for all $q_1,q_2 \in \Gamma(Q)$ if and only if (1) and (2) are satisfied.

In the same manner, for $q \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$, we compute

$$\Theta([\![\sigma_Q(q),\tau^\dagger]\!]) = (\partial_B \Delta_q \tau)^\uparrow$$

and

$$[\Theta(\sigma_Q(q)),\Theta(\tau^\dagger)] = [\widehat{\nabla}_q,(\partial_B\tau)^\uparrow] = (\nabla_q(\partial_B\tau))^\uparrow.$$

Thus, $\Theta([\![\sigma_Q(q),\tau^\dagger]\!]) = [\Theta(\sigma_Q(q)),\Theta(\tau^\dagger)]$ if and only if $\partial_B(\Delta_q\tau) = \nabla_q(\partial_B\tau)$. □

Proposition A.5. *The condition* (3) *of Lemma 3.2 is equivalent to* $R(q_1,q_2) = -R(q_2,q_1)$ *and* $[\![q_1,q_2]\!]_\sigma + [\![q_2,q_1]\!]_\sigma = 0$ *for* $q_1,q_2 \in \Gamma(Q)$.

Proof. Choose q_1,q_2 in $\Gamma(Q)$. Then we have

$$[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!] + [\![\sigma_Q(q_2),\sigma_Q(q_1)]\!] = \sigma_Q([\![q_1,q_2]\!]_\sigma + [\![q_2,q_1]\!]_\sigma) - \widetilde{R(q_1,q_2)} - \widetilde{R(q_2,q_1)}.$$

By the choice of the splitting, we have $\mathcal{D}\langle\sigma_Q(q_1),\sigma_Q(q_2)\rangle = \mathcal{D}(0) = 0$. Hence, (3) of Lemma 3.2 is true on horizontal lifts of sections of Q if and only if $R(q_1,q_2) = -R(q_2,q_1)$ and $[\![q_1,q_2]\!]_\sigma + [\![q_2,q_1]\!]_\sigma = 0$ for all $q_1,q_2 \in \Gamma(Q)$. Further, we have $[\![\sigma_Q(q),\tau^\dagger]\!] = (\Delta_q\tau)^\dagger$ and $[\![\tau^\dagger,\sigma_Q(q)]\!] = (-\Delta_q\tau + \rho_Q^*\boldsymbol{d}\langle\tau,q\rangle)^\dagger$ by definition. On core sections (3) is trivially satisfied since both the pairing and the bracket of two core sections vanish. □

Proposition A.6. *The derivation formula* (2) *in Lemma 3.2 is equivalent to the following*:

(1) Δ *is dual to* $[\![\cdot,\cdot]\!]_\sigma$, *that is* $[\![\cdot,\cdot]\!]_\sigma = [\![\cdot,\cdot]\!]_\Delta$,

(2) $[\![q_1,q_2]\!]_\sigma + [\![q_2,q_1]\!]_\sigma = 0$ *for all* $q_1,q_2 \in \Gamma(Q)$, *and*

(3) $R(q_1,q_2)^*q_3 = -R(q_1,q_3)^*q_2$ *for all* $q_1,q_2,q_3 \in \Gamma(Q)$.

Proof. We compute (CA2) for linear and core sections. First of all, the equations

$$\Theta(\tau_1^\dagger)\langle\tau_2^\dagger,\tau_3^\dagger\rangle = \langle[\![\tau_1^\dagger,\tau_2^\dagger]\!],\tau_3^\dagger\rangle + \langle\tau_2^\dagger,[\![\tau_1^\dagger,\tau_3^\dagger]\!]\rangle,$$
$$\Theta(\tau_1^\dagger)\langle\tau_2^\dagger,\sigma_Q(q)\rangle = \langle[\![\tau_1^\dagger,\tau_2^\dagger]\!],\sigma_Q(q)\rangle + \langle\tau_2^\dagger,[\![\tau_1^\dagger,\sigma_Q(q)]\!]\rangle$$

and

$$\Theta(\sigma_Q(q))\langle\tau_1^\dagger,\tau_2^\dagger\rangle = \langle[\![\sigma_Q(q),\tau_1^\dagger]\!],\tau_2^\dagger\rangle + \langle\tau_1^\dagger,[\![\sigma_Q(q),\tau_2^\dagger]\!]\rangle$$

are trivially satisfied for all $\tau_1,\tau_2,\tau_3 \in \Gamma(Q^*)$ and $q \in \Gamma(Q)$. Next, for $q_1,q_2 \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$, we have:

$$\begin{aligned}\Theta(\sigma_Q(q_1))\langle\sigma_Q(q_2),\tau^\dagger\rangle - \langle[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!],\tau^\dagger\rangle - \langle\sigma_Q(q_2),[\![\sigma_Q(q_1),\tau^\dagger]\!]\rangle&\\ = \widehat{\nabla}_{q_1}(q_B^*\langle q_2,\tau\rangle) - q_B^*\langle[\![q_1,q_2]\!]_\sigma,\tau\rangle - q_B^*\langle q_2,\Delta_{q_1}\tau\rangle&\\ = q_B^*\big(\rho_Q(q_1)\langle q_2,\tau\rangle - \langle[\![q_1,q_2]\!]_\sigma,\tau\rangle - \langle q_2,\Delta_{q_1}\tau\rangle\big)&\end{aligned}$$

Thus $\Theta(\sigma_Q(q_1))\langle\sigma_Q(q_2),\tau^\dagger\rangle = \langle[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!],\tau^\dagger\rangle + \langle\sigma_Q(q_2),[\![\sigma_Q(q_1),\tau^\dagger]\!]\rangle$ for all $q_1,q_2 \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$ if and only if Δ and $[\![\cdot,\cdot]\!]_\sigma$ are dual to each other. Using this, we compute

$$\begin{aligned}&\Theta(\tau^\dagger)\langle\sigma_Q(q_1),\sigma_Q(q_2)\rangle - \langle[\![\tau^\dagger,\sigma_Q(q_1)]\!],\sigma_Q(q_2)\rangle - \langle\sigma_Q(q_1),[\![\tau^\dagger,\sigma_Q(q_2)]\!]\rangle\\ &\quad = 0 - \langle-(\Delta_{q_1}\tau)^\dagger + (\rho_Q^*\boldsymbol{d}\langle q_1,\tau\rangle)^\dagger,\sigma_Q(q_2)\rangle - \langle\sigma_Q(q_1),-(\Delta_{q_2}\tau)^\dagger + (\rho_Q^*\boldsymbol{d}\langle q_2,\tau\rangle)^\dagger\rangle\\ &\quad = -q_B^*\langle[\![q_1,q_2]\!]_\sigma + [\![q_2,q_1]\!]_\sigma,\tau\rangle.\end{aligned}$$

Finally we have $\Theta(\sigma_Q(q_1))\langle\sigma_Q(q_2),\sigma_Q(q_3)\rangle = 0$ for all $q_1,q_2,q_3 \in \Gamma(Q)$, and $\langle[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!],\sigma_Q(q_3)\rangle = \ell_{-R(q_1,q_2)^*q_3}$. This shows that

$$\begin{aligned}&\Theta(\sigma_Q(q_1))\langle\sigma_Q(q_2),\sigma_Q(q_3)\rangle\\ &\qquad = \langle[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!],\sigma_Q(q_3)\rangle + \langle\sigma_Q(q_2),[\![\sigma_Q(q_1),\sigma_Q(q_3)]\!]\rangle\end{aligned}$$

if and only if $0 = -R(q_1,q_2)^*q_3 - R(q_1,q_3)^*q_2$. □

Proposition A.7. *Assume that* Δ *and* $[\![\cdot,\cdot]\!]_\sigma$ *are dual to each other. The Jacobi identity in Leibniz form for sections in* $\mathcal{S}$ *is equivalent to*

(1) $R(q_1,q_2)\circ\partial_B = R_\Delta(q_1,q_2)$,

$$\begin{aligned}(2)\quad R(q_1,[\![q_2,q_3]\!]_\Delta)-R(q_2,[\![q_1,q_3]\!]_\Delta)-R([\![q_1,q_2]\!]_\Delta),q_3)&\\ +\lozenge_{q_1}(R(q_2,q_3))-\lozenge_{q_2}(R(q_1,q_3))+\lozenge_{q_3}(R(q_1,q_2))&\\ =\nabla^*_\cdot(R(q_1,q_2)^*q_3)&\end{aligned}$$

for all $q_1,q_2,q_3\in\Gamma(Q)$.

If R is skew-symmetric as in (1) of Proposition A.5, then the second equation is $\boldsymbol{d}_{\nabla^*}\omega=0$ for $\omega\in\Omega^3(Q,B^*)$ defined by $\omega(q_1,q_2,q_3)=R(q_1,q_2)^*q_3$.

Proof. The Jacobi identity is trivially satisfied on core sections since the bracket of two core sections is 0. Similarly, for $\tau_1,\tau_2\in\Gamma(Q^*)$ and $q\in\Gamma(Q)$, we find $[\![\sigma_Q(q),[\![\tau_1^\dagger,\tau_2^\dagger]\!]]\!]=0$ and $[\![[\![\sigma_Q(q),\tau_1^\dagger]\!],\tau_2^\dagger]\!]+[\![\tau_1^\dagger,[\![\sigma_Q(q),\tau_2^\dagger]\!]]\!]=0$. We have

$$\begin{aligned}&[\![\sigma_Q(q_1),[\![\sigma_Q(q_2),\tau^\dagger]\!]]\!]-[\![\sigma_Q(q_2),[\![\sigma_Q(q_1),\tau^\dagger]\!]]\!]\\ &\qquad=[\![\sigma_Q(q_1),(\Delta_{q_2}\tau)^\dagger]\!]-[\![\sigma_Q(q_2),(\Delta_{q_1}\tau)^\dagger]\!]\\ &\qquad=(\Delta_{q_1}\Delta_{q_2}\tau)^\dagger-(\Delta_{q_2}\Delta_{q_1}\tau)^\dagger,\end{aligned}$$

and

$$\begin{aligned}[\![[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!],\tau^\dagger]\!]&=[\![\sigma_Q([\![q_1,q_2]\!]_\Delta)-\widetilde{R(q_1,q_2)},\tau^\dagger]\!]\\ &=(\Delta_{[\![q_1,q_2]\!]_\Delta}\tau)^\dagger+(R(q_1,q_2)(\partial_B\tau))^\dagger\end{aligned}$$

by Lemma 4.4. We now choose $q_1,q_2,q_3\in\Gamma(Q)$ and compute

$$\begin{aligned}&[\![[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!],\sigma_Q(q_3)]\!]\\ &\quad=[\![\sigma_Q([\![q_1,q_2]\!]_\Delta)-\widetilde{R(q_1,q_2)},\sigma_Q(q_3)]\!]\\ &\quad=\sigma_Q([\![[\![q_1,q_2]\!]_\Delta,q_3]\!]_\Delta)-\widetilde{R([\![q_1,q_2]\!]_\Delta,q_3)}-\mathcal{D}\ell_{\langle R(q_1,q_2)\cdot,q_3\rangle}+\widetilde{\lozenge_{q_3}R(q_1,q_2)}\\ &\quad=\sigma_Q([\![[\![q_1,q_2]\!]_\Delta,q_3]\!]_\Delta)-\widetilde{R([\![q_1,q_2]\!]_\Delta,q_3)}\\ &\qquad-\sigma_Q(\partial_B^*\langle R(q_1,q_2)\cdot,q_3\rangle)-\widetilde{\nabla^*_\cdot\langle R(q_1,q_2)\cdot,q_3\rangle}+\widetilde{\lozenge_{q_3}R(q_1,q_2)}\end{aligned}$$

and

$$\begin{aligned}&[\![\sigma_Q(q_2),[\![\sigma_Q(q_1),\sigma_Q(q_3)]\!]]\!]\\ &\qquad=[\![\sigma_Q(q_2),\sigma_Q([\![q_1,q_3]\!]_\Delta)-\widetilde{R(q_1,q_3)}]\!]\\ &\qquad=\sigma_Q([\![q_2,[\![q_1,q_3]\!]_\Delta]\!]_\Delta)-\widetilde{R(q_2,[\![q_1,q_3]\!]_\Delta)}-\widetilde{\lozenge_{q_2}R(q_1,q_3)}.\end{aligned}$$

We hence find that

$$\begin{aligned}&[\![[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!],\sigma_Q(q_3)]\!]+[\![\sigma_Q(q_2),[\![\sigma_Q(q_1),\sigma_Q(q_3)]\!]]\!]\\ &\qquad\qquad=[\![\sigma_Q(q_1),[\![\sigma_Q(q_2),\sigma_Q(q_3)]\!]]\!]\end{aligned}$$

if and only if

$$[\![[\![q_1,q_2]\!]_\Delta,q_3]\!]_\Delta+[\![q_2,[\![q_1,q_3]\!]_\Delta]\!]_\Delta=[\![q_1,[\![q_2,q_3]\!]_\Delta]\!]_\Delta+\partial_B^*\langle R(q_1,q_2)\cdot,q_3\rangle$$

and

$$R([\![q_1,q_2]\!]_\Delta,q_3)+\nabla^*_\cdot\langle R(q_1,q_2)\cdot,q_3\rangle-\lozenge_{q_3}R(q_1,q_2)\\ \qquad\qquad +R(q_2,[\![q_1,q_3]\!]_\Delta)+\lozenge_{q_2}R(q_1,q_3)\\ =R(q_1,[\![q_2,q_3]\!]_\Delta)+\lozenge_{q_1}R(q_2,q_3).$$

We conclude using (2) on page 156. □

A combination of Propositions A.3, A.4, A.5, A.6, A.7 and Lemma 3.2 proves Theorem 4.6.

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ALGORITHMIC HOMEOMORPHISM OF 3-MANIFOLDS AS A COROLLARY OF GEOMETRIZATION

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We prove two results, one semi-historical and the other new. The semi-historical result, which goes back to Thurston and Riley, is that the geometrization theorem implies that there is an algorithm for the homeomorphism problem for closed, oriented, triangulated 3-manifolds. We give a self-contained proof, with several variations at each stage, that uses only the statement of the geometrization theorem, basic hyperbolic geometry, and old results from combinatorial topology and computer science. For this result, we do not rely on normal surface theory, methods from geometric group theory, nor methods used to prove geometrization.

The new result is that the homeomorphism problem is elementary recursive, *i.e.*, that the computational complexity is bounded by a bounded tower of exponentials. This result relies on normal surface theory, Mostow rigidity, and bounds on the computational complexity of solving algebraic equations.

1. Introduction

In this paper, we will prove the following two theorems.

Theorem 1.1 (After Thurston [49]). *Suppose that M_1 and M_2 are two finite, simplicial complexes that represent closed, oriented 3-manifolds. Then, as a corollary of the geometrization theorem, it is recursive to determine if there is an orientation-preserving homeomorphism $M_1 \cong M_2$.*

Theorem 1.2. *The oriented homeomorphism problem for closed, oriented 3-manifolds is elementary recursive.*

Theorem 1.1 implies that the geometrization theorem is a classification of closed, oriented 3-manifolds by the standard of computer science, where the term *recursive* used here means the same thing as *decidable* or *computable*. Geometrization intuitively presents itself as a classification of closed 3-manifolds, or at least a big step towards one. However, the question of what counts as a “classification" in

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mathematics is generally not rigorous, even though it is typically a debate over rigorous results. The computability interpretation is thus important because it is rigorous, even though it is not by any means the only important standard of classification. (For instance, the set of twin primes is recursive, but they remain unclassified in the sense that it is not even proven that there are infinitely many.) Note that Thurston himself [49, Sec. 3] seriously addressed the relation between geometrization and computability.

We argue that Theorem 1.1 should largely be credited to Riley and Thurston from the 1970s, even though they did not publish a complete proof. (See Section 1.1 for more details.) To support this interpretation, we will prove Theorem 1.1 directly using hyperbolic geometry, and using other background results on computability and triangulations of manifolds that seem standard and germane. The most important results of the latter type are the Tarski-Seidenberg theorem, Theorem 2.8, that real algebraic equations can be solved recursively; the Kantorovich-Neuberger theorem on the convergence of Newton's method, Theorem 5.8; and the stellar and bistellar move theorems of Alexander, Newman, and Pachner, Theorems 3.2 and 3.3. Despite this restriction on methods, we give more than one argument for each of several stages of the proof. (For instance, although the papers of Neuberger and Pachner came after geometrization was formulated, the earlier results of Kantorovich in the former case and Alexander-Newman in the latter case suffice in context.)

In the intervening years, Jaco-Tollefson, Manning, Scott-Short, and others have published proofs of major parts of Theorem 1.1 [18; 29; 43; 4]. These approaches have various new ideas and implications, which is in keeping with Thurston's philosophy concerning the nature of progress in mathematics [50]. Even so, the status of Theorem 1.1 has remained unsettled. At one extreme, it has been interpreted as a folklore theorem and therefore standard knowledge, even if the proof is not elementary. At the other extreme, it has been interpreted as still an open problem. In the middle, one could argue that the published partial results piece together to make an entire proof. The problem with the middle position is that the total structure of an arbitrary closed, oriented 3-manifold is somewhat complicated; see Theorems 5.1, 5.2, and 5.3. So, one purpose of our proof of Theorem 1.1 is to give a complete proof in one paper, as requested by Aschenbrenner-Friedl-Wilton [4].

The intervening results also typically use either normal surface theory [24; 13] or geometric group theory [11; 46; 8]. While these methods certainly work, they arguably overshoot Theorem 1.1. Both theories are highly non-trivial in their own right, and they continued to be developed after the geometrization conjecture was stated. In particular the key results of Jaco-Tollefson [18] and Sela [46] came later. Sela's theorem applies to Gromov-hyperbolic groups, which are vastly more general than the Kleinian groups that arise in Theorem 1.1. Meanwhile Haken and Jaco-Tollefson prove sharper results than strictly necessary for their components of Theorem 1.1; namely, they establish algorithms with quantitative bounds on execution time. This brings us to Theorem 1.2.

In Theorem 1.2, an algorithm is *elementary recursive* if its execution time is bounded by a bounded tower of exponentials; for instance, time $O(2^{2^n})$. (See Section 2.2.) In contrast with Theorem 1.1, the proof of Theorem 1.2 does use normal surface theory, as well as Mostow rigidity, and improved bounds on the computational complexity of solving algebraic equations [10]. The connected-sum and JSJ decomposition stages of Theorem 1.2 were partly known. For instance, using similar methods, Mijatović [32; 31] established an elementary recursive bound on the number of Pachner moves needed to standardize either S^3 or a Seifert-fibered space with boundary.

The hyperbolic case of Theorem 1.2 is new. By contrast, Mijatović also established a primitive recursive bound on the number of Pachner moves needed to equate two hyperbolic, fiber-free, Haken 3-manifolds. However, primitive recursive is significantly weaker than elementary recursive; the Haken condition is also a significant restriction. Theorem 1.2 also has the advantage of combining a mixed set of methods to handle the full generality of closed, oriented 3-manifolds.

Remark. We leave the non-orientable versions of Theorems 1.1 and 1.2 for future work. This case includes new details such as 3-manifolds with essential, two-sided projective planes and Klein bottles. A more thorough result would also handle compact 3-manifolds with boundary.

1.1. ***History and discussion.*** As already mentioned, the geometrization conjecture has often been interpreted as a classification of closed 3-manifolds, and computability is one candidate standard of what it means to classify mathematical objects. In Thurston's famous survey of his results in the AMS Bulletin [49, Sec. 3], he says:

> Riley's work makes it clear that there is a rigorous, but not generally practical, algorithm for computing hyperbolic structures.

Thurston then sketches an algorithm which is similar to Manning's construction [29] in some ways and to our arguments in other ways. This passage, and some other aspects of the Bulletin article, support the conclusion that Thurston anticipated not only the statement of Theorem 1.1, but also its proof. The author also discussed Theorem 1.1 in personal communication with Thurston in the late 1990s.

At first glance, an algorithm that can only find a hyperbolic structure on a 3-manifold is both less general and weaker than Theorem 1.1. It is less general because a 3-manifold may also have non-hyperbolic components; it is weaker because the homeomorphism problem for two hyperbolic manifolds M_1 and M_2 takes more work than just finding their hyperbolic structures. However, in the theorem that it is recursive to geometrize a 3-manifold (Theorem 5.4), the hyperbolic pieces

(Lemma 5.7) are the hardest part. The geometric structure of the other pieces and the data to glue the pieces together are complicated to describe carefully, but the proof that this data is computable only requires moves on triangulations (Corollary 3.4), the principle that nested infinite loops can be combined into a single infinite loop (Proposition 2.5), and some facts about Seifert fibrations (Lemma 5.11 and Theorem 5.12).

In the second stage, the homeomorphism problem for hyperbolic 3-manifolds (Theorem 6.1) reduces to calculating isometries by Mostow rigidity, and a typical algorithm for this is similar to one for computing a hyperbolic structure. The homeomorphism problem for Seifert-fibered components and glued combinations of components (Section 6) requires little more than the ideas of Waldhausen [51] in his classification of graph manifolds.

Later in the Bulletin article [49, Sec. 6], Thurston gives a list of open problems and projects, including:

> 21. Develop a computer program to calculate hyperbolic structures on 3-manifolds.

Jeff Weeks' SnapPea [53] (now SnapPy [6]), which was originally written in the 1980s, met this challenge. It is fast and reliable in practice, it can also compute the isometries between two hyperbolic 3-manifolds, and it has been supremely useful for a lot of research in 3-manifold topology. SnapPea also supports the belief that the homeomorphism problem follows from geometrization, given its spectacular record in practice. However, its specific algorithms are not rigorous. SnapPea uses ideal triangulations of cusped 3-manifolds, together with Dehn fillings to make spun triangulations of closed 3-manifolds; it is only conjectured that such a structure always exists. SnapPea also uses non-rigorous methods to find suitable triangulations. In particular it uses limited-precision floating point arithmetic; it has no rigorous model of necessary precision as a function of geometric complexity. (Note that current versions of SnapPy can rigorously certify an answer, when its SnapPea engine finds one.) In contrast to the SnapPea data structure, and other reasons that ideal and spun triangulations are important, we will use triangulations with semi-ideal and finite tetrahedra to prove Theorem 1.1 (see Section 5.2.5).

To start the rigorous discussion of computable classification and the homeomorphism problem, we can say that closed 3-manifolds are classified if we can:

1. specify every closed 3-manifold by a finite data structure;

2. algorithmically generate a standard list of closed 3-manifolds without repetition; and

3. given any 3-manifold M, algorithmically identify the standard manifold M' such that $M \cong M'$.

For closed 3-manifolds, condition 1 is addressed by the fact that every 3-manifold has a unique smooth structure and a unique PL structure. As a result, we can describe a closed 3-manifold as a finite simplicial complex. Unlike in higher dimensions, it is easy to check whether a simplicial complex is a 3-manifold (Section 3). Conditions 2 and 3 are equivalent to an algorithm to determine whether two closed 3-manifolds M_1 and M_2 are homeomorphic by the following simple argument. In one direction, if both conditions 2 and 3 are satisfied, then Condition 3 immediately implies a homeomorphism algorithm. In the other direction, given a homeomorphism algorithm, we can lexicographically order all descriptions of all closed 3-manifolds according to condition 1, and then list only those examples that are not homeomorphic to any earlier example. This satisfies condition 2. Then given a description of a closed 3-manifold M, we can search the list in order to find the standard $M' \cong M$ to satisfy condition 3. (Haken calls this argument the "cheapological trick" [52, Sec. 4]. Arguably it is not a cheap trick after all, since it is similar to the nomenclature in tables of knots and 3-manifolds.)

As mentioned, Manning [29] and Scott and Short [43] give partial results toward Theorem 1.1, but they use more recent tools. In particular, Manning uses Sela's algorithm [46] for the isomorphism problem for word-hyperbolic groups, while Scott and Short use the theory of biautomatic groups [8].

Both Short-Scott and Aschenbrenner-Friedl-Wilton [4, Sec. 2.1] mention a particular subtlety in approaches to Theorem 1.1 that are based on analyzing the fundamental group $\pi_1(M)$ or the fundamental groups of its components. Namely, $\pi_1(M)$ is insensitive to the orientation of M. Worse, if

$$M \cong W_1 \# W_2 \# \dots \# W_n$$

is a decomposition of M into prime summands, then the orientation of each summand W_k can be chosen separately without changing $\pi_1(M)$. Or the summands can be lens spaces; two lens spaces can have the same fundamental group without even being unoriented homeomorphic. We surmount this subtlety by modelling all 3-manifolds and their components with triangulations that are decorated with orientations; see Section 5.2.

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2. Computability

We recommend Arora-Barak [3] and the Complexity Zoo [54] for modern introductions to models of computation and complexity classes.

2.1. ***Recursive and recursively enumerable problems.*** Let A be an *alphabet* (a finite set with at least two elements) and let A^* be the set of all finite words over that alphabet. A *decision problem* is a function

$$d : A^* \to \{\text{yes}, \text{no}\}.$$

A *function problem* is likewise a function $f : A^* \to A^*$, which can be multivalued. The input space A^* is equivalent to many other types of input by some suitable encoding: Finite sequences of strings, finite simplicial complexes, etc.

A decision problem or a function problem can be a *promise problem*, meaning that it is defined only on some subset of inputs $P \subseteq A^*$ which is called a *promise*. Whether two closed n-manifolds are PL homeomorphic is an example of a promise decision problem: The input consists of two simplicial complexes that are promised to be manifolds; then the yes/no decision is whether they are homeomorphic. (But see Proposition 3.1.)

A *decision algorithm* is a mathematical computer program, which can be modelled by a Turing machine (or some equivalent model of computation), that takes some input $x \in A^*$ and can do one of three things: (1) Terminate with the answer “yes", (2) terminate with the answer “no", or (3) continue in an infinite loop. Similarly, a function algorithm can terminate and report an output $y \in A^*$, or it can continue in an infinite loop. Given a multivalued function f, then a function algorithm is only required to calculate one of the values of $f(x)$ on input x.

A *complexity class* or *computability class* is some set of decision or function problems, typically defined by the existence of algorithms of some kind. For example, a decision problem d or a function problem f is *recursive* (or *computable* or *decidable*) if it is computed by an algorithm that always terminates. By definition, the complexity class R is the set of recursive decision problems. By abuse of notation, R can also denote the set of recursive, promise decision problems; or the set of recursive function problems, with or without a promise. The following proposition is elementary.

Proposition 2.1. *If d is a recursive promise problem, and if the promise itself is recursive, then d is a recursive non-promise problem if we let $d(x) =$ no when the promise is not satisfied.*

The complexity class RE is the set of *recursively enumerable* decision problems. These are problems with an algorithm that terminates with “yes" when the answer is yes; but if the answer is “no", the algorithm might not terminate. The complexity class coRE is defined in the same way as RE, but with yes and no switched. We review the following standard propositions and theorems.

Proposition 2.2. *A non-promise decision problem d is in* RE *if and only if there is an algorithm that lists all solutions $x_1, x_2, \ldots$ to $d(x) =$* yes *without repetition.*

Proposition 2.2 justifies the name "recursively enumerable" for the class RE. (Note that if the algorithm lists the solutions in non-decreasing order of length, $|x_1| \le |x_2| \le \ldots$, then $d \in \mathsf{R}$.) The proof is left as an exercise. Also, in the spirit of Proposition 2.2, a decision problem d can be identified with the set of solutions to $d(x) =$ yes; in this way we can call a set recursive, recursively enumerable, etc.

Proposition 2.3. $\mathsf{R} = \mathsf{RE} \cap \mathsf{coRE}$.

The proof of Proposition 2.3 is elementary but important: Given separate RE algorithms for both the "yes" and "no" answers, we can simply run them in parallel; one of them will finish. The proposition and its proof reveal the important point that a recursive algorithm might come with no bound whatsoever on its execution time.

Theorem 2.4 (Turing). *The halting problem is in* RE *but not in* R. *In particular*, $\mathsf{RE} \ne \mathsf{R}$.

Informally, the *halting problem* is the question of whether a given algorithm with a given input terminates. Let h be the halting decision problem, where the input x in the value $h(x)$ is an encoding of an algorithm and its input (or, traditionally, an encoding of a Turing machine). It is easy to show that h is RE-*complete* in the following sense: Given a problem $d \in \mathsf{RE}$, there is a recursive function f such that $d(x) = h(f(x))$ for any input x. Any other problem in RE with this same property is also called RE-complete.

The following proposition is important for recursively enumerable infinite searches. The interpretation of the proposition, which is conveyed by the proof, is that nested infinite loops can be reorganized into a single infinite loop.

Proposition 2.5. *Let G be a graph structure on A^*. If the edge set of G is recursively enumerable, then so is the set of pairs (x, y), where x and y are vertices in the same connected component of G.*

Proof. By Proposition 2.2, we can model a recursively enumerable set by an algorithm that lists its elements. The proposition states that the elements can be listed without repetition; but this is optional, since we can store all of the elements already listed and omit duplicates.

We use a recursive bijection f between the natural numbers $\mathbb{N}$ and $\mathbb{N}^*$, the set of finite sequences of elements of $\mathbb{N}$. We can express any element of $\mathbb{N}^*$ uniquely in an alphabet that consists of the ten digits and the comma symbol. We can then list of all of these strings first by length, and then in lexicographic order for each fixed length. We can then let $f(n)$ be the nth listed string.

We can now convert the value $f(n)$ to a finite path $(x_0, \dots, x_k)$ in the graph G, in such a way that every finite path is realized. If

$$f(n) = (n_0, \dots, n_k),$$

then we let x_0 be the n_0th string in A^*. For each $j > 0$, we let x_j be the n_jth neighbor of x_{j-1}. In order to find the n_jth neighbor of x_{j-1}, we list the elements of the edge set of G until the edge (x_{j-1}, x_j) arises as the jth edge from x_{j-1}. There is the technicality that x_{j-1} might not have an n_jth neighbor if it only has finitely many neighbors. To avoid this problem, we intersperse trivial edges of the form (x, x) infinitely many times, for every string $x \in A^*$, along with the non-trivial edges of G.

Since the algorithm finds every finite path in G, it finds every pair of vertices x and y in the same connected component. Thus, the set of such pairs is recursively enumerable. □

2.2. ***Elementary recursive problems.*** As mentioned after Proposition 2.3, a recursive algorithm need not have any explicit upper bound on its execution time, beyond the tautological bound that running it is a way to calculate how long it runs. This motivates smaller complexity classes that are defined by explicit bounds. The most common notation for a bound on the execution time of an algorithm is asymptotic notation as a function of the input length $n = |x|$ to a decision problem $d(x)$. For example, we could ask for a polynomial-time algorithm, by definition one that runs in time $O(n^k)$ for some fixed k.

We have two reasons to consider a fairly generous bound in this paper. First, the recursive class R is unfathomably generous, so any explicit bound can be considered a major improvement. Second, the computational complexity of a problem or algorithm depends somewhat on the specific computational model, but certain relatively generous complexity bounds are substantially model-independent.

We consider a traditional Turing machine first. By (informal) definition, a *Turing machine* is a finite-state “head" with an infinite linear memory tape, and deterministic dynamical behavior. We say that an algorithm is *elementary recursive* if it runs in time

$$O\left(\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}}_{k}\right)$$

for some constant k. We call the corresponding complexity class ER. By abuse of terminology, we use ER to refer to both decision problems and function problems, and to numerical bounds. (Note that if $f(n)$ can be computed in ER, then a running time bound of $O(f(n))$ is itself a subclass of ER.)

Without reviewing rigorous definitions, we list some of the many variations in the computational model that do not affect the class ER in the following proposition. The proposition is not really needed in this paper except to motivate ER as an important complexity class. The only tacit dependence is that a random access machine is somewhat closer to both intuitive descriptions of algorithms and actual computers than a Turing machine with a linear tape is.

Proposition 2.6. *Each of the following four computational models is the same as standard* ER.

1. *A Turing machine with an n-dimensional tape*, *or a random access tape addressable by a separate address tape*, *with an elementary recursive bound on computation time.*
2. *A Turing machine restricted to an elementary recursive bound on computational space and unrestricted computation time.*
3. *A randomized Turing machine whose answers are probably correct*, *with an elementary recursive bound on computation time.*
4. *A quantum Turing machine that can compute in quantum superposition*, *with an elementary recursive bound on computation time.*

Proof. Instead of a self-contained proof, we justify each case of the proposition with specific references to Arora-Barak [3].

1. This follows from Exercises 1.7 and 1.9 in Arora-Barak.
2. This follows from Theorem 4.2 in Arora-Barak.
3. This reduces to case 4 by the proof of Corollary 10.11 in Arora-Barak.
4. This reduces to case 2 by the proof of Theorem 10.23 in Arora-Barak. □

Remark. An elementary recursive bound is also a major improvement over another bound that is popular in logic and computer science: primitive recursive. An algorithm is *primitive recursive* if it runs in time $O(n[k]b)$ for some fixed k and b, where the kth operation $a[k]b$ is defined inductively as follows:

$$a[1]b = a+b \qquad a[2]b = ab \qquad a[3]b = a^b$$
$$a[k+1]b = \underbrace{a[k](a[k](\cdots(a[k]a)\cdots))}_{b}.$$

For example, the operation $a[4]b$, which is called *tetration*, is defined as a tower of exponentials of height b. The primitive recursive complexity class is denoted PR. We can organize PR into a complexity hierarchy by defining E_k to be the set of functions computable in time $O(n[k+1]b)$ for some fixed b. Then $\mathsf{E}_2 = \mathsf{P}$, $\mathsf{E}_3 = \mathsf{ER}$, E_4 consists of complexity bounds which are bounded towers of tetrations, etc.

2.3. *Computable numbers.* A *computable real number* $r \in \mathbb{R}$ is a real number with a computable sequence of bounding rational intervals. In other words, there is an algorithm that generates rational numbers $a_n, b_n \in \mathbb{Q}$ such that $x \in [a_n, b_n]$ and $b_n - a_n \to 0$. Many standard algorithms from numerical analysis, including field operations, integration of continuous functions, Newton's method, etc., have the property that if the input consists of computable numbers, then so does the output. One main limitation of computable real numbers is that inequality tests such as $a > b$ or $a \neq b$ are only recursively enumerable, not recursive. In other words, if $a \neq b$, then there is an algorithm to eventually confirm this fact and say which one is greater; but there is no terminating algorithm that always confirms that $a = b$.

We can avoid this shortcoming of the field of computable numbers by passing to a smaller subfield where equality is also recursive. In particular, we will use $\hat{\mathbb{Q}} = \mathbb{R} \cap \bar{\mathbb{Q}}$, the real algebraic closure of the rational numbers $\mathbb{Q}$, which has this property.

Theorem 2.7. *There is an encoding of the elements of $\hat{\mathbb{Q}}$ such that field operations, order relations, and conversion to computable real numbers are all recursive.*

One encoding of a real algebraic number x that can be used to prove Theorem 2.7 is to describe it by a minimal polynomial together with an isolating interval $x \in [a, b]$ with rational endpoints to distinguish x from its Galois conjugates. Note that the isolating interval may be made arbitrarily small since algebraic numbers are computable, for instance by Newton's method. Note also that a computable encoding of elements of $\hat{\mathbb{Q}}$ yields a computable encoding of elements of $\bar{\mathbb{Q}} \subseteq \mathbb{C}$ the field of all algebraic numbers.

Remark. The field of real algebraic numbers together with reliable equality testing is implemented in `Sage` [41].

Theorem 2.8 (Tarski-Seidenberg [47; 44]). *It is recursive to determine whether there is a solution to a finite list of polynomial equalities and inequalities with coefficients in $\hat{\mathbb{Q}}$ in finitely many variables; or to find a solution.*

Actually, Tarski and Seidenberg proved the stronger result that it is recursive to decide any assertion over $\mathbb{R}$ expressed with polynomial relations and first-order quantifiers.

Remark. Given Theorem 2.7, it is elementary that solvability of polynomial equations in $\hat{\mathbb{Q}}$ is in RE. However, the proof of Theorem 2.8 shows that the problem is in R directly without using Proposition 2.3.

3. Triangulations of manifolds and moves

In this section, we will analyze the form of the input to Theorem 1.1. We first assume some convenient syntax to describe any finite simplicial complex by a data string, so that it is easy to check whether the input is a valid simplicial complex Θ. We would like to know whether Θ describes a closed, orientable PL manifold. We take the convention that closed manifolds are connected; it is also easy to check whether any finite simplicial complex Θ is connected.

We will review that there is a routine algorithm to confirm whether a simplicial complex represents a closed, orientable 3-manifold. Thus Proposition 2.1 applies: we can view the homeomorphism problem as a non-promise problem. Actually, we will extend this to dimension 4 in Proposition 3.1, which is a much harder case than dimension 3 that we need.

We then discuss moves between triangulations of a manifold, mainly to establish Corollary 3.4. In light of Proposition 2.3, Corollary 3.4 is an easy half of Theorem 1.1, one that holds in any dimension n.

Proposition 3.1. *If Θ is a finite simplicial complex of dimension $n \leq 4$, then it is recursive to determine whether it is a closed PL n-manifold, and whether or not it is orientable.*

Proof. The proof is partly by induction on dimension n. The result is trivial if $n = 0$, where we need only check that Θ is a single point. Otherwise, we must check that the link Λ of every vertex of Θ is both a closed $(n-1)$-manifold and a PL n-sphere. The former condition is the inductive step. The latter condition requires an algorithm to recognize an $(n-1)$-sphere. If Λ is a closed 1-manifold, then it is immediately a 1-sphere, *i.e.*, a circle. If Λ is a closed 2-manifold, then we can compute its Euler characteristic. If Λ is a closed 3-manifold, then Theorem 1.1 implies that it is recursive to determine if Λ is a 3-sphere, although this result was obtained without geometrization by Rubinstein and Thompson (Theorem 8.7) [40; 48].

We can check that Θ is orientable (and orient it) algorithmically by computing its simplicial homology. □

The stellar and bistellar subdivision theorems establish that every two triangulations of a compact n-manifold, in particular a compact 3-manifold, are connected by a finite sequence of explicit moves. See Lickorish [28] for a modern treatment and a historical review.

Theorem 3.2 (Alexander-Newman). *If two finite simplicial complexes Θ_1 and Θ_2 are PL equivalent, then they are connected by a sequence of stellar subdivision moves and their inverses.*

Briefly, a *stellar* move in a simplicial complex Θ consists of replacing the star $\mathrm{st}(\Delta)$ of some simplex Δ in Θ with a cone over the subcomplex of simplices in $\mathrm{st}(\Delta)$ that do not contain Δ. Equivalently, the apex v of this cone is placed in the interior of Δ, and all simplices that contain Δ are subdivided to support the new vertex v.

Theorem 3.3 (Pachner). *If Θ_1 and Θ_2 are two triangulations of a compact, PL manifold M, then they are connected by bistellar moves.*

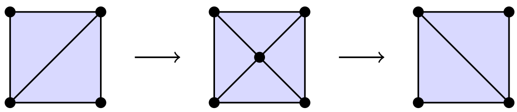

Figure 1. A bistellar move as the composition of a stellar move and an inverse stellar move.

A *bistellar* move of a triangulation of an n-manifold M consists of a stellation followed by the inverse of a different stellation at the same vertex. Equivalently, two triangulations of M differ by a bistellar move when there is a minimal “cobordism" between them consisting of a single $(n+1)$-simplex. (It is not strictly a cobordism in the sense of a connecting $(n+1)$-manifold.) In particular, a shellable triangulation of $M \times I$ yields a sequence of bistellar moves.

Lickorish points out that Newman conjectured and partially proved Theorem 3.3 in an earlier paper, before he and Alexander separately proved Theorem 3.2. Bistellar moves are also called Pachner moves, although arguably they should be called Newman-Pachner moves.

Theorem 3.3 also holds for ideal or semi-ideal triangulations of a compact 3-manifold with torus boundary components. (In other words, it holds for a 3-dimensional pseudomanifold with singular points with torus links, which are the ideal vertices.)

Theorems 3.2 and 3.3 each have the following corollary.

Corollary 3.4. *The PL homeomorphism problem for compact PL n-manifolds is in* RE.

Remark. There is a proof of Corollary 3.4 that works directly from the definition of PL equivalence without using Theorem 3.2 or 3.3, nor even Proposition 2.5. For each n, choose a linear embedding of an n-simplex $\Delta^n \subseteq \mathbb{R}^n$ with rational vertices (*i.e.*, vertices in $\mathbb{Q}^n$). Then a *geometric refinement* of Δ^n is a simplicial complex Θ with a homeomorphism onto Δ^n which is affine-linear on each simplex of Θ. Likewise a refinement of a simplicial complex Θ_1 is another simplicial complex Θ_2 with a homeomorphism $f : \Theta_2 \to \Theta_1$, such that f yields a geometric refinement of each simplex of Θ_1. By definition, two complexes Θ_1 and Θ_2 are PL equivalent if they share a refinement Θ_3. We can perturb any geometric refinement so that the new vertices are all at rational positions in their respective simplices. its vertices are all rational. The set of rational mutual refinements of two finite complexes Θ_1 and Θ_2 is recursive by direct verification. (In other words, given a simplicial complex Θ_3, and given rational target positions for its vertices in both Θ_1 and Θ_2, we can algorithmically check whether this data yields a mutual refinement.) Therefore the question of whether there exists a mutual refinement is directly recursively enumerable.

Proposition 3.5. *If Θ_1 is a finite simplicial complex with n_1 simplices (of arbitrary dimension) and $n_2 \geq n_1$, then it is recursive to produce a complete list of geometric subdivisions Θ_2 of Θ_1 with n_2 simplices.*

Proof. There are only finitely many simplicial complexes Θ_2 with n_2 simplices, and they can be generated recursively. For each candidate for Θ_2, there are only finitely many combinatorial choices for a function from the simplices of Θ_2 to the simplices of Θ_1. For each such choice, we can first check that the simplices of Θ_2 that land in a k-simplex $\Delta \in \Theta_1$ support a simplicial cycle that represents the fundamental class in $H^k(\Delta, \partial\Delta)$. We solve for each such cycle for all Δ (where each must be unique if Θ_2 indeed subdivides Θ_1). Then the constraint that each simplex of Θ_2 must be positively oriented in Θ_1 yields we obtain algebraic inequalities for the positions of all vertices. We can then apply Theorem 2.8 to see if there is a solution for those positions. □

We conclude this section with the following theorem which combines results of P.S. Novikov, Boone, Adian, Rabin, Markov, and S.P. Novikov [39]. We will not need this result; rather it stands in contrast to Theorem 1.1.

Theorem 3.6 (NBARMN). *The isomorphism problem for finitely presented groups, the PL homeomorphism problem for 4-manifolds, and the recognition of S^n among PL n-manifolds for each $n \geq 5$ are all* RE*-complete.*

It is not known whether either topological or PL recognition of S^4 is recursive.

Remark. The homeomorphism problem for PL n-manifolds in Theorem 3.6, or even recognition of S^n, needs to be handled with some care, for several reasons. First, because recognizing whether the input is a PL n-manifold is (by Theorem 3.6!) an uncomputable promise when $n \geq 6$. Second, because there are closed manifolds that are homeomorphic but not PL homeomorphic [23]. Third, because there are simplicial complexes that are not PL n-manifolds at all, but that are homeomorphic to S^n, for each $n \geq 5$ [7]. The proof of Theorem 3.6 dispenses with all of these concerns as follows. Given an input x to the halting problem $h(x)$ and an integer $n \geq 4$, there is an algorithm that constructs an n-manifold $M(x)$ such that:

1. $M(x)$ is manifestly a closed PL manifold.

2. $M(x)$ is PL homeomorphic to S^n when $n \geq 5$, or to a connected sum of copies of $S^2 \times S^2$ when $n = 4$, if and only if $M(x)$ is simply connected.

3. $M(x)$ is simply connected if and only if $h(x) = \text{yes}$.

Remark. By contrast with Theorem 3.6, the PL homeomorphism problem for simply connected n-manifolds with $n \geq 5$ is recursive [36].

4. Some notation

We summarize some notation for specific topological spaces, beyond the most standard notation that S^n is an n-sphere, D^n is an n-disk, P^n is real projective n-space, and $I = D^1$ is an interval.

We let $X \ltimes Y$ denote a fiber bundle with base X and fiber Y. Although the notation $X \tilde{\times} Y$ is reasonably standard for a twisted bundle, we prefer to write $X \ltimes Y$, for two reasons. First, because the notation specifies which side is the fiber; we can write $X \ltimes Y \cong Y \rtimes X$. Second, because a fiber bundle is analogous to a semidirect product in group theory.

We review Seifert's description of oriented Seifert-fibered spaces [45]. If F is a compact surface which may or may not be orientable, then there is a unique, canonically oriented I-bundle $F \ltimes I$. If F is orientable, then this I-bundle is simply $F \times I$; in this case we assume an orientation for the base F and the fiber I. We consider the double $F \ltimes S^1$ of $F \ltimes I$, which again when F is orientable is just $F \times S^1$.

If $p_1, p_2, \ldots, p_n$ are points of F, then we can apply a Dehn surgery with slope b_k/a_k to a solid torus neighborhood of the fiber over p_k in $F \ltimes S^1$, where a_k is a positive integer and b_k is a relatively prime integer of either sign. The resulting oriented 3-manifold N is thus constructed from its *Seifert data*, namely the multiset

$$\{F, (a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\}.$$

In general we interpret F as an orbifold. If $a_k \geq 2$, then we interpret $p_k \in F$ as an orbifold point of order a_k, and the circle over it is an *exceptional fiber*. By Seifert's classification, the integers a_k with $a_k \geq 2$ together with the residues $b_k \in \mathbb{Z}/a_k$ are all topological invariants of the fibration of N. If F and therefore N has boundary, then this is a complete set of invariants. If N is closed, then the Euler number

$$e(N) = \sum_k \frac{b_k}{a_k}$$

is the only additional necessary invariant. Thus there is a canonicalized version of the Seifert data in the form

$$\{F, b, (a_2, b_2), (a_3, b_3), \ldots, (a_n, b_n)\},$$

where b represents $(1, b)$ and otherwise $a_k \geq 2$ and $1 \leq b_k < a_k$. If F is non-compact, then b is irrelevant and we omit it in the canonical form.

With the notation of fiber bundles and Seifert-fibered spaces, we name these specific manifolds:

1. We use $S^1 \times S^1$ to denote the standard 2-torus, and T to denote an arbitrary 2-torus, *i.e.*, $T \cong S^1 \times S^1$.
2. $K^2 = S^1 \ltimes S^1$ is the 2-dimensional Klein bottle.
3. $L(m, n)$ is the lens space defined by the Seifert data $\{S^2, 0, (n, m)\}$.
4. $R(m, n)$ denotes the prism space defined by the Seifert data $\{P^2, 0, (m, n)\}$.

5. Geometrization is recursive

The goal of this section is to prove Theorem 5.4, which says that the geometric decomposition of a 3-manifold M is computable.

5.1. ***Statement of geometrization.*** We begin with three results that, together, are one formulation of the geometrization theorem for closed, oriented 3-manifolds.

Theorem 5.1 (Kneser-Milnor [24; 34]). *Every closed, oriented* 3*-manifold* (*other than* S^3) *is a connected sum of prime, closed, oriented* 3*-manifolds* (*none of which are* S^3). *The summands are unique up to oriented homeomorphism.*

We will adopt the convenience that a 3-sphere S^3 counts as a prime 3-manifold, notwithstanding that Theorem 5.1 would be easier to state if S^3 were instead interpreted as the "unit" in the terminology of unique factorization.

Theorem 5.2 (Jaco-Shalen-Johansson [19; 20]). *A closed, oriented, prime* 3*-manifold has a minimal collection of incompressible tori, unique up to isotopy and possibly empty, with the property that the complementary regions are either Seifert-fibered or atoroidal.*

Recall that a 3-manifold N which may have boundary is *atoroidal* if every essential torus in N is parallel to some boundary component of N. The decomposition in Theorem 5.2 is called the *JSJ decomposition*. We can call the tori *JSJ tori*, and the complementary regions *JSJ components*. We will use M to denote a general closed, oriented 3-manifold; then W to denote a prime summand of M; then N to denote a JSJ component of W.

Theorem 5.3 (Thurston-Hamilton-Perelman). *Suppose that N is an oriented, prime, atoroidal 3-manifold which is either closed or has torus boundary components. Then N is either Seifert-fibered, or it is closed and has a unique hyperbolic structure, or its interior N^* has a unique, complete hyperbolic structure with torus cusps.*

As everyone knows, Theorem 5.3 was conjectured and partly proven by Thurston [49], then fully proven by Perelman using the Ricci flow program of Hamilton [35]. (Note that Theorem 5.3 implicitly includes the Poincaré conjecture in the Seifert-fibered case.)

Remark. Mixing the JSJ decomposition with hyperbolization is a less pure approach than Thurston's decomposition into geometric components, but we find it convenient for Theorem 1.1. We could recognize spherical and Euclidean components with the same methods as hyperbolic components (Lemma 5.7), while several of the other Thurston geometries induce canonical Seifert fibrations. In fact, every Seifert-fibered 3-manifold or component is geometric. Conversely, every geometric 3-manifold or component is hyperbolic unless it is Seifert-fibered or a Sol manifold. Thus, the JSJ decomposition of a prime 3-manifold W differs from the minimal geometric decomposition only when W is Sol; in this case W is a torus bundle over a circle and has a JSJ torus which is one of the fibers.

5.2. *Statement of computational geometrization.*

Theorem 5.4. *If M is a triangulated 3-manifold, then it is recursive to compute a decorated triangulation which is adapted to its geometric decomposition.*

Before proving Theorem 5.4, we need to state it more precisely. When Θ is a *decorated, adapted triangulation* of M it means that:

1. M has a distinguished (but possibly empty) collection of disjoint, separating 2-spheres, each triangulated with 4 triangles in Θ, that separates it into prime summands $\{W\}$. Each W is closed; it inherits its triangulation from Θ and its holes are plugged with fresh tetrahedra.
2. The triangulation of each W supports a distinguished (but possibly empty) collection of disjoint thickened tori $T \times I$ and restricts to a shelled triangulation of each one. These thickened tori separate W into JSJ components $\{N\}$.
3. The tetrahedra at all stages are consistently oriented, to express an orientation of each summand W and each JSJ component N that is consistent with the orientation of M.
4. If N is Seifert-fibered with base F, then we make a triangulation which is adapted to Seifert's description of N by Dehn surgery on $F \times S^1$ when F is orientable, or Dehn surgery on a twisted bundle $F \ltimes S^1$ when F is non-orientable. This includes the case where $N = W = M$ is a 3-sphere.

5. If N is hyperbolic, then it is marked as the barycentric subdivision of a regular, adapted cellulation Λ. Λ comes from a geometric triangulation Λ^* of the interior N^*, in which each tetrahedron has at most one ideal vertex. If $\Delta \in \Lambda^*$ has an ideal vertex, then it is truncated to a triangular prism in Λ; if not, then it is kept in Λ. Each tetrahedron in Λ^* is also decorated with its dihedral angles.

We proceed to explain each stage of the definition.

5.2.1. *The prime decomposition.* Note that we take a triangulation to be a simplicial complex structure rather than a generalized triangulation. In geometric topology, for instance in the SnapPea census, a *generalized triangulation* is sometimes defined to be a CW-complex whose cells are simplices and whose attaching maps take simplices to simplices. In particular, a simplex in a generalized triangulation need not have distinct vertices and two simplices may have the same vertices. We can form a connected sum of two triangulated 3-manifolds by removing a single tetrahedron from each one and gluing the sphere boundaries.

5.2.2. *Shelled triangulations.* If X is a closed n-manifold with two triangulations Θ_0 and Θ_1, then a *shelled triangulation* of $X \times I$ is a simplicial complex $\Theta_{0,1}$ whose $(n+1)$-simplices are numbered. Taken in order, the $(n+1)$-simplices connect the triangulation Θ_0 of $X \times \{0\}$ to the triangulation Θ_1 of $X \times \{1\}$ via a sequence of bistellar moves. Note that this combinatorial restriction on $\Theta_{0,1}$ implies $\Theta_{0,1}$ is PL homeomorphic to $X \times I$. In other words, if we build $\Theta_{0,1}$ from a sequence of bistellar moves, and if $X \times \{0\}$ and $X \times \{1\}$ are disjoint in the result, then $\Theta_{0,1}$ is a triangulation of $X \times I$.

5.2.3. *Orientations.* To be precise, we can decorate each tetrahedron by ordering its vertices, where two orderings are equivalent if they differ by an even permutation.

5.2.4. *Seifert-fibered components.* We begin with preliminaries on cellulations and barycentric subdivisions that we will also need in Section 5.2.5.

A *cellulation* of a topological space X is a CW complex Λ with a homeomorphism to X. The complex Λ is *regular* if Λ is locally finite; and if the attaching map of each k-cell is an embedding in the $(k-1)$-skeleton, so that each closed k-cell of Λ is embedded in X.

We will use the following standard proposition to model regular CW complexes using triangulations.

Proposition 5.5 ([26, Sec. 10.3.5]). *Every regular CW complex Λ has a barycentric subdivision Θ which is a simplicial complex, and the spaces of Θ and Λ are homeomorphic.*

See Figure 2 for an example.

If N is a Seifert-fibered component, then as described in Section 4, it has a base orbifold F with one circle for each boundary torus of N. The fibration has

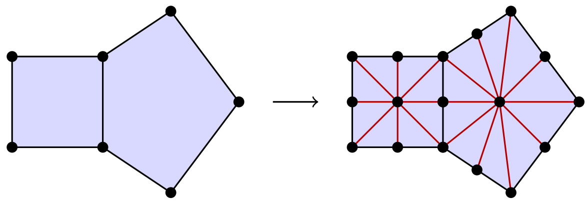

Figure 2. A barycentric subdivision of part of a cellulation of a surface.

canonical Seifert data

$$\{F, b, (a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\},$$

with b omitted when F or N has boundary. The data indicates surgery with slope b_k/a_k at the fiber over some $p_k \in F$ and (if it exists) surgery with slope b at $p_0 \in F$.

We choose a triangulation Θ_F of F such that each p_k (including p_0, if it exists) lies in the interior of a triangle, and such that all of these triangles are disjoint. We can lift Θ_F to a cellulation Λ_F such that the solid torus $\Delta \times S^1$ over each triangle Δ in Θ_F is tiled by two vertical triangular prisms. We take the barycentric subdivision of Λ_F to obtain a triangulation of $F \times I$ or $F \ltimes I$. If a triangle $\Delta \in \Lambda$ contains some p_k, we remove the solid torus $\Delta \times S^1$ (which is now triangulated with 72 tetrahedra) and glue it back using Dehn surgery. The gluing involves a homeomorphism of the boundary $\partial(\Delta \times S^1)$, which we implement with a shelled triangulation of a thickened torus, as in Section 5.2.2. The result is a triangulation of N, which we decorate with information about how it was constructed, so that the canonical Seifert data is part of the decoration.

5.2.5. *Hyperbolic components.* If the component N is hyperbolic, then we choose a *geometric triangulation* Θ^* of N^*, meaning one whose tetrahedra lift to geometric tetrahedra in the universal cover $\mathbb{H}^3$. Recall from the beginning of Section 5.2 that the geometry of each such tetrahedron is specified by giving its dihedral angles.

More precisely, if $\hat{N}$ is the compactification of N given by collapsing each torus boundary component to a point, we assume a continuous map

$$f : \Theta^* \to \hat{N}$$

such that the image $f(\Delta)$ of each combinatorial tetrahedron Δ lifts to a geometric tetrahedron in the standard compactification $\overline{\mathbb{H}^3}$ of hyperbolic space. If none of the vertices of $f(\Delta)$ are at infinity, then $f(\Delta)$ is *finite*; if they are all at infinity, then $f(\Delta)$ is *ideal*; and if some are at infinity, then $f(\Delta)$ is *semi-ideal*. We will assume that all of the simplices of our Θ^* are either finite or semi-ideal with one ideal vertex. If $N = N^*$ is closed, then all tetrahedra in Θ^* must be finite; if N has boundary components and thus N^* has cusps, then some of the tetrahedra must be semi-ideal.

Before proceeding further, we contrast this with some other models that have also been studied as geometric triangulations. In some treatments f is not a homeomorphism but only a homotopy equivalence. In the case we can still ask for the restriction of f to each tetrahedron Δ to be affine-linear in the Klein model of $\overline{\mathbb{H}^3}$. However, $f(\Delta)$ may be *degenerate*, meaning that it has zero volume, or it may be *flipped over*, meaning that it has negative signed volume relative to the orientation of Θ^* and the standard orientation of $\mathbb{H}^3$. In another variation, which is often used when N is closed, the inverse map $g : \hat{N} \to \Theta^*$ is defined, and a finite set of closed geodesic curves in $\hat{N}$ collapse to ideal points; but the inverse image of any open tetrahedron in Θ^* is still a geometric tetrahedron in $\hat{N}$. Such a structure is a *spun triangulation*, because a geodesic circle $C \subseteq N$ is approached by cusps of ideal tetrahedra that wind helically around it. In particular SnapPea uses spun triangulations.

Ideal geometric triangulations are especially desirable in computational hyperbolic geometry because they are rigid and algebraically the simplest. However, it is only a conjecture that every suitable hyperbolic manifold has an ideal, possibly spun geometric triangulation. Such a structure does always exist with degenerate or flipped-over tetrahedra, but these are less desirable. We will use finite and semi-ideal tetrahedra in order to avoid this impasse. The following proposition is then standard:

Proposition 5.6. *If N^* is a complete hyperbolic manifold which is either cusped or closed, then it has a geometric triangulation with finite and semi-ideal tetrahedra (none of which are spun, degenerate, or flipped over). Also, every semi-ideal tetrahedron has only one ideal vertex.*

Proof. We can choose a point $p \in N^*$ and consider the Voronoi tiling of its orbit in $\mathbb{H}^3$. Each Voronoi cell is a fundamental domain and yields a cellulation Λ_1 of $\hat{N}$. Λ_1 is not in general regular, but it has a barycentric subdivision Λ_2 which is regular. Moreover, each simplex of Λ_2 has at most one vertex of V and thus at most one ideal vertex. We can let $\Theta = \Lambda_3$ be a second barycentric subdivision, which is then a simplicial complex and still has the property that each tetrahedron has at most one ideal vertex. □

Given a semi-ideal triangulation of N^*, we can truncate the cusps so that each semi-ideal tetrahedron becomes a triangular prism, as in Figure 3. A barycentric subdivision of this cellulation is then the desired adapted triangulation.

5.3. ***Proof of Theorem 5.4.***

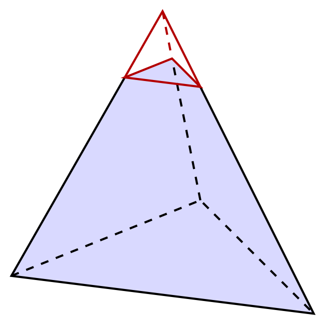

Figure 3. A tetrahedron truncated at one vertex.

Lemma 5.7. *It is recursive to find a geometric triangulation of a hyperbolic* 3-*manifold N which is either closed or has torus boundary components*, *using either of two descriptions of each dihedral angle* $0 < \alpha < \pi$ *of each tetrahedron*:

1. *Each imaginary exponential* $\exp(i\alpha)$ *is specified as an element of* $\bar{\mathbb{Q}}$.
2. *Each angle* α *is given as a computable real number.*

Hence, *it is in* RE *to determine if N is hyperbolic.*

Although the second case of Lemma 5.7 immediately follows from the first one, we will give a separate proof of each case. Moreover, even the weaker second case of Lemma 5.7 is sufficient to prove Theorem 1.1.

Remark. Manning [29, Thm. 5.2] also proves Lemma 5.7, but as a corollary of a harder result. His results show (without geometrization) that it is recursive to decide whether N is hyperbolic, when there is an algorithm for the word problem for $\pi_1(N)$. He also uses a single polyhedral fundamental domain to describe the geometry of N. Although this differs from a hyperbolic triangulation, which is what we use, the two models are somewhat interchangeable for our purposes.

Proof of case 1 of Lemma 5.7. Suppose that Θ^* is a geometric triangulation of N^*. We can model each tetrahedron $\Delta \in \Theta^*$ (non-uniquely) by choosing four vertices in the Poincaré upper half-space model, including one on the boundary if Δ is semi-ideal. (Note that the ideal vertices of Θ^* are marked in advance.) There is an algebraic formula for each finite edge length ℓ and each dihedral angle α of Δ, if these are represented by their exponential values $\exp(\ell)$ and $\exp(i\alpha)$. The main matching condition for Θ^* to be geometric is that if two tetrahedra share a finite edge, then the edge lengths agree; and the total dihedral angle around each edge equals 2π. The first condition is immediately an algebraic condition, although note that if an edge is semi-ideal, then it has infinite length and its length equation is vacuous.

The second condition is almost an algebraic condition since the product of the exponentiated angles must be 1; this shows that the total angle is a multiple of 2π, although not which one. However, this can be remedied with additional algebraic inequalities, recalling that we are allowed real algebraic equations for the real and imaginary parts $\cos(\alpha)$ and $\sin(\alpha)$ of each complex variable $\exp(i\alpha)$. Suppose that every edge of Θ^* has at most n incident tetrahedra. Then we can make a finite covering of the unit circle $S^1 \subseteq \mathbb{C}$ by rational rectangles, such that the radial projection of each rectangle onto the circle has length less than $2\pi/n$. We can then loop over choices for which rectangle contains each exponentiated angle $\exp(i\alpha)$. If each exponentiated angle is confined to such a rectangle, we can know whether the sum of the angles around an edge is specifically 2π and not some other multiple of 2π.

After forming algebraic equations for all of the geometric data, the equations have a solution in terms of real algebraic numbers when they have a solution at all. For any fixed triangulation Θ, we can thus use Theorem 2.7 (not Theorem 2.8; see the remark after the proof) to search for a solution and eventually find it, if it exists. We must also search over triangulations using Theorem 3.2 or Theorem 3.3. Since the result is a nested infinite search (over triangulations and then candidate geometric structures), we can apply Proposition 2.5. $\square$

Remark. Although an infinite search for a solution to algebraic gluing equations is preposterous in practice, it is good enough for an algorithm in RE. Alternatively, for each triangulation, we can apply the more difficult Theorem 2.8 to determine in R if there is a solution.

Remark. If we allowed geometric triangulations with fully ideal edges, then it would not be enough for the sum of the angles around such an edge e to be 2π. Since e goes to itself under hyperbolic translation as well as rotation, gluing together the tetrahedra that contain e could create a non-trivial translational holonomy. The two conditions together, that the total angle is 2π and the translational holonomy vanishes, are known as a Neumann-Zagier gluing relation [38].

Remark. Instead of calculating lengths and angles using positions of vertices in hyperbolic geometry, we can also relate them directly using formulas from hyperbolic and spherical trigonometry.

The separate proof of the second case of Lemma 5.7 works directly with computable numbers, in effect using numerical analysis to calculate better and better approximate solutions. In this approach, we need a criterion to know that an approximate solution is close to an exact one. Given a smooth multivariate equation $f(x) = 0$, the Newton-Kantorovich theorem [21] establishes a sufficient criterion for Newton's method to converge from an approximate solution x_0 to an exact solution

x_∞. Neuberger [37] points out that an ODE analogue of Newton's method, which is called the *continuous Newton's method*, simplifies the Newton-Kantorovich result.

Theorem 5.8 (Newton-Kantorovich-Neuberger [37, Thm. 2]). *Let $B_\epsilon(x_0) \subset \mathbb{R}^n$ be the open ball of radius $\epsilon > 0$ around $x_0 \in \mathbb{R}^n$, and let*

$$f : B_\epsilon(x_0) \to \mathbb{R}^n$$

be a C^2-smooth function with non-singular matrix derivative Df. Suppose that

$$||(Df(x))^{-1} f(x_0)|| < \epsilon \tag{1}$$

for all $x \in B_\epsilon(x_0)$, where $||\cdot||$ is the Euclidean norm on $\mathbb{R}^n$. Then there is a unique $x_\infty \in B_\epsilon(x_0)$ such that $f(x_\infty) = 0$. Also, given a solution x_∞ such that $Df(x_\infty)$ is non-singular, *equation* (1) *eventually holds as $x_0 \to x_\infty$, moreover with $\epsilon \to 0$.*

Although we will not reprove Theorem 5.8, we can discuss where the theorem and its proof come from. Newton's method to find a root of a univariate function $f : (a, b) \to \mathbb{R}$ begins at an approximate root $x_0 \in (a, b)$ and applies the iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

which in favorable cases converges to a solution x_∞ of the equation $f(x) = 0$. If f is multivariate as in Theorem 5.8, then this has the well-known matrix generalization

$$x_{n+1} = x_n - (Df(x_n))^{-1} f(x).$$

Finally in the continuous version, we let $x(0) = x_0$ and define the ODE

$$x'(t) = -(Df(x(t)))^{-1} f(x).$$

Then in favorable cases the limit

$$x_\infty = \lim_{t\to\infty} x(t)$$

is again a solution to $f(x) = 0$.

Remark. Although Neuberger's paper on the continuous Newton's method is more recent than Thurston's work, Kantorovich's earlier, more complicated formula also suffices for Lemma 5.7 and Theorem 1.1.

If the equation $f(x) = 0$ has a non-singular Jacobian Df in a neighborhood of a solution, as in Theorem 5.8, then the system of equations is also called *transverse* or *first-order rigid*. We will need a generalization of this concept. Given a smooth function

$$U \subseteq \mathbb{R}^n \qquad f : U \to \mathbb{R}^m,$$

where n and m need not be equal, if Df has constant rank k, then the image $f(U)$ is a manifold and f is a submersion onto its image. In this case the equation $f(x)=0$ is first-order rigid *except for* the directions parallel to the manifold $f^{-1}(0)$. By the implicit function theorem, we can discard some set of $n-k$ coordinates in the domain and project to some set of k coordinates in the target to achieve unconditional first-order rigidity that satisfies the hypotheses of Theorem 5.8.

To establish first-order rigidity in our case, we will need a corollary of the Calabi-Weil rigidity theorem.

Theorem 5.9 (Calabi-Weil [22, Sec. 8.10]). *If N is a closed, hyperbolic 3-manifold, then the induced representation of its fundamental group,*

$$\rho : \pi_1(N) \to \mathrm{Isom}(\mathbb{H}^3),$$

is first-order rigid except for conjugacy. (I.e., *it is infinitesimally rigid at the level of the first derivative.*) *The same is true if N is cusped, among representations that are parabolic at the torus cusps.*

Corollary 5.10 (Stated by Izmestiev [15, Sec. 1.3]). *If Θ is a geometric triangulation of a closed or cusped hyperbolic 3-manifold N^*, then it is first-order rigid except for motion of the non-ideal vertices.*

Since we could not find a proof of Corollary 5.10 in the literature, we provide one in Section 5.4.

Proof of case 2 of Lemma 5.7. We fix the model of each tetrahedron in the upper half space model so that it has exactly six degrees of freedom, or five if one of the vertices is ideal. After ordering the vertices v_1, v_2, v_3, v_4, we can put vertex v_1 at $(0, 0, 1)$; we can put vertex v_2 directly below it (or at $(0, 0, 0)$, allowing it to be the ideal vertex); and we can put v_3 at a position of the form $(a, 0, b)$. We approximate the positions of the vertices approximately with rational numbers. We can then approximate the lengths and angles of each tetrahedron in the same form, as well as the first and second derivatives of the lengths and angles as a function of the main variables, the separate positions of the vertices in the ideal models of the tetrahedra.

Suppose that there are n non-ideal vertices. By the implicit function theorem, any exact solution to the gluing equations for the tetrahedra can be perturbed so that some $3n$ of the coordinates are exactly rational. Also by the implicit function theorem, some $3n$ of the angle conditions are implied by the other angle conditions and can be omitted. Finally, the fixed coordinates and omitted angle conditions can be chosen so that the remaining system of constraints, which we can write abstractly as $f(x)=0$, has a non-singular Jacobian Df.

Moreover, the mapping f is real analytic with an explicit formula. Thus, given an approximate solution x_0 which is within ϵ of a true solution and ϵ is small enough, we can majorize $||(Df(x))^{-1}||$ on the ball $B_\epsilon(x_0)$ using Taylor series, to confirm equation (1). □

Just as Lemma 5.7 addresses the hyperbolic case of Theorem 5.4, the following lemma addresses the Seifert-fibered case.

Lemma 5.11. *It is recursive to find an adapted triangulation of a Seifert-fibered manifold N which is either closed or has torus boundary components. Hence, it is in* RE *to determine if it is Seifert-fibered.*

Proof. We can search through triangulations until we find one that is a barycentric subdivision of a cellulation by triangular prisms. It is then easy to check whether the prisms fit together following the rules in Section 5.2. □

Finally, a torus T that has matching Seifert-fibered structure on both sides is not needed and is not a JSJ torus. It is easy to see this case in the proof of Theorem 5.2. The more subtle possibility is that one or both sides might have more than one Seifert fibration. Fortunately this is rare for Seifert-fibered manifolds with boundary. It is addressed by the following result.

Theorem 5.12 (Waldhausen [16, Thm. VI.17 & Lem. VI.19]). *Let N be an oriented 3-manifold with non-empty boundary ∂N and which has at least one Seifert fibration. Then the fibration is uniquely determined up to isotopy by its restriction to ∂N, and is outright unique except the following cases:*

1. *If N is a solid torus $D^2 \times S^1$, then every fibration of ∂N extends to a fibration of N over a disk D^2 with at most one exceptional fiber.*
2. *If N is a thickened torus $S^1 \times S^1 \times I$, then every fibration is a trivial circle bundle over an annulus. There is such a fibration for every rational slope in a single torus $S^1 \times S^1$.*
3. *If N is a twisted I-bundle $K^2 \ltimes I$ over a Klein bottle K^2, then it has two non-isotopic fibrations. One fibration is over a Möbius strip with Seifert data $\{S^1 \ltimes I\}$, and one is over a disk D^2 with Seifert data $\{D^2, (2, 1), (2, 1)\}$.*

Proof of Theorem 5.4. We search over triangulations Θ of M using stellar or bistellar moves, and decorations of them, to find an adapted triangulation as described in Section 5.2. A decoration that shows that the triangulation is adapted consists of distinguished spheres and thickened tori, and a reverse barycentric subdivision in each JSJ component N to make triangular prisms in the Seifert-fibered case and a combination of once-truncated and ordinary tetrahedra in the hyperbolic case. Within this search, we search for geometric data to describe the hyperbolic structure of each N which is not Seifert-fibered. Since these are nested, infinite searches, we combine them using the RE search algorithm of Proposition 2.5. By the geometrization theorem, we will eventually find a Θ that fits the description of Section 5.2.

In the search, we need to veto a decorated triangulation whose 2-spheres do not represent a connected sum decomposition, or such that the tori in a summand do not represent a JSJ decomposition. We do this with the following checks:

1. All 2-spheres in the decoration of M are separating.

2. All tori in each summand W are essential.

3. Either M is a 3-sphere or no summand W is a 3-sphere.

4. Each summand W is prime.

5. No two tori in one summand W are parallel.

6. No torus T in a summand W has Seifert-fibered components on both sides that restrict to the same fibration of T.

7. No torus T in W has Seifert-fibered components on both sides that induce the same fibration of T, after refibering components.

Arguing each case separately, case 1 is straightforward.

In case 2, if a summand W is a torus sum of hyperbolic and Seifert-fibered components, and if some torus is inessential, then some torus bounds a Seifert-fibered solid torus with base D^2 and at most one exceptional fiber.

In case 3, we can recognize when M or a summand W is a 3-sphere by confirming that it is Seifert fibered with base S^2 and at most two exceptional fibers, and trivial homology. (This is assuming case 2, there are no inessential tori.)

In case 4, if a summand W is either closed hyperbolic or a torus sum of hyperbolic and Seifert-fibered components with essential tori, then it has trivial π_2 and is therefore prime. On the other hand, if W is closed Seifert-fibered, then it is prime unless it is $P^3 \# P^3$ with Seifert data $\{P^2, 0\}$ [16, Lem. VI.7].

In case 5, we check that no Seifert-fibered component N of a summand W is a thickened torus (with base an annulus and no singular fibers), unless W is a torus bundle over a circle.

Case 6 is straightforward.

By Theorem 5.12 and the comments after it, in case 7 we only have to consider two types of Seifert-fibered components, which can be recognized explicitly from any of their fibrations:

7a. $N \cong S^1 \times S^1 \times I$, or

7b. $N \cong K^2 \ltimes I$.

Case 7a is only possible if N is glued to itself to make W a torus bundle over a circle, $S^1 \ltimes (S^1 \times S^1)$, because we have already eliminated parallel tori. The torus is needed if and only if W is a Sol manifold. We can verify this case by confirming that the holonomy matrix in $\mathrm{SL}(2, \mathbb{Z})$ has distinct, real eigenvalues.

In case 7b, $N \cong K^2 \ltimes I$ only has one torus boundary component T, so its refibration does not affect any other torus. In this case the fibration of N may have Seifert data $\{S^1 \ltimes I\}$ or $\{D^2, (2, 1), (2, 1)\}$. The resulting binary choice may occur on one or both sides of T, and we veto Θ if the Seifert fibrations extend across T for any of these choices. □

5.4. ***Proof of Corollary 5.10.*** The idea of the proof is that we can convert a first-order deformation of a triangulation of N^* to a deformation of a representation of ρ, in much the same way that we can convert a triangulation to ρ in the first place.

Proof. In general, if Γ is a discrete group (such as the fundamental group of a topological space) and G is a Lie group, then we can describe a first-order deformation of a homomorphism $\rho : \Gamma \to G$ as a homomorphism

$$(\rho, \rho') : \Gamma \to G \ltimes \mathfrak{g}.$$

Here $\mathfrak{g}$ is the Lie algebra of G viewed as a group under addition, while $G \ltimes \mathfrak{g}$ is the semidirect product in which the non-normal subgroup G acts on the normal subgroup $\mathfrak{g}$ by conjugation. Also, (ρ, ρ') should reduce to ρ under the quotient map

$$\pi : G \ltimes \mathfrak{g} \to G.$$

Note that $G \ltimes \mathfrak{g}$ is also the total space of the tangent bundle TG. In other words, the extension ρ' is a choice of a tangent vector $\rho'(g) \in T_{\rho(g)}G$ for every $g \in \Gamma$, such that the pairs $(\rho(g), \rho'(g))$ together make a group homomorphism.

Suppose that $\Gamma = \Gamma_1(X)$ is the fundamental group of a based CW complex X. Then we can model ρ (non-uniquely) as a non-commutative cellular cocycle $\alpha \in C^1(X; G)$. Given ρ, we can likewise model the extension ρ' (also non-uniquely) as a commutative cocycle $\alpha' \in C^1(X; \mathfrak{g})$, where here $\mathfrak{g}$ is a coefficient system twisted by α.

Now let $X = N$, where N has a cellulation Θ that comes from a closed or cusped hyperbolic structure on N^* and a geometric triangulation Θ^*. If N is cusped and Θ^* is a semi-ideal triangulation, then we make Θ by truncating the ideal vertices of Θ^*. We then want to make a cocycle α from γ. To do this, we first choose a specific isometry $\tilde{N}^* \cong \mathbb{H}^3$. Then we choose an orthonormal tangent frame at each vertex of Θ. Given an edge $e \in \Theta$, we let $\alpha(e)$ be the element of $G = \mathrm{Isom}^+(\mathbb{H}^3)$ that takes the tail $\tilde{v}$ of a lift $\tilde{e}$ to the head $\tilde{w}$, and takes the lifted frame of $\tilde{v}$ to the lifted frame of $\tilde{w}$. If N^* is cusped, then we require that each truncation edge in N is assigned a parabolic element that fixes the corresponding ideal vertex in N^*.

In this setting, Theorem 5.9 says that $H^1(N; \mathfrak{g}) = 0$ in the closed case and $H^1(N, \partial N; \mathfrak{g}, \mathfrak{p}) = 0$ in the cusped case, where $\mathfrak{g}$ is the parabolic Lie subalgebra of $\mathfrak{g}$. The theorem is typically proved using de Rham cohomology rather than cellular cohomology, but these models of cohomology are isomorphic as usual. More explicitly, every 1-cocycle $\alpha' = \delta\beta$, where β is an $\mathfrak{g}$-valued 0-cochain on the vertices of Θ.

Finally, suppose that γ' is a first-order deformation of the hyperbolic structure γ of Θ^* that satisfies the first derivative of the gluing equations. Then we can lift γ' to a cocycle α' (non-uniquely) in the same way that γ lifts to α. Then Theorem 5.9 provides β, and β descends to a first-order motion of the vertices of Θ^* that induces the deformation γ'. □

6. Homeomorphism is recursive

In this section we will prove Theorem 1.1, postponing only the proof of Theorem 6.1 below until Section 7.

6.1. ***Connected sums.*** If M_1 and M_2 are two closed, oriented 3-manifolds given by triangulations, then by Theorem 5.4, we know the direct sum decompositions of each one into prime 3-manifolds. These summands can be freely permuted and can only be matched in finitely many ways. If we search over the ways to match them, we then reduce the oriented homeomorphism problem $M_1 \overset{?}{\cong} M_2$ to the oriented homeomorphism problem $W_1 \overset{?}{\cong} W_2$ for prime summands W_1 and W_2. To review, each summand W_k inherits an orientation from its parent M_k; in the reverse direction, there is no ambiguity in forming an oriented connected sum.

6.2. ***One JSJ component.*** We switch to the other end of geometric decompositions to analyze a single pair of JSJ components $N_1 \subseteq W_1 \subseteq M_1$ and $N_2 \subseteq W_2 \subseteq M_2$. We are interested not only in the isomorphism problem, but also in the effect of the mapping class group of a component N on the boundary ∂N.

Theorem 6.1. *Suppose that N is an oriented, hyperbolic JSJ summand such that N^* is either closed or cusped. Then the mapping class group of N is its isometry group. It is a finite group and its computation is recursive. If N_1 and N_2 are two such manifolds, then they are homeomorphic if and only if they are isometric, and recognizing this condition is recursive.*

Again, we will prove Theorem 6.1 in Section 7. Note that if N is hyperbolic and has torus boundary components, then each such component inherits a Euclidean structure from the hyperbolic structure on N^*.

Suppose instead that N is Seifert-fibered (and, as usual, oriented). Then in the direct sense the automorphism problem only matters for Theorem 1.1 when the JSJ graph is non-trivial and thus N has boundary. However, we will learn the relevant automorphism properties from an associated closed Seifert-fibered space.

Lemma 6.2. *Let N be a closed, oriented 3-manifold which is decorated with a Seifert fibration with Seifert data*

$$\{F, b, (a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\}.$$

Then:

1. *The exceptional fibers of N are freely permutable by automorphisms of the Seifert fibration, provided that the permutation preserves the orbifold number $a_k \geq 2$ and the residue $b_k \in \mathbb{Z}/a_k$ of each exceptional fiber.*
2. *Any finite set of regular fibers is freely permutable.*
3. *If the base F is orientable, then N has an orientation-preserving homeomorphism that inverts all fibers together, but they cannot be inverted separately.*
4. *If the base F is non-orientable, then given two disjoint finite sets $A, B \subseteq F$, N has a homeomorphism that inverts the fibers over A in place and fixes the fibers over B.*

Proof. Cases 1, 2, and 4 can all be established by isotopies of F that move points that correspond to the distinguished fibers. In case 4, using the hypothesis that F is non-orientable, we can move a point $p \in A$ around an orientation-reversing loop in F that stays away from B and from the rest of A.

Meanwhile in case 3, the fibration itself is orientable, which means that an orientation of any one fiber induces a canonical orientation of all fibers. On the other hand, Seifert's construction of the fibration via vertical Dehn surgery on $F \times S^1$ is invariant with respect to inverting the S^1 factor and simultaneously applying an orientation-reversing homeomorphism to F. □

We also need the counterpart to Theorem 5.12 for closed Seifert-fibered spaces.

Theorem 6.3 (Waldhausen [16, Thm. VI.17]). *If N is a closed, oriented Seifert-fibered 3-manifold, then its Seifert fibration is unique up to homeomorphism except in the following cases*:

1. *A Seifert-fibered space with base S^2 and at most two exceptional fibers is either a lens space $L(m, n)$, $S^2 \times S^1$, or S^3*
2. *A Seifert-fibered space with base P^2 and at most one exceptional fiber is either a lens space $L(4, n)$, a prism space $R(m, n)$, or $P^3 \# P^3$.*
3. *The space with Seifert data*

$$\{S^2, b, (2, 1), (2, 1), (a_1, b_1)\}$$

is a prism space $R(m, n)$.

4. *The twisted bundle $K^2 \ltimes S^1$ which is the double of $K^2 \ltimes I$ has the double of its two fibrations, namely the Seifert data $\{K^2, 0\}$ and the Seifert data*

$$\{S^2, 0, (2, 1), (2, 1), (2, 1), (2, 1)\}.$$

Theorem 6.3 comes with simple formulas for which lens space or prism space is obtained, which we omit. In particular, the answer is recursive and (as we will later want) elementary recursive.

6.3. ***The JSJ graph.*** If W is a prime 3-manifold, then its JSJ decomposition is modelled by a labelled graph Γ, whose vertices represent JSJ components and whose edges represent connecting tori. Each vertex is labelled by the homeomorphism type of its component, which is either Seifert-fibered or hyperbolic. In addition, each edge is decorated with gluing data and peripheral data which will be described precisely in the proof of Theorem 1.1 below.

Remark. This graph structure inspired the term *graph manifold* for a prime 3-manifold whose JSJ components are all Seifert-fibered [51]. This terminology is standard but ironic, since geometrization shows that the same graph concept is important for all prime 3-manifolds.

The labelled graph Γ is an invariant of W, which at first glance may seem like a complete invariant, provided that the homeomorphism problem for each JSJ component is recursive. However, it is not that simple, because we have to know the allowed permutations of the torus boundary components of a JSJ component N, and the allowed homeomorphisms of each torus boundary component. Finally, we need to deal with the special case that N is either $K^2 \ltimes I$ or $S^1 \times S^1 \times I$ and thus has more than one Seifert fibration.

Proof of Theorem 1.1. (Proof using case 1 of Lemma 5.7.) As explained in Section 6.1, it suffices to solve the homeomorphism problem $W_1 \overset{?}{\cong} W_2$ for prime 3-manifolds W_1 and W_2. The proof is divided into three steps. In steps 1 and 2, we let W be a prime 3-manifold and let Γ be its JSJ graph. It is recursive to calculate Γ and the isomorphism types of its vertices.

Step 1: We address the cases in which a JSJ component of W has more than one Seifert fibration. If any component is a $K^2 \ltimes I$, then its two fibrations (described in Theorem 5.12) are inequivalent; we choose one of them and use it for every occurrence of $K^2 \ltimes I$ in W. If a JSJ component N is a thickened torus, then as in the proof of Theorem 5.4, W is a Sol manifold and a torus bundle over a circle, $S^1 \ltimes (S^1 \times S^1)$. In this case the homeomorphism type of W is given by a pair of conjugacy classes in $\mathrm{SL}(2, \mathbb{Z})$, one for each orientation of the base circle. Recall that the conjugacy classes in $\mathrm{SL}(2,\mathbb{Z})$ can be classified with the aid of the isomorphism

$$\mathrm{PSL}(2,\mathbb{Z}) \cong C_2 * C_3.$$

If $g \in \mathrm{SL}(2,\mathbb{Z})$ has non-zero trace (which it does if W is Sol), then its conjugacy class is given by the sign of its trace and its reduced cyclic word in $C_2 * C_3$.

Step 2: We suppose that W is not a Sol torus bundle over a circle. If T is a JSJ torus in W and one side of T is a hyperbolic component N, then T inherits a Euclidean structure which we can normalize to have area 1. This Euclidean structure can be described by a quadratic form Q on the first homology $H_1(T) = H_1(T;\mathbb{Z})$, where $Q(c)$ is the square of the minimum length of $c \in H_1(T)$. Moreover, the coefficients of Q are real algebraic numbers computable from the geometry of N. On the other hand, if N is Seifert-fibered, then the induced fibration of T selects a *line* in $H_1(T)$, by which we mean a rank-one subgroup $L \subseteq H_1(T)$ with a torsion-free quotient $H_1(T)/L$.

Since T has two sides, it is then decorated by a pair of quadratic forms on $H_1(T)$, or a quadratic form and a line, or a pair of lines. In the third case when both sides of T are Seifert-fibered, the fibrations must be mismatched at T, so the two lines $L_1, L_2 \subseteq H_1(T)$ must be distinct. Hence they constitute a *rational line basis* in the sense that

$$H_1(T;\mathbb{Q}) = (L_1 \otimes \mathbb{Q}) \oplus (L_2 \otimes \mathbb{Q}).$$

We also obtain a rational line basis in the second case, when one side is Seifert-fibered and produces a line $L_1 = L$, and the other side is hyperbolic and produces a quadratic form Q. In this case, there exist a finite set of pairs of homology classes $\pm c \in H_1(T) \setminus L_1$ that minimize $Q(c)$. If there is only one such pair, we let L_2 be the line generated by $\pm c$. If there is more then one, we let L_2 be the line generated by the first such pair in the clockwise direction from L_1.

Note that each possible decoration of T induced by the geometry on both of its sides has a finite stabilizer in the oriented mapping class group $\mathrm{SL}(H_1(T)) \cong \mathrm{SL}(2,\mathbb{Z})$ of T. If we order the two sides of T, then the stabilizer usually has two elements; in rare cases it is a cyclic group of order 4 or 6.

If N is a Seifert-fibered component of W, then each of its torus boundary components is decorated by a rational line basis. We can thus make a closed Seifert-fibered space $\hat{N}$ by collapsing a circle fibration of each component $T \subseteq \partial N$ that represents the opposite line in $H_1(T)$, the one that does not come from N itself. Each torus component of ∂N becomes a distinguished fiber in $\hat{N}$ which may be either regular or exceptional.

Step 3: Suppose that W_1 and W_2 are two prime, closed, oriented 3-manifolds. If they do not have any JSJ tori, then each one is either closed hyperbolic or Seifert-fibered, and we can use Theorems 6.1 and 6.3 to tell if they are the same. Meanwhile if W_1 and W_2 are both Sol torus bundles, then we can use the algorithm in step 1 to determine if they are homeomorphic.

Otherwise we can assume that W_1 and W_2 each have at least on JSJ torus, and that each JSJ torus has a canonical decoration as described in step 2. To determine if W_1 and W_2 are homeomorphic, we search over isomorphisms $f : \Gamma_1 \to \Gamma_2$ between their JSJ graphs. For every pair of $T_1 \subseteq W_1$ and $T_2 \subseteq W_2$ that are matched by f, we search over mapping classes that preserve the canonical decorations of T_1 and T_2. In the innermost part of the search, we want to calculate whether the homeomorphisms of the JSJ tori extends to each matched pair of JSJ components $N_1 \subseteq W_1$ and $N_2 \subseteq W_2$. If N_1 and N_2 are hyperbolic, then we can use Theorem 6.1 to determine if the homeomorphism $\partial N_1 \cong \partial N_2$ extends. If they are both Seifert-fibered, then we can use Lemma 6.2 to determine whether the corresponding closed Seifert-fibered manifolds $\hat{N}_1$ and $\hat{N}_2$ have a homeomorphism that extends the given homeomorphism $\partial N_1 \cong \partial N_2$. Note that we can employ Lemma 6.2 because any relevant homeomorphism $N_1 \cong N_2$ preserves the fibration at the boundary, and is thus isotopic to a fibration-preserving homeomorphism by Theorem 5.12. □

Proof of Theorem 1.1. (Proof using case 2 of Lemma 5.7.) If the geometric data of each hyperbolic JSJ component N of a summand W is described with computable real numbers rather than real algebraic numbers, then the induced Euclidean structure on a JSJ torus $T \subseteq \partial N$ is only given by a convergent sequence of approximations. Thus, it is not possible to definitively calculate the isometries of T or the shortest cycles, as expressed with the quadratic form $Q(c)$. However, all non-isometries and all non-zero classes in $H_1(T)$ that are not shortest are eventually revealed. This yields an algorithm in coRE for the homeomorphism problem $M_1 \overset{?}{\cong} M_2$, which is enough to show that the problem is recursive per the discussion at the beginning of Section 7. □

7. Proofs of Theorem 6.1

In this section we will give several proofs of Theorem 6.1. Recall that Corollary 3.4 says that the existence of a PL homeomorphism $N_1 \cong N_2$ is in RE; it is also easy to check whether it preserves orientation. So, by Proposition 2.3, it suffices to show that homeomorphism is in coRE, although only one of the proofs will make use of this directly. By a similar argument, finding elements in the mapping class group of a single N is in RE; the remaining task is an algorithm to show that the list is complete.

Recall that if N has boundary, then its interior N^* is cusped and has a semi-ideal triangulation Θ^*. In this case, Θ is a cellulation in which semi-ideal tetrahedra are once truncated. We want to geometrize the truncation that produces Θ. We consider a horospheric truncation which is almost but not quite unique, with the following three properties:

1. The horosphere sections lie entirely within the semi-ideal tetrahedra of Θ^*, and therefore do not intersect each other.

2. For some common integer n, every horospheric torus has area 2^{-n}.

3. We do not use the smallest value of n that satisfies conditions 1 and 2.

For convenience, we let $N^* = N$ and $\Theta^* = \Theta$ if N is closed.

Some of the proofs make use of the following lemma.

Lemma 7.1. *It is recursive to obtain a lower bound in the injectivity radius of N and Θ.*

First proof. Suppose first that N is closed. For each vertex $v \in \Theta$, let U_v be the open star of Θ containing p. Then the collection $\{U_v\}$ is a finite open cover of N. It follows just from topology that there is some radius ϵ such that every ball of radius ϵ is contained in some U_v. For an explicit calculation, let Θ' be a barycentric subdivision of Θ, and for each $v \in \Theta$, let X_v be the closed star of $v \in \Theta'$; then the sets X_v are a closed cover. We can calculate or bound the distance from X_v to $N \setminus U_w$ for some $w \in \Theta$ with $X_v \subseteq U_w$. The minimum of all of these distances is thus a lower bound ϵ for the injectivity radius. $\square$

Second proof. In general we use the notation $B(p, r)$ for a hyperbolic ball of radius r centered at p.

Let r be the exact injectivity radius of N, and let p be a point on a closed geodesic of N of length $2r$. Then $p \in \Delta$ for some cell $\Delta \in \Theta$, and we can let ℓ be an upper bound of the diameter of Δ. Then in the universal cover

$$\tilde{N} \subseteq \tilde{N}^* \cong \mathbb{H}^3,$$

we obtain that at least $1/2r$ lifts of Δ intersect $B(p, 1)$, and thus at least this many copies of Δ are contained in $B(p, \ell + 1)$. Thus

$$\frac{1}{2r} \leq \frac{\mathrm{Vol}(B(p, \ell+1))}{\mathrm{Vol}(\Delta)},$$

hence

$$r > \frac{\mathrm{Vol}(\Delta)}{2\mathrm{Vol}(B(p, \ell+1))}. \tag{2}$$

We can calculate an upper bound of this form, if necessary using a lower bound for the numerator and an upper bound for the denominator, for every cell in Θ, since we do not know the position of the shortest geodesic loop in advance. $\square$

Third proof. This proof is a variation of the second proof using the entire diameter and volume of N. Jørgensen and Thurston proved that the set of possible volumes of N^* is well-ordered. In particular, there is one of least volume, so there is some constant $c > 0$ such that

$$\mathrm{Vol}(N^*) > c.$$

Our construction of the geometry of N spares more than half of the volume of N^*, so

$$\mathrm{Vol}(N) > \frac{\mathrm{Vol}(N^*)}{2} > \frac{c}{2} = c'.$$

We can obtain an upper bound ℓ on the diameter of all of N by adding bounds on the diameters of the cells in Θ. Then, we let D be a convex fundamental domain for N; it has the same volume and diameter at most 2ℓ. Thus we obtain an estimate similar to (2), but more robust:

$$r > \frac{\mathrm{Vol}(D)}{2\mathrm{Vol}(B(p,\ell+1))} > \frac{c'}{2\mathrm{Vol}(B(p,\ell+1))}. \qquad \square$$

Remark. Without an explicit bound on least-volume closed or cusped hyperbolic manifold, the third proof has the unusual feature of non-constructively proving that an algorithm exists, *i.e.*, without fully stating the algorithm. Meyerhoff [30] established the first lower bound

$$\mathrm{Vol}(N) \geq \frac{2}{5^4}$$

in the closed case. In the same paper, he and Jørgensen established

$$\mathrm{Vol}(N^*) \geq \frac{\sqrt{3}}{4} \Longrightarrow \mathrm{Vol}(N) \geq \frac{\sqrt{3}}{8}$$

in the cusped case. The exact minimum values are now known [9].

First proof of Theorem 6.1. This proof is similar to one given by Scott and Short [43]. We assume geometric triangulations Θ_1^* and Θ_2^* of N_1^* and N_2^*.

If N_1^* and N_2^* are homeomorphic and therefore isometric, then we can intersect the tetrahedra of Θ_1^* and Θ_2^* to make a tiling of $N_1 \cong N_2$ by various convex cells with 8 or fewer sides; we can then take a barycentric subdivision to make tetrahedra. We thus obtain a mutual refinement Θ_3 of Θ_1 and Θ_2. If we can bound the complexity of Θ_3, then we can find it with a finite search or show that it does not exist, rather than using stellar or bistellar moves in both the up and down directions.

Let $\Delta_1 \in \Theta_1^*$ and $\Delta_2 \in \Theta_2^*$ be two tetrahedra in the separate triangulations. In the universal cover $\tilde{N}_1^*$, any two lifts of Δ_1 and Δ_2 only intersect in a single cell with at most 8 sides. In N_1^* itself they can intersect many times; however, only as often as different lifts of Δ_1 intersect one fixed lift of Δ_2. If Δ_1 and/or Δ_2 are semi-ideal, then their lifts intersect if and only if their truncations do. There is a recursive volume bound on the number of possible intersections by the same argument as the second proof of Lemma 7.1.

Having bounded the necessary complexity of a mutual refinement Θ_3, we can now search over separate refinements Θ_3 of Θ_1 and Θ_4 of Θ_2 using Proposition 3.5, and look for an orientating-preserving simplicial isomorphism $\Theta_3 \cong \Theta_4$. The same method can be used to calculate the mapping class group of a single N. $\square$

Second proof. Suppose that X_1 and X_2 are two compact metric spaces, and suppose that for each $\epsilon > 0$ we have a way to make finite ϵ-nets S_1 and S_2 for X_1 and X_2, and calculate or approximate all distances within S_1 and within S_2. If X_1 and X_2 are isometric, then there is a function $f : S_1 \to S_2$ that changes distances by at most 2ϵ. On the other hand, if there is such a function for every ϵ, then X_1 and X_2 must be isometric.

In our case, we let $X_k = N_k$, where we make sure to use the same truncation area 2^{-n} to geometrize N_1 and N_2 given the geometries of N_1^* and N_2^*. We calculate a common lower bound δ on the injectivity radius.

We can choose some convenient coordinates inside each cell $\Delta \in \Theta_k$. We then have the ability to calculate geodesic segments in N_k that are made of geodesic segments in the separate tetrahedra. If Δ is truncated, then the geodesic segment might hug the truncation boundary for part of its length, but it still has a finite description. Without more work, we don't know which of these geodesics are shortest geodesics. However, if a geodesic is shorter than δ, then it is shortest. Taking $\delta \gg \epsilon \to 0$, we can make ϵ-nets of both N_1 and N_2 and look for approximate isometries between these ϵ-nets; it suffices to check distances below the fixed value δ.

More explicitly, we can use the covering by open stars S_v in the first proof of Lemma 7.1. There is a δ such that if $d(x, y) < \delta$, then x and y and even the connecting short geodesic are all in some open star.

This algorithm does not by itself ever prove that N_1 and N_2 are isometric, only that they aren't. Thus it shows that the homeomorphism problem is in coRE. This is good enough by Proposition 2.3 and Corollary 3.4.

The algorithm also does not by itself determine whether the isometry is orientation-preserving. However, this is very little extra work. Given $\epsilon \ll \delta$ and given ϵ-nets $S_1 \subseteq N_1$ and $S_2 \subseteq N_2$, we can let p_1, p_2, p_3, p_4 be 4 points in S_1 that lie in a ball of radius $\delta/2$ and that make an approximately regular tetrahedron Δ. If $f : S_1 \to S_2$ is an approximate isometry, then we can check whether f flips over Δ. If no orientation-preserving isometry exists, then when ϵ is small enough, either f will cease to exist or Δ will be flipped over.

We can use similar methods to find the mapping class group of a single N, since by Mostow rigidity it is also the isometry group of N. We assume that N

has boundary, which is technically short of the full generality of Theorem 6.1, but enough to prove Theorem 1.1. Just as with the method to check whether an approximate f preserves orientation, we can when ϵ is small enough compute the effect of f on $H_1(\partial N)$, which determines which isometry is close to f (if any). □

Third proof. In this proof, we restrict attention to case 1 of Lemma 5.7 and thus work over the ring $\hat{\mathbb{Q}}$ of real algebraic numbers. We assume real algebraic coordinates for $\mathbb{H}^3$ and for its isometry group $\mathrm{Isom}^+(\mathbb{H}^3)$; for example we can take $\mathbb{H}^3$ to be the set of positive, unit timelike vectors in $3+1$-dimensional Minkowski geometry, and we can take $\mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{SO}^+(3, 1)$. We again assume that N_1 and N_2 are made from N_1^* and N_2^* using a common truncation area 2^{-n}.

We assume geometric triangulations Θ_1^* and Θ_2^* of N_1^* and N_2^* with real algebraic descriptions. Using these triangulations, we can find finite, open coverings of N_1 and N_2 by metric balls $B(p, \epsilon)$, where each point p has a real algebraic position and the common radius is (a) also real algebraic, and (b) less than half of the injectivity radius of N_1 and N_2. Then we can give each ball the same algebraic coordinates as $\mathbb{H}^3$, and we can also calculate the relative position of every pair of balls as some element in $\mathrm{Isom}^+(\mathbb{H}^3)$. In other words, we obtain atlases of charts for N_1 and N_2 using the $\mathrm{Isom}^+(\mathbb{H}^3)$ pseudogroup. In fact, everything is constructed in the subgroup and sub-pseudogroup with real algebraic matrix entries.

If there is an isometry between N_1 and N_2, then their atlases combine into a larger atlas. There are only finitely many possible patterns of intersection between the balls of N_1 and the balls of N_2. For each such pattern, we obtain a finite system of algebraic equalities and inequalities, which says first that the intersection pattern is what is promised, and second that the gluing maps between the atlases are consistent. Theorem 2.8 then says that it is recursive to determine whether this system of equations has a solution. Since we work in the group $\mathrm{Isom}^+(\mathbb{H}^3)$, we are looking only for orientation-preserving isometries. □

8. Homeomorphism is in ER

We will use the basic fact that a finite composition of ER functions is in ER. In other words, if an algorithm has a bounded number of stages that expand its data by an exponential amount or otherwise by an ER amount, then it is still in ER.

8.1. ***The outer proof.*** In this section we will prove Theorem 1.2. The proof is a combination of the proof of Theorem 1.1 together with several computational improvements. We summarize these computational improvements in this section by stating some supporting theorems which we will prove ourselves (or prove by citation) with two main supporting tools. The first tool is normal surface theory, which we can use to find essential spheres and tori and Seifert fibrations. Note that Jaco, Letscher, and Rubinstein [17] sketched ideas that are similar to our proof.

The second tool is an ER version of Theorem 2.8 [10], which we use to bound the complexity of a geometric triangulation of a hyperbolic manifold, and the complexity of recognizing small Seifert-fibered spaces.

Theorem 8.1. *It is in* ER *to find and triangulate the prime summands* $\{W\}$ *of a closed*, *oriented* 3*-manifold* M, *to find and triangulate the JSJ components* $\{N\}$ *within each prime summand* W, *to find their JSJ graph* Γ, *and to recognize which components* N *are Seifert-fibered and find their fibrations.*

We will prove most of Theorem 8.1 in Section 8.2 using normal surface theory. Small Seifert-fibered spaces are a particularly difficult special case of Theorem 8.1 that we list as a separate theorem. Recall that a Seifert-fibered space is *small* if it is non-Haken (and therefore closed).

Theorem 8.2. *Recognizing small Seifert-fibered spaces is in* ER.

We will prove Theorem 8.2 in Section 8.5 using a combination of normal surface theory and algebraic methods. Note that Li [27] shows that recognizing small Seifert-fibered spaces with infinite π_1 is recursive, and his algorithm should be elementary recursive. However, we will use a different approach for this part of the theorem. Li also addresses the finite π_1 case in two different ways. Without assuming geometrization (which was still open at the time), he cites work of Rubinstein and Rannard-Rubinstein on small Seifert-fibered spaces. He also outlines a simplified argument for the finite π_1 case that depends on geometrization; we give a detailed argument which is in a similar spirit.

Theorem 8.3. *If a compact*, *oriented* 3*-manifold* N *has a closed or cusped hyperbolic structure*, *then it is in* ER *to find a geometric triangulation and specify its geometric data with algebraic numbers. The homeomorphism and automorphism problems are also both in* ER.

We will prove Theorem 8.3 in Section 8.4 using both Mostow rigidity and methods from algebraic geometry.

Proof of Theorem 1.2. We consider each stage of the proof of Theorem 1.1 in turn. The proof begins with a geometric recognition of a single closed, oriented 3-manifold M in Theorem 5.4. This is not elementary recursive as described, but we can replace it with Theorem 8.1 to find the direct sum and JSJ decomposition. Each JSJ component N which is not Seifert fibered must be hyperbolic, so we can apply Theorem 8.3, which is an ER version of Lemma 5.7, to calculate the hyperbolic structure of each such N. This calculation also yields a description of the Euclidean structure of each torus component of ∂N.

Finally given two closed, oriented 3-manifolds M_1, M_2, we first decompose them into summands. For each bijection among the summands, we want to calculate

$W_1 \overset{?}{\cong} W_2$ for each pair of matching summands. This is a calculation with JSJ graphs which is done in Section 6.3 to complete the proof of Theorem 1.1. This calculation is already elementary recursive, given the data computed for each JSJ component of each W_i, and given the fibration or Euclidean structure of each JSJ torus of each W_i. □

8.2. *Normal surfaces.* Let M be a compact 3-manifold with triangulation Θ. Recall that a *normal surface* $S \subseteq M$ intersects each tetrahedron $\Delta \in \Theta$ in 7 types of elementary disks, namely 4 types of triangles and 3 types of quadrilaterals. The surface $S = S_v$ is given by a vector $v \in \mathbb{Z}^{7t}_{\geq 0}$ that lists the number of each type of elementary disk. If v is such a vector, then S_v is embedded (and uniquely defined) provided that it only uses at most one type of quadrilateral in each tetrahedron. After specifying which type of quadrilateral is allowed in each tetrahedron, the normal surface equations then have a polytopal cone

$$C \subseteq \mathbb{Z}^{5t}_{\geq 0} \subseteq \mathbb{Z}^{7t}_{\geq 0}$$

of solutions. We define a *fundamental surface* S_v to be one whose vector $v \in C$ is not the sum of two other solutions in C. If S_v is non-orientable, then S_{2v} is its orientable double cover and we call it fundamental as well.

Lemma 8.4 (Haken). *The number of elementary disks in a fundamental surface in M is bounded above by an exponential in t.* (*Thus it is elementary recursive.*)

We can represent a normal surface S by listing all triangles and quadrilaterals in order in each tetrahedron $\Delta \in \Theta$. It is then easy to separate S into connected components and calculate the topology of each component. This is exponentially inefficient compared to algorithms such as Agol-Hass-Thurston [2], but it has no effect on whether the resulting algorithm is in ER.

We define a *complete set of essential 2-spheres* in a 3-manifold M to be a collection C such that cutting M along each 2-sphere in M and capping off the resulting boundary components produces irreducible 3-manifolds. Likewise a complete set of essential disks is a collection C of properly embedded disks which are not boundary parallel, such that the compression of every disk in C renders M boundary-incompressible.

Theorem 8.5 (Jaco-Tollefson). *Let M be a compact, oriented, triangulated* 3-*manifold. Then*:

1. *M has a collection of disjoint, fundamental surfaces which form a complete set of essential* 2-*spheres* [18, Thm. 5.2].
2. *If M has no essential* 2-*spheres, then it has a collection of disjoint, fundamental surfaces which form a complete set of essential disks* [18, Thm. 6.2].

3. *If M has no essential* 2*-spheres or disks*, *and it has an essential torus*, *then it has one which is fundamental* [18, Cor. 6.8].

4. *If M has no essential* 2*-spheres or disks*, *and it has an essential annulus*, *then it has one which is fundamental* [18, Cor. 6.8].

Jaco and Tollefson also show that each type of surface described in Theorem 8.5 is a vertex surface, which is a special case of a fundamental surface. Case 1 of Theorem 8.5 is stated for closed manifolds, but the proof is the same for manifolds with boundary. Finally, given Lemma 8.4, the surfaces produced by Theorem 8.5 all have an elementary recursive bound on their size.

We will use the following variation of Theorem 8.5, where we now define a *complete set of essential tori* in each summand W similarly, so that each complementary region in W is atoroidal. (A complete set of essential tori must include all of the JSJ tori of each W, but will be a strict superset when some of the Seifert-fibered JSJ have additional essential tori.)

Theorem 8.6 (Hass-K. [14]). *If M is a closed*, *oriented* 3*-manifold with a triangulation* Θ, *then it has a collection of disjoint normal surfaces which form a complete set of essential spheres and tori*, *such that the total number of elementary disks is bounded above by an exponential in t*.

Briefly, Theorem 8.6 uses a generalization of the normal surface equations which we call the *disjoint normal surface equations*. (They are similar to the crushed triangulation technique defined by Casson [17].) They are equations for a normal surface S which is disjoint from a fixed normal surface $R \subseteq M$. To prove Theorem 8.6, we find each surface S or T sequentially as a fundamental surface, relative to the union of previous surfaces.

Theorem 8.7 (Rubinstein [40], Thompson [48]). *Recognizing the* 3*-sphere S^3 is in* ER.

The proof of Theorem 8.7 uses a variant known as *almost normal surfaces* that are allowed one exceptional intersection with a tetrahedron that is either an octagon, or a triangle and a quadrilateral with a connecting annulus. The original papers only claim a recursive algorithm, but the algorithm is based on normal surface theory. In fact, the proof also uses disjoint normal surface equations. Schleimer [42] also refines the Rubinstein-Thompson algorithm to show that 3-sphere recognition is in the complexity class NP, which is a much better bound than just ER.

Proof of Theorem 8.1. As a first step, we check whether $M \cong S^3$ using Theorem 8.7. If not, we search over collections C of normal surfaces in M with a suitable elementary recursive complexity bound in order to find a set of surfaces that meets the conclusion of case 1 of Theorem 8.5. To test whether a given collection C is one that we want, we first calculate whether each surface in it is a 2-sphere. Then we can cut along all of the spheres (and retriangulate) and cap them to make a multiset of summands of M. For each non-separating sphere, we create a separate $S^2 \times S^1$ summand. What remains is a putative prime factorization $\{W\}$, but we must check whether the summands are irreducible and not S^3. We can use Theorem 8.7 to check that no summand W is S^3. If not, then we can again use case 1 of Theorem 8.5 to look for an essential 2-sphere in each W, and again use Theorem 8.7 to check whether it is essential.

For each summand W, we similarly search for a collection C that meets the conclusion of Theorem 8.6. We can check that each surface in C is a torus. We can then cut W along C to make a putative decomposition of W into atoroidal components $\{Q\}$. Geometrization gives us the following possibilities for each component Q:

1. Q has an essential disk, which necessarily cuts it into a ball. In this case, $Q \cong S^1 \times D^2$ is a solid torus.
2. Q has no essential disk, but it has a separating essential annulus that cuts it into two solid tori. In this case Q fibers over a disk with two exceptional fibers.
3. Q does not have a separating essential annulus, but it does have a non-separating essential annulus that cuts it into a solid torus. In this case Q fibers over an annulus or a Möbius strip with at most one exceptional fiber.
4. Q has a separating annulus that cuts it into two thickened tori. In this case $Q \cong S^1 \times F$, where F is a pair of pants.
5. Q has an essential torus, specifically an incompressible torus which is not boundary-parallel.
6. $Q = W$ is closed and has no essential torus. In this case Q is either hyperbolic or small Seifert-fibered.
7. Q has boundary, and it has no essential disk, annulus, or torus. In this case, Q is hyperbolic.

To see that this is an exhaustive list, first recall from geometrization that each Q is either hyperbolic or Seifert-fibered. (This is because a complete set of essential tori includes all of the JSJ tori, and the other essential tori are all vertical with respect to some fibration.) If Q is Seifert-fibered with boundary and is atoroidal, then with one exception its orbifold base is planar, and the total number of boundary circles plus exceptional fibers is at most three. The only exception is that $Q \cong K^2 \ltimes I$ has a Möbius strip base; but it also has its other fibration with Seifert data $\{D^2, (2, 1), (2, 1)\}$.

We claim that we can recognize each possibility for Q by a bounded number of applications of Theorem 8.5; the recognition algorithm is therefore in ER. Indeed, each case reduces to earlier cases when a putative essential surface is found. The most subtle case is case 5, where we can check whether a candidate torus in Q is essential by checking that it is not compressible (case 1) and does not bound a thickened torus (case 3). Note that if Q is a solid torus (case 1) or a thickened torus that is not glued to itself (case 3), or if Q has an essential torus (case 5), then the collection C in W should be rejected.

In case 6, we use Theorem 8.2 to determine if W is small Seifert-fibered, and if so, its homeomorphism type.

If Q is a thickened torus which is glued to itself, then W is a torus bundle over a circle, and we can find its monodromy matrix $A \in \mathrm{SL}(2,\mathbb{Z})$ with a homology calculation. We can solve the conjugacy problem in $\mathrm{SL}(2,\mathbb{Z})$ using the usual trick that $\mathrm{PSL}(2,\mathbb{Z}) \cong C_3 * C_2$, and thus determine the homeomorphism type of W. (Note that W may be either Sol, Nil, or Euclidean.)

Otherwise we can piece together the JSJ decomposition $\{N\}$ of W from the (often non-unique) atoroidal decomposition $\{Q\}$. In each remaining case Q has either 0 Seifert fibrations (if it is hyperbolic), or 2 fibrations (if it is $K^2 \ltimes I$), or 1 fibration (in all other Seifert-fibered cases). Using the recognition of Q, we can calculate its Seifert data and express the fibration of each boundary torus $T \subseteq \partial Q$ by the corresponding line $L \subseteq H_1(T)$. We can then piece together adjacent fibered components to make JSJ components, when the fibrations match. The Seifert data produced in this manner is not necessarily canonical, but canonicalizing it is straightforward. $\square$

8.3. *Algebraic algorithms.* We list several complexity bounds concerning algebraic numbers and solutions to algebraic equations.

Theorem 8.8 (Collins, Monk, Vorobiev-Grigoriev, Wüthrich [10, Thm. 4]). *Suppose that a set $S \subseteq \mathbb{R}^n$ is defined by a finite set of polynomial equalities and inequalities over $\mathbb{Q}$. Then it is in* ER *to calculate a representative finite set $F \subseteq \hat{\mathbb{Q}}^n$, with one point $p \in F$ in each connected component of S.*

Theorem 8.8 is of course an ER version of Theorem 2.8. In the statement of the theorem, each element $\alpha \in \hat{\mathbb{Q}}$ is described by its minimal polynomial $a(x) \in \mathbb{Z}[x]$ and an isolating interval $\alpha \in [b,c]$ that contains exactly one root of $a(x)$.

Lemma 8.9. *If $\alpha \in \hat{\mathbb{Q}}$ is a non-zero complex root of a polynomial $a(x) \in \mathbb{Z}[x]$, then there is an* ER *upper bound on $|\alpha|$ and $|\alpha^{-1}|$.*

Proof. Let $n = \deg a$ and write

$$a(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

We can assume without loss of generality that $a_0 \neq 0$. If $|\alpha| > \sum_k a_k$, then us that $|a_n \alpha^n|$ is larger than the total norm of all of the other terms, so by the triangle inequality, $a(\alpha) \neq 0$. This establishes $\sum_k a_k$ as an upper bound on $|\alpha|$. For the lower bound, we can observe that $\beta = \alpha^{-1}$ is a root of the polynomial

$$b(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = x^n a(x^{-1}).$$

We can thus repeat the argument. □

If $x_1, \ldots, x_n$ are algebraic numbers, then we say that an algebraic number z is an *integral primitive element* if each $x_k = f_k(z)$ for some integer polynomial $f_k \in \mathbb{Z}[x]$. It is a result of Galois that every finite set of algebraic numbers has a primitive element; we are interested in a computationally bounded version.

Theorem 8.10 (Koiran [25, Thm. 4]). *If $x_1, \ldots, x_n$ are algebraic numbers, then they have an integer primitive element z which can be computed in* ER, *and such that polynomials f_k with $f_k(z) = x_k$ can also be computed in* ER.

Koiran states the result in the form $x_k = f_k(z)/a_k$, where the denominator a_k is an elementary recursive integer. However, it is not difficult to modify z to eliminate these denominators. (Proof: Let $f \in \mathbb{Z}[x]$ be the minimal polynomial of z and let

$$z_1 = \frac{z}{f(0) \prod_k a_k}.$$

Then both $1/a_k$ and $f_k(z)$ are expressible as integer polynomials in z_1. Therefore, so is their product.)

Lemma 8.11. *If $h \in \mathbb{Z}[x]$ is an integer polynomial, then there is a prime p that can be computed in* ER *such that $h(x)$ has a root in $\mathbb{Z}/p$.*

Proof. If $d = \deg h$, then h attains the values ± 1 at most $2d$ times. Therefore there is an integer a with $|a| \leq d$ such that $h(a)$ is not ± 1, and we can let p be a prime divisor of $h(a)$. □

Using the results so far in this section, we obtain an elementary recursive version of Mal'cev's theorem, which says that finitely generated residually linear groups are residually finite. Our computational version requires a finitely presented group rather than just a finitely generated group.

Theorem 8.12. *Let Γ be a finitely presented group, let $g \in \Gamma \setminus \{1\}$ be a non-trivial element given by a word w in the generators of Γ, and suppose that there is a representation*

$$\rho : \Gamma \to \mathrm{GL}(n, \mathbb{C})$$

that distinguishes g from the identity. Then Γ admits a finite representation

$$\rho_p : \Gamma \to \mathrm{GL}(n, \mathbb{Z}/p)$$

that distinguishes g from the identity, where p is a prime number. Moreover, we can find such a p and ρ_p in ER *given the presentation of* Γ, *the word* w, *and the integer* n.

Proof. A function from the generators of Γ to $n \times n$ matrices forms a representation

$$\rho : \Gamma \to \mathrm{GL}(n, \mathbb{C})$$

if and only if the matrix entries satisfy equations that come from the relators of the presentation of Γ. We also want $\rho(g) \neq I$. To this end, we assume another matrix of variables Y and impose the condition

$$\mathrm{tr}(Y(\rho(g) - I)) = 1.$$

By hypothesis ρ and Y exist, and Theorem 8.8 then produces an algebraic, elementary recursive solution. By Theorem 8.10, the matrix entries are generated by an integer primitive element z, and by Lemma 8.11, we can replace z by a residue $\alpha \in \mathbb{Z}/p$ for some prime p computable in ER. We thus get a modular representation

$$\rho_p : \Gamma \to \mathrm{GL}(n, \mathbb{Z}/p)$$

and a matrix Y_p over $\mathbb{Z}/p$ again with the same properties. Since Y_p exists, $\rho_p(g)$ cannot be the identity. □

Remark. Assuming the Generalized Riemann Hypothesis, Koiran's work implies Theorem 8.12 with a much better bound, namely that $\log(p)$ can be bounded by a polynomial in the length of the presentation of Γ and the length of the word w.

8.4. *The hyperbolic case.* To prove Theorem 8.3, we will need a quick mutual corollary of Theorem 8.8 and the proof of Proposition 3.5.

Corollary 8.13. *If* Θ_1 *is a finite simplicial complex with* n_1 *simplices* (*of arbitrary dimension*) *and* $n_2 \geq n_1$, *then it is in* ER *to produce a complete list of geometric subdivisions* Θ_2 *of* Θ_1 *with* n_2 *simplices.*

Proof of Theorem 8.3. Let Θ be the input triangulation of N as a compact manifold, and let Θ^* be the result of adding a cone to each component of ∂N to make a semi-ideal combinatorial triangulation of N^*. The manifold N^* also has a hyperbolic structure which we interpret as a separate manifold. We rename the hyperbolic version X and assume a homeomorphism

$$f : N^* \to X.$$

We fix the vertices of N^* in the map f, and straighten all of the tetrahedra, to make a map g that represents Θ^* as a self-intersecting geometric triangulation of X. Since g is homotopic to f (or properly homotopic if N^* is not compact), it has (proper) degree 1. Since g need not be a homeomorphism, it may flatten or

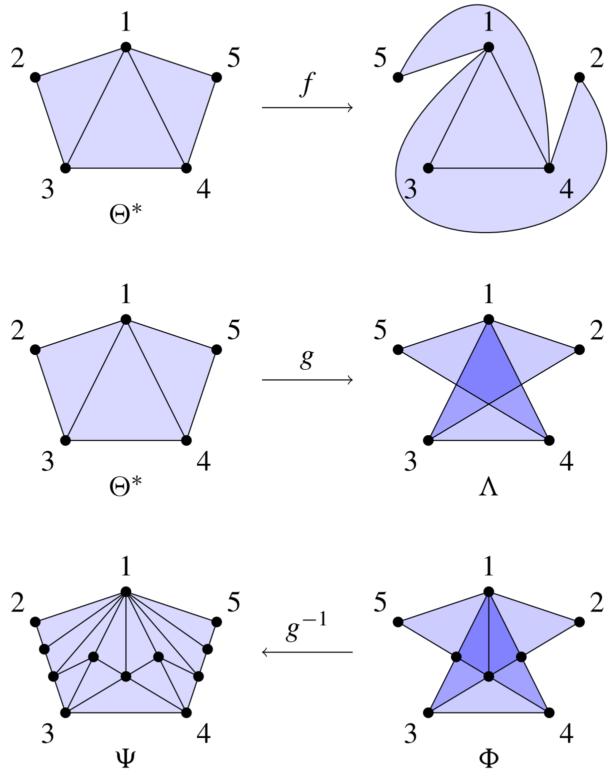

Figure 4. We straighten f to g, then subdivide the image, and finally subdivide the domain to make g a simplicial map. In the proof the image is barycentrically subdivided, but any refinement which is a triangulation suffices.

flip over some of its simplices. The self-intersections of $g(\Theta^*)$ yield a cellulation Λ of X with convex cells. Thus Λ has a barycentric subdivision Φ which is a geometric triangulation of X. Also let $\Psi = g^{-1}(\Phi)$. Then Ψ is a refinement of the triangulation Θ^*, and g is now a simplicial map from Ψ to Φ. See Figure 4. (The figure uses a simplicial refinement of the self-intersections which is simpler than barycentric subdivision; this is not important for the proof.)

Now suppose that we do not know the hyperbolic structure of N, only that it must have one because it prime, atoroidal, and acylindrical. If we are given Ψ as a combinatorial refinement of the triangulation Θ^*, then we can search for Φ as a simplicial quotient of Ψ, such that we can solve the hyperbolic gluing equations for Φ to recognize it as a geometric triangulation of a hyperbolic manifold X. We obtain a candidate map $g : N^* \to X$. If g has degree 1, and there is also a degree 1 map $h : X \to N^*$, then Mostow rigidity tells us that g and h are both homotopy equivalences and that X and N^* are homeomorphic. (Note that there can be a degree 1 map in one direction between two hyperbolic 3-manifolds that is not a homotopy equivalence [5], even though this cannot happen in the case of hyperbolic surfaces.) We can search for h by the same method of simplicial subdivision that we used to find g. This establishes an algorithm to calculate the hyperbolic structure of N^*.

We claim that a modified version of this algorithm is in ER. We first consider ER candidates for the map g. To do this, we make a non-commutative cocycle $\alpha \in C^1(N; G)$ as in the proof of Corollary 5.10, where

$$G = \mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{SO}^+(3, 1),$$

and with the extra restriction that α is parabolic on each component of ∂N. These cocycle equations are algebraic, so Theorem 8.8 guarantees a representative set of solutions. By Mostow or Calabi-Weil rigidity, one of components of the solution space yields a discrete homomorphism

$$\rho : \pi_1(N) \to \mathrm{Isom}^+(\mathbb{H}^3)$$

that describes the hyperbolic geometry of X. If we assign some point $p \in \mathbb{H}^3$ to one of the vertices of Θ, then in the closed case, its orbit under α is in ER and can be extended on each simplex of Θ to the map g. In the cusped case, there are also ideal vertices whose position on the sphere at infinity can be calculated from α as well.

If N has boundary, then we also want a truncated version of Θ^* which is larger than the original Θ, and slightly different from the horospheric truncation description in Section 7. If that $\Delta \in \Theta^*$ is semi-ideal, then let p be its ideal vertex, let F be the hyperplane containing the face of Δ opposite to p, and let F' be the hypersphere at distance $\log(2)$ from F which is on the same side as p. Then we truncate Δ with F' to make Δ'; or if $\Delta \in \Theta^*$ is a non-ideal tetrahedron, we let $\Delta' = \Delta$. We let $X' \subseteq X$ be the union of all Δ'. (In the closed case, we obtain $X' = X$.) X' can have a complicated shape because the truncations are usually mismatched, but we can calculate the positions of its vertices, and it is easy to confirm that it has at least half of the volume of X.

Our algorithm does not know which cocycle α gives us a desired g and we do not compute this directly. Instead, we can calculate an ER bound for its data complexity, using the complexity bounds in the statement of Theorem 8.8.

In particular, the existence of the map g gives us ER bounds on the parameters used in the third proof of Lemma 7.1. Using Lemma 8.9, the existence of g yields an ER upper bound on the diameter ℓ of X' and then a lower bound on its injectivity radius r.

We can now follow the first proof of Theorem 6.1. If $\Delta_1, \Delta_2 \in \Theta^*$ are two tetrahedra, then the intersection complexity of $g(\Delta_1)$ and $g(\Delta_2)$ is no worse than that of $g(\Delta'_1)$ and $g(\Delta'_2)$, and is bounded by an ER function of ℓ and r. This yields an ER bound on the complexity of the refinements Φ and Ψ. Recall that Ψ is a refinement of Θ^*, which is a slightly modified version of the input description of N. Having bounded the complexity of Ψ, we can search for it using Corollary 8.13 and solve for Φ and its geometry. We can also discard g if it does not have degree 1.

Thus far, the algorithm finds an ER collection of candidate maps $g : X^* \to N$ of degree 1, where N varies as well as g. At least one of these maps is a homotopy equivalence. Instead of finding an inverse h, We can repeat the algorithm to look for degree one maps among the target manifolds $\{N\}$. This induces a transitive relation among these manifolds. If N is chosen at the top of this relation, then the associated map $g : X^* \to N$ must be a homotopy equivalence.

To solve the homeomorphism problem, we find geometric triangulations Φ_1 and Φ_2 of the manifolds N_1^* and N_2^*. We can again follow the first proof of Theorem 6.1, except now with an ER bound on the complexity of Φ_1 and Φ_2, and we can again use Corollary 8.13. The same argument applies for the calculation of the isometry group of a single N^*. □

8.5. ***The small Seifert-fibered case.*** In this section we will prove Theorem 8.2.

Let N be a closed, oriented 3-manifold which has been recognized as irreducible and atoroidal by the relevant algorithm in the proof of Theorem 8.1. We want to distinguish between the case that N is small Seifert-fibered and the case that N is hyperbolic; and in the former case, find its homeomorphism type. By Theorem 8.7, we also may as well assume that $N \ncong S^3$. We divide the proof into two cases, according to whether $\pi_1(N)$ is finite or infinite. (Recall that N is spherical if and only if $\pi_1(N)$ is finite.)

Proposition 8.14. *It is in* ER *to determine if* $\pi_1(N)$ *is finite and compute its oriented homeomorphism type.*

Proof. We work from the fact [16, Sec. VI.11 & VI.16] that if $\pi_1(N)$ is finite, then N has a Seifert fibration whose base F is S^2 with at most two orbifolds points, or three orbifolds points of order a_1, a_2, and a_3 with

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} > 1.$$

Such a fibration of N lifts to the Hopf fibration of its universal cover S^3, and with the extra property that the action of $\pi_1(N)$ preserves an orientation of the Hopf fibers. Thus $\pi_1(N)$ can be realized as a finite subgroup of the unitary group U(2) that acts freely on the unit sphere $S^3 \subseteq \mathbb{C}^2$. Moreover, the 2-sphere S^2 which is the set of Hopf fibers is the orbifold universal cover of the base F. Thus the orbifold fundamental group $\pi_1(F)$ is the image of $\pi_1(N)$ in $\mathrm{PU}(2) \cong \mathrm{SO}(3)$.

If N has at most two exceptional fibers, then its the union of two solid tori and thus a lens space with a cyclic fundamental group. On the other hand, if N has three exceptional fibers, then $\pi_1(F)$ is a dihedral or Platonic subgroup of SO(3). In either of these latter two cases, the kernel of the projection $\pi_1(N) \to \pi_1(F)$ is the center $Z(\pi_1(N))$. This classification also tells us that if we pass to a cyclic subgroup of $\pi_1(F)$ and its inverse image in $\pi_1(N)$, we get an intermediate cover $\tilde{N}$ of N with $\pi_1(\tilde{N})$ abelian, so $\tilde{N}$ must be a lens space. Every dihedral or Platonic group has a cyclic subgroup of index at most 12.

We first calculate whether N is a lens space. We calculate $H_1(N)$ by applying the Smith normal form algorithm to its chain complex. If $H_1(N)$ is infinite, then N is not small Seifert-fibered. Otherwise the cardinality of $H_1(N)$ is elementary recursive. We can calculate whether N is a lens space by checking whether $H_1(N) \cong \mathbb{Z}/m$ is cyclic and calculating whether its abelian universal cover $\tilde{N}$ is isomorphic to S^3. To determine the parameter n in the homeomorphism type of $N \cong L(m,n)$, we can calculate the Reidemeister torsion of the twisted homology of N over the ring $\mathbb{Z}[\zeta]$, where ζ is an mth root of unity. This is a determinant calculation which is a priori elementary recursive. Note that this torsion determines the oriented homeomorphism type of N.

If N is spherical but not a lens space, then again it has a finite covering space $\tilde{N}$ of order at most 12 which is a lens space. We thus obtain an elementary recursive bound on the cardinality of $\pi_1(N)$. Using a presentation of $\pi_1(N)$ obtained from the triangulation of N, we can search exhaustively among surjective homomorphisms $\phi : \pi_1(N) \to \Gamma$, where Γ is a finite candidate for $\pi_1(N)$. For each such surjective homomorphism, we can build the corresponding covering space $\tilde{N}$ and calculate whether $\tilde{N} \cong S^3$. If this happens, then we know that $N \cong S^3/\Gamma$ as unoriented 3-manifolds.

Finally in the spherical case, we want to pass from the unoriented to the oriented homeomorphism type of N when N is spherical but not lens. (Note that every such N is chiral.) As a first warm-up, recall that we can distinguish the lens space $L(4,1)$ from its reverse $L(4,-1) = L(4,3)$ by computing its Reidemeister torsion. As a second warm-up, we consider the simplest prism space $R(1,2)$ whose fundamental group $\Gamma = \pi_1(R(1,2))$ is the quaternionic 8-element group. We can build $R(1,2)$ as a coset space inside SU(2):

$$\Gamma \subseteq \mathrm{SU}(2) \qquad R(1,2) \cong \mathrm{SU}(2)/\Gamma.$$

The group Γ has four cyclic subgroups of order 4 which are not conjugate in Γ itself, but which are conjugate in SU(2). Matching this calculation to the Seifert data, $R(1,2)$ has three double covers which are all-oriented homeomorphic to $L(4,1)$. We can thus calculate the orientation of N by calculating whether any double cover is $L(4,1)$ or $L(4,3)$.

If N is spherical but not lens, then again $\pi_1(N)/Z(\pi_1(N))$ is either dihedral, which is the prism case; or the isometry group of a Platonic solid: tetrahedral, octahedral, or icosahedral. In the case that $N \cong R(m,n)$ and m is odd, as well as in the Platonic cases, $\pi_1(N)$ has a unique subgroup isomorphic to the quaternionic group. Thus we can form the corresponding covering space $\tilde{N} \cong R(1,2)$ and calculate its orientation as in the second warmup. Mean while if $N \cong R(m,n)$ and m is even, then the center $Z(\pi_1(N)) \cong \mathbb{Z}/(2m)$ has a unique subgroup of order 4. Thus $\pi_1(N)$ has a canonically chosen cyclic subgroup of order 4, and we can again form $\tilde{N}$ and calculate whether it is $L(4,1)$ or $L(4,3)$. □

To prove Proposition 8.15, we will use a different combinatorial model of Seifert-fibered spaces than the one in Section 5.2.4. If N is Seifert-fibered with base F, then we can consider a triangulation Θ of F with the orbifold points placed at the vertices. For each triangle $\Delta \in \Theta$, we make a solid torus $\Delta \times S^1$ which we interpret as a chart for a circle bundle with structure group S^1. Then we can construct N with an atlas of charts of this type. (It is an atlas with closed charts rather than open charts, but this is valid in context.) When two triangles Δ_1 and Δ_2 intersect in an edge, we glue the charts together with a transition map

$$f_{12} : \Delta_1 \cap \Delta_2 \to S^1 \cong \mathbb{R}/\mathbb{Z}.$$

We can assume that each transition map f_{12} is affine-linear if lifted to $\mathbb{R}$, so that the endpoint values of f_{12} lie in $\mathbb{Q}/\mathbb{Z}$, and its slope is also in $\mathbb{Q}$. Moreover, it is not hard to convert the Seifert data for N into these transition functions, for any triangulation of F. Finally, note that if $p \in F$ is an orbifold point of order a and $\Delta \in \Theta$ is any triangle that has p as a vertex, then the gluing maps between charts glue the circle over p in such a way that it shortens by a factor of a and becomes a singular fiber of N.

Proposition 8.15. *It is in* ER *to determine if N is small Seifert-fibered with infinite $\pi_1(N)$, and if so, compute its oriented homeomorphism type.*

Proof. If $\pi_1(N)$ is infinite and N is small Seifert-fibered, then the base F of N is a 2-sphere with orbifold points of order $a_1 \geq a_2 \geq a_3$ with

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \leq 1.$$

In the equality case F is Euclidean and N is either Euclidean or Nil; otherwise F is hyperbolic and the geometry of N is either $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{Isom}}(\mathbb{H}^2)$. The orbifold fundamental group $\pi_1(F)$ is a von Dyck group $D(a_1,a_2,a_3)$ which is the orientation-preserving subgroup of index two in the corresponding triangle group $\Delta(a_1,a_2,a_3)$ in either $\mathrm{Isom}(\mathbb{E}^2)$ or $\mathrm{Isom}(\mathbb{H}^2)$, and $\pi_1(N)$ is a central extension of $D(a_1,a_2,a_3)$ by $\mathbb{Z}$.

We first consider the case in which F is Euclidean, which implies that (a_1, a_2, a_3) is either $(3, 3, 3)$, $(4, 4, 2)$, or $(6, 3, 2)$. In this case there is a homomorphism

$$\phi : \pi_1(N) \to G,$$

where the target group is respectively the dihedral group D_3, D_4, or D_6, such that the corresponding regular cover $\tilde{N}$ is a circle bundle over a torus. Given a putative choice for G and ϕ, we can construct the regular cover $\tilde{N}$ and apply the large Seifert-fibered case of Theorem 8.1 to recognize it. (Lest this look circular, only the small Seifert-fibered case of Theorem 8.1 needs the current Proposition 8.15.) If $\tilde{N}$ is indeed a circle bundle over a torus, then we know that N must be small Seifert-fibered with a Euclidean base, and the remaining question is to confirm that G and ϕ were correctly chosen and thus compute the specific Seifert data of N.

Either $\tilde{N}$ is $S^1 \times S^1 \times S^1$ (so that N itself is Euclidean), or it is a circle bundle over $S^1 \times S^1$ with a non-trivial Euler number and thus has Nil geometry. We thus obtain an explicit form of $\pi_1(\tilde{N})$ which is either $\mathbb{Z}^3$ or a central extension of $\mathbb{Z}^2$ by $\mathbb{Z}$. At the same time, since the recognition of the structure of $\tilde{N}$ is based on normal surface theory, it thus yields a retriangulation of $\tilde{N}$ in ER from the triangulation induced by the input triangulation of N to one that reveals the Seifert structure. Therefore we obtain an explicit (and elementary recursive) description of $\pi_1(N)$ as an extension of the finite group G by $\pi_1(\tilde{N})$. We can thus match $\pi_1(N)$ to the corresponding model Seifert-fibered space to determine the unoriented homeomorphism type of N. Finally we have to calculate the oriented homeomorphism type. In the Euclidean case, the recognition of $\tilde{N}$ gives us an orientation of $\pi_1(\tilde{N}) \cong \mathbb{Z}^3$. Similarly in the Nil case, when $\pi_1(\tilde{N})$ is an extension of $\mathbb{Z}^2$ by $\mathbb{Z}$, the orientation of $\tilde{N}$ still gives an orientation of both the center $\mathbb{Z}$ and the quotient $\mathbb{Z}^2$, up to switching both orientations. This lets us compare the given orientation of N to an orientation of the model to thus determine the oriented type of N.

The argument when the base F is hyperbolic is similar to the Euclidean case but more complicated. In this case, $\pi_1(N)$ has a non-trivial homomorphism to $\mathrm{Isom}(\mathbb{H}^2)$, which in turn embeds in $\mathrm{SL}(2, \mathbb{C})$, so Theorem 8.12 tells us that $\pi_1(N)$ has a non-trivial finite quotient G which we can find in ER even if we do not know the linear representation that explains that it exists.

We construct the finite cover $\tilde{N}$ of N corresponding to the quotient map $\phi : \pi_1(N) \to G$, and we apply part of Theorem 8.1 to determine if $\tilde{N}$ is large Seifert-fibered, and if so calculate its fibration and its base $\tilde{F}$. As before, we can first learn from this whether N is indeed small Seifert-fibered. Second, as before this part of Theorem 8.1 gives us an ER retriangulation from the initial triangulation of $\tilde{N}$ as a covering space of N, to a triangulation that reflects its Seifert fibration. In particular, we obtain a triangulation Θ of $\tilde{F}$ together with an atlas of charts to describe $\tilde{N}$. Third, we can choose an orientation of $\tilde{F}$ and an orientation of the circle fibers so that the two orientations together are consistent with the orientation of $\tilde{N}$ inherited from N. Fourth, using the orbifold point orders of $\tilde{F}$ and the cardinality of G, we obtain an ER upper bound on a_1, a_2, and a_3.

For each candidate for (a_1, a_2, a_3), the given representation of $D(a_1, a_2, a_3)$ in $\mathrm{Isom}(\mathbb{H}^2)$ is rigid. It is still rigid even as a representation of $\pi_1(N)$, because any nearby representation must still annihilate the kernel $Z(\pi_1(N))$. Passing to the covering space $\tilde{F}$, we obtain a preferred hyperbolic structure on $\tilde{F}$ and we can realize Θ as a geometric triangulation. (In two dimensions, every triangulation of a hyperbolic surface is geometric.) Now Theorem 8.8, combined with the fact that the retriangulation of $\tilde{N}$ is in ER, gives us an ER upper bound on the lengths of the edges of Θ. At the same time, we get a second triangulation Φ of $\tilde{F}$ by tiling it by lifts of the triangular fundamental domain of the triangle group $\Delta(a_1, a_2, a_3)$. The triangulation Φ is both geometric and $\pi_1(N)$-invariant. It is sometimes only a generalized triangulation in the sense that its 1-skeleton could have double edges or self-loops, but this doesn't matter for our arguments. Using the same arguments as in the proof of Theorem 8.3 in Section 8.4, Φ and Θ have a mutual refinement that can be found in ER. Thus we can search over retriangulations of $\tilde{F}$ in ER until we find one that is $\pi_1(N)$-invariant. We can then use this to compute the Seifert structure on N, moreover preserving the orientation information inherited from the fibration of $\tilde{N}$. □

9. Open problems

Theorem 1.2, together with the fact that ER is a fairly generous complexity class, suggests the following conjectures.

Conjecture 9.1. *If M is a closed, Riemannian 3-manifold, then Ricci flow with surgery on M can be accurately simulated in* ER.

In other words, we conjecture that Perelman's proof of geometrization can be placed in ER.

Conjecture 9.2. *Every closed, hyperbolic manifold N has a finite-sheeted Haken covering which is computable in* ER.

In other words, we conjecture that the statement of the virtual Haken conjecture, now the theorem of Agol et al [1], can be placed in ER. Maybe the known proof can be as well.

Conjecture 9.3. *Any two triangulations of a closed 3-manifold M have a mutual refinement computable in* ER.

Conjecture 9.3 does not follow from our proof of Theorem 1.2, because the algorithm in Theorem 8.3 only establishes a simplicial homotopy equivalence and then relies on Mostow rigidity. However, the rest of the proof of Theorem 1.2 uses a bounded number of normal surface dissections, which does establish an ER mutual refinement according to the arguments of Mijatović [31; 32]. Also, Conjecture 9.2 and the Haken case of Conjecture 9.3 would together imply the hyperbolic case of Conjecture 9.3, which would then imply the full conjecture. Mijatović [33] also established that any two triangulations of a fiber-free Haken 3-manifold have a primitive recursive mutual refinement.

Cases 3 and 4 of Proposition 2.6 are expected to be false for typical bounds on complexity that are better than ER. Thus, in discussing further improvements to Theorem 1.2, we should consider qualitative complexity classes, such as the famous NP, rather than just bounds on execution time. For one thing, ER is the union of an alternating, nested sequence of time and space complexity classes, as follows:

$$\mathsf{P} \subseteq \mathsf{PSPACE} \subseteq \mathsf{EXP} \subseteq \mathsf{EXPSPACE} \subseteq \mathsf{EEXP} \subseteq \cdots \ .$$

Here P is the set of decision problems that can be solved in deterministic polynomial time; PSPACE is solvability in polynomial space with unrestricted (but deterministic) computation time; EXP is deterministic time $\exp(\mathrm{poly}(n))$; etc. The author does not know where a careful version of our proof of Theorem 1.2 would land in this hierarchy.

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HARISH-CHANDRA MODULES FOR DIVERGENCE ZERO VECTOR FIELDS ON A TORUS

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The Lie algebra of divergence zero vector fields on a torus is an infinite-dimensional Lie algebra of skew derivations over the ring of Laurent polynomials. We consider the semidirect product of the Lie algebra of divergence zero vector fields on a torus with the algebra of Laurent polynomials. In this paper, we prove that a Harish-Chandra module of the universal central extension of the derived Lie subalgebra of this semidirect product is either a uniformly bounded module or a generalized highest weight module. We also classify all the generalized highest weight Harish-Chandra modules.

1. Introduction

Harish-Chandra modules, i.e., irreducible weight modules with finite-dimensional weight spaces, are no doubt one of the most important families in the study of the representation theory of infinite-dimensional Lie algebras. The classifications of Harish-Chandra modules over the Virasoro algebra ([Kaplansky and Santharoubane 1985; Mathieu 1992]), higher rank Virasoro algebras ([Su 2003; Lu and Zhao 2006]), and many other Lie algebras related to the Virasoro algebra have been achieved in [Guo et al. 2011; 2012; Lu and Zhao 2010; Liu and Jiang 2008; Mazorchuk 2000; Su 2004a; 2004b; Su et al. 2012; 2013; Wang and Tan 2007]. Let $A = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$ be the algebra of Laurent polynomials in commuting variables and B be the set of skew derivations of A. Let L be the universal central extension of the derived Lie subalgebra of the Lie algebra $A \rtimes B$. Set $\widetilde{L} = L \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$, where d_1, d_2 are two degree derivations. In this paper, we study Harish-Chandra modules over the Lie algebra $\widetilde{L} = L \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$, this Lie algebra is a generalization of the twisted Heisenberg–Virasoro algebra from rank one to rank two (see [Xue et al. 2006; Tan et al. 2015] for details). The structure of the Lie algebra L has been studied in [Xue et al. 2006]. Recently, the connection of the Lie algebra L with the vertex algebra

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has been established in [Guo and Wang 2016] and the representation theory of the Lie algebra L has been studied in [Tan et al. 2015; Guo and Liu 2019; Billig and Talboom 2018]. However, the classification of the Harish-Chandra modules over the Lie algebra $\widetilde{L}$ is unknown. We prove that a Harish-Chandra module of $\widetilde{L}$ is either a uniformly bounded module or a generalized highest weight module, and we classify the nonzero level Harish-Chandra modules of the Lie algebra $\widetilde{L}$. Based on these results, the classification of Harish-Chandra modules of $\widetilde{L}$ reduces to the classification of uniformly bounded modules of $\widetilde{L}$. In [Guo and Liu 2019], the uniformly bounded modules satisfying the condition that the torus subalgebra acting nonzero were classified. Another reason to study the Harish-Chandra modules of the Lie algebra $\widetilde{L}$ comes from the representation theory of the nullity 2 toroidal extended affine Lie algebras (see [Chen et al. 2018]). It was proved therein that the classification of irreducible integrable modules with finite-dimensional spaces of the nullity 2 toroidal extended affine Lie algebras of type A_1 can be reduced to the classification of Harish-Chandra modules of $\widetilde{L}$. This phenomenon is similar to the fact that the classification of irreducible integrable modules of the full toroidal Lie algebra can be reduced to the classification of irreducible $(\mathrm{Der}(A_n) \ltimes A_n)$-modules (see [Eswara Rao and Jiang 2005]), where

$$A_n = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}].$$

The techniques in this paper follow from [Lin and Tan 2006; Lin and Su 2013; Lu and Zhao 2006; Su 2003]. However, we want to point out that in [Lin and Tan 2006], the construction of the generalized highest weight modules of the Virasoro-like algebra is induced from the $\mathbb{Z}$-graded irreducible modules of a Heisenberg subalgebra, while in this paper, the construction of the generalized highest weight module of the Lie algebra $\widetilde{L}$ comes from the $\mathbb{Z}$-graded irreducible module of the subalgebra $\mathcal{H}_{\boldsymbol{b}_1}$ (see the definition in Section 2), which is the twist of three Heisenberg subalgebras. So we first need to classify the $\mathbb{Z}$-graded irreducible $\mathcal{H}_{\boldsymbol{b}_1}$-modules with finite-dimensional graded spaces, which we do in Propositions 2.6 and 2.8. For the classification of generalized highest weight Harish-Chandra modules of $\widetilde{L}$, we achieve this by considering the tensor product of the highest weight modules of the Lie algebra L with a torus. Moreover, we prove that these tensor modules of $\widetilde{L}$ are completely reducible, and every generalized highest weight Harish-Chandra module of $\widetilde{L}$ is isomorphic to one of the irreducible components of these tensor modules.

The paper is organized as follows. In Section 2, we prove that a Harish-Chandra module of $\widetilde{L}$ is either a uniformly bounded module or a generalized highest weight module. In Section 3, we prove that a nonzero level Harish-Chandra module of $\widetilde{L}$ is a generalized highest weight module. Then we characterize the generalized highest weight Harish-Chandra modules with nonzero level. In Section 4, we classify the generalized highest weight Harish-Chandra modules of $\widetilde{L}$.

Throughout this paper we use $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$ to denote the sets of complex numbers, integers, nonnegative integers and positive integers respectively. All the vector spaces mentioned in this paper are over $\mathbb{C}$. As usual, if $u_1, u_2, \ldots, u_k$ are elements of a certain vector space, we denote by $\langle u_1, u_2, \ldots, u_k\rangle$ the linear span of the elements $u_1, u_2, \ldots, u_k$ over $\mathbb{C}$. The universal enveloping algebra for a Lie algebra $\mathfrak{g}$ is denoted by $\mathcal{U}(\mathfrak{g})$ and $\mathrm{GL}_{n\times n}(\mathbb{Z})$ denotes the set of $n \times n$ invertible matrices with entries from $\mathbb{Z}$.

2. Harish-Chandra modules of $\widetilde{L}$

In this section, we first recall some basic definitions about Harish-Chandra modules of $\widetilde{L}$ and some results for Heisenberg algebras. Then we prove that a Harish-Chandra module of $\widetilde{L}$ is either a uniformly bounded module or a generalized highest weight module.

Let $\boldsymbol{e}_1 = (1, 0)$, $\boldsymbol{e}_2 = (0, 1)$, $\Gamma = \mathbb{Z}\boldsymbol{e}_1 + \mathbb{Z}\boldsymbol{e}_2$. Letting $(x_1, x_2), (y_1, y_2) \in \Gamma$, we define $(x_1, x_2) > (y_1, y_2)$ if and only if $x_1 > y_1$ and $x_2 > y_2$, and $(x_1, x_2) \geq (y_1, y_2)$ if and only if $x_1 \geq y_1$ and $x_2 \geq y_2$. For any $\boldsymbol{b}_1 = b_{11}\boldsymbol{e}_1 + b_{12}\boldsymbol{e}_2$, $\boldsymbol{b}_2 = b_{21}\boldsymbol{e}_1 + b_{22}\boldsymbol{e}_2 \in \Gamma$, we set

$$\det\begin{pmatrix}\boldsymbol{b}_1\\ \boldsymbol{b}_2\end{pmatrix} = b_{11}b_{22} - b_{12}b_{21}.$$

Now we recall the definition of the Lie algebra arising from the two-dimensional torus (also called the Heisenberg–Virasoro algebra of rank two). See [Xue et al. 2006] (cf. [Tan et al. 2015]) for details.

Definition 2.1. The *Heisenberg–Virasoro algebra of rank two* is the Lie algebra spanned by

$$\{t^{\boldsymbol{m}}, E(\boldsymbol{m}), K_i \mid \boldsymbol{m} \in \Gamma \setminus \{\boldsymbol{0}\}, i = 1, 2, 3, 4\}$$

with Lie bracket defined by

$$[t^{\boldsymbol{m}}, t^{\boldsymbol{n}}] = 0, \quad [K_i, L] = 0, \quad i = 1, 2, 3, 4,$$

$$[t^{\boldsymbol{m}}, E(\boldsymbol{n})] = \det\begin{pmatrix}\boldsymbol{n}\\ \boldsymbol{m}\end{pmatrix} t^{\boldsymbol{m}+\boldsymbol{n}} + \delta_{\boldsymbol{m}+\boldsymbol{n},0}\, h(\boldsymbol{m}),$$

$$[E(\boldsymbol{m}), E(\boldsymbol{n})] = \det\begin{pmatrix}\boldsymbol{n}\\ \boldsymbol{m}\end{pmatrix} E(\boldsymbol{m}+\boldsymbol{n}) + \delta_{\boldsymbol{m}+\boldsymbol{n},0}\, f(\boldsymbol{m}),$$

where $\boldsymbol{m} = m_1\boldsymbol{e}_1 + m_2\boldsymbol{e}_2$, $h(\boldsymbol{m}) = m_1K_1 + m_2K_2$, $f(\boldsymbol{m}) = m_1K_3 + m_2K_4$.

We denote this Lie algebra by L. Set $E(\boldsymbol{0}) = t^{\boldsymbol{0}} = 0$ for convenience. Obviously L is a $\mathbb{Z}^2$-graded Lie algebra and the subalgebra $\langle E(\boldsymbol{m}),\ K_3,\ K_4 \mid \boldsymbol{m} \in \Gamma \setminus \{\boldsymbol{0}\}\rangle$ of L is a Virasoro-like algebra. Let $\widetilde{L} = L \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$, where d_1, d_2 are two degree derivations defined by

$$[d_i, E(\boldsymbol{m})] = m_iE(\boldsymbol{m}), \quad [d_i, t^{\boldsymbol{m}}] = m_it^{\boldsymbol{m}}, \quad [d_i, K_j] = 0, \quad [d_1, d_2] = 0,$$

for $\boldsymbol{m} = m_1\boldsymbol{e}_1 + m_2\boldsymbol{e}_2 \in \Gamma$, $i = 1, 2$ and $j = 1, 2, 3, 4$. Lemma 2.2 is easy to check.

Lemma 2.2. *Let* $0 \neq \boldsymbol{b}_1 = b_{11}\boldsymbol{e}_1 + b_{12}\boldsymbol{e}_2 \in \Gamma$ *and* $\boldsymbol{b}_2 = b_{21}\boldsymbol{e}_1 + b_{22}\boldsymbol{e}_2 \in \Gamma$.

(1) *Then* $\langle E(\pm k\boldsymbol{b}_1),\ f(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle$ *and* $\langle E(k\boldsymbol{b}_1),\ t^{-k\boldsymbol{b}_1},\ h(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle$ *and* $\langle E(-k\boldsymbol{b}_1),\ t^{k\boldsymbol{b}_1},\ h(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle$ *are three Heisenberg subalgebras of* $\widetilde{L}$, *and*

(2) $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ *is a* $\mathbb{Z}$*-basis of* Γ *if and only if* $\det\binom{\boldsymbol{b}_1}{\boldsymbol{b}_2} = \pm 1$.

Now we recall some definitions related to the Harish-Chandra modules for $\widetilde{L}$. A *weight module* of $\widetilde{L}$ is a module V with weight space decomposition

$$V = \bigoplus_{\boldsymbol{\lambda} \in \mathbb{C}^6} V_{\boldsymbol{\lambda}},$$

where $V_{\boldsymbol{\lambda}} = \{v \in V \mid d_i v = \lambda_i v,\ K_j v = \lambda_{j+2} v,\ i = 1, 2,\ j = 1, 2, 3, 4\}$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_6) \in \mathbb{C}^6$. For a weight module V, we define the weight set of V by $\mathcal{P}(V) = \{\boldsymbol{\lambda} \in \mathbb{C}^6 \mid V_{\boldsymbol{\lambda}} \neq 0\}$. A weight module is said to be *quasifinite* if all weight spaces $V_{\boldsymbol{\lambda}}$ are finite-dimensional. Furthermore, if there exists a positive integer N such that $\dim V_{\boldsymbol{\lambda}} \leq N$ for all $\boldsymbol{\lambda} \in \mathbb{C}^6$, we call V a *uniformly bounded* module. An irreducible quasifinite weight module is called a *Harish-Chandra module*. Note that the centers K_1, K_2, K_3, K_4 of $\widetilde{L}$ act on an irreducible weight module V as scalars, i.e., $K_i.v = c_i v$ for certain $c_i \in \mathbb{C},\ i = 1, 2, 3, 4$, for all $v \in V$. And we call (c_1, c_2, c_3, c_4) the *level* of the module V. For simplicity of notation, we write $V_{(\lambda_1, \lambda_2)}$ instead of $V_{(\lambda_1, \dots, \lambda_6)}$ if the module V is irreducible, i.e., the level $(c_1, \dots, c_4)$ is fixed. One can easily see that there exist $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $\mathcal{P}(V) \subseteq (\lambda_1, \lambda_2) + \Gamma$ for an irreducible weight module V of $\widetilde{L}$. If there exists a $\mathbb{Z}$-basis $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ of Γ and $0 \neq v_{\boldsymbol{\lambda}} \in V_{\boldsymbol{\lambda}}$ such that $V = \mathcal{U}(\widetilde{L})v_{\boldsymbol{\lambda}}$ and $E(\boldsymbol{m})v_{\boldsymbol{\lambda}} = t^{\boldsymbol{m}}v_{\boldsymbol{\lambda}} = 0$, for all $\boldsymbol{m} \in \mathbb{Z}_+\boldsymbol{b}_1 + \mathbb{Z}_+\boldsymbol{b}_2$, we call V a *generalized highest weight module* with generalized highest weight $\boldsymbol{\lambda}$ corresponding to the $\mathbb{Z}$-basis B. The nonzero vector $v_{\boldsymbol{\lambda}}$ is called a *generalized highest weight vector corresponding to the* $\mathbb{Z}$*-basis* B, or simply *generalized highest weight vector*.

Let $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ be a $\mathbb{Z}$-basis of Γ and let

$$\mathcal{H}_{\boldsymbol{b}_1} = \langle E(k\boldsymbol{b}_1),\ t^{k\boldsymbol{b}_1},\ K_i \mid k \in \mathbb{Z} \setminus \{0\},\ i = 1, 2, 3, 4\rangle.$$

Denote

$$\begin{aligned}
&\widetilde{L}_0 = \mathcal{H}_{\boldsymbol{b}_1} \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2,\\
&\widetilde{L}_i = \langle E(m\boldsymbol{b}_1 + i\boldsymbol{b}_2), t^{m\boldsymbol{b}_1 + i\boldsymbol{b}_2} \mid m \in \mathbb{Z}\rangle, \quad i \neq 0,\\
&\widetilde{L}_+ = \oplus_{i>0}\widetilde{L}_i, \qquad \widetilde{L}_- = \oplus_{i<0}\widetilde{L}_i.
\end{aligned}$$

Then $\widetilde{L} = \widetilde{L}_+ \oplus \widetilde{L}_0 \oplus \widetilde{L}_-$. Let V be an irreducible weight $\widetilde{L}_0$-module. We extend V to be a $(\widetilde{L}_+ \oplus \widetilde{L}_0)$-module by defining $\widetilde{L}_+.V = 0$. Then we obtain the induced $\widetilde{L}$-module

$$\widetilde{M}(V) = \widetilde{M}(\boldsymbol{b}_1, \boldsymbol{b}_2, V) = \operatorname{Ind}_{\widetilde{L}_+ \oplus \widetilde{L}_0}^{\widetilde{L}} V = \mathcal{U}(\widetilde{L}) \otimes_{\mathcal{U}(\widetilde{L}_+ \oplus \widetilde{L}_0)} V.$$

It is clear that, as vector spaces,

$$\widetilde{M}(\boldsymbol{b}_1, \boldsymbol{b}_2, V) \simeq \mathcal{U}(\widetilde{L}_-) \otimes_{\mathbb{C}} V.$$

The $\widetilde{L}$-module $\widetilde{M}(\boldsymbol{b}_1, \boldsymbol{b}_2, V)$ has a unique maximal submodule $J(\boldsymbol{b}_1, \boldsymbol{b}_2, V)$ trivially intersecting with V. Then we obtain the unique irreducible quotient module

$$M(V) = M(\boldsymbol{b}_1, \boldsymbol{b}_2, V) = \widetilde{M}(\boldsymbol{b}_1, \boldsymbol{b}_2, V)/J(\boldsymbol{b}_1, \boldsymbol{b}_2, V).$$

It is clear that $M(V)$ is uniquely determined by the $\mathbb{Z}$-basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ of Γ and the $\widetilde{L}_0$-module V.

Remark 2.3. The irreducible $\widetilde{L}$-module $M(\boldsymbol{b}_1, \boldsymbol{b}_2, V)$ constructed above is a generalized highest weight module corresponding to the $\mathbb{Z}$-basis $\{\boldsymbol{b}_1 + \boldsymbol{b}_2, \boldsymbol{b}_1 + 2\boldsymbol{b}_2\}$ of Γ.

We recall some results about the $\mathbb{Z}$-graded module for Heisenberg Lie algebras.

For any $0 \neq \boldsymbol{b}_1 \in \Gamma$, denote the subalgebra $\langle E(\pm k\boldsymbol{b}_1),\ f(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle$ of $\widetilde{L}$ by $E_{\boldsymbol{b}_1}$. For any $E_{\boldsymbol{b}_1}$-module V, if the eigenvalue of $f(\boldsymbol{b}_1)$ is a scalar then we call it the *level* of V. Let

$$E^{\pm}_{\boldsymbol{b}_1} = \langle E(k\boldsymbol{b}_1) \mid \pm k \in \mathbb{N}\rangle.$$

For $0 \neq a \in \mathbb{C}$, let $\mathbb{C}v_a$ be a one-dimensional $(E^{\varepsilon}_{\boldsymbol{b}_1} \oplus \mathbb{C}f(\boldsymbol{b}_1))$-module such that $E^{\varepsilon}_{\boldsymbol{b}_1}.v_a = 0,\ f(\boldsymbol{b}_1).v_a = av_a,\ \ \varepsilon \in \{+, -\}$. Consider the induced $E_{\boldsymbol{b}_1}$-module

$$M^{\varepsilon}(a) = \mathcal{U}(E_{\boldsymbol{b}_1}) \otimes_{\mathcal{U}(E^{\varepsilon}_{\boldsymbol{b}_1} \oplus \mathbb{C}f(\boldsymbol{b}_1))} \mathbb{C}v_a$$

associated with a and ε (a is the level of $M^{\varepsilon}(a)$). Then the $E_{\boldsymbol{b}_1}$-module $M^{\varepsilon}(a)$ is irreducible.

The following result is due to Propositions 4.3(i) and 4.5 in [Futorny 1997].

Theorem 2.4. *If* $V = \bigoplus_{i\in\mathbb{Z}} V_i$ *is a* $\mathbb{Z}$*-graded* $E_{\boldsymbol{b}_1}$*-module of level* $0 \neq a \in \mathbb{C}$ *and* $\dim V_i < \infty$ *for at least one* $i \in \mathbb{Z}$ *then*

(1) *if* V *is an irreducible module then* $V \simeq M^{\varepsilon}(a)$ *for some* $\varepsilon \in \{+, -\}$;

(2) V *is completely reducible.*

Let $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ be a $\mathbb{Z}$-basis of Γ. For a $\mathcal{H}_{\boldsymbol{b}_1}$-module V, if $f(\boldsymbol{b}_1), h(\boldsymbol{b}_1), f(\boldsymbol{b}_2), h(\boldsymbol{b}_2)$ act as scalars $c_1, c_2, c_3, c_4 \in \mathbb{C}$, then we call (c_1, c_2, c_3, c_4) the *level* of the $\mathcal{H}_{\boldsymbol{b}_1}$-module V. Furthermore if $(c_1, c_2, c_3, c_4) = (0, 0, c_3, c_4)$, we say that V is a $\mathcal{H}_{\boldsymbol{b}_1}$-module of *level zero*. Otherwise, V is nonzero level. In the following, we will discuss the irreducible $\mathcal{H}_{\boldsymbol{b}_1}$-modules. First we recall the classification of $\mathbb{Z}$-graded irreducible $\mathcal{H}_{\boldsymbol{b}_1}$-modules of level zero. Then we classify the $\mathbb{Z}$-graded $\mathcal{H}_{\boldsymbol{b}_1}$-modules of nonzero level with finite-dimensional graded subspaces.

Set $T = \mathbb{C}[t^{\pm 1}]$. Let $\rho : \mathcal{H}_{\boldsymbol{b}_1} \to \mathbb{C}$ be a linear function with $\rho(f(\boldsymbol{b}_1)) = \rho(h(\boldsymbol{b}_1)) = 0$. We can define a $\mathcal{H}_{\boldsymbol{b}_1}$-module structure on T by

$$f(\boldsymbol{b}_1).t^n = 0, \qquad E(k\boldsymbol{b}_1).t^n = \rho(E(k\boldsymbol{b}_1))t^{k+n}, \tag{2-1}$$

$$h(\boldsymbol{b}_1).t^n = 0, \qquad t^{k\boldsymbol{b}_1}.t^n = \rho(t^{k\boldsymbol{b}_1})t^{k+n}, \tag{2-2}$$

$$f(\boldsymbol{b}_2).t^n = \rho(f(\boldsymbol{b}_2))t^n, \qquad h(\boldsymbol{b}_2).t^n = \rho(h(\boldsymbol{b}_2))t^n, \tag{2-3}$$

where $n \in \mathbb{Z}$, $k \in \mathbb{Z} \setminus \{0\}$. We denote

$$T_{\rho,i}(\mathcal{H}_{\boldsymbol{b}_1}) = \mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1}).t^i$$

the $\mathcal{H}_{\boldsymbol{b}_1}$-submodule of T generated by t^i for $i \in \mathbb{Z}$. And we write $T_{\rho,0}(\mathcal{H}_{\boldsymbol{b}_1})$ as $T_\rho(\mathcal{H}_{\boldsymbol{b}_1})$ for short. From the definition, we see that

$$T_{\rho,i}(\mathcal{H}_{\boldsymbol{b}_1}) \simeq T_{\rho,j}(\mathcal{H}_{\boldsymbol{b}_1}) \tag{2-4}$$

for $i, j \in \mathbb{Z}$ as $\mathcal{H}_{\boldsymbol{b}_1}$-modules.

Remark 2.5. For linear function $\rho : E_{\boldsymbol{b}_1} \to \mathbb{C}$ with $\rho(f(\boldsymbol{b}_1)) = 0$, we can define a $E_{\boldsymbol{b}_1}$-module structure on the Laurent polynomial ring T with the action given by (2-1). Similarly, let $T_{\rho,i}(E_{\boldsymbol{b}_1}) := \mathcal{U}(E_{\boldsymbol{b}_1}).t^i$ be the $E_{\boldsymbol{b}_1}$-submodule of T generated by t^i for $i \in \mathbb{Z}$. And we also write $T_{\rho,0}(E_{\boldsymbol{b}_1})$ as $T_\rho(E_{\boldsymbol{b}_1})$ for short.

Then we have the following results from Lemma 3.6 and Proposition 3.8 in [Chari 1986].

Proposition 2.6. (1) *The $\mathcal{H}_{\boldsymbol{b}_1}$-module $T_\rho(\mathcal{H}_{\boldsymbol{b}_1})$ (resp. $E_{\boldsymbol{b}_1}$-module $T_\rho(E_{\boldsymbol{b}_1})$) is irreducible if and only if $T_\rho(\mathcal{H}_{\boldsymbol{b}_1}) = T_r$ (resp. $T_\rho(E_{\boldsymbol{b}_1}) = T_r$) for some $r \in \mathbb{Z}_+$, where $T_0 = \mathbb{C}1$ and $T_r = \mathbb{C}[t^r, t^{-r}]$ if $r \in \mathbb{N}$.*

(2) *If V is a $\mathbb{Z}$-graded irreducible $\mathcal{H}_{\boldsymbol{b}_1}$-module (resp. $E_{\boldsymbol{b}_1}$-module) of level zero, then $V \simeq T_\rho(\mathcal{H}_{\boldsymbol{b}_1})$ for some linear function $\rho : \mathcal{H}_{\boldsymbol{b}_1} \to \mathbb{C}$ with $\rho(f(\boldsymbol{b}_1)) = \rho(h(\boldsymbol{b}_1)) = 0$ (resp. $V \simeq T_\rho(E_{\boldsymbol{b}_1})$ for some linear function $\rho : E_{\boldsymbol{b}_1} \to \mathbb{C}$ with $\rho(f(\boldsymbol{b}_1)) = 0$), and $T_\rho(\mathcal{H}_{\boldsymbol{b}_1}) = T_r$ (resp. $T_\rho(E_{\boldsymbol{b}_1}) = T_r$) for some $r \in \mathbb{Z}_+$.*

Remark 2.7. Since $\langle t^{k\boldsymbol{b}_1}, E(-k\boldsymbol{b}_1), h(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle$ and $\langle t^{-k\boldsymbol{b}_1}, E(k\boldsymbol{b}_1), h(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle$ are two Heisenberg Lie subalgebras of $\widetilde{L}$, Theorem 2.4 and Proposition 2.6 also hold for their corresponding $\mathbb{Z}$-graded irreducible modules.

For convenience, we let $\mathcal{E}_{\boldsymbol{b}_1}$ denote the set of all linear functions $\rho : \mathcal{H}_{\boldsymbol{b}_1} \to \mathbb{C}$ with $\rho(f(\boldsymbol{b}_1)) = \rho(h(\boldsymbol{b}_1)) = 0$ such that the $\mathcal{H}_{\boldsymbol{b}_1}$-module $T_\rho(\mathcal{H}_{\boldsymbol{b}_1})$ is irreducible. Let $t_{\boldsymbol{b}_1} = \langle t^{\pm k\boldsymbol{b}_1} \mid k \in \mathbb{N}\rangle$. Note that $t_{\boldsymbol{b}_1}$ is a centerless Heisenberg subalgebra of $\widetilde{L}$. Let $T_\rho(t_{\boldsymbol{b}_1})$ be the submodule of T generated by 1, where ρ is a linear function $\rho : t_{\boldsymbol{b}_1} \to \mathbb{C}$. The structure of the $t_{\boldsymbol{b}_1}$-module T is defined in a way similar to that of $E_{\boldsymbol{b}_1}$. In the following proposition we classify the $\mathbb{Z}$-graded irreducible $\mathcal{H}_{\boldsymbol{b}_1}$-modules with nonzero level.

Proposition 2.8. *Let $V=\bigoplus_{i\in\mathbb{Z}} V_i$ be a $\mathbb{Z}$-graded irreducible $\mathcal{H}_{\boldsymbol{b}_1}$-module with* $\dim V_i<\infty$ *for all $i\in\mathbb{Z}$. Suppose $f(\boldsymbol{b}_1).v=c_1v$, $h(\boldsymbol{b}_1).v=c_2v$, $f(\boldsymbol{b}_2).v=c_3v$ and $h(\boldsymbol{b}_2).v=c_4v$ for $v\in V$, where $c_1,c_2,c_3,c_4\in\mathbb{C}$ and $(c_1,c_2)\neq\mathbf{0}$.*

(1) *If $c_1\neq 0$ and $c_2\neq 0$, then*

$$V\simeq\mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1})\otimes_{\mathcal{U}(\langle E(k\boldsymbol{b}_1),\ t^{k\boldsymbol{b}_1},\ f(\boldsymbol{b}_i),\ h(\boldsymbol{b}_i)|k\in\mathbb{N},\ i=1,2\rangle)}\mathbb{C}1,$$

where $\langle E(k\boldsymbol{b}_1),\ t^{k\boldsymbol{b}_1}\mid k\in\mathbb{N}\rangle.1=0$, $f(\boldsymbol{b}_1).1=c_11$, $h(\boldsymbol{b}_1).1=c_21$, $f(\boldsymbol{b}_2).1=c_31$ and $h(\boldsymbol{b}_2).1=c_41$ or

$$V\simeq\mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1})\otimes_{\mathcal{U}(\langle E(-k\boldsymbol{b}_1),\ t^{-k\boldsymbol{b}_1},\ f(\boldsymbol{b}_i),\ h(\boldsymbol{b}_i)|k\in\mathbb{N},\ i=1,2\rangle)}\mathbb{C}1,$$

where $\langle E(-k\boldsymbol{b}_1),\ t^{-k\boldsymbol{b}_1}\mid k\in\mathbb{N}\rangle.1=0$, $f(\boldsymbol{b}_1).1=c_11$, $h(\boldsymbol{b}_1).1=c_21$, $f(\boldsymbol{b}_2).1=c_31$ and $h(\boldsymbol{b}_2).1=c_41$.

(2) *If $c_1\neq 0$ and $c_2=0$, then*

$$V\simeq T_\rho(t_{\boldsymbol{b}_1})\otimes M^\varepsilon(c_1),$$

for some linear function $\rho: t_{\boldsymbol{b}_1}\to\mathbb{C}$ such that $T_\rho(t_{\boldsymbol{b}_1})=T_r$ for some $r\in\mathbb{Z}_+$, where $M^\varepsilon(c_1)$ is the irreducible $E_{\boldsymbol{b}_1}$-module of level c_1, $\varepsilon\in\{+,-\}$.

(3) *If $c_1=0$ and $c_2\neq 0$, then*

$$V\simeq\mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1})\otimes_{\mathcal{U}(\langle E(k\boldsymbol{b}_1),\ t^{k\boldsymbol{b}_1},\ f(\boldsymbol{b}_i),\ h(\boldsymbol{b}_i)|k\in\mathbb{N},\ i=1,2\rangle)}\mathbb{C}1,$$

where $\langle E(k\boldsymbol{b}_1),\ t^{k\boldsymbol{b}_1}\mid k\in\mathbb{N}\rangle.1=0$, $f(\boldsymbol{b}_1).1=0$, $h(\boldsymbol{b}_1).1=c_21$, $f(\boldsymbol{b}_2).1=c_31$ and $h(\boldsymbol{b}_2).1=c_41$ or

$$V\simeq\mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1})\otimes_{\mathcal{U}(\langle E(-k\boldsymbol{b}_1),\ t^{-k\boldsymbol{b}_1},\ f(\boldsymbol{b}_i),\ h(\boldsymbol{b}_i)|k\in\mathbb{N},\ i=1,2\rangle)}\mathbb{C}1,$$

where $\langle E(-k\boldsymbol{b}_1),\ t^{-k\boldsymbol{b}_1}\mid k\in\mathbb{N}\rangle.1=0$, $f(\boldsymbol{b}_1).1=0$, $h(\boldsymbol{b}_1).1=c_21$, $f(\boldsymbol{b}_2).1=c_31$ and $h(\boldsymbol{b}_2).1=c_41$.

Proof. (1) If $c_1\neq 0$ and $c_2\neq 0$, by Theorem 2.4 we know that there exists some $0\neq v_0\in V_{i_0}$ for some $i_0\in\mathbb{Z}$ such that $E(k\boldsymbol{b}_1).v_0=0$ for any $k\in\mathbb{N}$ or $-k\in\mathbb{N}$. Without loss of generality, we assume $k\in\mathbb{N}$; then $\mathcal{U}(\langle E(-k\boldsymbol{b}_1)\mid k\in\mathbb{N}\rangle)v_0$ is an irreducible $E_{\boldsymbol{b}_1}$-module. Let

$$W:=\mathcal{U}(\langle t^{k\boldsymbol{b}_1},\ E(l\boldsymbol{b}_1),\ f(\boldsymbol{b}_1),\ h(\boldsymbol{b}_1)\mid k\in\mathbb{N},\ l\in\mathbb{Z}\setminus\{0\}\rangle)v_0\subseteq V.$$

Note that W as a $\mathbb{Z}$-graded $\langle t^{k\boldsymbol{b}_1},\ E(-k\boldsymbol{b}_1),\ h(\boldsymbol{b}_1)\mid k\in\mathbb{N}\rangle$-module is completely reducible. Then we have that

$$W=\Big(\bigoplus_{i\in I}\Big(\bigoplus_{m_i\in X_i}V^+_{i,m_i}\Big)\Big)\oplus\Big(\bigoplus_{j\in J}\Big(\bigoplus_{n_j\in Y_j}V^-_{j,n_j}\Big)\Big),$$

where

$$V^+_{i,m_i}=\mathcal{U}(\langle t^{k\boldsymbol{b}_1},\ E(-k\boldsymbol{b}_1),\ h(\boldsymbol{b}_1)\mid k\in\mathbb{N}\rangle)v_{i,m_i}\simeq M^+(c_2),$$

for some $0 \neq v_{i,m_i} \in V_i \cap W$ with $t^{k\boldsymbol{b}_1}.v_{i,m_i} = 0$ for all $k \in \mathbb{N}$, $i \in I$, $m_i \in X_i$, and

$$V^-_{j,n_j} = \mathcal{U}(\langle t^{k\boldsymbol{b}_1},\ E(-k\boldsymbol{b}_1),\ h(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle)u_{j,n_j} \simeq M^-(c_2),$$

where $0 \neq u_{j,n_j} \in V_j \cap W$ with $E(-k\boldsymbol{b}_1).u_{j,n_j} = 0$ for all $k \in \mathbb{N}$, $j \in J$, $n_j \in Y_j$, $I, J, X_i, Y_j \subseteq \mathbb{Z}$. Note that I has an upper bound, J has a lower bound and all X_i, Y_j are finite sets since $\dim V_n < \infty$ for all $n \in \mathbb{Z}$. Assume $J \neq \varnothing$; then there exists some nonzero vector $w_0 \in W \cap V_i$ such that $E(-k\boldsymbol{b}_1).w_0 = 0$ for all $k \in \mathbb{N}$ and some $i \in \mathbb{Z}$. Consider $W_0 = \mathcal{U}(E_{\boldsymbol{b}_1})w_0 \subseteq W$, then

$$W_0 = \mathcal{U}(\langle E(l\boldsymbol{b}_1) \mid l \in \mathbb{N}\rangle).w_0$$

and W_0 is a free $\mathcal{U}(\langle E(l\boldsymbol{b}_1) \mid l \in \mathbb{N}\rangle)$-module. On the other hand, since $w_0 \in W$, there exists $k \in \mathbb{N}$ such that $E(k\boldsymbol{b}_1).w_0 = 0$, which is a contradiction. Thus $J = \varnothing$ and $W = \bigoplus_{i\in I}\big(\bigoplus_{m_i \in X_i} V^+_{i,m_i}\big)$. Since I has an upper bound, there exists $0 \neq u_0 \in W \cap V_{i_1}$ for some $i_1 \in \mathbb{Z}$ such that $E(k\boldsymbol{b}_1).u_0 = t^{k\boldsymbol{b}_1}.u_0 = 0$ for all $k \in \mathbb{N}$. This shows that

$$V \simeq \mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1}) \otimes_{\mathcal{U}(\langle E(k\boldsymbol{b}_1),\ t^{k\boldsymbol{b}_1},\ f(\boldsymbol{b}_i),\ h(\boldsymbol{b}_i) \mid k\in\mathbb{N},\ i=1,2\rangle)} \mathbb{C}u_0.$$

Another case is similar.

(2) If $c_1 \neq 0$ and $c_2 = 0$, we can write

$$\mathcal{H}_{\boldsymbol{b}_1} = t_{\boldsymbol{b}_1} \oplus E_{\boldsymbol{b}_1} \oplus \mathbb{C}f(\boldsymbol{b}_2) \oplus \mathbb{C}h(\boldsymbol{b}_1) \oplus \mathbb{C}h(\boldsymbol{b}_2).$$

From Theorem 2.4, Proposition 2.6 and [Li 2004, Lemma 2.7], this result follows.

(3) If $c_1 = 0$ and $c_2 \neq 0$, by Theorem 2.4, V is completely reducible when we view V as a module of the two subalgebras $\langle t^{-k\boldsymbol{b}_1},\ E(k\boldsymbol{b}_1),\ h(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle$ and $\langle t^{k\boldsymbol{b}_1},\ E(-k\boldsymbol{b}_1),\ h(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle$. We write

$$V = \Big(\bigoplus_{i\in I}\Big(\bigoplus_{m_i\in X_i} V^+_{i,m_i}\Big)\Big) \oplus \Big(\bigoplus_{j\in J}\Big(\bigoplus_{n_j\in Y_j} V^-_{j,n_j}\Big)\Big)$$

when it is viewed as the module of the Lie algebra $\langle t^{-k\boldsymbol{b}_1},\ E(k\boldsymbol{b}_1),\ h(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle$, where $I, J, X_i, Y_j \subseteq \mathbb{Z}$,

$$V^+_{i,m_i} = \mathcal{U}(\langle t^{-k\boldsymbol{b}_1},\ E(k\boldsymbol{b}_1),\ h(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle)v_{i,m_i} \simeq M^+(c_2)$$

with $E(k\boldsymbol{b}_1).v_{i,m_i} = 0$ for all $k \in \mathbb{N}$, $i \in I$, $m_i \in X_i$, $0 \neq v_{i,m_i} \in V_i$ and

$$V^-_{j,n_j} = \mathcal{U}(\langle t^{-k\boldsymbol{b}_1},\ E(k\boldsymbol{b}_1),\ h(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle)u_{j,n_j} \simeq M^-(c_2)$$

with $t^{-k\boldsymbol{b}_1}.u_{j,n_j} = 0$ for all $k \in \mathbb{N}$, $j \in J$, $n_j \in Y_j$ and $0 \neq u_{j,n_j} \in V_j$. Similarly, we write

$$V = \Big(\bigoplus_{i\in I'}\Big(\bigoplus_{p_i\in X'_i} W^+_{i,p_i}\Big)\Big) \oplus \Big(\bigoplus_{j\in J'}\Big(\bigoplus_{q_j\in Y'_j} W^-_{j,q_j}\Big)\Big)$$

when it is viewed as the module of the Lie algebra $\langle t^{k\boldsymbol{b}_1},\ E(-k\boldsymbol{b}_1),\ h(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle$.

Note that both I and I' have upper bounds, J and J' have lower bounds and all X_i, Y_j, X'_i, Y'_j are finite sets as dim $V_n < \infty$ for all $n \in \mathbb{Z}$. If $I = \varnothing$, similar to the proof in (1), we get

$$V \simeq \mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1}) \otimes_{\mathcal{U}(\langle E(k\boldsymbol{b}_1),\ t^{k\boldsymbol{b}_1},\ f(\boldsymbol{b}_i),\ h(\boldsymbol{b}_i) | k\in\mathbb{N},\ i=1,2\rangle)} \mathbb{C}1,$$

where $\langle E(k\boldsymbol{b}_1),\ t^{k\boldsymbol{b}_1} \mid k \in \mathbb{N}\rangle.1 = 0$, $f(\boldsymbol{b}_1).1 = 0$, $h(\boldsymbol{b}_1).1 = c_2 1$, $f(\boldsymbol{b}_2).1 = c_3 1$ and $h(\boldsymbol{b}_2).1 = c_4 1$. Now suppose $I \neq \varnothing$. We can choose $0 \neq v_0 \in V^+_{i_0, m_{i_0}}$ for some $i_0 \in I$, $m_{i_0} \in X_{i_0}$ such that $E(k\boldsymbol{b}_1).v_0 = 0$ for all $k \in \mathbb{N}$. Then we have $v_0 = w_1 + w_2$, where

$$w_1 \in \Big(\bigoplus_{i\in I'}\Big(\bigoplus_{p_i\in X'_i} W^+_{i,p_i}\Big)\Big) \cap V_{i_0} \quad \text{and} \quad w_2 \in \Big(\bigoplus_{j\in J'}\Big(\bigoplus_{q_j\in Y'_j} W^-_{j,q_j}\Big)\Big) \cap V_{i_0}.$$

If $w_1 \neq 0$, we can choose large enough $k_0 \in \mathbb{N}$ such that $E(-k_0\boldsymbol{b}_1).w_2 = 0$ since J' has a lower bound. Since $\bigoplus_{i\in I'}\big(\bigoplus_{p_i\in X'_i} W^+_{i,p_i}\big)$ is a free $\langle E(-k\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle$-module, we have $0 \neq E(-k_0\boldsymbol{b}_1).w_1 = E(-k_0\boldsymbol{b}_1).v_0 \in \bigoplus_{i\in I'}\big(\bigoplus_{p_i\in X'_i} W^+_{i,p_i}\big)$ and $E(k\boldsymbol{b}_1).E(-k_0\boldsymbol{b}_1).v_0 = E(-k_0\boldsymbol{b}_1).E(k\boldsymbol{b}_1).v_0 = 0$ for all $k \in \mathbb{N}$. Now we claim that there exists $0 \neq v \in V_i$ for some $i \in \mathbb{Z}$ such that $t^{k\boldsymbol{b}_1}.v = E(k\boldsymbol{b}_1).v = 0$ for all $k \in \mathbb{N}$. In fact, if $t^{k\boldsymbol{b}_1}.(E(-k_0\boldsymbol{b}_1).v_0) = 0$ for all $k \in \mathbb{N}$, this is done by setting $v = E(-k_0\boldsymbol{b}_1).v_0$. If there exists $k_1 \in \mathbb{N}$ such that $t^{k_1\boldsymbol{b}_1}.(E(-k_0\boldsymbol{b}_1).v_0) \neq 0$, set $v_1 = t^{k_1\boldsymbol{b}_1}.(E(-k_0\boldsymbol{b}_1).v_0)$. We can repeat this process, and, since I' has an upper bound, we know that it will terminate after finitely many steps. This implies that

$$V \simeq \mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1}) \otimes_{\mathcal{U}(\langle E(k\boldsymbol{b}_1),\ t^{k\boldsymbol{b}_1},\ f(\boldsymbol{b}_i),\ h(\boldsymbol{b}_i) | k\in\mathbb{N},\ i=1,2\rangle)} \mathbb{C}1,$$

where $\langle E(k\boldsymbol{b}_1),\ t^{k\boldsymbol{b}_1} \mid k \in \mathbb{N}\rangle.1 = 0$, $f(\boldsymbol{b}_1).1 = 0$, $h(\boldsymbol{b}_1).1 = c_2 1$, $f(\boldsymbol{b}_2).1 = c_3 1$ and $h(\boldsymbol{b}_2).1 = c_4 1$. If $w_1 = 0$, i.e., $v_0 = w_2$, we know that there exists some $0 \neq u \in V_{j_0}$ for some $j_0 \in \mathbb{Z}$ such that $E(k\boldsymbol{b}_1).u = 0$ for $k \in \mathbb{Z} \setminus \{0\}$. In fact, if there exists $n_1 \in \mathbb{N}$ such that $E(-n_1\boldsymbol{b}_1)v_0 \neq 0$, set $u_1 = E(-n_1\boldsymbol{b}_1)v_0$. We also have $E(k\boldsymbol{b}_1).u_1 = 0$ for all $k \in \mathbb{N}$ since $f(\boldsymbol{b}_1).V = 0$. We can repeat this process, and, since J' has a lower bound, we know that it will terminate after finitely many steps. Then,

$$V \simeq \mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1}) \otimes_{\mathcal{U}(\langle E(m\boldsymbol{b}_1),\ f(\boldsymbol{b}_i),\ h(\boldsymbol{b}_i) | m\in\mathbb{Z}\setminus\{0\},\ i=1,2\rangle)} \mathbb{C}1,$$

where $E(m\boldsymbol{b}_1).1 = 0$ for all $m \in \mathbb{Z}\setminus\{0\}$, $f(\boldsymbol{b}_1).1 = 0$, $h(\boldsymbol{b}_1).1 = c_2 1$, $f(\boldsymbol{b}_2).1 = c_3 1$ and $h(\boldsymbol{b}_2).1 = c_4 1$. This contradicts the condition that dim $V_i < \infty$ for all $i \in \mathbb{Z}$. Then the conclusion follows. $\square$

Fix a $\mathbb{Z}$-basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ of Γ, $\boldsymbol{b}_1 = b_{11}\boldsymbol{e}_1 + b_{12}\boldsymbol{e}_2$, and $\lambda_1, \lambda_2 \in \mathbb{C}$. Any $\mathbb{Z}$-graded $\mathcal{H}_{\boldsymbol{b}_1}$-module $V = \oplus_{i\in\mathbb{Z}} V_i$ with fixed level can be extended to a weight module of $\widetilde{L}_0$ by defining

$$d_1 v_j = (\lambda_1 + j b_{11}) v_j, \quad d_2 v_j = (\lambda_2 + j b_{12}) v_j,$$

for $v_j \in V_j$, $j \in \mathbb{Z}$. One can easily see that the vector space V is a weight $\widetilde{L}_0$-module and $\mathcal{P}(V) \subseteq (\lambda_1, \lambda_2) + \mathbb{Z}\boldsymbol{b}_1$. For the $\mathbb{Z}$-graded irreducible $\mathcal{H}_{\boldsymbol{b}_1}$-modules given in Propositions 2.6 and 2.8, we let

$$V^+(\boldsymbol{c}) = \mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1}) \otimes_{\mathcal{U}(\langle E(k\boldsymbol{b}_1),\ t^{k\boldsymbol{b}_1},\ K_i | k\in\mathbb{N},\ i=1,2,3,4\rangle)} \mathbb{C}1,$$
$$V^-(\boldsymbol{c}) = \mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1}) \otimes_{\mathcal{U}(\langle E(-k\boldsymbol{b}_1),\ t^{-k\boldsymbol{b}_1},\ K_i | k\in\mathbb{N},\ i=1,2,3,4\rangle)} \mathbb{C}1,$$

$M^\varepsilon_\rho(\boldsymbol{c}) = T_\rho(t_{\boldsymbol{b}_1}) \otimes M^\varepsilon(c_1)$ and $T_\rho(\mathcal{H}_{\boldsymbol{b}_1})(\boldsymbol{c}) = T_\rho(\mathcal{H}_{\boldsymbol{b}_1})$. We can extend these modules to weight $\widetilde{L}_0$-modules by the above method, and then we denote the corresponding $\widetilde{L}_0$-module by $V^+(\boldsymbol{c}, \boldsymbol{\lambda})$, $V^-(\boldsymbol{c}, \boldsymbol{\lambda})$, $M^\varepsilon_\rho(\boldsymbol{c}, \boldsymbol{\lambda})$ and $T_\rho(\mathcal{H}_{\boldsymbol{b}_1})(\boldsymbol{c}, \boldsymbol{\lambda})$ respectively, where $\boldsymbol{c} = (c_1, c_2, c_3, c_4)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$, and $f(\boldsymbol{b}_1)$, $h(\boldsymbol{b}_1)$, $f(\boldsymbol{b}_2)$ and $h(\boldsymbol{b}_2)$ act as the scalars $c_1, c_2, c_3, c_4 \in \mathbb{C}$, respectively.

With this notation, the following results can be obtained from Propositions 2.6 and 2.8.

Corollary 2.9. *Let $V = \bigoplus_{i\in\mathbb{Z}} V_i$ be any irreducible weight module of $\widetilde{L}_0$ with* $\dim V_i < \infty$ *for all $i \in \mathbb{Z}$, and $f(\boldsymbol{b}_1).v = c_1 v$, $h(\boldsymbol{b}_1).v = c_2 v$, $f(\boldsymbol{b}_2).v = c_3 v$ and $h(\boldsymbol{b}_2).v = c_4 v$ for $v \in V$, where $V_i := V_{(\lambda_1,\lambda_2)+i\boldsymbol{b}_1}$ for some fixed $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{C}^2$.*

(1) *If $(c_1, c_2) \neq 0$, then $V \simeq V^\varepsilon(\boldsymbol{c}, \boldsymbol{\lambda})$ or $V \simeq M^\varepsilon_\rho(\boldsymbol{c}, \boldsymbol{\lambda})$ for some linear function $\rho : t_{\boldsymbol{b}_1} \to \mathbb{C}$ with $T_\rho(t_{\boldsymbol{b}_1}) = T_r$ for some $r \in \mathbb{Z}_+$ and $\varepsilon \in \{+, -\}$.*

(2) *If $(c_1, c_2) = 0$, then $V \simeq T_\rho(\mathcal{H}_{\boldsymbol{b}_1})(\boldsymbol{c}, \boldsymbol{\lambda})$ for some $\rho \in \mathcal{E}_{\boldsymbol{b}_1}$.*

The following lemma give the characterization of the irreducible weight modules of $\widetilde{L}$ with finite-dimensional weight spaces.

Lemma 2.10. *Let $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ be a $\mathbb{Z}$-basis of Γ. V is an irreducible weight module of $\widetilde{L}$ with finite-dimensional weight spaces and $f(\boldsymbol{b}_1), h(\boldsymbol{b}_1), f(\boldsymbol{b}_2), h(\boldsymbol{b}_2)$ act on V as scalars c_1, c_2, c_3, c_4 respectively. If there exist $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $V_{(\lambda_1,\lambda_2)} \neq 0$ and $\mathcal{P}(V) \cap ((\lambda_1, \lambda_2) + \mathbb{Z}\boldsymbol{b}_1 + \mathbb{N}\boldsymbol{b}_2) = \varnothing$, we have*:

(1) *If $c_1 = c_2 = 0$, $V \simeq M(\boldsymbol{b}_1, \boldsymbol{b}_2, T_\rho(\mathcal{H}_{\boldsymbol{b}_1})(\boldsymbol{c}, \boldsymbol{\lambda}))$ for some $\rho \in \mathcal{E}_{\boldsymbol{b}_1}$.*

(2) *If $c_1 \neq 0, c_2 = 0$, $V \simeq M(\boldsymbol{b}_1, \boldsymbol{b}_2, M^\varepsilon_\rho(\boldsymbol{c}, \boldsymbol{\lambda}))$ for some linear function $\rho : t_{\boldsymbol{b}_1} \to \mathbb{C}$ satisfying $T_\rho(t_{\boldsymbol{b}_1}) = T_r$ for some $r \in \mathbb{Z}_+$.*

(3) *If $c_2 \neq 0$, $V \simeq M(\boldsymbol{b}_1, \boldsymbol{b}_2, V^\varepsilon(\boldsymbol{c}, \boldsymbol{\lambda}))$, where $\varepsilon \in \{+, -\}$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$, $\boldsymbol{c} = (c_1, c_2, c_3, c_4)$.*

Proof. Let $W = \oplus_{i\in\mathbb{Z}} V_{(\lambda_1,\lambda_2)+i\boldsymbol{b}_1}$. Since $\mathcal{P}(V) \cap ((\lambda_1, \lambda_2) + \mathbb{Z}\boldsymbol{b}_1 + \mathbb{N}\boldsymbol{b}_2) = \varnothing$, we see that W is an irreducible $\widetilde{L}_0$ weight module and $\widetilde{L}_+ W = 0$. Thus by the construction of $\widetilde{M}(\boldsymbol{b}_1, \boldsymbol{b}_2, W)$ and the PBW theorem, there exists an epimorphism φ from $\widetilde{M}(\boldsymbol{b}_1, \boldsymbol{b}_2, W)$ to V such that $\varphi|_W = \mathrm{id}_W$. Therefore, the lemma follows from Corollary 2.9 and the irreducibility of V. □

Using the same notation as in Lemma 2.10, the following lemma shows that the cases (2) and (3) of Lemma 2.10 don't occur.

Lemma 2.11. *For any $\mathbb{Z}$-basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ of Γ, neither $M(\boldsymbol{b}_1, \boldsymbol{b}_2, M_\rho^\varepsilon(\boldsymbol{c}, \boldsymbol{\lambda}))$ nor $M(\boldsymbol{b}_1, \boldsymbol{b}_2, V^\varepsilon(\boldsymbol{c}, \boldsymbol{\lambda}))$ is a Harish-Chandra module.*

Proof. Using the notation in Lemma 2.10, for the case that $f(\boldsymbol{b}_1)$ acts as the scalar $c_1 \neq 0$, the lemma follows from Lemma 2.6 in [Lin and Tan 2006]. So we only need to consider the case where $c_1 = 0$, $c_2 \neq 0$. Without loss of generality, we may assume that there exists a weight vector $0 \neq v_0 \in V^\varepsilon(\boldsymbol{c}, \boldsymbol{\lambda})$ such that $E(k\boldsymbol{b}_1)v_0 = t^{k\boldsymbol{b}_1}v_0 = 0$ and $E(-k\boldsymbol{b}_1)v_0 \neq 0$ and $t^{-k\boldsymbol{b}_1}v_0 \neq 0$ for all $k \in \mathbb{N}$ (see Proposition 2.8(3)). For any $n \in \mathbb{N}$, we can choose $k_j \in \mathbb{Z}$, $1 \le j \le n$ with $0 < k_1 < k_2 < \cdots < k_n$ such that $h(-k_j\boldsymbol{b}_1 + \boldsymbol{b}_2)v_0 \neq 0$ for $1 \le j \le n$. We claim that

$$\{E(k_j\boldsymbol{b}_1 - \boldsymbol{b}_2)t^{-k_j\boldsymbol{b}_1}v_0 \mid 1 \le j \le n\} \subseteq M(\boldsymbol{b}_1, \boldsymbol{b}_2, V^\varepsilon(\boldsymbol{c}, \boldsymbol{\lambda}))_{(\lambda_1,\lambda_2)-\boldsymbol{b}_2}$$

is a set of linear independent vectors, therefore the conclusion follows. In fact, if $\sum_{j=1}^n a_j E(k_j\boldsymbol{b}_1 - \boldsymbol{b}_2)t^{-k_j\boldsymbol{b}_1}v_0 = 0$, then

$$\begin{aligned} 0 &= t^{-k_1\boldsymbol{b}_1+\boldsymbol{b}_2} \sum_{j=1}^n a_j E(k_j\boldsymbol{b}_1 - \boldsymbol{b}_2)t^{-k_j\boldsymbol{b}_1}v_0 \\ &= a_1 h(-k_1\boldsymbol{b}_1 + \boldsymbol{b}_2)t^{-k_1\boldsymbol{b}_1}v_0 + \sum_{j=2}^n a_j \det\begin{pmatrix} k_j\boldsymbol{b}_1 - \boldsymbol{b}_2 \\ -k_1\boldsymbol{b}_1 + \boldsymbol{b}_2 \end{pmatrix} t^{(k_j-k_1)\boldsymbol{b}_1} t^{-k_j\boldsymbol{b}_1}v_0. \end{aligned}$$

Since $h(-k_1\boldsymbol{b}_1 + \boldsymbol{b}_2) \neq 0$, this implies $a_1 = 0$. Similarly, we can prove $a_2 = a_3 = \cdots = a_n = 0$. Therefore the conclusion follows. □

From Lemmas 2.10 and 2.11, we have:

Proposition 2.12. *Let $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ be a $\mathbb{Z}$-basis of Γ and let V be a Harish-Chandra module of $\widetilde{L}$. If there exist $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $\mathcal{P}(V) \cap ((\lambda_1, \lambda_2) + \mathbb{Z}\boldsymbol{b}_1 + \mathbb{N}\boldsymbol{b}_2) = \varnothing$ and $V_{(\lambda_1,\lambda_2)} \neq 0$, then $V \simeq M(\boldsymbol{b}_1, \boldsymbol{b}_2, T_\rho(\mathcal{H}_{\boldsymbol{b}_1})(\boldsymbol{c}, \boldsymbol{\lambda}))$ for some $\rho \in \mathcal{E}_{\boldsymbol{b}_1}$.*

Remark 2.13. If V is a Harish-Chandra module V of $\widetilde{L}$ satisfying the conditions in Proposition 2.12, then $\boldsymbol{c} = (0, 0, c_3, c_4)$, i.e., $f(\boldsymbol{b}_1)$, $h(\boldsymbol{b}_1)$ act trivially.

As one of the main results in this paper, we prove that a Harish-Chandra module of $\widetilde{L}$ is either a generalized highest weight module or a uniformly bounded module. First, we need the following lemma.

Lemma 2.14. *An irreducible weight $\widetilde{L}$-module V is a generalized highest weight module if there is a $\mathbb{Z}$-basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ of Γ and a weight vector $v \neq 0$ such that $E(\boldsymbol{b}_1)v = E(\boldsymbol{b}_2)v = t^{\boldsymbol{b}_1}v = 0$.*

Proof. Since there is a weight vector $v \neq 0$, such that $E(\boldsymbol{b}_1)v = E(\boldsymbol{b}_2)v = t^{\boldsymbol{b}_1}v = 0$, by induction, we have

$$E(\boldsymbol{m})v = t^{\boldsymbol{m}}v = 0$$

for $\boldsymbol{m} \in \mathbb{N}\boldsymbol{b}_1 + \mathbb{N}\boldsymbol{b}_2$. Therefore, we have

$$E(\boldsymbol{m})v = t^{\boldsymbol{m}}v = 0$$

for $\boldsymbol{m} \in \mathbb{Z}_+\boldsymbol{b}'_1 + \mathbb{Z}_+\boldsymbol{b}'_2$, where $\boldsymbol{b}'_1 = 2\boldsymbol{b}_1 + \boldsymbol{b}_2$, $\boldsymbol{b}'_2 = 3\boldsymbol{b}_1 + \boldsymbol{b}_2 \in \Gamma$. It is obvious that $\{\boldsymbol{b}'_1, \boldsymbol{b}'_2\}$ is a $\mathbb{Z}$-basis of Γ. Now we get that V is a generalized highest weight module since V is irreducible. □

Proposition 2.15. *A Harish-Chandra module V of $\widetilde{L}$ is either a generalized highest weight module or a uniformly bounded module.*

Proof. Let $(\lambda_1, \lambda_2) \in \mathcal{P}(V)$ and let $V_{\boldsymbol{b}} := V_{(\lambda_1,\lambda_2)+\boldsymbol{b}}$ for $\boldsymbol{b} \in \Gamma$. Then $V = \bigoplus_{\boldsymbol{b}\in\Gamma} V_{\boldsymbol{b}}$. If V is not a generalized highest weight module, for $\boldsymbol{m} = (m_1, m_2) \in \Gamma$, consider the linear maps $E(-m_1\boldsymbol{e}_1 + \boldsymbol{e}_2) : V_{(m_1,m_2)} \to V_{(0,m_2+1)}$, $E((1-m_1)\boldsymbol{e}_1 + \boldsymbol{e}_2) : V_{(m_1,m_2)} \to V_{(1,m_2+1)}$ and $t^{-m_1\boldsymbol{e}_1+\boldsymbol{e}_2} : V_{(m_1,m_2)} \to V_{(0,m_2+1)}$. By Lemma 2.14, we have

$$\ker E(-m_1\boldsymbol{e}_1 + \boldsymbol{e}_2) \cap \ker E((1-m_1)\boldsymbol{e}_1 + \boldsymbol{e}_2) \cap \ker t^{-m_1\boldsymbol{e}_1+\boldsymbol{e}_2} = 0.$$

This shows that

$$\dim V_{(m_1,m_2)} \le 2 \dim V_{(0,m_2+1)} + \dim V_{(1,m_2+1)}.$$

Now we consider the linear maps $E(-\boldsymbol{e}_1 + (1-m_2)\boldsymbol{e}_2) : V_{(0,m_2+1)} \to V_{(-1,2)}$, $E(-\boldsymbol{e}_1 - m_2\boldsymbol{e}_2) : V_{(0,m_2+1)} \to V_{(-1,1)}$ and $t^{-\boldsymbol{e}_1-m_2\boldsymbol{e}_2} : V_{(0,m_2+1)} \to V_{(-1,1)}$. By the same reasoning, we get

$$\dim V_{(0,m_2+1)} \le 2 \dim V_{(-1,1)} + \dim V_{(-1,2)}.$$

Similarly, we have

$$\dim V_{(1,m_2+1)} \le 2 \dim V_{(0,1)} + \dim V_{(0,2)}.$$

Thus, V is a uniformly bounded module. □

3. Nonzero level Harish-Chandra modules of $\widetilde{L}$

In this section, we study the nonzero level Harish-Chandra module V of $\widetilde{L}$, which satisfies $K_i.v = c_i v$ for $v \in V$, $\mathbf{0} \neq (c_1, c_2, c_3, c_4) \in \mathbb{C}^4$.

We denote

$$[p, q] = \{x \mid x \in \mathbb{Z}, p \le x \le q\}$$

and similarly for $(-\infty, p]$, $[q, \infty)$ and $(-\infty, +\infty)$. First, we have:

Theorem 3.1. *If V is a nonzero level Harish-Chandra module of $\widetilde{L}$, then V is a generalized highest weight module.*

Proof. Without loss of generality, we may assume the center element K_1 acts as $0 \neq c_1 \in \mathbb{C}$. Let $(\lambda_1, \lambda_2) \in \mathcal{P}(V)$. Set $W_0 := \oplus_{i\in\mathbb{Z}} V_{(\lambda_1,\lambda_2)+i\boldsymbol{e}_1} \neq 0$. From Theorem 2.4, we see that W_0 as a $\langle E(k\boldsymbol{e}_1), t^{-k\boldsymbol{e}_1}, K_1 \mid k \in \mathbb{N}\rangle$-module is completely reducible. Also from Theorem 2.4, we know that V is not a uniformly bounded module. Thus V is a generalized highest weight module. □

Corollary 3.2. *If V is a uniformly bounded Harish-Chandra module of $\widetilde{L}$, then $K_i.v = 0$ for $v \in V$, $i = 1, 2, 3, 4$.*

We assume that $V = \bigoplus_{\boldsymbol{n}\in\Gamma} V_{\boldsymbol{\lambda}+\boldsymbol{n}}$ is a nontrivial generalized highest weight Harish-Chandra $\widetilde{L}$-module with generalized highest weight $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ corresponding to a $\mathbb{Z}$-basis $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ of Γ. Without loss of generality, we assume $\boldsymbol{\lambda} = \boldsymbol{0}$.

Lemma 3.3. (1) *For any $v \in V$, there exists $p > 0$ such that $E(i\boldsymbol{b}_1 + j\boldsymbol{b}_2)v = t^{i\boldsymbol{b}_1+j\boldsymbol{b}_2}v = 0$ for all $(i, j) \geq (p, p)$.*

(2) *For any $0 \neq v \in V$, $(m_1, m_2) > \boldsymbol{0}$, we have $E(-m_1\boldsymbol{b}_1 - m_2\boldsymbol{b}_2)v \neq 0$.*

(3) *If $\boldsymbol{b} := i_1\boldsymbol{b}_1 + i_2\boldsymbol{b}_2 \in \mathcal{P}(V)$, then for any $(m_1, m_2) > \boldsymbol{0}$, there exists $m \geq 0$ such that $\{x \in \mathbb{Z} \mid \boldsymbol{b} + x\boldsymbol{a} \in \mathcal{P}(V)\} = (-\infty, m]$, where $\boldsymbol{a} = m_1\boldsymbol{b}_1 + m_2\boldsymbol{b}_2$.*

Proof. Let v_0 be the generalized highest weight vector of V corresponding to the $\mathbb{Z}$-basis B.

(1) Since $v = uv_0$ for some $u \in \mathcal{U}(\widetilde{L})$, u can be written as a linear combination of elements of the form $u_{m,n} = t^{i_1\boldsymbol{b}_1+j_1\boldsymbol{b}_2} \cdots t^{i_m\boldsymbol{b}_1+j_m\boldsymbol{b}_2} E(k_1\boldsymbol{b}_1+l_1\boldsymbol{b}_2) \cdots E(k_n\boldsymbol{b}_1+l_n\boldsymbol{b}_2)$. Without loss of generality, we may assume $u = u_{m,n}$. Take

$$p_1 = -\Sigma_{i_s<0} i_s - \Sigma_{k_t<0} k_t + 1, \quad p_2 = -\Sigma_{j_s<0} j_s - \Sigma_{l_t<0} l_t + 1.$$

Fix $m \in \mathbb{Z}_+$. By induction on n, one gets $E(i\boldsymbol{b}_1 + j\boldsymbol{b}_2)v = t^{i\boldsymbol{b}_1+j\boldsymbol{b}_2}v = 0$ for all $(i, j) \geq (p_1, p_2)$. Take $p = \max\{p_1, p_2\}$. Then the result follows.

(2) Suppose $E(-m_1\boldsymbol{b}_1 - m_2\boldsymbol{b}_2)v = 0$ for some $0 \neq v \in V$ and some $(m_1, m_2) > \boldsymbol{0}$. Let p be as in the proof of (1). Then one gets

$$E(-m_1\boldsymbol{b}_1 - m_2\boldsymbol{b}_2)v = E(\boldsymbol{b}_1 + p(m_1\boldsymbol{b}_1 + m_2\boldsymbol{b}_2))v = E(\boldsymbol{b}_2 + p(m_1\boldsymbol{b}_1 + m_2\boldsymbol{b}_2))v = 0$$

$$t^{\boldsymbol{b}_1+p(m_1\boldsymbol{b}_1+m_2\boldsymbol{b}_2)}v = t^{\boldsymbol{b}_2+p(m_1\boldsymbol{b}_1+m_2\boldsymbol{b}_2)}v = 0.$$

Note that the Lie algebra L is generated by these elements, so we have $Lv = 0$, which contradicts V being a nontrivial irreducible module.

(3) See Lemma 3.2 in [Lin and Tan 2006]. □

The following lemma follows from Lemma 3.3 and the proof is given in [Lin and Tan 2006].

Lemma 3.4. *There exists a $\mathbb{Z}$-basis $B' = \{\boldsymbol{b}'_1, \boldsymbol{b}'_2\}$ of Γ such that:*

(1) *V is a generalized highest weight module with generalized highest weight $\boldsymbol{0}$ corresponding to the $\mathbb{Z}$-basis B'.*

(2) $\{\mathbb{Z}_+\boldsymbol{b}'_1 + \mathbb{Z}_+\boldsymbol{b}'_2\} \cap \mathcal{P}(V) = \boldsymbol{0}$.

(3) $\{-\mathbb{Z}_+\boldsymbol{b}'_1 - \mathbb{Z}_+\boldsymbol{b}'_2\} \subseteq \mathcal{P}(V)$.

(4) *If $i_1\boldsymbol{b}'_1 + i_2\boldsymbol{b}'_2 \notin \mathcal{P}(V)$, then $k_1\boldsymbol{b}'_1 + k_2\boldsymbol{b}'_2 \notin \mathcal{P}(V)$ for $(k_1, k_2) \geq (i_1, i_2)$.*

(5) *If* $i_1\boldsymbol{b}'_1+i_2\boldsymbol{b}'_2\in\mathcal{P}(V)$, *then* $k_1\boldsymbol{b}'_1+k_2\boldsymbol{b}'_2\in\mathcal{P}(V)$ *for* $(k_1,k_2)\le(i_1,i_2)$.

(6) *For any* $\mathbf{0}\ne(k_1,k_2)\ge\mathbf{0}$, $(i_1,i_2)\in\Gamma$, *we have*

$$\{x\in\mathbb{Z}\mid i_1\boldsymbol{b}'_1+i_2\boldsymbol{b}'_2+x(k_1\boldsymbol{b}'_1+k_2\boldsymbol{b}'_2)\in\mathcal{P}(V)\}=(-\infty,m]$$

for some $m\in\mathbb{Z}$.

From now on, we assume that V is a nontrivial generalized highest weight Harish-Chandra module with generalized highest weight $\mathbf{0}$ corresponding to the $\mathbb{Z}$-basis $B=\{\boldsymbol{b}_1,\boldsymbol{b}_2\}$ and B satisfies the properties in Lemma 3.4. To characterize the nontrivial generalized highest weight Harish-Chandra module V of $\widetilde{L}$, we need the following lemmas due to [Lin and Tan 2006] (cf. [Lu and Zhao 2006; Su 2003]).

Lemma 3.5. *If there exist an integer* $s>0$ *and* $(i_1,i_2),(k_1,k_2)\in\Gamma$ *such that* k_1, k_2 *are coprime, and*

$$\{i_1\boldsymbol{b}_1+i_2\boldsymbol{b}_2+x_1s\boldsymbol{b}_1+x_2s\boldsymbol{b}_2\mid(x_1,x_2)\in\Gamma,k_1x_1+k_2x_2=0\}\cap\mathcal{P}(V)=\varnothing,$$

then $V\simeq M(\boldsymbol{b}'_1,\boldsymbol{b}'_2,T_\rho(\mathcal{H}_{\boldsymbol{b}'_1})(\boldsymbol{c},\boldsymbol{\lambda}))$ *for some* $\mathbb{Z}$*-basis* $\{\boldsymbol{b}'_1,\boldsymbol{b}'_2\}$ *of* Γ *and some* $\rho\in\mathcal{E}_{\boldsymbol{b}'_1}$, *where* $f(\boldsymbol{b}'_1),h(\boldsymbol{b}'_1),f(\boldsymbol{b}'_2),h(\boldsymbol{b}'_2)$ *act as scalars* $c_1=0,c_2=0,c_3,c_4$ *respectively and* $\boldsymbol{c}=(c_1,c_2,c_3,c_4)$.

Lemma 3.6. *If there exist* (i_1,i_2), $(0,0)\ne(k_1,k_2)\in\Gamma$ *such that*

$$\{i_1\boldsymbol{b}_1+i_2\boldsymbol{b}_2+x(k_1\boldsymbol{b}_1+k_2\boldsymbol{b}_2)\mid x\in\mathbb{Z}\}\cap\mathcal{P}(V)=\varnothing,$$

then $V\simeq M(\boldsymbol{b}'_1,\boldsymbol{b}'_2,T_\rho(\mathcal{H}_{\boldsymbol{b}'_1})(\boldsymbol{c},\boldsymbol{\lambda}))$ *for some* $\mathbb{Z}$*-basis* $\{\boldsymbol{b}'_1,\boldsymbol{b}'_2\}$ *of* Γ *and some* $\rho\in\mathcal{E}_{\boldsymbol{b}'_1}$.

Lemma 3.7. *If there exist* $(0,0)\ne(m,n)\in\Gamma$, $(i,j)\in\Gamma$, $p,q\in\mathbb{Z}$ *such that*

$$\{x\in\mathbb{Z}\mid i\boldsymbol{b}_1+j\boldsymbol{b}_2+x(m\boldsymbol{b}_1+n\boldsymbol{b}_2)\in\mathcal{P}(V)\}\supseteq(-\infty,p]\cup[q,\infty),$$

then $V\simeq M(\boldsymbol{b}'_1,\boldsymbol{b}'_2,T_\rho(\mathcal{H}_{\boldsymbol{b}'_1})(\boldsymbol{c},\boldsymbol{\lambda}))$ *for some* $\mathbb{Z}$*-basis* $\{\boldsymbol{b}'_1,\boldsymbol{b}'_2\}$ *of* Γ *and some* $\rho\in\mathcal{E}_{\boldsymbol{b}'_1}$.

Lemma 3.8. *If there exist* $(i,j),(k,l)\in\Gamma$ *and* $x_1,x_2,x_3\in\mathbb{Z}$ *with* $x_1<x_2<x_3$ *such that*

$$i\boldsymbol{b}_1+j\boldsymbol{b}_2+x_1(k\boldsymbol{b}_1+l\boldsymbol{b}_2)\notin\mathcal{P}(V),\tag{3-1}$$

$$i\boldsymbol{b}_1+j\boldsymbol{b}_2+x_2(k\boldsymbol{b}_1+l\boldsymbol{b}_2)\in\mathcal{P}(V),\tag{3-2}$$

$$i\boldsymbol{b}_1+j\boldsymbol{b}_2+x_3(k\boldsymbol{b}_1+l\boldsymbol{b}_2)\notin\mathcal{P}(V),\tag{3-3}$$

then $V\simeq M(\boldsymbol{b}'_1,\boldsymbol{b}'_2,T_\rho(\mathcal{H}_{\boldsymbol{b}'_1})(\boldsymbol{c},\boldsymbol{\lambda}))$ *for some* $\mathbb{Z}$*-basis* $\{\boldsymbol{b}'_1,\boldsymbol{b}'_2\}$ *of* Γ *and some* $\rho\in\mathcal{E}_{\boldsymbol{b}'_1}$.

Proof. Without loss of generality, we may assume k, l are coprime. Thus we can choose $(m, n) \in \Gamma$ with $kn - lm = 1$. Let $\boldsymbol{b}'_1 = k\boldsymbol{b}_1 + l\boldsymbol{b}_2$ and let $\boldsymbol{b}'_2 = m\boldsymbol{b}_1 + n\boldsymbol{b}_2$; then $\{\boldsymbol{b}'_1, \boldsymbol{b}'_2\}$ is a $\mathbb{Z}$-basis of Γ. Replacing x_2 by the largest $x < x_3$ with $i\boldsymbol{b}_1 + j\boldsymbol{b}_2 + x(k\boldsymbol{b}_1 + l\boldsymbol{b}_2) \in \mathcal{P}(V)$, then replacing x_3 by $x_2 + 1$ and (i, j) by $(i, j) + x_2(k, l)$, we can assume

$$x_1 < x_2 = 0 < x_3 = 1. \tag{3-4}$$

We may assume that there exists $s \in \mathbb{Z}$ with

$$i\boldsymbol{b}_1 + j\boldsymbol{b}_2 + \boldsymbol{b}'_2 + s\boldsymbol{b}'_1 = (i + m)\boldsymbol{b}_1 + (j + n)\boldsymbol{b}_2 + s(k\boldsymbol{b}_1 + l\boldsymbol{b}_2) \notin \mathcal{P}(V). \tag{3-5}$$

Otherwise, by Lemma 3.7, we are done. Thus by (3-1)–(3-5), we have

$$\begin{aligned}
E(x_1\boldsymbol{b}'_1)v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} &= E(x_1(k\boldsymbol{b}_1 + l\boldsymbol{b}_2))v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} = 0,\\
t^{x_1\boldsymbol{b}'_1}v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} &= t^{x_1(k\boldsymbol{b}_1+l\boldsymbol{b}_2)}v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} = 0,\\
E(\boldsymbol{b}'_1)v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} &= E(k\boldsymbol{b}_1 + l\boldsymbol{b}_2)v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} = 0,\\
t^{\boldsymbol{b}'_1}v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} &= t^{k\boldsymbol{b}_1+l\boldsymbol{b}_2}v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} = 0,\\
E(\boldsymbol{b}'_2 + s\boldsymbol{b}'_1)v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} &= 0,\\
t^{\boldsymbol{b}'_2+s\boldsymbol{b}'_1}v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} &= 0,
\end{aligned}$$

where $0 \neq v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} \in V_{i\boldsymbol{b}_1+j\boldsymbol{b}_2}$. Note that since $x_1 < 0$, we have that

$$\{E(p\boldsymbol{b}'_1 + q\boldsymbol{b}'_2),\ t^{p\boldsymbol{b}'_1+q\boldsymbol{b}'_2} \mid p \in \mathbb{Z}, q \in \mathbb{N}\}$$

belongs to the subalgebra generated by

$$\{E(x_1\boldsymbol{b}'_1),\ E(\boldsymbol{b}'_1),\ E(\boldsymbol{b}'_2 + s\boldsymbol{b}'_1),\ t^{x_1\boldsymbol{b}'_1},\ t^{\boldsymbol{b}'_1},\ t^{\boldsymbol{b}'_2+s\boldsymbol{b}'_1}\}.$$

We obtain $E(p\boldsymbol{b}'_1 + q\boldsymbol{b}'_2)v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} = t^{p\boldsymbol{b}'_1+q\boldsymbol{b}'_2}v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} = 0$ for $p \in \mathbb{Z}, q \in \mathbb{N}$. Since $\{\boldsymbol{b}'_1, \boldsymbol{b}'_2\}$ is a $\mathbb{Z}$-basis of Γ and V is irreducible, from the PBW theorem, we have $V = \mathcal{U}(\widetilde{L})v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2}$ and

$$\{i\boldsymbol{b}_1 + j\boldsymbol{b}_2 + \mathbb{Z}\boldsymbol{b}'_1 + \mathbb{N}\boldsymbol{b}'_2\} \cap \mathcal{P}(V) = \varnothing.$$

Thus the result follows from Proposition 2.12. □

Lemma 3.9. *If there exist $i > 0$, $j < 0$ and $0 \neq v_{\boldsymbol{a}} \in V_{\boldsymbol{a}}$, $\boldsymbol{a} \in \mathbb{C}^2$, $\boldsymbol{b} = m\boldsymbol{b}_1 + n\boldsymbol{b}_2 \neq \boldsymbol{0}$, such that $E(i\boldsymbol{b})v_{\boldsymbol{a}} = 0$, $E(j\boldsymbol{b})v_{\boldsymbol{a}} = 0$, then $V \simeq M(\boldsymbol{b}'_1, \boldsymbol{b}'_2, T_\rho(\mathcal{H}_{\boldsymbol{b}'_1})(\boldsymbol{c}, \boldsymbol{\lambda}))$ for some $\mathbb{Z}$-basis $\{\boldsymbol{b}'_1, \boldsymbol{b}'_2\}$ of Γ and some $\rho \in \mathcal{E}_{\boldsymbol{b}'_1}$.*

Proof. Write $(m, n) = s(m', n')$ with m', n' coprime and $s \geq 1$. Then we can choose $(m_2, n_2) \in \Gamma$ with $n'm_2 - m'n_2 = 1$. Let $\boldsymbol{b}'_1 = m'\boldsymbol{b}_1 + n'\boldsymbol{b}_2$, $\boldsymbol{b}'_2 = m_2\boldsymbol{b}_1 + n_2\boldsymbol{b}_2$; then $\{\boldsymbol{b}'_1, \boldsymbol{b}'_2\}$ is a $\mathbb{Z}$-basis of Γ. Fix any $0 \neq q \in \mathbb{Z}$.

Case 1: If $\{\boldsymbol{a} + q\boldsymbol{b}'_2 + x\boldsymbol{b}'_1 \mid x \in \mathbb{Z}\} \cap \mathcal{P}(V) = \varnothing$, then, by Lemma 3.7, we are done.

Case 2: If there exist integers $x_1 < x_2 < x_3$ with $\boldsymbol{a} + q\boldsymbol{b}'_2 + x_2\boldsymbol{b}'_1 \in \mathcal{P}(V)$ and $\boldsymbol{a} + q\boldsymbol{b}'_2 + x_i\boldsymbol{b}'_1 \notin \mathcal{P}(V),\ i = 1, 3$, then, by Lemma 3.9, we are done.

Case 3: If there exist $m, n \in \mathbb{Z}$ with

$$(-\infty, m] \cup [n, \infty) \subseteq \{x \in \mathbb{Z} \mid \boldsymbol{a} + q\boldsymbol{b}'_2 + x\boldsymbol{b}'_1 \in \mathcal{P}(V)\},$$

then, by Lemma 3.8, we are done.

Now if the above three cases don't occur, we know that there exists some integer p_q such that $A_q := \{x \in \mathbb{Z} \mid \boldsymbol{a} + q\boldsymbol{b}'_2 + x\boldsymbol{b}'_1 \in \mathcal{P}(V)\} = (-\infty, p_q]$ or $[p_q, \infty)$. We first assume $A_q = (-\infty, p_q]$. Thus

$$E(q\boldsymbol{b}'_2 - jxs\boldsymbol{b}'_1 \pm \boldsymbol{b}'_1)v_{\boldsymbol{a}} = t^{q\boldsymbol{b}'_2 - jxs\boldsymbol{b}'_1 \pm \boldsymbol{b}'_1} v_{\boldsymbol{a}} = 0$$

for a sufficiently large integer $x > 0$. Since $E(j\boldsymbol{b})v_{\boldsymbol{a}} = E(js\boldsymbol{b}'_1)v_{\boldsymbol{a}} = 0$, we can obtain

$$E(q\boldsymbol{b}'_2 \pm \boldsymbol{b}'_1)v_{\boldsymbol{a}} = t^{q\boldsymbol{b}'_2 \pm \boldsymbol{b}'_1} v_{\boldsymbol{a}} = 0.$$

If $A_q = [p_q, \infty)$, by a similar argument, we can also obtain

$$E(q\boldsymbol{b}'_2 \pm \boldsymbol{b}'_1)v_{\boldsymbol{a}} = t^{q\boldsymbol{b}'_2 \pm \boldsymbol{b}'_1} v_{\boldsymbol{a}} = 0.$$

This implies

$$E(\pm(\boldsymbol{b}'_1 + \boldsymbol{b}'_2))v_{\boldsymbol{a}} = E(\pm(\boldsymbol{b}'_1 + 2\boldsymbol{b}'_2))v_{\boldsymbol{a}} = 0, \quad t^{\pm(\boldsymbol{b}'_1 + \boldsymbol{b}'_2)} v_{\boldsymbol{a}} = t^{\pm(\boldsymbol{b}'_1 + 2\boldsymbol{b}'_2)} v_{\boldsymbol{a}} = 0.$$

Since $\{\boldsymbol{b}'_1 + \boldsymbol{b}'_2, \boldsymbol{b}'_1 + 2\boldsymbol{b}'_2\}$ is a $\mathbb{Z}$-basis of Γ, L is generated by

$$\{E(\pm(\boldsymbol{b}'_1 + \boldsymbol{b}'_2)), E(\pm(\boldsymbol{b}'_1 + 2\boldsymbol{b}'_2)), t^{\pm(\boldsymbol{b}'_1 + \boldsymbol{b}'_2)}, t^{\pm(\boldsymbol{b}'_1 + 2\boldsymbol{b}'_2)}\}.$$

Thus $V = \mathcal{U}(\widetilde{L})v_{\boldsymbol{a}}$ is a trivial module, which is a contradiction. □

The following proposition gives the characterization of the nontrivial generalized highest weight Harish-Chandra module.

Proposition 3.10. *If V is a nontrivial generalized highest weight Harish-Chandra $\widetilde{L}$-module with generalized highest weight $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ corresponding to a $\mathbb{Z}$-basis $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ of Γ, then $V \simeq M(\boldsymbol{b}'_1, \boldsymbol{b}'_2, T_\rho(\mathcal{H}_{\boldsymbol{b}'_1})(\boldsymbol{c}, \lambda))$ for some $\mathbb{Z}$-basis $\{\boldsymbol{b}'_1, \boldsymbol{b}'_2\}$ of Γ and some $\rho \in \mathcal{E}_{\boldsymbol{b}'_1}$.*

Proof. From Lemma 3.9 and the proof of Proposition 3.9 in [Lin and Tan 2006], we can obtain our result. □

Together with Theorem 3.1 and Proposition 3.10, we have:

Theorem 3.11. *If V is a nonzero level Harish-Chandra $\widetilde{L}$-module, then*

$$V \simeq M(\boldsymbol{b}'_1, \boldsymbol{b}'_2, T_\rho(\mathcal{H}_{\boldsymbol{b}'_1})(\boldsymbol{c}, \boldsymbol{\lambda}))$$

for some $\mathbb{Z}$-basis $\{\boldsymbol{b}'_1, \boldsymbol{b}'_2\}$ of Γ and some $\rho \in \mathcal{E}_{\boldsymbol{b}'_1}$, $\boldsymbol{\lambda} \in \mathbb{C}^2$.

4. Classification of generalized highest weight Harish-Chandra $\widetilde{L}$-modules

In this section, we will provide the classification of generalized highest weight Harish-Chandra modules of $\widetilde{L}$ by using the highest weight modules of L. From Proposition 3.10, we only need to find in which case the irreducible generalized highest weight $\widetilde{L}$-module $M(\boldsymbol{b}_1, \boldsymbol{b}_2, T_\rho(\mathcal{H}_{\boldsymbol{b}_1})(\boldsymbol{c}, \boldsymbol{\lambda}))$ is a Harish-Chandra module.

First we give a triangular decomposition of L and construct a class of $\mathbb{Z}$-graded irreducible highest weight modules of L. Recall that

$$\widetilde{L}_i = \langle E(m\boldsymbol{b}_1 + i\boldsymbol{b}_2),\ t^{m\boldsymbol{b}_1+i\boldsymbol{b}_2} \mid m \in \mathbb{Z}\rangle, \quad i \in \mathbb{Z} \setminus \{0\},$$

and

$$\widetilde{L}_+ = \bigoplus_{i>0} \widetilde{L}_i, \quad \widetilde{L}_- = \bigoplus_{i<0} \widetilde{L}_i.$$

Then $L = \widetilde{L}_+ \oplus \mathcal{H}_{\boldsymbol{b}_1} \oplus \widetilde{L}_-$.

Remark 4.1. In this section, we call a L-module V a highest weight module (corresponding to the $\mathbb{Z}$-basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$) if there exists a nonzero $v \in V$ such that $V = \mathcal{U}(L)v$ and $\widetilde{L}_+.v = 0$.

For any linear function $\rho : \mathcal{H}_{\boldsymbol{b}_1} \to \mathbb{C}$ with $\rho(f(\boldsymbol{b}_1)) = \rho(h(\boldsymbol{b}_1)) = 0$, we define a one-dimensional $(\mathcal{H}_{\boldsymbol{b}_1} \oplus \widetilde{L}_+)$-module $\mathbb{C}v_0$ as follows:

$$\widetilde{L}_+.v_0 = 0, \quad x.v_0 = \rho(x)v_0, \quad x \in \mathcal{H}_{\boldsymbol{b}_1}. \tag{4-1}$$

Then we have an induced L-module

$$\overline{V}(\rho) = \operatorname{Ind}^{L}_{\mathcal{H}_{\boldsymbol{b}_1}\oplus\widetilde{L}_+} \mathbb{C}v_0 = \mathcal{U}(L) \otimes_{\mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1}\oplus\widetilde{L}_+)} \mathbb{C}v_0. \tag{4-2}$$

We see that $\overline{V}(\rho)$ is a $\mathbb{Z}$-graded module. It is clear that $\overline{V}(\rho)$ has a unique maximal $\mathbb{Z}$-graded submodule $J(\rho)$. Then we obtain a $\mathbb{Z}$-graded irreducible highest weight L-module

$$V(\rho) = \overline{V}(\rho)/J(\rho) = \oplus_{i\in\mathbb{Z}} V(\rho)_i,$$

where, for $i \in \mathbb{Z}$,

$$V(\rho)_i = \operatorname{Span}_{\mathbb{C}}\Big\{ E(i_1\boldsymbol{b}_1 + j_1\boldsymbol{b}_2)E(i_2\boldsymbol{b}_1 + j_2\boldsymbol{b}_2)\cdots E(i_m\boldsymbol{b}_1 + j_m\boldsymbol{b}_2)t^{s_1\boldsymbol{b}_1+k_1\boldsymbol{b}_2}\cdots t^{s_n\boldsymbol{b}_1+k_n\boldsymbol{b}_2}v_0 \ \Big|\ m, n \in \mathbb{Z}_+,\ \sum_{p=1}^{m} j_p + \sum_{p=1}^{n} k_p = i \Big\}.$$

We call $V(\rho)_i$ for $i \in \mathbb{Z}$ the weight space of the L-module $V(\rho)$. If $\dim V(\rho)_i < \infty$, we say that the weight space $V(\rho)_i$ is finite-dimensional.

For later use, we need a conception of an exp-polynomial function. Recall from [Billig and Zhao 2004] that a function $f:\mathbb{Z}\to\mathbb{C}$ is said to be *exp-polynomial* if it can be written as a finite sum

$$f(n)=\sum c_{m,a}n^m a^n,$$

for some $c_{m,a}\in\mathbb{C}$, $m\in\mathbb{Z}_+$ and $0\neq a\in\mathbb{C}$.

The following lemma is due to [Wilson 2008].

Lemma 4.2. *A function $f:\mathbb{Z}\to\mathbb{C}$ is an* exp-polynomial function *if and only if there exist $a_0,\dots,a_n\in\mathbb{C}$ with $a_0a_n\neq 0$, such that*

$$\sum_{i=0}^{n}a_i f(m+i)=0,$$

for all $m\in\mathbb{Z}$.

Remark 4.3. In general, for fixed $a_0,\dots,a_n\in\mathbb{C}$ with $a_0a_n\neq 0$, the exp-polynomial function f satisfying $\sum_{i=0}^{n}a_i f(m+i)=0$, for all $m\in\mathbb{Z}$, is not unique.

Then we have the following result.

Proposition 4.4. *Suppose the linear function $\rho:\mathcal{H}_{\boldsymbol{b}_1}\to\mathbb{C}$ such that $\rho(f(\boldsymbol{b}_1))=\rho(h(\boldsymbol{b}_1))=0$. Then the $\mathbb{Z}$-graded L-module $V(\rho)$ has finite-dimensional weight spaces if and only if there exist two exp-polynomials $g_j:\mathbb{Z}\to\mathbb{C}$ satisfying $\sum_{i=0}^{n}a_i g_j(k+i)=0$ for $j=1,2,\ k\in\mathbb{Z},\ a_i\in\mathbb{C},\ a_0a_n\neq 0$ and*

$$g_1(0)=\det\begin{pmatrix}\boldsymbol{b}_1\\ \boldsymbol{b}_2\end{pmatrix}\rho(f(\boldsymbol{b}_2)),\qquad g_2(0)=\det\begin{pmatrix}\boldsymbol{b}_1\\ \boldsymbol{b}_2\end{pmatrix}\rho(h(\boldsymbol{b}_2)),$$

$$g_1(m)=\rho(mE(m\boldsymbol{b}_1)),\qquad g_2(m)=\rho(mt^{m\boldsymbol{b}_1}),\ m\in\mathbb{Z}\setminus\{0\}.$$

Proof. First, we define two linear maps $\phi_1,\phi_2:\mathbb{C}[t_1^{\pm1},t_2^{\pm1}]\to L$ by

$$\phi_i(t_1^{m_1}t_2^{m_2})=\begin{cases}E(m_1\boldsymbol{b}_1+m_2\boldsymbol{b}_2) & \text{if } i=1,\\ t^{m_1\boldsymbol{b}_1+m_2\boldsymbol{b}_2} & \text{if } i=2.\end{cases}$$

If $V(\rho)$ has finite-dimensional weight spaces, since $\dim V(\rho)_{-1}<\infty$ and

$$\phi_1(t_1^i t_2^{-1})v_0\in V(\rho)_{-1}$$

for all $i\in\mathbb{Z}$, there exists $k\in\mathbb{Z}$ and a nonzero polynomial $P(t_1)=\sum_{i=0}^{n}a_i t_1^i\in\mathbb{C}[t_1]$ with $a_0a_n\neq 0$ such that

$$\phi_1(t_2^{-1}t_1^k P(t_1))v_0=0.$$

Applying $\phi_i(t_1^s t_2)$ for any $s\in\mathbb{Z},\ i=1,2$ to the above equation respectively, we get

$$\left(\sum_{i=0}^{n}a_i(k+s+i)E((k+s+i)\boldsymbol{b}_1)+\det\begin{pmatrix}\boldsymbol{b}_1\\ \boldsymbol{b}_2\end{pmatrix}a_{-k-s}f(\boldsymbol{b}_2)\right).v_0=0,\tag{4-3}$$

and

$$\left(\sum_{i=0}^{n} a_i(k+s+i)t^{(k+s+i)\boldsymbol{b}_1} + \det\begin{pmatrix}\boldsymbol{b}_1\\ \boldsymbol{b}_2\end{pmatrix} a_{-k-s}h(\boldsymbol{b}_2)\right).v_0 = 0, \tag{4-4}$$

where $a_{-k-s} = 0$ if $-k-s \notin \{0, 1, \ldots, n\}$. Set $g_1 : \mathbb{Z} \to \mathbb{C}$ such that $g_1(0) = \det\binom{\boldsymbol{b}_1}{\boldsymbol{b}_2}\rho(f(\boldsymbol{b}_2))$ and $g_1(m) = \rho(mE(m\boldsymbol{b}_1))$ for $m \in \mathbb{Z}\setminus\{0\}$. Then (4-3) becomes

$$\sum_{i=0}^{n} a_i g_1(m+i) = 0, \quad \text{for all } m \in \mathbb{Z}.$$

Set $g_2 : \mathbb{Z} \to \mathbb{C}$ such that $g_2(0) = \det\binom{\boldsymbol{b}_1}{\boldsymbol{b}_2}\rho(h(\boldsymbol{b}_2))$ and $g_2(m) = \rho(mt^{m\boldsymbol{b}_1})$ for $m \in \mathbb{Z}\setminus\{0\}$. Then (4-4) becomes

$$\sum_{i=0}^{n} a_i g_2(m+i) = 0, \quad \text{for all } m \in \mathbb{Z}.$$

From Lemma 4.2, we have that g_1, g_2 are exp-polynomial functions.

Conversely, we use Theorem 1.7 in [Billig and Zhao 2004] to prove that the $\mathbb{Z}$-graded L-module $V(\rho)$ has finite-dimensional weight spaces, i.e., $\dim V(\rho)_i < \infty$ for all $i \in \mathbb{Z}$. Since $\mathbb{C}v_0$ is a one-dimensional $\mathcal{H}_{\boldsymbol{b}_1}$-module with exp-polynomial action, i.e., $\mathcal{H}_{\boldsymbol{b}_1}$ acts on $\mathbb{C}v_0$ through two exp-polynomials g_1, g_2, and $\widetilde{L}_+.v_0 = 0$, from Theorem 1.7 in [Billig and Zhao 2004], we just need to prove that L is $\mathbb{Z}$-extragraded (see Definition 1.4 of the same work). Set the index sets $X_i = \{(1, i), (2, i)\}$ for $i \in \mathbb{Z}\setminus\{0\}$ and $X_0 = \{(i, 0) \mid i = 1, 2, \ldots, 6\}$. For $i \in \mathbb{Z}\setminus\{0\}$, let

$$\mathcal{L}^i_{\boldsymbol{k}}(j) = \begin{cases} E(i\mathbf{b_2} + j\mathbf{b_1}), & \boldsymbol{k} = (1, i),\ j \in \mathbb{Z},\\ t^{i\mathbf{b_2}+j\mathbf{b_1}}, & \boldsymbol{k} = (2, i),\ j \in \mathbb{Z},\end{cases}$$

and

$$\mathcal{L}^0_{\boldsymbol{k}}(j) = \begin{cases} jE(j\mathbf{b_1}), & \boldsymbol{k} = (1, 0),\ j \neq 0,\\ jt^{j\mathbf{b_1}}, & \boldsymbol{k} = (2, 0),\ j \neq 0,\\ K_i, & \boldsymbol{k} = (i+2, 0),\ i = 1, 2, 3, 4,\ j = 0.\end{cases}$$

Claim 1: L is a $\mathbb{Z}$-graded exp-polynomial Lie algebra (see Definition 1.2 in [Billig and Zhao 2004]).

In fact, let $L = \bigoplus_{j\in\mathbb{Z}} L(j)$, where $L(j) = \langle \mathcal{L}^i_{\boldsymbol{k}}(j) \mid i \in \mathbb{Z},\ \boldsymbol{k} \in X_i\rangle$. $[L(j_1), L(j_2)] \subseteq L(j_1 + j_2)$ for $j_1, j_2 \in \mathbb{Z}$. Thus L is $\mathbb{Z}$-graded, and it is straightforward to check that L is an exp-polynomial Lie algebra with the distinguished spanning set $\{\mathcal{L}^i_{\boldsymbol{k}}(j) \mid \boldsymbol{k} \in X_i,\ i, j \in \mathbb{Z}\}$.

Claim 2: The $\mathbb{Z}$-graded exp-polynomial Lie algebra L is $\mathbb{Z}$-extragraded.

In fact, let $L = \bigoplus_{i\in\mathbb{Z}} L^{(i)}$, where $L^{(i)} = \langle \mathcal{L}^i_{\boldsymbol{k}}(j) \mid j \in \mathbb{Z},\ \boldsymbol{k} \in X_i\rangle$. $[L^{(i_1)}, L^{(i_2)}] \subseteq L^{(i_1+i_2)}$ for $i_1, i_2 \in \mathbb{Z}$, i.e., L has another $\mathbb{Z}$-gradation. □

For linear function $\rho : \mathcal{H}_{\boldsymbol{b}_1} \to \mathbb{C}$ with $\rho(f(\boldsymbol{b}_1)) = \rho(h(\boldsymbol{b}_1)) = 0$, we say that ρ is an *exp-polynomial function over* $\mathcal{H}_{\boldsymbol{b}_1}$ if there exist $a_0, \dots, a_n \in \mathbb{C},\ a_0 a_n \neq 0$ and two exp-polynomials g_0, g_1 given by

$$g_1(0) = \det \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix} \rho(f(\boldsymbol{b}_2)), \quad g_2(0) = \det \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix} \rho(h(\boldsymbol{b}_2)),$$

and

$$g_1(m) = \rho(mE(m\boldsymbol{b}_1)), \quad g_2(m) = \rho(mt^{m\boldsymbol{b}_1})$$

for all $m \in \mathbb{Z} \setminus \{0\}$ such that $\sum_{i=0}^{n} a_i g_j(k+i) = 0$ for $j = 1, 2,\ k \in \mathbb{Z}$.

Let

$$\begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix}^{-1} = \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} \in \mathrm{GL}_{2\times 2}(\mathbb{Z}).$$

Set $\widetilde{d_1} = p_1 d_1 + p_2 d_2,\ \widetilde{d_2} = q_1 d_1 + q_2 d_2$. Then we have

$$[\widetilde{d_i}, E(m_1\boldsymbol{b}_1 + m_2\boldsymbol{b}_2)] = m_i E(m\boldsymbol{b}_1 + n\boldsymbol{b}_2), \quad [\widetilde{d_i}, t^{m_1\boldsymbol{b}_1 + m_2\boldsymbol{b}_2}] = m_i t^{m_1\boldsymbol{b}_1 + m_2\boldsymbol{b}_2}$$

for $i = 1, 2,\ m_1, m_2 \in \mathbb{Z}$.

Now we construct a class of $\mathbb{Z}^2$-graded irreducible generalized highest weight $\widetilde{L}$-modules by using the above $\mathbb{Z}$-graded highest weight L-module $V(\rho)$. For any linear function $\rho : \mathcal{H}_{\boldsymbol{b}_1} \to \mathbb{C}$ with $\rho(f(\boldsymbol{b}_1)) = \rho(h(\boldsymbol{b}_1)) = 0$, we set

$$\widehat{V}(\rho) = V(\rho) \otimes \mathbb{C}[t^{\pm 1}]$$

and define the actions of $\widetilde{L}$ on $\widehat{V}(\rho)$ as

$$\begin{aligned} E(m\boldsymbol{b}_1 + n\boldsymbol{b}_2).(v \otimes t^k) &= (E(m\boldsymbol{b}_1 + n\boldsymbol{b}_2).v) \otimes t^{m+k}, \\ t^{m\boldsymbol{b}_1 + n\boldsymbol{b}_2}.(v \otimes t^k) &= (t^{m\boldsymbol{b}_1 + n\boldsymbol{b}_2}.v) \otimes t^{m+k}, \\ \widetilde{d_1}.(v \otimes t^k) &= k(v \otimes t^k), \\ \widetilde{d_2}.(v \otimes t^k) &= j(v \otimes t^k), \\ K_i.(v \otimes t^k) &= (K_i.v) \otimes t^k \end{aligned}$$

for $(m, n) \in \mathbb{Z}^2 \setminus \{\boldsymbol{0}\},\ v \in V(\rho)_j,\ j \in \mathbb{Z},\ i = 1, 2, 3, 4$. It is clear that $\widehat{V}(\rho)$ is a $\mathbb{Z}^2$-graded $\widetilde{L}$-module, and

$$\widehat{V}(\rho) = \bigoplus_{m,n \in \mathbb{Z}} \widehat{V}(\rho)_{(m,n)},$$

where $\widehat{V}(\rho)_{(m,n)} = V(\rho)_m \otimes t^n$. We call $\widehat{V}(\rho)_{(m,n)}$, $m, n \in \mathbb{Z}$ weight spaces of the module $\widehat{V}(\rho)$ with respect to $\widetilde{d_1}, \widetilde{d_2}$.

Let $W(i)$ be the $\widetilde{L}$-submodule of $\widehat{V}(\rho)$ generated by $v_0 \otimes t^i,\ i \in \mathbb{Z}$, where v_0 is defined in (4-1).

Lemma 4.5. *Let $\rho \in \mathcal{E}_{\boldsymbol{b}_1}$ and $W(i)$ be a $\mathbb{Z}^2$-graded irreducible $\widetilde{L}$-submodule of $\widehat{V}(\rho)$.*

(1) *If $T_\rho(\mathcal{H}_{\boldsymbol{b}_1}) = T_0$, then $\widehat{V}(\rho) = \bigoplus_{i\in\mathbb{Z}} W(i)$.*

(2) *If $T_\rho(\mathcal{H}_{\boldsymbol{b}_1}) = T_r$ for some $r \in \mathbb{N}$, then $\widehat{V}(\rho) = \bigoplus_{i=0}^{r-1} W(i)$.*

Proof. We need to notice the following two facts. First, any nonzero $\widetilde{L}$-submodule of $\widehat{V}(\rho)$ contains $v_0 \otimes t^i$ for some $i \in \mathbb{Z}$. Second, the two $\widetilde{L}$-submodules $W(m) = W(n)$ if and only if $t^{m-n} \in T_r$, where $T_r = T_\rho(\mathcal{H}_{\boldsymbol{b}_1})$, $r \in \mathbb{Z}_+$. For (1), that $W(i)$ is an $\mathbb{Z}^2$-graded irreducible $\widetilde{L}$-module follows from $V(\rho)$ being an irreducible L-module. For (2), let M be a nonzero submodule of the $\widetilde{L}$-module $W(i)$; then $v_0 \otimes t^n \in M$ for some $n \in \mathbb{Z}$. Since $\mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1})(v_0 \otimes t^i) = v_0 \otimes (T_r \cdot t^i)$ and $v_0 \otimes t^n \in \mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1})(v_0 \otimes t^i)$, we have $t^n \in T_r \cdot t^i$. This implies that $v_0 \otimes t^i \in W(n) \subseteq M$, i.e., $W(i) \subseteq M$. Thus $M = W(i)$, which shows that $W(i)$ is irreducible. □

For $\rho \in \mathcal{E}_{\boldsymbol{b}_1}$, we know that there exists a unique maximal $\mathbb{Z}^2$-graded submodule $J(i)$ of $\widehat{V}(\rho)$ which insects $W(i)$ trivially by Lemma 4.5. Then we get the $\mathbb{Z}^2$-graded irreducible $\widetilde{L}$-module

$$\widehat{V}(\rho, i) = \widehat{V}(\rho)/J(i) \simeq W(i).$$

Remark 4.6. (1) From Lemma 3.3 in [Wilson 2008], we see $\rho \in \mathcal{E}_{\boldsymbol{b}_1}$ if ρ is an exp-polynomial function over $\mathcal{H}_{\boldsymbol{b}_1}$.

(2) For $\rho \in \mathcal{E}_{\boldsymbol{b}_1}$, $W(i) \simeq W(j)$ as an $\widetilde{L}$-module up to a shift of the action of $\widetilde{d}_1$, $i, j \in \mathbb{Z}$ from (2-4) and Lemma 4.5.

Lemma 4.7. (1) *For any linear function $\rho : \mathcal{H}_{\boldsymbol{b}_1} \to \mathbb{C}$ with $\rho(f(\boldsymbol{b}_1)) = \rho(h(\boldsymbol{b}_1)) = 0$, the $\widetilde{L}$-module $\widehat{V}(\rho)$ has finite-dimensional weight spaces if and only if L-module $V(\rho)$ has finite weight spaces.*

(2) *For $\rho \in \mathcal{E}_{\boldsymbol{b}_1}$, $M(\boldsymbol{b}_1, \boldsymbol{b}_2, T_\rho(\mathcal{H}_{\boldsymbol{b}_1})(\boldsymbol{c}, \boldsymbol{\lambda})) \simeq \widehat{V}(\rho, 0)$ as an $\widetilde{L}$-module up to scalar shifts of the actions of d_1, d_2.*

Proof. (1) Since $\widehat{V}(\rho)_{(m,n)} = V(\rho)_m \otimes t^n$, $m, n \in \mathbb{Z}$, the first assertion is obvious.

(2) For any $\widetilde{L}$-module W, it is clear that we can modify the actions of d_1 and d_2. In fact, let σ be the corresponding representation of this $\widetilde{L}$-module W. Set $\pi(x) = \sigma(x)$ for $x \in L$, and $\pi(d_i) = \sigma(d_i) + a_i \operatorname{Id}_W$ for some fixed $a_i \in \mathbb{C}$, $i = 1, 2$. Obviously, $\pi : \widetilde{L} \to \mathrm{gl}(W)$ is a representation of $\widetilde{L}$, i.e., one can define a new $\widetilde{L}$-module structure on W through shifting the actions of d_1, d_2. Note that $\mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1}).(v_0 \otimes 1) \simeq T_\rho(\mathcal{H}_{\boldsymbol{b}_1})$ for $\rho \in \mathcal{E}_{\boldsymbol{b}_1}$ and $\widetilde{L}_+.(\mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1}).(v_0 \otimes 1)) = 0$. Then the result follows from the irreducibility of $\widehat{V}(\rho, 0)$. □

By Lemma 4.7, together with Proposition 4.4, we obtain the main result in this section.

Theorem 4.8. *For $\rho \in \mathcal{E}_{\boldsymbol{b}_1}$, the irreducible generalized highest weight $\widetilde{L}$-module $M(\boldsymbol{b}_1, \boldsymbol{b}_2, T_\rho(\mathcal{H}_{\boldsymbol{b}_1})(\boldsymbol{c}, \boldsymbol{\lambda}))$ is a Harish-Chandra module if and only if ρ is an exp-polynomial function over $\mathcal{H}_{\boldsymbol{b}_1}$.*

Remark 4.9. If ρ is an exp-polynomial function over $\mathcal{H}_{\boldsymbol{b}_1}$, then the generalized highest weight Harish-Chandra $M(\boldsymbol{b}_1, \boldsymbol{b}_2, T_\rho(\mathcal{H}_{\boldsymbol{b}_1})(\boldsymbol{c}, \boldsymbol{\lambda}))$ is a one-dimensional trivial module if and only if $\rho = 0$, i.e., $T_\rho(\mathcal{H}_{\boldsymbol{b}_1}) = T_0$, $\boldsymbol{c} = \boldsymbol{0}$.

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WEIGHTED ESTIMATES FOR ROUGH SINGULAR INTEGRALS WITH APPLICATIONS TO ANGULAR INTEGRABILITY

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We study certain singular integral operators, as well as their corresponding truncated maximal operators, along polynomial curves. Assuming that the kernels of operators are rough not only on the unit sphere but also on the radial direction, we establish certain weighted estimates for these operators. As applications, we obtain that these operators are bounded on the mixed radial-angular spaces $L^p_{|x|}L^{\tilde{p}}_{\theta}(\mathbb{R}^n)$ and on the vector-valued mixed radial-angular spaces $L^p_{|x|}L^{\tilde{p}}_{\theta}(\mathbb{R}^n, \ell^{\tilde{p}})$. The bounds are independent of the coefficients of the polynomials in the definition of the operators. Our results we obtained improve theorems of Antonio Córdoba (2016) and Piero D'Ancona and Renato Lucà (2016).

1. Introduction

Let $\mathbb{R}^n$ be the Euclidean space of dimension n and S^{n-1} denote the unit sphere in $\mathbb{R}^n$ ($n \geq 2$) equipped with the normalized Lebesgue measure $d\sigma$. The mixed radial-angular spaces $L^p_{|x|}L^{\tilde{p}}_{\theta}(\mathbb{R}^n)$, $1 \leq p, \tilde{p} \leq \infty$, consist of all functions u satisfying $\|u\|_{L^p_{|x|}L^{\tilde{p}}_{\theta}(\mathbb{R}^n)} < \infty$, where

$$\|u\|_{L^p_{|x|}L^{\tilde{p}}_{\theta}(\mathbb{R}^n)} := \left(\int_0^\infty \|u(\rho\,\cdot)\|^p_{L^{\tilde{p}}(\mathrm{S}^{n-1})} \rho^{n-1}\, d\rho \right)^{1/p},$$
$$\|u\|_{L^\infty_{|x|}L^{\tilde{p}}_{\theta}(\mathbb{R}^n)} := \sup_{\rho>0} \|u(\rho\,\cdot)\|_{L^{\tilde{p}}(\mathrm{S}^{n-1})}.$$

The spaces $L^p_{|x|}L^{\tilde{p}}_{\theta}(\mathbb{R}^n)$ have the following easy properties:

(i) If $p = \tilde{p}$ and $1 \leq p \leq \infty$, then

$$\|u\|_{L^p_{|x|}L^{\tilde{p}}_{\theta}(\mathbb{R}^n)} = \|u\|_{L^p(\mathbb{R}^n)}. \tag{1-1}$$

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(ii) If u is a radial function on $\mathbb{R}^n$ and $1 \le p \le \infty$, $1 \le \tilde{p} \le \infty$, then

$$\|u\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)} \simeq \|u\|_{L^p(\mathbb{R}^n)}.$$

(iii) If $1 \le \tilde{p}_1 \le \tilde{p}_2 \le \infty$ and $1 \le p \le \infty$, then

$$\|u\|_{L^p_{|x|}L^{\tilde{p}_1}_\theta(\mathbb{R}^n)} \le C_{n,p,\tilde{p}_1,\tilde{p}_2}\|u\|_{L^p_{|x|}L^{\tilde{p}_2}_\theta(\mathbb{R}^n)}.$$

Here the notation $A \simeq B$ means that there are positive constants C, C' such that $A \le CB$ and $B \le C'A$.

One might think that the mixed radial-angular space $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)$ is merely a formal extension of the Lebesgue space L^p, but recently it has been successfully used in studying Strichartz estimates and dispersive equations (see [Cho and Ozawa 2009; Cacciafesta and D'Ancona 2013; Fang and Wang 2011; Lucà 2014; Machihara et al. 2005; Ozawa and Rogers 2013; Sterbenz 2005; Tao 2000]). Also, it plays active roles in the theory of singular integral operator. Córdoba [2016] proved that the singular integral

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y)\frac{\Omega(y)}{|y|^n}\,dy, \tag{1-2}$$

where Ω is a homogeneous function of degree zero, is bounded on $L^p_{|x|}L^2_\theta(\mathbb{R}^n)$ for all $1 < p < \infty$, provided that $\Omega \in \mathcal{C}^1(\mathrm{S}^{n-1})$ and satisfies

$$\int_{\mathrm{S}^{n-1}} \Omega(y)\,d\sigma(y) = 0. \tag{1-3}$$

D'Ancona and Lucà [2016] then used the argument in Córdoba's Theorem 2.1 to extend the above result:

Theorem A. *Let $\Omega \in \mathcal{C}^1(\mathrm{S}^{n-1})$ satisfy* (1-3) *and $1 < p < \infty$, $1 < \tilde{p} < \infty$. Then*

$$\|T_\Omega f\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)} \le C_{\Omega,p,\tilde{p}}\|f\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)}.$$

Recently, Cacciafesta and Lucà [2016, Theorem 1.1] extended Theorem A to the weighted setting.

On the other hand, it is a long-time interesting topic to study the rough singular integral operators. Precisely, by assuming that $\Omega \in L\log L(\mathrm{S}^{n-1})$, Calderón and Zygmund [1956] proved that T_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Their proof is based on the rotation method to reduce the operator T_Ω to the directional Hilbert transform so that the well-known Riesz theorem can be applied. Fefferman [1979] considered another singular integral

$$T_{h,\Omega} f(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y)\frac{h(|y|)\Omega(y)}{|y|^n}\,dy, \tag{1-4}$$

where Ω is given as in (1-2) and $h(\cdot) \in L^\infty(\mathbb{R}_+)$ with $\mathbb{R}_+ := (0, \infty)$. Clearly, the operator T_Ω corresponds to the special case of $T_{h,\Omega}$ for $h(t) \equiv 1$. Fefferman discovered that the Calderón–Zygmund rotation method is no longer available if the operator $T_{h,\Omega}$ is also rough in the radial direction, for instance $h \in L^\infty$, so that new methods must be addressed. In his fundamental work on $T_{h,\Omega}$, Fefferman [1979] proved that $T_{h,\Omega}$ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ if $\Omega \in \mathrm{Lip}_\alpha(\mathrm{S}^{n-1})$ for some $\alpha > 0$ and $h \in L^\infty(\mathbb{R}_+)$. Afterwards, Namazi [1986] improved Fefferman's result by assuming $\Omega \in L^q(\mathrm{S}^{n-1})$ for $q > 1$ instead of $\Omega \in \mathrm{Lip}_\alpha(\mathrm{S}^{n-1})$. Subsequently, Duoandikoetxea and Rubio de Francia [1986] used the Littlewood–Paley theory to improve the above results to the case $\Omega \in L^q(\mathrm{S}^{n-1})$ for any $q > 1$ and $h \in \Delta_2(\mathbb{R}_+)$. Here $\Delta_\gamma(\mathbb{R}_+)$, $\gamma > 0$, is the set of all measurable functions h defined on $\mathbb{R}_+$ satisfying

$$\|h\|_{\Delta_\gamma(\mathbb{R}_+)} := \sup_{R>0}\left(\frac{1}{R}\int_0^R |h(t)|^\gamma \, dt\right)^{1/\gamma} < \infty.$$

In the same article, they also studied the $L^p(\mathbb{R}^n)$ boundedness for the maximal operator

$$T^*_{h,\Omega} f(x) = \sup_{\epsilon>0}\left|\int_{|y|>\epsilon} f(x-y)\frac{h(|y|)\Omega(y)}{|y|^n}\, dy\right|.$$

These results were improved and extended by many authors (see [Al-Salman and Pan 2002; Fan and Pan 1997; Liu 2014; Liu et al. 2016; Sato 2009]). It is worth remarking the following inclusion relations:

$$\mathcal{C}^1(\mathrm{S}^{n-1}) \subsetneq \mathrm{Lip}_\alpha(\mathrm{S}^{n-1}) \subsetneq L^q(\mathrm{S}^{n-1}), \tag{1-5}$$

$$L^\infty(\mathbb{R}_+) = \Delta_\infty(\mathbb{R}_+) \subsetneq \Delta_{\gamma_2}(\mathbb{R}_+) \subsetneq \Delta_{\gamma_1}(\mathbb{R}_+) \quad \text{for } 1 \le \gamma_1 < \gamma_2 < \infty. \tag{1-6}$$

In light of the above background and observation, a question that arises naturally is the following:

Question B. Are $T_{h,\Omega}$ and $T^*_{h,\Omega}$ bounded on $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)$ $(p \neq \tilde{p})$ if $\Omega \in L^q(\mathrm{S}^{n-1})$ and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $1 < q,\ \gamma \le \infty$?

In this paper we will give an affirmative answer to the above question by treating a family of operators that are even broader than $T_{h,\Omega}$ and $T^*_{h,\Omega}$. To be more precise, let h, Ω be given as in (1-4) and $P_N(t)$ be a real polynomial on $\mathbb{R}$ of degree N satisfying $P(0) = 0$. The corresponding singular integral operator T_{P_N} and the related maximal singular integral operator $T^*_{P_N}$ along the "polynomial curve" P_N on $\mathbb{R}^n$ are defined by

$$T_{P_N} f(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - P_N(|y|)y')\frac{h(|y|)\Omega(y)}{|y|^n}\, dy,$$

$$T^*_{P_N} f(x) = \sup_{\epsilon>0}\left|\int_{|y|>\epsilon} f(x - P_N(|y|)y')\frac{h(|y|)\Omega(y)}{|y|^n}\, dy\right|,$$

where $y' = y/|y|$ for $y \neq 0$. Clearly, $T_{P_N} = T_{h,\Omega}$ and $T^*_{P_N} = T^*_{h,\Omega}$ if $P_N(t) = t$.

In order to obtain the $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)$ boundedness of $T_{h,\Omega}$ and $T^*_{h,\Omega}$, we will establish some weighted inequalities. Our main results can be stated as follows.

Theorem 1.1. *Let $P_N(t)$ be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P_N(0) = 0$. Suppose that $\Omega \in L^q(\mathrm{S}^{n-1})$ satisfies* (1-3) *and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $1 < q, \gamma \leq \infty$:*

(i) *Let $2 \leq p < \infty$. Then for any nonnegative measurable function u on $\mathbb{R}^n$, the following inequality holds*:

$$\|T_{P_N} f\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,p,s,N} \|f\|_{L^p(\Lambda_{N,s}u)} \quad \forall s > 1. \tag{1-7}$$

(ii) *Let $1 < p < 2$ and $\{t_k\}_{k\in\mathbb{N}}$ be a sequence of positive numbers satisfying $t_1 = 2/p$ and*

$$\frac{1}{t_{k+1}} = \frac{1}{t_k} + \frac{p}{2}\Big(1 - \frac{1}{t_k}\Big).$$

Then for any nonnegative measurable function u on $\mathbb{R}^n$ and any fixed $k \in \mathbb{N}$, the following inequality holds:

$$\|T_{P_N} f\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,p,s,N,t_k} \|f\|_{L^p(\Lambda_{N,s}u)} \quad \forall s > t_k. \tag{1-8}$$

Here and throughout the paper we use $C_{h,\Omega,\alpha,\beta,\gamma,\dots}$ to denote positive constants that depend on the functions Ω, h and parameters $\alpha, \beta, \gamma, \dots$ appearing either in the definitions of the operators or in the statements of the theorems. In particular, they are independent of the coefficients of the polynomial P_N in the definition of T_{P_N}. In (1-8) *we also used the notation*

$$\Lambda_{N,s}u = \begin{cases} \mathrm{M}^N_s u + \mathrm{M}^2_s \widetilde{\mathrm{M}^N_s u} + H_{N,s}u & \text{if } 1 < p < 2, \\ L_{N,s}u & \text{if } 2 \leq p < \infty, \end{cases}$$

$$L_{\lambda,s}u = \sum_{i=0}^{\lambda} \mathrm{M}^{\lambda+1-i}_s M^{\tilde{\sigma}}_{i,s} \mathrm{M}_s u, \qquad H_\lambda u = \sum_{i=1}^{\lambda} \mathrm{M}^2 M^{\tilde{\sigma}}_i \mathrm{M}^{\lambda+1-i} u \quad \forall 1 \leq \lambda \leq N$$

and $M^{\tilde{\sigma}}_{\lambda,s}u = (M^{\tilde{\sigma}}_\lambda(u^s))^{1/s}$, $\mathrm{M}^k_s u = (\mathrm{M}^k u^s)^{1/s}$ for any $k \in \mathbb{N}$, $H_{\lambda,s}u = (H_\lambda u^s)^{1/s}$. Here M^k denotes the Hardy–Littlewood maximal operator M iterated k times for all $k \in \mathbb{N}$ and $M^{\tilde{\sigma}}_\lambda$ is a maximal operator given as in the proof of Theorem 1.1.

Theorem 1.2. *Let $P_N(t)$ be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P_N(0) = 0$. Suppose that $\Omega \in L^q(\mathrm{S}^{n-1})$ satisfies* (1-3) *and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $1 < q, \gamma \leq \infty$:*

(i) *Let $2 \leq p < \infty$. Then for any nonnegative measurable function u on $\mathbb{R}^n$, the following inequality holds*:

$$\|T^*_{P_N} f\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,p,s} \|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s u + \Theta_{N,s}\mathrm{M}^2_s u)} \quad \forall s > 1. \tag{1-9}$$

(ii) *Let* $1 < p < 2$ *and* $\{t_k\}_{k\in\mathbb{N}}$ *be given as in Theorem 1.1. Then for any nonnegative measurable function* u *on* $\mathbb{R}^n$ *and any fixed* $k \in \mathbb{N}$, *the following inequality holds*:

$$\|T^*_{P_N} f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,s,N,t_k} \|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s u+\Theta_{N,s}\mathrm{M}^2_s u)} \quad \forall s > t_k. \tag{1-10}$$

Here

$$\Theta_{N,s}u = \begin{cases} \mathrm{M}^N_s u + \mathrm{M}^2_s\widetilde{\mathrm{M}^N_s u} + H_{N,s}u & \text{if } 1 < p < 2, \\ \mathrm{M}^N_s u + L_{N,s}u + I_{N,s}u + J_{N,s}u & \text{if } 2 \le p < \infty, \end{cases}$$

where $L_{N,s}$ *is given as in Theorem 1.1 and*

$$I_{\lambda,s}u = \sum_{i=1}^{\lambda} \mathrm{M}_s M^{\tilde\sigma}_{i,s}\mathrm{M}^{\lambda-i}_s u, \quad J_{\lambda,s}u = \sum_{i=1}^{\lambda} \mathrm{M}^2_s M^{\tilde\sigma}_{i-1,s}\mathrm{M}^{\lambda-i}_s u \quad \forall 1 \le \lambda \le N.$$

As applications of Theorems 1.1 and 1.2, we obtain the $L^p_{|x|}L^{\tilde p}_\theta(\mathbb{R}^n)$ boundedness of the operators T_{P_N} and $T^*_{P_N}$ in following results.

Corollary 1.3. *Let* $P_N(t)$ *be a real polynomial on* $\mathbb{R}$ *of degree* N *and satisfy* $P_N(0) = 0$. *Suppose that* $\Omega \in L^q(\mathrm{S}^{n-1})$ *satisfies* (1-3) *and* $h \in \Delta_\gamma(\mathbb{R}_+)$ *for some* $1 < q,\ \gamma \le \infty$. *Then for* $1 < p < \infty$ *and* $1 < \tilde p < \infty$, *the following inequalities hold*:

$$\|T_{P_N} f\|_{L^p_{|x|}L^{\tilde p}_\theta(\mathbb{R}^n)} \le C_{h,\Omega,q,\gamma,p,\tilde p,N} \|f\|_{L^p_{|x|}L^{\tilde p}_\theta(\mathbb{R}^n)}, \tag{1-11}$$

$$\begin{aligned}\Big\|\Big(\sum_{j\in\mathbb{Z}} |T_{P_N} f_j|^{\tilde p}\Big)^{1/\tilde p}\Big\|_{L^p_{|x|}L^{\tilde p}_\theta(\mathbb{R}^n)} \\ \le C_{h,\Omega,q,\gamma,p,\tilde p,N} \Big\|\Big(\sum_{j\in\mathbb{Z}} |f_j|^{\tilde p}\Big)^{1/\tilde p}\Big\|_{L^p_{|x|}L^{\tilde p}_\theta(\mathbb{R}^n)},\end{aligned} \tag{1-12}$$

where the constants $C_{h,\Omega,q,\gamma,p,\tilde p,N} > 0$ *are independent of the coefficients of* P_N.

Corollary 1.4. *Let* $P_N(t)$ *be a real polynomial on* $\mathbb{R}$ *of degree* N *and satisfy* $P_N(0) = 0$. *Suppose that* $\Omega \in L^q(\mathrm{S}^{n-1})$ *satisfies* (1-3) *and* $h \in \Delta_\gamma(\mathbb{R}_+)$ *for some* $1 < q,\ \gamma \le \infty$. *Then for* $1 < \tilde p \le p < \infty$, *the following inequalities hold*:

$$\|T^*_{P_N} f\|_{L^p_{|x|}L^{\tilde p}_\theta(\mathbb{R}^n)} \le C_{h,\Omega,q,\gamma,p,\tilde p,N} \|f\|_{L^p_{|x|}L^{\tilde p}_\theta(\mathbb{R}^n)}, \tag{1-13}$$

$$\begin{aligned}\Big\|\Big(\sum_{j\in\mathbb{Z}} |T^*_{P_N} f_j|^{\tilde p}\Big)^{1/\tilde p}\Big\|_{L^p_{|x|}L^{\tilde p}_\theta(\mathbb{R}^n)} \\ \le C_{h,\Omega,q,\gamma,p,\tilde p,N} \Big\|\Big(\sum_{j\in\mathbb{Z}} |f_j|^{\tilde p}\Big)^{1/\tilde p}\Big\|_{L^p_{|x|}L^{\tilde p}_\theta(\mathbb{R}^n)},\end{aligned} \tag{1-14}$$

where the constants $C_{h,\Omega,q,\gamma,p,\tilde p,N} > 0$ *are independent of the coefficients of* P_N.

Corollary 1.5. *Let $P_N(t)$ be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P_N(0)=0$. Suppose that $\Omega \in L^q(\mathrm{S}^{n-1})$ satisfies* (1-3) *and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $1 < q,\ \gamma \le \infty$:*

(i) *If $1 < p < \infty$ and $1 < \tilde{p} < \infty$, then*

$$\Big\|\Big(\sum_{j\in\mathbb{Z}} |T_{P_N} f_j|^{\tilde{p}}\Big)^{1/\tilde{p}}\Big\|_{L^p(\mathbb{R}^n)} \le C_{h,\Omega,q,\gamma,p,\tilde{p},N} \Big\|\Big(\sum_{j\in\mathbb{Z}} |f_j|^{\tilde{p}}\Big)^{1/\tilde{p}}\Big\|_{L^p(\mathbb{R}^n)}. \tag{1-15}$$

(ii) *If $1 < \tilde{p} \le p < \infty$, then*

$$\Big\|\Big(\sum_{j\in\mathbb{Z}} |T^*_{P_N} f_j|^{\tilde{p}}\Big)^{1/\tilde{p}}\Big\|_{L^p(\mathbb{R}^n)} \le C_{h,\Omega,q,\gamma,p,\tilde{p},N} \Big\|\Big(\sum_{j\in\mathbb{Z}} |f_j|^{\tilde{p}}\Big)^{1/\tilde{p}}\Big\|_{L^p(\mathbb{R}^n)}. \tag{1-16}$$

The above constants $C_{h,\Omega,q,\gamma,p,\tilde{p},N} > 0$ are independent of the coefficients of P_N.

Remark 1.6. Corollary 1.3 improves and generalizes Theorem A by (1-5) and (1-6).

The rest of this paper is organized as follows. Section 2 contains a key criterion, which says that certain weighted norm inequalities for an operator will automatically imply its boundedness on the mixed radial-angular spaces, vector-valued mixed radial-angular spaces, and vector-valued inequalities. The main results of this paper will be proved in Section 3, where the proofs of Corollaries 1.3–1.5 are based on Theorems 1.1 and 1.2 and the criterion established in Section 2 (see Proposition 2.1). Finally, we will discuss several corresponding results concerning the Hardy–Littlewood maximal operator, Calderón–Zygmund operators, and the singular integral operators with Grafakos–Stefanov kernels. We would like to remark that the main idea in the proofs of our results is a combination of ideas and arguments from [Córdoba 2016; D'Ancona and Lucà 2016; Hofmann 1993; Liu 2014].

Throughout this note, for any $p \in (1,\infty)$, we let p' denote the dual exponent to p defined as $1/p + 1/p' = 1$. In what follows, for any function f, we define $\tilde{f}$ by $\tilde{f}(x) = f(-x)$. We denote by M^k the Hardy–Littlewood maximal operator M iterated k times for all $k = 1, 2, \ldots$ Specifically, $\mathrm{M}^k = \mathrm{M}$ when $k=1$. For $s > 1$, we denote $\mathrm{M}_s u = (\mathrm{M}u^s)^{1/s}$. For $f \in L^p(u)$, we set

$$\|f\|_{L^p(u)} = \Big(\int_{\mathbb{R}^n} |f(x)|^p u(x)\,dx\Big)^{1/p}.$$

2. A criterion

To prove our main results, we need the following proposition, which is of interest in its own right.

Proposition 2.1. *Let $1 < p < \infty$ and $\{t_k\}_{k\in\mathbb{N}}$ be a strictly decreasing sequence of positive numbers satisfying $\lim_{k\to\infty} t_k = 1$. Assume that T is a linear or sublinear operator such that*

$$\|Tf\|_{L^p(u)} \le C_{p,s,t_k} \|f\|_{L^p(\mathcal{G}_s(u))} \quad \forall s > t_k \tag{2-1}$$

for any fixed positive integer k and any nonnegative measurable function u on $\mathbb{R}^n$, where $\mathcal{G}_s$ is a bounded operator from $L^q(\mathbb{R}^n)$ to itself for any $q \in (s,\infty)$ with $s > t_k$. Then for any $p < q < \infty$, the following inequalities hold:

$$\|Tf\|_{L^q_{|x|}L^p_\theta(\mathbb{R}^n)} \le C_{p,q}\|f\|_{L^q_{|x|}L^p_\theta(\mathbb{R}^n)}, \tag{2-2}$$

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|Tf_j|^p\Big)^{1/p}\Big\|_{L^q_{|x|}L^p_\theta(\mathbb{R}^n)} \le C_{p,q}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^p\Big)^{1/p}\Big\|_{L^q_{|x|}L^p_\theta(\mathbb{R}^n)}, \tag{2-3}$$

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|Tf_j|^p\Big)^{1/p}\Big\|_{L^q(\mathbb{R}^n)} \le C_{p,q}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^p\Big)^{1/p}\Big\|_{L^q(\mathbb{R}^n)}. \tag{2-4}$$

Proof. We only prove (2-2) since (2-3) and (2-4) can be proved similarly. The argument in the proof of (2-2) is essentially the same as in the proof of [D'Ancona and Lucà 2016, Theorem 2.6]. Let $1 < p < q < \infty$ and write $r = q/(q-p)$. Fix a number s in the interval $(1, r)$ and choose k_0 as the smallest integer for which we have $t_{k_0} < s$. We denote by X the set of all $g \in \mathcal{S}(\mathbb{R})$ with $\int_0^\infty g^r(\rho)\rho^{n-1}\,d\rho \le 1$. By polar coordinates, we write

$$\begin{aligned}\|Tf\|^p_{L^q_{|x|}L^p_\theta(\mathbb{R}^n)} &= \bigg(\int_0^\infty\Big(\int_{\mathbb{S}^{n-1}}|Tf(\rho\theta)|^p d\sigma(\theta)\Big)^{q/p}\rho^{n-1}\,d\rho\bigg)^{p/q}\\ &= \sup_{g\in X}\int_0^\infty\int_{\mathbb{S}^{n-1}}|Tf(\rho\theta)|^p g(\rho)\rho^{n-1}\,d\sigma(\theta)\,d\rho\\ &= \sup_{g\in X}\int_{\mathbb{R}^n}|Tf(x)|^p g(|x|)\,dx.\end{aligned} \tag{2-5}$$

Fix $g \in X$. Let $I(g) := \int_{\mathbb{R}^n}|Tf(x)|^p g(|x|)\,dx$ and $h(x) = g(|x|)$. Changing variables, we obtain from (2-1) and Hölder's inequality that

$$\begin{aligned}I(g) &\le C_{p,s,t_{k_0}}\int_{\mathbb{R}^n}|f(x)|^p\mathcal{G}_s(h)(x)\,dx\\ &\le C_{p,s,t_{k_0}}\int_0^\infty\int_{\mathbb{S}^{n-1}}|f(\rho\theta)|^p\,d\sigma(\theta)\,\mathcal{G}_s(g)(\rho)\rho^{n-1}\,d\rho\\ &\le C_{p,s,t_{k_0}}\bigg(\int_0^\infty\Big(\int_{\mathbb{S}^{n-1}}|f(\rho\theta)|^p d\sigma(\theta)\Big)^{q/p}\rho^{n-1}d\rho\bigg)^{p/q}\\ &\qquad\qquad\times\bigg(\int_0^\infty(\mathcal{G}_s(g)(\rho))^r\rho^{n-1}\,d\rho\bigg)^{1/r}\end{aligned}$$

$$\leq C_{p,q} \| f \|^{p}_{L^{q}_{|x|} L^{p}_{\theta}(\mathbb{R}^n)} \| \mathcal{G}_s(h) \|_{L^r(\mathbb{R}^n)}$$
$$\leq C_{p,q} \| f \|^{p}_{L^{q}_{|x|} L^{p}_{\theta}(\mathbb{R}^n)},$$

which, together with (2-5), yields (2-2). □

3. Proofs of main results

In this section we shall present the proofs of Theorems 1.1 and 1.2 and Corollaries 1.3–1.5. In what follows, we may assume, without loss of generality, that $P_N(t) = \sum_{i=1}^{N} a_i t^i$ with $a_i \neq 0$. We also let $P_\lambda(t) = \sum_{i=1}^{\lambda} a_i t^i$ for $\lambda = 1, 2, \ldots, N$ and $P_0(t) = 0$.

Proof of Theorem 1.1. For $\lambda \in \{1, 2, \ldots, N\}$, we define two families of measures $\{\sigma_{k,\lambda}\}_{k\in\mathbb{Z}}$ and $\{|\sigma_{k,\lambda}|\}_{k\in\mathbb{Z}}$ respectively by

$$\int_{\mathbb{R}^n} f(x)\, d\sigma_{k,\lambda}(x) = \int_{2^k<|x|\leq 2^{k+1}} f(P_\lambda(|x|)x') \frac{h(|x|)\Omega(x)}{|x|^n}\, dx$$

and

$$\int_{\mathbb{R}^n} f(x)\, d|\sigma_{k,\lambda}|(x) = \int_{2^k<|x|\leq 2^{k+1}} f(P_\lambda(|x|)x') \frac{|h(|x|)\Omega(x)|}{|x|^n}\, dx.$$

We also define the maximal operators M_λ^σ and $M_\lambda^{\tilde{\sigma}}$ respectively by

$$M_\lambda^\sigma f(x) = \sup_{k\in\mathbb{Z}} ||\sigma_{k,\lambda}| * f(x)|$$

and

$$M_\lambda^{\tilde{\sigma}} f(x) = \sup_{k\in\mathbb{Z}} ||\tilde{\sigma}_{k,\lambda}| * f(x)|,$$

where

$$\int_{\mathbb{R}^n} f(x)\, d|\tilde{\sigma}_{k,\lambda}|(x) = \int_{\mathbb{R}^n} f(-x)\, d|\sigma_{k,\lambda}|(x).$$

One can easily verify that

$$M_0^\sigma f(x) \leq C_{h,\Omega,q,\gamma} |f(x)|, \tag{3-1}$$

$$M_\lambda^{\tilde{\sigma}} f(x) = M_\lambda^\sigma \tilde{f}(x), \tag{3-2}$$

$$T_{P_N} f(x) = \sum_{k\in\mathbb{Z}} \sigma_{k,N} * f(x). \tag{3-3}$$

Also, from [Liu 2014, Lemma 2.2] and a direct computation, one has

$$\max\big\{|\hat{\sigma}_{k,\lambda}(\xi) - \hat{\sigma}_{k,\lambda-1}(\xi)|, |\widehat{|\sigma_{k,\lambda}|}(\xi) - \widehat{|\sigma_{k,\lambda-1}|}(\xi)|\big\} \leq C_{h,\Omega,q,\gamma} \min\{1, |2^{k\lambda} a_\lambda \xi|\}, \tag{3-4}$$

$$\max\big\{|\hat{\sigma}_{k,\lambda}(\xi)|, |\widehat{|\sigma_{k,\lambda}|}(\xi)|\big\} \leq C_{h,\Omega,q,\gamma} (\min\{1, |2^{k\lambda} a_\lambda \xi|^{-1}\})^{1/(4\lambda q'\gamma')}. \tag{3-5}$$

We shall prove Theorem 1.1 by considering the following three steps:

Step 1: The bounds for M_λ^σ. We want to show that

$$\|M_\lambda^\sigma f\|_{L^p(\mathbb{R}^n)} \le C_{h,\Omega,q,\gamma}\|f\|_{L^p(\mathbb{R}^n)} \tag{3-6}$$

for all $0 \le \lambda \le N$ and $1 < p < \infty$. It is obvious that (3-6) holds for $\lambda = 0$ by (3-1). Choose a nonnegative function $\phi \in \mathcal{S}(\mathbb{R}^n)$ supported in $\{|t| \le 1\}$ satisfying $\phi(t) = 1$ when $|t| < \frac{1}{2}$. For $\lambda \in \{1, 2, \dots, N\}$, we define the family of functions $\{\psi_{k,\lambda}\}_{k\in\mathbb{Z}}$ via the Fourier transform $\widehat{\psi}_{k,\lambda}(\xi) = \phi(2^{k\lambda}|a_\lambda\xi|)$. Define the family of Borel measures $\{\omega_{k,\lambda}\}_{k\in\mathbb{Z}}$ on $\mathbb{R}^n$ by

$$\hat{\omega}_{k,\lambda}(\xi) = \widehat{|\sigma_{k,\lambda}|}(\xi) - \psi_{k,\lambda}(\xi)\widehat{|\sigma_{k,\lambda-1}|}(\xi). \tag{3-7}$$

One easily checks that (or see [Liu 2014])

$$|\hat{\omega}_{k,\lambda}(x)| \le C_{h,\Omega,q,\gamma}(\min\{1, |2^{k\lambda}a_\lambda x|, |2^{k\lambda}a_\lambda x|^{-1}\})^{1/(4\lambda q'\gamma')}, \tag{3-8}$$

$$M_\lambda^\omega f(x) \le M_\lambda^\sigma|f|(x) + \mathrm{M}M_{\lambda-1}^\sigma|f|(x), \tag{3-9}$$

$$M_\lambda^\sigma f(x) \le \mathrm{M}M_{\lambda-1}^\sigma|f|(x) + G_\lambda^\omega f(x), \tag{3-10}$$

where $M_\lambda^\omega f(x) = \sup_{k\in\mathbb{Z}} ||\omega_{k,\lambda}| * f(x)|$ and $G_\lambda^\omega f(x) = \left(\sum_{k\in\mathbb{Z}} |\omega_{k,\lambda} * f(x)|^2\right)^{1/2}$. By (3-1), (3-8)–(3-10) and a standard iteration argument from [Duoandikoetxea and Rubio de Francia 1986], we can obtain (3-6) for all $1 \le \lambda \le N$. The details are omitted.

Step 2: The proof of (i) of Theorem 1.1. For $1 \le \lambda \le N$ and $s > 1$, let $\Lambda_{\lambda,s}$ be given as in Theorem 1.1. Let ϕ be given as above. We define the family of functions $\{\Phi_\lambda\}_{\lambda=1}^N$ by

$$\Phi_\lambda(\xi) = \prod_{j=\lambda}^N \phi(|2^{kj}a_j\xi|).$$

For $1 \le \lambda \le N$, define the Borel measures $\{\mu_{k,\lambda}\}_{k\in\mathbb{Z}}$ on $\mathbb{R}^n$ by

$$\hat{\mu}_{k,\lambda}(\xi) = \hat{\sigma}_{k,\lambda}(\xi)\Phi_{\lambda+1}(\xi) - \hat{\sigma}_{k,\lambda-1}(\xi)\Phi_\lambda(\xi).$$

Here we use the convention $\prod_{j\in\varnothing} a_j = 1$. One can easily check that (or see [Liu 2014])

$$\sigma_{k,N} = \sum_{\lambda=1}^N \mu_{k,\lambda}, \tag{3-11}$$

$$M_\lambda^\mu f(x) \le \mathrm{M}^{N-\lambda}M_\lambda^\sigma|f|(x) + \mathrm{M}^{N-\lambda+1}M_{\lambda-1}^\sigma|f|(x), \tag{3-12}$$

$$|\hat{\mu}_{k,\lambda}(x)| \le C_{h,\Omega,q,\gamma}(\min\{1, |2^{k\lambda}a_\lambda x|, |2^{k\lambda}a_\lambda x|^{-1}\})^{1/(4\lambda q'\gamma')}. \tag{3-13}$$

Equation (3-3) and (3-11) clearly yield that

$$T_{P_N} f(x) = \sum_{k\in\mathbb{Z}}\sum_{\lambda=1}^N \mu_{k,\lambda} * f(x) = \sum_{\lambda=1}^N\sum_{k\in\mathbb{Z}} \mu_{k,\lambda} * f(x) =: \sum_{\lambda=1}^N T_\lambda f(x). \tag{3-14}$$

Notice that $u \le \mathrm{M}_s u$, $\mathrm{M}_s u \in A_1$ (see [Coifman and Rochberg 1980]), and

$$\sum_{\lambda=1}^{N} \mathrm{M}_s M_{\lambda,s}^{\tilde{\mu}} \mathrm{M}_s u \le \sum_{\lambda=1}^{N} (\mathrm{M}_s^{N+1-\lambda} M_{\lambda,s}^{\tilde{\sigma}} \mathrm{M}_s u + \mathrm{M}_s^{N+2-\lambda} M_{\lambda-1,s}^{\tilde{\sigma}} \mathrm{M}_s u) \le L_{N,s} u$$

by (3-12). Therefore, (1-7) reduces to the following inequality:

$$\|T_\lambda f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s} \|f\|_{L^p(\Lambda_{N,s} u)} \tag{3-15}$$

for all $1 \le \lambda \le N$, $2 \le p < \infty$, $s > 1$ and any nonnegative measurable function u on $\mathbb{R}^n$.

We now prove (3-15). For $1 \le \lambda \le N$, let $\Psi_\lambda(t) \in \mathcal{C}_c^\infty\big(\big(\frac{1}{4}, 1\big)\big)$ such that $0 \le \Psi_\lambda \le 1$ and $\sum_{k \in \mathbb{Z}} (\Psi_\lambda(2^{k\lambda} |a_\lambda \xi|))^3 = 1$. Define the Fourier multiplier operators $\{S_{k,\lambda}\}_{k \in \mathbb{Z}}$ by $S_{k,\lambda} f(x) = \Theta_{k,\lambda} * f(x)$, where $\widehat{\Theta}_{k,\lambda}(\xi) = \Psi_\lambda(2^{k\lambda} |a_\lambda \xi|)$. It was shown in [Hofmann 1993] that

$$\Big\| \Big(\sum_{k \in \mathbb{Z}} |S_{k,\lambda} f|^2 \Big)^{1/2} \Big\|_{L^p(w)} \le C_{p,w,\lambda} \|f\|_{L^p(w)} \tag{3-16}$$

and

$$\Big\| \sum_{k \in \mathbb{Z}} S_{k,\lambda} f_k \Big\|_{L^p(w)} \le C_{p,w,\lambda} \Big\| \Big(\sum_{k \in \mathbb{Z}} |f_k|^2 \Big)^{1/2} \Big\|_{L^p(w)} \tag{3-17}$$

for all $1 < p < \infty$ and $w \in A_p$.

Write

$$\begin{aligned} T_\lambda f(x) &= \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} S_{j+k,\lambda}^3 (\mu_{k,\lambda} * f)(x) \\ &= \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} S_{j+k,\lambda}^3 (\mu_{k,\lambda} * f)(x) =: \sum_{j \in \mathbb{Z}} T_{\lambda,j} f(x). \end{aligned} \tag{3-18}$$

By (3-13) and Plancherel's theorem, it holds that

$$\int_{\mathbb{R}^n} |\mu_{k,\lambda} * S_{j+k,\lambda} w(x)|^2 \, dx \le C_{h,\Omega,q,\gamma} 2^{-|j|/(2q'\gamma')} \int_{\mathbb{R}^n} |w(x)|^2 \, dx \tag{3-19}$$

for an arbitrary function w on $\mathbb{R}^n$. Fix a nonnegative measurable function u on $\mathbb{R}^n$. It is easy to see that

$$\begin{aligned} &\int_{\mathbb{R}^n} |\mu_{k,\lambda} * S_{j+k,\lambda} w(x)|^2 u^s(x) \, dx \\ &\qquad \le \|\mu_{k,\lambda}\| \|\Theta_{j+k,\lambda}\|_{L^1(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\mu_{k,\lambda}| * |\Theta_{j+k,\lambda}| * |w|^2(x) u^s(x) \, dx \\ &\qquad \le C_{h,\Omega,q,\gamma} \int_{\mathbb{R}^n} |w(x)|^2 \mathrm{M} M_\lambda^{\tilde{\mu}} u^s(x) \, dx \end{aligned} \tag{3-20}$$

for any $s > 1$. By (3-19) and (3-20) and the interpolation of L^2-spaces with a change of measure [Bergh and Löfström 1976, Theorem 5.4.1], we obtain

$$\text{(3-21)}\quad \int_{\mathbb{R}^n} |\mu_{k,\lambda} * S_{j+k,\lambda} w(x)|^2 u(x)\,dx \\ \le C_{h,\Omega,q,\gamma,s} 2^{-(1-1/s)/(2q'\gamma')|j|} \int_{\mathbb{R}^n} |w(x)|^2 \mathrm{M}_s M^{\tilde{\mu}}_{\lambda,s} u(x)\,dx$$

for any $s > 1$. By (3-21) with $w = S_{j+k,\lambda} f$ and (3-16), it follows that

$$\begin{aligned}
\|T_{\lambda,j} f\|^2_{L^2(u)} &= \Big\| \sum_{k\in\mathbb{Z}} S^3_{j+k,\lambda} \mu_{k,\lambda} * f \Big\|^2_{L^2(u)} \\
&\le C_\lambda \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} |\mu_{k,\lambda} * S^2_{j+k,\lambda} f(x)|^2 u(x)\,dx \\
&\le C_{h,\Omega,q,\gamma,\lambda,s} 2^{-(1-1/s)/(2q'\gamma')|j|} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} |S_{j+k,\lambda} f(x)|^2 \mathrm{M}_s M^{\tilde{\mu}}_{\lambda,s} u(x)\,dx \\
&\le C_{h,\Omega,q,\gamma,\lambda,s} 2^{-(1-1/s)/(2q'\gamma')|j|} \|f\|^2_{L^2(\mathrm{M}_s M^{\tilde{\mu}}_{\lambda,s} u)}.
\end{aligned}$$

Hence we obtain

$$\text{(3-22)}\qquad \|T_{\lambda,j} f\|_{L^2(u)} \le C_{h,\Omega,q,\gamma,\lambda,s} 2^{-(1-1/s)/(4q'\gamma')|j|} \|f\|_{L^2(\mathrm{M}_s M^{\tilde{\mu}}_{\lambda,s} u)}.$$

Next, we shall only prove

$$\text{(3-23)}\qquad \|T_{\lambda,j} f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s} \|f\|_{L^p(\mathrm{M}_s M^{\tilde{\mu}}_{\lambda,s} u)}$$

for all $2 < p < \infty$. Actually, by (3-22), (3-23), and an interpolation (see [Bergh and Löfström 1976, Corollary 5.5.4]), one has

$$\text{(3-24)}\qquad \|T_{\lambda,j} f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s} 2^{-\alpha(p,q,\gamma,s)|j|} \|f\|_{L^p(\mathrm{M}_s M^{\tilde{\mu}}_{\lambda,s} u)}$$

for $2 \le p < \infty$ and $s > 1$. Here $\alpha(p,q,\gamma,s) > 0$ depends only on p, q, γ, and s. Equation (3-24) together with (3-18) yields (3-15).

To prove (3-23), it suffices to show that

$$\text{(3-25)}\quad \Big\| \Big(\sum_{k\in\mathbb{Z}} |\mu_{k,\lambda} * g_k|^2 \Big)^{1/2} \Big\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,s} \Big\| \Big(\sum_{k\in\mathbb{Z}} |g_k|^2 \Big)^{1/2} \Big\|_{L^p(M^{\tilde{\mu}}_{\lambda,s} u)}$$

for all $2 < p < \infty$ and any $s > 1$. In fact, by (3-16), (3-17), (3-25), and the fact $M^{\tilde{\mu}}_{\lambda,s} u \le \mathrm{M}_s M^{\tilde{\mu}}_{\lambda,s} u$, it holds that

$$\|T_{\lambda,j} f\|_{L^p(u)} = \Big\| \sum_{k\in\mathbb{Z}} S^3_{j+k,\lambda} \mu_{k,\lambda} * f \Big\|_{L^p(u)}$$

$$\leq C_{p,\lambda}\Big\|\Big(\sum_{k\in\mathbb{Z}}|\mu_{k,\lambda}*S^2_{j+k,\lambda}f|^2\Big)^{1/2}\Big\|_{L^p(u)}$$
$$\leq C_{h,\Omega,q,\gamma,p,\lambda,s}\Big\|\Big(\sum_{k\in\mathbb{Z}}|S^2_{j+k,\lambda}f|^2\Big)^{1/2}\Big\|_{L^p(M^{\tilde{\mu}}_{\lambda,s}u)}$$
$$\leq C_{h,\Omega,q,\gamma,p,\lambda,s}\|f\|_{L^p(\mathrm{M}_s M^{\tilde{\mu}}_{\lambda,s}u)}$$

for all $2<p<\infty$ and any $s>1$. This yields (3-23).

Below we prove (3-25). Fix $2<p<\infty$. By duality we can choose a function $v\in L^{(p/2)'}(u)$ with unit norm such that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\mu_{k,\lambda}*g_k|^2\Big)^{1/2}\Big\|^2_{L^p(u)}=\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}|\mu_{k,\lambda}*g_k(x)|^2\cdot v(x)u(x)\,dx.$$

This together with the fact that $\|\mu_{k,\lambda}\|\leq C_{h,\Omega,q,\gamma}$ yields that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\mu_{k,\lambda}*g_k|^2\Big)^{1/2}\Big\|^2_{L^p(u)}\leq C_{h,\Omega,q,\gamma}\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}|g_k(x)|^2||\tilde{\mu}_{k,\lambda}|*(vu)(x)|\,dx. \tag{3-26}$$

Fix $s>1$ and let $r=\frac{1}{2}ps$. Hölder's inequality yields

$$||\tilde{\mu}_{k,\lambda}|*(vu)|\leq(|\tilde{\mu}_{k,\lambda}|*u^s)^{1/r}(|\tilde{\mu}_{k,\lambda}|*(u^{r'/(p/2)'}v^{r'}))^{1/r'}. \tag{3-27}$$

By Hölder's inequality with exponents $\frac{1}{2}p$ and $\big(\frac{1}{2}p\big)'$ again and (3-26), (3-27), it holds that

$$\begin{aligned}\Big\|\Big(\sum_{k\in\mathbb{Z}}|\mu_{k,\lambda}*g_k|^2\Big)^{1/2}\Big\|^2_{L^p(u)}&\\ \leq C_{h,\Omega,q,\gamma}&\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}|g_k(x)|^2(M^{\tilde{\mu}}_\lambda u^s)^{1/r}M^{\tilde{\mu}}_\lambda(u^{r'/(p/2)'}v^{r'}))^{1/r'}(x)\,dx\\ \leq C_{h,\Omega,q,\gamma}&\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|^2_{L^p(M^{\tilde{\mu}}_{\lambda,s}u)}\|M^{\tilde{\mu}}_\lambda(u^{r'/(p/2)'}v^{r'})\|^{1/r'}_{L^{(p/2)'/r'}(\mathbb{R}^n)}.\end{aligned} \tag{3-28}$$

By (3-1), (3-6), and (3-12), one has

$$\|M^{\tilde{\mu}}_\lambda f\|_{L^t(\mathbb{R}^n)}\leq C_{h,\Omega,q,\gamma,t}\|f\|_{L^t(\mathbb{R}^n)}\quad\forall 1<t<\infty. \tag{3-29}$$

Since $\frac{1}{2}p = r/s < r$, then $\left(\frac{1}{2}p\right)' > r'$. Equation (3-29) leads to

$$\|M_\lambda^{\tilde{\mu}}(u^{r'/(p/2)'}v^{r'})\|^{1/r'}_{L^{(p/2)'/r'}(\mathbb{R}^n)} \\ \le C_{h,\Omega,q,\gamma,p,s}\|u^{r'/(p/2)'}v^{r'}\|^{1/r'}_{L^{(p/2)'/r'}(\mathbb{R}^n)} \le C_{h,\Omega,q,\gamma,p,s}.$$

This together with (3-28) yields (3-25) and completes the proof of (i) of Theorem 1.1.

Step 3: The proof of (ii) of Theorem 1.1. For $1 \le \lambda \le N$ and $s > 1$, let $\Lambda_{\lambda,s}$, $H_{\lambda,s}$, and $\{t_k\}_{k\in\mathbb{N}}$ be given as in Theorem 1.1. To prove (1-8), it suffices to show that

$$\|T_\lambda f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s,t_k}\|f\|_{L^p(\Lambda_{N,s}u)} \quad \forall s > t_k \tag{3-30}$$

for all $1 \le \lambda \le N$, $1 < p < 2$, all $k \in \mathbb{N}$, and any nonnegative measurable function u on $\mathbb{R}^n$. Actually, (3-30) reduces to the following

$$\int_{\mathbb{R}^n} |T_\lambda f(x)|^p u^{1/s}(x)\,dx \\ \le C_{h,\Omega,q,\gamma,p,\lambda,s,t_k}\int_{\mathbb{R}^n}|f(x)|^p(\mathrm{M}^N u + \mathrm{M}^2\widetilde{\mathrm{M}^N u} + H_N u)^{1/s}(x)\,dx \quad \forall s > t_k \tag{3-31}$$

for all $1 \le \lambda \le N$, $1 < p < 2$, all $k \in \mathbb{N}$, and any nonnegative measurable function u on $\mathbb{R}^n$. To this end, we substitute u^s for u in (3-31). With this substitution, the weight on the left becomes u and the weight on the right is not more than $\mathrm{M}_s^N u + \mathrm{M}_s^2\widetilde{\mathrm{M}_s^N u} + H_{N,s}u$.

We now prove (3-31). Fix $s > 1$ and a nonnegative measurable function u on $\mathbb{R}^n$. It follows from (3-10) that

$$\int_{\mathbb{R}^n}(M_\lambda^\sigma f(x))^p u^{1/s}(x)\,dx \\ \le \int_{\mathbb{R}^n}(\mathrm{M}M_{\lambda-1}^\sigma|f|(x))^p u^{1/s}(x)\,dx + \int_{\mathbb{R}^n}(G_\lambda^\omega f(x))^p u^{1/s}(x)\,dx \tag{3-32}$$

for all $1 < p < 2$. Hence by the well-known Fefferman–Stein inequality for M (see (3-102) below) we have

$$\int_{\mathbb{R}^n}(\mathrm{M}M_{\lambda-1}^\sigma|f|)(x))^p u^{1/s}(x)\,dx \\ \le C_p\|M_{\lambda-1}^\sigma|f|\|^p_{L^p(\mathrm{M}u^{1/s})} \le C_p\|M_{\lambda-1}^\sigma|f|\|^p_{L^p((\mathrm{M}u)^{1/s})} \tag{3-33}$$

for $1 < p < \infty$. Next, we consider $G_\lambda^\omega f$. By Minkowski's inequality, we obtain

$$G_\lambda^w f(x) = \biggl(\sum_{k\in\mathbb{Z}}\biggl|\omega_{k,\lambda} * \sum_{j\in\mathbb{Z}} S^3_{j+k,\lambda}f(x)\biggr|^2\biggr)^{1/2} \\ \le \sum_{j\in\mathbb{Z}}\biggl(\sum_{k\in\mathbb{Z}}|\omega_{k,\lambda} * S^3_{j+k,\lambda}f(x)|^2\biggr)^{1/2}$$

$$=: \sum_{j\in\mathbb{Z}} G_{\lambda,j} f(x).$$

It follows that

$$\|G_\lambda^\omega f\|_{L^p(u^{1/s})} \le \sum_{j\in\mathbb{Z}} \|G_{\lambda,j} f\|_{L^p(u^{1/s})} \tag{3-34}$$

for all $1 < p < \infty$. It is obvious to see that

$$\|\omega_{k,\lambda} * f\|_{L^\infty(\mathbb{R}^n)} \le C_{h,\Omega,q,\gamma} \|f\|_{L^\infty(\mathbb{R}^n)}, \tag{3-35}$$

$$\|\omega_{k,\lambda} * f\|_{L^1(u)} \le C\|f\|_{L^1(M_\lambda^{\tilde\sigma} u + M_{\lambda-1}^{\tilde\sigma}\mathrm{M}u)} \le C\|f\|_{L^1(\mathrm{M}M_\lambda^{\tilde\sigma} u + \mathrm{M}M_{\lambda-1}^{\tilde\sigma}\mathrm{M}u)}. \tag{3-36}$$

Thus, by interpolating between (3-35) and (3-36), we obtain

$$\|\omega_{k,\lambda} * f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p} \|f\|_{L^p(\mathrm{M}M_\lambda^{\tilde\sigma} u + \mathrm{M}M_{\lambda-1}^{\tilde\sigma}\mathrm{M}u)} \tag{3-37}$$

for all $1 < p < 2$. It follows from (3-37) that

$$\begin{aligned}\int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |\omega_{k,\lambda} * f_k(x)|^p u(x)\,dx \\ \le C_{h,\Omega,q,\gamma,p} \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |f_k(x)|^p (\mathrm{M}M_\lambda^{\tilde\sigma} u + \mathrm{M}M_{\lambda-1}^{\tilde\sigma}\mathrm{M}u)(x)\,dx\end{aligned} \tag{3-38}$$

for all $1 < p < 2$. On the other hand, we get from (3-6) and (3-9) that

$$\int_{\mathbb{R}^n} (\sup_{k\in\mathbb{Z}} |\omega_{k,\lambda} * f_k(x)|)^p\,dx \le C_{h,\Omega,q,\gamma,p} \int_{\mathbb{R}^n} (\sup_{k\in\mathbb{Z}} |f_k(x)|)^p\,dx \tag{3-39}$$

for all $1 < p < 2$. An interpolation between (3-38) and (3-39) now yields

$$\begin{aligned}\int_{\mathbb{R}^n} \Big(\sum_{k\in\mathbb{Z}} |\omega_{k,\lambda} * f_k(x)|^2\Big)^{p/2} u^{1/t_1}(x)\,dx \\ \le C_{h,\Omega,q,\gamma,p,t_1} \int_{\mathbb{R}^n} \Big(\sum_{k\in\mathbb{Z}} |f_k(x)|^2\Big)^{p/2} (\mathrm{M}M_\lambda^{\tilde\sigma} u + \mathrm{M}M_{\lambda-1}^{\tilde\sigma}\mathrm{M}u)^{1/t_1}(x)\,dx\end{aligned} \tag{3-40}$$

for all $1 < p < 2$, where $t_1 = 2/p$. Substitute u^{t_1} for u in (3-40), we obtain that

$$\begin{aligned}\int_{\mathbb{R}^n} \Big(\sum_{k\in\mathbb{Z}} |\omega_{k,\lambda} * f_k(x)|^2\Big)^{p/2} u(x)\,dx \\ \le C_{h,\Omega,q,\gamma,p,t_1} \int_{\mathbb{R}^n} \Big(\sum_{k\in\mathbb{Z}} |f_k(x)|^2\Big)^{p/2} (\mathrm{M}M_\lambda^{\tilde\sigma} u^{t_1} + \mathrm{M}M_{\lambda-1}^{\tilde\sigma}\mathrm{M}u^{t_1})^{1/t_1}(x)\,dx \\ \le C_{h,\Omega,q,\gamma,p,t_1} \int_{\mathbb{R}^n} \Big(\sum_{k\in\mathbb{Z}} |f_k(x)|^2\Big)^{p/2} (\mathrm{M}_{t_1} M_{\lambda,t_1}^{\tilde\sigma} u + \mathrm{M}_{t_1} M_{\lambda-1,t_1}^{\tilde\sigma}\mathrm{M}_{t_1}u)(x)\,dx.\end{aligned} \tag{3-41}$$

Since $\mathrm{M}_{t_1} M^{\tilde{\sigma}}_{\lambda,t_1} u + \mathrm{M}_{t_1} M^{\tilde{\sigma}}_{\lambda-1,t_1} \mathrm{M}_{t_1} u \in A_1$, by the weighted Littlewood–Paley theory, (3-41) yields that

$$
\begin{aligned}
\text{(3-42)}\quad & \|G_{\lambda,j} f\|_{L^p(u)} \\
&= \bigg\| \bigg(\sum_{k\in\mathbb{Z}} |\omega_{k,\lambda} * S^3_{j+k,\lambda} f|^2 \bigg)^{1/2} \bigg\|_{L^p(u)} \\
&\le C_{h,\Omega,q,\gamma,p,t_1} \bigg\| \bigg(\sum_{k\in\mathbb{Z}} |S^3_{j+k,\lambda} f|^2 \bigg)^{1/2} \bigg\|_{L^p(\mathrm{M}_{t_1} M^{\tilde{\sigma}}_{\lambda,t_1} u + \mathrm{M}_{t_1} M^{\tilde{\sigma}}_{\lambda-1,t_1} \mathrm{M}_{t_1} u)} \\
&\le C_{h,\Omega,q,\gamma,p,\lambda,t_1} \|f\|_{L^p(\mathrm{M}_{t_1} M^{\tilde{\sigma}}_{\lambda,t_1} u + \mathrm{M}_{t_1} M^{\tilde{\sigma}}_{\lambda-1,t_1} \mathrm{M}_{t_1} u)}
\end{aligned}
$$

for all $1 < p < 2$. Substituting u^{1/t_1} for u in (3-42), one finds

$$
\text{(3-43)}\qquad \|G_{\lambda,j} f\|_{L^p(u^{1/t_1})} \le C_{h,\Omega,q,\gamma,p,\lambda,t_1} \|f\|_{L^p((\mathrm{M} M^{\tilde{\sigma}}_{\lambda} u + \mathrm{M} M^{\tilde{\sigma}}_{\lambda-1} \mathrm{M} u)^{1/t_1})}
$$

for all $1 < p < 2$. On the other hand, by (3-8) and the arguments similar to those used in deriving (3-21),

$$
\begin{aligned}
\text{(3-44)}\quad & \int_{\mathbb{R}^n} |\omega_{k,\lambda} * S_{j+k,\lambda} w(x)|^2 u(x)\,dx \\
&\qquad \le C_{h,\Omega,q,\gamma,s} 2^{-(1-1/s)/(2q'\gamma')|j|} \int_{\mathbb{R}^n} |w(x)|^2 \mathrm{M}_s M^{\tilde{\omega}}_{\lambda,s} u(x)\,dx
\end{aligned}
$$

for any function w and any $s > 1$. By (3-44) with $w = S^2_{j+k,\lambda} f$ and (3-17), we obtain that

$$
\begin{aligned}
\text{(3-45)}\quad & \|G_{\lambda,j} f\|^2_{L^2(u)} \\
&= \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |\omega_{k,\lambda} * S^3_{j+k,\lambda} f(x)|^2 u(x)\,dx \\
&\le \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} |\omega_{k,\lambda} * S^3_{j+k,\lambda} f(x)|^2 u(x)\,dx \\
&\le C_{h,\Omega,q,\gamma,s} 2^{-(1-1/s)/(2q'\gamma')|j|} \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |S^2_{j+k,\lambda} f(x)|^2 \mathrm{M}_s M^{\tilde{\omega}}_{\lambda,s} u(x)\,dx \\
&\le C_{h,\Omega,q,\gamma,s} 2^{-(1-1/s)/(2q'\gamma')|j|} \int_{\mathbb{R}^n} |f(x)|^2 \mathrm{M}_s M^{\tilde{\omega}}_{\lambda,s} u(x)\,dx.
\end{aligned}
$$

Take $s = t_1$. Substituting u^{1/t_1} for u in (3-45), we get

$$
\text{(3-46)}\qquad \|G_{\lambda,j} f\|_{L^2(u^{1/t_1})} \le C_{h,\Omega,q,\gamma,\lambda,t_1} 2^{-(1-1/t_1)/(4q'\gamma')|j|} \|f\|_{L^2((\mathrm{M} M^{\tilde{\omega}}_{\lambda} u)^{1/t_1})}.
$$

It follows from (3-9) that

$$
\text{(3-47)}\qquad \mathrm{M} M^{\tilde{\omega}}_{\lambda} u \le \mathrm{M} M^{\tilde{\sigma}}_{\lambda} |u| + \mathrm{M}^2 M^{\tilde{\sigma}}_{\lambda-1} |u| \le \mathrm{M} M^{\tilde{\sigma}}_{\lambda} \mathrm{M} u + \mathrm{M}^2 M^{\tilde{\sigma}}_{\lambda-1} \mathrm{M} u.
$$

Formula (3-47) together with (3-46) implies that

$$\text{(3-48)}\quad \|G_{\lambda,j}f\|_{L^2(u^{1/t_1})} \le C_{h,\Omega,q,\gamma,\lambda,t_1}2^{-(1-1/t_1)/(4q'\gamma')|j|}\|f\|_{L^2((\mathrm{M}M_\lambda^{\tilde\sigma}\mathrm{M}u+\mathrm{M}^2M_{\lambda-1}^{\tilde\sigma}\mathrm{M}u)^{1/t_1})}.$$

By an interpolation between (3-43) and (3-48), we obtain

$$\text{(3-49)}\quad \|G_{\lambda,j}f\|_{L^p(u^{1/t_1})} \le C_{h,\Omega,q,\gamma,p,\lambda,t_1}2^{-\beta(p,q,\gamma,t_1)|j|}\|f\|_{L^p((\mathrm{M}M_\lambda^{\tilde\sigma}\mathrm{M}u+\mathrm{M}^2M_{\lambda-1}^{\tilde\sigma}\mathrm{M}u)^{1/t_1})}$$

for all $1 < p < 2$, where $\beta(p,q,\gamma,t_1) > 0$. We get from (3-49) and (3-34) that

$$\text{(3-50)}\quad \|G_\lambda^\omega f\|_{L^p(u^{1/t_1})} \le C_{h,\Omega,q,\gamma,p,\lambda,t_1}\|f\|_{L^p((\mathrm{M}M_\lambda^{\tilde\sigma}\mathrm{M}u+\mathrm{M}^2M_{\lambda-1}^{\tilde\sigma}\mathrm{M}u)^{1/t_1})}$$

for all $1 < p < 2$. Combining (3-50) with (3-32), (3-33), we now have

$$\text{(3-51)}\quad \int_{\mathbb{R}^n}(M_\lambda^\sigma f(x))^p u^{1/t_1}(x)\,dx \le C_{h,\Omega,q,\gamma,p,\lambda,t_1}\biggl(\int_{\mathbb{R}^n}(M_{\lambda-1}^\sigma|f|(x))^p(\mathrm{M}u)^{1/t_1}(x)\,dx + \int_{\mathbb{R}^n}|f(x)|^p(\mathrm{M}M_\lambda^{\tilde\sigma}\mathrm{M}u+\mathrm{M}^2M_{\lambda-1}^{\tilde\sigma}\mathrm{M}u)^{1/t_1}(x)\,dx\biggr)$$

for all $1 < p < 2$. We want to show that

$$\text{(3-52)}\quad \int_{\mathbb{R}^n}(M_\lambda^\sigma f(x))^p u^{1/t_1}(x)\,dx \le C_{h,\Omega,q,\gamma,p,\lambda,t_1}\int_{\mathbb{R}^n}|f(x)|^p(\mathrm{M}^\lambda u+\mathrm{M}^2\widetilde{\mathrm{M}^\lambda}u+H_\lambda u)^{1/t_1}(x)\,dx$$

for all $1 \le \lambda \le N$ and $1 < p < 2$. When $\lambda = 1$, (3-1) and (3-51) imply

$$\begin{aligned}\int_{\mathbb{R}^n}(M_1^\sigma f(x))^p u^{1/t_1}(x)\,dx &\le C_{h,\Omega,q,\gamma,p,t_1}\biggl(\int_{\mathbb{R}^n}(M_0^\sigma|f|(x))^p(\mathrm{M}u)^{1/t_1}(x)\,dx \\ &\qquad + \int_{\mathbb{R}^n}|f(x)|^p(\mathrm{M}M_1^{\tilde\sigma}\mathrm{M}u+\mathrm{M}^2M_0^{\tilde\sigma}\mathrm{M}u)^{1/t_1}(x)\,dx\biggr) \\ &\le C_{h,\Omega,q,\gamma,p,t_1}\int_{\mathbb{R}^n}|f(x)|^p(\mathrm{M}u+\mathrm{M}^2\widetilde{\mathrm{M}u}+\mathrm{M}M_1^{\tilde\sigma}\mathrm{M}u)^{1/t_1}(x)\,dx \\ &\le C_{h,\Omega,q,\gamma,p,t_1}\int_{\mathbb{R}^n}|f(x)|^p(\mathrm{M}u+\mathrm{M}^2\widetilde{\mathrm{M}u}+H_1u)^{1/t_1}(x)\,dx\end{aligned}$$

for any $1 < p < 2$. This yields (3-52) for $\lambda = 1$. Assume that (3-52) holds for $\lambda = \iota - 1$ with $\iota \in \{2, \dots, N\}$. We obtain, from (3-51) and our assumption,

$$\begin{aligned}
&\int_{\mathbb{R}^n} (M_\iota^\sigma f(x))^p u^{1/t_1}(x)\,dx \\
&\quad\le C_{h,\Omega,q,\gamma,p,\iota,t_1}\biggl(\int_{\mathbb{R}^n} (M_{\iota-1}^\sigma |f|(x))^p (\mathrm{M}u)^{1/t_1}(x)\,dx \\
&\qquad\qquad + \int_{\mathbb{R}^n} |f(x)|^p (\mathrm{M}M_\iota^{\tilde\sigma}\mathrm{M}u + \mathrm{M}^2 M_{\iota-1}^{\tilde\sigma}\mathrm{M}u)^{1/t_1}(x)\,dx\biggr) \\
&\quad\le C_{h,\Omega,q,\gamma,p,\iota,t_1}\biggl(\int_{\mathbb{R}^n} |f(x)|^p (\mathrm{M}^\iota\mathrm{M}u + \mathrm{M}^2\widetilde{\mathrm{M}^{\iota-1}\mathrm{M}}u + H_{\iota-1}\mathrm{M}u)^{1/t_1}(x)\,dx \\
&\qquad\qquad + \int_{\mathbb{R}^n} |f(x)|^p (\mathrm{M}M_\iota^{\tilde\sigma}\mathrm{M}u + \mathrm{M}^2 M_{\iota-1}^{\tilde\sigma}\mathrm{M}u)^{1/t_1}(x)\,dx\biggr) \\
&\quad\le C_{h,\Omega,q,\gamma,p,\iota,t_1}\int_{\mathbb{R}^n} |f(x)|^p (\mathrm{M}^\iota u + \mathrm{M}^2\widetilde{\mathrm{M}^\iota}u + H_\iota\mathrm{M}u)^{1/t_1}(x)\,dx
\end{aligned}$$

for all $1 < p < 2$. This yields (3-52) for $\lambda = \iota$. Thus, (3-52) is proved. Inequality (3-52) together with (3-9) and (3-33) yields that

$$\begin{aligned}
(3\text{-}53)\quad &\int_{\mathbb{R}^n} (\sup_{k\in\mathbb{Z}} |\omega_{k,\lambda} * f(x)|)^p u^{1/t_1}(x)\,dx \\
&\quad\le \int_{\mathbb{R}^n} (M_\lambda^\omega |f|(x))^p u^{1/t_1}(x)\,dx \\
&\quad\le \int_{\mathbb{R}^n} (M_\lambda^\sigma |f|(x))^p u^{1/t_1}(x)\,dx + \int_{\mathbb{R}^n} (\mathrm{M}M_{\lambda-1}^\sigma |f|(x))^p u^{1/t_1}(x)\,dx \\
&\quad\le C_{h,\Omega,q,\gamma,p,\lambda,t_1}\int_{\mathbb{R}^n} |f(x)|^p (\mathrm{M}^{\lambda+1}u + \mathrm{M}^2\widetilde{\mathrm{M}^\lambda}u + H_\lambda\mathrm{M}u)^{1/t_1}(x)\,dx
\end{aligned}$$

for all $1 < p < 2$. Since $\mathrm{M}M_\lambda^{\tilde\sigma}u + \mathrm{M}M_{\lambda-1}^{\tilde\sigma}\mathrm{M}u \le H_\lambda u$, an interpolation between (3-38) and (3-53) yields

$$\begin{aligned}
(3\text{-}54)\quad &\int_{\mathbb{R}^n} \biggl(\sum_{k\in\mathbb{Z}} |\omega_{k,\lambda} * f_k(x)|^2\biggr)^{p/2} u^{1/t_2}(x)\,dx \\
&\quad\le C_{h,\Omega,q,\gamma,p,\lambda,t_2}\int_{\mathbb{R}^n} \biggl(\sum_{k\in\mathbb{Z}} |f_k(x)|^2\biggr)^{p/2} (\mathrm{M}^\lambda u + \mathrm{M}^2\widetilde{\mathrm{M}^\lambda}u + H_\lambda u)^{1/t_2}(x)\,dx
\end{aligned}$$

for all $1 < p < 2$, where $1/t_2 = 1/t_1 + \frac{1}{2}p(1 - 1/t_1)$. Using (3-54) and arguments similar to those used in deriving (3-52), we obtain

$$\begin{aligned}
(3\text{-}55)\quad &\int_{\mathbb{R}^n} (M_\lambda^\sigma f(x))^p u^{1/t_2}(x)\,dx \\
&\qquad\le C_{h,\Omega,q,\gamma,p,\lambda,t_2}\int_{\mathbb{R}^n} |f(x)|^p (\mathrm{M}^\lambda u + \mathrm{M}^2\widetilde{\mathrm{M}^\lambda}u + H_\lambda u)^{1/t_2}(x)\,dx
\end{aligned}$$

for all $1 \le \lambda \le N$ and $1 < p < 2$. As the same reason as above, we can obtain a strictly decreasing sequence $\{t_k\}_{k\in\mathbb{N}}$ by the recursion formula

$$t_1 = \frac{2}{p}, \quad \frac{1}{t_{k+1}} = \frac{1}{t_k} + \frac{p}{2}\Big(1 - \frac{1}{t_k}\Big), \quad k = 2, 3, \dots$$

such that

$$\text{(3-56)} \quad \int_{\mathbb{R}^n} (M_\lambda^\sigma f(x))^p u^{1/t_k}(x)\,dx \le C_{h,\Omega,q,\gamma,p,\lambda,t_k} \int_{\mathbb{R}^n} |f(x)|^p (\mathrm{M}^\lambda u + \mathrm{M}^2 \widetilde{\mathrm{M}^\lambda} u + H_\lambda u)^{1/t_k}(x)\,dx$$

for all $1 \le \lambda \le N$, $1 < p < 2$, and all $k \in \mathbb{N}$. Using (3-12), (3-56), and the well-known Fefferman–Stein inequality for M (see (3-102) below), we have

$$\begin{aligned}
\text{(3-57)} \quad & \int_{\mathbb{R}^n} (M_\lambda^\mu f(x))^p u^{1/t_k}(x)\,dx \\
&\le \int_{\mathbb{R}^n} (\mathrm{M}^{N-\lambda} M_\lambda^\sigma |f|(x))^p u^{1/t_k}(x)\,dx + \int_{\mathbb{R}^n} (\mathrm{M}^{N-\lambda+1} M_{\lambda-1}^\sigma |f|(x))^p u^{1/t_k}(x)\,dx \\
&\le C_p \Big(\int_{\mathbb{R}^n} (M_\lambda^\sigma |f|(x))^p (\mathrm{M}^{N-\lambda} u)^{1/t_k}(x)\,dx + \int_{\mathbb{R}^n} (M_{\lambda-1}^\sigma |f|(x))^p (\mathrm{M}^{N-\lambda+1} u)^{1/t_k}(x)\,dx \Big) \\
&\le C_{h,\Omega,q,\gamma,p,\lambda,t_k} \Big(\int_{\mathbb{R}^n} |f(x)|^p \Big(\mathrm{M}^\lambda (\mathrm{M}^{N-\lambda} u) + \mathrm{M}^2 \mathrm{M}^\lambda \widetilde{(\mathrm{M}^{N-\lambda}} u) + H_\lambda (\mathrm{M}^{N-\lambda} u) \Big)^{1/t_k}(x)\,dx \\
&\quad + \int_{\mathbb{R}^n} |f(x)|^p \Big(\mathrm{M}^{\lambda-1} (\mathrm{M}^{N-\lambda+1} u) + \mathrm{M}^2 \mathrm{M}^{\lambda-1} \widetilde{(\mathrm{M}^{N-\lambda+1}} u) + H_{\lambda-1} (\mathrm{M}^{N-\lambda+1} u) \Big)^{1/t_k}(x)\,dx \Big) \\
&\le C_{h,\Omega,q,\gamma,p,\lambda,t_k} \int_{\mathbb{R}^n} |f(x)|^p (\mathrm{M}^N u + \mathrm{M}^2 \widetilde{\mathrm{M}^N} u + H_N u)^{1/t_k}(x)\,dx.
\end{aligned}$$

By (3-57) and the lemma in [Zhang 2008, p.1574] we can get (3-31). □

Proof of Theorem 1.2. For $1 \le \lambda \le N$, let $\Theta_{\lambda,s}$ be given as in Theorem 1.2. We shall prove Theorem 1.2 by combining the method used in the proof of [Zhang 2008, Theorem 1.2] with ideas from [Duoandikoetxea and Rubio de Francia 1986; Fan et al. 1999]. For any $\epsilon > 0$, there exists an integer k such that $2^{k-1} \le \epsilon < 2^k$. We now write

$$\text{(3-58)} \qquad T_{P_N}^* f(x) \le M_N^\sigma f(x) + \sup_{k\in\mathbb{Z}} \Big| \sum_{j=k}^{\infty} \sigma_{j,N} * f(x) \Big|.$$

We shall prove Theorem 1.2 by considering the following two steps:

Step 1: The proof of (i) of Theorem 1.2. By (3-58), to prove (1-9), it suffices to show that

$$\|M_N^\sigma f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,N,s}\|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s u)} \tag{3-59}$$

and

$$\Big\|\sup_{k\in\mathbb{Z}}\Big|\sum_{j=k}^{\infty}\sigma_{j,N}*f\Big|\Big\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,N,s}\|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s u+\Theta_{N,s}\mathrm{M}_s^2 u)} \tag{3-60}$$

for all $2 \le p < \infty$, $s > 1$, and any nonnegative measurable function u on $\mathbb{R}^n$.

Let us first prove (3-59). Fix $u \in A_1$ and $1 \le \lambda \le N$. By arguments similar to those used in deriving (3-25),

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\omega_{k,\lambda}*g_k|^2\Big)^{1/2}\Big\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,s}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(M^{\tilde{\omega}}_{\lambda,s}u)} \tag{3-61}$$

for all $2 < p < \infty$ and any $s > 1$. It follows from (3-61) that

$$\begin{aligned}\|G_{\lambda,j}f\|_{L^p(u)} &= \Big\|\Big(\sum_{k\in\mathbb{Z}}|\omega_{k,\lambda}*S^3_{j+k}f|^2\Big)^{1/2}\Big\|_{L^p(u)}\\ &\le C_{h,\Omega,q,\gamma,p,s}\Big\|\Big(\sum_{k\in\mathbb{Z}}|S^3_{j+k}f|^2\Big)^{1/2}\Big\|_{L^p(\mathrm{M}_s M^{\tilde{\omega}}_{\lambda,s}u)}\\ &\le C_{h,\Omega,q,\gamma,p,\lambda,s}\|f\|_{L^p(\mathrm{M}_s M^{\tilde{\omega}}_{\lambda,s}u)}\end{aligned} \tag{3-62}$$

for all $2 < p < \infty$ and any $s > 1$. In the last inequality of (3-62), we used the weighted Littlewood–Paley theory and the fact that $\mathrm{M}_s M^{\tilde{\omega}}_{\lambda,s}u \in A_1$. On the other hand, it follows from (3-45) that

$$\|G_{\lambda,j}f\|_{L^2(u)} \le C_{h,\Omega,q,\gamma,\lambda,s}2^{-(1-1/s)/(4q'\gamma')|j|}\|f\|_{L^2(\mathrm{M}_s M^{\tilde{\omega}}_{\lambda,s}u)}. \tag{3-63}$$

By (3-62), (3-63), and an interpolation (see [Bergh and Löfström 1976, Corollary 5.5.4]), we have

$$\|G_{\lambda,j}f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s}2^{-\vartheta(p,q,\gamma,s)|j|}\|f\|_{L^p(\mathrm{M}_s M^{\tilde{\omega}}_{\lambda,s}u)} \tag{3-64}$$

for all $2 \le p < \infty$ and $s > 1$, where $\vartheta(p,q,\gamma,s) > 0$ depends on p, q, γ and s. Combining (3-64) with (3-34) yields that

$$\|G^\omega_\lambda f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s}\|f\|_{L^p(\mathrm{M}_s M^{\tilde{\omega}}_s u)} \tag{3-65}$$

for all $2 \le p < \infty$ and $s > 1$. We get from (3-9) that

$$\mathrm{M}_s M^{\tilde{\omega}}_{\lambda,s}u \le C(\mathrm{M}_s M^{\tilde{\sigma}}_{\lambda,s}|u| + \mathrm{M}_s^2 M^{\tilde{\sigma}}_{\lambda-1,s}|u|). \tag{3-66}$$

Inequality (3-66) together with (3-65) yields

$$\|G_\lambda^{\omega} f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s}\|f\|_{L^p(\mathrm{M}_s M_{\lambda,s}^{\tilde{\sigma}}|u|+\mathrm{M}_s^2 M_{\lambda-1,s}^{\tilde{\sigma}}|u|)} \tag{3-67}$$

for all $2 \le p < \infty$ and $s > 1$. On the other hand, from (3-10), (3-67), and the well-known Fefferman–Stein inequality for M (see (3-102) below) we have

$$\begin{aligned}\|M_\lambda^{\sigma} f\|_{L^p(u)} &\le \|\mathrm{M} M_{\lambda-1}^{\sigma}|f|\|_{L^p(u)} + \|G_\lambda^{\omega} f\|_{L^p(u)}\\ &\le C_p\|M_{\lambda-1}^{\sigma}|f|\|_{L^p(\mathrm{M}u)} + C_{h,\Omega,q,\gamma,p,\lambda,s}\|f\|_{L^p(\mathrm{M}_s M_{\lambda,s}^{\tilde{\sigma}}|u|+\mathrm{M}_s^2 M_{\lambda-1,s}^{\tilde{\sigma}}|u|)}\end{aligned} \tag{3-68}$$

for all $2 \le p < \infty$ and any $s > 1$. Formula (3-68) together with (3-1), (3-2), and an induction argument implies that

$$\|M_\lambda^{\sigma} f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s}\|f\|_{L^p(\mathrm{M}^\lambda u+I_{\lambda,s}u+J_{\lambda,s}u)} \quad \forall 1 \le \lambda \le N. \tag{3-69}$$

Since $u \le \mathrm{M}_s u$ and $\mathrm{M}_s u \le A_1$, (3-69) leads to

$$\|M_\lambda^{\sigma} f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s}\|f\|_{L^p(\mathrm{M}_s^{\lambda+1}u+I_{\lambda,s}\mathrm{M}_s u+J_{\lambda,s}\mathrm{M}_s u)} \tag{3-70}$$

for all $2 \le p < \infty$, $s > 1$, and any nonnegative function u on $\mathbb{R}^n$. Inequality (3-70) yields that (3-59) holds for all $2 \le p < \infty$.

We now prove (3-60). It follows from (3-11) that

$$\sup_{k\in\mathbb{Z}}\Bigl|\sum_{j=k}^{\infty}\sigma_{j,N} * f(x)\Bigr| \le \sum_{\lambda=1}^{N}\sup_{k\in\mathbb{Z}}\Bigl|\sum_{j=k}^{\infty}\mu_{j,\lambda} * f(x)\Bigr| =: \sum_{\lambda=1}^{N} T_\lambda^* f(x). \tag{3-71}$$

Fix $1 \le \lambda \le N$, let $\psi_{k,\lambda}$ be given as in (3-7). We write

$$\sum_{j=k}^{\infty}\mu_{j,\lambda}*f(x) = \psi_{k,\lambda}*T_\lambda f(x) - \psi_{k,\lambda}*\sum_{j=-\infty}^{k-1}\mu_{j,\lambda}*f(x) + (\delta-\psi_{k,\lambda})*\sum_{j=k}^{\infty}\mu_{j,\lambda}*f(x).$$

Here, δ is the Dirac-delta and T_λ is given as in (3-14). It follows that

$$\begin{aligned}T_\lambda^* f(x) &\le \sup_{k\in\mathbb{Z}}|\psi_{k,\lambda} * T_\lambda f(x)| + \sup_{k\in\mathbb{Z}}\Bigl|\psi_{k,\lambda} * \sum_{j=-\infty}^{k-1}\mu_{j,\lambda} * f(x)\Bigr|\\ &\qquad + \sup_{k\in\mathbb{Z}}\Bigl|(\delta-\psi_{k,\lambda}) * \sum_{j=k}^{\infty}\mu_{j,\lambda} * f(x)\Bigr|\\ &=: A_{1,\lambda} f(x) + A_{2,\lambda} f(x) + A_{3,\lambda} f(x).\end{aligned} \tag{3-72}$$

For $A_{1,\lambda} f$, by the well-known Fefferman–Stein inequality for M (see (3-102) below) and (3-15), we obtain

$$\begin{aligned} \|A_{1,\lambda} f\|_{L^p(u)} &\le \|\mathrm{M}(T_\lambda f)\|_{L^p(u)} \le C_p \|T_\lambda f\|_{L^p(\mathrm{M}u)} \\ &\le C_{h,\Omega,q,\gamma,p,\lambda,s} \|f\|_{L^p(\Lambda_{N,s}\mathrm{M}u)} \\ &\le C_{h,\Omega,q,\gamma,p,\lambda,s} \|f\|_{L^p(\Theta_{N,s}\mathrm{M}u)} \end{aligned} \tag{3-73}$$

for all $2 \le p < \infty$, $s > 1$, and any nonnegative measurable function u on $\mathbb{R}^n$.

For $A_{2,\lambda} f$, it is clear that

$$\begin{aligned} A_{2,\lambda} f(x) &= \sup_{k\in\mathbb{Z}} \Big| \sum_{j=1}^{\infty} \psi_{k,\lambda} * \mu_{k-j,\lambda} * f(x) \Big| \\ &\le \sum_{j=1}^{\infty} \sup_{k\in\mathbb{Z}} |\psi_{k,\lambda} * \mu_{k-j,\lambda} * f(x)| =: \sum_{j=1}^{\infty} I_j f(x). \end{aligned}$$

Consequently,

$$\|A_{2,\lambda} f\|_{L^p(u)} \le \sum_{j=1}^{\infty} \|I_j f\|_{L^p(u)} \tag{3-74}$$

for all $1 < p < \infty$ and any nonnegative measurable function u on $\mathbb{R}^n$. We shall show that

$$\|I_j f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s} \|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s^2 u)} \tag{3-75}$$

for all $2 \le p < \infty$, any $s > 1$, and any nonnegative measurable function u on $\mathbb{R}^n$. We get by the well-known Fefferman–Stein inequality for M (see (3-102) below), (3-12), and (3-70), that

$$\begin{aligned} &\|I_j f\|_{L^p(u)} \\ &\quad \le \|\mathrm{M} M_\lambda^{\mu} |f|\|_{L^p(u)} \le C_p \|M_\lambda^{\mu} |f|\|_{L^p(\mathrm{M}u)} \\ &\quad \le C_p (\|M_\lambda^{\sigma} |f|\|_{L^p(\mathrm{M}^{N-\lambda+1}u)} + \|M_{\lambda-1}^{\sigma} |f|\|_{L^p(\mathrm{M}^{N-\lambda+2}u)}) \\ &\quad \le C_{h,\Omega,q,\gamma,p,\lambda,s} \|f\|_{L^p(\mathrm{M}_s^{N+2}u + I_{\lambda,s}\mathrm{M}_s^{N-\lambda+2}u + I_{\lambda,s}\mathrm{M}_s^{N-\lambda+3}u + J_{\lambda,s}\mathrm{M}_s^{N-\lambda+2}u + J_{\lambda-1,s}\mathrm{M}_s^{N-\lambda+3}u)} \\ &\quad \le C_{h,\Omega,q,\gamma,p,\lambda,s} \|f\|_{L^p(\mathrm{M}_s^{N+2}u + I_{N,s}\mathrm{M}_s^2 u + J_{N,s}\mathrm{M}_s^2 u)} \end{aligned}$$

for all $2 \le p < \infty$, any $s > 1$, and any nonnegative measurable function u on $\mathbb{R}^n$. This proves (3-75).

On the other hand, by (3-13) and Plancherel's theorem, we get

$$\begin{aligned}\|I_j f\|^2_{L^2(\mathbb{R}^n)} &\le \Big\|\Big(\sum_{k\in\mathbb{Z}} |\psi_{k,\lambda} * \mu_{k-j,\lambda} * f|^2\Big)^{1/2}\Big\|^2_{L^2(\mathbb{R}^n)}\\ &\le \sum_{k\in\mathbb{Z}} \int_{\{|a_\lambda \xi|\le 2^{-k\lambda}\}} |\hat{\mu}_{k-j,\lambda}(\xi)|^2 |\hat{f}(\xi)|^2\, d\xi\\ &\le C\int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |\hat{\mu}_{k-j,\lambda}(\xi)|^2 \chi_{\{|a_\lambda\xi|\le 2^{-k\lambda}\}} |\hat{f}(\xi)|^2\, d\xi\\ &\le C_{h,\Omega,q,\gamma} \sup_{\xi\in\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |a_\lambda 2^{\lambda(k-j)}\xi|^{1/(2\lambda q'\gamma')} \chi_{\{|a_\lambda\xi|\le 2^{-k\lambda}\}} \|f\|^2_{L^2(\mathbb{R}^n)}\\ &\le C^{-j/(2q'\gamma')}_{h,\Omega,q,\gamma} \sup_{\xi\in\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |2^{k\lambda} a_\lambda \xi|^{1/(2\lambda q'\gamma')} \chi_{\{|a_\lambda\xi|\le 2^{-k\lambda}\}} \|f\|^2_{L^2(\mathbb{R}^n)}\\ &\le C_{h,\Omega,q,\gamma} 2^{-j/(2q'\gamma')} \|f\|^2_{L^2(\mathbb{R}^n)},\end{aligned}$$

where the last inequality is obtained by the properties of the lacunary sequence. It follows that

$$\|I_j f\|_{L^2(\mathbb{R}^n)} \le C_{h,\Omega,q,\gamma} 2^{-j/(4q'\gamma')} \|f\|_{L^2(\mathbb{R}^n)}. \tag{3-76}$$

On the other hand, by (3-75) with $p = 2$ and u replacing u^s, we get

$$\|I_j f\|_{L^2(u^s)} \le C_{h,\Omega,q,\gamma,\lambda,s} \|f\|_{L^2(\Theta_{N,s} \mathrm{M}^2_s u^s)}. \tag{3-77}$$

An interpolation between (3-76) and (3-77) yields

$$\begin{aligned}\|I_j f\|_{L^2(u)} &\le C_{h,\Omega,q,\gamma,\lambda,s} 2^{-(1-1/s)/(4q'\gamma')j} \|f\|_{L^2((\Theta_{N,s}\mathrm{M}^2_s u^s)^{1/s})}\\ &\le C_{h,\Omega,q,\gamma,\lambda,s} 2^{-(1-1/s)/(4q'\gamma')j} \|f\|_{L^2(\Theta_{N,s^2}\mathrm{M}^2_{s^2} u)}.\end{aligned} \tag{3-78}$$

Take s^2 replacing s. Formula (3-78) leads to

$$\|I_j f\|_{L^2(u)} \le C_{h,\Omega,q,\gamma,\lambda,s} 2^{-(1-1/\sqrt{s})/(4q'\gamma')} \|f\|_{L^2(\Theta_{N,s}\mathrm{M}^2_s u)}. \tag{3-79}$$

Interpolating between (3-79) and (3-75) (see [Bergh and Löfström 1976, Corollary 5.5.4]) yields

$$\|I_j f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s} 2^{-\varsigma(p,q,\gamma,s)j} \|f\|_{L^p(\Theta_{N,s}\mathrm{M}^2_s u)}, \tag{3-80}$$

for all $2 \le p < \infty$, where $\varsigma(p, q, \gamma, s) > 0$. Thus, we get from (3-80) and (3-74),

$$\|A_{2,\lambda} f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s} \|f\|_{L^p(\Theta_{N,s}\mathrm{M}^2_s u)} \tag{3-81}$$

for all $2 \le p < \infty$, $s > 1$, and any nonnegative measurable function u on $\mathbb{R}^n$.

Finally we estimate $A_{3,\lambda} f$. Obviously,

$$\begin{aligned} A_{3,\lambda} f(x) &= \sup_{k\in\mathbb{Z}}\Big|\sum_{j=0}^{\infty}(\delta-\psi_{k,\lambda})*\mu_{k+j,\lambda}*f(x)\Big| \\ &\le \sum_{j=0}^{\infty}\sup_{k\in\mathbb{Z}}|(\delta-\psi_{k,\lambda})*\mu_{k+j,\lambda}*f(x)| \\ &=: \sum_{j=0}^{\infty} J_j f(x). \end{aligned}$$

It follows that

$$\|A_{3,\lambda} f\|_{L^p(u)} \le \sum_{j=0}^{\infty}\|J_j f\|_{L^p(u)} \tag{3-82}$$

for all $1 < p < \infty$ and any nonnegative measurable function u on $\mathbb{R}^n$. By the argument similar to those used to derive (3-75),

$$\|J_j f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s}\|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s^2 u)} \tag{3-83}$$

for all $2 \le p < \infty$, any $s > 1$, and any nonnegative measurable function u on $\mathbb{R}^n$. On the other hand, using (3-13) and the Plancherel theorem, we can obtain

$$\begin{aligned} \|J_j f\|^2_{L^2(\mathbb{R}^n)} &\le \bigg\|\bigg(\sum_{k\in\mathbb{Z}}|(\delta-\psi_{k,\lambda})*\mu_{j+k,\lambda}*f|^2\bigg)^{1/2}\bigg\|^2_{L^2(\mathbb{R}^n)} \\ &\le \sum_{k\in\mathbb{Z}}\int_{\{2^{k\lambda}a_\lambda\xi|\ge 1\}}|\hat{\mu}_{j+k,\lambda}(\xi)|^2|\hat{f}(\xi)|^2\,d\xi \\ &\le \sum_{k\in\mathbb{Z}}\sum_{i=-\infty}^{k}\int_{\{2^{-\lambda i}\le|a_\lambda\xi|<2^{-\lambda(i-1)}\}}|\hat{\mu}_{j+k,\lambda}(\xi)|^2|\hat{f}(\xi)|^2\,d\xi \\ &\le C_{h,\Omega,q,\gamma}\sum_{k\in\mathbb{Z}}\sum_{i=-\infty}^{k}2^{-(j+k-i)/(2q'\gamma')}\int_{\{2^{-\lambda i}\le|a_\lambda\xi|<2^{-\lambda(i-1)}\}}|\hat{f}(\xi)|^2\,d\xi \\ &\le C_{h,\Omega,q,\gamma}2^{-j/(2q'\gamma')}\sum_{i=0}^{\infty}2^{-i/(2q'\gamma')}\|f\|^2_{L^2(\mathbb{R}^n)} \\ &\le C_{h,\Omega,q,\gamma}2^{-j/(2q'\gamma')}\|f\|^2_{L^2(\mathbb{R}^n)}. \end{aligned}$$

It follows that

$$\|J_j f\|_{L^2(\mathbb{R}^n)} \le C_{h,\Omega,q,\gamma}2^{-j/(4q'\gamma')}\|f\|_{L^2(\mathbb{R}^n)}. \tag{3-84}$$

By (3-83), (3-84), and arguments similar to those used in deriving (3-80),

$$\|J_j f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s} 2^{-\tau(p,q,\gamma,s)j} \|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s^2 u)} \tag{3-85}$$

for all $2 \le p < \infty$ and $s > 1$, where $\tau(p,q,\gamma,s) > 0$. Equation (3-85) together with (3-82) yields

$$\|A_{3,\lambda} f\|_{L^p(u)} \le C_{h,\Omega,q,\lambda,\gamma} \|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s^2 u)} \tag{3-86}$$

for all $2 \le p < \infty$ and $s > 1$. Then (3-60) follows from (3-71)–(3-73), (3-81), and (3-86). □

Step 2: The proof of (ii) of Theorem 1.2. Let $1 < p < 2$ and $\{t_k\}_{k\in\mathbb{N}}$ be given as in Theorem 1.2. Fix a nonnegative measurable function u on $\mathbb{R}^n$. By (3-58), to prove (1-10), it suffices to show that

$$\|M_N^\sigma f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,N,s,t_k} \|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s u)} \quad \forall s > t_k \tag{3-87}$$

and

$$\begin{aligned}\Bigl\|\sup_{k\in\mathbb{Z}}\Bigl|\sum_{j=k}^{\infty}\sigma_{j,N} * f\Bigr|\Bigr\|_{L^p(u)} \\ \le C_{h,\Omega,q,\gamma,p,N,s,t_k} \|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s u+\Theta_{N,s}\mathrm{M}_s^2 u)} \quad \forall s > t_k\end{aligned} \tag{3-88}$$

for all $k \in \mathbb{N}$.

We first prove (3-87). Fix $k \in \mathbb{N}$. Substitute u^{t_k} for u in (3-56), one has

$$\begin{aligned}\int_{\mathbb{R}^n} (M_\lambda^\sigma f(x))^p u(x)\,dx \\ \le C_{h,\Omega,q,\gamma,p,\lambda,t_k} \int_{\mathbb{R}^n} |f(x)|^p (\mathrm{M}^\lambda u^{t_k} + \mathrm{M}^2\widetilde{\mathrm{M}^\lambda u^{t_k}} + H_\lambda u^{t_k})^{1/t_k}(x)\,dx\end{aligned} \tag{3-89}$$

for all $1 \le \lambda \le N$. Notice that

$$(\mathrm{M}^\lambda u^{t_k} + \mathrm{M}^2\widetilde{\mathrm{M}^\lambda u^{t_k}} + H_\lambda u^{t_k})^{1/t_k}(x) \le C_{s,t_k}(\mathrm{M}^\lambda u^s + \mathrm{M}^2\widetilde{\mathrm{M}^\lambda u^s} + H_\lambda u^s)^{1/s}(x)$$

for any $s > t_k$ by Hölder's inequality. Then (3-89) yields that

$$\begin{aligned}\int_{\mathbb{R}^n} (M_\lambda^\sigma f(x))^p u(x)\,dx \\ \le C_{h,\Omega,q,\gamma,p,\lambda,t_k} \int_{\mathbb{R}^n} |f(x)|^p (\mathrm{M}_s^\lambda u + \mathrm{M}_s^2\widetilde{\mathrm{M}_s^\lambda} u + H_{\lambda,s} u)(x)\,dx \quad \forall s > t_k\end{aligned} \tag{3-90}$$

holds for all $1 \le \lambda \le N$ and any fixed positive integer k, which gives (3-87).

Below we prove (3-88). For $A_{1,\lambda} f$, by the well-known Fefferman–Stein inequality for M (see (3-102) below) and (3-30), we obtain

$$\begin{aligned}\|A_{1,\lambda} f\|_{L^p(u)} &\le C\|\mathrm{M}(T_\lambda f)\|_{L^p(u)} \le C_p\|T_\lambda f\|_{L^p(\mathrm{M}u)}\\ &\le C_{h,\Omega,q,\gamma,p,\lambda,s,t_k}\|f\|_{L^p(\Lambda_{N,s}\mathrm{M}u)}\\ &\le C_{h,\Omega,q,\gamma,p,\lambda,s,t_k}\|f\|_{L^p(\Theta_{N,s}\mathrm{M}u)} \quad \forall s > t_k\end{aligned} \tag{3-91}$$

for any fixed positive integer k.

For $A_{2,\lambda} f$, it follows from the well-known Fefferman–Stein inequality for M (see (3-102) below), (3-12), and (3-91) that

$$\begin{aligned}&\|I_j f\|_{L^p(u)}\\ &\quad\le C\|\mathrm{M}M_\lambda^\mu f\|_{L^p(u)} \le C_p\|M_\lambda^\mu f\|_{L^p(\mathrm{M}u)}\\ &\quad\le C_p(\|M_\lambda^\sigma|f|\|_{L^p(\mathrm{M}^{N-\lambda+1}u)} + \|M_{\lambda-1}^\sigma|f|\|_{L^p(\mathrm{M}^{N-\lambda+2}u)})\\ &\quad\le C_{h,\Omega,q,\gamma,p,\lambda,s,t_k}\|f\|_{L^p(\mathrm{M}_s^N Mu+\mathrm{M}_s^2\widetilde{\mathrm{M}_s^N\mathrm{M}u}+H_{\lambda,s}\mathrm{M}^{N-\lambda+1}u+H_{\lambda-1,s}\mathrm{M}^{N-\lambda+2}u)}\\ &\quad\le C_{h,\Omega,q,\gamma,p,\lambda,s,t_k}\|f\|_{L^p(\mathrm{M}_s^N\mathrm{M}u+\mathrm{M}_s^2\widetilde{\mathrm{M}_s^N\mathrm{M}u}+H_{N,s}\mathrm{M}u)}\\ &\quad\le C_{h,\Omega,q,\gamma,p,\lambda,s,t_k}\|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s^2u)} \quad \forall s > t_k\end{aligned} \tag{3-92}$$

for any fixed positive integer k. Interpolating between (3-79) and (3-92) (see [Bergh and Löfström 1976, Corollary 5.5.4]) yields

$$\|I_j f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s,t_k}2^{-\delta(p,q,\gamma,s)j}\|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s^2u)} \quad \forall s > t_k, \tag{3-93}$$

where $\delta(p, q, \gamma, s) > 0$. Thus, we get from (3-93) and (3-74) that

$$\|A_{2,\lambda} f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s,t_k}\|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s^2u)} \quad \forall s > t_k \tag{3-94}$$

for any fixed positive integer k.

For $A_{3,\lambda} f$, by the argument similar to those used to derive (3-92),

$$\|J_j f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s,t_k}\|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s^2u)} \quad \forall s > t_k \tag{3-95}$$

for any fixed positive integer k. By (3-95), (3-84), and arguments similar to those used in deriving (3-85),

$$\|J_j f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s,t_k}2^{-o(p,q,\gamma,s)j}\|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s^2u)} \quad \forall s > t_k \tag{3-96}$$

for any fixed positive integer k, where $o(p, q, \gamma, s) > 0$. Inequality (3-96) together with (3-82) yields

$$\|A_{3,\lambda} f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s,t_k}\|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s^2u)} \quad \forall s > t_k \tag{3-97}$$

for any fixed positive integer k. Then (3-88) follows from (3-71), (3-72), (3-92), (3-94), and (3-97).

We now turn to proving Corollaries 1.3–1.5.

Proof of Corollary 1.3. By (3-6), one finds that

$$\|\Lambda_{N,s} f\|_{L^r(\mathbb{R}^n)} \le C_{h,\Omega,q,\gamma} \|f\|_{L^r(\mathbb{R}^n)} \tag{3-98}$$

for any $1 < s < \infty$ and $r > s$. We let $\{t_k\}$ be the sequence as in (ii) of Theorem 1.1 when $1 < p < 2$, and, for the sake of convenience, we set $\{t_k\} = \{1 + 1/k\}$ when $2 \le p < \infty$. It is clear that $\{t_k\}_{k\in\mathbb{N}}$ is strictly decreasing and $\lim_{k\to\infty} t_k = 1$. It follows from (1-7) and (1-8) that for $1 < p < \infty$, it holds that

$$\|T_{P_N} f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,s,N,t_k} \|f\|_{L^p(\Lambda_{N,s} u)} \quad \forall s > t_k \tag{3-99}$$

for any fixed positive integer k and any nonnegative measurable function u on $\mathbb{R}^n$. By (3-98), (3-99), and Proposition 2.1, we have (1-11) and (1-12) for the case of $1 < \tilde{p} < p < \infty$. On the other hand, by duality, we can obtain (1-11) and (1-12) for the case of $1 < p < \tilde{p} < \infty$. Taking $u = 1$, we obtain $\Lambda_{N,s} u \le C$. This together with (3-99) yields that T_{P_N} is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. It leads to (1-11) for the case $p = \tilde{p}$ by (1-1). The inequality (1-12) for the case $p = \tilde{p}$ follows from (1-1) and (1-15). This completes the proof of Corollary 1.3. □

Proof of Corollary 1.4. By (3-6), one finds that

$$\|\Theta_{N,s}\mathrm{M}_s u + \Theta_{N,s}\mathrm{M}_s^2 u\|_{L^r(\mathbb{R}^n)} \le C_{h,\Omega,q,\gamma} \|u\|_{L^r(\mathbb{R}^n)} \tag{3-100}$$

for any $1 < s < \infty$ and $r > s$. We let $\{t_k\}$ be the sequence as in (ii) of Theorem 1.1 when $1 < p < 2$, and, for the sake of convenience, we set $\{t_k\} = \{1 + 1/k\}$ when $2 \le p < \infty$. Then Theorem 1.2 yields

$$\|T^*_{P_N} f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,s,N,t_k} \|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s u + \Theta_{N,s}\mathrm{M}_s^2 u)} \quad \forall s > t_k \tag{3-101}$$

for any fixed positive integer k and any nonnegative measurable function u on $\mathbb{R}^n$. By (3-100), (3-101), and Proposition 2.1, we obtain (1-13) and (1-14) for the case $1 < \tilde{p} < p < \infty$. It was known that $T^*_{P_N}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. This together with (1-1) yields (1-13) for the case of $p = \tilde{p}$. The inequality (1-14) for the case of $p = \tilde{p}$ follows from (1-1) and (1-16). This finishes the proof of Corollary 1.4. □

Proof of Corollary 1.5. By (3-98), (3-99), and Proposition 2.1, we obtain (1-15) for the case of $1 < \tilde{p} < p < \infty$. On the other hand, a duality argument yields (1-15) for the case of $1 < p < \tilde{p} < \infty$. Inequality (1-15) for the case $p = \tilde{p}$ follows from (3-98), (3-99), and the L^p bounds for T_{P_N}. Similarly, we can obtain (1-16) for the case of $1 < \tilde{p} \le p < \infty$ by (3-100), (3-101), and the L^p boundedness of $T^*_{P_N}$. This completes the proof of Corollary 1.5. □

We want to make a few remarks before ending the paper. Since Proposition 2.1 plays a crucial rule to show the boundedness of an operator T on the mixed radial-angular space $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)$, we expect to establish a suitable weighted L^p inequality for T. To this end, for the operator $T_{h,\Omega}$, we need to treat some technical difficulties for different assumptions on Ω. This is a key step, but is definitely not trivial. For instance, we have no idea how to establish a suitable weighted L^p inequality for $T_{h,\Omega}$, although the $L^p(\mathbb{R}^n)$ boundedness of $T_{h,\Omega}$ is well known, if Ω is a function in the function class $L\log L(\mathrm{S}^{n-1})$. For the singular integral T_Ω, another roughness assumption on Ω is that Ω lies in the Grafakos–Stefanov class $\mathcal{F}_\alpha(\mathrm{S}^{n-1})$, where

$$\mathcal{F}_\alpha(\mathrm{S}^{n-1}) \\ := \left\{\Omega \in L^1(\mathrm{S}^{n-1}) : \sup_{\xi\in \mathrm{S}^{n-1}} \int_{\mathrm{S}^{n-1}} |\Omega(y')| \left(\log \frac{1}{|\xi\cdot y'|}\right)^\alpha d\sigma(y') < \infty\right\} \quad \text{for} \quad \alpha > 0,$$

and this class was originally introduced by Grafakos and Stefanov [1998] in the study of L^p boundedness of T_Ω. With the help of the established weighted L^p inequality for T_Ω (see [Zhang 2008, Lemma 2]) applying [Zhang 2008, Theorems 1 and 2], and Proposition 2.1, we can show that both T_Ω and its maximal operator T^*_Ω are bounded on $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)$ for any $1 < p < \infty$ and $1 < \tilde{p} < \infty$ provided $\Omega \in \mathcal{F}_\alpha(\mathrm{S}^{n-1})$ for all $\alpha > 1$.

Not only for rough singular integrals, Proposition 2.1 also works for all linear or sublinear operators. The Hardy–Littlewood maximal function M is bounded on $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)$, based on Proposition 2.1 and the well-known Fefferman–Stein [1971] weighted norm inequality

$$\|\mathrm{M}f\|_{L^p(u)} \le C_p\|f\|_{L^p(\mathrm{M}u)}. \tag{3-102}$$

Also, any Calderón–Zygmund operator T is bounded on $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)$ for any $1 < p < \infty$ and $1 < \tilde{p} < \infty$ because of Proposition 2.1 and the well-known inequality

$$\|\mathrm{T}f\|_{L^p(u)} \le C_p\|f\|_{L^p(\mathrm{M}_s u)}, \quad s > 1.$$

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∞-TILTING THEORY

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We define the notion of an infinitely generated tilting object of infinite homological dimension in an abelian category. A one-to-one correspondence between ∞-tilting objects in complete, cocomplete abelian categories with an injective cogenerator and ∞-cotilting objects in complete, cocomplete abelian categories with a projective generator is constructed. We also introduce ∞-tilting pairs, consisting of an ∞-tilting object and its ∞-tilting class, and obtain a bijective correspondence between ∞-tilting and ∞-cotilting pairs. Finally, we discuss the related derived equivalences and t-structures.

Introduction

The phrase "tilting theory" is often used to refer to a well-developed general machinery for producing equivalences between triangulated categories (see [Angeleri Hügel et al. 2007] for an introduction, history and applications). Such equivalences are often represented by a distinguished object, a so-called tilting object, and it is crucial to most of the theory that such a tilting object is homologically small. If A is an abelian category with exact coproducts (e.g., a category of modules over a ring or sheaves on a topological space) and T is a tilting object, the smallness typically translates at least to the assumptions that T is finitely generated and of finite projective dimension.

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In this paper we introduce and systematically develop ∞-tilting theory, where all homological smallness assumptions are dropped. This brings under one roof various concepts and results from the literature:

(1) Wakamatsu tilting modules [Mantese and Reiten 2004; Wakamatsu 1988; 1990] over finite-dimensional algebras.

(2) Semidualizing bimodules and the Foxby equivalence [Christensen 2000; Holm and White 2007; Positselski 2017a].

(3) The comodule-contramodule [Positselski 2010, Section 0.2 and Chapter 5] and the semimodule-semicontramodule [Positselski 2010, Sections 0.3.7 and 6.3] correspondences.

These results come with rather different motivations: from criteria for stable equivalences of finite-dimensional self-injective algebras in (1), through Gorenstein homological algebra in (2), to the representation theory of infinite-dimensional Lie algebras (e.g., the Virasoro or Kac–Moody algebras) in (3).

A part of the work has been done in our previous paper [Positselski and Šťovíček 2019], where we explained how the finite generation assumption can be naturally dropped with help of additive monads and, in several cases of interest, with topological rings.

In this paper we focus on dropping the assumption of finite homological dimension. It turns out that we still obtain triangulated equivalences and (co)tilting t-structures, but in general not for the conventional derived categories of two abelian categories, but rather for a so-called *pseudo-coderived* category of one of them and a *pseudo-contraderived* category of the other.

Here, a pseudo-coderived category of an abelian category A is a certain triangulated category D to which A fully embeds as the heart of a t-structure and such that $\mathrm{Ext}^i_{\mathsf{A}}(X, Y)$ is canonically isomorphic to $\mathrm{Hom}_{\mathsf{D}}(X, Y[i])$ for all $X, Y \in \mathsf{A}$ and $i \geq 0$. The term "pseudo-coderived" comes from the fact that, under reasonable assumptions satisfied in particular in the situations (1)–(3) above, the pseudo-coderived category is an intermediate Verdier quotient between the conventional derived category $\mathsf{D}(\mathsf{A})$ and the coderived category $\mathsf{D}^{\mathrm{co}}(\mathsf{A})$ (which is none other than the homotopy category $\mathsf{Hot}(\mathsf{A}_{\mathsf{inj}})$ of complexes of injective objects if A is a locally Noetherian Grothendieck category). A pseudo-contraderived category has formally dual properties. Pseudo-co/contraderived categories are in fact not determined uniquely by their abelian hearts, but depend on a certain parameter, so that we often do not get just a single triangulated equivalence, but rather a family of compatible triangulated equivalences. We refer to [Positselski 2017a] for an in-depth discussion of this new class of triangulated categories.

To put our results into context, we briefly recall the history of tilting theory, which evolved through a series of successive generalizations in several directions.

The definition of what is now known as a finitely generated tilting module of projective dimension 1 over a finite-dimensional associative algebra first appeared in [Happel and Ringel 1982] (see also [Bongartz 1981]), which built upon [Brenner and Butler 1980]. The main result was the so-called *tilting theorem*, or the *Brenner–Butler theorem*, establishing equivalences between certain additive subcategories of the categories of finitely generated modules over an algebra R and over the endomorphism algebra S of a tilting R-module. Happel [1987] proved that a tilting module induces a triangulated equivalence between the derived categories of finitely generated R-modules and S-modules.

Finitely presented tilting modules of projective dimension 1 over arbitrary rings were discussed by Colby and Fuller [1990], while finitely presented tilting modules of arbitrary finite projective dimension n were studied by Miyashita [1986] and Cline, Parshall and Scott [Cline et al. 1986]. The tilting theorem (for categories of infinitely generated modules) was proved in [Miyashita 1986], and the related derived equivalence was constructed in [Cline et al. 1986]. Infinitely generated tilting modules of projective dimension 1 (now also known as *big* 1*-tilting modules*) were defined by Colpi and Trlifaj [1995]. The tilting theorem for self-small tilting objects of projective dimension 1 in Grothendieck abelian categories was obtained by Colpi [1999]. Cotilting modules of injective dimension 1 were introduced by Colby and Fuller [1990] and Colpi, D'Este and Tonolo [Colpi et al. 1997a] (see also [Colpi et al. 1997b]). Finally, infinitely generated tilting modules of projective dimension n and cotilting modules of injective dimension n (*big n-tilting* and *n-cotilting modules*) were defined by Angeleri Hügel and Coelho [2001] and characterized by Bazzoni [2004].

The main results of the infinitely generated tilting theory claim that all n-tilting modules are of finite type [Bazzoni and Šťovíček 2007] and all n-cotilting modules are pure-injective [Šťovíček 2006]. The tilting theorem for big 1-tilting modules was obtained in some form by Gregorio and Tonolo [2001]. Another approach, based on a previous work by Facchini, was developed by Bazzoni [2010], who also proved that the derived category of R-modules is equivalent to a full subcategory and a quotient category of the derived category of S-modules when S is the endomorphism ring of a big 1-tilting R-module. This was extended to big n-tilting modules by Bazzoni, Mantese, and Tonolo [Bazzoni et al. 2011]. A correspondence between n-cotilting modules and small n-tilting objects in Grothendieck abelian categories together with the related derived equivalence were constructed in [Šťovíček 2014a]. Big n-tilting objects in abelian categories were defined and the related derived equivalence was obtained by Nicolás, Saorín and Zvonareva [Nicolás et al. 2019] and Fiorot, Mattiello and Saorín [Fiorot et al. 2017] (see also Psaroudakis and Vitória [2018]). Finally, a correspondence between big n-tilting and n-cotilting objects in abelian categories was constructed in [Positselski and Šťovíček 2019].

The main innovation in [Positselski and Šťovíček 2019], which allows one to obtain very naturally derived equivalences from big n-tilting objects and which is based on the ideas previously developed in [Šťovíček 2014a; Psaroudakis and Vitória 2018; Nicolás et al. 2019; Positselski and Rosický 2017], is that to a big tilting object T in an abelian category A one can assign a richer structure than its ring of endomorphisms $\mathrm{Hom}_{\mathsf{A}}(T, T)$. For any set X, consider the set of all morphisms $T \to T^{(X)}$ in A, where $T^{(X)}$ denotes the coproduct of X copies of T. Then the endofunctor $X \mapsto \mathrm{Hom}_{\mathsf{A}}(T, T^{(X)})$ is a monad on the category of sets. The tilting heart B corresponding to the tilting object $T \in \mathsf{A}$ is the abelian category of all algebras (which we also call modules) over this monad. In many naturally occurring situations, one can equip $\mathrm{Hom}_{\mathsf{A}}(T, T)$ with a complete and separated topology so that $\mathrm{Hom}_{\mathsf{A}}(T, T^{(X)})$ identifies with families of elements of $\mathrm{Hom}_{\mathsf{A}}(T, T)$ indexed by X which converge to zero.

A notion of a finitely generated tilting module of infinite projective dimension (now known as *Wakamatsu tilting modules*) was introduced in the representation theory of finite-dimensional algebras by Wakamatsu in [Wakamatsu 1988; 1990] and it was studied further by Mantese and Reiten in [2004].

In the present paper we work out a common generalization of two lines of thought described above, namely of big n-tilting/cotilting modules and finite-dimensional Wakamatsu tilting modules. We develop a theory of big tilting and cotilting objects of possibly infinite homological dimension in abelian categories. Our goal is also to put on a rigorous footing the discussion of "∞-tilting objects" in [Positselski and Šťovíček 2019, Sections 10.1, 10.2 and 10.3]. The structure of the paper is as follows.

To a complete, cocomplete abelian category A with an injective cogenerator J and an ∞-tilting object T we associate in Section 2 a complete, cocomplete abelian category B with a projective generator P and an ∞-cotilting object W. We do so in such a way that, up to equivalence, this induces a bijective correspondence between the triples (A, T, J) and (B, P, W).

In order to obtain the announced version of derived equivalences, we need to associate to each ∞-tilting object a certain coresolving subcategory $\mathsf{E} \subset \mathsf{A}$ which plays the role of the tilting class in [Positselski and Šťovíček 2019]. This is discussed in Section 3. Such a class E is in general not unique, but the possible choices form a complete lattice with respect to the inclusion. Having chosen E, we already obtain a uniquely determined full subcategory $\mathsf{F} \subset \mathsf{B}$ which plays the role of a cotilting class, and equivalences $\mathsf{E} \simeq \mathsf{F}$ and $\mathsf{D}(\mathsf{E}) \simeq \mathsf{D}(\mathsf{F})$. Each of $\mathsf{D}(\mathsf{E})$ and $\mathsf{D}(\mathsf{F})$ comes naturally equipped with two t-structures, and the two abelian categories A and B are the hearts of these two t-structures (see Section 5).

If, moreover, the ∞-tilting class E is closed under coproducts in A and the ∞-cotilting class F is closed under products in B, then we show in Section 4 that

$\mathsf{D}(\mathsf{E})$ is a pseudo-coderived category and $\mathsf{D}(\mathsf{F})$ is a pseudo-contraderived category in the sense of [Positselski 2017a]. Although the above closure properties of E and F are not automatic in our setup, they are satisfied for our motivating classes of examples mentioned above and examined in detail in Section 6.

1. The tilted and cotilted abelian categories

Given an additive category C with set-indexed coproducts and an object $M \in \mathsf{C}$, we denote by $\mathsf{Add}(M) \subset \mathsf{C}$ the full subcategory formed by the direct summands of coproducts of copies of M in C. Similarly, given an additive category C with set-indexed products and an object $L \in \mathsf{C}$, we denote by $\mathsf{Prod}(L) \subset \mathsf{C}$ the full subcategory formed by the direct summands of products of copies of L in C. Given a set X, the coproduct of X copies of M is denoted by $M^{(X)} \in \mathsf{Add}(M)$ and the product of X copies of L is denoted by $L^X \in \mathsf{Prod}(L)$.

We say that an additive category is *idempotent-complete* (or, in other terminology, Karoubian or pseudo-abelian) if it contains the images of all idempotent endomorphisms of its objects.

Theorem 1.1. (a) *Let C be an idempotent-complete additive category with coproducts and $M \in \mathsf{C}$ be an object. Then there exists a unique abelian category B with enough projective objects such that the full subcategory of projective objects $\mathsf{B}_{\mathsf{proj}} \subset \mathsf{B}$ is equivalent to the full subcategory $\mathsf{Add}(M) \subset \mathsf{C}$. The abelian category B has products, coproducts, and a natural projective generator $P \in \mathsf{B}_{\mathsf{proj}}$ corresponding to the object $M \in \mathsf{Add}(M)$.*

(b) *Let C be an idempotent-complete additive category with products and $L \in \mathsf{C}$ be an object. Then there exists a unique abelian category A with enough injective objects such that the full subcategory of injective objects $\mathsf{A}_{\mathsf{inj}} \subset \mathsf{A}$ is equivalent to the full subcategory $\mathsf{Prod}(L) \subset \mathsf{C}$. The abelian category A has products, coproducts, and a natural injective cogenerator $J \in \mathsf{A}_{\mathsf{inj}}$ corresponding to the object $L \in \mathsf{Prod}(L)$.*

Proof. (a) The category B is unique, because an abelian category with enough projective objects is determined by its full subcategory of projective objects [Šťovíček 2014a, proof of Theorem 6.2; Positselski 2015, proof of Theorem 3.6].

To prove existence, one can construct B as the category of finitely presented (coherent) contravariant functors on $\mathsf{Add}(M)$. This category can be also described as the quotient category of the category $\mathsf{Add}(M)^2$ of morphisms in $\mathsf{Add}(M)$ by the ideal of all morphisms in $\mathsf{Add}(M)^2$ which factorize through objects of the full subcategory in $\mathsf{Add}(M)^2$ consisting of all the split epimorphisms in $\mathsf{Add}(M)$ [Beligiannis 2000]. The category B is abelian, because the additive category $\mathsf{Add}(M)$ is right coherent (has weak kernels) [Freyd 1966, Corollary 1.5; Krause 1998, Lemma 2.2, Proposition 2.3; Beligiannis 2000, Proposition 4.5(1); Krause 2002, Lemma 1(1)] (see also [Fiorot 2016, Appendix B] for a further discussion and references). Indeed, if $f : M' \to M''$ is a morphism in $\mathsf{Add}(M)$ and X is the set of all morphisms $M \to M'$ whose composition with f vanishes, then the natural morphism $M^{(X)} \to M'$ is a weak kernel of f in $\mathsf{Add}(M)$ (cf. [Krause 2002, Lemma 2(1)]).

Even more explicitly, B is the category of modules over the monad

$$\mathbb{T} : X \mapsto \mathrm{Hom}_{\mathsf{C}}(M, M^{(X)})$$

on the category of sets (we call modules here what are often called monadic $\mathbb{T}$-algebras, since they generalize ordinary modules over a ring; see the discussions in the introduction to [Positselski and Rosický 2017], [Positselski 2017b, Lemma 1.1 and Example 1.2(2)], and [Positselski and Šťovíček 2019, Sections 6.1 and 6.3]).

Coproducts in the category of coherent functors exist [Krause 2002, Lemma 1(2)]; more generally, whenever the category of projective objects $\mathsf{B}_{\mathsf{proj}}$ in an abelian category B with enough projective objects has coproducts, the coproducts in B can be constructed in terms of the coproducts in $\mathsf{B}_{\mathsf{proj}}$ (and the embedding functor $\mathsf{B}_{\mathsf{proj}} \to \mathsf{B}$ preserves coproducts). Products exist in the category of algebras over every monad $\mathbb{T} : \mathsf{Sets} \to \mathsf{Sets}$ and are preserved by the forgetful functor from the category of $\mathbb{T}$-algebras to Sets (coproducts also exist in the category of $\mathbb{T}$-algebras, but are not preserved by the forgetful functor). The natural projective generator $P \in \mathsf{B}_{\mathsf{proj}}$ is the free $\mathbb{T}$-algebra/module with one generator.

(b) This is dual to the proof of (a). Explicitly, A is the opposite category to the category of coherent covariant functors on $\mathsf{Prod}(L)$, or the opposite category to the category of modules over the monad $\mathbb{T} : X \mapsto \mathrm{Hom}_{\mathsf{C}}(L^X, L)$ on the category of sets. □

We will use the notation $\mathsf{B} = \sigma_M(\mathsf{C})$ and $\mathsf{A} = \pi_L(\mathsf{C})$. Assuming that $M \in \mathsf{C}$ is a "tilting object" in one sense or another (cf. Section 2), one can call B the *abelian category tilted from* C *at* M. Similarly, assuming that $L \in \mathsf{C}$ is a "cotilting object" in some sense, one can call A the *abelian category cotilted from* C *at* L.

Now let us assume that C is an abelian category. Then, in the context of Theorem 1.1(a), the additive embedding functor $\Phi_{\mathsf{proj}} : \mathsf{B}_{\mathsf{proj}} \simeq \mathsf{Add}(M) \to \mathsf{C}$ can be uniquely extended to a right exact functor $\Phi : \mathsf{B} \to \mathsf{C}$. To compute the object $\Phi(B) \in \mathsf{C}$ for a given object $B \in \mathsf{B}$, one can present B as the cokernel of a morphism of projective objects $f : P'' \to P'$ in B and put $\Phi(B) = \operatorname{coker} \Phi_{\mathsf{proj}}(f)$.

The additive embedding functor $\mathsf{Add}(M) \simeq \mathsf{B}_{\mathsf{proj}} \to \mathsf{B}$ can be extended to a (left exact) functor $\Psi : \mathsf{C} \to \mathsf{B}$ right adjoint to Φ. Representing the objects of B as modules over the monad $\mathbb{T} : X \mapsto \mathrm{Hom}_{\mathsf{C}}(M, M^{(X)})$ on the category of sets, one can compute the functor Ψ as the functor $N \mapsto \mathrm{Hom}_{\mathsf{C}}(M, N)$, with the $\mathbb{T}$-module structure on the set $\mathrm{Hom}_{\mathsf{C}}(M, N)$ constructed as explained in [Positselski and Šťovíček 2019, Section 6.3] (in this case $\mathrm{Hom}_{\mathsf{C}}(M, N)$ of course carries the structure of an abelian group, even a right $\mathbb{T}(*)$-module, where $*$ stands for a one-element set).

Indeed, let us show that the functor Φ is left adjoint to Ψ. First of all, the natural projective generator $P \in \mathsf{B}$ (corresponding to the object $M \in \mathsf{Add}(M)$) corepresents the forgetful functor from the category $\mathsf{B} \simeq \mathbb{T}\text{–}\mathsf{mod}$ to the category of sets or abelian groups, that is, for any object $B \in \mathbb{T}\text{–}\mathsf{mod}$ one has $\mathrm{Hom}_{\mathsf{B}}(P, B) \simeq B$. In particular, for any object $N \in \mathsf{C}$ we have a natural isomorphism of the Hom groups

$$\mathrm{Hom}_{\mathsf{B}}(P, \Psi(N)) = \mathrm{Hom}_{\mathsf{B}}(P, \mathrm{Hom}_{\mathsf{C}}(M, N)) \simeq \mathrm{Hom}_{\mathsf{C}}(M, N) = \mathrm{Hom}_{\mathsf{C}}(\Phi(P), N).$$

Hence for any set X there are natural isomorphisms

$$\mathrm{Hom}_{\mathsf{B}}(P^{(X)}, \Psi(N)) \simeq \mathrm{Hom}_{\mathsf{C}}(M, N)^X \simeq \mathrm{Hom}_{\mathsf{C}}(M^{(X)}, N) \simeq \mathrm{Hom}_{\mathsf{C}}(\Phi(P^{(X)}), N).$$

Passing to the direct summands, we get a natural isomorphism of the Hom groups

$$\mathrm{Hom}_{\mathsf{B}}(P', \Psi(N)) \simeq \mathrm{Hom}_{\mathsf{C}}(\Phi(P'), N)$$

for all objects $P' \in \mathsf{B}_{\mathsf{proj}}$ and $N \in \mathsf{C}$. This isomorphism is clearly functorial in an object $N \in \mathsf{C}$; and the construction of the action of the monad $\mathbb{T}$ on the set $\Psi(N) = \mathrm{Hom}_{\mathsf{C}}(M, N)$ in [Positselski and Šťovíček 2019, proof of Proposition 6.2 and Remark 6.4] is designed so as to make these isomorphisms compatible with all the morphisms $P'' \to P'$ in the category $\mathsf{B}_{\mathsf{proj}} \simeq \mathsf{Add}(M)$. Finally, both the contravariant functors $\mathrm{Hom}_{\mathsf{B}}(-, \Psi(N))$ and $\mathrm{Hom}_{\mathsf{C}}(\Phi(-), N)$ take the cokernels of morphisms in B to the kernels of morphisms of abelian groups, so our isomorphism of the Hom groups extends from $P' \in \mathsf{B}_{\mathsf{proj}}$ to all objects $B \in \mathsf{B}$.

Similarly, in the context of Theorem 1.1(b), the additive embedding functor $\Psi_{\mathsf{inj}} : \mathsf{A}_{\mathsf{inj}} \simeq \mathsf{Prod}(L) \to \mathsf{C}$ can be uniquely extended to a left exact functor $\Psi : \mathsf{A} \to \mathsf{C}$. The additive embedding functor $\mathsf{Prod}(L) \simeq \mathsf{A}_{\mathsf{inj}} \to \mathsf{A}$ can be extended to a (right exact) functor $\Phi : \mathsf{C} \to \mathsf{A}$ left adjoint to Ψ.

For more explicit descriptions of abelian categories B arising in connection with objects M in more specific classes of additive categories C in Theorem 1.1(a), we refer to [Positselski and Šťovíček 2019, Theorems 7.1, 9.9, and 9.11, and Proposition 9.1].

The following question will be addressed in Section 2: given an abelian category A with coproducts and an object $M \in \mathsf{A}$, under which assumptions is there an object L in the abelian category $\mathsf{B} = \sigma_M(\mathsf{A})$ such that $\pi_L(\mathsf{B}) = \mathsf{A}$? Similarly, given an abelian category B with products and an object $L \in \mathsf{B}$, under which assumptions is there an object M in the abelian category $\mathsf{A} = \pi_L(\mathsf{B})$ such that $\sigma_M(\mathsf{A}) = \mathsf{B}$?

2. ∞-tilting-cotilting correspondence

Let A be an abelian category with coproducts. We will say that an object $T \in \mathsf{A}$ is *weakly tilting* if one has

$$\mathrm{Ext}^i_{\mathsf{A}}(T, T^{(X)}) = 0 \quad \text{for all sets } X \text{ and all integers } i > 0.$$

Given two objects $T' \in \mathsf{Add}(T) \subset \mathsf{A}$ and $A \in \mathsf{A}$, a morphism $t : T' \to A$ is said to be an $\mathsf{Add}(T)$-*precover* if every morphism $t'' : T'' \to A$ with $T'' \in \mathsf{Add}(T)$ factorizes through the morphism t. Equivalently, this means that the map of abelian groups $\operatorname{Hom}_{\mathsf{A}}(T, t) : \operatorname{Hom}_{\mathsf{A}}(T, T') \to \operatorname{Hom}_{\mathsf{A}}(T, A)$ is surjective. For every object $A \in \mathsf{A}$, the natural morphism $T^{(\operatorname{Hom}_{\mathsf{A}}(T,A))} \to A$ is an $\mathsf{Add}(T)$-precover.

Let $T \in \mathsf{A}$ be a weakly tilting object. By the definition, the full subcategory $\mathsf{E}_{\mathsf{max}}(T) \subset \mathsf{A}$ consists of all the objects $E \in \mathsf{A}$ satisfying the following two conditions:

($\mathrm{i}_{\mathsf{max}}$) $\operatorname{Ext}^i_{\mathsf{A}}(T, E) = 0$ for all $i > 0$.

($\mathrm{ii}_{\mathsf{max}}$) There exists an exact sequence

$$\cdots \to T_2 \to T_1 \to T_0 \to E \to 0$$

in A such that $T_j \in \mathsf{Add}(T)$ for all $j \geq 0$ and the sequence remains exact after applying the functor $\operatorname{Hom}_{\mathsf{A}}(T, -)$.

Notice that the condition of exactness of the sequence of abelian groups obtained by applying $\operatorname{Hom}_{\mathsf{A}}(T, -)$ in ($\mathrm{ii}_{\mathsf{max}}$) can be equivalently restated as the condition that the images Z_j of the morphisms $T_{j+1} \to T_j$ satisfy $\operatorname{Ext}^1_{\mathsf{A}}(T, Z_j) = 0$ for all $j \geq 0$. In this case, assuming ($\mathrm{i}_{\mathsf{max}}$), one also has $\operatorname{Ext}^i_{\mathsf{A}}(T, Z_j) = 0$ for all $j \geq 0$ and $i > 0$. As ($\mathrm{ii}_{\mathsf{max}}$) is obviously satisfied for Z_j, it follows that $Z_j \in \mathsf{E}_{\mathsf{max}}(T)$ for all $j \geq 0$.

Conversely, given a short exact sequence $0 \to Z_0 \to T_0 \to E \to 0$ with E satisfying ($\mathrm{i}_{\mathsf{max}}$), Z_0 satisfying ($\mathrm{ii}_{\mathsf{max}}$), $T_0 \in \mathsf{Add}(T)$, and $\operatorname{Hom}_{\mathsf{A}}(T, T_0) \to \operatorname{Hom}_{\mathsf{A}}(T, E)$ a surjective map, one clearly has $E \in \mathsf{E}_{\mathsf{max}}(T)$.

The following lemma is a generalization of [Wakamatsu 1990, Proposition 2.6].

Lemma 2.1. *For any weakly tilting object $T \in \mathsf{A}$, the full subcategory $\mathsf{E}_{\mathsf{max}}(T)$ in the abelian category A is closed under*

(a) *extensions*,

(b) *the cokernels of monomorphisms*,

(c) *the kernels of those epimorphisms which remain epimorphisms after applying the functor $\operatorname{Hom}_{\mathsf{A}}(T, -)$, and*

(d) *direct summands.*

Proof. To prove parts (a)–(c), consider a short exact sequence $0 \to E' \to E \to E'' \to 0$ in the abelian category A.

(a) Clearly, the object E satisfies the condition ($\mathrm{i}_{\mathsf{max}}$) whenever the objects E' and E'' do. Suppose that $T'_0 \to E'$ and $T''_0 \to E''$ are epimorphisms onto the objects E' and E'' from objects $T'_0, T''_0 \in \mathsf{Add}(T)$ that remain epimorphisms after applying the functor $\operatorname{Hom}_{\mathsf{A}}(T, -)$. Since $\operatorname{Ext}^1_{\mathsf{A}}(T''_0, E') = 0$, the morphism $T''_0 \to E''$ can be lifted to a morphism $T''_0 \to E$. Hence we obtain a morphism from the split short exact sequence $0 \to T'_0 \to T'_0 \oplus T''_0 \to T''_0 \to 0$ to the short exact sequence

$0 \to E' \to E \to E'' \to 0$. Being an epimorphism at the leftmost and rightmost terms, this morphism of short exact sequences is also an epimorphism at the middle term. The short sequence of kernels $0 \to Z_0' \to Z_0 \to Z_0'' \to 0$ is then also exact, and the vanishing of $\mathrm{Ext}^i_{\mathsf{A}}(T, Z_0')$ and $\mathrm{Ext}^i_{\mathsf{A}}(T, Z_0'')$ implies the same of $\mathrm{Ext}^i_{\mathsf{A}}(T, Z_0)$. We can thus proceed with the construction of a resolution as in ($\mathrm{ii}_{\mathsf{max}}$) inductively.

(b) Clearly, the object E'' satisfies the condition ($\mathrm{i}_{\mathsf{max}}$) whenever the objects E' and E do. Moreover, the epimorphism $E \to E''$ remains an epimorphism after applying $\mathrm{Hom}_{\mathsf{A}}(T, -)$, since $\mathrm{Ext}^1_{\mathsf{A}}(T, E') = 0$. Let $T_0 \to E$ be an epimorphism onto E from an object $T_0 \in \mathsf{Add}(T)$ that remains an epimorphism after applying $\mathrm{Hom}_{\mathsf{A}}(T, -)$. Then the composition $T_0 \to E \to E''$ has the same property. Let Z_0 and Z_0'' be the kernels of the epimorphisms $T_0 \to E$ and $T_0 \to E''$. Then there is a short exact sequence $0 \to Z_0 \to Z_0'' \to E' \to 0$. Assuming that $Z_0 \in \mathsf{E}_{\mathsf{max}}(T)$, one can apply part (a) in order to conclude that $Z_0'' \in \mathsf{E}_{\mathsf{max}}(T)$; hence $E'' \in \mathsf{E}_{\mathsf{max}}(T)$.

(c) Let us first show that the kernel of every $\mathsf{Add}(T)$-precover $t' \colon T' \to E$ belongs to $\mathsf{E}_{\mathsf{max}}(T)$ whenever $E \in \mathsf{E}_{\mathsf{max}}(T)$. By the definition, there exists an $\mathsf{Add}(T)$-precover $t_0 : T_0 \to E$ with the kernel Z_0 belonging to $\mathsf{E}_{\mathsf{max}}(T)$. Consider the following pullback diagram:

$$\begin{tikzcd}
 & Z' \arrow[r, equal] \arrow[d, tail] & Z' \arrow[d, tail] \\
Z_0 \arrow[r, tail] \arrow[d, equal] & S \arrow[r, two heads] \arrow[d, two heads] & T' \arrow[d, two heads, "t'"] \\
Z_0 \arrow[r, tail] & T_0 \arrow[r, two heads, "t_0"'] & E
\end{tikzcd}$$

As Z_0 and T' are in $\mathsf{E}_{\mathsf{max}}(T)$, we have $S \in \mathsf{E}_{\mathsf{max}}(T)$ by part (a). Furthermore, since t' stays an epimorphism after applying $\mathrm{Hom}_{\mathsf{A}}(T, -)$ and T', E satisfy ($\mathrm{i}_{\mathsf{max}}$), it follows that Z' satisfies ($\mathrm{i}_{\mathsf{max}}$) and the middle column splits. Hence there exists a short exact sequence $0 \to T_0 \to S \to Z' \to 0$ and $Z' \in \mathsf{E}_{\mathsf{max}}(T)$ by part (b).

Now we can return to our short exact sequence $0 \to E' \to E \to E'' \to 0$. Clearly, if the objects E and E'' satisfy ($\mathrm{i}_{\mathsf{max}}$) and the map $\mathrm{Hom}_{\mathsf{A}}(T, E) \to \mathrm{Hom}_{\mathsf{A}}(T, E'')$ is surjective, then the object E' also satisfies ($\mathrm{i}_{\mathsf{max}}$). Furthermore, if $T_0 \to E$ is an $\mathsf{Add}(T)$-precover with the kernel Z_0 and if Z_0'' is the kernel of the composition $T_0 \to E \to E''$, then $Z_0,\ Z_0'' \in \mathsf{E}_{\mathsf{max}}(T)$ by the previous paragraph. It remains to apply part (b) to the short exact sequence $0 \to Z_0 \to Z_0'' \to E' \to 0$ in order to conclude that $E' \in \mathsf{E}_{\mathsf{max}}(T)$.

(d) Let E' and E'' be two objects in A for which $E = E' \oplus E'' \in \mathsf{E}_{\mathsf{max}}(T)$. Then it is obvious that E' and E'' satisfy ($\mathrm{i}_{\mathsf{max}}$). Starting from the exact sequence ($\mathrm{ii}_{\mathsf{max}}$) for the object E, we will simultaneously construct similar exact sequences for the two objects E' and E''. Applying the construction of part (b) to the short exact sequence $0 \to E' \to E \to E'' \to 0$, we get an epic $\mathsf{Add}(T)$-precover $T_0 \to E''$ with the kernel Z_0'' included in a short exact sequence $0 \to Z_0 \to Z_0'' \to E' \to 0$. Applying the same construction to the short exact sequence $0 \to E'' \to E \to E' \to 0$, we have an epic $\mathsf{Add}(T)$-precover $T_0 \to E'$ with the kernel Z_0' included in a short exact sequence $0 \to Z_0 \to Z_0' \to E'' \to 0$.

Continuing with an epic $\mathsf{Add}(T)$-precover $T_1 \to Z_0$ and applying the construction of part (a), we obtain an epic $\mathsf{Add}(T)$-precover $T_1 \oplus T_0 \to Z_0''$ with the kernel Z_1'' included in a short exact sequence $0 \to Z_1 \to Z_1'' \to Z_0' \to 0$. Proceeding in this way, we obtain an epic $\mathsf{Add}(T)$-precover $T_2 \oplus T_1 \oplus T_0 \to Z_1''$ with the kernel Z_2'' included in a short exact sequence $0 \to Z_2 \to Z_2'' \to Z_1' \to 0$, an epic $\mathsf{Add}(T)$-precover $T_3 \oplus T_2 \oplus T_1 \oplus T_0 \to Z_2''$, etc. Hence we obtain a long exact sequence satisfying the requirements of ($\mathrm{ii}_{\mathsf{max}}$) for E'' of the form

$$\cdots \to T_2 \oplus T_1 \oplus T_0 \to T_0 \oplus T_1 \to T_0 \to E'' \to 0,$$

and there is a similar sequence of the same form for E'. □

It follows from Lemma 2.1(c) that, given an object $E \in \mathsf{E}_{\mathsf{max}}(T)$, one can construct an exact sequence ($\mathrm{ii}_{\mathsf{max}}$) for it by choosing an arbitrary $\mathsf{Add}(T)$-precover $T_0 \to E$, taking its kernel Z_0, choosing an arbitrary $\mathsf{Add}(T)$-precover $T_1 \to Z_0$, etc. Whichever $\mathsf{Add}(T)$-precovers one chooses, all the subsequent $\mathsf{Add}(T)$-precovers will be epimorphisms, so one will not encounter any problems in this process.

In view of Lemma 2.1(a), for any weakly tilting object $T \in \mathsf{A}$, the full subcategory $\mathsf{E}_{\mathsf{max}}(T) \subset \mathsf{A}$ inherits a Quillen exact category structure from the abelian category A. There are enough projective objects in the exact category $\mathsf{E}_{\mathsf{max}}(T)$, and the full subcategory of projective objects in $\mathsf{E}_{\mathsf{max}}(T)$ coincides with $\mathsf{Add}(T) \subset \mathsf{E}_{\mathsf{max}}(T) \subset \mathsf{A}$.

Given a full subcategory E of an idempotent complete exact category A, we will call E a *coresolving* subcategory provided that

(a) E is closed under extensions, cokernels of admissible monomorphisms, and direct summands in A, and

(b) E is cogenerating in A, i.e., each $A \in \mathsf{A}$ admits an admissible monomorphism $A \to E$ in A with $E \in \mathsf{E}$.

Coresolving subcategories provide a suitable framework to speak of coresolution dimensions of objects [Šťovíček 2014a, §2; Auslander and Bridger 1969, Chapter 3].

Let now A be an abelian category with set-indexed products and an injective cogenerator $J \in \mathsf{A}$. Then set-indexed coproducts exist and are exact in A [Positselski and Šťovíček 2019, Section 2]. The full subcategory of injective objects in A can be described as $\mathsf{A}_{\mathsf{inj}} = \mathsf{Prod}(J)$.

We will say that an object $T \in \mathsf{A}$ is *∞-tilting* (or *big Wakamatsu tilting*) if T is weakly tilting and $\mathsf{A}_{\mathsf{inj}} \subset \mathsf{E}_{\mathsf{max}}(T)$. In this case, the full subcategory $\mathsf{E}_{\mathsf{max}}(T) \subset \mathsf{A}$ is coresolving, there are enough injective objects in the exact category $\mathsf{E}_{\mathsf{max}}(T)$, and these are precisely the injective objects of the ambient abelian category A.

Now let us present the dual definitions. Let B be an abelian category with products. We will say that an object $W \in \mathsf{B}$ is *weakly cotilting* if one has

$$\mathrm{Ext}^i_{\mathsf{B}}(W^X, W) = 0 \quad \text{for all sets } X \text{ and all integers } i > 0.$$

Let $W \in \mathsf{B}$ be a weakly cotilting object. By the definition, the full subcategory $\mathsf{F}_{\mathsf{max}}(W) \subset \mathsf{B}$ consists of all the objects $F \in \mathsf{B}$ satisfying the following two conditions:

($\mathrm{i}^*_{\mathsf{max}}$) $\mathrm{Ext}^i_{\mathsf{B}}(F, W) = 0$ for all $i > 0$.

($\mathrm{ii}^*_{\mathsf{max}}$) There exists an exact sequence

$$0 \to F \to W^0 \to W^1 \to W^2 \to \cdots$$

in B such that $W^j \in \mathsf{Prod}(W)$ for all $j \geq 0$ and the sequence remains exact after applying the contravariant functor $\mathrm{Hom}_{\mathsf{B}}(-, W)$.

Lemma 2.2. *For any weakly cotilting object* $W \in \mathsf{B}$, *the full subcategory* $\mathsf{F}_{\mathsf{max}}(T)$ *in the abelian category* B *is closed under*

(a) *extensions*,

(b) *the kernels of epimorphisms*,

(c) *the cokernels of those monomorphisms which are transformed into surjective maps by the contravariant functor* $\mathrm{Hom}_{\mathsf{B}}(-, W)$, *and*

(d) *direct summands.*

Proof. Dual to Lemma 2.1. □

The definition of a $\mathsf{Prod}(W)$*-preenvelope* in B is dual to the above definition of an $\mathsf{Add}(T)$-precover in A. The morphism $F \to W^0$ in an exact sequence ($\mathrm{ii}^*_{\mathsf{max}}$) is a $\mathsf{Prod}(W)$-preenvelope. Denoting the cokernel of this morphism by Z^0, the morphism $Z^0 \to W^1$ is also a $\mathsf{Prod}(W)$-preenvelope, etc.

Conversely, it follows from Lemma 2.2(c) that, given any object $F \in \mathsf{F}_{\mathsf{max}}(W)$, one can construct an exact sequence ($\mathrm{ii}^*_{\mathsf{max}}$) for it by choosing some arbitrary $\mathsf{Prod}(W)$-preenvelope $F \to W^0$, taking its cokernel Z^0, choosing an arbitrary $\mathsf{Prod}(W)$-preenvelope $Z^0 \to W^1$, etc. Whichever $\mathsf{Prod}(W)$-preenvelopes one chooses in this process, all the subsequent $\mathsf{Prod}(W)$-preenvelopes will be monomorphisms, so one will not encounter any problems.

In view of Lemma 2.2(a), for any weakly cotilting object $W \in \mathsf{B}$, the full subcategory $\mathsf{F}_{\mathsf{max}}(W) \subset \mathsf{B}$ inherits an exact category structure from the abelian category B. There are enough injective objects in the exact category $\mathsf{F}_{\mathsf{max}}(W)$, and the full subcategory of injective objects in $\mathsf{F}_{\mathsf{max}}(W)$ coincides with $\mathsf{Prod}(W)$.

$$\begin{tikzcd}
 & \mathsf{B} \arrow[rr, bend left, "\Phi"] \arrow[rr, phantom, "\perp"] & & \mathsf{A} \arrow[ll, bend left, "\Psi"] & \\
P,\ W \in & \mathsf{F}_{\mathsf{max}}(W) \arrow[u, "\cup"] \arrow[rr, bend left, "\Phi|_{\mathsf{F}_{\mathsf{max}}(W)}"] \arrow[rr, phantom, "\sim"] & & \mathsf{E}_{\mathsf{max}}(T) \arrow[u, "\cup"'] \arrow[ll, bend left, "\Psi|_{\mathsf{E}_{\mathsf{max}}(T)}"] & \ni T,\ J
\end{tikzcd}$$

Figure 1. Illustration of the ∞-tilting-cotilting correspondence (see Theorems 2.3 and 2.4 and Corollary 2.5).

Let B be an abelian category with set-indexed coproducts and a projective generator $P \in \mathsf{B}$. Then set-indexed products exist and are exact in B. The full subcategory of projective objects in B can be described as $\mathsf{B}_{\mathsf{proj}} = \mathsf{Add}(P)$.

We will say that an object $W \in \mathsf{B}$ is *∞-cotilting* (or *big Wakamatsu cotilting*) if W is weakly cotilting and $\mathsf{B}_{\mathsf{proj}} \subset \mathsf{F}_{\mathsf{max}}(T)$.

When the object W is ∞-cotilting, the full subcategory $\mathsf{F}_{\mathsf{max}}(W) \subset \mathsf{B}$ is *resolving* (i.e., generating and closed under extensions, kernels of epimorphisms and direct summands). In this case, there are enough projective objects in the exact category $\mathsf{F}_{\mathsf{max}}(W)$, and these are precisely the projective objects of the ambient abelian category B.

Theorem 2.3. *Let* A *be a complete, cocomplete abelian category with an injective cogenerator J and an ∞-tilting object $T \in \mathsf{A}$. Put* $\mathsf{B} = \sigma_T(\mathsf{A})$, *and let* $\Phi : \mathsf{B} \to \mathsf{A}$ *be the right exact functor identifying the full subcategory of projective objects* $\mathsf{B}_{\mathsf{proj}} \subset \mathsf{B}$ *with the full subcategory* $\mathsf{Add}(T) \subset \mathsf{A}$. *Let* $\Psi : \mathsf{A} \to \mathsf{B}$ *be the left exact functor right adjoint to* Φ; *so $P = \Psi(T)$ is a projective generator of* B. *Set* $W = \Psi(J) \in \mathsf{B}$.

Then W is an ∞-cotilting object in B, *and the restrictions of the functors Ψ and Φ induce a pair of inverse equivalences of exact categories between* $\mathsf{E}_{\mathsf{max}}(T)$ *and* $\mathsf{F}_{\mathsf{max}}(W)$ (*see Figure 1*), *which identify the ∞-tilting object $T \in \mathsf{A}$ with the projective generator $P \in \mathsf{B}$ and the ∞-cotilting object $W \in \mathsf{B}$ with the injective cogenerator $J \in \mathsf{A}$.*

Proof. The functor $\Psi|_{\mathsf{E}_{\mathsf{max}}}(T) : \mathsf{E}_{\mathsf{max}}(T) \to \mathsf{B}$ is exact, because the functor Ψ can be computed as $\mathrm{Hom}_{\mathsf{A}}(T, -)$, and the condition ($\mathrm{i}_{\mathsf{max}}$) is imposed.

To check that the functor $\Psi|_{\mathsf{E}_{\mathsf{max}}(T)}$ is fully faithful, one can choose for any two objects E' and $E'' \in \mathsf{E}_{\mathsf{max}}(T)$ two initial fragments $T_1' \to T_0' \to E' \to 0$ and $T_1'' \to T_0'' \to E'' \to 0$ of exact sequences ($\mathrm{ii}_{\mathsf{max}}$). The two sequences being exact in the exact category $\mathsf{E}_{\mathsf{max}}(T)$ and the objects of $\mathsf{Add}(T)$ being projective in $\mathsf{E}_{\mathsf{max}}(T)$, one can compute the group $\mathrm{Hom}_{\mathsf{A}}(E', E'')$ as the group of all morphisms $T_0' \to T_0''$ forming a commutative square with some morphism $T_1' \to T_1''$, modulo those morphisms that come from some morphism $T_0' \to T_1''$. The functor Ψ takes the exact sequences $T_1^{(k)} \to T_0^{(k)} \to E^{(k)} \to 0$, $k = 1, 2$, to exact sequences $\Psi(T_1^{(k)}) \to \Psi(T_0^{(k)}) \to \Psi(E^{(k)}) \to 0$ with the objects $\Psi(T_j^{(k)})$ belonging to $\mathsf{B}_{\mathsf{proj}}$, so the groups $\mathrm{Hom}_{\mathsf{B}}(\Psi(E'), \Psi(E''))$ can be computed similarly in terms of morphisms between the objects $\Psi(T_j^{(k)})$. It remains to recall that the functor $\Psi|_{\mathsf{Add}(T)}$ is fully faithful (see Section 1).

Furthermore, since the functor $\Psi|_{\mathsf{E}_{\mathsf{max}}(T)}$ is exact and fully faithful, and takes the projective objects of $\mathsf{E}_{\mathsf{max}}(T)$ to projective objects in B, and since there are enough projectives in $\mathsf{E}_{\mathsf{max}}(T)$, it follows that the functor $\Psi|_{\mathsf{E}_{\mathsf{max}}(T)}$ induces isomorphisms of the Ext groups

$$\mathrm{Ext}^i_{\mathsf{E}_{\mathsf{max}}(T)}(E', E'') \simeq \mathrm{Ext}^i_{\mathsf{B}}(\Psi(E'), \Psi(E''))$$

for all objects E' and $E'' \in \mathsf{E}_{\mathsf{max}}(T)$ and all $i \geq 0$. Similarly, as there are enough injectives in $\mathsf{E}_{\mathsf{max}}(T)$ and the injectives of $\mathsf{E}_{\mathsf{max}}(T)$ are injective in A, one has

$$\mathrm{Ext}^i_{\mathsf{E}_{\mathsf{max}}(T)}(E', E'') \simeq \mathrm{Ext}^i_{\mathsf{A}}(E', E''), \qquad E', E'' \in \mathsf{E}_{\mathsf{max}}(T), \quad i \geq 0.$$

The functor Ψ, being right adjoint, preserves products; so the equations $\mathsf{Prod}(J) = \mathsf{A}_{\mathsf{inj}}$ and $W = \Psi(J)$ imply $\mathsf{Prod}(W) = \Psi(\mathsf{A}_{\mathsf{inj}})$. In particular, $W^X = \Psi(J^X)$ for any set X. As $\mathsf{A}_{\mathsf{inj}} \subset \mathsf{E}_{\mathsf{max}}(T)$ and $\mathrm{Ext}^i_{\mathsf{A}}(J^X, J) = 0$ for $i > 0$, it follows that $\mathrm{Ext}^i_{\mathsf{B}}(W^X, W) = 0$. So the object $W \in \mathsf{B}$ is weakly cotilting.

Moreover, for the same reasons one has $\mathrm{Ext}^i_{\mathsf{B}}(\Psi(E), W) = 0$ for all $E \in \mathsf{E}_{\mathsf{max}}(T)$ and $i > 0$. In other words, the objects $\Psi(E) \in \mathsf{B}$ satisfy the condition (i$^*_{\mathsf{max}}$). Let us show that they also satisfy (ii$^*_{\mathsf{max}}$), that is $\Psi(\mathsf{E}_{\mathsf{max}}(T)) \subset \mathsf{F}_{\mathsf{max}}(W)$. Let $0 \to E \to J^0 \to J^1 \to J^2 \to \cdots$ be an injective coresolution of E in A. In view of Lemma 2.1(b), this coresolution is an acyclic complex in the exact category $\mathsf{E}_{\mathsf{max}}(T)$. The object $J \in \mathsf{A}$ being injective, this coresolution is taken to an acyclic complex of abelian groups by the contravariant functor $\mathrm{Hom}_{\mathsf{A}}(-, J)$. Hence, applying the fully faithful exact functor $\Psi|_{\mathsf{E}_{\mathsf{max}}(T)}$, we obtain a coresolution (ii$^*_{\mathsf{max}}$) for the object $\Psi(E)$. Thus $\mathsf{B}_{\mathsf{proj}} = \Psi(\mathsf{Add}(T)) \subset \Psi(\mathsf{E}_{\mathsf{max}}(T)) \subset \mathsf{F}_{\mathsf{max}}(W)$, and we have shown that the object W is ∞-cotilting in B.

There are enough injective objects in the category A, and the left exact functor Ψ establishes an equivalence $\mathsf{A}_{\mathsf{inj}} \simeq \mathsf{Prod}(W)$. Hence we have $\mathsf{A} = \pi_W(\mathsf{B})$. The assertions dual to what we have already proved now tell that the functor Φ is exact and fully faithful in restriction to $\mathsf{F}_{\mathsf{max}}(W)$ and that $\Phi(\mathsf{F}_{\mathsf{max}}(W)) \subset \mathsf{E}_{\mathsf{max}}(T)$. Being an adjoint pair of exact and fully faithful functors, $\Psi|_{\mathsf{E}_{\mathsf{max}}(T)}$ and $\Phi|_{\mathsf{F}_{\mathsf{max}}(W)}$ are equivalences of the exact categories $\mathsf{E}_{\mathsf{max}}(T)$ and $\mathsf{F}_{\mathsf{max}}(W)$. □

Theorem 2.4. *Let* B *be a complete, cocomplete abelian category with a projective generator* P *and an* ∞*-cotilting object* $W \in \mathsf{B}$. *Put* $\mathsf{A} = \pi_W(\mathsf{B})$, *and let* $\Psi : \mathsf{A} \to \mathsf{B}$ *be the left exact functor identifying the full subcategory of injective objects* $\mathsf{A}_{\mathsf{inj}} \subset \mathsf{A}$

with the full subcategory $\mathsf{Prod}(W) \subset \mathsf{B}$. *Let* $\Phi : \mathsf{B} \to \mathsf{A}$ *be the right exact functor left adjoint to* Ψ; *so* $J = \Phi(W)$ *is a injective cogenerator of* A. *Set* $T = \Phi(P) \in \mathsf{A}$.

Then T *is an* ∞*-tilting object in* A, *and the restrictions of the functors* Φ *and* Ψ *induce a pair of inverse equivalences of exact categories between* $\mathsf{F}_{\mathsf{max}}(W)$ *and* $\mathsf{E}_{\mathsf{max}}(T)$ (*see Figure 1*), *which identify the* ∞*-cotilting object* $W \in \mathsf{B}$ *with the injective cogenerator* $J \in \mathsf{A}$ *and the* ∞*-tilting object* $T \in \mathsf{A}$ *with the projective generator* $P \in \mathsf{B}$.

Proof. Dual to Theorem 2.3. □

Corollary 2.5. *The constructions of Theorems 2.3 and 2.4 establish a one-to-one correspondence between equivalence classes of*

(1) *complete*, *cocomplete abelian categories* A *with an injective cogenerator* J *and an* ∞*-tilting object* T, *and*
(2) *complete*, *cocomplete abelian categories* B *with a projective generator* P *and an* ∞*-cotilting object* W. □

3. ∞-tilting and ∞-cotilting pairs

As above, let A be an abelian category with set-indexed products and an injective cogenerator $J \in \mathsf{A}$. Let $T \in \mathsf{A}$ be an object and $\mathsf{E} \subset \mathsf{A}$ be a full subcategory. We will say that (T, E) is an ∞*-tilting pair* in A if the following conditions hold:

(i) $\mathsf{A}_{\mathsf{inj}} \subset \mathsf{E}$.
(ii) $\mathsf{Add}(T) \subset \mathsf{E}$.
(iii) $\mathrm{Ext}^1_{\mathsf{A}}(T, E) = 0$ for all $E \in \mathsf{E}$.
(iv) E is closed under the cokernels of monomorphisms and extensions in A.
(v) Every $\mathsf{Add}(T)$-precover $T' \to E$ of an object $E \in \mathsf{E}$ is an epimorphism in A with the kernel belonging to E.

Due to the condition (iv), the full subcategory $\mathsf{E} \subset \mathsf{A}$ inherits an exact category structure from the abelian category A. According to the condition (i), there are enough injective objects in the exact category E, and these are precisely the injective objects of the ambient abelian category A, that is, $\mathsf{E}_{\mathsf{inj}} = \mathsf{A}_{\mathsf{inj}}$.

It follows from the condition (iii) together with the condition (i) and the first part of the condition (iv) that

$$\mathrm{Ext}^i_{\mathsf{A}}(T, E) = 0 \quad \text{for all } E \in \mathsf{E} \text{ and all integers } i > 0.$$

Hence, in view of the condition (ii), the object $T \in \mathsf{A}$ has to be weakly tilting.

From the conditions (ii) and (iii) we see that the objects of $\mathsf{Add}(T)$ are projective in the exact category E. It follows from the condition (v) that there are enough projective objects belonging to $\mathsf{Add}(T)$ in E. Hence there are enough projective objects in E and the class of all projective objects in E coincides with $\mathsf{Add}(T)$, that is $\mathsf{E}_{\mathsf{proj}} = \mathsf{Add}(T)$.

Now it is clear that all the objects $E \in \mathsf{E}$ satisfy the conditions ($\mathrm{i}_{\mathsf{max}}$) and ($\mathrm{ii}_{\mathsf{max}}$); so we have $\mathsf{E} \subset \mathsf{E}_{\mathsf{max}}(T) \subset \mathsf{A}$. From the condition (i) we conclude that $\mathsf{A}_{\mathsf{inj}} \subset \mathsf{E}_{\mathsf{max}}(T)$. Thus the object $T \in \mathsf{A}$ has to be ∞-tilting. Conversely, according to Lemma 2.1, for any ∞-tilting object $T \in \mathsf{A}$, the pair $(T, \mathsf{E}_{\mathsf{max}}(T))$ is an ∞-tilting pair in A. To summarize, we have shown the following.

Lemma 3.1. *Let* A *be a complete, cocomplete abelian category with an injective cogenerator. Then an object* $T \in \mathsf{A}$ *is a part of an* ∞*-tilting pair* (T, E) *in* A *if and only if it is an* ∞*-tilting object. The full subcategory* $\mathsf{E} = \mathsf{E}_{\mathsf{max}}(T)$ *is the maximal of all full subcategories* $\mathsf{E} \subset \mathsf{A}$ *forming an* ∞*-tilting pair with* $T \in \mathsf{A}$. □

In general, we do not assume that E is closed under direct summands. However, we can add that assumption whenever convenient (e.g., in Sections 4 or 5):

Lemma 3.2. *If* (T, E) *is an* ∞*-tilting pair in* A *and* E' *is the closure of* E *under direct summands, then* (T, E') *is also an* ∞*-tilting pair and* E' *is a coresolving subcategory in* A.

Proof. The conditions (i)–(iii) are obviously true for E'. To prove (iv), suppose that we have an exact sequence

$$0 \to E_1' \to E_1 \to E_1'' \to 0$$

with $E_1', E_1'' \in \mathsf{E}'$, i.e., there exist $E_2', E_2'' \in \mathsf{A}$ such that $E' = E_1' \oplus E_2'$ and $E'' = E_1'' \oplus E_2''$ belong to E. Then $E_1 \oplus E_2' \oplus E_2''$ is an extension of E' by E'' in A, and hence $E_1 \in \mathsf{E}'$. Similarly, if $f_1 : E_1' \to E_1$ is a monomorphism in A with $E_1', E_1 \in \mathsf{E}'$, then there is a split monomorphism $f_2 : E_2' \to E_2$ such that $f_1 \oplus f_2$ is a monomorphism in A between objects of E. Finally, to prove (v), it suffices to note that if $E = E_1 \oplus E_2 \in \mathsf{E}$ and if $t_1 : T_1 \to E_1$ and $t_2 : T_2 \to E_2$ are $\mathsf{Add}(T)$-precovers, then also $t_1 \oplus t_2 : T_1 \oplus T_2 \to E$ is an $\mathsf{Add}(T)$-precover. □

Now we present the dual definitions. Let B be an abelian category with set-indexed coproducts and a projective generator $P \in \mathsf{B}$. Let $W \in \mathsf{B}$ be an object and $\mathsf{F} \subset \mathsf{B}$ be a full subcategory. We will say that (W, F) is an ∞-*cotilting pair* in B if the following conditions hold:

- (i*) $\mathsf{B}_{\mathsf{proj}} \subset \mathsf{F}$.
- (ii*) $\mathsf{Prod}(W) \subset \mathsf{F}$.
- (iii*) $\mathrm{Ext}^1_{\mathsf{B}}(F, W) = 0$ for all $F \in \mathsf{F}$.
- (iv*) F is closed under the kernels of epimorphisms and extensions in B.
- (v*) Every $\mathsf{Prod}(W)$-preenvelope $F \to W'$ of an object $F \in \mathsf{F}$ is a monomorphism in B with the cokernel belonging to F.

As above, it follows from the conditions (i*)–(v*) that

$$\mathrm{Ext}^i_{\mathsf{B}}(F, W) = 0 \quad \text{for all } F \in \mathsf{F} \text{ and all integers } i > 0,$$

the object $W \in \mathsf{B}$ is weakly cotilting, and the full subcategory $\mathsf{F} \subset \mathsf{B}$ inherits an exact category structure from the abelian category B. The exact category F has both enough projective and enough injective objects; the full subcategories of projective and injective objects in F are described as $\mathsf{F}_{\mathsf{proj}} = \mathsf{B}_{\mathsf{proj}}$ and $\mathsf{F}_{\mathsf{inj}} = \mathsf{Prod}(W)$. Moreover, as before one also has:

Lemma 3.3. *Let* B *be a complete, cocomplete abelian category with a projective generator. Then an object* $W \in \mathsf{B}$ *is a part of an* ∞*-cotilting pair* (W, F) *in* B *if and only if it is an* ∞*-cotilting object. The full subcategory* $\mathsf{F} = \mathsf{F}_{\mathsf{max}}(W)$ *is the maximal of all full subcategories* $\mathsf{F} \subset \mathsf{B}$ *forming an* ∞*-cotilting pair with* $W \in \mathsf{B}$.

Moreover, if (W, F) *is an* ∞*-cotilting pair and* F' *is the closure of* F *under direct summands, then* (W, F') *is also an* ∞*-cotilting pair and* F' *is a resolving subcategory of* B.

Proof. This is dual to Lemmas 3.1 and 3.2. □

The ∞-tilting-cotilting correspondence from the last section now extends to one between ∞-tilting and ∞-cotilting pairs.

Proposition 3.4. *In the context of Corollary 2.5 (see also Figure 1), the assignments* $\mathsf{F} = \Psi(\mathsf{E})$ *and* $\mathsf{E} = \Phi(\mathsf{F})$ *establish a bijective correspondence between*

(1) *the full subcategories* $\mathsf{E} \subset \mathsf{E}_{\mathsf{max}}(T)$ *forming an* ∞*-tilting pair with* $T \in \mathsf{A}$ *and*

(2) *the full subcategories* $\mathsf{F} \subset \mathsf{F}_{\mathsf{max}}(W)$ *forming an* ∞*-cotilting pair with* $W \in \mathsf{B}$.

Proof. Let (T, E) be an ∞-tilting pair in the category A. Put $\mathsf{F} = \Psi(\mathsf{E})$. We have to show that (W, F) is an ∞-cotilting pair in the category B.

Indeed, the condition (i*) follows from (ii) and the condition (ii*) follows from (i), as $\mathsf{B}_{\mathsf{proj}} = \Psi(\mathsf{Add}(T))$ and $\mathsf{Prod}(W) = \Psi(\mathsf{A}_{\mathsf{inj}})$. The condition (iii*) holds, since $\mathsf{F} = \Psi(\mathsf{E}) \subset \Psi(\mathsf{E}_{\mathsf{max}}(T)) = \mathsf{F}_{\mathsf{max}}(W)$ and all $F \in \mathsf{F}_{\mathsf{max}}(W)$ satisfy (iii*).

The full subcategory F is closed under extensions in $\mathsf{F}_{\mathsf{max}}(W)$, because $\Psi : \mathsf{E}_{\mathsf{max}}(T) \to \mathsf{F}_{\mathsf{max}}(W)$ is an equivalence of exact categories and the full subcategory E is closed under extensions in $\mathsf{E}_{\mathsf{max}}(W)$. Since the full subcategory $\mathsf{F}_{\mathsf{max}}(W)$ is closed under extensions in B by Lemma 2.2(a), it follows that F is closed under extensions in B.

Let $f : F' \to F''$ be an epimorphism in B between two objects $F', F'' \in \mathsf{F}$. Then there exists a morphism $e : E' \to E''$ in E such that $F^{(s)} \simeq \Psi(E^{(s)})$, $s = 1, 2$, and $f = \Psi(e)$. The map of abelian groups $\mathrm{Hom}_{\mathsf{E}}(T, e)$ is surjective, since the map $\mathrm{Hom}_{\mathsf{F}}(P, f)$ is and $P = \Psi(T)$. Let $T_0 \to E'$ be an $\mathsf{Add}(T)$-precover; then the composition $T_0 \to E' \to E''$ is also an $\mathsf{Add}(T)$-precover. Denote the kernels of the morphisms $T_0 \to E'$ and $T_0 \to E''$ by Z'_0 and Z''_0, respectively. Then $Z'_0, Z''_0 \in \mathsf{E}$ by the condition (v) and the natural morphism $Z_0' \to Z_0''$ is a monomorphism. Hence $\ker(e) = \operatorname{coker}(Z_0' \to Z_0'') \in \mathsf{E}$ by the condition (iv) and $\ker(f) = \Psi(\ker e) \in \mathsf{F}$. This proves that the full subcategory $\mathsf{F} \subset \mathsf{B}$ is closed under the kernels of epimorphisms in B and finishes the proof of the condition (iv*).

To prove the condition (v*), let $f : F \to W^0$ be a $\mathsf{Prod}(W)$-preenvelope of an object $F \in \mathsf{F}$. Then $f = \Psi(e)$, where $e : E \to J^0$ is a morphism in E and $J^0 \in \mathsf{A}_{\mathsf{inj}}$. The map $\operatorname{Hom}_{\mathsf{E}}(e, J)$ is surjective, since the map $\operatorname{Hom}_{\mathsf{F}}(f, W)$ is and $W = \Psi(J)$. Since J is an injective cogenerator of A, it follows that e is a monomorphism in A. By the condition (iv), the cokernel of e belongs to E, so e is a monomorphism in E. Since the functor $\Psi|_{\mathsf{E}}$ is exact, it follows that f is a monomorphism in B with the cokernel belonging to F.

To summarize these arguments, the conditions (iv) and (v) essentially say that the full subcategory E is closed under the cokernels of monomorphisms, extensions, and kernels of epimorphisms in $\mathsf{E}_{\mathsf{max}}(T)$, while the conditions (iv*) and (v*) mean that the full subcategory F is closed under the kernels of epimorphisms, extensions, and cokernels of monomorphisms in $\mathsf{F}_{\mathsf{max}}(W)$. □

Corollary 3.5. *The constructions of Theorems 2.3 and 2.4 and Proposition 3.4 establish a one-to-one correspondence between equivalence classes of*

(1) *quadruples* $(\mathsf{A}, \mathsf{E}, T, J)$, *where* A *is a complete, cocomplete abelian category with an injective cogenerator* J *and* (T, E) *is an* ∞*-tilting pair in* A, *and*

(2) *quadruples* $(\mathsf{B}, \mathsf{F}, P, W)$, *where* B *is a complete, cocomplete abelian category with a projective generator* P *and* (W, F) *is an* ∞*-cotilting pair in* B.

In this correspondence, the exact categories E *and* F *are naturally equivalent,* $\mathsf{E} \simeq \mathsf{F}$, *and the equivalence identifies* T *with* P *and* W *with* J. □

In general, there can be many classes E which form an ∞-tilting pair with a given ∞-tilting object $T \in \mathsf{A}$. Thanks to the following lemma, we know that they form a complete lattice.

Lemma 3.6. *Let* A *be a complete, cocomplete abelian category with an injective cogenerator and let* $T \in \mathsf{A}$ *be an* ∞*-tilting object. If* $\mathsf{E}_i \subset \mathsf{A}$, $i \in I$, *is a collection of full subcategories such that* (T, E_i) *is an* ∞*-tilting pair for each* $i \in I$, *then* (T, E) *is an* ∞*-tilting pair with* $\mathsf{E} = \bigcap_{i \in I} \mathsf{E}_i$.

Dually, if B *is a complete, cocomplete abelian category with a projective generator,* $W \in \mathsf{B}$ *is an* ∞*-cotilting object and* (W, F_j) *are* ∞*-cotilting pairs,* $j \in J$, *then* (W, F) *is an* ∞*-cotilting pair with* $\mathsf{F} = \bigcap_{j \in J} \mathsf{F}_j$.

Proof. It is straightforward to check that each of the conditions (i)–(v) and (i*)–(v*) is preserved by intersections of classes. □

Example 3.7. In particular, whenever T is an ∞-tilting object in A, there exists a unique minimal full subcategory $\mathsf{E}_{\mathsf{min}}(T) \subset \mathsf{A}$ for which $(T, \mathsf{E}_{\mathsf{min}}(T))$ is an ∞-tilting pair in A. In fact, the full subcategory $\mathsf{E}_{\mathsf{min}}(T)$ consists of all the objects in A that can be obtained from the objects of $\mathsf{A}_{\mathsf{inj}} \subset \mathsf{E}_{\mathsf{min}}(\mathsf{A})$ and $\mathsf{Add}(T) \subset \mathsf{E}_{\mathsf{min}}(\mathsf{A})$ by applying iteratively the operations of the passage to the cokernel of a monomorphism, an extension, or the kernel of an $\mathsf{Add}(T)$-precover. For every ∞-tilting pair (T, E) in A, one then has $\mathsf{E}_{\mathsf{min}}(T) \subset \mathsf{E}$.

Similarly, whenever W is an ∞-cotilting object in B, there exists a unique minimal full subcategory $\mathsf{F}_{\mathsf{min}}(W) \subset \mathsf{B}$ such that $(W, \mathsf{F}_{\mathsf{min}}(W))$ is an ∞-cotilting pair in B. For every ∞-cotilting pair (W, F) in B, one has $\mathsf{F}_{\mathsf{min}}(W) \subset \mathsf{F}$.

In the situation of Corollary 2.5 (and Figure 1), the full subcategories $\mathsf{E}_{\mathsf{min}}(T) \subset \mathsf{A}$ and $\mathsf{F}_{\mathsf{min}}(W) \subset \mathsf{B}$ are transformed into each other by the functors Ψ and Φ, that is

$$\mathsf{F}_{\mathsf{min}}(W) = \Psi(\mathsf{E}_{\mathsf{min}}(T)) \quad \text{and} \quad \mathsf{E}_{\mathsf{min}}(T) = \Phi(\mathsf{F}_{\mathsf{min}}(W)).$$

Remark 3.8. There is a certain similarity between our results in Sections 1–3 of this paper and those in [Enomoto 2017, Sections 2–3]. Let us explain the connection and the differences between our approaches. The paper [Enomoto 2017] is a far-reaching development of the traditional point of view in Wakamatsu tilting theory, in which finitely generated modules over Artinian algebras are the main objects of study. Enomoto [2017] works with skeletally small exact categories, and essentially never considers infinite products or coproducts. The definition of a projective generator in [Enomoto 2017, paragraph before Corollary 2.14] presumes an exact category with enough projective objects in which every projective object is a direct summand of a *finite* direct sum of copies of the (single) generator.

Nevertheless, the generality level in [Enomoto 2017, Sections 2–3] exceeds that of our exposition. In particular, our Lemma 2.2 is but a particular case of [Enomoto 2017, Proposition 3.2] (while our Lemma 2.1 is dual). Enomoto achieves this generality by working with arbitrary (skeletally small) additive categories in place of our classes $\mathsf{Add}(T)$ and $\mathsf{Prod}(W)$. An exact category playing the role of our F is generally denoted by $\mathcal{E}$ in [Enomoto 2017], an additive category playing the role of our $\mathsf{Add}(T) = \mathsf{F}_{\mathsf{proj}} = \mathsf{B}_{\mathsf{proj}}$ is denoted by $\mathcal{C}$, an additive category in the role of our $\mathsf{Prod}(W) = \mathsf{F}_{\mathsf{inj}}$ is denoted by $\mathcal{W}$, and the exact category in the role of our $\mathsf{F}_{\mathsf{max}}(W)$ is denoted by $\mathsf{X}_{\mathcal{W}}$. (The reader should be warned that Enomoto calls "Wakamatsu tilting" what we would call "Wakamatsu cotilting" or "∞-cotilting".)

Finally, in the role of our abelian category B, Enomoto has an exact category which he denotes by $\mathsf{mod}\,\mathcal{C}$. This difference occurs because our observation that the category $\mathsf{Add}(T)$ always has weak kernels (as pointed out in the proof of Theorem 1.1) has no counterpart in [Enomoto 2017].

4. ∞-tilting-cotilting derived equivalences

Unlike in [Positselski and Šťovíček 2019, Sections 3–5], in our present situation the coresolution dimensions of objects of the category A with respect to its coresolving

$$\begin{tikzcd}
\mathsf{Hot}(\mathsf{A}_{\mathsf{inj}}) \arrow[d] & \mathsf{Hot}(\mathsf{B}_{\mathsf{proj}}) \arrow[d] \\
\mathsf{D}(\mathsf{E}) \arrow[r, equal] \arrow[d] & \mathsf{D}(\mathsf{F}) \arrow[d] \\
\mathsf{D}(\mathsf{A}) & \mathsf{D}(\mathsf{B})
\end{tikzcd}$$

Figure 2. Equivalences induced by ∞-tilting/∞-cotilting pairs in general.

subcategory E or $\mathsf{E}_{\mathsf{max}}(T)$ can well be infinite, and so can the resolution dimensions of objects of the category B with respect to its resolving subcategory F or $\mathsf{F}_{\mathsf{max}}(W)$. Hence the equivalence of exact categories $\mathsf{E} \simeq \mathsf{F}$ does not generally lead to any equivalence between the derived categories $\mathsf{D}(\mathsf{A})$ and $\mathsf{D}(\mathsf{B})$. All one can say is that there is the commutative diagram formed by triangulated functors and a triangulated equivalence in Figure 2. Here $\mathsf{Hot}(\mathsf{A}_{\mathsf{inj}})$ and $\mathsf{Hot}(\mathsf{B}_{\mathsf{proj}})$ are the homotopy categories of (unbounded complexes in) the additive categories $\mathsf{A}_{\mathsf{inj}}$ and $\mathsf{B}_{\mathsf{proj}}$, while $\mathsf{D}(\mathsf{E})$ and $\mathsf{D}(\mathsf{F})$ are the (unbounded) derived categories of the exact categories E and F, and $\mathsf{D}(\mathsf{A})$ and $\mathsf{D}(\mathsf{B})$ are the similar derived categories of the abelian categories A and B.

If A is a Grothendieck category, the canonical functor $\mathsf{Hot}(\mathsf{A}_{\mathsf{inj}}) \to \mathsf{D}(\mathsf{A})$ in the left-hand side column of Figure 2 is a Verdier quotient functor. This follows, e.g., from [Alonso Tarrío et al. 2000, Theorem 5.4]. If the full subcategories $\mathsf{E} \subset \mathsf{A}$ and $\mathsf{F} \subset \mathsf{B}$ have additional closure properties, we will obtain a similar diagram below where all the functors are Verdier quotients.

One issue here is that, unlike for tilting modules of finite projective dimension, the class E in the definition of a tilting pair need not be closed under coproducts in A (cf. [Positselski and Šťovíček 2019, Lemma 5.3]). Dually, the class F need not be closed under products. There are some elementary relations between the closure properties of E and F, however.

Lemma 4.1. *In the context of Corollary 3.5, if the full subcategory* $\mathsf{E} \subset \mathsf{A}$ *is closed under products, then the full subcategory* $\mathsf{F} \subset \mathsf{B}$ *is closed under products. If the full subcategory* $\mathsf{F} \subset \mathsf{B}$ *is closed under coproducts, then the full subcategory* $\mathsf{E} \subset \mathsf{A}$ *is closed under coproducts.*

Proof. The first assertion holds, since $\mathsf{F} = \Psi(\mathsf{E})$ and the functor $\Psi : \mathsf{A} \to \mathsf{B}$ preserves products (see Figure 1). The second assertion holds, since $\mathsf{E} = \Phi(\mathsf{F})$ and the functor $\Phi : \mathsf{B} \to \mathsf{A}$ preserves coproducts. □

It would be interesting to know whether the converse assertions to those of Lemma 4.1 are true.

Suppose now that E is a part of an ∞-tilting pair in a complete, cocomplete abelian category A with an injective cogenerator. Then A has exact coproducts ([Mitchell 1965, Exercise III.2]) and, if E is closed under coproducts, E has exact coproducts too. In such a situation the following definition from [Positselski 2010, Sections 2.1 and 4.1] or [Positselski 2012, Section A.1] applies and gives a more adequate replacement of $\mathsf{Hot}(\mathsf{A}_{\mathsf{inj}}) = \mathsf{Hot}(\mathsf{E}_{\mathsf{inj}})$ in Figure 2.

If E is an exact category with arbitrary coproducts which are exact, we call a complex *coacyclic* if it belongs to the smallest localizing subcategory of $\mathsf{Hot}(\mathsf{E})$ which contains the total complexes of short exact sequences of complexes over E. The *coderived category* of E, which we denote by $\mathsf{D}^{\mathsf{co}}(\mathsf{E})$, is defined as the Verdier quotient category of $\mathsf{Hot}(\mathsf{E})$ by the subcategory of coacyclic complexes.

Note that it follows from the above definition that each coacyclic complex is exact and, thus, we have a Verdier quotient functor $\mathsf{D}^{\mathsf{co}}(\mathsf{E}) \to \mathsf{D}(\mathsf{E})$. On the other hand, if E in addition has enough injectives, the natural functor $\mathsf{Hot}(\mathsf{E}_{\mathsf{inj}}) \to \mathsf{D}^{\mathsf{co}}(\mathsf{E})$ is fully faithful by [Positselski 2012, Lemma A.1.3]. To summarize, we have triangulated functors

$$\mathsf{Hot}(\mathsf{E}_{\mathsf{inj}}) \rightarrowtail \mathsf{D}^{\mathsf{co}}(\mathsf{E}) \twoheadrightarrow \mathsf{D}(\mathsf{E}),$$

where the first functor is fully faithful and the second one is a Verdier quotient. In fact, the fully faithful functor was proved to be an equivalence in some cases [Positselski 2017c, Theorem 2.4].

If F is an exact category with arbitrary products which are exact, the class of *contraacyclic* complexes in $\mathsf{Hot}(\mathsf{F})$ and the *contraderived* category $\mathsf{D}^{\mathsf{ctr}}(\mathsf{F})$ of F are defined dually, and we have triangulated functors

$$\mathsf{Hot}(\mathsf{F}_{\mathsf{proj}}) \rightarrowtail \mathsf{D}^{\mathsf{ctr}}(\mathsf{F}) \twoheadrightarrow \mathsf{D}(\mathsf{F}).$$

As above, the fully faithful functor in the leftmost arrow is known to be an equivalence in some cases [Positselski 2017c, Theorem 4.4(b)].

Now we can state the main result of the section (see also Figure 3 below).

Proposition 4.2. (a) *Let A be an exact category where set-indexed coproducts exist and are exact, and let $\mathsf{E} \subset \mathsf{A}$ be a coresolving subcategory closed under coproducts. Then the functor between the coderived categories $\mathsf{D}^{\mathsf{co}}(\mathsf{E}) \to \mathsf{D}^{\mathsf{co}}(\mathsf{A})$ induced by the embedding of exact categories $\mathsf{E} \to \mathsf{A}$ is a triangulated equivalence. The triangulated functor between the conventional derived categories $\mathsf{D}(\mathsf{E}) \to \mathsf{D}(\mathsf{A})$ induced by the same exact embedding is a Verdier quotient functor.*

(b) *Let B be an exact category where set-indexed products exist and are exact, and let $\mathsf{F} \subset \mathsf{B}$ be a resolving subcategory closed under products. Then the functor between the contraderived categories $\mathsf{D}^{\mathsf{ctr}}(\mathsf{F}) \to \mathsf{D}^{\mathsf{ctr}}(\mathsf{B})$ induced by the embedding of exact categories $\mathsf{F} \to \mathsf{B}$ is a triangulated equivalence. The triangulated functor between the conventional derived categories $\mathsf{D}(\mathsf{F}) \to \mathsf{D}(\mathsf{B})$ induced by the same exact embedding is a Verdier quotient functor.*

$$\begin{tikzcd}
\mathsf{D}^{\mathsf{co}}(\mathsf{A}) \arrow[d, two heads] & \mathsf{D}^{\mathsf{ctr}}(\mathsf{B}) \arrow[d, two heads] \\
\mathsf{D}(\mathsf{E}) \arrow[r, equal] \arrow[d, two heads] & \mathsf{D}(\mathsf{F}) \arrow[d, two heads] \\
\mathsf{D}(\mathsf{A}) & \mathsf{D}(\mathsf{B})
\end{tikzcd}$$

Figure 3. Equivalences induced by ∞-tilting/∞-cotilting pairs when E is closed under coproducts and F under products.

Proof. The first assertion of part (b) is [Positselski 2012, Proposition A.3.1(b)], and the first assertion of part (a) is the dual result.

To prove the second assertion of part (a), notice that we have a commutative diagram of triangulated functors $\mathsf{D}^{\mathsf{co}}(\mathsf{E}) = \mathsf{D}^{\mathsf{co}}(\mathsf{A}) \to \mathsf{D}(\mathsf{E}) \to \mathsf{D}(\mathsf{A})$, where both the functors $\mathsf{D}^{\mathsf{co}}(\mathsf{A}) \to \mathsf{D}(\mathsf{E})$ and $\mathsf{D}^{\mathsf{co}}(\mathsf{A}) \to \mathsf{D}(\mathsf{A})$ are Verdier quotient functors. It follows that the functor $\mathsf{D}(\mathsf{E}) \to \mathsf{D}(\mathsf{A})$ is also a Verdier quotient. $\square$

In particular, Proposition 4.2 tells that, when in the situation of Corollary 3.5 the full subcategory $\mathsf{E} \subset \mathsf{A}$ is closed under coproducts and the full subcategory $\mathsf{F} \subset \mathsf{B}$ is closed under products, we have a commutative diagram formed by Verdier quotient functors and a triangulated equivalence as in Figure 3.

Remark 4.3. Let A be an exact category with exact coproducts, and let $\mathsf{E}' \subset \mathsf{E}'' \subset \mathsf{A}$ be two coresolving subcategories closed under coproducts. Then one has $\mathsf{D}^{\mathsf{co}}(\mathsf{E}') \simeq \mathsf{D}^{\mathsf{co}}(\mathsf{E}'') \simeq \mathsf{D}^{\mathsf{co}}(\mathsf{A})$, while the natural functors between the conventional derived

$$\begin{tikzcd}
\mathsf{D}^{\mathsf{co}}(\mathsf{A}) \arrow[d, two heads] & \mathsf{D}^{\mathsf{ctr}}(\mathsf{B}) \arrow[d, two heads] \\
\mathsf{D}(\mathsf{E}') \arrow[r, equal] \arrow[d, two heads] & \mathsf{D}(\mathsf{F}') \arrow[d, two heads] \\
\mathsf{D}(\mathsf{E}'') \arrow[r, equal] \arrow[d, two heads] & \mathsf{D}(\mathsf{F}'') \arrow[d, two heads] \\
\mathsf{D}(\mathsf{A}) & \mathsf{D}(\mathsf{B})
\end{tikzcd}$$

Figure 4. Compatible equivalences for different choices of ∞-tilting/∞-cotilting pairs.

categories $\mathsf{D}(\mathsf{E}') \to \mathsf{D}(\mathsf{E}'') \to \mathsf{D}(\mathsf{A})$ are Verdier quotient functors. Thus, when a coproduct-closed coresolving subcategory is being enlarged, its derived category gets deflated. In other words, the larger the subcategory $\mathsf{E} \subset \mathsf{A}$, the smaller its derived category $\mathsf{D}(\mathsf{E})$.

Similarly, let B be an exact category with exact products, and let $\mathsf{F}' \subset \mathsf{F}'' \subset \mathsf{B}$ be two resolving subcategories closed under products. Then one has

$$\mathsf{D}^{\mathsf{ctr}}(\mathsf{F}') \simeq \mathsf{D}^{\mathsf{ctr}}(\mathsf{F}'') \simeq \mathsf{D}^{\mathsf{ctr}}(\mathsf{B}),$$

while the natural functors between the conventional derived categories

$$\mathsf{D}(\mathsf{F}') \to \mathsf{D}(\mathsf{F}'') \to \mathsf{D}(\mathsf{B})$$

are Verdier quotient functors.

In particular, when in the situation of Proposition 3.4 there are two ∞-tilting pairs (T, E') and (T, E'') with $\mathsf{E}' \subset \mathsf{E}'' \subset \mathsf{A}$, and the corresponding two ∞-cotilting pairs are (W, F') and (W, F''), so $\mathsf{F}' \subset \mathsf{F}'' \subset \mathsf{B}$, we obtain the commutative diagram of Verdier quotient functors and triangulated equivalences as in Figure 4. We refer to [Positselski 2017a, Section 1] for a further discussion.

5. ∞-tilting and ∞-cotilting t-structures

The aim of the section is to lift the canonical t-structures from $\mathsf{D}(\mathsf{A})$ and $\mathsf{D}(\mathsf{B})$ to $\mathsf{D}(\mathsf{E})$ and $\mathsf{D}(\mathsf{F})$, respectively, in Figures 2 or 3 in the previous section. By doing this, we obtain a picture very similar to the classical tilting theory, where both A and B can be viewed as full subcategories of $\mathsf{D}(\mathsf{E})$ such that $\mathsf{E} = \mathsf{A} \cap \mathsf{B}$ (since $\mathsf{E} \simeq \mathsf{F}$, we of course obtain the same picture in $\mathsf{D}(\mathsf{F})$).

We start with a lemma showing that t-structures can be lifted with respect to certain triangulated functors with partial adjoints.

Lemma 5.1. *Let D and $'\mathsf{D}$ be triangulated categories and let $(\mathsf{D}^{\le 0}, \mathsf{D}^{\ge 0})$ be a t-structure on D. Let $F : {}'\mathsf{D} \to \mathsf{D}$ be a triangulated functor such that a right adjoint functor to F is defined on $\mathsf{D}^{\ge 0} \subset \mathsf{D}$, that is, for every object $X \in \mathsf{D}^{\ge 0}$ there exists an object $G(X) \in {}'\mathsf{D}$ such that the functors $\mathrm{Hom}_{\mathsf{D}}(F(-), X)$ and $\mathrm{Hom}_{'\mathsf{D}}(-, G(X))$ are isomorphic on $'\mathsf{D}$. Assume that the adjunction morphism $\varepsilon_X : FG(X) \to X$ is an isomorphism in D for all objects $X \in \mathsf{D}^{\ge 0}$.*

Set ${}'\mathsf{D}^{\le 0} = F^{-1}(\mathsf{D}^{\le 0}) \subset {}'\mathsf{D}$ to be the full preimage of $\mathsf{D}^{\le 0}$ under F and ${}'\mathsf{D}^{\ge 0} = G(\mathsf{D}^{\ge 0}) \subset {}'\mathsf{D}$ to be the essential image of $\mathsf{D}^{\ge 0}$ under G. Then the pair of full subcategories $({}'\mathsf{D}^{\le 0}, {}'\mathsf{D}^{\ge 0})$ is a t-structure on $'\mathsf{D}$. The functors F and G restrict to mutually inverse equivalences between the abelian hearts $\mathsf{A} = \mathsf{D}^{\le 0} \cap \mathsf{D}^{\ge 0} \subset \mathsf{D}$ and ${}'\mathsf{A} = {}'\mathsf{D}^{\le 0} \cap {}'\mathsf{D}^{\ge 0} \subset {}'\mathsf{D}$ of the two t-structures.

Proof. One can easily check that the functor G commutes with the shift functors $[-1]$ on $'\mathsf{D}$ and D (since the functor F does). Let us show that $\mathrm{Hom}_{'\mathsf{D}}({}'X, {}'Y) = 0$

for all $'X \in {}'\mathsf{D}^{\leq 0}$ and $'Y \in {}'\mathsf{D}^{\geq 1}$. Indeed, we have $F('X) \in \mathsf{D}^{\leq 0}$ and $'Y = G(Y)$ for some $Y \in \mathsf{D}^{\geq 1}$. Hence

$$\mathrm{Hom}_{'\mathsf{D}}('X, 'Y) = \mathrm{Hom}_{'\mathsf{D}}('X, G(Y)) = \mathrm{Hom}_{\mathsf{D}}(F('X), Y) = 0.$$

Now let $'X \in {}'\mathsf{D}$ be an arbitrary object. Set $X = F('X) \in \mathsf{D}$, and consider a distinguished triangle

$$\tau_{\leq 0}X \to X \to \tau_{\geq 1}X \to (\tau_{\leq 0}X)[1] \tag{1}$$

in D with $\tau_{\leq 0}X \in \mathsf{D}^{\leq 0}$ and $\tau_{\geq 1}X \in \mathsf{D}^{\geq 1}$. Put $\tau_{\geq 1}{}'X = G(\tau_{\geq 1}X) \in {}'\mathsf{D}^{\geq 1}$. Then the morphism $F('X) = X \to \tau_{\geq 1}X$ in D corresponds to a certain morphism

$$'X \to G(\tau_{\geq 1}X) = \tau_{\geq 1}{}'X$$

in $'\mathsf{D}$. Denote by $\tau_{\leq 0}{}'X$ a cocone of the latter morphism, so that we have a distinguished triangle

$$\tau_{\leq 0}{}'X \to {}'X \to \tau_{\geq 1}{}'X \to (\tau_{\leq 0}{}'X)[1] \tag{2}$$

in $'\mathsf{D}$. Applying the functor F to the morphism $'X \to \tau_{\geq 1}{}'X$ produces the morphism $X = F('X) \to F(\tau_{\geq 1}('X)) = FG(\tau_{\geq 1}(X)) = \tau_{\geq 1}(X)$. Thus the triangulated functor F takes the distinguished triangle (2) to the distinguished triangle (1), and it follows that the object $F(\tau_{\leq 0}{}'X)$ is isomorphic to $\tau_{\leq 0}X$. In other words, we have $\tau_{\leq 0}{}'X \in {}'\mathsf{D}^{\leq 0}$ and $\tau_{\geq 1}{}'X \in {}'\mathsf{D}^{\geq 1} := {}'\mathsf{D}^{\geq 0}[-1]$ in (2). It follows that $('\mathsf{D}^{\leq 0}, {}'\mathsf{D}^{\geq 0})$ is a t-structure.

Furthermore, the functors F and G restrict to an equivalence between the coaisles $\mathsf{D}^{\geq 0} \subset \mathsf{D}$ and $'\mathsf{D}^{\geq 0} \subset {}'\mathsf{D}$. Indeed, if $'X \in {}'\mathsf{D}^{\geq 0}$, then $'X = G(X)$ for some $X \in \mathsf{D}^{\geq 0}$ and $F('X) = FG(X) = X \in \mathsf{D}^{\geq 0}$. Thus the functor F restricts to $F : {}'\mathsf{D}^{\geq 0} \to \mathsf{D}^{\geq 0}$, the functor $G : \mathsf{D}^{\geq 0} \to {}'\mathsf{D}^{\geq 0}$ is its (honest) right adjoint, and the composition

$$\mathsf{D}^{\geq 0} \xrightarrow{G} {}'\mathsf{D}^{\geq 0} \xrightarrow{F} \mathsf{D}^{\geq 0}$$

is the identity functor by assumption. Hence the functor G is fully faithful; and its essential image coincides with $'\mathsf{D}^{\geq 0}$ by the definition. Finally, for any $'X \in {}'\mathsf{D}^{\geq 0}$ we have $'X \in {}'\mathsf{A}$ if and only if $F('X) \in \mathsf{A}$, because we have $'X \in {}'\mathsf{D}^{\leq 0}$ if and only if $F('X) \in \mathsf{D}^{\leq 0}$. □

Remark 5.2. In the special case where the functor $F : {}'\mathsf{D} \to \mathsf{D}$ from the former lemma is a part of a recollement

$$\begin{tikzcd} \mathsf{D} \arrow[r, bend left, tail] \arrow[r, bend right, tail, "G"'] & {}'\mathsf{D} \arrow[l, "F"'] \arrow[r, bend left] \arrow[r, bend right] & {}''\mathsf{D}, \arrow[l, tail] \end{tikzcd}$$

the t-structure $('\mathsf{D}^{\leq 0}, {}'\mathsf{D}^{\geq 0})$ coincides with the result of gluing $(\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 0})$ with the trivial t-structure $(''\mathsf{D}, 0)$ on $''\mathsf{D}$ in the sense of [Beĭlinson et al. 1982, Théorème 1.4.10].

We recall that for any t-structure $(\mathsf{D}^{\le 0}, \mathsf{D}^{\ge 0})$ on a triangulated category D with the abelian heart $\mathsf{A} = \mathsf{D}^{\le 0} \cap \mathsf{D}^{\ge 0} \subset \mathsf{D}$ there are natural maps

$$(3)\quad \theta^i_{\mathsf{A},\mathsf{D}} = \theta^i_{\mathsf{A},\mathsf{D}}(X,Y) : \operatorname{Ext}^i_{\mathsf{A}}(X,Y) \to \operatorname{Hom}_{\mathsf{D}}(X, Y[i]) \quad \text{for all } X,\ Y \in \mathsf{A},\ i \ge 0.$$

A t-structure $(\mathsf{D}^{\le 0}, \mathsf{D}^{\ge 0})$ is said to be *of the derived type* if the maps $\theta^i_{\mathsf{A},\mathsf{D}}(X,Y)$ are isomorphisms for all $X, Y \in \mathsf{A}$ and $i \ge 0$ (see [Beĭlinson et al. 1982, Remarque 3.1.17; Positselski 2011, Corollary A.17; Positselski and Šťovíček 2019, Section 4] for further details).

Lemma 5.3. *In the context of Lemma 5.1, the t-structure* $('\mathsf{D}^{\le 0}, '\mathsf{D}^{\ge 0})$ *on the triangulated category* $'\mathsf{D}$ *is of the derived type if and only if the t-structure* $(\mathsf{D}^{\le 0}, \mathsf{D}^{\ge 0})$ *on the triangulated category* D *is.*

Proof. According to Lemma 5.1, the functor $F : {'\mathsf{A}} \to \mathsf{A}$ is an equivalence of categories. So, according to the proof of Lemma 5.1, is the functor $F : {'\mathsf{D}^{\ge 0}} \to \mathsf{D}^{\ge 0}$. It remains to observe that the domain of the map (3) is an Ext group computed in the abelian heart of the t-structure, while the codomain is a Hom group in the coaisle: $\operatorname{Hom}_{\mathsf{D}}(X, Y[i]) = \operatorname{Hom}_{\mathsf{D}}(X[-i], Y)$, and both the objects $X[-i]$ and Y belong to $\mathsf{D}^{\ge 0}$. □

Lemma 5.4 describes the situation in which we want to apply Lemma 5.1.

Lemma 5.4. *Let* A *be an abelian category and* $\mathsf{E} \subset \mathsf{A}$ *be a coresolving subcategory, viewed as an exact category with the exact category structure inherited from* A. *Then the functor between the derived categories of bounded below complexes* $\mathsf{D}^+(\mathsf{E}) \to \mathsf{D}^+(\mathsf{A})$ *induced by the exact embedding functor* $\mathsf{E} \to \mathsf{A}$ *is a triangulated equivalence. The inverse functor to this equivalence* $\mathsf{D}(\mathsf{A}) \supset \mathsf{D}^+(\mathsf{A}) \to \mathsf{D}^+(\mathsf{E}) \subset \mathsf{D}(\mathsf{E})$ *is a partially defined right adjoint functor* (*in a sense analogous to the statement of Lemma 5.1*) *to the functor between the unbounded derived categories* $\mathsf{D}(\mathsf{E}) \to \mathsf{D}(\mathsf{A})$ *induced by the exact embedding* $\mathsf{E} \to \mathsf{A}$.

Proof. For any bounded below complex $A^\bullet$ in A there exists a bounded below complex $E^\bullet$ in E together with a quasi-isomorphism $A^\bullet \to E^\bullet$ of complexes in A [Hartshorne 1966, Lemma I.4.6(1)]. Thus, the functor $\mathsf{D}^+(\mathsf{E}) \to \mathsf{D}^+(\mathsf{A})$ is essentially surjective.

Since E is closed under the cokernels of monomorphisms, any bounded below complex in E that is acyclic in A is also acyclic in E. From this we will deduce that for any complex $E^\bullet$ in E and any bounded below complex $F^\bullet$ in E the natural map

$$(4)\qquad \operatorname{Hom}_{\mathsf{D}(\mathsf{E})}(E^\bullet, F^\bullet) \to \operatorname{Hom}_{\mathsf{D}(\mathsf{A})}(E^\bullet, F^\bullet)$$

is an isomorphism, which implies both that the functor $\mathsf{D}^+(\mathsf{E}) \to \mathsf{D}^+(\mathsf{A})$ is fully faithful (hence, a triangulated equivalence) and that the inverse functor to it is partially right adjoint to the canonical functor $\mathsf{D}(\mathsf{E}) \to \mathsf{D}(\mathsf{A})$.

Indeed, an arbitrary morphism $E^\bullet \to F^\bullet$ in the derived category $\mathsf{D}(\mathsf{A})$ can be represented by a fraction of morphisms of complexes $E^\bullet \to X^\bullet \leftarrow F^\bullet$, where $X^\bullet$ is a complex in A and $F^\bullet \to X^\bullet$ is a quasi-isomorphism of complexes in A. Now the complex $X^\bullet$ is acyclic in low cohomological degrees, so for $n \ll 0$ the natural morphism from $X^\bullet$ to its canonical truncation $X^\bullet \to \tau_{\geq n} X^\bullet$ is a quasi-isomorphism of complexes in A. The complex $\tau_{\geq n} X^\bullet$ is bounded below, so there exists a bounded below complex $G^\bullet$ in E together with a quasi-isomorphism $\tau_{\geq n} X^\bullet \to G^\bullet$ of complexes in A. Then the composition $F^\bullet \to X^\bullet \to \tau_{\geq n} X^\bullet \to G^\bullet$ is a quasi-isomorphism of complexes in the exact category E. This allows us to represent our morphism $E^\bullet \to F^\bullet$ in $\mathsf{D}(\mathsf{A})$ by a fraction $E^\bullet \to G^\bullet \leftarrow F^\bullet$ of morphisms of complexes in E. This proves surjectivity of the map (4).

The injectivity is similar. If a fraction $E^\bullet \to X^\bullet \leftarrow F^\bullet$ vanishes in the group $\mathrm{Hom}_{\mathsf{D}(\mathsf{A})}(E^\bullet, F^\bullet)$, then there exists a quasi-isomorphism $X^\bullet \to G^\bullet$ of complexes in A such that $E^\bullet \to X^\bullet \to G^\bullet$ is null-homotopic. As above, we can choose $X^\bullet \to G^\bullet$ so that $G^\bullet$ is a bounded below complex in E, and it follows that the fraction vanishes in $\mathrm{Hom}_{\mathsf{D}(\mathsf{E})}(E^\bullet, F^\bullet)$ as well. □

Given an abelian category A with a coresolving subcategory $\mathsf{E} \subset \mathsf{A}$, for any complex $E^\bullet$ in E we denote by $H^n_{\mathsf{A}}(E^\bullet) \in \mathsf{A}$ the cohomology objects of the complex $E^\bullet$ viewed as a complex in A. Consider the following two full subcategories in the unbounded derived category $\mathsf{D}(\mathsf{E})$:

- $\mathsf{D}^{\leq 0}_{\mathsf{A}}(\mathsf{E}) \subset \mathsf{D}(\mathsf{E})$ is the full subcategory of all complexes $E^\bullet$ in E such that $H^n_{\mathsf{A}}(E^\bullet) = 0$ for all $n > 0$.
- $\mathsf{D}^{\geq 0}(\mathsf{E}) \subset \mathsf{D}(\mathsf{E})$ is the full subcategory of all objects in $\mathsf{D}(\mathsf{E})$ that can be represented by complexes $E^\bullet$ in E with $E^n = 0$ for all $n < 0$.

As in the usual notation, for any $n \in \mathbb{Z}$ we set $\mathsf{D}^{\leq n}_{\mathsf{A}}(\mathsf{E}) = \mathsf{D}^{\leq 0}_{\mathsf{A}}(\mathsf{E})[-n] \subset \mathsf{D}(\mathsf{E})$ and $\mathsf{D}^{\geq n}(\mathsf{E}) = \mathsf{D}^{\geq 0}(\mathsf{E})[-n] \subset \mathsf{D}(\mathsf{E})$.

Proposition 5.5. *Let* A *be an abelian category and* $\mathsf{E} \subset \mathsf{A}$ *be a coresolving subcategory. Then the pair of full subcategories* $(\mathsf{D}^{\leq 0}_{\mathsf{A}}(\mathsf{E}), \mathsf{D}^{\geq 0}(\mathsf{E}))$ *is a t-structure on the unbounded derived category* $\mathsf{D}(\mathsf{E})$ *of the exact category* E. *Moreover, this is a t-structure of the derived type, and the triangulated functor* $\mathsf{D}(\mathsf{E}) \to \mathsf{D}(\mathsf{A})$ *induced by the exact embedding* $\mathsf{E} \to \mathsf{A}$ *identifies its heart* $\mathsf{D}^{\leq 0}_{\mathsf{A}}(\mathsf{E}) \cap \mathsf{D}^{\geq 0}(\mathsf{E})$ *with the abelian category* A.

Proof. We apply Lemma 5.1 to the situation described in Lemma 5.4, where $'\mathsf{D} = \mathsf{D}(\mathsf{E})$, $\mathsf{D} = \mathsf{D}(\mathsf{A})$, and $F : \mathsf{D}(\mathsf{E}) \to \mathsf{D}(\mathsf{A})$ is the canonical functor. Moreover, we set $(\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 0})$ to be the canonical t-structure on $\mathsf{D}(\mathsf{A})$, which is certainly of the derived type. Then $G = F|^{-1}_{\mathsf{D}^{\geq 0}(\mathsf{E})} : \mathsf{D}^{\geq 0} \to \mathsf{D}^{\geq 0}(\mathsf{E}) \subset \mathsf{D}(\mathsf{E})$ is a partially defined right adjoint to F in the sense of Lemma 5.1 and $(\mathsf{D}^{\leq 0}_{\mathsf{A}}(\mathsf{E}), \mathsf{D}^{\geq 0}(\mathsf{E}))$ is precisely the lifted t-structure from the conclusion of the lemma. It is of the derived type by Lemma 5.3.

For clarity, we summarize the construction of the t-structure truncations $\tau^{\mathsf{E}}_{\le 0}E^\bullet$ and $\tau^{\mathsf{E}}_{\ge 1}E^\bullet$ for a given complex $E^\bullet$ over E. One first considers its canonical truncation $\tau^{\mathsf{A}}_{\ge 1}E^\bullet$ as a complex in A, in the standard t-structure on $\mathsf{D}(\mathsf{A})$. So $\tau^{\mathsf{A}}_{\ge 1}E^\bullet$ is a complex in A with the terms concentrated in the cohomological degrees ≥ 1; hence there exists a complex $F^\bullet$ in E with the terms concentrated in the cohomological degrees ≥ 1 endowed with a quasi-isomorphism $\tau^{\mathsf{A}}_{\ge 1}E^\bullet \to F^\bullet$ of complexes in A. One sets $\tau^{\mathsf{E}}_{\ge 1}E^\bullet = F^\bullet$, and $\tau^{\mathsf{E}}_{\le 0}E^\bullet$ is a cocone of the morphism of complexes $E^\bullet \to F^\bullet$ in $\mathsf{D}(\mathsf{E})$. □

Remark 5.6. It is useful to look into (non)degeneracy properties of the t-structure $(\mathsf{D}^{\le 0}_{\mathsf{A}}(\mathsf{E}), \mathsf{D}^{\ge 0}(\mathsf{E}))$ on $\mathsf{D}(\mathsf{E})$. The intersection $\bigcap_{n\ge 0}\mathsf{D}^{\ge n}(\mathsf{E}) \subset \mathsf{D}(\mathsf{E})$ consists of some bounded below complexes in E with vanishing cohomology in A. All such complexes are acyclic in E, so this intersection is a zero category. On the other hand, the intersection $\bigcap_{n\le 0}\mathsf{D}^{\le n}_{\mathsf{A}}(\mathsf{E}) \subset \mathsf{D}(\mathsf{E})$ consists of all the complexes in E with vanishing cohomology in A. This is precisely the kernel of the triangulated functor $\mathsf{D}(\mathsf{E}) \to \mathsf{D}(\mathsf{A})$, and it can very well be nontrivial. Indeed, let k be a field, $\mathsf{A} = k[x]/(x^2)$–mod and $\mathsf{E} = \mathsf{A}_{\mathsf{inj}}$ (see also Example 6.3 below). Since the complex

$$\cdots \to k[x]/(x^2) \xrightarrow{x} k[x]/(x^2) \xrightarrow{x} k[x]/(x^2) \to \cdots$$

is acyclic but not contractible, it is nonzero in $\mathsf{D}(\mathsf{E}) = \mathsf{Hot}(\mathsf{A}_{\mathsf{inj}})$, but it becomes zero in $\mathsf{D}(\mathsf{A})$.

Let us formulate the dual assertions. Given an abelian category B with a resolving subcategory $\mathsf{F} \subset \mathsf{B}$, for any complex $F^\bullet$ in F we denote by $H^n_{\mathsf{B}}(F^\bullet) \in \mathsf{B}$ the cohomology objects of the complex $F^\bullet$ viewed as a complex in B. Consider the following two subcategories in the unbounded derived category $\mathsf{D}(\mathsf{F})$:

- $\mathsf{D}^{\le 0}(\mathsf{F}) \subset \mathsf{D}(\mathsf{F})$ is the full subcategory of all objects in $\mathsf{D}(\mathsf{F})$ that can be represented by complexes $F^\bullet$ in F with $F^n = 0$ for all $n > 0$.
- $\mathsf{D}^{\ge 0}_{\mathsf{B}}(\mathsf{F}) \subset \mathsf{D}(\mathsf{F})$ is the full subcategory of all complexes $F^\bullet$ in F such that $H^n_{\mathsf{B}}(F^\bullet) = 0$ for all $n < 0$.

Proposition 5.7. *Let* B *be an abelian category and* $\mathsf{F} \subset \mathsf{B}$ *be a resolving subcategory. Then the pair of full subcategories* $(\mathsf{D}^{\le 0}(\mathsf{F}), \mathsf{D}^{\ge 0}_{\mathsf{B}}(\mathsf{F}))$ *is a t-structure on the unbounded derived category* $\mathsf{D}(\mathsf{F})$ *of the exact category* F*. Moreover, this is a t-structure of the derived type, and the triangulated functor* $\mathsf{D}(\mathsf{F}) \to \mathsf{D}(\mathsf{B})$ *induced by the exact embedding* $\mathsf{F} \to \mathsf{B}$ *identifies its heart* $\mathsf{D}^{\le 0}(\mathsf{F}) \cap \mathsf{D}^{\ge 0}_{\mathsf{B}}(\mathsf{F})$ *with the abelian category* B.

Proof. Dual to Proposition 5.5. □

Now we are well-equipped for the discussion of ∞-tilting and ∞-cotilting t-structures. Let A be a complete, cocomplete abelian category with an injective cogenerator J and an ∞-tilting pair (T, E), and let B be the corresponding complete, cocomplete abelian category with a projective generator P and an ∞-cotilting pair (W, F), as in Corollary 3.5. Suppose further for convenience that E, and hence also F, are idempotent complete. Then the exact category $\mathsf{E} \simeq \mathsf{F}$ is simultaneously a coresolving subcategory in A and a resolving subcategory in B.

Thus we have two t-structures

$$(\mathsf{D}^{\leq 0}_{\mathsf{A}}(\mathsf{E}),\ \mathsf{D}^{\geq 0}(\mathsf{E})) \quad \text{and} \quad (\mathsf{D}^{\leq 0}(\mathsf{F}),\ \mathsf{D}^{\geq 0}_{\mathsf{B}}(\mathsf{F}))$$

on the unbounded derived category $\mathsf{D}(\mathsf{E}) = \mathsf{D} = \mathsf{D}(\mathsf{F})$. The hearts of these t-structures are the abelian categories A and B, respectively.

From the point of view of the category A, the t-structure $(\mathsf{D}^{\leq 0}_{\mathsf{A}}(\mathsf{E}),\ \mathsf{D}^{\geq 0}(\mathsf{E}))$ on the triangulated category D can be called the *standard t-structure*, and the t-structure $(\mathsf{D}^{\leq 0}(\mathsf{F}),\ \mathsf{D}^{\geq 0}_{\mathsf{B}}(\mathsf{F}))$ is the ∞*-tilting t-structure*. Looking from the point of view of the category B, the t-structure $(\mathsf{D}^{\leq 0}(\mathsf{F}),\ \mathsf{D}^{\geq 0}_{\mathsf{B}}(\mathsf{F}))$ on the triangulated category D is the *standard t-structure*, and the t-structure $(\mathsf{D}^{\leq 0}_{\mathsf{A}}(\mathsf{E}),\ \mathsf{D}^{\geq 0}(\mathsf{E}))$ is the ∞*-cotilting t-structure*. The abelian category B is the ∞*-tilting heart*, and the abelian category A is the ∞*-cotilting heart*.

6. Examples

Example 6.1. Let A be a complete, cocomplete abelian category with an injective cogenerator J and an ∞-tilting object $T \in \mathsf{A}$, and let B be the corresponding complete, cocomplete abelian category with a projective generator P and an ∞-cotilting object $W \in \mathsf{B}$, as in Corollary 2.5. In this context, if *both* the projective dimension of the ∞-tilting object $T \in \mathsf{A}$ and the injective dimension of the ∞-cotilting object $W \in \mathsf{B}$ are finite, then they are equal to each other, $\mathrm{pd}_{\mathsf{A}}\, T = n = \mathrm{id}_{\mathsf{B}}\, W$. Furthermore, this holds if and only if the object $T \in \mathsf{A}$ is n-tilting if and only if the object $W \in \mathsf{B}$ is n-cotilting (both in the sense of [Positselski and Šťovíček 2019, Sections 2 and 4]).

Indeed, suppose that $\mathrm{pd}_{\mathsf{A}}\, T < \infty$ and $\mathrm{id}_{\mathsf{B}}\, W < \infty$ and denote by n the maximum of the two values. Then the left exact functor $\Psi : \mathsf{A} \to \mathsf{B}$ has finite homological dimension, since it can be computed as the functor $\mathrm{Hom}_{\mathsf{A}}(T, -)$; and the right exact functor $\Phi : \mathsf{B} \to \mathsf{A}$ has finite homological dimension, since it can be computed as the functor $\mathrm{Hom}_{\mathsf{B}}(-, W)^{\mathsf{op}}$. Denote by $\mathsf{E}_T \subset \mathsf{A}$ the full subcategory of all objects $E \in \mathsf{A}$ such that $\mathrm{Ext}^i_{\mathsf{A}}(T, E) = 0$ for all $i > 0$, and by $\mathsf{F}_W \subset \mathsf{B}$ the full subcategory of all objects $F \in \mathsf{B}$ such that $\mathrm{Ext}^i_{\mathsf{B}}(F, W) = 0$ for all $i > 0$. (By the definition, we have $\mathsf{E}_{\max}(T) \subset \mathsf{E}_T$ and $\mathsf{F}_{\max}(W) \subset \mathsf{F}_W$.)

Then the functor Ψ is exact on the exact category E_T and the functor Φ is exact on the exact category F_W. The full subcategory E_T is coresolving in A, and the full subcategory F_W is resolving in B, with both the (co)resolution dimensions bounded by the finite constant n. The latter fact is due to the observation that, thanks to a simple dimension-shifting argument, any n-th cosyzygy object in A belongs to E_T and any n-th syzygy object in B belongs to F_W.

Let us show that the functors Φ and Ψ restrict to mutually inverse equivalences between the exact categories E_T and F_W. Given an object $E \in \mathsf{E}_T$, choose an exact sequence $0 \to E \to J^0 \to \cdots \to J^{d-1} \to E' \to 0$ in A with $J^i \in \mathsf{A}_{\mathsf{inj}}$ with $d \geq \max(n, 2)$. Then the sequence

$$0 \to \Psi(E) \to \Psi(J^0) \to \cdots \to \Psi(J^{d-1}) \to \Psi(E') \to 0$$

is exact in B, and the objects $\Psi(J^i)$ belong to the full subcategory $\mathsf{Prod}(W) \subset \mathsf{F}_W \subset \mathsf{B}$. Hence $\Psi(E) \in \mathsf{F}_W$ by dimension shifting.

Furthermore, we have $E' \in \mathsf{E}_T$, hence $\Psi(E') \in \mathsf{F}_W$. It follows that the sequence $0 \to \Phi\Psi(E) \to \Phi\Psi(J^0) \to \cdots \to \Phi\Psi(J^{d-1}) \to \Phi\Psi(E') \to 0$ is exact in A. Since the adjunction morphisms $\Phi\Psi(J^i) \to J^i$ are isomorphisms for i equal to 0 and 1, so is the adjunction morphism $\Phi\Psi(E) \to E$. Similarly one shows that $\Phi(F) \in \mathsf{E}_T$ for all $F \in \mathsf{F}_W$, and the adjunction morphism $F \to \Psi\Phi(F)$ is an isomorphism.

According to [Bondal and van den Bergh 2003, Lemmas 5.4.1, 5.4.2; Fiorot et al. 2017, Proposition 1.5; Positselski and Šťovíček 2019, Theorem 5.5], the triangulated functors $\mathsf{D}(\mathsf{E}_T) \to \mathsf{D}(\mathsf{A})$ and $\mathsf{D}(\mathsf{F}_W) \to \mathsf{D}(\mathsf{B})$ induced by the exact embedding functors $\mathsf{E}_T \to \mathsf{A}$ and $\mathsf{F}_W \to \mathsf{B}$ are equivalences of triangulated categories. Thus we obtain a triangulated equivalence

$$\mathsf{D}(\mathsf{A}) \simeq \mathsf{D}(\mathsf{E}_T) = \mathsf{D}(\mathsf{F}_W) \simeq \mathsf{D}(\mathsf{B}).$$

Applying, e.g., [Positselski and Šťovíček 2019, Proposition 2.5 and Corollary 4.4(b)], one can conclude that the conditions (i)–(iii) and (i*)–(iii*) of [Positselski and Šťovíček 2019, Sections 2 and 4] hold for T and W, respectively. That is, T is n-tilting, W is n-cotilting and, moreover, $\operatorname{pd}_{\mathsf{A}} T = n = \operatorname{id}_{\mathsf{B}} W$ by [Positselski and Šťovíček 2019, Corollary 4.12].

Following [Positselski and Šťovíček 2019, Lemma 5.1], the two conditions ($\mathrm{i}_{\mathsf{max}}$) and ($\mathrm{ii}_{\mathsf{max}}$) defining the full subcategory $\mathsf{E}_{\mathsf{max}}(T) \subset \mathsf{A}$ are equivalent in this case. Similarly, the two conditions ($\mathrm{i}^*_{\mathsf{max}}$) and ($\mathrm{ii}^*_{\mathsf{max}}$) defining the full subcategory $\mathsf{F}_{\mathsf{max}}(W) \subset \mathsf{B}$ are equivalent. So either one of the two conditions is sufficient to define these classes in the n-(co)tilting case, and we actually have $\mathsf{E}_{\mathsf{max}}(T) = \mathsf{E}_T$ and $\mathsf{F}_{\mathsf{max}}(W) = \mathsf{F}_W$. It is only in the ∞-(co)tilting situation that we need to impose both the conditions. The full subcategory $\mathsf{E} = \mathsf{E}_{\mathsf{max}}(T)$ is the *n-tilting class* of an n-tilting object T, and the full subcategory $\mathsf{F} = \mathsf{F}_{\mathsf{max}}(W)$ is the *n-cotilting class* of an n-cotilting object W, as discussed in [Positselski and Šťovíček 2019, Sections 3–4]. According to [Positselski and Šťovíček 2019, Lemma 5.3 and Remark 5.4], both the full subcategories E and F are closed under both the infinite products and coproducts in A and B.

Finally, note that if T is n-tilting, then W is n-cotilting and vice versa by [Positselski and Šťovíček 2019, Corollary 4.12]. Thus, both the projective dimension of T and the injective dimension of W need to be finite for either of the two objects to be n-(co)tilting.

Example 6.2. Let A be a complete, cocomplete abelian category with an injective cogenerator J and an ∞-tilting pair (T, E), and let B be the corresponding complete, cocomplete abelian category with a projective generator P and an ∞-cotilting pair (W, F), as in Corollary 3.5. Suppose that the full subcategory $\mathsf{E} \subset \mathsf{A}$ is closed under coproducts and the full subcategory $\mathsf{F} \subset \mathsf{B}$ is closed under products. Then, by Proposition 4.2, the triangulated functors $\mathsf{D}(\mathsf{E}) \to \mathsf{D}(\mathsf{A})$ and $\mathsf{D}(\mathsf{F}) \to \mathsf{D}(\mathsf{B})$ induced by the exact embeddings $\mathsf{E} \to \mathsf{A}$ and $\mathsf{F} \to \mathsf{B}$ are Verdier quotient functors.

Assume that only *one* of the objects T and W has finite homological dimension, or more specifically, that $\mathrm{pd}_{\mathsf{A}} T < \infty$. Then the left exact functor $\Psi : \mathsf{A} \to \mathsf{B}$ has finite homological dimension and the full subcategory

$$\mathsf{E}_T = \{E \in \mathsf{A} \mid \mathrm{Ext}^i_{\mathsf{A}}(T, E) = 0 \quad \text{for all } i > 0\}$$

of A has finite coresolution dimension, as in the previous example. In particular, the complex $\Psi(E^\bullet)$ is acyclic in B for any complex $E^\bullet$ in the category E that is acyclic in A. So the composition of triangulated functors $\mathsf{D}(\mathsf{E}) \simeq \mathsf{D}(\mathsf{F}) \to \mathsf{D}(\mathsf{B})$ factorizes through the Verdier quotient functor $\mathsf{D}(\mathsf{E}) \to \mathsf{D}(\mathsf{A})$, or in other words, the triangulated equivalence $\mathsf{D}(\mathsf{E}) \simeq \mathsf{D}(\mathsf{F})$ descends to a triangulated functor $\mathsf{D}(\mathsf{A}) \to \mathsf{D}(\mathsf{B})$ in Figure 3. This is also a Verdier quotient functor (since such is the functor $\mathsf{D}(\mathsf{F}) \to \mathsf{D}(\mathsf{B})$).

Similarly, assume that $\mathrm{id}_{\mathsf{B}} W < \infty$. Then the right exact functor $\Phi : \mathsf{B} \to \mathsf{A}$ has finite homological dimension. In particular, the complex $\Phi(F^\bullet)$ is acyclic in A for any complex $F^\bullet$ in the category F that is acyclic in B. Hence the triangulated equivalence $\mathsf{D}(\mathsf{F}) \simeq \mathsf{D}(\mathsf{E})$ descends to a triangulated Verdier quotient functor $\mathsf{D}(\mathsf{B}) \to \mathsf{D}(\mathsf{A})$.

In the representation theory of finite-dimensional algebras, it is an open problem whether a finite-dimensional ∞-tilting module of finite projective dimension is already n-tilting for some n. It goes under the name of the Wakamatsu tilting conjecture, and it is a member of a family of long standing so-called homological conjectures for finite-dimensional algebras [Mantese and Reiten 2004, Section 4; Beligiannis and Reiten 2007, §IV.3].

Example 6.3. Let A be a locally Noetherian Grothendieck abelian category (see [Positselski and Šťovíček 2019, Section 10.2]). Choose an injective object $J \in \mathsf{A}$ such that $\mathsf{A}_{\mathsf{inj}} = \mathsf{Add}(J)$; then it follows that J is an injective cogenerator of A, and one also has $\mathsf{A}_{\mathsf{inj}} = \mathsf{Prod}(J)$. Set $T = J$ and $\mathsf{E} = \mathsf{A}_{\mathsf{inj}} \subset \mathsf{A}$. Then (T, E) is an ∞-tilting pair in A.

In the corresponding abelian category B with a natural projective generator P [Positselski 2015, Theorem 3.6], one has $\mathsf{B}_{\mathsf{proj}} = \mathsf{Add}(P) = \mathsf{Prod}(P)$ (see also Lemma 4.1). The related ∞-cotilting pair in B is (W, F), where $W = P$ and $\mathsf{F} = \mathsf{B}_{\mathsf{proj}}$. So both the full subcategories $\mathsf{E} \subset \mathsf{A}$ and $\mathsf{F} \subset \mathsf{B}$ are closed under both the products and coproducts. As always in the context of Corollary 3.5, one has an equivalence of additive/exact categories $\mathsf{E} \simeq \mathsf{F}$.

The derived category $\mathsf{D}(\mathsf{E})$ is simply the homotopy category $\mathsf{Hot}(\mathsf{A}_{\mathsf{inj}})$; it is equivalent to the coderived category $\mathsf{D}^{\mathsf{co}}(\mathsf{A})$ (see the argument for [Positselski 2017c, Theorem 2.4]). The derived category $\mathsf{D}(\mathsf{F})$ is simply the homotopy category $\mathsf{Hot}(\mathsf{B}_{\mathsf{proj}})$; it is equivalent to the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathsf{B})$ (cf. [Positselski 2017c, Theorem 4.4(b)] and [Positselski 2012, Corollary A.6.2]). Hence the derived equivalence

$$\mathsf{D}^{\mathsf{co}}(\mathsf{A}) \simeq \mathsf{Hot}(\mathsf{A}_{\mathsf{inj}}) = \mathsf{Hot}(\mathsf{B}_{\mathsf{proj}}) \simeq \mathsf{D}^{\mathsf{ctr}}(\mathsf{B}).$$

These are the *minimal* ∞-tilting and ∞-cotilting pair for the ∞-tilting object $T \in \mathsf{A}$ and the ∞-cotilting object $W \in \mathsf{B}$, in the sense of Example 3.7: one has $\mathsf{E}_{\mathsf{min}}(T) = \mathsf{E} = \mathsf{A}_{\mathsf{inj}}$ and $\mathsf{F}_{\mathsf{min}}(W) = \mathsf{F} = \mathsf{B}_{\mathsf{proj}}$.

Example 6.4. In the context of the previous example, it is also instructive to consider the *maximal* ∞-tilting pair $(T, \mathsf{E}_{\mathsf{max}}(T))$ for the ∞-tilting object $T = J$ in the category A and the maximal ∞-cotilting pair $(W, \mathsf{F}_{\mathsf{max}}(W))$ for the ∞-cotilting object $W = P$ in the category B.

The full subcategory $\mathsf{E}_{\mathsf{max}}(T) \subset \mathsf{A}$ consists of all the objects $E \in \mathsf{A}$ for which there exists an unbounded acyclic complex of injective objects

$$\cdots \to J^{-2} \to J^{-1} \to J^{0} \to J^{1} \to J^{2} \to \cdots$$

such that the complex $\mathrm{Hom}_{\mathsf{A}}(J, J^{\bullet})$ is acyclic and E is the image of the morphism $J^{-1} \to J^{0}$. This is known as the full subcategory of *Gorenstein injective objects* in the abelian category A.

Similarly, the full subcategory $\mathsf{F}_{\mathsf{max}}(W) \subset \mathsf{B}$ consists of all the objects $F \in \mathsf{B}$ for which there exists an unbounded acyclic complex of projective objects

$$\cdots \to P_{2} \to P_{1} \to P_{0} \to P_{-1} \to P_{-2} \to \cdots$$

such that the complex $\mathrm{Hom}_{\mathsf{B}}(P_{\bullet}, P)$ is acyclic and F is the image of the morphism $P_0 \to P_{-1}$. This is known as the full subcategory of *Gorenstein projective objects* in the abelian category B (cf. [Enomoto 2017, Definition 3.7]).

Hence we can conclude from Theorems 2.3 and 2.4 that the exact categories of Gorenstein injective objects in A and Gorenstein projective objects in B are naturally equivalent.

If A has a generating set of objects of finite projective dimension, then, by [Gillespie 2017, Theorem 5.7] or [Šťovíček 2014b, Lemma 7.2], the class of acyclic complexes of injectives is closed under products (although products may not be exact in A). In particular, $\mathsf{E}_{\mathsf{max}}(T) \subset \mathsf{A}$ is closed under products, and so is $\mathsf{F}_{\mathsf{max}}(W) \subset \mathsf{B}$ by Lemma 4.1. Dually, if B has a cogenerating set of objects of finite injective dimension, then both $\mathsf{E}_{\mathsf{max}}(T) \subset \mathsf{A}$ and $\mathsf{F}_{\mathsf{max}}(W) \subset \mathsf{B}$ are closed under coproducts.

In particular, if A is the category of quasi-coherent sheaves on a quasi-compact semiseparated scheme X, then any quasi-coherent sheaf on X is the quotient of one of the so-called very flat quasi-coherent sheaves [Positselski 2012, Lemma 4.1.1] (see [Murfet 2006, Section 2.4] or [Efimov and Positselski 2015, Lemma A.1] for the more widely known, but weaker assertion with flat sheaves in place of the very flat ones). If X is covered by n affine open subschemes, then the projective dimension of any very flat quasi-coherent sheaf, as an object of A, does not exceed n, as one can show using a Čech resolution for the affine covering, together with the fact that the projective dimension of a very flat module does not exceed 1 (cf. [Positselski and Slávik 2017, properties (VF5) and (VF6)]). Thus the class of acyclic complexes of injectives is closed under products in A. If X is also Noetherian, then A is a locally Noetherian category, and the discussion in the previous paragraph applies.

Example 6.5. In the context of Examples 6.3 and 6.4, one can say that a locally Noetherian Grothendieck abelian category A is *n-Gorenstein* if the ∞-tilting object $T = J$ is n-tilting. This means that $\mathrm{pd}_{\mathsf{A}}\, T = \mathrm{id}_{\mathsf{B}}\, W \le n$ (cf. [Positselski and Šťovíček 2019, Theorem 10.3]).

In this case, we have the minimal ∞-tilting and ∞-cotilting pair $(T,\ \mathsf{E}_{\mathsf{min}}(T))$ and $(W,\ \mathsf{F}_{\mathsf{min}}(W))$ with $\mathsf{E}_{\mathsf{min}}(T) = \mathsf{A}_{\mathsf{inj}}$ and $\mathsf{F}_{\mathsf{min}}(W) = \mathsf{B}_{\mathsf{proj}}$, as in Example 6.3. We also have the maximal ∞-tilting and ∞-cotilting pair $(T,\ \mathsf{E}_{\mathsf{max}}(T))$ and $(W,\ \mathsf{F}_{\mathsf{max}}(W))$ with $\mathsf{E}_{\mathsf{max}}(T) = \mathsf{E}_T$ being the n-tilting class of the n-tilting object $T \in \mathsf{A}$ (consisting of all the Gorenstein injectives in A) and $\mathsf{F}_{\mathsf{max}}(W) = \mathsf{F}_W$ being the n-cotilting class of the n-cotilting object $W \in \mathsf{B}$ (consisting of all the Gorenstein projectives in B), as in Examples 6.1 and 6.4.

The two related derived equivalences (as in Section 4) form a commutative diagram with the natural Verdier quotient functors as follows:

$$\begin{tikzcd}
\mathsf{D}^{\mathsf{co}}(\mathsf{A}) \arrow[r, equal] \arrow[d, two heads] & \mathsf{Hot}(\mathsf{A}_{\mathsf{inj}}) \arrow[r, equal] \arrow[d, two heads] & \mathsf{Hot}(\mathsf{B}_{\mathsf{proj}}) \arrow[r, equal] \arrow[d, two heads] & \mathsf{D}^{\mathsf{ctr}}(\mathsf{B}) \arrow[d, two heads] \\
\mathsf{D}(\mathsf{A}) \arrow[r, equal] & \mathsf{D}(\mathsf{E}_T) \arrow[r, equal] & \mathsf{D}(\mathsf{F}_W) \arrow[r, equal] & \mathsf{D}(\mathsf{B})
\end{tikzcd}$$

Example 6.6. Let A and B be associative rings, and let C be an A-B-bimodule. One says that C is a *semidualizing bimodule* (in the terminology of [Holm and White 2007]) or a *pseudo-dualizing bimodule* (in the terminology of [Positselski 2017a], which we adopt here) for the rings A and B if the following conditions are satisfied:

- The left A-module C has a projective resolution by finitely generated projective left A-modules, and the right B-module C has a projective resolution by finitely generated projective right B-modules.
- The homothety maps $A \to \operatorname{Ext}^*_{B^{\mathrm{op}}}(C, C)$ and $B^{\mathrm{op}} \to \operatorname{Ext}^*_A(C, C)$ are isomorphisms of graded rings (where B^{op} denotes the opposite ring to B).

This definition is (essentially) obtained by dropping the finite injective dimension condition in the definition of a dualizing module over a pair of associative rings.

Let C be a pseudo-dualizing A-B-bimodule. Set $\mathsf{A} = A\text{–}\mathsf{mod}$ and $\mathsf{B} = B\text{–}\mathsf{mod}$ to be the abelian categories of left modules over the rings A and B. Then $T = C$ is a (finitely generated) ∞-tilting object in A. The related maximal ∞-tilting class $\mathsf{E}_{\mathsf{max}}(T) \subset \mathsf{A}$ is known as the *Bass class* [Holm and White 2007, Theorem 6.1], and it contains the injective left A-modules by [Holm and White 2007, Lemma 4.1].

The corresponding tilted abelian category is $\sigma_T(\mathsf{A}) = \mathsf{B}$, and its natural projective generator is $P = B$. Choosing $J = \operatorname{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ as the injective cogenerator of A, the corresponding ∞-cotilting object in B is $W = \operatorname{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z})$. The related maximal ∞-cotilting class $\mathsf{F}_{\mathsf{max}}(W) \subset \mathsf{B}$ is known as the *Auslander class* [Holm and White 2007, Theorem 2]. The objects of the full subcategory $\mathsf{Add}(T) \subset \mathsf{A}$ are called *C-projectives* in [Holm and White 2007], and the objects of the full subcategory $\mathsf{Prod}(W) \subset \mathsf{B}$ are called *C-injectives*. The equivalence of exact categories

$$\mathsf{E}_{\mathsf{max}}(T) \simeq \mathsf{F}_{\mathsf{max}}(W)$$

is a part of what is known as the *Foxby equivalence* [Holm and White 2007, Theorem 1 or Proposition 4.1].

Both the full subcategories $\mathsf{E}_{\mathsf{max}}(T) \subset \mathsf{A}$ and $\mathsf{F}_{\mathsf{max}}(W) \subset \mathsf{B}$ are closed under both the infinite products and coproducts [Holm and White 2007, Proposition 4.2], so the results of our Section 4 apply and provide a commutative diagram of a triangulated equivalence and Verdier quotient functors:

$$\begin{tikzcd}
\mathsf{D}^{\mathsf{co}}(A\text{–}\mathsf{mod}) \arrow[d, two heads] & \mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod}) \arrow[d, two heads] \\
\mathsf{D}(\mathsf{E}_{\mathsf{max}}(T)) \arrow[r, equal] \arrow[d, two heads] & \mathsf{D}(\mathsf{F}_{\mathsf{max}}(W)) \arrow[d, two heads] \\
\mathsf{D}(A\text{–}\mathsf{mod}) & \mathsf{D}(B\text{–}\mathsf{mod})
\end{tikzcd}$$

The paper [Positselski 2017a] is devoted to generalizing this theory to the case of a pseudo-dualizing *complex* of bimodules. In particular, (a coproduct and product-closed version of) the *minimal* ∞-tilting and ∞-cotilting classes for T and W are discussed in [Positselski 2017a, Section 5].

Example 6.7. Let $\mathcal{C}$ be a coassociative, counital coring over an associative ring A (see [Positselski and Šťovíček 2019, Section 10.3]). Assume that $\mathcal{C}$ is a projective left and a flat right A-module. Let $\mathsf{A} = \mathcal{C}\text{–comod}$ be the category of left $\mathcal{C}$-comodules; it is a Grothendieck abelian category. Set $T \in \mathsf{A}$ to be the cofree left $\mathcal{C}$-comodule $T = \mathcal{C}$. We claim that T is an ∞-tilting object in A.

Indeed, it was explained in [Positselski and Šťovíček 2019, Section 10.3] that T is weakly tilting, so it remains to show that the injective objects of A satisfy the condition (ii$_{\mathsf{max}}$). A left $\mathcal{C}$-comodule is injective if and only if it is a direct summand of a $\mathcal{C}$-comodule $\mathcal{C} \otimes_A I$ coinduced from an injective left A-module I [Positselski 2010, Sections 1.1.2 and 5.1.5]. Now applying the coinduction functor $\mathcal{C} \otimes_A -$ to a projective resolution of the A-module I produces an $\mathsf{Add}(T)$-resolution of the $\mathcal{C}$-comodule $\mathcal{C} \otimes_A I$ as in (ii$_{\mathsf{max}}$). This resolution remains exact after applying the functor $\mathrm{Hom}_{\mathsf{A}}(T, -)$, because $\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{C} \otimes_A V) \simeq \mathrm{Hom}_A(\mathcal{C}, V)$ and $\mathcal{C}$ is a projective left A-module.

The abelian category $\mathsf{B} = \sigma_T(\mathsf{A})$ is the category of left $\mathcal{C}$-contramodules, $\mathsf{B} = \mathcal{C}\text{–contra}$ [Positselski and Šťovíček 2019, Section 10.3]. The natural projective generator is $P = \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{C}) = \mathrm{Hom}_A(\mathcal{C}, A)$. Given an injective cogenerator I of the category of left A-modules, one can choose $J = \mathcal{C} \otimes_A I$ as the injective cogenerator of $\mathsf{A} = \mathcal{C}\text{–comod}$; then the related cotilting object in $\mathsf{B} = \mathcal{C}\text{–contra}$ is $W = \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{C} \otimes_A I) = \mathrm{Hom}_A(\mathcal{C}, I)$.

When the left homological dimension of the ring A is finite, we can describe the minimal class $\mathsf{E}_{\mathsf{min}}(T)$ which forms an ∞-tilting pair with T (Example 3.7) more explicitly as the full subcategory $\mathsf{E} \subset \mathsf{A}$ of all $\mathcal{C}/A$-injective left $\mathcal{C}$-comodules [Positselski 2010, Sections 5.1.4 and 5.3], [Positselski 2015, Section 3.4]. Here, a left $\mathcal{C}$-comodule $\mathcal{M}$ is called *$\mathcal{C}/A$-injective* if $\mathrm{Ext}^i_{\mathcal{C}}(\mathcal{L}, \mathcal{M}) = 0$ for all $i > 0$ and all left $\mathcal{C}$-comodules $\mathcal{L}$ with projective underlying left A-modules. The class E is coresolving and contains all the coinduced $\mathcal{C}$-comodules $\mathcal{C} \otimes_A M$, $M \in A\text{–mod}$. In particular, $\mathsf{Add}(T) \subset \mathsf{E}$, objects of $\mathsf{Add}(T)$ are by definition projective in E, and by [Positselski 2010, Lemma 5.2(a) and the proof of Lemma 5.3.2(a)], there are enough such projectives. On the other hand, E has enough injectives, $\mathsf{E}_{\mathsf{inj}} = \mathsf{A}_{\mathsf{inj}}$, and the proof of [Positselski 2010, Theorem 5.3] reveals that any object of E has finite injective dimension bounded by the left homological dimension of A. Now we can use the following observation.

Lemma 6.8. *Let (T, E) be an ∞-tilting pair in a complete, cocomplete abelian category A with an injective cogenerator. If E has finite homological dimension as an exact category, then $\mathsf{E} = \mathsf{E}_{\mathsf{min}}(T)$.*

Proof. If n is the homological dimension of E, then any object $E \in \mathsf{E}$ admits a long exact sequence

$$0 \to E \to J^0 \to \cdots \to J^n \to 0$$

in E with $J^0, \dots, J^n \in \mathsf{E}_{\mathsf{inj}} = \mathsf{A}_{\mathsf{inj}}$. Since this sequence remains exact after applying $\mathrm{Hom}_{\mathsf{A}}(T, -)$, it follows that $E \in \mathsf{E}_{\mathsf{min}}(T)$ by the conditions (i) and (v) from Section 3. □

Dually, the full subcategory $\mathsf{F} \subset \mathsf{B}$ in the related ∞-cotilting pair (W, F) consists of all the $\mathcal{C}/A$-projective left $\mathcal{C}$-contramodules. This is analogously the minimal ∞-cotilting pair for the ∞-cotilting object $W \in \mathsf{B}$. Since the class of $\mathcal{C}/A$-injective comodules is closed under products and the class of $\mathcal{C}/A$-projective contramodules is closed under coproducts, both the full subcategories $\mathsf{E} \subset \mathsf{A}$ and $\mathsf{F} \subset \mathsf{B}$ are closed under both the infinite products and coproducts by Lemma 4.1. The related derived equivalence [Positselski 2010, Section 5.4] is

$$\mathsf{D}^{\mathsf{co}}(\mathcal{C}\text{–}\mathsf{comod}) \simeq \mathsf{D}(\mathsf{E}) = \mathsf{D}(\mathsf{F}) \simeq \mathsf{D}^{\mathsf{ctr}}(\mathcal{C}\text{–}\mathsf{contra}).$$

For comparison, when $\mathcal{C}$ is a left Gorenstein coring in the sense of [Positselski and Šťovíček 2019, Section 10.3], i.e., $T \in \mathsf{A}$ is an n-tilting object, considering the corresponding tilting and cotilting classes $\mathsf{E}_{\mathsf{max}}(T) \subset \mathsf{A}$ and $\mathsf{F}_{\mathsf{max}}(W) \subset \mathsf{B}$ produces a triangulated equivalence between the conventional derived categories, $\mathsf{D}(\mathcal{C}\text{–}\mathsf{comod}) \simeq \mathsf{D}(\mathcal{C}\text{–}\mathsf{contra})$.

Example 6.9. The case of a coassociative coalgebra $\mathcal{C}$ over a field k is a common particular case of Examples 6.3, 6.4 and 6.7. It is also a particular case of Example 6.10.

In this case, one has $\mathsf{A} = \mathcal{C}\text{–}\mathsf{comod}$ and $\mathsf{B} = \mathcal{C}\text{–}\mathsf{contra}$. The ∞-tilting object $T = \mathcal{C} = J$ is the natural injective cogenerator of the locally Noetherian Grothendieck abelian category A, and the ∞-cotilting object $W = \mathcal{C}^* = \mathrm{Hom}_k(\mathcal{C}, k) = P$ is the natural projective generator of the abelian category B.

When $\mathcal{C}$ is a Gorenstein coalgebra, we are in the situation of Example 6.5 (see [Positselski and Šťovíček 2019, Section 10.1]).

Example 6.10. Let $\mathcal{S}$ be a semiassociative, semiunital semialgebra over a coassociative, counital coalgebra $\mathcal{C}$ over a field k (see [Positselski and Šťovíček 2019, Section 10.3]). Assume that $\mathcal{S}$ is an injective left and right $\mathcal{C}$-comodule. Let $\mathsf{A} = \mathcal{S}\text{–}\mathsf{simod}$ be the category of left $\mathcal{S}$-semimodules; it is a Grothendieck abelian category. Set $T \in \mathsf{A}$ to be the semifree left $\mathcal{S}$-semimodule $T = \mathcal{S}$, and take $\mathsf{E} \subset \mathsf{A}$ to be the full subcategory of all left $\mathcal{S}$-semimodules whose underlying left $\mathcal{C}$-comodules are injective, $\mathsf{E} = \mathcal{S}\text{–}\mathsf{simod}_{\mathcal{C}-\mathsf{inj}}$. Then (T, E) is an ∞-tilting pair in A.

The related abelian category $\mathsf{B} = \sigma_T(\mathsf{A})$ is the category of left $\mathcal{S}$-semicontramodules, $\mathsf{B} = \mathcal{S}\text{–}\mathsf{sicntr}$ [Positselski and Šťovíček 2019, Section 10.3]. The natural projective generator is $P = \mathrm{Hom}_{\mathcal{S}}(\mathcal{S}, \mathcal{S}) \in \mathcal{S}\text{–}\mathsf{sicntr}$. The full subcategory $\mathsf{F} = \Psi(\mathsf{E}) \subset \mathsf{B}$ consists of all left $\mathcal{S}$-semicontramodules whose underlying left $\mathcal{C}$-contramodules are projective, $\mathsf{F} = \mathcal{S}\text{–}\mathsf{sicntr}_{\mathcal{C}-\mathsf{proj}}$. The ∞-cotilting object $W \in \mathsf{B}$ corresponding to the natural choice of an injective cogenerator $J \in \mathsf{A}$ is $W = \mathcal{S}^* =$

$\mathrm{Hom}_k(\mathcal{S}, k) \in \mathcal{S}\text{–}\mathsf{sicntr}$. Both the full subcategories $\mathsf{E} \subset \mathsf{A}$ and $\mathsf{F} \subset \mathsf{B}$ are closed under both the products and coproducts. A detailed discussion of the equivalence of exact categories $\mathcal{S}\text{–}\mathsf{simod}_{\mathcal{C}-\mathsf{inj}} \simeq \mathcal{S}\text{–}\mathsf{sicntr}_{\mathcal{C}-\mathsf{proj}}$ can be found in [Positselski 2015, Section 3.5].

The derived category $\mathsf{D}(\mathsf{E})$ of the exact category E is called the *semiderived category of left $\mathcal{S}$-semimodules* and denoted by $\mathsf{D}(\mathcal{S}\text{–}\mathsf{simod}_{\mathcal{C}-\mathsf{inj}}) = \mathsf{D}^{\mathsf{si}}(\mathcal{S}\text{–}\mathsf{simod})$ [Positselski 2010, Section 0.3.3]. Generally speaking, it is properly intermediate between the coderived category $\mathsf{D}^{\mathsf{co}}(\mathcal{S}\text{–}\mathsf{simod})$ and the derived category $\mathsf{D}(\mathcal{S}\text{–}\mathsf{simod})$. Similarly, the derived category $\mathsf{D}(\mathsf{F})$ of the exact category F is called the *semiderived category of left $\mathcal{S}$-semicontramodules* and denoted by $\mathsf{D}(\mathcal{S}\text{–}\mathsf{sicntr}_{\mathcal{C}-\mathsf{proj}}) = \mathsf{D}^{\mathsf{si}}(\mathcal{S}\text{–}\mathsf{sicntr})$ [Positselski 2010, Section 0.3.6]. Generally speaking, it is properly intermediate between the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathcal{S}\text{–}\mathsf{sicntr})$ and the derived category $\mathsf{D}(\mathcal{S}\text{–}\mathsf{sicntr})$.

The triangulated equivalence $\mathsf{D}^{\mathsf{si}}(\mathcal{S}\text{–}\mathsf{simod}) \simeq \mathsf{D}^{\mathsf{si}}(\mathcal{S}\text{–}\mathsf{sicntr})$ is called the *derived semimodule-semicontramodule correspondence* [Positselski 2010, Sections 0.3.7 and 6.3]. For an application to representation theory of infinite-dimensional Lie algebras (such as the Virasoro or Kac–Moody algebras), see [Positselski 2010, Corollary D.3.1].

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THE "QUANTUM" TURÁN PROBLEM FOR OPERATOR SYSTEMS

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Let $\mathcal{V}$ be a linear subspace of $M_n(\mathbb{C})$ which contains the identity matrix and is stable under Hermitian transpose. A "quantum k-clique" for $\mathcal{V}$ is a rank k orthogonal projection $P \in M_n(\mathbb{C})$ for which $\dim(P\mathcal{V}P) = k^2$, and a "quantum k-anticlique" is a rank k orthogonal projection for which $\dim(P\mathcal{V}P) = 1$. We give upper and lower bounds both for the largest dimension of $\mathcal{V}$ which would ensure the existence of a quantum k-anticlique, and for the smallest dimension of $\mathcal{V}$ which would ensure the existence of a quantum k-clique.

1. Background

In finite dimensions, an *operator system* is a linear subspace $\mathcal{V}$ of $M_n(\mathbb{C})$ with the properties

- $I_n \in \mathcal{V}$,
- $A \in \mathcal{V} \Rightarrow A^* \in \mathcal{V}$,

where I_n is the $n \times n$ identity matrix and A^* is the Hermitian transpose of A.

A natural class of examples arises from graphs with vertex set $\{1, \ldots, n\}$. Given such a graph G, we can define an operator system

$$\mathcal{V}_G = \mathrm{span}\{E_{ij} : i = j \text{ or } i \text{ is adjacent to } j\},$$

where E_{ij} is the $n \times n$ matrix with a 1 in the (i, j) entry and 0's elsewhere. Note that the symmetry of the edge set of G is reflected in the stability of $\mathcal{V}_G$ under Hermitian transpose. (These are precisely the operator systems which are bimodules over the diagonal subalgebra of $M_n(\mathbb{C})$.)

Operator systems have been studied by C*-algebraists for decades, but only recently have they begun to be thought of as being, in some way, a matrix or "quantum" analog of graphs. More generally, we can regard the notion of a linear subspace of $M_n(\mathbb{C})$ as a linearization of the notion of a subset of $\{1, \ldots, n\}^2$, i.e., a relation on the set $\{1, \ldots, n\}$. The two conditions which define operator systems

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are then matrix versions of reflexivity and symmetry, so that an operator system becomes a matrix version of a reflexive, symmetric relation on a set — which is effectively the same as a graph on that set.[1] This point of view was developed in [Weaver 2012; 2015].

The term "quantum" is supported by the fact that operator systems appear in the theory of quantum error correction, playing a role exactly analogous to the role played by ordinary graphs in classical error correction [Duan et al. 2013]. In the classical case we have a *confusability graph* which tells us when two transmitted signals could be received as the same signal, and in the quantum case we have a *confusability operator system* which tells us when two transmitted states could be received as the same state. The two settings even have a natural common generalization; see [Weaver 2015].

The first paper to demonstrate that there could be a "quantum graph theory" for operator systems was [Duan et al. 2013], where, driven by the needs of quantum error correction, a "quantum Lovász number" was defined for an arbitrary operator system, in analogy to the classical Lovász number of a graph. The error correction perspective on quantum graphs was developed further in [Stahlke 2016].

The present paper is a sequel to [Weaver 2017], where an operator system version of Ramsey's theorem was proven. This result involves quantum versions of graph-theoretic cliques and anticliques. The theory of error correction tells us what a quantum anticlique should be, because in classical error correction a "code" is realized as an anticlique in the confusability graph, whereas in quantum error correction a "code" is realized as an orthogonal projection $P \in M_n(\mathbb{C})$ satisfying $PAP = \lambda P$ for all A belonging to the confusability operator system $\mathcal{V}$. Equivalently, this condition can be stated as $\dim(P\mathcal{V}P) = 1$, where $P\mathcal{V}P = \{PAP : A \in \mathcal{V}\}$.

Observe that if $P \in M_n(\mathbb{C})$ is any orthogonal projection (i.e., $P = P^2 = P^*$) and $\mathcal{V} \subseteq M_n(\mathbb{C})$ is any operator system, then $P\mathcal{V}P$ is effectively a set of linear transformations from $\mathrm{ran}(P)$ to itself, and the condition that P should be a code is that this set should be minimal, consisting only of the scalar multiples of the identity operator on $\mathrm{ran}(P)$. If these are the anticliques of $\mathcal{V}$, then it is natural to take the cliques of $\mathcal{V}$ to be the orthogonal projections P for which $P\mathcal{V}P$ is maximal, i.e., it consists of all linear operators from $\mathrm{ran}(P)$ to itself. This can also be expressed by saying that $\dim(P\mathcal{V}P) = k^2$. We therefore make the following definition.

Definition 1.1. Let $\mathcal{V} \subseteq M_n(\mathbb{C})$ be an operator system. A rank k orthogonal projection $P \in M_n(\mathbb{C})$ is a *quantum k-anticlique* for $\mathcal{V}$ if $\dim(P\mathcal{V}P) = 1$, and a *quantum k-clique* for $\mathcal{V}$ if $\dim(P\mathcal{V}P) = k^2$.

[1] There is an obvious 1-1 correspondence between graphs on a vertex set V and reflexive, symmetric relations on V. This correspondence is more natural if we adopt the convention that graphs must have a loop at each vertex; in the error correction setting discussed below, where an edge between two vertices expresses that they are "sufficiently close", this is in fact a good convention.

In general, if we identify $PM_n(\mathbb{C})P$ with $M_k(\mathbb{C})$, where $k = \mathrm{rank}(P)$, then $P\mathcal{V}P$ becomes an operator system in $M_k(\mathbb{C})$. This is the *induced operator system* which is analogous to a subgraph induced on a subset of the vertex set of a graph. (Some intuition for this analogy is given in [Weaver 2015], again with the natural common generalization mentioned earlier.) Thus P is a quantum clique if the induced operator system is a full matrix algebra and it is a quantum anticlique if the induced operator system is trivial.

The classical theorem of Ramsey states that for any k there exists n such that every graph with n vertices has either a k-clique or a k-anticlique. The quantum Ramsey theorem proven in [Weaver 2017] states that for any k there exists n such that every operator system in $M_n(\mathbb{C})$ has either a quantum k-clique or a quantum k-anticlique. The most surprising aspect of this result is that in the quantum setting n grows polynomially in k, not exponentially as in the classical case. (The specific value given in [Weaver 2017] is $n = 8k^{11}$, but this is surely not optimal. An easy lower bound is

$$n = (k-1)(k^2-1) = k^3 - k^2 - k + 1,$$

obtained by taking $r = k^2 - 1$ in the construction described in Proposition 2.1 below.) A quantum Ramsey theorem for infinite-dimensional operator systems was proven in [Kennedy et al. 2017].

Michael Jury suggested to me the problem of finding a version of Turán's theorem for operator systems. The classical theorem of Turán gives the maximum number of edges a graph with n vertices can have without having any $(k+1)$-cliques; by taking edge complements, we see that $\binom{n}{2}$ minus this number is the minimum number of edges a graph with n vertices can have without having any $(k+1)$-anticliques. The analogous questions for operator systems are: what is the maximum dimension $T^{\uparrow}(n,k)$ of an operator system in $M_n(\mathbb{C})$ having no quantum $(k+1)$-cliques, and what is the minimum dimension $T^{\downarrow}(n,k)$ of an operator system in $M_n(\mathbb{C})$ having no quantum $(k+1)$-anticliques? These two questions constitute a “quantum Turán problem”. The goal of this paper is not to give exact answers to them, but merely to provide upper and lower bounds for both values. Specifically, we prove

$$\sqrt{\frac{n}{k}} < T^{\downarrow}(n,k) \le \left\lceil \frac{n}{k} \right\rceil \quad \text{and} \quad 2(k-1)n - (k-1)^2 + 3 \le T^{\uparrow}(n,k) < 16(k+1)^8 n.$$

Because, unlike the classical case, there is no natural symmetry between quantum cliques and quantum anticliques, we are really dealing with two distinct questions. Broadly speaking, it is *easy* to find quantum cliques and *hard* to find quantum anticliques. This is dramatically illustrated by the fact that our upper bound on the maximum dimension of an operator system having no quantum $(k+1)$-cliques is linear in n. As there are n^2 available dimensions in $M_n(\mathbb{C})$, this means that when n is large compared to k one needs only a comparatively small number of dimensions to guarantee that quantum $(k+1)$-cliques exist. In contrast, the upper bound on the lower quantum Turán number is $\lceil \frac{n}{k} \rceil$, meaning that $\dim(\mathcal{V})$ has to be even smaller than this to ensure that quantum $(k+1)$-anticliques exist.

2. Lower quantum Turán numbers

We define the *lower quantum Turán number* $T^{\downarrow}(n,k)$ to be the smallest number d such that some operator system in $M_n(\mathbb{C})$ whose dimension is d has no quantum $(k+1)$-anticliques.

Every rank 1 projection is always both a quantum 1-anticlique and a quantum 1-clique for any operator system, so let us assume throughout that $k \geq 1$.

Classically, a graph on n vertices which lacks $(k+1)$-anticliques, and has the minimum number of edges for doing so, looks like a disjoint union of k many cliques of equal or nearly equal size. So a natural guess for an operator system in $M_n(\mathbb{C})$ which lacks quantum $(k+1)$-anticliques and has the smallest possible dimension is a direct sum of k many matrix algebras of equal or nearly equal size, $\mathcal{V} = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_{k-1}}(\mathbb{C})$. This operator system indeed has no quantum $(k+1)$-anticliques; in fact, it has no quantum 2-anticliques because it contains the diagonal operator system D_n, which itself has no quantum 2-anticliques [Weaver 2017, Proposition 2.1]. But this shows that this $\mathcal{V}$ is far from being minimal: its dimension is approximately n^2/k, whereas the dimension of D_n is n. Quantum $(k+1)$-anticliques for $k > 1$ can be blocked using even fewer dimensions.

Proposition 2.1. *Let $P_1, \ldots, P_r$ be orthogonal projections in $M_n(\mathbb{C})$, each of rank at most k, satisfying $P_1 + \cdots + P_r = I_n$. Then the operator system $\mathcal{V} = \mathrm{span}(P_1, \ldots, P_r)$ has no quantum $(k+1)$-anticliques.*

Proof. Let P be a rank $k+1$ orthogonal projection in $M_n(\mathbb{C})$ and assume that $P\mathcal{V}P = \mathbb{C} \cdot P$. For each i, the matrix PP_iP has rank at most $\mathrm{rank}(P_i) \leq k$, so the only way it can be a scalar multiple of P is for it to be zero. But this implies that $P = P(P_1 + \cdots + P_r)P = 0$, a contradiction. □

(If $r = \dim(\mathcal{V}) \leq k^2 - 1$ and each P_i has rank $k-1$, then this operator system has neither quantum k-cliques nor quantum k-anticliques, explaining a parenthetical comment made in the introduction.)

Note that $P_1 + \cdots + P_r = I_n$ implies that the ranges of the P_i are orthogonal and their direct sum is $\mathbb{C}^n$. The minimum value of r for which there exist r projections, each of rank at most k, which sum to I_n is therefore $\lceil \frac{n}{k} \rceil$. Thus the following corollary is immediate.

Corollary 2.2. $$T^{\downarrow}(n,k) \leq \lceil \tfrac{n}{k} \rceil.$$

If $r = \lceil \frac{n}{k} \rceil$ then the operator system described in Proposition 2.1 is minimal in the sense that every operator system properly contained in it does have a quantum

$(k+1)$-anticlique. In order to prove this, it will be useful to have the following alternative characterization of quantum anticliques. (This characterization is implicit in [Knill et al. 2000].)

Lemma 2.3. *Let $\mathcal{V} \subseteq M_n(\mathbb{C})$ be an operator system. Then $\mathcal{V}$ has a quantum k-anticlique if and only if there exists an orthonormal set $\{v_1, \ldots, v_k\}$ in $\mathbb{C}^n$ such that for every Hermitian $A \in \mathcal{V}$*

$$\langle Av_i, v_j\rangle = 0 \qquad \text{and} \qquad \langle Av_i, v_i\rangle = \langle Av_j, v_j\rangle$$

whenever $i \neq j$.

Proof. If there is a quantum k-anticlique P for $\mathcal{V}$ then any orthonormal basis $\{v_1, \ldots, v_k\}$ of its range is easily seen to have the stated properties, since $PAP = \lambda P$ implies $\langle Av_i, v_j\rangle = \langle PAPv_i, v_j\rangle = \lambda\langle v_i, v_j\rangle$ for all i and j. Conversely, suppose we are given a set $\{v_1, \ldots, v_k\}$ satisfying the conditions of the lemma and let P be the orthogonal projection onto its span. Since every matrix in $\mathcal{V}$ is a linear combination of two Hermitian matrices in $\mathcal{V}$, the stated equations will be true of any matrix in $\mathcal{V}$. So fix a matrix $A \in \mathcal{V}$ and let λ be the common value of the inner products $\langle Av_i, v_i\rangle$. Then $P = \sum v_i v_i^*$ and so

$$PAP = \sum_{i,j} v_i v_i^* A v_j v_j^* = \sum \lambda v_i v_i^* = \lambda P,$$

since $v_i^* A v_j = \langle Av_j, v_i\rangle$ is 0 when $i \neq j$ and λ when $i = j$. Thus PAP is a scalar multiple of P for every $A \in \mathcal{V}$, i.e., P is a quantum anticlique. □

Proposition 2.4. *Let $P_1, \ldots, P_r$ be orthogonal projections in $M_n(\mathbb{C})$ satisfying $P_1 + \cdots + P_r = I_n$. Then any operator system properly contained in $\mathcal{V} = \operatorname{span}(P_1, \ldots, P_r)$ has a quantum k-anticlique where k is the sum of the two smallest ranks of the P_i's.*

Proof. Let $\mathcal{V}_0$ be an operator system properly contained in $\mathcal{V}$. Its Hermitian part $\mathcal{V}_0^h$ has the form

$$\mathcal{V}_0^h = \left\{\sum a_i P_i : \vec{a} = (a_1, \ldots, a_r) \in E\right\},$$

where E is some proper subspace of $\mathbb{R}^r$ which includes the vector $(1, \ldots, 1)$ (since we require $I_n \in \mathcal{V}_0$). So we can find a nonzero $\vec{b} \in \mathbb{R}^r$ such that $\vec{a} \cdot \vec{b} = 0$ for all $\vec{a} \in E$. Since $(1, \ldots, 1) \in E$, it follows that $\vec{b}$ contains both strictly positive and strictly negative components; by rearranging, we can assume that $b_1, \ldots, b_j > 0$ and $b_{j+1}, \ldots, b_r \leq 0$. We can also assume that $b_1 + \cdots + b_j = -b_{j+1} - \cdots - b_r = 1$.

For each i let $e_{i,1}, \ldots, e_{i,\mathrm{rank}(P_i)}$ be an orthonormal basis of $\operatorname{ran}(P_i)$. Let k_1 be the smallest rank among $P_1, \ldots, P_j$ and let k_2 be the smallest rank among $P_{j+1}, \ldots, P_r$, so that $k \leq k_1 + k_2$. Then for $1 \leq l \leq k_1$ set $v_l = \sqrt{b_1}e_{1,l} + \cdots + \sqrt{b_j}e_{j,l}$, and for $1 \leq l \leq k_2$ set $v_{k_1+l} = \sqrt{-b_{j+1}}e_{j+1,l} + \cdots + \sqrt{-b_r}e_{r,l}$. The vectors v_l form an orthonormal set of size k_1+k_2. For any $A = a_1P_1 + \cdots + a_rP_r \in \mathcal{V}_0^h$ we then have $\langle Av_l, v_{l'}\rangle = 0$ whenever $l \neq l'$, and for any $1 \leq l \leq k_1$ and $k_1+1 \leq l' \leq k_1+k_2$ we also have

$$\langle Av_l, v_l\rangle = a_1b_1 + \cdots + a_jb_j = -a_{j+1}b_{j+1} - \cdots - a_rb_r = \langle Av_{l'}, v_{l'}\rangle.$$

So Lemma 2.3 implies that $\mathcal{V}_0$ has a quantum (k_1+k_2)-anticlique. □

Corollary 2.5. *Let $\mathcal{V}$ be the operator system from Proposition 2.1 and assume that $r = \lceil \frac{n}{k} \rceil$. Then every operator system properly contained in $\mathcal{V}$ has a quantum $(k+1)$-anticlique.*

Proof. Any family of projections, each of rank at most k, which sums to I_n must contain at least $r = \lceil \frac{n}{k} \rceil$ members. Thus if it contains exactly this many members then the sum of the two smallest ranks of the P_i's must be at least $k+1$, as otherwise these two projections could be replaced by a single projection of rank at most k. The conclusion now follows from Proposition 2.4. □

It is not to be expected that the kind of minimality expressed in Corollary 2.5 can only happen at dimension $\lceil \frac{n}{k} \rceil$. The following is an easy counterexample.

Example 2.6. Take $n = 6$ and suppose $P_1 + P_2 = I_6$ where $\mathrm{rank}(P_1) = \mathrm{rank}(P_2) = 3$. Then by Propositions 2.1 and 2.4, $\mathrm{span}(P_1, P_2)$ is a two-dimensional operator system with no quantum 4-anticliques, but every operator system properly contained in it (there is only one, namely $\mathbb{C} \cdot I_6$) has a quantum 4-anticlique.

Alternatively, suppose $Q_1 + Q_2 + Q_3 = I_6$, where $\mathrm{rank}(Q_1) = \mathrm{rank}(Q_2) = \mathrm{rank}(Q_3) = 2$. Then $\mathrm{span}(Q_1, Q_2, Q_3)$ is a three-dimensional operator system which has no quantum 4-anticliques (Proposition 2.1), but I claim that any operator system properly contained in it does have a quantum 4-anticlique. To see this, note first that any two-dimensional operator system in $M_6(\mathbb{C})$ equals $\mathrm{span}(I_6, A)$ for some Hermitian matrix A, and if it is contained in $\mathrm{span}(Q_1, Q_2, Q_3)$ then we can write $A = aQ_1 + bQ_2 + cQ_3$ for some $a, b, c \in \mathbb{R}$. Without loss of generality assume $a \leq b \leq c$. If either $a = b$ or $b = c$ then the existence of a quantum 4-anticlique is immediate: just take $Q_1 + Q_2$ or $Q_2 + Q_3$. Otherwise let $\{e_{i,1}, e_{i,2}\}$ be an orthonormal basis for $\mathrm{ran}(Q_i)$ $(i = 1, 2, 3)$ and set $\alpha = (c-b)/(c-a)$ and $\gamma = (b-a)/(c-a)$, so that $\alpha + \gamma = 1$ and $a\alpha + c\gamma = b$. Then the set $S = \{\sqrt{\alpha}e_{1,1} + \sqrt{\gamma}e_{3,1}, \sqrt{\alpha}e_{1,2} + \sqrt{\gamma}e_{3,2}, e_{2,1}, e_{2,2}\}$ satisfies the conditions given in Lemma 2.3, so $\mathrm{span}(I_6, A)$ has a quantum 4-anticlique.

In general, for any Hermitian $A \in M_n(\mathbb{C})$, a straightforward modification of the argument used in this example shows that we can always find a quantum $\lceil \frac{n}{2} \rceil$-anticlique for the two-dimensional operator system $\mathcal{V} = \mathrm{span}(I_n, A)$. Let us record this fact:

Proposition 2.7. *Let $A \in M_n(\mathbb{C})$ be Hermitian. Then* $\operatorname{span}(I_n, A)$ *has a quantum* $\lceil \frac{n}{2} \rceil$*-anticlique.*

This is proven by ordering the eigenvalues of A as $\lambda_1 \leq \cdots \leq \lambda_n$, then letting $r = \lceil \frac{n}{2} \rceil$ and for $1 \leq i \leq r-1$ finding a convex combination $\alpha_i\lambda_i + \alpha_{r+i}\lambda_{r+i} = \lambda_r$, and then applying Lemma 2.3 to the vectors $\sqrt{\alpha_i}v_i + \sqrt{\alpha_{r+i}}v_{r+i}$ plus the one additional vector v_r, where v_i is the eigenvector belonging to λ_i.

In Example 2.6 this number is improved to $\lceil \frac{n}{2} \rceil + 1$ because the two middle eigenvalues of A are equal and their corresponding eigenvectors can both be used separately.

Actually, Turán's theorem does not just give the minimum number of edges in a $(k+1)$-anticliqueless graph on n vertices, it explicitly describes the structure of such a graph with that minimum number of edges — and there is only one up to isomorphism. I do not know whether $\lceil \frac{n}{k} \rceil$ is the minimum dimension of a quantum $(k+1)$-anticliqueless operator system in $M_n(\mathbb{C})$, but the operator system described in Proposition 2.1 with $r = \lceil \frac{n}{k} \rceil$ is not the only quantum $(k+1)$-anticliqueless operator system of that dimension. We can see this from the following extension of Proposition 2.1.

Proposition 2.8. *Let $A_1, \ldots, A_r$ be positive matrices in $M_n(\mathbb{C})$, each of rank at most k, and suppose that the dimension of* $\ker\bigl(\sum A_i\bigr)$ *is also at most k. Then the operator system* $\mathcal{V} = \operatorname{span}(I_n, A_1, \ldots, A_r)$ *has no quantum $(k+1)$-anticliques.*

Proof. As in the proof of Proposition 2.1, if we assume that P is a quantum $(k+1)$-anticlique for $\mathcal{V}$ then comparing ranks shows that $PA_iP = 0$ for all i. Thus $P\bigl(\sum A_i\bigr)P = 0$, which implies that $\bigl(\sum A_i\bigr)^{1/2}P = 0$ and hence that $\bigl(\sum A_i\bigr)P = \bigl(\sum A_i\bigr)^{1/2}\bigl(\sum A_i\bigr)^{1/2}P = 0$. This shows that $\operatorname{ran}(P)$ is contained in $\ker\bigl(\sum A_i\bigr)$, which contradicts the hypothesis that $\dim\bigl(\ker\bigl(\sum A_i\bigr)\bigr) \leq k$. □

Thus there are many operator systems of dimension $\lceil \frac{n}{k} \rceil$ which have no quantum $(k+1)$-anticliques. Indeed, if $A_1, \ldots, A_r$ are positive matrices of rank k, where $r = \lceil \frac{n}{k} \rceil - 1$, then *generically* the kernel of their sum will have dimension at most k and Proposition 2.8 will apply.

Now let us turn to lower bounds for $T^{\downarrow}(n, k)$. The next pair of results are basically Theorems 3 and 4 of [Knill et al. 2000], with two small improvements. For the reader's convenience I include the full proofs.

Lemma 2.9. *Let $\mathcal{V}$ be an operator system in $M_n(\mathbb{C})$ and let $d =$* $\dim(\mathcal{V})$. *Assume every matrix in $\mathcal{V}$ is diagonal. If $(k-1)d + 1 \leq n$ then $\mathcal{V}$ has a quantum k-anticlique.*

Proof. Write $\mathcal{V} = \operatorname{span}(A_1, \ldots, A_d)$ with each A_i Hermitian and $A_1 = I_n$. Then for each $1 \leq j \leq n$ let $\vec{b}_j \in \mathbb{R}^{d-1}$ be the vector whose components are the (j, j) entries of $A_2, \ldots, A_d$. That is, $\vec{b}_j$ is the sequence of eigenvalues of the A_i, excepting $A_1 = I_n$, belonging to the j-th standard basis vector e_j. By a theorem of

Tverberg [1966; 1981], if $n \geq kd - (d-1) = (k-1)d + 1$, then the index set $\{1, \ldots, n\}$ can be partitioned into k blocks $S_1, \ldots, S_k$ such that the convex hulls of the sets $\{\vec{b}_j : j \in S_l\} \subset \mathbb{R}^{d-1}$, for $1 \leq l \leq k$, have nonempty intersection. That is, we can find a single point $\vec{b} \in \mathbb{R}^{d-1}$ such that for each $1 \leq l \leq k$ some convex combination $\sum_{j \in S_l} \mu_j \vec{b}_j$ equals $\vec{b}$. Letting $v_l = \sum_{j \in S_l} \sqrt{\mu_j} e_j$, we then have that $\langle A_i v_l, v_{l'} \rangle = 0$ whenever $l \neq l'$, for any i (even $i = 1$), and if $i \neq 1$ then $\langle A_i v_l, v_l \rangle$ equals the i-th component of $\vec{b}$, while $\langle A_1 v_l, v_l \rangle = 1$ for any l. So $\mathcal{V}$ has a quantum k-anticlique by Lemma 2.3. $\square$

Theorem 2.10. *Let $\mathcal{V}$ be an operator system in $M_n(\mathbb{C})$ and let $d = \dim(\mathcal{V})$. If $(k-1)d + 1 \leq \lceil \frac{n}{d-1} \rceil$ then $\mathcal{V}$ has a quantum k-anticlique.*

Proof. We reduce to Lemma 2.9 by compressing $\mathcal{V}$ to an operator system which contains only diagonal matrices. To do this, write $\mathcal{V} = \mathrm{span}(A_1, \ldots, A_d)$ with each A_i Hermitian and $A_1 = I_n$. Start the construction by letting v_1 be a norm 1 eigenvector of A_2. Then let $E_1 = \mathcal{V}v_1 = \{Bv_1 : B \in \mathcal{V}\}$ and let P_1 be the orthogonal projection onto $E_1^\perp$. Having constructed v_j, E_j, and P_j, let $v_{j+1} \in \mathrm{ran}(P_j)$ be a norm 1 eigenvector for $P_j A_2 P_j$, let $E_{j+1} = P_j \mathcal{V} v_{j+1}$, and let P_{j+1} be the orthogonal projection onto $(E_1 + \cdots + E_j)^\perp$. Continue until all of $\mathbb{C}^n$ is exhausted.

Since v_j is an eigenvector for $P_{j-1} A_2 P_{j-1}$ (setting $P_0 = I_n$), and also since $P_{j-1} A_1 P_{j-1} v_j = v_j$, it follows that the dimension of E_j is at most $d-1$. Thus we have a sequence $(v_1, \ldots, v_r)$ with $r \geq \lceil \frac{n}{d-1} \rceil$. Also, by construction $A_i v_j$ is orthogonal to $v_{j'}$ when $j < j'$, for any i. Thus if P is the orthogonal projection onto the span of the v_j's, then the matrices PA_iP are diagonal with respect to the v_j basis. In other words, $P\mathcal{V}P$ satisfies the hypotheses of Lemma 2.9 with $r \geq \lceil \frac{n}{d-1} \rceil$ in place of n. So $(k-1)d + 1 \leq \lceil \frac{n}{d-1} \rceil$ implies that $P\mathcal{V}P$ has a quantum k-anticlique, and hence that $\mathcal{V}$ does as well. $\square$

The only novel aspects of these two proofs are (1) elimination of the first coordinates of the vectors $\vec{b}_j$ in Lemma 2.9 and (2) our choice of v_j to be an eigenvector of $P_{j-1} A_2 P_{j-1}$. Both yield small improvements on the inequality that has to be assumed, meaning that in both cases the inequality is slightly weakened.

Replacing $\lceil \frac{n}{d-1} \rceil$ with $\frac{n}{d-1}$ yields, if anything, a stronger condition on k. So $(k-1)d + 1 \leq n/(d-1)$ implies that $\mathcal{V}$ has a quantum k-anticlique. Substituting $k+1$ for k and solving for d yields the condition $d \leq (k-1+\sqrt{(k+1)^2 + 4kn})/2k$, and thus any operator system whose dimension is at most this value must have a quantum $(k+1)$-anticlique. As

$$\sqrt{\frac{n}{k}} = \frac{\sqrt{4kn}}{2k} \leq \frac{k-1+\sqrt{(k+1)^2+4kn}}{2k},$$

$T^\downarrow(n,k)$ must be larger than this value. Together with Corollary 2.2, this yields the following estimate.

Theorem 2.11. $$\sqrt{\frac{n}{k}} < T^{\downarrow}(n,k) \leq \left\lceil \frac{n}{k} \right\rceil.$$

The more precise lower bound $\bigl(k-1+\sqrt{(k+1)^2+4kn}\,\bigr)/2k$ is only marginally better than $\sqrt{n/k}$. But when $k=1$ it improves $T^{\downarrow}(n,1)>\sqrt{n}$ to $T^{\downarrow}(n,1)>\sqrt{n+1}$.

The obvious inefficiency in the proof of Theorem 2.10, where we start by compressing to a diagonal operator system, plus the minimality demonstrated in Corollary 2.5, make it natural to conjecture that the lower quantum Turán number $T^{\downarrow}(n,k)$ exactly equals $\lceil \frac{n}{k} \rceil$. When $n=3$ and $k=1$, Proposition 2.7 and Corollary 2.2 yield $T^{\downarrow}(3,1)=3$; so the first interesting case is $n=4$, $k=1$, when Theorem 2.11 yields $3 \leq T^{\downarrow}(4,1) \leq 4$ and the natural conjecture is $T^{\downarrow}(4,1)=4$, i.e., that every three-dimensional operator system in $M_4(\mathbb{C})$ has a quantum 2-anticlique. But even this special case seems hard. I have only been able to prove two partial positive results. The first is an immediate consequence of either Corollary 2.5 or Lemma 2.9. (It can also be inferred from Theorem 2.14 below.)

Proposition 2.12. *Let $\mathcal{V}$ be an operator system in $M_4(\mathbb{C})$ consisting of diagonal matrices, and whose dimension is at most* 3. *Then $\mathcal{V}$ has a quantum* 2*-anticlique.*

The other partial result is more substantive. Its content resides almost entirely in the next lemma, which is a slightly modified version of a theorem of Bryant [2017].

Lemma 2.13. *Let $B_1, B_2, B_3 \in M_2(\mathbb{C})$ with B_1 and B_3 Hermitian and let $a \geq 1$. Then there exist $\lambda \in [0,1]$ and $U \in \mathrm{SU}(2)$ such that*

$$SB_1S + CUB_2S + SB_2^*U^*C + CUB_3U^*C$$

is a scalar multiple of I_2, where

$$S = \mathrm{diag}(\sqrt{\lambda}, \sqrt{\lambda/a}) \quad \textit{and} \quad C = \mathrm{diag}(\sqrt{1-\lambda}, \sqrt{1-\lambda/a}).$$

Proof. For any unit vector $\vec{z} = (z_0, z_1) \in \mathbb{C}^2$ we have a special unitary matrix $U_{\vec{z}} = \left[\begin{smallmatrix} z_0 & -\bar{z}_1 \\ z_1 & \bar{z}_0 \end{smallmatrix}\right]$, and this identifies the 3-sphere S^3 with SU(2). Define $f_\lambda : \mathrm{SU}(2) \to M_2(\mathbb{C})^h$ by $f_\lambda(U) = SB_1S + CUB_2S + SB_2^*U^*C + CUB_3U^*C$, with S and C as given above. (Recall that $M_2(\mathbb{C})^h$ is the Hermitian part of $M_2(\mathbb{C})$.) Also define $g : M_2(\mathbb{C})^h \to \mathbb{R} \oplus \mathbb{C}$ by

$$g(A) = (a_{11} - a_{22}, 2a_{12})$$

where $A = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]$. Note that g is real-linear and $g(A)=0$ if and only if A is a scalar multiple of I_2. Finally, let $F_\lambda : \mathrm{SU}(2) \to \mathbb{R} \oplus \mathbb{C}$ be the map $F_\lambda = g \circ f_\lambda$.

If B_3 is a scalar multiple of I_2 then $f_0(U) = UB_3U^*$ is a scalar multiple of I_2 for any $U \in \mathrm{SU}(2)$, and we are done. So assume this is not the case. Since $S^2 + C^2 = I_2$, adding a scalar multiple of I_2 to both B_1 and B_3 does not change the problem, so we can assume one of the eigenvalues of B_3 is 0. Multiplying B_1, B_2, and B_3 by a nonzero scalar, we can assume the other eigenvalue is 1. We can then find $V \in \mathrm{SU}(2)$ such that $VB_3V^* = \mathrm{diag}(1, 0)$, and if U solves the problem for B_1, VB_2, and VB_3V^* then UV solves the problem for B_1, B_2, and B_3. So we may assume $B_3 = \mathrm{diag}(1, 0)$.

To reach a contradiction, suppose $F_\lambda(U) \neq 0$ for all $\lambda \in [0, 1]$ and $U \in \mathrm{SU}(2)$. Then we can define $\tilde{F}_\lambda : \mathrm{SU}(2) \to S^2 \subset \mathbb{R} \oplus \mathbb{C} \cong \mathbb{R}^3$ by $\tilde{F}_\lambda(U) = F_\lambda(U)/|F_\lambda(U)|$.

The family of maps $\tilde{F}_\lambda$ constitutes a homotopy from $\tilde{F}_0$ to $\tilde{F}_1$. Now $S = 0$ and $C = I_2$ when $\lambda = 0$, so that $f_0(U) = UB_3U^*$. Recalling that we have reduced to the case where $B_3 = \mathrm{diag}(1, 0)$, a short computation shows that

$$\tilde{F}_0(U_{\bar{z}}) = F_0(U_{\bar{z}}) = (|z_0|^2 - |z_1|^2, 2z_0\bar{z}_1),$$

i.e., it is the Hopf map from S^3 to S^2.

This map is homotopically nontrivial, so to generate a contradiction we need only to show that $\tilde{F}_1$ is null homotopic. When $\lambda = 1$ we have $C = \mathrm{diag}(0, a')$ with $a' = \sqrt{1 - 1/a}$. So $F_1(U) \in \mathbb{R} \oplus \mathbb{C}$ is a constant (namely, $g(SB_1S)$) plus something real-linear in the entries of U (namely, $g(CUB_2S + SB_2^*U^*C)$) plus something in $\mathbb{R}\oplus 0$ (namely, $g(CUB_3U^*C)$). Letting $X = \{U \in \mathrm{SU}(2) : F_1(U) \in \mathbb{R}\oplus 0\}$, it follows that X is the intersection of $\mathrm{SU}(2) \cong S^3$ with an affine real-linear subspace of

$$\left\{ \begin{bmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix} : \alpha, \beta \in \mathbb{C} \right\} \cong \mathbb{R}^4$$

whose real dimension is at least 2. Thus X is connected, and therefore its image under F_1 in $\mathbb{R} \oplus 0$ is connected. Since this image does not contain 0, it must therefore lie entirely in $(0, \infty) \oplus 0$ or $(-\infty, 0) \oplus 0$; in either case, the image of $\tilde{F}_1$ cannot be all of S^2 and so $\tilde{F}_1$ must be null homotopic. This contradicts the homotopic nontriviality of $\tilde{F}_0$, and we conclude that $g(f_\lambda(U)) = F_\lambda(U) \in \mathbb{R} \oplus \mathbb{C}$ must be 0 for some $\lambda \in [0, 1]$ and $U \in \mathrm{SU}(2)$. So $f_\lambda(U)$ is a scalar multiple of I_2 for this λ and U. □

Theorem 2.14. *Let $A, B \in M_4(\mathbb{C})$ be Hermitian and assume A has a repeated eigenvalue. Then $\mathcal{V} = \mathrm{span}(I_4, A, B)$ has a quantum 2-anticlique.*

Proof. If A has a triple eigenvalue then there is a rank 3 orthogonal projection P such that PAP is a scalar multiple of P. We can then identify $PM_4(\mathbb{C})P$ with $M_3(\mathbb{C})$ and invoke Proposition 2.7 to infer that $\mathrm{span}(I_3, PBP)$ has a quantum 2-anticlique Q. This Q will then be a quantum 2-anticlique for $\mathcal{V}$.

So assume A has an eigenvalue of multiplicity exactly 2. By adding a scalar multiple of I_4 to A, we can assume that this eigenvalue is 0. There are now two cases to consider. First, suppose the two nonzero eigenvalues of A have opposite sign. Without loss of generality say $A = \mathrm{diag}(a, -b, 0, 0)$ with $a, b > 0$. Multiplying A

by a nonzero scalar, we can also assume that $\frac{1}{a}+\frac{1}{b}=1$. Then let

$$W=\begin{bmatrix} \frac{1}{\sqrt{a}} & 0 & 0 \\ \frac{1}{\sqrt{b}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so that $W^*W=I_3$ and $W^*AW=0$. Again by Proposition 2.7, $\mathrm{span}(I_3, W^*BW)\subset M_3(\mathbb{C})$ has a quantum 2-anticlique Q, and $P=WQW^*$ is then a quantum 2-anticlique for $\mathcal{V}$.

In the other case, the two nonzero eigenvalues of A have the same sign. Multiplying by a scalar and diagonalizing, we can assume that $A=\mathrm{diag}(1,a,0,0)$ with $a\geq 1$. In this basis write $B=\left[\begin{smallmatrix} B_1 & B_2^* \\ B_2 & B_3\end{smallmatrix}\right]$ with $B_1, B_2, B_3\in M_2(\mathbb{C})$ and B_1 and B_3 Hermitian. Then find λ and U as in Lemma 2.13 and define

$$P=\begin{bmatrix} S^2 & SCU \\ U^*SC & U^*C^2U\end{bmatrix},$$

with S and C as in the statement of that lemma. A computation now shows that both PAP and PBP are scalar multiples of P. To see that $\mathrm{rank}(P)=2$, observe that P is unitarily conjugate to $\left[\begin{smallmatrix} S^2 & SC \\ SC & C^2\end{smallmatrix}\right]$, which after interchanging the middle two basis vectors is the direct sum of

$$\begin{bmatrix}\sqrt{\lambda} \\ \sqrt{1-\lambda}\end{bmatrix}\begin{bmatrix}\sqrt{\lambda} & \sqrt{1-\lambda}\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}\sqrt{\lambda/a} \\ \sqrt{1-\lambda/a}\end{bmatrix}\begin{bmatrix}\sqrt{\lambda/a} & \sqrt{1-\lambda/a}\end{bmatrix}. \qquad \square$$

In other words, any three-dimensional operator system in $M_4(\mathbb{C})$ has a quantum 2-anticlique provided it contains a nonscalar matrix that has a repeated eigenvalue. Unfortunately, for generic Hermitian $A, B\in M_4(\mathbb{C})$ the operator system $\mathrm{span}(I_4, A, B)$ does not have this property [Bryant 2018].

3. Upper quantum Turán numbers

We define the *upper quantum Turán number* $T^{\uparrow}(n,k)$ to be the largest number d such that some operator system in $M_n(\mathbb{C})$ whose dimension is d has no quantum $(k+1)$-cliques. As before, we restrict attention to the case $k\geq 1$.

Evaluating $T^{\uparrow}(n,k)$ and $T^{\downarrow}(n,k)$ are very different problems. In general there is no natural “quantum” analog of edge complementation which would interchange quantum cliques and anticliques. In finite dimensions we can consider the orthocomplement $\mathcal{V}^{\perp}$ of an operator system $\mathcal{V}\subseteq M_n(\mathbb{C})$ relative to the Hilbert–Schmidt inner product $\langle A, B\rangle=\mathrm{tr}(AB^*)$, but it will not contain I_n. In order to produce a “complementary” operator system we could define $\mathcal{V}^{\dagger}=\mathcal{V}^{\perp}+\mathbb{C}\cdot I_n$, and this is a genuine complementation operation in the sense that $\mathcal{V}^{\dagger\dagger}=\mathcal{V}$. This operation transforms quantum anticliques into quantum cliques, but not vice versa (incidentally making precise the idea that anticliques are more special than cliques). We can infer from this fact that $T^{\uparrow}(n,k) \le n^2+1-T^{\downarrow}(n,k)$, but this upper bound is terrible compared to the one proven below.[2]

For $k=1$, evaluation of $T^{\uparrow}(n,1)$ is not trivial, but it is completely solved:

Theorem 3.1 [Weaver 2017, Theorem 3.3]. *For any $n \ge 2$, $T^{\uparrow}(n,1)=3$.*

(The cited result only states that $T^{\uparrow}(n,1)<4$, but the reverse inequality follows from the trivial lower bound $T^{\uparrow}(n,k) \ge (k+1)^2-1$. If $\dim(\mathcal{V})<(k+1)^2$ then $\mathcal{V}$ obviously cannot have any quantum $(k+1)$-cliques.)

In contrast, it follows from [Weaver 2017, Proposition 2.3] that $T^{\uparrow}(n,2)\to\infty$ as $n\to\infty$. The example which shows this can be described more abstractly, in a way that generalizes to larger values of k.

Proposition 3.2. *Let Q be an orthogonal projection in $M_n(\mathbb{C})$ of rank $n-k+1$. Then the operator system*

$$\mathcal{V}_Q=\{A\in M_n(\mathbb{C}) : QAQ \text{ is a scalar multiple of } Q\}$$

has no quantum $(k+1)$-cliques. Indeed, no two-dimensional extension of $\mathcal{V}_Q$ has any quantum $(k+1)$-cliques, but every three-dimensional extension of $\mathcal{V}_Q$ does have a quantum $(k+1)$-clique.

Proof. Let P be a rank $k+1$ orthogonal projection. Then since $\mathrm{rank}(P)+\mathrm{rank}(Q)=n+2$ there is a rank 2 orthogonal projection P_0 which lies below both P and Q. Since Q is a quantum anticlique for $\mathcal{V}_Q$, so is P_0, i.e., $\dim(P_0\mathcal{V}_QP_0)=1$. Thus any two-dimensional extension $\mathcal{V}'_Q$ of $\mathcal{V}_Q$ must satisfy $\dim(P_0\mathcal{V}'_QP_0)\le 3$, so that P_0 cannot be a quantum 2-clique for $\mathcal{V}'_Q$. This implies that P cannot be a quantum $(k+1)$-clique for $\mathcal{V}'_Q$.

Now let $\mathcal{V}''_Q$ be a three-dimensional extension of $\mathcal{V}_Q$. Then $\dim(Q\mathcal{V}''_QQ)=4$. (Consider the map $F : A\mapsto QAQ$ from $M_n(\mathbb{C})$ to $QM_n(\mathbb{C})Q$. We have $\mathcal{V}_Q=\ker(F)+\mathbb{C}\cdot I_n$, so if $\mathcal{V}$ is a d-dimensional extension of $\mathcal{V}_Q$ then $\dim(F(\mathcal{V}))=d+1$.) So by Theorem 3.1 $Q\mathcal{V}''_QQ$ has a quantum 2-clique Q_0, and I claim that the projection $P=(I-Q)+Q_0$ is then a quantum $(k+1)$-clique for $\mathcal{V}''_Q$. To see this, let $A\in M_n(\mathbb{C})$ be any matrix which satisfies $PAP=A$; we must show that $A\in P\mathcal{V}''_QP$.

[2] Maybe *quantum clique for $\mathcal{V}$* should be redefined to simply mean *quantum anticlique for $\mathcal{V}^{\dagger}$*? This would automatically introduce a symmetry between quantum cliques and quantum anticliques, but it suffers from two drawbacks: first, it does not generalize to the infinite-dimensional setting, and second, the quantum Ramsey theorem from [Weaver 2017] would fail. According to [Weaver 2017, Proposition 2.1] the diagonal operator system D_n has no quantum 2-anticliques, but $D_n^{\dagger}$ also has no quantum 2-anticliques. (Suppose P is a quantum 2-anticlique for $D_n^{\dagger}$. Let v_1 and v_2 be orthonormal vectors in $\mathrm{ran}(P)$ and consider the operator $A : v\mapsto \langle v, v_1\rangle v_2$. Then $A=PAP$ and $\mathrm{tr}(A)=0$, so that $\mathrm{tr}(AB^*)=\mathrm{tr}(APB^*P)=0$ for all $B\in D_n^{\dagger}$, which implies that $A\in D_n$. But A cannot belong to D_n because it does not commute with A^*, which is a contradiction.)

Since Q_0 is a quantum 2-clique for $\mathcal{V}''_Q$, we can find $B_0 \in \mathcal{V}''_Q$ such that $Q_0 B_0 Q_0 = Q_0 A Q_0$. Let $B_1 = Q B_0 Q$; then $Q(B_1 - B_0)Q = 0$ and so $B_1 - B_0 \in \mathcal{V}_Q$, which implies $B_1 \in \mathcal{V}''_Q$. Similarly, $Q(A - Q_0 A Q_0)Q = Q(PAP - Q_0 A Q_0)Q = 0$ so $A - Q_0 A Q_0 \in \mathcal{V}_Q$, and finally $B = A - Q_0 A Q_0 + B_1$ belongs to $\mathcal{V}''_Q$ and satisfies $PBP = A$. Thus we have shown that $P\mathcal{V}''_Q P$ contains A, as desired. □

The last part of Proposition 3.2 shows that the operator systems $\mathcal{V}'_Q$ are maximal for not having any quantum $(k+1)$-cliques. Of course, this does not rule out the possibility that other operator systems whose dimensions are larger could lack quantum $(k+1)$-cliques.

If Q is diagonalized as $Q = \mathrm{diag}(0, \ldots, 0, 1, \ldots, 1)$ (with $k-1$ zeros and $n-k+1$ ones) then $\mathcal{V}_Q$ appears as the set of matrices whose restriction to the bottom right $(n-k+1) \times (n-k+1)$ corner is a scalar multiple of the $(n-k+1) \times (n-k+1)$ identity matrix, and which can be anything on the top and left $(k-1) \times n$ and $n \times (k-1)$ strips. Thus $\dim(\mathcal{V}_Q) = 2(k-1)n - (k-1)^2 + 1$ and we infer the following corollary.

Corollary 3.3. $T^{\uparrow}(n,k) \geq 2(k-1)n - (k-1)^2 + 3$.

The classical analog of the operator system $\mathcal{V}_Q$ is the graph on n vertices which is the edge complement of a single $(n-k+1)$-clique. In other words, the only missing edges are those both of whose endpoints lie within a fixed set of $n-k+1$ vertices. Such a graph contains no $(k+1)$-cliques, but the number of edges it has is linear in n, whereas the classical Turán numbers grow like n^2.

We could try to get a better lower bound by considering the matrix analog of a $(k+1)$-cliqueless graph with the maximal number of edges. This graph is the edge complement of a disjoint union of k many cliques of equal or nearly equal size. The matrix analog would be the operator system $\{A \in M_n(\mathbb{C}) : P_i A P_i$ is a scalar multiple of P_i for $1 \leq i \leq k\}$ where $P_1, \ldots, P_k$ are orthogonal projections of equal or nearly equal rank which sum to I_n. But this idea does not work because this operator system typically does have quantum $(k+1)$-cliques. This is most simply illustrated in the case $k = 2$ when we are dealing with a “complete bipartite” operator system which might be expected to have no quantum 3-cliques. This expectation fails badly, however:

Proposition 3.4. *Let $\mathcal{V}_0 \subset M_{2k}(\mathbb{C})$ be the set of matrices of the form $\left[\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right]$ with $A, B \in M_k(\mathbb{C})$, and let $\mathcal{V}$ be the operator system $\mathcal{V} = \mathcal{V}_0 + \mathbb{C} \cdot I_{2k}$. Then $\mathcal{V}$ has a quantum k-clique.*

Proof. Let $E = \{v \oplus v : v \in \mathbb{C}^k\} \subset \mathbb{C}^{2k}$ and let P be the orthogonal projection onto E. Any linear operator from E to itself has the form $(v \oplus v) \mapsto (Av \oplus Av)$ for some $A \in M_k(\mathbb{C})$. But for any $A \in M_k(\mathbb{C})$ the matrix $A' = \left[\begin{smallmatrix} 0 & A \\ A & 0 \end{smallmatrix}\right]$ satisfies

$$(PA'P)(v \oplus v) = A'(v \oplus v) = Av \oplus Av,$$

so that $P\mathcal{V}P$ contains every linear operator from E to itself. That is, P is a quantum k-clique. □

In fact, linearity in n is the most we can ask for in a lower bound on $T^{\uparrow}(n, k)$, because — incredibly — we can give an upper bound on $T^{\uparrow}(n, k)$ which is also linear in n. The argument uses the following result from [Weaver 2017]. Let (e_i) be the standard basis of $\mathbb{C}^n$.

Lemma 3.5 [Weaver 2017, Lemma 4.4]. *Let $n = k^4 + k^3 + k - 1$ and let $\mathcal{V}$ be an operator system contained in $M_n(\mathbb{C})$. Suppose $\mathcal{V}$ contains matrices $A_1, \ldots, A_{k^4+k^3}$ such that for each i we have $\langle A_i e_i, e_{i+1}\rangle \neq 0$, and also $\langle A_i e_r, e_s\rangle = 0$ whenever $\max(r, s) > i + 1$ and $r \neq s$. Then $\mathcal{V}$ has a quantum k-clique.*

We need this lemma to prove the next result, which is extracted from the proof of [Weaver 2017, Theorem 4.5]. For the reader's convenience I include the proof here.

Lemma 3.6. *Let $\mathcal{V}$ be an operator system in $M_n(\mathbb{C})$ and suppose that for each nonzero $v \in \mathbb{C}^n$ we have $\dim(\mathcal{V}v) \geq 8k^8$. Then $\mathcal{V}$ has a quantum k-clique.*

Proof. Let v_1 be any nonzero vector in $\mathbb{C}^n$ and find $A_1 \in \mathcal{V}$ such that $v_2 = A_1 v_1$ is nonzero and orthogonal to v_1. Then find $A_2 \in \mathcal{V}$ such that $v_3 = A_2 v_2$ is nonzero and orthogonal to each of v_1, $A_1 v_1$, $A_1^* v_1$, $A_1 v_2$, and $A_1^* v_2$. Continue in this way, at the r-th step finding $A_r \in \mathcal{V}$ such that $v_{r+1} = A_r v_r$ is nonzero and orthogonal to the span of the vectors v_1 and $A_i v_j$ and $A_i^* v_j$ for $i < r$ and $j \leq r$. The dimension of this span is at most $2r^2 - 2r + 1$, so as long as $r \leq 2k^4$ its dimension is less than $8k^8$ and a suitable matrix A_r can be found. Compressing to the span of the v_i for $1 \leq i \leq k^4 + k^3 + k - 1$ then puts us in the situation of Lemma 3.5, so there is a quantum k-clique by that result. □

Theorem 3.7. *Let $\mathcal{V}$ be an operator system in $M_n(\mathbb{C})$ of dimension at least $16k^8 n$. Then $\mathcal{V}$ has a quantum k-clique.*

Proof. Fix k; the proof goes by induction on n. The smallest sensible value of n is $n = 16k^8$; as for smaller values of n the dimension of $\mathcal{V}$ is at most $n^2 < 16k^8 n$. When n exactly equals $16k^8$, the only way to have $\dim(\mathcal{V}) \geq 16k^8 n$ is if $\mathcal{V} = M_n(\mathbb{C})$, so it certainly has a quantum k-clique. In the induction step, first suppose that there exists a nonzero vector $v \in \mathbb{C}^n$ such that $\dim(\mathcal{V}v) < 8k^8$. Let P be the rank $n - 1$ orthogonal projection onto the orthocomplement of $\mathbb{C} \cdot v$ in $\mathbb{C}^n$. If $A \in \mathcal{V}$ satisfies $PAP = 0$ then, with respect to an orthonormal basis of which v is the first element, A is the sum of a matrix which is zero except on the leftmost column and a matrix which is zero except on the topmost row. Since $\dim(\mathcal{V}v) < 8k^8$, it follows that the set $\{A \in \mathcal{V} : PAP = 0\}$ has dimension at most $16k^8$. Thus

$$\dim(P\mathcal{V}P) \geq \dim(\mathcal{V}) - 16k^8 \geq 16k^8(n - 1),$$

and the induction hypothesis tells us that $P\mathcal{V}P$ has a quantum k-clique, so $\mathcal{V}$ does as well.

Otherwise, for every nonzero vector $v \in \mathbb{C}^n$ we have $\dim(\mathcal{V}v) \geq 8k^8$, and then $\mathcal{V}$ has a quantum k-clique by Lemma 3.6. □

Putting this together with Corollary 3.3 yields the promised bounds on $T^{\uparrow}(n, k)$.

Corollary 3.8. $2(k-1)n - (k-1)^2 + 3 \leq T^{\uparrow}(n, k) < 16(k+1)^8 n$.

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BCOV TORSION AND DEGENERATIONS OF CALABI–YAU MANIFOLDS

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The Fang–Lu formula is an identity relating the Weil–Petersson metric, the generalized Hodge metrics and the BCOV torsion on the moduli space of polarized Calabi–Yau manifolds. In this note, we extend this formula to the compactification of the moduli space of polarized Calabi–Yau manifolds assuming the logarithm of BCOV torsion is locally L^1-integrable. On the other hand, we use this extended formula to study global numerical properties for polarized families of Calabi–Yau manifolds.

1. Introduction

Reidemeister torsion (R-torsion) is an invariant that can distinguish between closed manifolds which are homotopy equivalent but not homeomorphic. Analytic torsion (or Ray–Singer torsion) is an invariant of Riemannian manifolds defined by Ray and Singer [1971; 1973] as an analytic analogue of Reidemeister torsion. These two torsions naturally coincide, which is known as the Cheeger–Müller theorem [Müller 1978; Cheeger 1979].

Bershadsky, Cecotti, Ooguri and Vafa [Bershadsky et al. 1993; 1994] conjectured an equivalence between the physical quantity F_1 of a Calabi–Yau threefold and a linear combination of the holomorphic analytic torsion, which is now called BCOV torsion. Motivated by their conjecture, Fang, Lu and Yoshikawa [Fang et al. 2008] considered a modification of BCOV torsion, called the BCOV invariant, and conducted a detailed study of the asymptotic behavior of that invariant for Calabi–Yau threefold. See also [Yoshikawa 2015; 2017].

By using the curvature formula for Quillen metrics, Bershadsky, Cecotti, Ooguri and Vafa [Bismut et al. 1988a; 1988b; 1988c] obtained a variational formula for the BCOV torsion of Ricci-flat Calabi–Yau manifolds. (In our terminology, a compact connected Kähler manifold X is called a *Calabi–Yau* manifold if $H^q(X, \mathcal{O}_X) = 0$ for $0 < q < \dim X$ and $K_X \cong \mathcal{O}_X$.) Fang and Lu [2005] expressed the variation of the BCOV torsion T of Ricci-flat Calabi–Yau manifolds as a linear combination

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of the Weil–Petersson metric ω_{WP} and the generalized Hodge metrics ω_{H^i} (see Section 2 for precise definitions):

$$\sum_{i=1}^{n}(-1)^i\omega_{\mathrm{H}^i}-\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log T=\frac{\chi}{12}\omega_{\mathrm{WP}},\tag{1-1}$$

where χ is the Euler characteristic number. As has been pointed out in [Fang and Lu 2005], it is an interesting problem to extend this formula to the compactified moduli space. The main aim of this note is to establish this extension. In fact, let $\mathcal{M}$ be a connected component of the moduli space of polarized Calabi–Yau manifolds. Since $\mathcal{M}$ is quasiprojective, the smooth part $\mathcal{M}_{\mathrm{reg}}$ of $\mathcal{M}$ admits a compactification $\overline{\mathcal{M}}$, where $\overline{\mathcal{M}}$ is a smooth projective manifold such that $D=\overline{\mathcal{M}}-\mathcal{M}_{\mathrm{reg}}$ is a simple normal crossing divisor. Suppose that $D=\overline{\mathcal{M}}-\mathcal{M}_{\mathrm{reg}}=\sum_v D_v$, where the sum is locally finite and each D_v is a irreducible hypersurface of $\overline{\mathcal{M}}$. We will show that the following holds:

Theorem 1.1. *Assume* $\log T$ *is locally* L^1*-integrable on* $\overline{\mathcal{M}}$*. For each component* D_v *of the boundary divisor* D, *we set*

$$a_v:=\lim_{p\to D_v}\frac{\log T}{\log|f_v|^2},\tag{1-2}$$

where $p\in\mathcal{M}_{\mathrm{reg}}$ *and* f_v *is a local defining function of the hypersurface* D_v*. Then the following equation of currents on* $\overline{\mathcal{M}}$ *holds*:

$$dd^c\log T+\sum_{i=1}^{n}(-1)^{i-1}T_{\omega_{\mathrm{H}^i}}+\frac{\chi}{12}T_{\omega_{\mathrm{WP}}}=\sum_v a_v[D_v],\tag{1-3}$$

where the currents $T_{\omega_{\mathrm{WP}}}$ *and* $T_{\omega_{\mathrm{H}^i}}$ *are the trivial extensions of* ω_{WP} *and* ω_{H^i} *from* $\mathcal{M}_{\mathrm{reg}}$ *to* $\overline{\mathcal{M}}$, *and for each* v, $[D_v]=\int_{D_v}$ *is the current associated to the hypersurface* D_v.

For polarized families of Calabi–Yau manifolds we have a similar result. In fact, let $\mathcal{X}$ be a smooth projective variety of dimension $n+m$, let S be a smooth projective variety of dimension m and let $f:\mathcal{X}\to S$ be a surjective, flat holomorphic map with generic fiber Calabi–Yau n-fold. Let $f^0:\mathcal{X}^0\to S^0$ be the smooth part of f and assume that the discriminant locus $E:=S\setminus S^0$ of f is a simple normal crossing divisor.

Theorem 1.2. *Assume* $\log T$ *is locally* L^1*-integrable on* S*. Let* $E=\sum_v E_v$ *be the irreducible decomposition of* E *and set*

$$a_v:=\lim_{p\to E_v}\frac{\log T}{\log|f_v|^2},\tag{1-4}$$

where $p\in S^0$ *and* f_v *is a local defining function of the hypersurface* E_v*. Then the* *following equation of currents on S holds*:

$$dd^c \log T + \sum_{i=1}^{n}(-1)^{i-1} T_{\omega_{\mathrm{H}^i}} + \frac{\chi}{12} T_{\omega_{\mathrm{WP}}} = \sum_v a_v [E_v], \tag{1-5}$$

where the currents $T_{\omega_{\mathrm{WP}}}$ and $T_{\omega_{\mathrm{H}^i}}$ are the trivial extensions of ω_{WP} and ω_{H^i} from S^0 to S, $[E_v] = \int_{E_v}$ is the current associated to the hypersurface E_v and χ is the topological Euler number of a general fiber of f.

At first glance, Theorem 1.2 is a direct consequence of Theorem 1.1. However, this is not the case. First of all, unlike the space $\mathcal{M}$, the compactification $\overline{\mathcal{M}}$ itself is not a moduli space, so we do not have an induced map $S \to \overline{\mathcal{M}}$. Secondly, the pull back of currents may not be well-defined in general [Demailly 2012, pp. 18]. This is why we state them as two separate theorems. Nevertheless, Theorem 1.2 can be proved in the same way as Theorem 1.1. Hence we will only give the proof of Theorem 1.1 and leave the proof of Theorem 1.2 to the readers.

The current equation (1-5) in the case where $n = 3$, $m = 1$ was proved in [Fang et al. 2008, Theorem 10.1]. The proof of these two theorems is a modification of the arguments in [Fang et al. 2008, Section 7]. There are two crucial points here. The first is the assumption that $\log T$ is locally L^1-integrable.[1] This assumption is natural in the sense that it has been shown to be valid when $n = 3$ [Fang et al. 2008, Theorem 9.1]. Under this assumption, we will show that $\log T$ has at most logarithmic growth (see (3-18)) when approaching the boundary divisor D (in Theorem 1.1) and the discriminant locus E (in Theorem 1.2) so that the asymptotic values a_v in these two theorems make sense. The second is that each term in the Fang–Lu formula (1-1) is bounded by the Poincaré metric near the boundary of the moduli space [Fang and Lu 2005, Theorem A.1].

Next, we use an observation in [Liu and Xia 2019] to get the following:

Corollary 1.3. *We assume the same conditions as in Theorem 1.2. Suppose that the local monodromies of the polarized family $f^0 : \mathcal{X}^0 \to S^0$ are all unipotent. Then*

$$\sum_{i=1}^{n}(-1)^{i-1} \sum_{0\le p\le i} p(\deg P\mathcal{H}_e^{p,i-p} + \deg P\mathcal{H}_e^{p-1,i-p-1} + \cdots) + \frac{\chi}{12}\deg P\mathcal{H}_e^{n,0} = \sum_v a_v \deg E_v, \tag{1-6}$$

where $P\mathcal{H}_e^{p,q} \to S$ denotes the canonical Deligne extensions of the Hodge bundles $P\mathcal{H}^{p,q} = PR^q f^0_* \Omega^p_{\mathcal{X}^0/S^0} \to S^0$. In particular, for polarized families of Calabi–Yau

[1] The recent work of Eriksson, Freixas and Mourougane has provided further evidence for this assumption. See [Eriksson et al. 2018a, Theorem B].

3-folds, we have

$$(\chi+36)\deg P\mathcal{H}_e^{3,0}+12\deg P\mathcal{H}_e^{2,1}=12\sum_v a_v \deg E_v, \tag{1-7}$$

and for polarized families of Calabi–Yau 4-folds, we have

$$(\chi-48)\deg P\mathcal{H}_e^{4,0}-24\deg P\mathcal{H}_e^{3,1}+12\deg P\mathcal{H}_e^{2,1}=12\sum_v a_v \deg E_v. \tag{1-8}$$

The identity (1-7) for $m=1$ was proved in [Liu and Xia 2019, Theorem 4.4]. These identities are closely related to the Grothendieck–Riemann–Roch theorem and it is our belief that the asymptotic value a_v can be read out directly from the data of the corresponding singular fiber, see [Liu and Xia 2019; Eriksson et al. 2018b; Green et al. 2009]. Moreover, combining (1-8) with the Arakelov inequality in [Liu and Xia 2019, Theorem 5.1], we get:

Corollary 1.4. *We assume the same conditions as in Theorem 1.2 and we let $n=4, m=1$. If $f:\mathcal{X}\to S$ is not isotrivial and the local monodromies of the polarized family $f^0:\mathcal{X}^0\to S^0$ are all unipotent, then*

$$\begin{aligned}1+\frac{\sum_v a_v-\deg P\mathcal{H}_e^{2,1}}{2\deg P\mathcal{H}_e^{4,0}}&\le\frac{\chi}{24}\\ &\le\frac{2\pi(2g-2+s)(h^{3,1}+4)+\sum_v a_v-\deg P\mathcal{H}_e^{2,1}}{2\deg P\mathcal{H}_e^{4,0}}.\end{aligned} \tag{1-9}$$

This article is structured as follows. In Section 2 we recall basic definitions and fix our notations. In Section 3 we give the proofs of our main results. In Section 4 we apply our main results to polarized families of Calabi–Yau manifolds.

2. Preliminaries

2A. ***BCOV torsion.*** Let (M,g) be a compact Kähler manifold of dimension n with Kähler form ω. Let $\square_{p,q}=(\bar\partial+\bar\partial^*)^2$ be the $\bar\partial$-Laplacian acting on C^∞ (p,q)-forms on M or equivalently $(0,q)$-forms on M with values in Ω_M^p, where Ω_M^1 is the holomorphic cotangent bundle of M and $\Omega_M^p:=\Lambda^p\Omega_M^1$. Let $\{\lambda_j\}$ be the eigenvalues of $\square_{p,q}$; then it is well known that $0=\lambda_0\le\lambda_1\le\lambda_2\le\cdots\le\lambda_j\longrightarrow+\infty$. The *spectral zeta function* of $\square_{p,q}$ is defined as

$$\zeta_{p,q}(s):=\sum_{\lambda_j>0}\lambda_j^{-s}, \tag{2-1}$$

where the multiplicities of the eigenvalues are taken into account.[2] Then $\zeta_{p,q}(s)$ converges on the half-plane $\{s \in \mathbb{C};\ \Re s > \dim M\}$, extends to a meromorphic function on $\mathbb{C}$, and is holomorphic at $s = 0$. From [Ray and Singer 1973], the (holomorphic) *analytic torsion* of (M, Ω_M^p) is the real number defined as

$$\tau(M, \Omega_M^p) := \exp\biggl\{-\sum_{q\geq 0}(-1)^q q\, \zeta'_{p,q}(0)\biggr\}. \tag{2-2}$$

Note that $\tau(M, \Omega_M^p)$ depends not only on the complex structure of M but also on the metric g.

Bershadsky, Cecotti, Ooguri and Vafa [Bershadsky et al. 1994] introduced the following combination of analytic torsion:

Definition 2.1. The *BCOV torsion* of (M, g) is the real number defined as

$$T_{\mathrm{BCOV}}(M, g) := \prod_{p\geq 0} \tau(M, \Omega_M^p)^{p(-1)^p} = \exp\biggl\{-\sum_{p,q\geq 0}(-1)^{p+q} pq\, \zeta'_{p,q}(0)\biggr\}. \tag{2-3}$$

Recall that a compact connected Kähler manifold X is said to be *Calabi–Yau* if $H^q(X, \mathcal{O}_X) = 0$ for $0 < q < \dim X$ and $K_X \cong \mathcal{O}_X$, where K_X is the canonical line bundle of X. In general, $T_{\mathrm{BCOV}}(M, g)$ is only a spectrum invariant. But when M is a Calabi–Yau threefold, it is possible to construct a holomorphic invariant of M from $T_{\mathrm{BCOV}}(M, g)$ by multiplying by a correction factor, see [Fang et al. 2008]. In this note we will only consider BCOV torsion with respect to the unique Ricci-flat metric [Yau 1978] on a polarized Calabi–Yau manifold.

2B. ***The moduli space of the polarized Calabi–Yau manifold.*** Let $(X, [\omega])$ be a polarized Calabi–Yau manifold of dimension $n \geq 3$, that is, X is a Calabi–Yau manifold and $[\omega] = c_1(L) \in H^2(X, \mathbb{Z})$ is the first Chern class of an ample line bundle L on X. Let $\mathcal{M}$ be the (coarse) moduli space of the polarized Calabi–Yau manifold $(X, [\omega])$. Locally, $\mathcal{M}$ is identified as a finite discrete quotient of the local versal deformation space Def (Kuranishi space) of X. By the Bogomolov–Tian–Todorov theorem [Tian 1987; Todorov 1989], the base space Def of the Kuranishi family

$$\pi : (\mathfrak{X}, X) \to (\mathrm{Def}, 0) \tag{2-4}$$

is smooth (a priori it is only a complex analytic space). Indeed, Def is an open subset of the linear space $H^1(X, \Theta_X)$, where Θ_X is the holomorphic tangent bundle of X, so we may assume Def is contractible. Since $H^0(X, \Theta_X) \cong H^0(X, \Omega_X^{n-1}) = 0$, the Kuranishi family π is universal. Define

$$H^1(X, \Theta_X)_\omega := \{\theta \in H^1(X, \Theta_X) \mid \theta \lrcorner \omega = 0 \in H^2(X, \mathcal{O}_X)\}, \tag{2-5}$$

[2] By this we mean that if an eigenvalue λ_j has multiplicity k, then it will appear k times in the series $\sum_{\lambda_j>0} \lambda_j^{-s}$.

and $\mathrm{Def}_\omega := \mathrm{Def} \cap H^1(X, \Theta_X)_\omega$. Then each point $\theta \in \mathrm{Def}_\omega$ stands for a fiber of the Kuranishi family $\pi : (\mathfrak{X}, X) \to (\mathrm{Def}, 0)$ with a polarization

$$[\omega_\theta] = [\omega] \in H^2(X_\theta, \mathbb{Z}) = H^2(X, \mathbb{Z})$$

[Tian 1987]. In fact, since $H^2(X, \mathcal{O}_X) = 0$, we have $\mathrm{Def}_\omega = \mathrm{Def}$. We thus have the associated period mapping $\mathrm{Def} \to D$, where D is the period domain, i.e., the classifying space of polarized (weight k) Hodge structure of $(X, [\omega])$. $\mathrm{Def} \to D$ is a holomorphic mapping, and is an immersion (local Torelli) for $k = n$, see [Griffiths 1968, Corollary 3.6]. We remark that $\mathcal{M}$ is a complex orbifold and is locally covered by Def, so when we work with metrics, and curvatures of $\mathcal{M}$, we can treat these notions on Def instead.

We recall some natural metrics on $\mathcal{M}$, see [Fang and Lu 2005] for more detail.

2C. *Weil–Petersson metric.* Let $t \in \mathrm{Def}$. The Kodaira–Spencer map is now an isomorphism

$$\rho : T_t \mathrm{Def} \xrightarrow{\cong} H^1(X_t, \Theta_t), \tag{2-6}$$

where Θ_t is the holomorphic tangent bundle of X_t.

Let $(t_1, \dots, t_m)$ be a local holomorphic coordinate system of $\mathcal{M}$; we define a Hermitian inner product on $T_t\mathcal{M}$ by

$$\left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial \bar{t}_j}\right)_{\mathrm{WP}} = \int_{X_t} A^{\alpha}_{i\bar{\beta}} \cdot \overline{A^{\gamma}_{j\bar{\delta}}} g^{\delta\bar{\beta}} g_{\alpha\bar{\gamma}} \, dV_{X_t}, \tag{2-7}$$

where $A_i = A^{\alpha}_{i\bar{\beta}}(\partial/\partial t_\alpha) \otimes d\bar{t}^{\beta}$ $(i = 1, \dots, m)$ are the harmonic representations of $\rho(\partial/\partial t_i)$. This inner product on each T_tU for $t \in \mathcal{M}$ gives a Hermitian metric on the moduli space $\mathcal{M}$, which is called the Weil–Petersson metric. Equipped with the Weil–Petersson metric, $\mathcal{M}$ is a Kähler orbifold.

Let Ω be a (nonzero) holomorphic $(n, 0)$-form on X_t. Define $\Omega \lrcorner \rho(\partial/\partial t_i)$ to be the contraction of Ω and $\rho(\partial/\partial t_i)$. The Weil–Petersson metric can be rewritten (see [Tian 1987] as

$$\left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial \bar{t}_j}\right)_{\mathrm{WP}} = -\frac{\int_{X_t} \Omega \lrcorner \rho\big(\frac{\partial}{\partial t_i}\big) \wedge \overline{\Omega \lrcorner \rho\big(\frac{\partial}{\partial t_j}\big)}}{\int_{X_t} \Omega \wedge \overline{\Omega}}. \tag{2-8}$$

2D. *Generalized Hodge metrics and Hodge bundles.* Recall that, for all $0 \le k \le n$ and $p+q = k$ there are natural holomorphic vector bundles $P\mathcal{H}^{p,q} := PR^q\pi_*\Omega^p_{\mathfrak{X}/\mathrm{Def}}$, called Hodge bundles, on Def (hence on $\mathcal{M}$), whose fiber is

$$(PR^q\pi_*\Omega^p_{\mathfrak{X}/\mathrm{Def}})_t = PH^q(X_t, \Omega^p_{X_t}), \tag{2-9}$$

where $PH^q(X_t, \Omega^p_{X_t})$ is the primitive cohomology of $(X_t, [\omega_t])$. By abuse of notation, we will always use the same symbol $P\mathcal{H}^{p,q}$ to denote the Hodge bundle on Def and on $\mathcal{M}$.

By differentiating harmonic representatives, we have a holomorphic bundle map

$$\frac{\partial}{\partial t_i} : P\mathcal{F}^p \to P\mathcal{H}/P\mathcal{F}^p, \tag{2-10}$$

where $P\mathcal{F}^p = P\mathcal{H}^{p,k-p} \oplus P\mathcal{H}^{p+1,k-p-1} \oplus \cdots \oplus P\mathcal{H}^{k,0}$ and $P\mathcal{H} := PR^k\pi_*(\mathbb{C})$. In this way, we get a natural holomorphic bundle map

$$T(\mathrm{Def}) \to \bigoplus_{1\le p\le k} \mathrm{Hom}\,(P\mathcal{F}^p, P\mathcal{H}/P\mathcal{F}^p). \tag{2-11}$$

We remark that this bundle map is just the differential of the period mapping $\mathrm{Def} \to D$. There are natural metrics on the Hodge bundles $P\mathcal{F}^p$, and hence on each $\mathrm{Hom}\,(P\mathcal{F}^p, P\mathcal{H}/P\mathcal{F}^p)$, induced by the Riemann–Hodge bilinear relations. Let h_{PH^k} be the pull back of the metric on

$$\bigoplus_{1\le p\le k} \mathrm{Hom}\,(P\mathcal{F}^p, P\mathcal{H}/P\mathcal{F}^p).$$

Then, by (2-11), we have that h_{PH^k} is semipositive. We use ω_{PH^k} to denote the associated (1, 1)-form of the Hermitian symmetric bilinear form h_{PH^k} for all $k \le n$. We let

$$\omega_{\mathrm{H}^k} := \omega_{PH^k} + \omega_{PH^{k-2}} + \cdots . \tag{2-12}$$

We call both ω_{H^k} and ω_{PH^k} generalized Hodge metrics. We note that when $k = n$, ω_{PH^n} is a positive (1, 1)-form by local Torelli. It's just the pull back of the usual Hodge metric on D and we call $\omega_{\mathrm{H}} := \omega_{PH^n}$ the Hodge metric.

2E. *Weil–Petersson form and generalized Hodge forms.* Let $f : \mathcal{X} \to S$ be a smooth polarized family of Calabi–Yau manifolds with $(X_0 := f^{-1}(0), [\omega_0]) \cong (X, [\omega])$, where $0 \in S$ and $[\omega_0]$ is the polarization of X_0 induced from that of $\mathcal{X}$. Since $\mathcal{M}$ is the moduli space of the polarized Calabi–Yau manifold $(X, [\omega])$, we have a natural commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{\ \phi\ } & D/\Gamma \\ \Big\downarrow{\scriptstyle\psi} & & \Big\downarrow \\ \mathcal{M} & \xrightarrow{\ \varphi\ } & D/G_{\mathbb{Z}} \end{array}$$

where Γ is the monodromy group of the family f, and $G_{\mathbb{Z}} = \mathrm{Aut}(H_{\mathbb{Z}}, Q)$, see [Griffiths 1984] for more information. To keep our notation simple, we will just use ω_{WP} and ω_{H^i} to denote the pull back forms $\psi^*\omega_{\mathrm{WP}}$ and $\psi^*\omega_{\mathrm{H}^i}$, and call them the Weil–Petersson form and generalized Hodge forms on S, respectively. Similarly, we will use $P\mathcal{H}^{p,q}$ to denote the pull back bundle $\psi^* P\mathcal{H}^{p,q}$, which is isomorphic to $PR^q f_* \Omega^p_{\mathcal{X}/S}$, and call them Hodge bundles on S.

3. Proof of Theorem 1.1 and Theorem 1.2

After some preparations in Section 3A and 3B, we will prove Theorem 1.1 and Theorem 1.2 in Section 3C and 3D respectively.

3A. ***Extension of dd^c-closed functions.*** In this subsection we show that dd^c-closed functions can be locally extended through a hypersurface when some suitable growth conditions are satisfied.[3] The proof is a slight modification to that of the one variable case [Fang et al. 2008, Section 7.2]. We use Δ to denote the unit open disc in the complex plane, Δ^* to denote the punctured unit disc and $\mathcal{O}$ to denote the sheaf of holomorphic functions. Recall that $d^c := (\sqrt{-1}/4\pi)(\bar\partial - \partial)$ and $dd^c = (\sqrt{-1}/2\pi)\partial\bar\partial$. It is easy to see that the real or imaginary part of a holomorphic function is dd^c-closed and in the one dimensional case dd^c-closed functions are just harmonic functions.

Lemma 3.1. *Let $H(z)$ be a real-valued dd^c-closed function on $U \times \Delta^*$, where U is an open subset in $\mathbb{C}^k$. Then we have that*:

(1) *There exist $c \in \mathbb{R}$, $F(z) \in \mathcal{O}(U \times \Delta^*)$ and a real-valued dd^c-closed function g on U with*

$$H(z) = c \log|t|^2 + 2\operatorname{Re} F(z) + g(u),$$

where $z = (u, t) = (u_1, \dots, u_k, t) \in U \times \Delta^*$.

(2) *If for any fixed $u \in U$, there exist $\gamma = \gamma(u) \in \mathbb{R}$ such that $|t|^\gamma e^{H(z)}$ is a locally L^1-integrable function on Δ , then for the function F in* (1), *we have $F(z) \in \mathcal{O}(U \times \Delta)$.*

(3) *If for any fixed $u \in U$, $H(u, t) = O(\log(-\log|t|))$ as $t \to 0$, then for the functions F, g in* (1), *we have $H(z) = 2\operatorname{Re} F(z) + g(u)$ and $H(z)$ extends to a dd^c-closed function on $U \times \Delta$.*

Proof. (1) Since $H(z)$ is dd^c-closed, we know that for all i, j, the derivatives $\partial H/\partial u_i$ and $f(z) := \partial H/\partial t$ are holomorphic functions on $U \times \Delta^*$. Let $f(z) = f(u, t) = \sum_{n\in\mathbb{Z}} a_n(u)\, t^n$ be the Laurent expansion of f with respect to t and define $F(z) \in \mathcal{O}(U \times \Delta^*)$ by $F(z) := \sum_{n\neq -1} a_n(u)\, t^{n+1}/(n+1)$. Noting that $f = a_{-1}(u)/t + \partial F/\partial t$ and using the fact that $H(z)$ is real-valued, we have

$$d_t H(z) = \frac{\partial H}{\partial t} dt + \frac{\partial H}{\partial \bar t} d\bar t = f\,dt + \bar f\, d\bar t = a_{-1}\frac{dt}{t} + \bar a_{-1}\frac{d\bar t}{\bar t} + d_t F + d_t \bar F, \tag{3-1}$$

[3] For more background material of Lemma 3.1, we refer the readers to [Axler 1986; Kodaira 2007].

where d_t means differential with respect to the t variable. Integrating both sides over the circle $|t| = 1/2$, for all $u \in U$ we get $a_{-1}(u) \in \mathbb{R}$ by the Stokes' theorem, and we conclude that $a_{-1}(u)$ is a real constant because we know that it is holomorphic. So $d_t H(z) = a_{-1} d \log|t|^2 + 2d_t\{\mathrm{Re}\, F(z)\}$ and $H(z) = a_{-1} \log|t|^2 + 2\mathrm{Re}\, F(z) + g(u)$ for some real-valued dd^c-closed function g on U. This proves (1).

(2) First we fix $u \in U$. Then $e^{H(z)} = |t|^{2c} |e^{F(z)}|^2 e^{g(u)}$. By assumption, there exists $\gamma \in \mathbb{R}$ such that

$$e^{g(u)} \int_{|t|<1/2} |t|^{\gamma+2c}\, |e^{F(z)}|^2 \sqrt{-1}\, dt \wedge d\bar{t} < +\infty, \tag{3-2}$$

where F, g and c are as given in (1). Since $e^{F(z)}$ is holomorphic in $t \in \Delta^*$, we deduce from (3-2) that $e^{F(z)}$ is a meromorphic function in $t \in \Delta$. Set $F_u(t) := F(z)$. There exist $N \in \mathbb{Z}$ and a nowhere vanishing holomorphic function $\epsilon_u(t) \in \mathcal{O}(\Delta)$ with $e^{F(z)} = t^N \epsilon_u(t)$, where $\epsilon_u(t)$ is holomorphic in u. Then $F'_u(t) = N t^{-1} + \epsilon'_u(t) \epsilon_u(t)^{-1}$. Since $e^{F(z)}$, and thus $F_u(t)$ is a meromorphic function in $t \in \Delta$, the residue of $F'_u(t)$ must vanish, i.e., $N = 0$. Therefore $F'_u(t) = \epsilon'_u(t) \epsilon_u(t)^{-1}$. Since $\epsilon_u(t) \in \mathcal{O}(\Delta)$ and is nowhere vanishing, from the Laurent series expansions of $F_u(t)$ and $F'_u(t)$ on Δ^* we deduce that $t = 0$ is a removable singularity of the function $F_u(t)$. Hence, for any $t \in \Delta$ we may set

$$F_u(t) := \int_0^t \frac{\epsilon'_u(s)}{\epsilon_u(s)}\, ds + F_u(0), \quad \text{where } F_u(0) := \lim_{t \to 0} F_u(t).$$

It follows that $F(z) \in \mathcal{O}(U \times \Delta)$.

(3) From the assumption, we easily see that for any fixed $u \in U$, there exists $\gamma \in \mathbb{R}$ such that $|t|^\gamma e^{H(u,t)} \in L^1_{\mathrm{loc}}(\Delta)$. Indeed, it suffices to choose any positive $\gamma \in \mathbb{R}$. By (1) and (2), we have $F(z) \in \mathcal{O}(U \times \Delta)$ and so $H(z) - c \log|t|^2 = 2\,\mathrm{Re}\, F(z) + g(u)$ is a dd^c-closed function on $U \times \Delta$, where F, g and c are as given in (1). Since $H(u,t) = O(\log(-\log|t|))$ as $t \to 0$ and $\lim_{t \to 0}(\log|t|^2 / \log(-\log|t|)) = \infty$, we get $c = 0$. This completes the proof. □

Lemma 3.2. *Let $T(z) = T(u,t)$ be a positive function on $U \times \Delta^*$ such that for any fixed $u \in U$, $T(u,t)$ is locally L^1-integrable on Δ. Let $P(z)$ be a real-valued function on $U \times \Delta^*$ such that for any fixed $u \in U$, $P(z) = P(u,t) \le C(-\log|t|)$, where $C = C(u) \in \mathbb{R}$ is a constant, and if $\log T(z) + P(z)$ is dd^c-closed on $U \times \Delta^*$, then there exists $a \in \mathbb{R}$ such that for any fixed $u \in U$,*

$$\log T(u,t) = a \log|t|^2 + O(|P(u,t)|), \qquad t \to 0. \tag{3-3}$$

Proof. Set $H(z) = \log T(z) + P(z)$. By Lemma 3.1 (1), there exist $a \in \mathbb{R}$ and $F(z) \in \mathcal{O}(U \times \Delta^*)$ and a real-valued dd^c-closed function g on U with $H(z) = a \log|t|^2 + 2\,\mathrm{Re}\, F(z) + g(u)$. Since for any fixed $u \in U$, $P(u,t) \le C(-\log|t|)$, we have $\log T(z) = H(z) - P(z) \geq H(z) - C(-\log|t|)$. By our assumption on $T(z)$, we have that for any fixed $u \in U$,

$$\int_{|t|<1/2} |t|^C e^{H(u,t)} \sqrt{-1}\, dt \wedge d\bar{t} \leq \int_{|t|<1/2} T(u,t) \sqrt{-1}\, dt \wedge d\bar{t} < +\infty. \tag{3-4}$$

By Lemma 3.1 (2), we deduce that $F(z) \in \mathcal{O}(U \times \Delta)$, and hence $\log T(u,t) = H(z) - P(z) = a \log|t|^2 + O(|P(u,t)|)$, $t \to 0$. □

Lemma 3.3. *Let $T(z) = T(u,t)$ be a positive function on $U \times \Delta^*$ such that for any fixed $u \in U$, $T(u,t)$ is locally L^1-integrable on Δ. Let $P(z)$ be a real-valued function on $U \times \Delta^*$ such that for any fixed $u \in U$, $P(u,t) = O(\log(-\log|t|))$ as $t \to 0$, and if $\log T(z) + P(z) + f(u)$ is dd^c-closed on $U \times \Delta^*$ where $f(u)$ is a smooth function independent of t, then*:

(1) *There exists $a \in \mathbb{R}$ such that for any fixed $u \in U$,*

$$\log T(u,t) = a \log|t|^2 + O(\log(-\log|t|)), \qquad t \to 0. \tag{3-5}$$

(2) $\log T(z) + P(z) + f(u) - a \log|t|^2$ *extends to a dd^c-closed function on $U \times \Delta$.*

Proof. First we fix $u \in U$. By assumption, $P(z) + f(u) = O(\log(-\log|t|))$ as $t \to 0$, and so $P(z) + f(u) \leq C\,(-\log|t|)$ for some constant C. Applying Lemma 3.2, we get

$$\log T(u,t) = a \log|t|^2 + O(|P(u,t) + f(u)|) = a \log|t|^2 + O(\log(-\log|t|)), \tag{3-6}$$

as $t \to 0$. This proves (1). For (2), we set $H(z) = \log T(z) + P(z) + f(u) - a \log|t|^2$. Then $H(z) = O(\log(-\log|t|))$ as $t \to 0$. It follows from Lemma 3.1 (3) that $H(z)$ extends to a dd^c-closed function on $U \times \Delta$. □

3B. ***The Poincaré metric and trivial current extensions of* (1, 1)*-forms.*** The Poincaré metric on $\Delta^k \times \Delta^{*l}$ is defined by

$$\omega_{\mathrm{P}} = \sqrt{-1} \sum_{j=1}^{k} dz_j \wedge d\bar{z}_j + \sqrt{-1} \sum_{j=k+1}^{k+l} \frac{dz_j \wedge d\bar{z}_j}{|z_j|^2 \cdot \log^2 |z_j|^2}. \tag{3-7}$$

Let M be a complex manifold of dimension n and u a $(1,1)$-current on M. Then u is *positive* if $u(\eta \wedge \bar{\eta}) \geq 0$ for all $\eta \in A_c^{n-1,0}(M)$, where $A_c^{n-1,0}(M)$ is the vector space of smooth $(n-1,0)$-forms on M with compact support. A real-valued function $f \in L^1_{\mathrm{loc}}(M)$ is *plurisubharmonic* if f is upper semicontinuous and $dd^c f$ is positive as currents on M.

A C^∞ real $(1,1)$-form α on $\Delta^k \times \Delta^{*l}$ has *Poincaré growth* if there exists $C > 0$ with

$$-C\,\omega_{\mathrm{P}} \leq \alpha \leq C\,\omega_{\mathrm{P}}. \tag{3-8}$$

In this case, the coefficient of α lies in $L^1_{\mathrm{loc}}(\Delta^k \times \Delta^{*l})$. If α has Poincaré growth, the $(1,1)$-current T_α on Δ^{k+l} defined by

$$T_\alpha(\psi) := \int_{\Delta^{k+l}} \alpha \wedge \psi, \qquad \psi \in A_c^{k+l-1,k+l-1}(\Delta^{k+l}), \tag{3-9}$$

is called the *trivial extension* of α from $\Delta^k \times \Delta^{*l}$ to Δ^{k+l}. By definition, if α is semipositive, is d-closed and has Poincaré growth on $\Delta^k \times \Delta^{*l}$, then its trivial extension T_α is a closed, positive $(1,1)$-current on Δ^{k+l}. Note also that

$$dd^c\left\{2\pi\left(\sum_{j=1}^k |z_j|^2 - \sum_{j=1}^l \log(-\log|z_{k+j}|^2)\right)\right\} = T_{\omega_{\mathrm{P}}} \tag{3-10}$$

as currents on Δ^{k+l}.

We will follow the convention that $\widehat{z_{k+i}}$ means z_{k+i} has been omitted.

Proposition 3.4. *Let ω_i, $i = 1, \dots, N$, be closed, semipositive $(1,1)$-forms on $\Delta^k \times \Delta^{*l}$ with $\omega_i \le C\omega_{\mathrm{P}}$ for some positive constant C. Let T be a positive smooth function on $\Delta^k \times \Delta^{*l}$ such that $\log T$ is a locally L^1-integrable function on Δ^{k+l} and $dd^c \log T + \sum_{i=1}^N A_i\omega_i = 0$ on $\Delta^k \times \Delta^{*l}$. Then we have that:*

(1) *ω_i extend trivially to closed positive $(1,1)$-currents T_{ω_i} on Δ^{k+l}.*

(2) *There exist constants $a_i \in \mathbb{R}$ such that for any fixed $(z_1, \dots, \widehat{z_{k+i}}, \dots, z_{k+l}) \in \Delta^k \times \Delta^{*l-1}$, and for $i = 1, \dots, l$,*

$$\log T = a_i \log|z_{k+i}|^2 + O(\log(-\log|z_{k+i}|)), \quad z_{k+i} \to 0. \tag{3-11}$$

(3) *The following equation of currents on Δ^{k+l} holds:*

$$dd^c \log T + \sum_{i=1}^N A_i T_{\omega_i} = \sum_{i=1}^l a_i dd^c \log|z_{k+i}|^2. \tag{3-12}$$

Proof. The proof of (1) is obvious since all the ω_i are semipositive, d-closed and have Poincaré growth on $\Delta^k \times \Delta^{*l}$. For (2), by the dd^c-Poincaré lemma [Griffiths and Harris 1978, pp. 387], there exists a plurisubharmonic function $\varphi_i \in \mathrm{Psh}(\Delta^{k+l})$ with $dd^c\varphi_i = T_{\omega_i}$ for each $i = 1, \dots, N$. From $\omega_i \le C\omega_{\mathrm{P}}$ we infer that

$$dd^c\left\{2\pi C\left(\sum_{j=1}^k |z_j|^2 - \sum_{j=1}^l \log(-\log|z_{k+j}|^2)\right) - \varphi_i\right\} = CT_{\omega_{\mathrm{P}}} - T_{\omega_i} \ge 0, \tag{3-13}$$

where $T_{\omega_{\mathrm{P}}}$ is the trivial current extension of the Poincaré metric ω_{P} from $\Delta^k \times \Delta^{*l}$ to Δ^{k+l}. Therefore $2\pi C\big(\sum_{j=1}^k |z_j|^2 - \sum_{j=1}^l \log(-\log|z_{k+j}|^2)\big) - \varphi_i$ is a plurisubharmonic function on Δ^{k+l}; combining this with the fact $\varphi_i \in \mathrm{Psh}(\Delta^{k+l})$, it follows that there exist constants C_1, C_2, such that

$$2\pi C\Bigl(\sum_{j=1}^{k}|z_j|^2-\sum_{j=1}^{l}\log(-\log|z_{k+j}|^2)\Bigr)-C_1\le\varphi_i\le C_2\quad\text{on } \Delta^{k+l}\bigl(\tfrac12\bigr), \tag{3-14}$$

where $\Delta^{k+l}\bigl(\frac12\bigr):=\{z=(z_1,\dots,z_{k+l})\in\Delta^{k+l}:|z_j|<\frac12,\ j=1,\dots,k+l\}$, and if we set $P=\sum_{i=1}^{N}A_i\varphi_i$, we easily see that for any fixed $(z_1,\dots,\widehat{z_{k+i}},\dots,z_{k+l})\in\Delta^{k+l-1}\bigl(\frac12\bigr)$, $i\ge1$,

$$-C_3\log(-\log|z_{k+i}|)\le P\le C_3\log(-\log|z_{k+i}|) \tag{3-15}$$

for small $|z_{k+i}|$ and some constant $C_3=C_3(z_1,\dots,\widehat{z_{k+i}},\dots,z_{k+l})$. Now since $\log T+P$ is dd^c-closed and $P_{z_1,\dots,\widehat{z_{k+i}},\dots,z_{k+l}}(z_{k+i})=O(\log(-\log|z_{k+i}|))$, $z_{k+i}\to0$ for each $i=1,\dots,l$, we get that (2) follows from Lemma 3.3 (1). By applying Lemma 3.3 (2) with $f=0$, it follows that $\log T+P-a_1\log|z_{k+1}|^2$ extends to a dd^c-closed function on $\Delta^{k+1}\times\Delta^{*l-1}$. Next by applying Lemma 3.3 (2) with $f=a_1\log|z_{k+1}|^2$, it follows that $\log T+P-a_1\log|z_{k+1}|^2-a_2\log|z_{k+2}|^2$ extends to a dd^c-closed function on $\Delta^{k+2}\times\Delta^{*l-2}$. Continuing this way, (3) follows easily. □

3C. ***Current equation on the compactification of the moduli space of polarized Calabi–Yau manifolds.*** In this subsection we prove a current equation which extends the Fang–Lu formula to the compactification of the moduli space of polarized Calabi–Yau manifolds. Let $\mathcal{M}$ be a connected component of the moduli space of polarized Calabi–Yau manifolds. By [Viehweg 1995], $\mathcal{M}$ is quasiprojective. The smooth part $\mathcal{M}_{\text{reg}}$ of $\mathcal{M}$ admits a compactification $\overline{\mathcal{M}}$, where $\overline{\mathcal{M}}$ is a smooth projective manifold such that $D=\overline{\mathcal{M}}-\mathcal{M}_{\text{reg}}$ is a simple normal crossing divisor. On $\mathcal{M}_{\text{reg}}$, there exists a Kähler metric ω that is locally equivalent to the Poincaré metric ω_p [Zucker 1979]. More precisely, we can choose a local coordinate chart $(W;z_1,\dots,z_m)$ of $\overline{\mathcal{M}}$ such that

(1) D is given by $z_{k+1}\cdots z_m=0$;

(2) $W\cong\Delta^m$;

(3) $\mathcal{M}_{\text{reg}}\cap W\cong\Delta^k\times\Delta^{*m-k}$;

(4) $\omega\sim\omega_{\text{P}}=\sqrt{-1}\sum_{j=1}^{k}dz_j\wedge d\bar z_j+\sqrt{-1}\sum_{j=k+1}^{m}\dfrac{dz_j\wedge d\bar z_j}{|z_j|^2\log^2|z_j|^2}$.

It is known from [Fang and Lu 2005, Theorem A.1] that

$$\omega_{\text{H}^i}\le C\omega_{\text{P}},\quad i=0,\dots,n\quad\text{and}\quad\omega_{\text{WP}}\le C\omega_{\text{P}}, \tag{3-16}$$

for some positive constant C depending on the lower bound of $\text{Ric}(\omega)$, the dimension m of $\mathcal{M}$ and the dimension n of Calabi–Yau manifolds. Next, recall that on $\mathcal{M}$, we have the Fang–Lu formula [2005, Theorem 1.1]:

$$\sum_{i=1}^{n}(-1)^i\omega_{\mathrm{H}^i}-\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log T=\frac{\chi}{12}\omega_{\mathrm{WP}}, \tag{3-17}$$

where ω_{H^i} are the generalized Hodge metrics defined in Section 2D, T is the BCOV torsion with respect to the unique Ricci-flat metric in the integral cohomology class that defines the polarization and χ is the Euler number. Note that T is a smooth positive function on $\mathcal{M}$.

Because of (3-16), we may let $T_{\omega_{\mathrm{H}^i}}$, $T_{\Omega_{\mathrm{WP}}}$ be the trivial current extensions of ω_{H^i}, ω_{WP} from $\mathcal{M}_{\mathrm{reg}}$ to $\overline{\mathcal{M}}$. $\overline{\mathcal{M}}$ can be covered by coordinate charts $\{(W\cong\Delta^m;z_1,\dots,z_m)\}$ such that either $W\cap D=\varnothing$ or $W\cap\mathcal{M}_{\mathrm{reg}}\cong\Delta^k\times\Delta^{*m-k}$ and D is given by $z_{k+1}\cdots z_m=0$. In the second case, by (3-16), (3-17) and Proposition 3.4 (2), we know that if $\log T$ is locally L^1-integrable on $\overline{\mathcal{M}}$ then there exist constants $a_i\in\mathbb{R}$ such that for any fixed $(z_1,\dots,\widehat{z_i},\dots,z_{k+l})\in\Delta^k\times\Delta^{*l}$,

$$\log T=a_i\log|z_i|^2+O(\log(-\log|z_i|)),\quad z_i\to 0,\ \text{for all } i\geq k+1. \tag{3-18}$$

It’s easy to see that

$$a_i=\lim_{z_i\to 0}\frac{\log T}{\log|z_i|^2}. \tag{3-19}$$

Suppose now that $D=\overline{\mathcal{M}}-\mathcal{M}_{\mathrm{reg}}=\sum_v D_v$, where the sum is locally finite and each D_v is an irreducible hypersurface of $\overline{\mathcal{M}}$. We may associate a real number a_v with each component D_v of the divisor D by setting

$$a_v:=\lim_{p\to D_v}\frac{\log T}{\log|f_v|^2}, \tag{3-20}$$

where $p\in\mathcal{M}_{\mathrm{reg}}$ and f_v is a local defining function of the hypersurface D_v. This is well defined because of (3-18) and (3-19).

Theorem 3.5 (=Theorem 1.1). *Assume* $\log T$ *is locally* L^1*-integrable on* $\overline{\mathcal{M}}$. *Then the following equation of currents on* $\overline{\mathcal{M}}$ *holds*:

$$dd^c\log T+\sum_{i=1}^{n}(-1)^{i-1}T_{\omega_{\mathrm{H}^i}}+\frac{\chi}{12}T_{\omega_{\mathrm{WP}}}=\sum_v a_v[D_v], \tag{3-21}$$

where for each v, $[D_v]=\int_{D_v}$ *is the current associated to the hypersurface* D_v.

Proof. The statement is local. On a coordinate chart W with $W\cap D=\varnothing$, the equality (3-21) reduces to (3-17) because $\operatorname{supp}\big(\sum_v a_v[D_v]\big)\subseteq D$. So it’s enough to prove it on coordinate charts W with $W\cap D\neq\varnothing$. First, it follows immediately from (3-16), (3-17) and Proposition 3.4 (3) that

$$dd^c \log T + \sum_{i=1}^{n} (-1)^{i-1} T_{\omega_{\mathrm{H}^i}} + \frac{\chi}{12} T_{\omega_{\mathrm{WP}}} = \sum_{i=k+1}^{m} a_i dd^c \log |z_i|^2 \tag{3-22}$$

holds on coordinate chart $(W; z_1, \ldots, z_m)$ of $\overline{\mathcal{M}}$ such that D is given by $W \cong \Delta^m$, $z_{k+1} \cdots z_m = 0$ and $\mathcal{M}_{\mathrm{reg}} \cap W \cong \Delta^k \times \Delta^{*m-k}$. Now, (3-21) follows from (3-20), (3-22) and the Poincaré–Lelong formula. □

Since $\overline{\mathcal{M}}$ is projective, we may set ω_{FS} to be the Kähler form of the Fubini–Study metric restricted to $\overline{\mathcal{M}}$.

Corollary 3.6. *Under the same conditions as in Theorem 3.5 we have*

$$\int_{\overline{\mathcal{M}}} \left(\sum_{i=1}^{n} (-1)^{i-1} T_{\omega_{\mathrm{H}^i}} + \frac{\chi}{12} T_{\omega_{\mathrm{WP}}} \right) \wedge \omega_{\mathrm{FS}}^{m-1} = \sum_{v} a_v \deg D_v, \tag{3-23}$$

where $\deg D_v := \int_{\overline{\mathcal{M}}} c_1(D_v) \wedge \omega_{\mathrm{FS}}^{m-1}$ *and* $c_1(D_v)$ *is the first Chern class of the line bundle associated to* D_v.

Proof. We note that $dd^c \log T \wedge \omega_{\mathrm{FS}}^{m-1}$ is d-exact on $\overline{\mathcal{M}}$. By integrating the identity (3-21) over $\overline{\mathcal{M}}$, (3-23) follows immediately. □

3D. ***Current equation for polarized family of Calabi–Yau manifolds.*** In this subsection, we prove a similar current equation for a polarized family of Calabi–Yau manifolds.

Let $\mathcal{X}$ be a smooth projective variety of dimension $n+m$, let S be a smooth projective variety of dimension m and let $f : \mathcal{X} \to S$ be a surjective, flat holomorphic map with generic fiber a Calabi–Yau n-fold. Let $f^0 : \mathcal{X}^0 \to S^0$ be the smooth part of f, that is, each fiber $f^{-1}(s)$ is smooth for $s \in S^0$ and singular for $s \notin S^0$, $\mathcal{X}^0 = f^{-1}(S^0)$ and f^0 is the restriction of f to $\mathcal{X}^0$. With the polarization induced from the embedding of $\mathcal{X}$ into projective space, $f^0 : \mathcal{X}^0 \to S^0$ becomes a smooth polarized family of Calabi–Yau manifolds. We further assume that the discriminant locus $E := S \setminus S^0$ of f is a simple normal crossing divisor.

We make the following two observations. First, by pulling back the Fang–Lu formula via the induced holomorphic map $\psi : S^0 \to \mathcal{M}$, we know that (3-17) still holds on S^0:

$$\sum_{i=1}^{n} (-1)^i \omega_{\mathrm{H}^i} - \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log T = \frac{\chi}{12} \omega_{\mathrm{WP}}, \tag{3-24}$$

where, by our conventions (see Section 2E), we use the same symbols to denote the forms after pulling back. In particular, T is the BCOV torsion of the fibers of $f^0 : \mathcal{X}^0 \to S^0$. Second, from the proof of [Fang and Lu 2005, Theorem A.1], we easily see that (3-16) still holds near E. So we can proceed as in Theorem 3.5 and get the following:

Theorem 3.7 (=Theorem 1.2). *Assume* $\log T$ *is locally* L^1*-integrable on* S. *Let* $E=\sum_v E_v$ *be the irreducible decomposition of* E *and set*

$$a_v := \lim_{p\to E_v} \frac{\log T}{\log|f_v|^2}, \tag{3-25}$$

where $p\in S^0$ *and* f_v *is a local defining function of the hypersurface* E_v. *Then the following equation of currents on* S *holds*:

$$dd^c \log T + \sum_{i=1}^{n}(-1)^{i-1} T_{\omega_{\mathrm{H}^i}} + \frac{\chi}{12} T_{\omega_{\mathrm{WP}}} = \sum_v a_v[E_v], \tag{3-26}$$

where the currents $T_{\omega_{\mathrm{WP}}}$ *and* $T_{\omega_{\mathrm{H}^i}}$ *are the trivial extensions of* ω_{WP} *and* ω_{H^i} *from* S^0 *to* S, $[E_v]=\int_{E_v}$ *is the current associated to the hypersurface* E_v *and* χ *is the topological Euler number of a general fiber of* f.

Proof. The proof is almost the same as Theorem 3.5 and is omitted. □

Remark 3.8. For $n=3$, the BCOV torsion T here differs from the BCOV invariant τ in [Fang et al. 2008] by a positive constant depending only on the polarization [Yoshikawa 2017, pp. 284]; it follows that the asymptotic value a_v is the same for T and τ.

4. Applications to polarized families of Calabi–Yau manifolds

Now we want to integrate (3-26) over S to get global numerical results, but before that we make the following useful observations as in [Liu and Xia 2019].

Firstly, if we assume the local monodromies of the polarized family $f^0:\mathcal{X}^0\to S^0$ (with the polarization induced from the projective embedding of $\mathcal{X}$) are all unipotent, there is then the canonical Deligne extensions $P\mathcal{H}_e^{p,q}$ of the Hodge bundles $P\mathcal{H}^{p,q}$ from S^0 to S. Recall that the Hodge bundles are defined by $P\mathcal{H}^{p,q} := PR^q\pi_*\Omega^p_{\mathcal{X}^0/S^0}$ and $(PR^q\pi_*\Omega^p_{\mathcal{X}^0/S^0})_t = PH^q(X_t,\Omega^p_{X_t})$ for any $t\in S$, where $PH^q(X_t,\Omega^p_{X_t})$ is the primitive cohomology of $(X_t,[\omega_t])$. The first Chern forms of $P\mathcal{H}^{p,q}$ are related to the Weil–Petersson form and generalized Hodge forms ([Tian 1987, Theorem 2; Fang and Lu 2005, Proposition 2.8]):

$$\begin{aligned} \omega_{\mathrm{WP}} &= c_1(P\mathcal{H}^{n,0}),\\ \omega_{\mathrm{H}^i} &= \sum_{0\le p\le i} p\big(c_1(P\mathcal{H}^{p,i-p}) + c_1(P\mathcal{H}^{p-1,i-p-1}) + \cdots\big). \end{aligned} \tag{4-1}$$

Secondly, it follows from the work of Cattani, Kaplan and Schmid [Cattani et al. 1986, Corollary 5.23] that:

(i) The forms $c_1(P\mathcal{H}^{p,q})$ are integrable and they define closed, (1, 1)-currents $\overline{c_1(P\mathcal{H}^{p,q})}$ (trivial extensions) on the completion S.

(ii) $\overline{c_1(P\mathcal{H}^{p,q})} = c_1(P\mathcal{H}_e^{p,q})$ in $H^2_{\mathrm{DR}}(S)$.

Hence we have:

Lemma 4.1. *As cohomology classes in $H^2_{\mathrm{DR}}(S)$, we have*

$$T_{\omega_{\mathrm{WP}}} = \overline{c_1(P\mathcal{H}^{n,0})} = c_1(P\mathcal{H}_e^{n,0}), \tag{4-2}$$

and

$$T_{\omega_{\mathrm{H}^i}} = \sum_{0\le p\le i} p\big(\overline{c_1(P\mathcal{H}^{p,i-p})} + \overline{c_1(P\mathcal{H}^{p-1,i-p-1})} + \cdots\big) \tag{4-3}$$

$$= \sum_{0\le p\le i} p\big(c_1(P\mathcal{H}_e^{p,i-p}) + c_1(P\mathcal{H}_e^{p-1,i-p-1}) + \cdots\big). \tag{4-4}$$

Proof. This follows from (4-1) and (ii) above. □

Now we can get the following:

Corollary 4.2. *We assume the same conditions as in Theorem 3.7. Suppose that the local monodromies of the polarized family $f^0 : \mathcal{X}^0 \to S^0$ are all unipotent. Then*

$$\sum_{i=1}^{n}(-1)^{i-1}\sum_{0\le p\le i} p(\deg P\mathcal{H}_e^{p,i-p}+\deg P\mathcal{H}_e^{p-1,i-p-1}+\cdots)+\frac{\chi}{12}\deg P\mathcal{H}_e^{n,0} = \sum_v a_v \deg E_v. \tag{4-5}$$

In particular, for polarized family of Calabi–Yau 3*-folds, we have*

$$(\chi+36)\deg P\mathcal{H}_e^{3,0} + 12\deg P\mathcal{H}_e^{2,1} = 12\sum_v a_v \deg E_v, \tag{4-6}$$

and for polarized family of Calabi–Yau 4*-folds, we have*

$$(\chi-48)\deg P\mathcal{H}_e^{4,0} - 24\deg P\mathcal{H}_e^{3,1} + 12\deg P\mathcal{H}_e^{2,1} = 12\sum_v a_v \deg E_v. \tag{4-7}$$

Proof. By Theorem 3.7 and Lemma 4.1, we have the following equality of cohomology classes in $H^2_{\mathrm{DR}}(S)$:

$$dd^c\log T + \sum_{i=1}^{n}(-1)^{i-1}\sum_{0\le p\le i} p(c_1(P\mathcal{H}_e^{p,i-p})+c_1(P\mathcal{H}_e^{p-1,i-p-1})+\cdots)+\frac{\chi}{12}c_1(P\mathcal{H}_e^{n,0}) = \sum_v a_v[E_v]. \tag{4-8}$$

Since S is projective, let ω_{FS} be the restriction of the Kähler form of the Fubini–Study metric to S, by integration on S, and noting that $dd^c \log T \wedge \omega_{\mathrm{FS}}^{m-1}$ is d-exact, (4-5) follows. □

Now assume $m = 1$, that is, S is a compact Riemann surface. In this case, $E = \{s_v\}$ is a set of points and we have $\deg s_v = 1$, for all v. We see that for any n, $12 \sum_v a_v$ is an integer which is not clear from the definition. In fact, for Calabi–Yau 3-folds, each individual a_v is known to be a rational number [Yoshikawa 2015, Theorem 0.1]. On the other hand, by applying the Grothendieck–Riemann–Roch formula to the family $f : \mathcal{X} \to S$ (see [Green et al. 2009] for $n = 3$), the sum $12 \sum_v a_v$ can be given a geometrical meaning. For example, if the family f has only semistable singular fibers and is relatively minimal, then $12 \sum_v a_v$ is equal to an expression involving the Euler characteristic number and intersection data of the singular fibers. In [Liu and Xia 2019], we conjectured that each individual a_v should be determined only by the data of the corresponding singular fiber. This statement is true for ordinary double point singularities. Indeed, Yoshikawa [2015, Theorem 5.2] proved that $a_v = \frac{1}{6} \#\mathrm{Sing}\, X_{s_v}$, where $\#\mathrm{Sing}\, X_{s_v}$ is the number of ordinary double points on the singular fiber X_{s_v}. In this direction, Eriksson, Freixas and Mourougane [Eriksson et al. 2018b] obtained interesting results under the assumption that the total space $\mathcal{X}$ has trivial canonical bundle which is equivalent to f being relatively minimal because they work locally.

We now use the identity (4-7) to prove an Euler number bound for a polarized family of Calabi–Yau 4-folds over a compact Riemann surface:

Corollary 4.3. *We assume the same conditions as in Theorem 3.7 and we let* $n = 4, m = 1$. *If* $f : \mathcal{X} \to S$ *is not isotrivial and the local monodromies of the polarized family* $f^0 : \mathcal{X}^0 \to S^0$ *are all unipotent, then*

$$1+\frac{\sum_v a_v-\deg P\mathcal{H}_e^{2,1}}{2\deg P\mathcal{H}_e^{4,0}} \le \frac{\chi}{24} \le \frac{2\pi(2g-2+s)(h^{3,1}+4)+\sum_v a_v-\deg P\mathcal{H}_e^{2,1}}{2\deg P\mathcal{H}_e^{4,0}}. \tag{4-9}$$

Proof. We have the following inequality [Liu and Xia 2019, Theorem 5.1]:

$$0 < \deg P\mathcal{H}_e^{4,0} \le 2\deg P\mathcal{H}_e^{4,0} + \deg P\mathcal{H}_e^{3,1} \le \pi(2g-2+s)(h^{3,1}+4). \tag{4-10}$$

Now (4-9) follows from (4-7) and (4-10). □

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CLASSIFICATION OF GRADIENT EXPANDING AND STEADY RICCI SOLITONS

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In this paper, we prove some classification theorems for gradient expanding and steady Ricci solitons. We show that a complete noncompact radially Ricci flat (i.e., $\mathrm{Ric}(\nabla f, \nabla f) = 0$) gradient expanding Ricci soliton with nonnegative Ricci curvature is a finite quotient of $\mathbb{R}^n$. Moreover, we prove that a complete noncompact gradient expanding Ricci soliton with $\mathrm{Ric} \geq 0$ and $\mathrm{div}^4\mathrm{Rm} = 0$ is a finite quotient of $\mathbb{R}^n$. For a nontrivial complete noncompact radially Ricci flat (i.e., $\mathrm{Ric}(\nabla f, \nabla f) = 0$) gradient steady Ricci soliton with $\int |\nabla R|^2 e^{\alpha f} < +\infty$ for some $\alpha \in \mathbb{R}$, we show that it is Einstein with vanishing Ricci curvature or a quotient of $\mathbb{R}^n$ or of the product $\mathbb{R}^k \times N^{n-k}$ with $1 \leq k \leq n-1$, where N is Einstein with vanishing Ricci curvature.

1. Introduction

A complete Riemannian manifold (M^n, g) is called a gradient Ricci soliton if there exists a smooth function f on M^n such that the Ricci tensor Ric of the metric g satisfies the equation

$$\mathrm{Ric} + \mathrm{Hess}\, f = \lambda g \tag{1-1}$$

for some constant λ. For $\lambda > 0$ the Ricci soliton is shrinking, for $\lambda = 0$ it is steady and for $\lambda < 0$ expanding.

An Einstein manifold with constant potential function is called a *trivial* gradient Ricci soliton. When $f = \frac{\lambda}{2}|x|^2$ on $\mathbb{R}^n$, $\mathrm{Hess}\, f = \lambda g$ and therefore yields a gradient soliton where the background metric is flat. This example is called a *Gaussian* soliton.

Taking a product $N \times \mathbb{R}^k$ with N being Einstein with Einstein constant λ and $f = \frac{\lambda}{2}|x|^2$ on $\mathbb{R}^k$ yields a mixed gradient soliton. A gradient soliton is *rigid* if it is of the type $N \times_\Gamma \mathbb{R}^k$, where Γ acts freely on N and by orthogonal transformations on $\mathbb{R}^k$ (no translational components).

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Throughout the paper, for gradient expanding Ricci solitons we normalize the constant $\lambda = -\frac{1}{2}$ so that (1-1) becomes

$$\mathrm{Ric} + \mathrm{Hess}\, f = -\tfrac{1}{2} g. \tag{1-2}$$

In this paper, we shall focus our attention on gradient expanding and steady Ricci solitons (M^n, g, f). It turns out that a compact gradient steady or expanding Ricci soliton is necessarily an Einstein metric (see [Hamilton 1995; Ivey 1994]).

Some properties of gradient expanding Ricci solitons have been proved in recent years. G. Catino, P. Mastrolia and D. D. Monticelli [Catino et al. 2017] showed that a gradient expanding Ricci soliton with nonnegative Ricci curvature and fourth order divergence-free Weyl tensor has harmonic Weyl curvature. H. D. Cao et al. [2014] estimated the potential function f of a complete noncompact gradient expanding soliton with nonnegative Ricci curvature, that is $-f$ is of quadratic growth. They also showed that the condition of $\mathrm{Ric} \geq 0$ can be relaxed to $\mathrm{Rc} \geq -\big(\frac{1}{2} - \varepsilon\big) g$ for any small $\varepsilon > 0$.

A 3-dimensional complete gradient expanding Ricci soliton with constant scalar curvature is classified and indeed it is a finite quotient of $\mathbb{R}^3$, $\mathbb{H}^2 \times \mathbb{R}$, and $\mathbb{H}^3$ (see [Petersen and Wylie 2010]). For a 3-dimensional complete gradient expanding Ricci soliton with nonnegative Ricci curvature and the scalar curvature $R \in L^1(M^3)$, Catino, Mastrolia and Monticelli [Catino et al. 2016] showed that it is isometric to a quotient of the Gaussian soliton $\mathbb{R}^3$. Moreover, Cao et al. [2014] proved that a 3-dimensional complete expanding gradient Ricci solitons with divergence-free Bach tensor and nonnegative Ricci curvature is rotationally symmetric. For the n-dimensional case, they also proved that a complete Bach-flat gradient expanding Ricci soliton with nonnegative Ricci curvature is rotationally symmetric.

Some properties of gradient steady Ricci solitons are as follows: P. Petersen and W. Wylie [2009] proved that a gradient steady soliton whose scalar curvature achieves its minimum is Ricci flat. Moreover, if f is not constant then it is a product of a Ricci flat manifold with $\mathbb{R}$. O. Munteanu and N. Sesum [2013] showed that any gradient steady Ricci soliton has at least linear volume growth and at most growth rate of $e^{\sqrt{r}}$. Moreover, they proved that a gradient steady Ricci soliton has at most one nonparabolic end. P. Wu [2013] proved that the infimum of the potential function of a gradient steady Ricci soliton must decay linearly.

Cao et al. [2014] proved that a 3-dimensional gradient steady Ricci soliton with divergence-free Bach tensor is either flat or isometric to the Bryant soliton up to a scaling factor. Catino, Mastrolia and Monticelli [Catino et al. 2016] proved that a 3-dimensional complete gradient steady Ricci soliton with

$$\liminf_{r \to +\infty} \frac{1}{r} \int_{B_r(O)} R = 0$$

is isometric to a quotient of $\mathbb{R}^3$ or $\mathbb{R} \times \Sigma^2$, where Σ^2 is the cigar soliton. Under the condition of κ-noncollapsed, S. Brendle [2013] proved that a 3-dimensional complete nonflat gradient steady Ricci soliton is isometric to the Bryant soliton up to scaling.

In higher dimensions, Cao and Q. Chen [2012] proved that an n-dimensional complete noncompact locally conformally flat gradient steady Ricci soliton is either flat or isometric to the Bryant soliton. Moreover, Cao et al. [2014] showed that a Bach-flat gradient steady Ricci soliton with positive Ricci curvature such that the scalar curvature R attains its maximum at some interior point is isometric to the Bryant soliton up to a scaling factor. Brendle [2014] proved that a steady gradient Ricci soliton of dimension n ($n \geq 4$) is rotationally symmetric if it has positive sectional curvature and is asymptotically cylindrical. In particular, it is isometric to the Bryant soliton up to scaling.

The aim of this paper is to obtain some classification theorems of gradient expanding and steady Ricci solitons. In order to state our results precisely, we introduce the following definitions for the Riemannian curvature:

$$(\operatorname{div} \mathrm{Rm})_{ijk} := \nabla_l R_{ijkl}, \qquad (\operatorname{div}^2 \mathrm{Rm})_{ik} := \nabla_j \nabla_l R_{ijkl},$$
$$(\operatorname{div}^3 \mathrm{Rm})_{i} := \nabla_k \nabla_j \nabla_l R_{ijkl}, \qquad \operatorname{div}^4 \mathrm{Rm} := \nabla_i \nabla_k \nabla_j \nabla_l R_{ijkl}.$$

The main results of this paper are the following theorems for gradient expanding and steady Ricci solitons.

For a complete noncompact gradient expanding Ricci soliton with nonnegative Ricci curvature, we have the following classification theorem.

Theorem 1.1. *Let (M^n, g, f) be a complete noncompact gradient expanding Ricci soliton with nonnegative Ricci curvature. Then, under any of the additional conditions*

(i) *(M^n, g, f) is radially Ricci flat, or*

(ii) $\operatorname{div}^4 \mathrm{Rm} = 0$, *or*

(iii) $\operatorname{tr} \operatorname{div}^2 \mathrm{Rm} = 0$,

(M^n, g, f) is a finite quotient of the Gaussian expanding soliton $\mathbb{R}^n$.

For a nontrivial complete noncompact gradient steady Ricci soliton, we will prove the following classification theorem.

Theorem 1.2. *Let (M^n, g, f) be a nontrivial complete noncompact gradient steady Ricci soliton. Then, under any of the additional conditions*

(i) *(M^n, g, f) is radially Ricci flat and $\int |\nabla R| e^{\alpha f} < +\infty$ for some $\alpha \in \mathbb{R}$, or*

(ii) $\operatorname{div}^4 \mathrm{Rm} = 0$ *and $\int |\mathrm{Rm}| e^{\alpha f} < +\infty$ for some $\alpha \neq 0$, or*

(iii) $\operatorname{tr} \operatorname{div}^2 \mathrm{Rm} = 0$ *and $\int |\nabla R| e^{\alpha f} < +\infty$ for some $\alpha \neq 0$,*

(M^n, g, f) is Einstein with vanishing Ricci curvature or a quotient of $\mathbb{R}^n$ or of the product $\mathbb{R}^k \times N^{n-k}$ with $1 \le k \le n-1$, where N is Einstein with vanishing Ricci curvature.

Remark 1.3. As will be clear from the proof, the scalar assumptions on the vanishing of $\operatorname{div}^4 \mathrm{Rm}$ in Theorem 1.1 and Theorem 1.2 can be trivially relaxed to a (suitable) inequality. The condition of $\operatorname{div}^4 \mathrm{Rm} = 0$ in Theorem 1.1 can be relaxed to $\int \operatorname{div}^4 \mathrm{Rm}\, e^f \le 0$. Moreover, the condition of $\operatorname{div}^4 \mathrm{Rm} = 0$ in Theorem 1.2 can be relaxed to $\int \operatorname{div}^4 \mathrm{Rm}\, e^{\alpha f} \le 0$ for some $\alpha \ne 0$.

The rest of this paper is organized as follows. In Section 2, we recall some background material which will be needed in the proof of the main theorems. In Section 3, we prove an integral identity for complete noncompact gradient expanding Ricci solitons with nonnegative Ricci curvature. In Section 4, we finish the proof of Theorem 1.1. In Section 5, we deal with Theorem 1.2. In the Appendix, we show that a complete noncompact gradient expanding or steady Ricci soliton with $\operatorname{div}^3 \mathrm{Rm}(\nabla f) = 0$ is rigid.

2. Preliminaries

We recall the following formulas for gradient Ricci solitons.

Proposition 2.1 [Yang and Zhang 2017]. *Let (M^n, g, f) be a gradient Ricci soliton. We have the following identities*:

$$(\operatorname{div}^2 \mathrm{Rm})_{ik} = 2\lambda R_{ik} + \nabla_l R_{ik} \nabla_l f - \tfrac{1}{2}\nabla_i \nabla_k R - R_{ik}^2 - R_{ijkl} R_{jl}, \tag{2-1}$$

$$(\operatorname{div}^3 \mathrm{Rm})_i = -R_{ijkl} \nabla_k R_{jl}, \tag{2-2}$$

$$(\operatorname{div}^3 \mathrm{Rm})(\nabla f) = -\tfrac{1}{2} |\operatorname{div} \mathrm{Rm}|^2. \tag{2-3}$$

Next we list the results that will be needed in the proof of the main theorems.

Lemma 2.2 [Cao et al. 2014]. *Let (M^n, g_{ij}, f) $(n \ge 3)$ be a complete noncompact gradient expanding soliton with nonnegative Ricci curvature* $\mathrm{Rc} \ge 0$. *Then there exist some constants $c_1 > 0$ and $c_2 > 0$ such that the potential function f satisfies the estimates*

$$\tfrac{1}{4}\big(r(x) - c_1\big)^2 - c_2 \le -f(x) \le \tfrac{1}{4}\big(r(x) + 2\sqrt{-f(O)}\,\big)^2, \tag{2-4}$$

where $r(x)$ is the distance function from any fixed base point in M^n. In particular, f is a strictly concave exhaustion function achieving its maximum at some interior point O, which we take as the base point, and the underlying manifold M^n is diffeomorphic to $\mathbb{R}^n$.

Lemma 2.3 [Petersen and Wylie 2009]. *The following conditions for a gradient expanding soliton* $\mathrm{Ric} + \mathrm{Hess}\, f = \lambda g$ *all imply that the metric is radially flat and has constant scalar curvature.*

(1) *The scalar curvature is constant and* $\sec(E, \nabla f) \le 0$.

(2) *The scalar curvature is constant and* $\lambda g \le \mathrm{Ric} \le 0$.

(3) *The curvature tensor is harmonic.*

(4) $\mathrm{Ric} \le 0$ *and* $\sec(E, \nabla f) = 0$.

Lemma 2.4 [Petersen and Wylie 2009]. *A gradient soliton is rigid if and only if it has constant scalar curvature and is radially flat*, *that is*, $\sec(E, \nabla f) = 0$.

3. An integral identity for gradient expanding Ricci solitons

We prove a useful integral identity (see Lemma 3.2 below), which will be needed in the proof of Theorem 1.1. The first step is to obtain the following proposition.

Proposition 3.1. *Let* (M^n, g, f) *be a complete noncompact gradient expanding Ricci soliton with* $\mathrm{Ric} \ge 0$; *then*

$$\left|\int \nabla_{\nabla f} R e^f\right| < +\infty. \tag{3-1}$$

Proof. Since $\mathrm{Ric} \ge 0$, $|\mathrm{Ric}| \le R$ and the Bishop comparison theorem implies that the volume of a geodesic ball is at most Euclidean growth. By Lemma 2.2, $-f$ is of quadratic growth. Note that $R + |\nabla f|^2 + f = \mathrm{Const.}$, $|\mathrm{Ric}||\nabla f|^2$ of at most polynomial growth. Therefore, we have

$$\left|\int \nabla_{\nabla f} R e^f\right| = 2\left|\int \mathrm{Ric}(\nabla f, \nabla f) e^f\right| \le 2\int |\mathrm{Ric}||\nabla f|^2 e^f < +\infty. \qquad \square$$

Lemma 3.2. *Let* (M^n, g, f) *be a complete noncompact gradient expanding Ricci soliton with* $\mathrm{Ric} \ge 0$; *then we have*

$$\int \nabla_{\nabla f} R e^f = -\int \Delta R e^f. \tag{3-2}$$

Proof. Let $\phi(t) = 1$ on $(0, s)$, $\phi(t) = \frac{2s-t}{s}$ on $(s, 2s)$ and $\phi \equiv 0$ on $[2s, +\infty)$ for any fixed $s > 0$. Since $\mathrm{Ric} \ge 0$, Lemma 2.2 implies that $-f$ is of quadratic growth. Therefore, $\phi(-f)$ has compact support for any fixed $s > 0$. Define the compact set $D(s) := \{x \in M^n \mid -f(x) \le s\}$.

By direct computation, we have

$$\begin{aligned}\int \nabla_{\nabla f} R \phi(-f) e^f &= \int \langle \nabla R, \nabla e^f\rangle \phi(-f) \\ &= -\int \Delta R \phi(-f) e^f + \int \langle \nabla R, \nabla f\rangle \phi'(-f) e^f \\ &= -\int \Delta R \phi(-f) e^f - \frac{1}{s}\int_{D(2s)\setminus D(s)} \nabla_{\nabla f} R e^f.\end{aligned} \tag{3-3}$$

It follows from Proposition 3.1 that

$$\lim_{s\to+\infty}\frac{1}{s}\int_{D(2s)\setminus D(s)}\nabla_{\nabla f}Re^{f}=0.$$

Therefore, (3-2) follows by taking $s\to+\infty$ in (3-3). □

4. Proof of the main result for gradient expanding Ricci solitons

In this section, we prove Theorem 1.1.

Theorem 4.1. *Let (M^n, g, f) be a complete noncompact gradient expanding Ricci soliton with $\mathrm{Ric}\geq 0$ and $\mathrm{Ric}(\nabla f,\nabla f)=0$. Then (M^n, g, f) is a finite quotient of the Gaussian expanding soliton $\mathbb{R}^n$.*

Proof. Since $\nabla_{\nabla f}R=2\,\mathrm{Ric}(\nabla f,\nabla f)=0$, Lemma 3.2 implies that

$$\int \Delta Re^{f}=0. \tag{4-1}$$

Noting that $\Delta_f R=-R-2|\mathrm{Ric}|^2$ and $\Delta_f R=\Delta R-\nabla_{\nabla f}R=\Delta R$, we have

$$\Delta R=-R-2|\mathrm{Ric}|^2. \tag{4-2}$$

Applying (4-2) to (4-1), we obtain

$$\int|\mathrm{Ric}|^2e^{f}=-\frac{1}{2}\int Re^{f}. \tag{4-3}$$

Since $\mathrm{Ric}\geq 0$, $R\geq 0$. From (4-3), we know that $|\mathrm{Ric}|=0$ on M^n, i.e., (M^n, g, f) has vanishing Ricci curvature.

Hence, condition (2) in Lemma 2.3 holds. It follows from Lemma 2.3 that (M^n, g, f) is radially flat and has constant scalar curvature. By Lemma 2.4, we have (M^n, g, f) is rigid. Since $\mathrm{Ric}=0$ on M^n, (M^n, g, f) is a finite quotient of the Gaussian expanding soliton $\mathbb{R}^n$. □

Theorem 4.2. *Let (M^n, g, f) be a complete noncompact gradient expanding Ricci soliton. If $\mathrm{div}^4\,\mathrm{Rm}=0$ and $\mathrm{Ric}\geq 0$, then (M^n, g, f) is a finite quotient of the Gaussian expanding soliton $\mathbb{R}^n$.*

Proof. Let $\phi(t)=1$ on $[0,s]$, $\phi(t)=\frac{2s-t}{s}$ on $(s,2s)$ and $\phi\equiv 0$ on $[2s,+\infty)$ for any fixed $s>0$. Since $\mathrm{Ric}\geq 0$, Lemma 2.2 implies that $-f$ is of quadratic growth. Therefore, $\phi(-f)$ has compact support for any fixed $s>0$. Define the compact set $D(s):=\{x\in M^n \mid -f(x)\leq s\}$.

Integrating by parts, we have

$$\begin{aligned}\int \operatorname{div}^4 \mathrm{Rm}\,\phi(-f)e^f &= -\int \operatorname{div}^3 \mathrm{Rm}(\nabla f)\phi(-f)e^f + \int \operatorname{div}^3 Rm(\nabla f)\phi'(-f)e^f \\ &= \frac{1}{2}\int |\operatorname{div}\mathrm{Rm}|^2\phi(-f)e^f + \frac{1}{2s}\int_{D(2s)\setminus D(s)} |\operatorname{div}\mathrm{Rm}|^2 e^f,\end{aligned} \tag{4-4}$$

where we used (2-3) in the second equality.

Since $\operatorname{div}^4 \mathrm{Rm} = 0$, $\phi(-f) \geq 0$ on M^n, it follows from (4-4) that

$$\int |\operatorname{div}\mathrm{Rm}|^2\phi(-f)e^f = 0. \tag{4-5}$$

Note that $\phi(-f) = 1$ on the compact set $D(s) := \{x \in M^n \mid -f(x) \leq s\}$. From (4-5), we know that

$$\int_{D(s)} |\operatorname{div}\mathrm{Rm}|^2 e^f = 0. \tag{4-6}$$

Taking $s \to +\infty$ in (4-6), we have

$$\int |\operatorname{div}\mathrm{Rm}|^2 e^f = 0,$$

that is, $|\operatorname{div}\mathrm{Rm}| = 0$ on M^n.

Note that

$$\begin{aligned} 0 = \nabla_l R_{ijkl} &= \nabla_j R_{ik} - \nabla_i R_{jk} \\ &= -\nabla_j\nabla_i\nabla_k f + \nabla_i\nabla_j\nabla_k f \\ &= R_{ijkl}\nabla_l f. \end{aligned} \tag{4-7}$$

It follows that M^n is radially flat.

Tracing $\operatorname{div}\mathrm{Rm}$, we have

$$0 = g^{ik}\nabla_l R_{ijkl} = \nabla_l R_{jl} = \tfrac{1}{2}\nabla_j R, \tag{4-8}$$

that is, M^n has a constant scalar curvature.

By Lemma 2.4, we have (M^n, g, f) is rigid. Since $\mathrm{Ric} \geq 0$, we conclude that (M^n, g, f) is a finite quotient of the Gaussian expanding soliton $\mathbb{R}^n$. □

Theorem 4.3. *Let (M^n, g, f) be a complete noncompact gradient expanding Ricci soliton with* $\mathrm{Ric} \geq 0$ *and* $\operatorname{tr}\operatorname{div}^2 \mathrm{Rm} = 0$ *then (M^n, g, f) is a finite quotient of the Gaussian expanding soliton $\mathbb{R}^n$.*

Proof. Tracing (2-1), we have

$$0 = g^{ik}(\operatorname{div}^2 \mathrm{Rm})_{ik} = -R + \nabla_{\nabla f} R - \tfrac{1}{2}\Delta R - 2|\mathrm{Ric}|^2 \tag{4-9}$$
$$= -R + \tfrac{1}{2}\Delta R - \Delta_f R - 2|\mathrm{Ric}|^2$$
$$= \tfrac{1}{2}\Delta R,$$

where we used the fact that $\Delta_f R = -R - 2|\mathrm{Ric}|^2$.

It follows that

$$\Delta R = 0. \tag{4-10}$$

Applying (4-10) to Lemma 3.2, we obtain that $\int \Delta R e^f = \int \nabla_{\nabla f} R e^f = 0$. Noting that $\Delta R - \nabla_{\nabla f} R = \Delta_f R = -R - 2|\mathrm{Ric}|^2$, we have

$$\int |\mathrm{Ric}|^2 e^f = -\frac{1}{2}\int R e^f \leq 0. \tag{4-11}$$

It follows that $|\mathrm{Ric}| = 0$ on M^n, i.e., (M^n, g, f) has vanishing Ricci curvature.

Hence, condition (2) in Lemma 2.3 holds. It follows from Lemma 2.3 that (M^n, g, f) is radially flat and has constant scalar curvature. By Lemma 2.4, we have that (M^n, g, f) is rigid. Since $\mathrm{Ric} = 0$ on M^n, (M^n, g, f) is a finite quotient of the Gaussian expanding soliton $\mathbb{R}^n$. □

Theorem 1.1 follows directly from Theorems 4.1–4.3.

5. Proof of the main result for gradient steady Ricci solitons

In this section, we prove Theorem 1.2.

Theorem 5.1. *Let (M^n, g, f) be a nontrivial complete noncompact radially Ricci flat (i.e., $\mathrm{Ric}(\nabla f, \nabla f) = 0$) gradient steady Ricci soliton with $\int |\nabla R| e^{\alpha f} < +\infty$ for some $\alpha \in \mathbb{R}$. Then (M^n, g, f) is Einstein with vanishing Ricci curvature or a quotient of $\mathbb{R}^n$ or of the product $\mathbb{R}^k \times N^{n-k}$ with $1 \leq k \leq n-1$, where N is Einstein with vanishing Ricci curvature.*

Proof. Let B_r be a geodesic ball with radius r and let ν be the unit outward normal vector field to ∂B_r. Integrating by parts, we obtain

$$(\alpha+1)\int_{B_r} \nabla_{\nabla f} R e^{\alpha f} = \int_{B_r} \langle \nabla R, \nabla e^{\alpha f}\rangle + \int_{B_r} \nabla_{\nabla f} R e^{\alpha f} \tag{5-1}$$
$$= \int_{\partial B_r} \nabla_\nu R e^{\alpha f} - \int_{B_r} \Delta R e^{\alpha f} + \int_{B_r} \nabla_{\nabla f} R e^{\alpha f}$$
$$= \int_{\partial B_r} \nabla_\nu R e^{\alpha f} - \int_{B_r} \Delta_f R e^{\alpha f}$$
$$= \int_{\partial B_r} \nabla_\nu R e^{\alpha f} + 2\int_{B_r} |\mathrm{Ric}|^2 e^{\alpha f},$$

where we used the fact that $\Delta_f R = -2|\mathrm{Ric}|^2$.

Note that $\nabla_{\nabla f} R = 2\,\mathrm{Ric}(\nabla f, \nabla f) = 0$. It follows from (5-1) that

$$\int_{B_r} |\mathrm{Ric}|^2 e^{\alpha f} = -\int_{\partial B_r} \nabla_\nu R e^{\alpha f} \le \int_{\partial B_r} |\nabla R| e^{\alpha f}. \tag{5-2}$$

Since $\int |\nabla R| e^{\alpha f} < +\infty$, we have

$$\lim_{r\to+\infty} \int_{\partial B_r} |\nabla R| e^{\alpha f} = 0. \tag{5-3}$$

Taking $r \to +\infty$ in (5-2) and using (5-3), we obtain

$$\int |\mathrm{Ric}|^2 e^{\alpha f} = 0, \tag{5-4}$$

that is, $|\mathrm{Ric}| = 0$ on M^n. It follows that (M^n, g, f) has vanishing scalar curvature.

Moreover, we have

$$\begin{aligned} R_{ijkl}\nabla_l f &= \nabla_i\nabla_j\nabla_k f - \nabla_j\nabla_i\nabla_k f \\ &= -\nabla_i R_{jk} + \nabla_j R_{ik} = 0, \end{aligned} \tag{5-5}$$

where we used (1-1) in the second equality and $\mathrm{Ric} = 0$ on M^n. It follows that $\sec(E, \nabla f) = R_{ijkl} E_i E_k \nabla_j f \nabla_l f = 0$, i.e., (M^n, g, f) is radially flat.

Since M has vanishing scalar curvature and is radially flat, Lemma 2.4 implies (M^n, g, f) is rigid. To conclude, (M^n, g, f) is Einstein with vanishing Ricci curvature or a quotient of $\mathbb{R}^n$ or of the product $\mathbb{R}^k \times N^{n-k}$ with $1 \le k \le n-1$, where N is Einstein with vanishing Ricci curvature. □

Theorem 5.2. *Let (M^n, g, f) be a nontrivial complete noncompact gradient steady Ricci soliton with $\int |\mathrm{Rm}| e^{\alpha f} < +\infty$ for some $\alpha \ne 0$. If in addition* $\mathrm{div}^4\,\mathrm{Rm} = 0$, *then it is Einstein with vanishing Ricci curvature or a quotient of $\mathbb{R}^n$ or of the product $\mathbb{R}^k \times N^{n-k}$ with $1 \le k \le n-1$, where N is Einstein with vanishing Ricci curvature.*

Proof. Let B_r be a geodesic ball with radius r and let ν be the outward unit normal vector field to ∂B_r. Integrating by parts, we obtain

$$\begin{aligned} \int_{B_r} \mathrm{div}^4\,\mathrm{Rm}\, e^{\alpha f} &\equiv \int_{B_r} \nabla_i\nabla_k\nabla_j\nabla_l R_{ijkl} e^{\alpha f} \\ &= \int_{\partial B_r} \nabla_k\nabla_j\nabla_l R_{ijkl}\nu_i e^{\alpha f} - \alpha \int_{B_r} \nabla_k\nabla_j\nabla_l R_{ijkl}\nabla_i f e^{\alpha f} \\ &= -\int_{\partial B_r} R_{ijkl}\nabla_k R_{jl}\nu_i e^{\alpha f} + \frac{\alpha}{2}\int_{B_r} |\mathrm{div}\,\mathrm{Rm}|^2 e^{\alpha f}, \end{aligned}$$

where we used (2-2) and (2-3) in the last equality.

Since $\operatorname{div}^4 \mathrm{Rm} = 0$, we have

$$\frac{\alpha}{2}\int_{B_r} |\operatorname{div}\mathrm{Rm}|^2 e^{\alpha f} = \int_{\partial B_r} R_{ijkl}\nabla_k R_{jl}\nu_i e^{\alpha f}. \tag{5-6}$$

Next, we prove

$$\lim_{r\to+\infty}\int_{\partial B_r} R_{ijkl}\nabla_k R_{jl}\nu_i e^{\alpha f} = 0. \tag{5-7}$$

By direct computation, we have

$$\begin{aligned} \nabla_p R_{lkjp} &= \nabla_k R_{lj} - \nabla_l R_{kj} \\ &= -\nabla_k\nabla_l\nabla_j f + \nabla_l\nabla_k\nabla_j f \\ &= R_{lkjp}\nabla_p f, \end{aligned} \tag{5-8}$$

where we used the second Bianchi identity in the first equality and (1-1) in the second.

Noting that $R \ge 0$ (cf. B. L. Chen [2009]) and $R + |\nabla f|^2 = \text{Const.}$, we have that $|\nabla f|$ is bounded. By direct computation, we obtain

$$\begin{aligned} \left|\int R_{ijkl}\nabla_k R_{jl}\nu_i e^{\alpha f}\right| &= \frac{1}{2}\left|\int R_{ijkl}(\nabla_k R_{jl} - \nabla_l R_{jk})\nu_i e^{\alpha f}\right| \\ &= \frac{1}{2}\left|\int R_{ijkl}\nabla_p R_{lkjp}\nu_i e^{\alpha f}\right| \\ &= \frac{1}{2}\left|\int R_{ijkl}R_{lkjp}\nabla_p f\,\nu_i e^{\alpha f}\right| \\ &\le \frac{1}{2}\int |\mathrm{Rm}|^2|\nabla f| e^{\alpha f} \\ &\le c\int |\mathrm{Rm}|^2 e^{\alpha f} < +\infty, \end{aligned}$$

where we used (5-8) in the third equality and the assumption of

$$\int |\mathrm{Rm}|^2 e^{\alpha f} < +\infty$$

in the last. Then (5-7) follows.

Taking $r \to +\infty$ in (5-6), we have

$$\int |\operatorname{div}\mathrm{Rm}|^2 e^{\alpha f} = 0,$$

that is, $|\operatorname{div}\mathrm{Rm}| = 0$ on M.

By direct computation, we have

$$R_{ijkl}\nabla_l f = \nabla_i\nabla_j\nabla_k f - \nabla_j\nabla_i\nabla_k f = -\nabla_i R_{jk} + \nabla_j R_{ik} = \nabla_l R_{ijkl} = 0, \tag{5-9}$$

where we used (1-1) in the second equality and $\mathrm{Ric} = 0$ on M^n. It follows that $\sec(E, \nabla f) = R_{ijkl} E_i E_k \nabla_j f \nabla_l f = 0$, i.e., (M^n, g, f) is radially flat.

Moreover, we have

$$\nabla_l R = 2\nabla_j R_{jl} = 2g^{ik}\nabla_l R_{ijkl} = 0, \tag{5-10}$$

that is, R is a constant on M^n.

Since M^n is radially flat and has a constant scalar curvature, Lemma 2.4 implies that (M^n, g) is rigid. To conclude, (M^n, g, f) is Einstein with vanishing Ricci curvature or a quotient of $\mathbb{R}^n$ or of the product $\mathbb{R}^k \times N^{n-k}$ with $1 \le k \le n-1$, where N is Einstein with vanishing Ricci curvature. □

Theorem 5.3. *Let (M^n, g, f) be a nontrivial complete noncompact gradient steady Ricci soliton with $\int |\nabla R| e^{\alpha f} < +\infty$ for some $\alpha \neq 0$. If in addition* $\mathrm{tr}\,\mathrm{div}^2\,\mathrm{Rm} = 0$, *then (M^n, g, f) is a quotient of $\mathbb{R}^n$ or of the product $\mathbb{R}^k \times N^{n-k}$ with $1 \le k \le n-1$, where N is Einstein with vanishing Ricci curvature.*

Proof. From the proof of Theorem 5.1, we only need to show that M^n has vanishing Ricci curvature.

Let B_r be a geodesic ball with radius r and let ν be the unit outward normal vector field to ∂B_r. Integrating by parts, we obtain

$$\alpha \int_{B_r} \nabla_{\nabla f} R e^{\alpha f} = \int_{B_r} \langle \nabla R, \nabla e^{\alpha f}\rangle = \int_{\partial B_r} \nabla_\nu R e^{\alpha f} - \int_{B_r} \Delta R e^{\alpha f}. \tag{5-11}$$

Tracing (2-1), we have

$$g^{ik}(\mathrm{div}^2\,\mathrm{Rm})_{ik} = \nabla_{\nabla f} R - \tfrac{1}{2}\Delta R - 2|\mathrm{Ric}|^2 = \tfrac{1}{2}\Delta R - \Delta_f R - 2|\mathrm{Ric}|^2 = \tfrac{1}{2}\Delta R, \tag{5-12}$$

where we used $\Delta_f R = -2|\mathrm{Ric}|^2$.

Since $\mathrm{tr}\,\mathrm{div}^2\,\mathrm{Rm} = 0$, it follows from (5-12) that

$$\Delta R = 0. \tag{5-13}$$

On the other hand,

$$g^{ik}(\operatorname{div}^2 \mathrm{Rm})_{ik} = \nabla_{\nabla f} R - \tfrac{1}{2}\Delta R - 2|\mathrm{Ric}|^2 = \tfrac{1}{2}\nabla_{\nabla f} R - \tfrac{1}{2}\Delta_f R - 2|\mathrm{Ric}|^2 = \tfrac{1}{2}\nabla_{\nabla f} R - |\mathrm{Ric}|^2, \tag{5-14}$$

where we used the fact that $\Delta_f R = -2|\mathrm{Ric}|^2$.

It follows from $\operatorname{tr}\operatorname{div}^2 \mathrm{Rm} = 0$ and (5-14) that

$$\nabla_{\nabla f} R = 2|\mathrm{Ric}|^2. \tag{5-15}$$

Applying (5-13) and (5-15) to (5-11), and noting that $\alpha \neq 0$, we obtain

$$\int_{B_r} |\mathrm{Ric}|^2 e^{\alpha f} = \frac{1}{2\alpha}\int_{\partial B_r} \nabla_\nu R e^{\alpha f} \le \frac{1}{2|\alpha|}\int_{\partial B_r} |\nabla R| e^{\alpha f}. \tag{5-16}$$

Since $\int |\nabla R| e^{\alpha f} < +\infty$, we have

$$\lim_{r\to+\infty} \int_{\partial B_r} \nabla_\nu R e^{\alpha f} = 0.$$

By taking $r \to +\infty$ in (5-16), we obtain

$$\int |\mathrm{Ric}|^2 e^{\alpha f} = 0, \tag{5-17}$$

that is, $|\mathrm{Ric}| = 0$ on M^n, i.e., (M^n, g, f) has vanishing Ricci curvature.

This completes the proof of Theorem 5.3. □

Theorem 1.2 follows directly from Theorems 5.1–5.3.

Appendix

We prove a rigid result for complete noncompact gradient steady and expanding Ricci solitons in this section. It was implicitly proved by Yang and Zhang [2017]. For readers' convenience, we include a proof here.

Theorem A.1. *Let (M^n, g, f) be a complete noncompact gradient steady or expanding Ricci soliton with* $\operatorname{div}^3 \mathrm{Rm}(\nabla f) = 0$; *then (M^n, g, f) is rigid.*

Proof. Since $\operatorname{div}^3 \mathrm{Rm}(\nabla f) = 0$, it follows from (2-3) that

$$|\operatorname{div} \mathrm{Rm}|^2 = -2\operatorname{div}^3 \mathrm{Rm}(\nabla f) = 0.$$

Therefore, M is radially flat. Moreover, we have

$$\nabla_i R = 2\nabla_l R_{il} = -2g^{jk}\nabla_l R_{ijkl} = 0,$$

that is, R is a constant on M.

Since M^n is radially flat and has constant scalar curvature, Lemma 2.4 implies that (M^n, g, f) is rigid. $\square$

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