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JEFFREY D. ADLER AND DIPENDRA PRASAD

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MULTIPLICITY UPON RESTRICTION TO THE DERIVED SUBGROUP

JEFFREY D. ADLER AND DIPENDRA PRASAD

We present a conjecture on multiplicity of irreducible representations of a subgroup H contained in the irreducible representations of a group G, with G and H having the same derived groups. We point out some consequences of the conjecture, and verification of some of the consequences. We give an explicit example of multiplicity 2 upon restriction, as well as certain theorems in the context of classical groups where the multiplicity is 1.

1. Introduction

Suppose k is a local field, G is a connected reductive k-group, G' is a subgroup of G containing the derived group, and π is a smooth, irreducible, complex representation of $\mathrm{G}(k)$. In an earlier work [Adler and Prasad 2006], we showed that for many choices of G, the restriction $\mathrm{Res}^{\mathrm{G}(k)}_{\mathrm{G}'(k)}\,\pi$ decomposes without multiplicity.

A number of years ago, in the process of identifying situations where multiplicity one did not hold, one of us discovered an example of a depth-zero supercuspidal representation of $\mathrm{GU}(2d, 2d)$, a k-quasisplit group, whose restriction to $\mathrm{SU}(2d, 2d)$ decomposes with multiplicity two, and the other formulated a conjecture in the form of a reciprocity law involving enhanced Langlands parameters. In this paper, we present both the example and the conjecture, together with some consequences of the latter, and a verification of some of those consequences. Besides these, the paper proves several results by elementary means involving classical groups where multiplicity one holds.

A complete analysis of decomposition of the unitary principal series for $\mathrm{U}(n, n)$ and its restriction to $\mathrm{SU}(n, n)$ was done by Keys [1987], who also phrased his results in terms of "reciprocity" theorems for R-groups; in particular, he found cases of multiplicity greater than one.

After presenting our conjecture (Section 2), we give some of the heuristics behind it. In the formulation of the conjecture, we have considered a more general situation than that of a subgroup. We consider G_1 and G_2 to be two connected reductive groups over a local field k, and $\lambda : \mathrm{G}_1 \to \mathrm{G}_2$ to be a k-homomorphism

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that is a central isogeny when restricted to their derived subgroups, allowing us to "restrict" representations of $G_2(k)$ to $G_1(k)$. Since under such a homomorphism λ, the image of $G_1(k)$ is a normal subgroup of $G_2(k)$ with abelian quotient, all the irreducible representations of $G_1(k)$ which appear in this restriction problem for a given irreducible representation of $G_2(k)$ appear with the same multiplicity. In Section 3, we verify that for our conjectural multiplicity, this relationship does indeed hold. We show (Section 4) that if the conjecture is true for tempered representations, then via the Langlands classification it holds for all representations.

Our conjecture (for $\lambda : G_1 \to G_2$ a k-homomorphism) implies multiplicity one in situations where Langlands parameters for G_1 have abelian component groups. We list a few such situations in Section 5, and prove multiplicity one for restriction from $GU(n)$ to $U(n)$ (Section 6). Along the way, we prove multiplicity one in some other cases where it follows from elementary considerations. In Section 7, we present an example of a depth-zero supercuspidal representation of quasisplit $GU(2d, 2d)$ that decomposes with multiplicity two upon restriction to $SU(2d, 2d)$. Finally (Section 8), we give a general procedure for constructing higher multiplicities.

2. The conjecture on multiplicities

Let G_1^{qs} and G_2^{qs} be two connected quasisplit reductive groups over a local field k and let $\lambda : G_1^{qs} \to G_2^{qs}$ be a k-homomorphism that is a central isogeny when restricted to their derived subgroups. In what follows we will be twisting G_1^{qs} by a cohomology class in $H^1(\mathrm{Gal}(\bar{k}/k), G_1^{qs}(\bar{k}))$ to construct a pure inner form G_1 of G_1^{qs}. Simultaneously, by twisting G_2^{qs} by the image of this class under the map $H^1(\mathrm{Gal}(\bar{k}/k), G_1^{qs}(\bar{k})) \to H^1(\mathrm{Gal}(\bar{k}/k), G_2^{qs}(\bar{k}))$, we will have a pure inner form G_2 of G_2^{qs}, together with a map of algebraic groups that we will still call $\lambda : G_1 \to G_2$, which will appear in considerations below, all coming from an element of $H^1(\mathrm{Gal}(\bar{k}/k), G_1^{qs}(\bar{k}))$.

The map $\lambda : G_1 \to G_2$ gives rise to a "restriction" map from representations of $G_2(k)$ to those of $G_1(k)$, and from [Silberger 1979] one knows that the restriction of an irreducible representation of $G_2(k)$ is a finite direct sum of irreducible representations of $G_1(k)$. In particular, we obtain a functor $\lambda^\star : \mathcal{R}_{\mathrm{fin}}(G_2(k)) \to \mathcal{R}_{\mathrm{fin}}(G_1(k))$, where $\mathcal{R}_{\mathrm{fin}}(H)$ denotes the category of smooth, finite-length representations of a group H.

Let ${}^L G_1 = \widehat{G}_1 \rtimes W_k'$ and ${}^L G_2 = \widehat{G}_2 \rtimes W_k'$ be the L-groups associated to the quasisplit reductive groups G_1^{qs} and G_2^{qs} respectively. The map $\lambda : G_1^{qs} \to G_2^{qs}$ also gives rise to a homomorphism of L-groups,

$$ {}^L\lambda : {}^L G_2 \to {}^L G_1, $$

as well as a homomorphism of their centers,

$$ {}^L\lambda : Z(\widehat{G}_2)^{W_k} \to Z(\widehat{G}_1)^{W_k}. $$

It follows, in particular, that a character χ_1 of $\pi_0(Z(\widehat{\mathrm{G}}_1)^{W_k})$ gives rise to a character χ_2 of $\pi_0(Z(\widehat{\mathrm{G}}_2)^{W_k})$ which, by the Kottwitz isomorphism (assuming k to be nonarchimedean at this point),

$$H^1(\mathrm{Gal}(\bar{k}/k), \mathrm{G}_i^{\mathrm{qs}}(\bar{k})) \cong \mathrm{Hom}(\pi_0(Z(\widehat{\mathrm{G}}_i)^{W_k}), \mathbf{Q}/\mathrm{Z}),$$

constructs pure inner forms G_1 of $\mathrm{G}_1^{\mathrm{qs}}$ and G_2 of $\mathrm{G}_2^{\mathrm{qs}}$, together with a map $\lambda : \mathrm{G}_1 \to \mathrm{G}_2$ as before.

Let $\varphi_2 : W_k' \to {}^L\mathrm{G}_2$, and $\varphi_1 = {}^L\lambda \circ \varphi_2 : W_k' \to {}^L\mathrm{G}_1$ be associated Langlands parameters, where $W_k' = W_k \times \mathrm{SL}_2(\mathbf{C})$, with W_k the Weil group of k. Then ${}^L\lambda$ gives rise to a homomorphism of centralizers of the images of the parameters φ_1 with values in ${}^L\mathrm{G}_1$ and φ_2 with values in ${}^L\mathrm{G}_2$, and also a homomorphism of the groups of connected components of their centralizers:

$$\pi_0({}^L\lambda) : \pi_0(Z_{\widehat{\mathrm{G}}_2}(\varphi_2)) \to \pi_0(Z_{\widehat{\mathrm{G}}_1}(\varphi_1)).$$

This allows one to "restrict" representations of $\pi_0(Z_{\widehat{\mathrm{G}}_1}(\varphi_1))$ to representations of $\pi_0(Z_{\widehat{\mathrm{G}}_2}(\varphi_2))$, giving rise to the restriction functor

$$\lambda_\star : K_0(\pi_0(Z_{\widehat{\mathrm{G}}_1}(\varphi_1))) \to K_0(\pi_0(Z_{\widehat{\mathrm{G}}_2}(\varphi_2))),$$

where $K_0(H)$ denotes the Grothendieck group of finite-length representations of a group H.

The formulation of our conjecture below presumes that the local Langlands correspondence involving enhanced Langlands parameters has been achieved, giving rise to a bijection between enhanced Langlands parameters and the set of isomorphism classes of irreducible admissible representations of all pure inner forms of quasisplit groups. This will be needed for *both* of the groups G_1 and G_2; it is possible on the other hand that one could reverse this role, and use the conjectural multiplicity formula to construct an enhanced Langlands parametrization for G_2, knowing it for G_1.

Conjecture 1. (a) *Let* G_1 *and* G_2 *be two connected reductive groups over a local field k and let* $\lambda : \mathrm{G}_1 \to \mathrm{G}_2$ *be a k-homomorphism that is a central isogeny when restricted to their derived subgroups. For* $i = 1, 2$, *let* π_i *be an irreducible admissible representation of* $\mathrm{G}_i(k)$ *with Langlands parameter* φ_i. *Let*

$$\mathrm{m}(\pi_2, \pi_1) := \dim \mathrm{Hom}_{\mathrm{G}_1(k)}[\pi_1, \lambda^\star\pi_2] = \dim \mathrm{Hom}_{\mathrm{G}_1(k)}[\lambda^\star\pi_2, \pi_1].$$

Then $\mathrm{m}(\pi_2, \pi_1) = 0$ *unless* $\varphi_1 = {}^L\lambda \circ \varphi_2$.

(b) *Let* $\mathrm{G}_1^{\mathrm{qs}}$ *and* $\mathrm{G}_2^{\mathrm{qs}}$ *be two connected reductive quasisplit groups over a local field k and let* $\lambda : \mathrm{G}_1^{\mathrm{qs}} \to \mathrm{G}_2^{\mathrm{qs}}$ *be a k-homomorphism that is a central isogeny when restricted to their derived subgroups. Let* φ_1 *and* φ_2 *be Langlands parameters associated to the groups* $\mathrm{G}_1^{\mathrm{qs}}$ *and* $\mathrm{G}_2^{\mathrm{qs}}$ *with* $\varphi_1 = {}^L\lambda \circ \varphi_2$, *and let* χ_i *be characters of their component groups* $\pi_0(Z_{\widehat{\mathrm{G}}_i}(\varphi_i))$. *Then, if* $\mathrm{Hom}_{\pi_0(Z(\varphi_2))}[\chi_2, \lambda_\star\chi_1]$ *is nonzero, the* *characters χ_i define pure inner forms G_i of $\mathrm{G}_i^{\mathrm{qs}}$ together with a k-homomorphism, $\lambda : \mathrm{G}_1 \to \mathrm{G}_2$, as discussed earlier. Then if $\pi_i = \pi(\varphi_i, \chi_i)$ are the corresponding irreducible admissible representations of $\mathrm{G}_i(k)$, we have*

$$\mathrm{m}(\pi_2, \pi_1) = \dim \mathrm{Hom}_{\pi_0(Z(\varphi_2))}[\chi_2, \lambda_\star \chi_1].$$

The main heuristic for the conjectural multiplicity is the following.

(1) For any L-packet $\{\pi\}$ on any reductive group $\mathrm{G}(k)$ defined by a parameter φ (that is, $\{\pi\} = \{\pi_{(\varphi,\chi)}\}$ where one takes those characters χ of the component group which have a particular restriction to $Z(\widehat{\mathrm{G}})^{W_k}$ defining the group $\mathrm{G}(k)$ assumed to be a pure inner form of a fixed quasisplit group G^{qs}),

$$\sum_{\chi} \chi(1)\Theta(\pi_{(\varphi,\chi)})$$

is a stable distribution on $\mathrm{G}(k)$. Here, for any admissible representation π we are letting $\Theta(\pi)$ denote its character, regarded as a distribution on $\mathrm{G}(k)$.

(2) For a homomorphism $\lambda : \mathrm{G}_1 \to \mathrm{G}_2$ of reductive groups over k which is an isogeny when restricted to their derived subgroups, the pullback of a stable distribution on $\mathrm{G}_2(k)$ is a stable distribution on $\mathrm{G}_1(k)$.

(3) The restriction to $\mathrm{G}_1(k)$ of an irreducible representation π_2 of $\mathrm{G}_2(k)$ is a finite-length (completely reducible) representation of $\mathrm{G}_1(k)$, whose irreducible components are all in the same L-packet. This L-packet for $\mathrm{G}_1(k)$ depends only on the L-packet for $\mathrm{G}_2(k)$ containing π_2. If the Langlands parameter of our L-packet for $\mathrm{G}_2(k)$ is $\varphi_2 : W_k' \to {}^L\mathrm{G}_2$, then the Langlands parameter of our L-packet for $\mathrm{G}_1(k)$ is $\varphi_1 := {}^L\lambda \circ \varphi_2 : W_k' \to {}^L\mathrm{G}_1$. (This is part (a) of the conjecture.)

(4) If Conjecture 1 is true, then the pullback from $\mathrm{G}_2(k)$ to $\mathrm{G}_1(k)$ of the distribution

$$\sum_{\chi_2} \chi_2(1)\Theta(\pi_{(\varphi_2,\chi_2)}),$$

where the sum is taken over those characters χ_2 of the component group which have a particular restriction to $Z(\widehat{G}_2)^{W_k}$ defining the group $\mathrm{G}_2(k)$ assumed to be a pure inner form of a fixed quasisplit group $\mathrm{G}_2^{\mathrm{qs}}(k)$, is a stable distribution on $\mathrm{G}_1(k)$ as we check now.

By Conjecture 1, the pullback of the distribution $\Theta_{\pi_2} = \Theta(\pi_{(\varphi_2,\chi_2)})$ on $\mathrm{G}_2(k)$ to $\mathrm{G}_1(k)$ is

$$\sum_{\pi_1} m(\pi_2, \pi_1)\Theta(\pi_1) = \sum_{\chi_1} \Theta(\pi_{(\varphi_1,\chi_1)}) \dim \mathrm{Hom}_{\pi_0(Z(\varphi_2))}[\chi_2, \lambda_\star \chi_1].$$

Therefore, the pullback to $G_1(k)$ of the distribution $\sum_{\chi_2} \chi_2(1)\Theta(\pi_{(\varphi_2,\chi_2)})$ on $G_2(k)$ is (assuming Conjecture 1)

$$\sum_{\chi_1,\chi_2} \chi_2(1)\Theta(\pi_{(\varphi_1,\chi_1)}) \dim \mathrm{Hom}_{\pi_0(Z(\varphi_2))}[\chi_2, \lambda_\star\chi_1],$$

which is the same as

$$\sum_{\chi_1,\chi_2} \Theta(\pi_{(\varphi_1,\chi_1)}) \dim \mathrm{Hom}_{\pi_0(Z(\varphi_2))}[\chi_2(1)\chi_2, \lambda_\star\chi_1],$$

where the sum is taken over all pairs of characters χ_1, χ_2 with particular restrictions to $Z(\widehat{G}_1)^{W_k}$ and $Z(\widehat{G}_2)^{W_k}$. Observe that those characters χ_2 whose restrictions to $Z(\widehat{G}_2)^{W_k}$ are not compatible with the restriction of χ_1 to $Z(\widehat{G}_1)^{W_k}$ contribute 0 to the sum. Therefore, we can take the sum over all χ_2. The sum then is the same as

$$(*) \qquad \sum_{\chi_1} \Theta(\pi_{(\varphi_1,\chi_1)}) \dim \mathrm{Hom}_{\pi_0(Z(\varphi_2))}[R, \lambda_\star\chi_1],$$

where $R = \sum \chi_2(1)\chi_2$ is the regular representation of $\pi_0(Z(\varphi_2))$.

By Schur orthogonality,

$$\dim \mathrm{Hom}_{\pi_0(Z(\varphi_2))}[\chi_2, \lambda_\star\chi_1] = \frac{1}{|\pi_0(Z(\varphi_2))|} \sum_{g\in\pi_0(Z(\varphi_2))} \chi_1(\lambda^\star g)\overline{\chi}_2(g),$$

where $\lambda^\star$ denotes the map $\pi_0(^L\lambda)\colon \pi_0(Z(\varphi_2)) \to \pi_0(Z(\varphi_1))$. So

$$\dim \mathrm{Hom}_{\pi_0(Z(\varphi_2))}[R, \lambda_\star\chi_1] = \frac{1}{|\pi_0(Z(\varphi_2))|} \sum_{g\in\pi_0(Z(\varphi_2))} \chi_1(\lambda^\star g)\chi_R(g),$$

where R is the regular representation of $\pi_0(Z(\varphi_2))$ and χ_R its character, thus

$$\chi_R(g) = \begin{cases} 0 & \text{if } g \text{ is not the identity,} \\ |\pi_0(Z(\varphi_2))| & \text{if } g \text{ is the identity.} \end{cases}$$

Therefore,

$$\dim \mathrm{Hom}_{\pi_0(Z(\varphi_2))}[R, \lambda_\star\chi_1] = \chi_1(1).$$

By $(*)$ it follows that the pullback of the distribution $\sum_{\chi_2} \chi_2(1)\Theta(\pi_{(\varphi_2,\chi_2)})$ on $G_2(k)$ to $G_1(k)$ is equal to $\sum_{\chi_1} \chi_1(1)\Theta(\pi_{(\varphi_1,\chi_1)})$, where the sum is taken over those χ_1 with a given restriction to $Z(\widehat{G}_1)^{W_k}$. Thus the pullback of the distribution $\sum_{\chi_2} \chi_2(1)\Theta(\pi_{(\varphi_2,\chi_2)})$ on $G_2(k)$ to $G_1(k)$ is a stable distribution on $G_1(k)$ which is what we set out to prove.

Remark 2. A weaker version of our conjecture says that the pullback to $G_1(k)$ of the stable character $\sum_\chi \chi(1)\Theta_\chi$ on $G_2(k)$ is $\sum_\mu \mu(1)\Theta_\mu$ on $G_1(k)$, where both of the sums are over the characters of component groups defining fixed pure inner forms that are G_2 and G_1, respectively.

3. Some remarks on the multiplicity formula

Conjecture 1 relating $\mathrm{m}(\pi_2, \pi_1)$ with $\dim \mathrm{Hom}_{\pi_0(Z(\varphi_2))}[\lambda_\star\chi_1, \chi_2]$ can be considered as a set of assertions keeping π_2 fixed and varying π_1, or keeping π_1 fixed and varying π_2, say, inside an L-packet for $\mathrm{G}_2(k)$. It is easy to see that for G_1 and G_2 two reductive groups over a local field k, and $\lambda : \mathrm{G}_1 \to \mathrm{G}_2$, a k-homomorphism that is a central isogeny when restricted to their derived subgroups, the image of $\mathrm{G}_1(k)$ inside $\mathrm{G}_2(k)$ is a normal subgroup, and therefore every irreducible representation of $\mathrm{G}_1(k)$ that appears inside a given irreducible representation π_2 of $\mathrm{G}_2(k)$ does so with the same multiplicity (depending, of course, on π_2). This section aims to prove this as a consequence of our Conjecture 1.

This section is meant to prove that $\dim \mathrm{Hom}_{\pi_0(Z(\varphi_2))}[\lambda_\star\chi_1, \chi_2]$ remains constant when χ_2 is a fixed character of $\pi_0(Z(\varphi_2))$ but χ_1 varies among characters of $\pi_0(Z(\varphi_1))$. This is achieved by combining Corollary 4 with Lemma 5. We begin with the following lemma whose straightforward proof will be omitted.

Lemma 3. *Let N be a normal subgroup of a finite group G with $A = G/N$ an abelian group. Let π be an irreducible representation of N. Then any two irreducible representations π_1 and π_2 of G containing π on restriction to N are twists of each other by characters of G/N, i.e.,*

$$\pi_2 \cong \pi_1 \otimes \chi,$$

for $\chi : G/N \to \mathbb{C}^\times$.

Corollary 4. *If N is a normal subgroup of a group G with $A = G/N$ a finite abelian group, and π an irreducible representation of N, then all irreducible G-submodules of $\mathrm{Ind}_N^G(\pi)$ appear in it with the same multiplicity.*

Lemma 5. *Let G_1 and G_2 be two connected reductive groups over a local field k and let $\lambda\colon \mathrm{G}_1 \to \mathrm{G}_2$ be a k-homomorphism that is a central isogeny when restricted to their derived subgroups, and giving rise to a homomorphism ${}^L\lambda\colon {}^L\mathrm{G}_2 \to {}^L\mathrm{G}_1$ of the L-groups. Let $\varphi_2 : W_k' \to {}^L\mathrm{G}_2$, and $\varphi_1 = {}^L\lambda \circ \varphi_2 : W_k' \to {}^L\mathrm{G}_1$ be associated Langlands parameters. Then for the associated homomorphism of finite groups $\lambda^\star\colon \pi_0(Z_{\widehat{\mathrm{G}}_2}(\varphi_2)) \to \pi_0(Z_{\widehat{\mathrm{G}}_1}(\varphi_1))$, the image is normal with abelian cokernel.*

Proof. It suffices to prove the lemma separately in the two cases:

(1) $\lambda\colon \mathrm{G}_1 \to \mathrm{G}_2$ is injective as a homomorphism of algebraic groups.

(2) $\lambda\colon \mathrm{G}_1 \to \mathrm{G}_2$ is surjective as a homomorphism of algebraic groups.

We will address only the first case, the other being very similar.

Assume then that $\lambda\colon \mathrm{G}_1 \to \mathrm{G}_2$ is injective, and thus $\widehat{\lambda}\colon \widehat{\mathrm{G}}_2 \to \widehat{\mathrm{G}}_1$ is surjective with kernel, say, $\widehat{Z}$. Use $\varphi_2\colon W_k' \to {}^L\mathrm{G}_2$ and $\varphi_1 = {}^L\lambda \circ \varphi_2 : W_k' \to {}^L\mathrm{G}_1$ to give $\widehat{\mathrm{G}}_2$

and $\widehat{G}_1$, a W_k'-group structure, such that we have an exact sequence of W_k'-groups,

$$1 \to \widehat{Z} \to \widehat{G}_2 \to \widehat{G}_1 \to 1.$$

This gives rise to a long exact sequence of W_k'-cohomology sets:

$$1 \to \widehat{Z}^{W_k'} \to \widehat{G}_2^{W_k'} \to \widehat{G}_1^{W_k'} \to H^1(W_k', \widehat{Z}) \to \cdots .$$

Equivalently, we have the exact sequence of groups,

$$1 \to Z_{\widehat{G}_2}(\varphi_2)/\widehat{Z}^{W_k'} \to Z_{\widehat{G}_1}(\varphi_1) \to A \to 1,$$

where A is a subgroup of $H^1(W_k', \widehat{Z})$, a locally compact abelian group. Taking π_0 of the terms in the above exact sequence which all fit together in a long exact sequence of π_i's (higher homotopy groups), the assertion in the lemma follows on noting that if $E_1 \to E_2$ is a surjective map of locally compact and locally connected topological groups, then the induced map $\pi_0(E_1) \to \pi_0(E_2)$ is also surjective. □

4. Reduction of the conjecture to the case of tempered representations

As before, let G_1 and G_2 be two reductive groups over a local field k, and let $\lambda : G_1 \to G_2$ be a k-homomorphism that is a central isogeny when restricted to their derived subgroups, giving rise to the restriction functor

$$\lambda^\star : \mathcal{R}_{\text{fin}}(G_2(k)) \to \mathcal{R}_{\text{fin}}(G_1(k)).$$

Lemma 6. *Let V be a finite-length representation of* $G_2(k)$ *with maximal semisimple quotient Q. Then $\lambda^\star Q$ is the maximal semisimple quotient of $\lambda^\star V$, a finite-length representation of $G_1(k)$.*

Proof. It suffices to observe that a finite-length representation of $G_2(k)$ is semisimple if and only if its image under $\lambda^\star$ is a finite-length, semisimple representation of $G_1(k)$. If $Z(G_1)(k)\cdot G_1(k)$ is of finite index in $G_2(k)$, such as when k is of characteristic zero, then this is easy to see. By a theorem of Silberger [1979], irreducible representations of $G_2(k)$ remain finite-length semisimple representations when restricted to $G_1(k)$, and the lemma follows in general. □

To set up the next result, let $P_2 = M_2N_2$ be a Levi factorization of a parabolic subgroup in G_2. If we let $P_1 = \lambda^{-1}(P_2)$, $M_1 = \lambda^{-1}(M_2)$, and $N_1 = \lambda^{-1}(N_2)$, then $P_1 = M_1N_1$ is a Levi factorization of a parabolic subgroup in G_1. Then $\lambda : M_1 \to M_2$ gives us a restriction functor $\mathcal{R}_{\text{fin}}(M_2(k)) \to \mathcal{R}_{\text{fin}}(M_1(k))$ that we will also denote by $\lambda^\star$. Since λ gives an isomorphism $G_1(k)/P_1(k) \to G_2(k)/P_2(k)$, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{R}_{\text{fin}}(\mathrm{G}_2(k)) & \xrightarrow{\lambda^\star} & \mathcal{R}_{\text{fin}}(\mathrm{G}_1(k)) \\
{\scriptstyle \mathrm{Ind}_{P_2(k)}^{\mathrm{G}_2(k)}}\uparrow & & \uparrow{\scriptstyle \mathrm{Ind}_{P_1(k)}^{\mathrm{G}_1(k)}} \\
\mathcal{R}_{\text{fin}}(M_2(k)) & \xrightarrow{\lambda^\star} & \mathcal{R}_{\text{fin}}(M_1(k))
\end{array}$$

Lemma 7. *Let σ_2 be an irreducible, essentially tempered representation of $M_2(k)$ with strictly positive exponents along the center $\mathrm{Z}(M_2)(k)$ of $M_2(k)$. Write*

$$\lambda^\star \sigma_2 = \sum_\alpha m_\alpha \sigma_{1,\alpha},$$

a sum of irreducible, essentially tempered representations of $M_1(k)$ with (finite) multiplicities m_α. Let π_2 be the Langlands quotient of the standard module $\mathrm{Ind}_{P_2(k)}^{\mathrm{G}_2(k)} \sigma_2$, and $\pi_{1,\alpha}$ the Langlands quotients of $\mathrm{Ind}_{P_1(k)}^{\mathrm{G}_1(k)} \sigma_{1,\alpha}$. Then

$$\lambda^\star \pi_2 = \sum_\alpha m_\alpha \pi_{1,\alpha}.$$

Proof. Clearly,

$$\lambda^\star \mathrm{Ind}_{P_2(k)}^{\mathrm{G}_2(k)} \sigma_2 = \mathrm{Ind}_{P_1(k)}^{\mathrm{G}_1(k)} \lambda^\star \sigma_2 = \sum_\alpha m_\alpha \mathrm{Ind}_{P_1(k)}^{\mathrm{G}_1(k)} \sigma_{1,\alpha}.$$

Since "taking maximal semisimple quotient" commutes with direct sum, our result follows from Lemma 6. □

Corollary 8. *If Conjecture 1 is true for tempered representations, then it is true in general.*

Proof. Every representation π_2 of $\mathrm{G}_2(k)$ can be realized as a Langlands quotient of a standard module $\mathrm{Ind}_{P_2(k)}^{\mathrm{G}_2(k)} \sigma_2$ for an essentially tempered representation σ_2 of $M_2(k)$. The Langlands parameter $\varphi_2 \colon W'_F \to {}^L\mathrm{G}_2$ for π_2 is the same as the Langlands parameter φ_2 for σ_2 considered as a map $W'_F \xrightarrow{\varphi_2} {}^L M_2 \to {}^L\mathrm{G}_2$. The component groups of these parameters, and thus the representations of these component groups, correspond as discussed in [Prasad 2019, §5]. Therefore, our result is a consequence of Lemma 7. □

5. Consequences of the conjecture

If the group of connected components $\pi_0(Z_{\widehat{\mathrm{G}}_1}(\varphi_1))$ is known to be abelian, as is the case when G_1 is any of the groups SL_n, U_n, SO_n, and Sp_n, then our conjecture predicts that for any homomorphism $\lambda \colon \mathrm{G}_1 \to \mathrm{G}_2$ of connected reductive algebraic groups that is an isomorphism up to center (i.e., $\bar{\lambda} \colon \mathrm{G}_1/\mathrm{Z}_1 \to \mathrm{G}_2/\mathrm{Z}_2$ is an isomorphism of algebraic groups, where Z_i is the center of G_i), any irreducible representation of $\mathrm{G}_2(k)$ when restricted via λ to $\mathrm{G}_1(k)$ decomposes as a sum of irreducible representations of $\mathrm{G}_1(k)$ with multiplicity ≤ 1.

We note that by our earlier work [Adler and Prasad 2006], we know that multiplicity is ≤ 1 whenever the pair $(\mathrm{G}_1, \mathrm{G}_2)$ is $(\mathrm{SL}_n, \mathrm{GL}_n)$, or (when the characteristic of k is not two) either $(\mathrm{O}_n, \mathrm{GO}_n)$ or $(\mathrm{Sp}_n, \mathrm{GSp}_n)$. In the next section, we will see that multiplicity ≤ 1 also holds for $(\mathrm{U}_n, \mathrm{GU}_n)$. The paper [Gee and Taïbi 2018] shows that multiplicity ≤ 1 holds for the pair $(\mathrm{SO}_n, \mathrm{GSO}_n)$ if k has characteristic zero.

6. Generalities on restriction to unitary and special unitary groups

Let E/k denote a separable quadratic extension of nonarchimedean local fields, $N = N_{E/k}$ the norm map from $E^\times$ to $k^\times$, and E_1 the kernel of this map.

Let B denote a nondegenerate E/k-hermitian form on some E-vector space V of some dimension r. Then we can form algebraic groups $\mathrm{SU}(V, B)$, $\mathrm{U}(V, B)$, and $\mathrm{GU}(V, B)$ whose k-points consist respectively of the elements of $\mathrm{SL}(r, E)$ that preserve B; the elements of $\mathrm{GL}(r, E)$ that preserve B; and the elements of $\mathrm{GL}(r, E)$ that preserve B up to a scalar in $k^\times$. The group $\mathrm{GU}(V, B)$ comes equipped with a map $\mu\colon \mathrm{GU}(V, B) \to \mathrm{GL}_1$ called the *similitude character*. We will write our algebraic groups as $\mathrm{SU}(r)$, $\mathrm{U}(r)$, and $\mathrm{GU}(r)$ when V and B are understood.

If G is a group, H is a subgroup, and $G/Z(G)H$ is cyclic, then every irreducible representation of G restricts to H without multiplicity. How far can we exploit this fact?

Theorem 9. *Let p be the residual characteristic of k.*

(a) *All irreducible representations of* $\mathrm{GU}(r)(k)$ *decompose without multiplicity upon restriction to* $\mathrm{U}(r)(k)$. *Such a restriction is irreducible when r is odd, and has at most two components when r is even.*

(b) *All irreducible representations of* $\mathrm{U}(r)(k)$ *decompose without multiplicity upon restriction to* $\mathrm{SU}(r)(k)$ *when r is coprime to p, or $k = \mathbf{Q}_p$ (p odd).*

(c) *All irreducible representations of* $\mathrm{GU}(r)(k)$ *decompose without multiplicity upon restriction to* $\mathrm{SU}(r)(k)$ *when r is odd and coprime to p.*

Proof. (a) Let $\mu\colon \mathrm{GU}(r) \to \mathrm{GL}(1)$ denote the similitude character. Clearly the group $\mathrm{GU}(r)$ contains the scalar matrices eI_r for all $e \in E^\times$, and for such matrices the similitude is $N_{E/k}(e)$. Therefore, the image under μ of the center of $\mathrm{GU}(r)(k)$ is $N_{E/k}(E^\times)$, so μ thus gives an isomorphism

$$\frac{\mathrm{GU}(r)}{Z(\mathrm{GU}(r))\,\mathrm{U}(r)} \xrightarrow{\sim} \frac{\mathrm{Im}(\mu)}{N(E^\times)}.$$

A scalar $a \in k^\times$ is a similitude for some linear transformation g of V if and only if for all $v, w \in V$, we have that $B(gv, gw) = a \cdot B(v, w)$. That is, B and $a \cdot B$

are equivalent Hermitian forms. It is known that two Hermitian forms over a non-archimedean local field k are equivalent if and only if their discriminants, which are elements of $k^\times/N(E^\times)$, are the same. Therefore, B and aB are equivalent if and only if $\operatorname{disc} B = a^r \operatorname{disc} B$ in $k^\times/N(E^\times) \cong \mathbf{Z}/2$. Thus, if r is even, then B and aB are equivalent for a an arbitrary element of $k^\times$, but if r is odd, then a must lie in $N(E^\times)$. Thus,

$$\frac{\mathrm{GU}(r)}{\mathrm{Z}(\mathrm{GU}(r))\,\mathrm{U}(r)} \cong \mathbf{Z}/2 \quad \text{or} \quad \{1\}.$$

(b) Let R_E and P_E denote the ring of integers and prime ideal for E. The determinant character gives us an isomorphism,

$$\det\colon \frac{\mathrm{U}(r)(k)}{Z(\mathrm{U}(r))(k)\,\mathrm{SU}(r)(k)} \xrightarrow{\sim} \frac{E_1}{(E_1)^r}.$$

As an abstract group, E_1 inherits a direct product decomposition from $R_E^\times \cong k_E^\times \times (1+P_E)$. Thus, E_1 is a direct product of a cyclic group (of order coprime to p) and a pro-p-group A, implying that E_1/E_1^r is cyclic if and only A/A^r is cyclic. But this latter quotient is trivial if r is coprime to p, and is cyclic if $k = \mathbf{Q}_p$ (p odd).

(c) This follows from the previous two parts of the theorem. □

7. An example of multiplicity upon restriction

Let ϖ be a uniformizer of k, E/k an unramified quadratic extension, R_k and R_E the rings of integers in k and E, and $\mathfrak{f}$ and $\mathfrak{f}_E$ the residue fields. Let V be a $4d$-dimensional hermitian space over E, with hyperbolic basis $\{e_1, f_1, \ldots, e_{2d}, f_{2d}\}$. Thus, $\langle e_i, f_i\rangle = 1$ for all $1 \le i \le 2d$, and all the other products being 0. Let $\mathrm{U}(V)$ be the corresponding unitary group. Define the lattice $\mathcal{L}$ in E by

$$\mathcal{L} = \operatorname{span}_{R_E}\{e_1, f_1, \ldots, e_d, f_d, \varpi e_{d+1}, f_{d+1}, \ldots, \varpi e_{2d}, f_{2d}\}.$$

Clearly, $\mathcal{L}^\vee := \{v \in V \mid \langle v, \ell\rangle \in R_E \text{ for all } \ell \in \mathcal{L}\}$ is given by

$$\mathcal{L}^\vee = \operatorname{span}_{R_E}\{e_1, f_1, \ldots, e_d, f_d, e_{d+1}, \varpi^{-1} f_{d+1}, \ldots, e_{2d}, \varpi^{-1} f_{2d}\}.$$

Observe that

$$\varpi\mathcal{L}^\vee \subseteq \mathcal{L} \subseteq \mathcal{L}^\vee,$$

and $\mathcal{L}^\vee/\mathcal{L}$ and $\mathcal{L}/\varpi\mathcal{L}^\vee$ are $2d$-dimensional hermitian spaces over $\mathfrak{f}_E$ with natural hermitian structures. For example, given two elements ℓ_1 and ℓ_2 in $\mathcal{L}^\vee$ with images $\overline{\ell}_1$ and $\overline{\ell}_2$ in $\mathcal{L}^\vee/\mathcal{L}$, the hermitian structure on $\mathcal{L}^\vee/\mathcal{L}$ is defined by having $\langle\overline{\ell}_1, \overline{\ell}_2\rangle$ as the image of $\varpi\langle\ell_1, \ell_2\rangle$ (which belongs to R_E) in $\mathfrak{f}_E$.

Define $K = \mathrm{U}(\mathcal{L})$ to be the stabilizer of the lattice $\mathcal{L}$ in $\mathrm{U}(V)$, i.e., $\mathrm{U}(\mathcal{L}) = \{g \in \mathrm{U}(V) \mid g\ell \in \mathcal{L} \text{ for all } \ell \in \mathcal{L}\}$. If an element of $\mathrm{U}(V)$ preserves $\mathcal{L}$, then it clearly preserves $\mathcal{L}^\vee$ and $\varpi\mathcal{L}$, giving a map $\mathrm{U}(\mathcal{L}) \to \mathrm{U}(2d, \mathfrak{f}) \times \mathrm{U}(2d, \mathfrak{f})$. Similarly, we have a map $\mathrm{SU}(\mathcal{L}) \to \mathrm{S}(\mathrm{U}(2d) \times \mathrm{U}(2d))(\mathfrak{f})$.

Let $g_0 \in \mathrm{GU}(V)$ be defined (for $i \leq d$) by

$$e_i \mapsto e_{d+i}, \quad f_i \mapsto \varpi^{-1} f_{d+i}, \quad e_{d+i} \mapsto \varpi^{-1} e_i, \quad f_{d+i} \mapsto f_i.$$

Clearly, g_0 has similitude factor ϖ^{-1}, and $g_0\mathcal{L} = \mathcal{L}^\vee$. Therefore, we have

$$g_0 \,\mathrm{U}(\mathcal{L}) g_0^{-1} = \mathrm{U}(\mathcal{L}^\vee).$$

Thus conjugation by g_0 induces an isomorphism of $\mathrm{U}(\mathcal{L})$ into $\mathrm{U}(\mathcal{L}^\vee)$, making the diagram

$$\begin{array}{ccc} \mathrm{U}(\mathcal{L}) & \xrightarrow{\quad g_0 \quad} & \mathrm{U}(\mathcal{L}^\vee) \\ \downarrow & & \downarrow \\ \mathrm{U}(2d, \mathfrak{f}) \times \mathrm{U}(2d, \mathfrak{f}) & \xrightarrow{\ j\ } & \mathrm{U}(2d, \mathfrak{f}) \times \mathrm{U}(2d, \mathfrak{f}) \end{array}$$

commute, where $j(x, y) = (y, x)$.

Theorem 10. *Let ρ be any irreducible cuspidal representation of $\mathrm{U}(2d)(\mathfrak{f})$ such that $\rho \not\cong \rho\chi$, where χ is a quadratic character of $\mathrm{U}(2d)(\mathfrak{f})$ trivial on $\mathrm{SU}(2d)(\mathfrak{f})$. Let $\sigma := \mathrm{infl}(\rho \otimes \rho\chi)$ denote the inflation of $\rho \otimes \rho\chi$ from $(\mathrm{U}(2d) \times \mathrm{U}(2d))(\mathfrak{f})$ to $\mathrm{U}(\mathcal{L})$ and let $\pi = \text{c-Ind}_{\mathrm{U}(\mathcal{L})}^{\mathrm{U}(V)} \sigma$. Then $\pi \oplus \pi^{g_0}$ extends to an irreducible representation $\widetilde{\pi}$ of $\mathrm{GU}(V)$ whose restriction to $\mathrm{SU}(V)$ decomposes with multiplicity two.*

Proof. From [Moy and Prasad 1996, Proposition 6.6], π is an irreducible, supercuspidal representation of $\mathrm{U}(V)$. Let π also denote one of its extensions to $Z(\mathrm{GU}(V))\,\mathrm{U}(V)$. From the last sentence of [Moy and Prasad 1994, Theorem 5.2], $\pi^{g_0} \not\cong \pi$, so the sum $\pi \oplus \pi^{g_0}$ extends to an irreducible (also supercuspidal) representation $\widetilde{\pi}$ of $\mathrm{GU}(V)$. By the induction-restriction formula (observe that by the explicit description of $\mathrm{U}(\mathcal{L})$, $\det : \mathrm{U}(\mathcal{L}) \to E_1$ is surjective, and hence $\mathrm{U}(\mathcal{L})\,\mathrm{SU}(V) = \mathrm{U}(V)$),

$$\pi|_{\mathrm{SU}(V)} = \text{c-Ind}_{\mathrm{SU}(\mathcal{L})}^{\mathrm{SU}(V)}(\sigma|_{\mathrm{SU}(\mathcal{L})}),$$

$$\pi^{g_0}|_{\mathrm{SU}(V)} = \text{c-Ind}_{\mathrm{SU}(\mathcal{L})}^{\mathrm{SU}(V)}(\sigma^{g_0}|_{\mathrm{SU}(\mathcal{L})}).$$

Since $\rho \otimes \rho\chi \cong \rho\chi \otimes \rho$ as representations of $\mathrm{S}(\mathrm{U}(2d) \times \mathrm{U}(2d))(\mathfrak{f})$, we have that $\sigma \cong \sigma^{g_0}$ as representations of $\mathrm{SU}(\mathcal{L})$, so

$$\widetilde{\pi}|_{\mathrm{SU}(V)} = (\pi \oplus \pi^{g_0})|_{\mathrm{SU}(V)} = 2 \cdot \text{c-Ind}_{\mathrm{SU}(\mathcal{L})}^{\mathrm{SU}(V)}(\sigma|_{\mathrm{SU}(\mathcal{L})}). \qquad \square$$

In order to have an example of multiplicity at least two, it is thus sufficient to find a representation ρ of $\mathrm{U}(2d)(\mathfrak{f})$ such that $\rho \not\cong \rho\chi$, as in the theorem. In fact, most irreducible Deligne–Lusztig cuspidal representations of $\mathrm{U}(2d)(\mathfrak{f})$ will have this property, as they restrict irreducibly to $\mathrm{SU}(2d)(\mathfrak{f})$.

Remark 11. In a future work, we will expand upon the example in Theorem 10, whose essence is the following. Given a supercuspidal representation of $\mathrm{G}_2(k)$ whose restriction to $\mathrm{G}_1(k)$ has regular components (in the sense of Kaletha [2016]), then the components occur with multiplicity one. (Nevins [2015] already verified this for many cases.) If the components are not regular, then higher multiplicities can occur.

Our example begins with ρ, an irreducible cuspidal representation of $\mathrm{U}(2d)(\mathfrak{f})$ that arises via Deligne–Lusztig induction from a character θ of the group of $\mathfrak{f}$-points of an anisotropic torus $\mathsf{T} \subset \mathrm{U}(2d)$. Suppose also that the restriction of θ to $\mathsf{T}(\mathfrak{f}) \cap \mathrm{SU}(2d)(\mathfrak{f})$ remains regular so that the restriction of ρ to $\mathrm{SU}(2d)(\mathfrak{f})$ remains irreducible. The torus $\mathsf{T} \times \mathsf{T} \subset \mathrm{U}(2d) \times \mathrm{U}(2d)$ lifts to give an unramified torus $T \subset \mathrm{GU}(V)$, and the character $\theta \otimes \theta\chi$ can be inflated and extended to give a character Θ of T. The representation $\widetilde{\pi}$ of $\mathrm{GU}(V)$ that we have constructed in the theorem is a regular supercuspidal representation in the sense of Kaletha [2016], but the irreducible components of its restriction to $\mathrm{SU}(V)$ are not since our character Θ of T, when restricted to $T \cap \mathrm{SU}(V)$, is not regular because of the presence of the element $g_0 \in \mathrm{GU}(V)$.

For depth-zero supercuspidal representations of quasisplit unitary groups, the parahoric that we have used is the only one that can lead to higher multiplicities.

8. Generalities on constructing higher multiplicities

In this section, we discuss some generalities underlying the example of the previous section, which will be useful for constructing higher multiplicities in general.

Let G be a group, and N a normal subgroup of G such that

$$G/N \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2.$$

A good example to keep in mind is $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, the quaternion group of order 8, and $N = \{\pm 1\}$. Let ω_1 and ω_2 be two distinct, nontrivial characters of G that are trivial on N.

Suppose π is an irreducible representation of G such that

$$\pi \cong \pi \otimes \omega_1 \cong \pi \otimes \omega_2.$$

By [Gelbart and Knapp 1982, §2], $\pi|_N$ must be one of

(1) a sum of four inequivalent, irreducible representations, or

(2) a sum of two copies of an irreducible representation.

Deciding which of these two options we have is a subtle question, and this is what we wish to do here.

Let $N_1 = \ker\{\omega_1 \colon G \to \mathbf{Z}/2\}$, so that $G \supset N_1 \supset N$. Because $\pi \cong \pi \otimes \omega_1$, $\pi|_{N_1}$ is equal to $\pi_1 \oplus \pi_2$, a sum of inequivalent, irreducible representations. Further, since $\pi \cong \pi \otimes \omega_2$, we have

$$(\pi_1 \oplus \pi_2) \cong (\pi_1 \oplus \pi_2) \otimes \omega_{21},$$

where ω_{21} is equal to $\omega_2|_{N_1}$, a nontrivial character of N_1 of order 2. Therefore, we have the following two possibilities:

(i) $\pi_1 \cong \pi_1 \otimes \omega_{21}$.

(ii) $\pi_2 \cong \pi_1 \otimes \omega_{21}$.

In case (i), π_1, which is an irreducible representation of N_1, decomposes when restricted to N into two inequivalent irreducible representations, and therefore π has at least two inequivalent irreducible subrepresentations when restricted to N; hence, in case (i),

$$\pi|_N = \text{a sum of 4 inequivalent, irreducible representations.}$$

In case (ii), clearly $\pi|_N$ is twice an irreducible representation.

How does one then construct an example of an irreducible representation π of G for which $\pi|_N$ is twice an irreducible representation? We start with an irreducible representation π_1 of N_1 such that the following equivalent conditions hold:

(i) π_1 does not extend to a representation of G.

(ii) $\pi_1^g \ncong \pi_1$ for some $g \in G$.

Given such a representation π_1 of N_1, next we must ensure that

$$\pi_1^g \cong \pi_1 \otimes \omega_{21} \quad \text{for } g \in G \setminus N.$$

If we understand N_1, together with the action of G on the representations of N_1, then the condition

$$\pi_1^g \cong \pi_1 \otimes \omega_{21} \ncong \pi_1$$

is checkable, constructing an irreducible representation $\pi = \mathrm{Ind}_{N_1}^G \pi_1$ of G such that

$$\pi|_N = 2\pi_1|_N.$$

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JEFFREY D. ADLER
DEPARTMENT OF MATHEMATICS AND STATISTICS
AMERICAN UNIVERSITY
WASHINGTON, DC
UNITED STATES
jadler@american.edu

DIPENDRA PRASAD
INDIAN INSTITUTE OF TECHNOLOGY
MUMBAI
INDIA
and
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
MUMBAI
INDIA
and
LABORATORY OF MODERN ALGEBRA AND APPLICATIONS
SAINT-PETERSBURG STATE UNIVERSITY
RUSSIA
prasad.dipendra@gmail.com

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