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We show that the order of h-torsion homology classes in the Bar-Natan deformation of Khovanov homology with $\mathbb{Z}/2\mathbb{Z}$-coefficients is a lower bound for the unknotting number. This is not a bound for the slice genus, unlike most lower bounds for the unknotting number, and only vanishes for the unknot. We give examples of knots for which this is a better lower bound than $|s(K)/2|$, where $s(K)$ is the Rasmussen s invariant defined by the Bar-Natan spectral sequence.

1. Introduction

Khovanov [2000] introduced a knot (and link) invariant which categorifies the Jones polynomial, now known as *Khovanov homology*. This invariant is constructed by applying a specific TQFT to the cube of resolutions corresponding to a projection of the knot. Using a different TQFT, Bar-Natan [2005] defined a deformation of Khovanov homology, called Bar-Natan homology. Let $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. This invariant is a bigraded $\mathbb{F}[h]$-module, and in this paper we introduce a lower bound for the unknotting number in terms of the order of *h-torsion* elements.

For a knot K, let $\mathcal{H}_{BN}(K)$ denote the Bar-Natan homology of K. A homology class $\alpha \in \mathcal{H}_{BN}(K)$ is called *torsion* if $h^n.\alpha = 0$ for a positive integer n. The smallest n with this property is called the *order* of α, denoted by $\mathrm{ord}(\alpha)$. Let $T_{BN}(K)$ denote the set of torsion classes in $\mathcal{H}_{BN}(K)$.

Definition 1.1. For a knot K in $\mathbb{R}^3$, we define

$$\mathfrak{u}(K) := \max_{\alpha \in T_{BN}(K)} \mathrm{ord}(\alpha).$$

Theorem 1.2. *For any knot K, $\mathfrak{u}(K)$ is a lower bound for the unknotting number of K.*

Let K_+ and K_- be knot diagrams that differ in a single crossing c. Orient them

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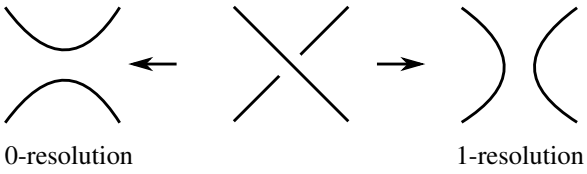

Figure 1. The two possible resolutions of a crossing.

such that c is a positive crossing in K_+ and a negative crossing in K_-. We prove Theorem 1.2 by introducing chain maps

$$f_c^+ : \mathcal{C}_{BN}(K_+) \to \mathcal{C}_{BN}(K_-) \quad \text{and} \quad f_c^- : \mathcal{C}_{BN}(K_-) \to \mathcal{C}_{BN}(K_+) \tag{1}$$

such that the induced maps by $f_c^- \circ f_c^+$ and $f_c^+ \circ f_c^-$ on $\mathcal{H}_{BN}(K_+)$ and $\mathcal{H}_{BN}(K_-)$, respectively, are equal to multiplication by h. In [Alishahi and Dowlin 2018], we introduce similar chain maps for the Lee homology and prove the knight move conjecture [Khovanov 2000; Bar-Natan 2002] for knots with unknotting number smaller than 3.

Despite the algebraic definition of the chain maps (1), we show that they can be described in terms of the cobordism maps associated to specific cobordisms from K_+ to $K_-\#H$, and K_- to $K_+\#mH$, where H is the right-handed Hopf link and mH is its mirror. In [Alishahi and Eftekhary 2016], we use corresponding cobordism maps for the knot Floer homology to deduce a lower bound for the unknotting number, in terms of the order of torsion classes in variants of knot Floer homology.

This paper is organized as follows. Section 2 reviews the Bar-Natan chain complex and collects some results we will need later. Section 3 proves Theorem 1.2. Section 4 gives a geometric description, using cobordism maps, for the chain maps, defined algebraically, in the process of proving Theorem 1.2 in Section 3. Finally, Section 5 gives examples of knots for which our invariant (Definition 1.1) is a better lower bound compared to the s-invariant, i.e., $\mathfrak{u}(K) > |s(K)|/2$.

2. Background

In this section, we review the Bar-Natan chain complex, describe its module structure and discuss some of its basic properties.

Bar-Natan's deformation of Khovanov homology. Let K be an oriented knot or link diagram in $\mathbb{R}^2$ with n crossings. Denote the set of crossings in K by $\mathfrak{C} = \{c_1, \dots, c_n\}$. Each crossing can be resolved in two different ways, the 0-resolution and the 1-resolution; see Figure 1.

For any vertex v of $\{0, 1\}^n$, let K_v denote the complete resolution obtained by replacing the crossing c_i by its v_i-resolution. Let k_v denote the number of connected components of K_v.

There is a partial order on $\{0,1\}^n$ by setting $u \le v$ if $u_i \le v_i$ for all $1 \le i \le n$. Denote $u \lessdot v$ if $u < v$ and $|v| - |u| = 1$, where $|v|$ denotes $\sum_i v_i$. Corresponding to each edge of the cube, i.e., a pair $u \lessdot v$, there is an embedded cobordism in $\mathbb{R}^2 \times [0,1]$ from K_u to K_v, constructed by attaching an embedded one-handle near the crossing c_i where $u_i < v_i$. If $k_u > k_v$, the cobordism *merges* two circles, and otherwise *splits* two circles.

Set $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. Let $\boldsymbol{A}$ denote the 2-dimensional Frobenius algebra over $\mathbb{F}[h]$ with basis $\{x_+, x_-\}$ and multiplication and comultiplication defined as:

$$\begin{aligned}
&x_+ \otimes x_+ \overset{m}{\mapsto} x_+, \\
&x_- \otimes x_+ \overset{m}{\mapsto} x_-, \qquad & x_+ &\overset{\Delta}{\mapsto} x_+ \otimes x_- + x_- \otimes x_+ + h x_+ \otimes x_+, \\
&x_+ \otimes x_- \overset{m}{\mapsto} x_-, \qquad & x_- &\overset{\Delta}{\mapsto} x_- \otimes x_-. \\
&x_- \otimes x_- \overset{m}{\mapsto} h x_-,
\end{aligned}$$

The Bar-Natan chain complex is obtained by applying the (1+1)-dimensional TQFT corresponding to $\boldsymbol{A}$ to the above cube of cobordisms for K. More precisely, corresponding to a vertex $v \in \{0,1\}^n$, a Khovanov generator is a labeling of the circles in K_v by x_+ or x_-. The module $\mathcal{C}_{BN}(K_v)$ is defined as the free $\mathbb{F}[h]$-module generated by the Khovanov generators corresponding to v and

$$\mathcal{C}_{BN}(K) := \bigoplus_{v \in \{0,1\}^n} \mathcal{C}_{BN}(K_v).$$

The differential δ_{BN} decomposes along the edges; for any $u \lessdot v$ the component

$$\delta_{BN}^{u,v} : \mathcal{C}_{BN}(K_u) \to \mathcal{C}_{BN}(K_v)$$

is defined by the multiplication if K_v is obtained from K_u by merging two circles; otherwise, it is defined by the comultiplication. The Bar-Natan chain complex, $(\mathcal{C}_{BN}(K), \delta_{BN})$, was studied by Bar-Natan [2005]; the homology is denoted by $\mathcal{H}_{BN}(K)$. For simplicity, we denote the differential by δ.

The chain complex is bigraded by the homological grading gr_h, and an internal grading gr_q, called *quantum* grading. The homological grading for each summand $\mathcal{C}_{BN}(K_v)$ of $\mathcal{C}_{BN}(K)$ is given by $|v| - n_-$, where $n_\bullet$ denotes the number of $\bullet$-crossings in K for $\bullet \in \{+,-\}$. The quantum grading for each Khovanov generator x at a vertex v is given by

$$\mathrm{gr}_q(x) = n_+ - 2n_- + |v| + k_v^+ - k_v^-,$$

where $k_v^\bullet$ denotes the number of circles labeled by $x_\bullet$ in K_v, for $\bullet = +, -$. Furthermore, the formal variable h has homological grading 0 and quantum grading -2. Note that if K is a knot, the bigraded module $\mathcal{H}_{BN}(K)$ is independent of the choice of orientation on K.

Module structure on Bar-Natan homology and basepoint action. Let K be a knot diagram and p be a point on K away from the crossings. The choice of p induces a module structure on the Khovanov homology of K described in [Khovanov 2003]. We recall this structure for Bar-Natan homology. Choose a small unknot U near p and disjoint from K such that merging the unknot with K gives a knot or link diagram isotopic to K. Then, attaching the corresponding embedded one-handle to $K \sqcup U$ gives an embedded cobordism in $\mathbb{R}^3 \times I$ from $K \sqcup U$ to K, and its associated cobordism map, denoted by m_p,

$$m_p : \mathcal{C}_{BN}(K \sqcup U) = \mathcal{C}_{BN}(K) \otimes_{\mathbb{F}[h]} \boldsymbol{A} \to \mathcal{C}_{BN}(K),$$

is given by the multiplication map m of $\boldsymbol{A}$. More precisely, for a Khovanov generator $x \in \mathcal{C}_{BN}(K_v)$, $m_p(x \otimes x_\bullet)$ is the Khovanov generator obtained from x by multiplying the label of the circle containing p with $x_\bullet$.

Similarly, let

$$\Delta_p : \mathcal{C}_{BN}(K) \to \mathcal{C}_{BN}(K) \otimes_{\mathbb{F}[h]} \boldsymbol{A}$$

denote the cobordism map associated to the inverse cobordism from K to $K \sqcup U$.

If K is related to another knot diagram K' by a Reidemeister move away from the basepoint p, it is straightforward that the chain homotopy equivalence between $\mathcal{C}_{BN}(K)$ and $\mathcal{C}_{BN}(K')$, defined in [Bar-Natan 2005], commutes with m_p. On the other hand, any Reidemeister move which crosses p is equivalent to a sequence of Reidemeister moves away from p. So it induces an $\boldsymbol{A}$-module structure on the Bar-Natan homology of the underlying knot K.

For a point p on K, let

$$\boldsymbol{x}_p : \mathcal{C}_{BN}(K) \to \mathcal{C}_{BN}(K)$$

be the chain map $\boldsymbol{x}_p(a) = m_p(a \otimes x_-)$, defined as in [Hedden and Ni 2013]. Therefore, for a Khovanov generator $x \in \mathcal{C}_{BN}(K_v)$ if the circle containing p is labeled by x_+ then $\boldsymbol{x}_p(x)$ is the Khovanov generator obtained from x by changing the label of this circle to x_-, otherwise $\boldsymbol{x}_p(x) = hx$. Thus, $\boldsymbol{x}_p \circ \boldsymbol{x}_p = h\boldsymbol{x}_p$ and $\boldsymbol{x}_p$ reduces the quantum grading by 2.

Lemma 2.1. *For any point $p \in K$ away from the crossings and $a \in \mathcal{C}_{BN}(K)$,*

$$\Delta_p(a) = a \otimes x_- + (\boldsymbol{x}_p(a) + ha) \otimes x_+.$$

Proof. It is enough to check this relation for each Khovanov generator $x \in \mathcal{C}_{BN}(K_v)$. If the circle containing p is labeled by x_+, then by definition

$$\Delta_p(x) = x \otimes x_- + \boldsymbol{x}_p(x) \otimes x_+ + hx \otimes x_+ = x \otimes x_- + (\boldsymbol{x}_p(x) + hx) \otimes x_+.$$

If the circle containing p is labeled by x_-, then $\boldsymbol{x}_p(x) + hx = 0$, and by definition

$$\Delta_p(x) = x \otimes x_-. \qquad \square$$

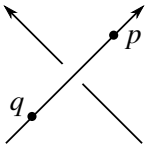

Figure 2. Two points lying on opposite sides of a single crossing.

In contrast to the Khovanov homology, the chain homotopy type of the module multiplication map $\boldsymbol{x}_p$ is not independent of the marked point p. In fact, [Hedden and Ni 2013, Lemma 2.3] may be generalized to describe the difference of $\boldsymbol{x}_p$ and $\boldsymbol{x}_q$ when p and q lie on the opposite sides of a crossing as follows.

Lemma 2.2. *Let $p, q \in K$ be points away from the crossings that lie on the opposite sides of a single crossing as in Figure 2. Then, $\boldsymbol{x}_p + \boldsymbol{x}_q$ is homotopy equivalent to multiplication by h.*

Before proving this lemma, we review some of the properties of the cobordism maps associated with oriented *saddle* moves. Assume K and L are oriented link diagrams such that L is obtained from K by an oriented saddle move as in Figure 3. This saddle move represents an oriented, embedded saddle cobordism in $\mathbb{R}^3 \times [0, 1]$ from K to L. Let

$$\mathfrak{f} : \mathcal{C}_{BN}(K) \to \mathcal{C}_{BN}(L)$$

denote the chain map on the Bar-Natan chain complex associated with this cobordism.

Lemma 2.3. *For any point p on K away from the crossings,*

$$\mathfrak{f} \circ \boldsymbol{x}_p = \boldsymbol{x}_p \circ \mathfrak{f}.$$

Proof. To prove this, it suffices to check two elementary cases:

(1) K is one circle and L is a disjoint union of two circles, and so $\mathfrak{f} = \Delta$. Then,

$$\Delta \circ \boldsymbol{x}_p(x_+) = x_- \otimes x_- = \boldsymbol{x}_q(\Delta x_+) = \boldsymbol{x}_{q'}(\Delta x_+)$$

and

$$\Delta \circ \boldsymbol{x}_p(x_-) = hx_- \otimes x_- = \boldsymbol{x}_q(x_- \otimes x_-) = \boldsymbol{x}_{q'}(x_- \otimes x_-).$$

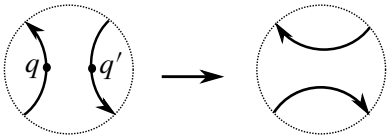

Figure 3. Saddle move: q and q' are the attaching points of the corresponding one-handle.

(2) K is a disjoint union of two circles and L is one circle, and so $\mathfrak{f} = m$. Then,

$$\begin{aligned}
\boldsymbol{x}_p \circ m(x_+ \otimes x_+) &= \ \ x_- \ \ = m \circ \boldsymbol{x}_p(x_+ \otimes x_+),\\
\boldsymbol{x}_p \circ m(x_+ \otimes x_-) &= hx_- = m \circ \boldsymbol{x}_p(x_+ \otimes x_-),\\
\boldsymbol{x}_p \circ m(x_- \otimes x_+) &= hx_- = m \circ \boldsymbol{x}_p(x_- \otimes x_+),\\
\boldsymbol{x}_p \circ m(x_- \otimes x_-) &= h^2x_- = m \circ \boldsymbol{x}_p(x_- \otimes x_-).
\end{aligned}$$ □

Let $\bar{\mathfrak{f}} : \mathcal{C}_{BN}(L) \to \mathcal{C}_{BN}(K)$ be the chain map associated with the inverse saddle move.

Lemma 2.4. *With the above notation fixed, for any $a \in \mathcal{C}_{BN}(K)$ we have*

$$\bar{\mathfrak{f}} \circ \mathfrak{f}(a) = ha + \boldsymbol{x}_q(a) + \boldsymbol{x}_{q'}(a). \tag{2}$$

Proof. Given a vertex $v \in \{0,1\}^n$, if q and q' lie on the same connected component of the complete resolution K_v, then $\boldsymbol{x}_q|_{\mathcal{C}_{BN}(K_v)} = \boldsymbol{x}_{q'}|_{\mathcal{C}_{BN}(K_v)}$. Thus, for any $a \in \mathcal{C}_{BN}(K_v)$, (2) follows from

$$m\Delta(x_+) = hx_+ \quad \text{and} \quad m\Delta(x_-) = hx_-.$$

Otherwise, if q and q' belong to distinct connected components of K_v, for any Khovanov generator $x \in \mathcal{C}_{BN}(K_v)$ the statement follows from one of the following relations, depending on the labels of the circles containing q and q':

$$\begin{aligned}
&\Delta m(x_+ \otimes x_+) = hx_+ \otimes x_+ + x_- \otimes x_+ + x_+ \otimes x_-,\\
&\Delta m(x_- \otimes x_+) = x_- \otimes x_- = hx_- \otimes x_+ + m(x_- \otimes x_-) \otimes x_+ + x_- \otimes x_-,\\
&\Delta m(x_+ \otimes x_-) = x_- \otimes x_- = hx_+ \otimes x_- + x_- \otimes x_- + x_+ \otimes m(x_- \otimes x_-),\\
&\Delta m(x_- \otimes x_-) = hx_- \otimes x_- = hx_- \otimes x_- + m(x_- \otimes x_-) \otimes x_- + x_- \otimes m(x_- \otimes x_-).
\end{aligned}$$ □

Proof of Lemma 2.2. The proof is similar to the proof of [Hedden and Ni 2013, Lemma 2.3]. Denote the crossing between p and q by c. Let $K_\bullet$ be the diagram obtained from K by applying the $\bullet$-resolution at c. One may orient K_0 and K_1 such that K_1 is obtained from K_0 by an oriented saddle move, and up to appropriate grading shifts, $\mathcal{C}_{BN}(K)$ is given by the mapping cone

$$\mathfrak{f} : \mathcal{C}_{BN}(K_0) \to \mathcal{C}_{BN}(K_1),$$

where $\mathfrak{f}$ is the corresponding cobordism map. Let $\bar{\mathfrak{f}}$ denote the cobordism map associated with the inverse cobordism from K_1 to K_0. Under this decomposition we define

$$H(a_0, a_1) := (\bar{\mathfrak{f}}(a_1), 0).$$

Then,

$$\begin{aligned}\delta H(a_0, a_1) + H\delta(a_0, a_1) &= \delta(\bar{\mathfrak{f}}(a_1), 0) + H(\delta a_0, \mathfrak{f}(a_0) + \delta a_1)\\ &= (\delta\bar{\mathfrak{f}}(a_1), \mathfrak{f}\bar{\mathfrak{f}}(a_1)) + (\bar{\mathfrak{f}}(\mathfrak{f}(a_0) + \delta a_1), 0)\\ &= (\bar{\mathfrak{f}}\mathfrak{f}(a_0), \mathfrak{f}\bar{\mathfrak{f}}(a_1)),\end{aligned}$$

and thus it follows from Lemma 2.4 that H is a chain homotopy between $\boldsymbol{x}_p + \boldsymbol{x}_q$ and multiplication by h. □

Corollary 2.5. *Assume K and L are oriented link diagrams so that L is obtained from K by an oriented saddle move. If the attaching points of the saddle lie on the same connected component of K, then $\bar{\mathfrak{f}} \circ \mathfrak{f}$ is chain homotopic to multiplication by h. As before, $\mathfrak{f}$ and $\bar{\mathfrak{f}}$ denote the chain maps associated with the corresponding saddle cobordism and its inverse, respectively.*

Proof. Let p and q be the attaching points of the saddle. Consider an arc $\alpha \subset K$ connecting p to q. Moving the point p along α, it would cross an even number of crossings until it gets to q; thus Lemma 2.2 implies that $\boldsymbol{x}_p$ is homotopy equivalent to $\boldsymbol{x}_q$. Then, by Lemma 2.4 we have that $\bar{\mathfrak{f}}\mathfrak{f}$ is homotopy equivalent to multiplication by h. □

3. Lower bound for unknotting number

The goal of this section is to prove Theorem 1.2.

Let C be a chain complex of $\mathbb{F}[h]$-modules. Recall that a homology class $\alpha \in \mathrm{H}_\star(C)$ is called *torsion* if $h^n\alpha = 0$ for some positive n, and the smallest such n is called the *order* of α, denoted by $\mathrm{ord}(\alpha)$. Let $T(C)$ be the set of torsion homology classes in $\mathrm{H}_\star(C)$ and define

$$\mathfrak{u}(C) := \max_{\alpha \in T(C)} \mathrm{ord}(\alpha).$$

Lemma 3.1. *Given chain complexes C and C' of $\mathbb{F}[h]$-modules, together with chain maps*

$$f : C \to C' \quad \text{and} \quad g : C' \to C$$

so that both $f_\star \circ g_\star$ and $g_\star \circ f_\star$ are equal to multiplication by h^n for some $n > 0$, then

$$|\mathfrak{u}(C) - \mathfrak{u}(C')| \le n.$$

Proof. For any homology class $\alpha \in T(C)$ we have $f_\star(\alpha) \in T(C')$, and it follows from $g_\star \circ f_\star(\alpha) = h^n\alpha$ that

$$\mathrm{ord}(h^n\alpha) \le \mathrm{ord}(f_\star(\alpha)) \le \mathrm{ord}(\alpha).$$

Thus, $\mathrm{ord}(\alpha) \le \mathrm{ord}(f_\star(\alpha)) + n$, and so $\mathfrak{u}(C) \le \mathfrak{u}(C') + n$. Similarly, $\mathfrak{u}(C') \le \mathfrak{u}(C) + n$, which proves the claim. □

Suppose K_+ and K_- are oriented knot diagrams so that K_- is obtained from K_+ by changing one positive crossing, denoted by c, to a negative crossing. For $i = 0, 1$, denote the i-resolution of K_+ at the crossing c by K_i. We orient K_0 and K_1 such that they are related by an oriented saddle move. Let $\mathfrak{f}$ and $\bar{\mathfrak{f}}$ denote the cobordism maps corresponding to the saddle cobordism from K_0 to K_1 and its inverse from K_1 to K_0, respectively. The Bar-Natan chain complexes $\mathcal{C}_{BN}(K_+)$ and $\mathcal{C}_{BN}(K_-)$, up to grading shifts, are given by the mapping cones of $\mathfrak{f}$ and $\bar{\mathfrak{f}}$, respectively.

Choose the points p and q on the opposite sides of the crossing c, as in Figure 2 and define

$$\begin{aligned} &f_c^+ : \mathcal{C}_{BN}(K_+) \to \mathcal{C}_{BN}(K_-), && f_c^- : \mathcal{C}_{BN}(K_-) \to \mathcal{C}_{BN}(K_+), \\ &f_c^+(a_0, a_1) = ((\boldsymbol{x}_p + \boldsymbol{x}_q)(a_1), a_0), && f_c^-(a_1, a_0) = ((\boldsymbol{x}_p + \boldsymbol{x}_q)(a_0), a_1), \end{aligned} \tag{3}$$

where $a_i \in \mathcal{C}_{BN}(K_i)$.

Lemma 3.2. *Both f_c^+ and f_c^- are chain maps.*

Proof. Let $\delta_\bullet$ denote the differential of $\mathcal{C}_{BN}(K_\bullet)$ for $\bullet \in \{0, 1, +, -\}$. Then,

$$f_c^+\delta_+(a_0, a_1) = f_c^+(\delta_0 a_0, \delta_1 a_1 + \mathfrak{f}(a_0)) = ((\boldsymbol{x}_p + \boldsymbol{x}_q)(\delta_1 a_1 + \mathfrak{f}(a_0)), \delta_0 a_0).$$

Both $\mathfrak{f}$ and $\bar{\mathfrak{f}}$ commute with $\boldsymbol{x}_p + \boldsymbol{x}_q$ by Lemma 2.3. On the other hand, for any $v \in \{0, 1\}^{n-1}$, the points p and q lie on the same component of either K_{0v} or K_{1v}, the complete resolutions of K_0 and K_1 by v, respectively. Thus, $\boldsymbol{x}_p + \boldsymbol{x}_q$ vanishes on one of $\mathcal{C}_{BN}(K_{0v})$ or $\mathcal{C}_{BN}(K_{1v})$, and so

$$\mathfrak{f} \circ (\boldsymbol{x}_p + \boldsymbol{x}_q) = (\boldsymbol{x}_p + \boldsymbol{x}_q) \circ \mathfrak{f} = \bar{\mathfrak{f}} \circ (\boldsymbol{x}_p + \boldsymbol{x}_q) = (\boldsymbol{x}_p + \boldsymbol{x}_q) \circ \bar{\mathfrak{f}} = 0.$$

As a result, we get

$$f_c^+\delta_+(a_0, a_1) = ((\boldsymbol{x}_p + \boldsymbol{x}_q)\delta_1(a_1), \delta_0(a_0))$$

and

$$\delta_- f_c^+(a_0, a_1) = \delta_-((\boldsymbol{x}_p + \boldsymbol{x}_q)a_1, a_0) = ((\boldsymbol{x}_p + \boldsymbol{x}_q)\delta_1(a_1), \delta_0(a_0)).$$

Therefore, f_c^+ is a chain map. The proof for f_c^- is similar. □

Corollary 3.3. *With the above notation fixed,* $|\mathfrak{u}(K_+) - \mathfrak{u}(K_-)| \leq 1$.

Proof. By Lemma 2.2 the induced maps on homology by both $f_c^+ \circ f_c^-$ and $f_c^- \circ f_c^+$ are equal to multiplication by h. Thus the claim follows from Lemma 3.1. □

Proof of Theorem 1.2. Consider a diagram for K such that we get a diagram for the unknot after switching N crossings $\{c_1, \ldots, c_N\}$, where N is the unknotting number of K. Abusing the notation we denote the diagram by K. For any $i = 1, \ldots, N$, let K_i be the diagram obtained from K after switching the crossings $c_1, \ldots, c_i$. The diagrams K_{i-1} and K_i differ in a single crossing for each i, so it follows from Corollary 3.3 that $|\mathfrak{u}(K_{i-1}) - \mathfrak{u}(K_i)| \leq 1$. Thus, $|\mathfrak{u}(K) - \mathfrak{u}(\text{Unknot})| = \mathfrak{u}(K) \leq N$. □

Setting $h = 1$, one may think of $(\mathcal{C}_{BN}(K), \delta)$ as a filtered chain complex of $\mathbb{F}$-modules, where the differential increases homological grading by 1 and does not decrease quantum grading. This gives a spectral sequence from $\mathrm{Kh}(K) = \mathrm{Kh}(K; \mathbb{F})$ to $\mathbb{F} \oplus \mathbb{F}$, called the Bar-Natan spectral sequence [Turner 2006].

Lemma 3.4. *For any knot K, if the Bar-Natan spectral sequence collapses in the n-th page, then $\mathfrak{u}(K) = n - 1$.*

Proof. The power of h induces a filtration on the chain complex $(\mathcal{C}_{BN}(K), \delta)$. This gives a spectral sequence, closely related to the Bar-Natan spectral sequence, from $\mathrm{Kh}(K) \otimes_{\mathbb{F}} \mathbb{F}[h]$ to $\mathcal{H}_{BN}(K)$. For each k, the k-th page of this spectral sequence has a free part F^k and a torsion part T^k. Inductively, it is straightforward to prove that $F^k = E^k_{BN} \otimes_{\mathbb{F}} \mathbb{F}[h]$, where E^k_{BN} denotes the k-th page of the Bar-Natan spectral sequence, and its differential on the k-th page is obtained by multiplying the differential on E^k_{BN} by h^k. So, the maximum order of elements in T^k is $k - 1$. If this spectral collapses in the n-th page, $T^n = T_{BN}(K)$ implies that $\mathfrak{u}(K) = n - 1$. Since, this spectral sequence collapses in the same page as the Bar-Natan spectral sequence, we are done. □

Lemma 3.5. *The Bar-Natan spectral sequence for a knot K collapses in the first page if and only if K is the unknot.*

Proof. It follows from [Kronheimer and Mrowka 2011] that K is the unknot if and only if the reduced Khovanov homology of K, $\mathrm{Khr}(K) = \mathrm{Khr}(K; \mathbb{F})$, is isomorphic to $\mathbb{F}$. Since, $\mathrm{Kh}(K) \cong \mathrm{Khr}(K) \oplus \mathrm{Khr}(K)$ we conclude that $\mathrm{Kh}(K) \cong \mathbb{F} \oplus \mathbb{F}$, and so $E^1 = E^\infty$ if and only if K is the unknot. □

Corollary 3.6. *If K is not the unknot, then $\mathfrak{u}(K) > 0$.*

Proof. If K is nontrivial, by Lemma 3.5, the Bar-Natan spectral sequence for K collapses in n-th page for some $n > 1$. Then, Lemma 3.4 implies $\mathfrak{u}(K) = n - 1 > 0$. □

4. A geometric interpretation of the chain maps

Suppose K and K' are oriented pointed knots, i.e., oriented knots with marked points on them, such that K' is obtained from K by a sequence of crossing changes. To any such sequence, Eftekhary and the author associate a decorated cobordism from K to a connected sum of K' with some right- or left-handed Hopf links. Then, by the corresponding cobordism maps for knot Floer homology [Alishahi and Eftekhary 2016, Section 8.2], we define chain maps between the knot Floer chain complexes of K and K' satisfying the assumptions of Lemma 3.1 [Alishahi and Eftekhary 2016, Section 8.3]. As a result, one gets a lower bound for the unknotting number in term of the *u-torsion* in knot Floer homology.

Following the approach in [Alishahi and Eftekhary 2016], one may use cobordism maps for Bar-Natan homology to define chain maps between $\mathcal{C}_{BN}(K)$ and $\mathcal{C}_{BN}(K')$

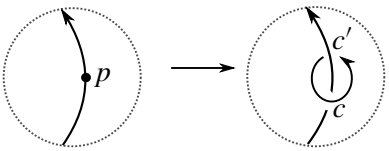

Figure 4. Connected sum with right-handed Hopf link at p.

which satisfy the assumptions of Lemma 3.1. The goal of this section is to show that for any crossing change process, these chain maps are equal to the ones defined in Section 3.

Connected sum with a Hopf link. A connected sum formula for Khovanov homology has been studied in [Khovanov 2000]. In this section, we recall a special case of this formula, taking a connected sum with the Hopf link, for the Bar-Natan homology.

Let H be the right-handed Hopf link. Choose an arbitrary point $p \in K$. We obtain an oriented diagram L for the link $K\#H$ by changing K locally in a neighborhood of p, as in Figure 4.

Denote the new crossings by c and c'. For $i, j \in \{0, 1\}$, let L_{ij} denote the diagram obtained from L by applying i- and j-resolutions at the crossings c and c', respectively. We orient these diagrams such that their orientations coincide with the orientation of K outside the above neighborhood of p. The oriented diagrams L_{01} and L_{10} are isotopic to K, while L_{00} and L_{11} are isotopic to a disjoint union of K with an unknot near the point p. The chain complex $\mathcal{C}_{BN}(L)$ is given by the mapping cone

$$\begin{array}{ccc} \mathcal{C}_{BN}(L_{10}) = \mathcal{C}_{BN}(K) & \xrightarrow{\Delta_p} & \mathcal{C}_{BN}(L_{11}) = \mathcal{C}_{BN}(K) \otimes_{\mathbb{F}[h]} \boldsymbol{A} \\ \uparrow{\scriptstyle m_p} & & \uparrow{\scriptstyle \Delta_p} \\ \mathcal{C}_{BN}(L_{00}) = \mathcal{C}_{BN}(K) \otimes_{\mathbb{F}[h]} \boldsymbol{A} & \xrightarrow{m_p} & \mathcal{C}_{BN}(L_{01}) = \mathcal{C}_{BN}(K) \end{array}$$

From now on, we use this decomposition to write any $a \in \mathcal{C}_{BN}(L)$ as $a = (a_{00}, a_{01}, a_{10}, a_{11})$ where $a_{ij} \in \mathcal{C}_{BN}(L_{ij})$ and so $a_{00}, a_{11} \in \mathcal{C}_{BN}(K) \otimes \boldsymbol{A}$ while $a_{10}, a_{01} \in \mathcal{C}_{BN}(K)$.

We define chain maps $i : \mathcal{C}_{BN}(K) \to \mathcal{C}_{BN}(L)$ and $p : \mathcal{C}_{BN}(L) \to \mathcal{C}_{BN}(K)$ as

$$i(a) = (0, 0, 0, a \otimes x_+) \quad \text{and} \quad p(a_{00}, a_{01}, a_{10}, a_{11}) = a_{00}^-, \tag{4}$$

where

$$a_{00} = a_{00}^+ \otimes x_+ + a_{00}^- \otimes x_-.$$

It is clear that both i and p are chain maps.

Lemma 4.1. *The sequence*

$$0 \to \mathcal{H}_{BN}(K) \xrightarrow{i_\star} \mathcal{H}_{BN}(L) \xrightarrow{p_\star} \mathcal{H}_{BN}(K) \to 0$$

is a split exact sequence.

Proof. First, we prove that $\mathrm{im}(i_\star) = \ker(p_\star)$. Consider $\alpha \in \ker(p_\star)$ and let $a \in \mathcal{C}_{BN}(L)$ be a cycle representing α. Since, $p(a) = a^-_{00} = \delta b$ for some $b \in \mathcal{C}_{BN}(K)$, after adding $\delta(b \otimes x_-, 0, 0, 0)$ to a, we may assume that $a^-_{00} = 0$. Then, $\delta a = 0$ implies that

$$0 = m_p(a_{00}) + \delta a_{01} = a^+_{00} + \delta a_{01},$$

and thus

$$a + \delta(a_{01} \otimes x_+, 0, 0, 0) = (0, 0, a_{01} + a_{10}, a_{11}).$$

Again, it follows from $\delta a = 0$ and Lemma 2.1 that

$$\delta a_{11} = \Delta_p(a_{10} + a_{01}) = (a_{10} + a_{01}) \otimes x_- + (\boldsymbol{x}_p(a_{10} + a_{01}) + h(a_{10} + a_{01})) \otimes x_+.$$

Therefore, $\delta a^-_{11} = a_{10} + a_{01}$ where $a_{11} = a^+_{11} \otimes x_+ + a^-_{11} \otimes x_-$ and so

$$(0, 0, a_{10}+a_{01}, a_{11})+\delta(0, 0, a^-_{11}, 0)=(0, 0, 0, a_{11}+\Delta_p a^-_{11})=i(a^+_{11}+\boldsymbol{x}_p(a^-_{11})+ha^-_{11}).$$

Note that the last equality follows from Lemma 2.1.

Next, let

$$r : \mathcal{C}_{BN}(L) \to \mathcal{C}_{BN}(K) \quad \text{and} \quad s : \mathcal{C}_{BN}(K) \to \mathcal{C}_{BN}(L)$$

be the chain maps defined as

$$r(a_{00}, a_{01}, a_{10}, a_{11})=a^+_{11}+\boldsymbol{x}_p(a^-_{11})+ha^-_{11} \text{ and } s(a)=(\boldsymbol{x}_p(a)\otimes x_+ + a\otimes x_-, 0, 0, 0),$$

where $a_{11} = a^+_{11} \otimes x_+ + a^-_{11} \otimes x_-$. It is clear that r and s are chain maps such that $r \circ i = \mathrm{id}$ and $p \circ s = \mathrm{id}$. So $i_\star$ and $p_\star$ are injective and surjective, respectively, and the sequence splits. □

The homomorphism $p_\star$ preserves the homological grading and decreases the quantum grading by 1, while $i_\star$ increases the homological grading by 2 and the quantum grading by 5.

Combining the chain maps i, p, r and s, we define chain maps

$$\mathsf{i} : \mathcal{C}_{BN}(K) \to \mathcal{C}_{BN}(L) \quad \text{and} \quad \mathsf{p} : \mathcal{C}_{BN}(L) \to \mathcal{C}_{BN}(K)$$

as

$$\mathsf{i} = i + s \quad \text{and} \quad \mathsf{p} = p + r. \tag{5}$$

Similarly, there exist chain maps i and p for the connected sum of K with the left-handed Hopf link, $K\#mH$, so that the induced homomorphisms on homology give a split exact sequence. The only difference is that $p_\star$ increases the homological

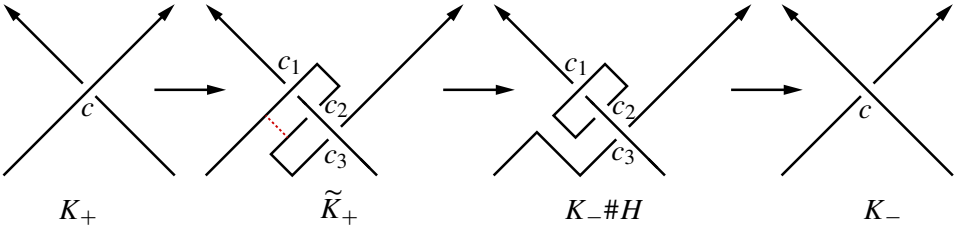

Figure 5. Changing a positive crossing to a negative crossing with an oriented saddle move followed by the removal of a Hopf link summand.

grading by 2 and quantum grading by 5, while $i_\star$ preserves the homological grading and decreases the quantum grading by 1. Also, we define analogous chain maps

$$\mathsf{i} : \mathcal{C}_{BN}(K) \to \mathcal{C}_{BN}(K\#mH) \quad \text{and} \quad \mathsf{p} : \mathcal{C}_{BN}(K\#mH) \to \mathcal{C}_{BN}(K).$$

Crossing change and cobordism maps. As before, let K_+ be an oriented knot diagram with a specific positive crossing c, and K_- be the oriented knot diagram obtained from K_+ by changing c into a negative crossing. As in Figure 5, after a Reidemeister II move near the crossing c on K_+, followed by an oriented saddle move, one gets a diagram for $K_-\#H$. Here, H is the right-handed Hopf link.

As in Figure 5, we denote the knot diagram obtained from K_+ by the specified Reidemeister II move by $\widetilde{K}_+$. Further, let $\mathsf{h} : \mathcal{C}_{BN}(K_+) \to \mathcal{C}_{BN}(\widetilde{K}_+)$ and $\widetilde{\mathsf{h}} : \mathcal{C}_{BN}(\widetilde{K}_+) \to \mathcal{C}_{BN}(K_+)$ be the chain homotopy equivalences corresponding to this move as defined in [Bar-Natan 2005, Section 4.3]. For the reader's convenience, we recall the definition of h and $\widetilde{\mathsf{h}}$. For $i, j = 0, 1$, let $\mathcal{C}^{ij}_{BN}(\widetilde{K}_+)$ denote the direct sum of the summands of $\mathcal{C}_{BN}(\widetilde{K}_+)$ corresponding to vertices v of the cube so that $v(c_2) = i$ and $v(c_3) = j$. Also, let $\mathsf{h}^{ij} = \pi_{ij} \circ \mathsf{h}$ and $\widetilde{\mathsf{h}}^{ij} = \widetilde{\mathsf{h}} \circ \iota_{ij}$ where π_{ij} is the projection of $\mathcal{C}_{BN}(\widetilde{K}_+)$ on $\mathcal{C}^{ij}_{BN}(\widetilde{K}_+)$ and ι_{ij} is the inclusion of $\mathcal{C}^{ij}_{BN}(\widetilde{K}_+)$ in $\mathcal{C}_{BN}(\widetilde{K}_+)$. Then, $\mathsf{h}^{00} = \mathsf{h}^{11} = \widetilde{\mathsf{h}}^{00} = \widetilde{\mathsf{h}}^{11} = 0$, $\mathsf{h}^{10} = \widetilde{\mathsf{h}}^{10} = \mathrm{id}$ and $\mathsf{h}^{01} = g \otimes x_+$, where g is the cobordism map corresponding to the saddle move along the dashed curve in Figure 6. In other words, h^{01} is a chain map for the cobordism which is the union of a saddle and a cup. Finally, $\widetilde{\mathsf{h}}^{01}$ is the cobordism map for the inverse of the aforementioned saddle move union a cap; see Figure 6.

Let $\mathfrak{f} : \mathcal{C}_{BN}(\widetilde{K}_+) \to \mathcal{C}_{BN}(K_-\#H)$ and $\bar{\mathfrak{f}} : \mathcal{C}_{BN}(K_-\#H) \to \mathcal{C}_{BN}(\widetilde{K}_+)$ denote the cobordism maps for the saddle move in Figure 5.

Theorem 4.2. *With the above notation fixed,* $f_c^+ = \mathsf{p} \circ \mathfrak{f} \circ \mathsf{h}$ *and* $f_c^- = \widetilde{\mathsf{h}} \circ \bar{\mathfrak{f}} \circ \mathsf{i}$, *where* i *and* p *are the chain maps defined in* (5).

Proof. We prove both equalities by looking at the cube of resolutions for the three crossings c_1, c_2 and c_3. For $i = 0, 1$, let $\mathcal{C}^i_{BN}(K_+)$ and $\mathcal{C}^i_{BN}(K_-)$ denote the summands of $\mathcal{C}_{BN}(K_+)$ and $\mathcal{C}_{BN}(K_-)$, respectively, corresponding to the i-resolution at c.

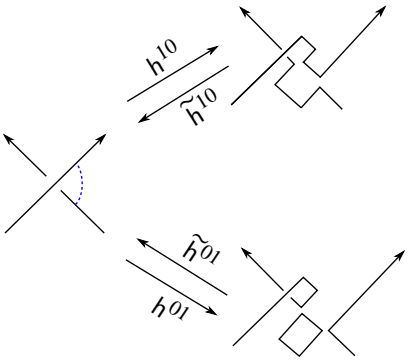

Figure 6. $\mathsf{h}^{10} = \widetilde{\mathsf{h}}^{10} = \mathrm{id}$, while h^{01} and $\widetilde{\mathsf{h}}^{01}$ are the maps associated with the cobordism corresponding to the saddle move on the dashed curve union a cup and its inverse, respectively.

Assume $a = (a_0, a_1)$ is an element in $\mathcal{C}_{BN}(K)$ such that $a_i \in \mathcal{C}^i_{BN}(K_+)$. It follows from the definition of h that $\mathsf{h}^{ii}(a_j) = 0$ for any $i, j = 0, 1$. Further, considering the definition of p, it is enough to compute $\mathfrak{f}(\mathsf{h}^{01}(a_0))$ and $\mathfrak{f}(\mathsf{h}^{10}(a_1))$. As in Figure 7, $\mathsf{h}^{01}(a_0) = \Delta_p(a_0) \otimes x_+$ and so

$$\mathfrak{f}(\mathsf{h}^{01}(a_0)) = \Delta_p(a_0) = a_0 \otimes x_- + (\boldsymbol{x}_p(a_0) + h a_0) \otimes x_+ \tag{6}$$

by Lemma 2.1.

On the other hand, $\mathsf{h}^{10}(a_1) = a_1$ and, as Figure 8 shows,

$$\mathfrak{f}(\mathsf{h}^{10}(a_1)) = \Delta_q(a_1) = a_1 \otimes x_- + (\boldsymbol{x}_q(a_1) + h a_1) \otimes x_+. \tag{7}$$

Thus, by (6) and (7) and the definition of p, we have

$$\mathsf{p}(\mathfrak{f} \circ \mathsf{h}(a)) = (\boldsymbol{x}_p(a_1) + \boldsymbol{x}_q(a_1), a_0).$$

Let $a = (a_0, a_1)$ be an element in $\mathcal{C}_{BN}(K_-)$ where $a_i \in \mathcal{C}^i_{BN}(K_-)$ for $i = 0, 1$. By definition,

$$\mathsf{i}(a_0) = (\boldsymbol{x}_p(a_0) \otimes x_+ + a_0 \otimes x_-, 0, 0, a_0 \otimes x_+),$$
$$\mathsf{i}(a_1) = (\boldsymbol{x}_q(a_1) \otimes x_+ + a_1 \otimes x_-, 0, 0, a_1 \otimes x_+).$$

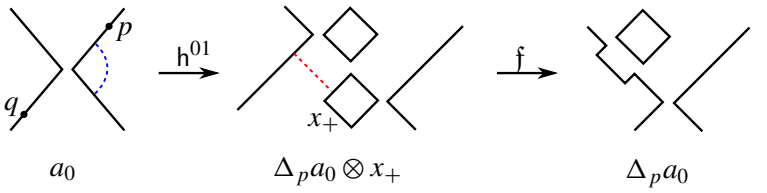

Figure 7. Evaluation of $\mathfrak{f} \circ \mathsf{h}^{01}$ over an element $a_0 \in \mathcal{C}^0_{BN}(K_+)$.

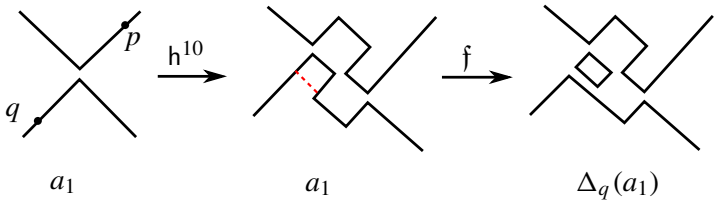

Figure 8. Evaluation of $\mathfrak{f}\circ\mathfrak{h}^{10}$ over an element $a_1 \in \mathcal{C}^1_{BN}(K_+)$.

The only nonzero components of $\widetilde{\mathfrak{h}}$ are $\widetilde{\mathfrak{h}}^{01}$ and $\widetilde{\mathfrak{h}}^{10}$, so

$$\widetilde{\mathfrak{h}}\circ\bar{\mathfrak{f}}\circ\mathfrak{i}(a) = (\widetilde{\mathfrak{h}}^{01}\circ\bar{\mathfrak{f}}(\boldsymbol{x}_q(a_1)\otimes x_+ + a_1\otimes x_-), \widetilde{\mathfrak{h}}^{10}\circ\bar{\mathfrak{f}}(a_0\otimes x_+)).$$

By Figure 8, we have

$$\widetilde{\mathfrak{h}}^{10}\circ\bar{\mathfrak{f}}(a_0\otimes x_+) = \widetilde{\mathfrak{h}}^{10}(a_0) = a_0,$$

and by Figure 7,

$$\begin{aligned}
&\widetilde{\mathfrak{h}}^{01}\circ\bar{\mathfrak{f}}(\boldsymbol{x}_q(a_1)\otimes x_+ + a_1\otimes x_-)\\
&\quad = \widetilde{\mathfrak{h}}^{01}(\Delta_q(\boldsymbol{x}_q(a_1)\otimes x_+ + a_1\otimes x_-))\\
&\quad = \widetilde{\mathfrak{h}}^{01}((\boldsymbol{x}_q(a_1)\otimes x_-)\otimes x_+ + (a_1\otimes x_- + (\boldsymbol{x}_q(a_1)+ha_1)\otimes x_+)\otimes x_-)\\
&\quad = \boldsymbol{x}_q(a_1)+\boldsymbol{x}_p(a_1). \qquad \square
\end{aligned}$$

Remark 4.3. Similarly, one may change K_- by a Reidemeister II move near c to get a diagram $\widetilde{K}_-$, so that $K_+\#mH$ is obtained from $\widetilde{K}_-$ by a saddle move. Let $\mathfrak{h}:\mathcal{C}_{BN}(K_-)\to\mathcal{C}_{BN}(\widetilde{K}_-)$ and $\widetilde{\mathfrak{h}}:\mathcal{C}_{BN}(\widetilde{K}_-)\to\mathcal{C}_{BN}(K_-)$ be the corresponding chain homotopy equivalences and

$$\mathfrak{f}:\mathcal{C}_{BN}(\widetilde{K}_-)\to\mathcal{C}_{BN}(K_+\#mH)$$

and

$$\bar{\mathfrak{f}}:\mathcal{C}_{BN}(K_+\#mH)\to\mathcal{C}_{BN}(\widetilde{K}_-)$$

be the cobordism maps for the saddle move. Then, by the same argument, one can show that $f_c^- = \mathfrak{p}\circ\mathfrak{f}\circ\mathfrak{h}$ and $f_c^+ = \widetilde{\mathfrak{h}}\circ\bar{\mathfrak{f}}\circ\mathfrak{i}$.

5. Examples

The Rasmussen's s invariant gives a lower bound for the slice genus, $|s(K)|/2$, and thus the unknotting number [Rasmussen 2010]. We used Cotton Seed's package, Knotkit [Seed 2011], to compute $\mathfrak{u}$ and s (defined using the Bar-Natan spectral sequence) for some knots with more than 12 crossings. We found some examples where $\mathfrak{u}$ is a better lower bound compared to $|s|/2$; for instance, $|s|/2$ for the knots $13n689$, $13n1166$, $13n2504$ and $13n2807$ is equal to 1, while $\mathfrak{u}$ is equal to 2.

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