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COMPLEMENTED COPIES OF $c_0(\tau)$ IN TENSOR PRODUCTS OF $L_p[0, 1]$

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Let X be a Banach space and τ an infinite cardinal. We show that if τ has uncountable cofinality, $p \in [1, \infty)$, and either the Lebesgue–Bochner space $L_p([0, 1], X)$ or the injective tensor product $L_p[0, 1]\widehat{\otimes}_\varepsilon X$ contains a complemented copy of $c_0(\tau)$, then so does X. We show also that if $p \in (1, \infty)$ and the projective tensor product $L_p[0, 1]\widehat{\otimes}_\pi X$ contains a complemented copy of $c_0(\tau)$, then so does X.

1. Introduction and preliminaries

We use standard set-theoretical and Banach space theory terminology as may be found, e.g., in [Jech 2003] and [Johnson and Lindenstrauss 2001]. We denote by B_X the closed unit ball of the Banach space X. If X and Y are Banach spaces, we denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from X to Y and by $\mathcal{K}(X, Y)$ the subspace of all compact linear operators. We say that Y contains a copy (resp. a complemented copy) of X, and write $X \hookrightarrow Y$ (resp. $X \overset{c}{\hookrightarrow} Y$), if X is isomorphic to a subspace (resp. complemented subspace) of Y. The *density character* of X, denoted by $\text{dens}(X)$, is the smallest cardinality of a dense subset of X.

A Banach space X has the *bounded approximation property* if there exists $\lambda > 0$ such that, for every compact subset K of X and every $\varepsilon > 0$, there exists a finite rank operator $T : X \to X$ such that $\|T\| \le \lambda$ and $\|x - T(x)\| < \varepsilon$ for every $x \in K$.

We shall denote the projective and injective tensor norms by $\|\cdot\|_\pi$ and $\|\cdot\|_\varepsilon$, respectively. The projective (resp. injective) tensor product of X and Y is the completion of $X \otimes Y$ with respect to $\|\cdot\|_\pi$ (resp. $\|\cdot\|_\varepsilon$) and will be denoted by $X\widehat{\otimes}_\pi Y$ (resp. $X\widehat{\otimes}_\varepsilon Y$).

For a nonempty set Γ, $c_0(\Gamma)$ denotes the Banach space of all real-valued maps f on Γ with the property that for each $\varepsilon > 0$, the set $\{\gamma \in \Gamma : |f(\gamma)| \ge \varepsilon\}$ is finite, equipped with the supremum norm. We will refer to $c_0(\Gamma)$ as $c_0(\tau)$ when

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the cardinality of Γ (denoted by $|\Gamma|$) is equal to τ. This space will be denoted by c_0 when $\tau = \aleph_0$. By $\ell_\infty(\Gamma)$ we will denote the Banach space of all bounded real-valued maps on Γ, with the supremum norm. This space will be denoted by ℓ_∞ when $\Gamma = \mathbb{N}$.

Given X a Banach space and $p \in [1, \infty)$, we denote by $L_p([0, 1], X)$ the Lebesgue–Bochner space of all (classes of equivalence of) measurable functions $f : [0, 1] \to X$ such that the scalar function $\|f\|^p$ is integrable, equipped with the complete norm

$$\|f\|_p = \left[\int_0^1 \|f(t)\|^p \, dt\right]^{\frac{1}{p}}.$$

These spaces will be denoted by $L_p[0, 1]$ when $X = \mathbb{R}$.

A measurable function $f : [0, 1] \to X$ is *essentially bounded* if there exists $\varepsilon > 0$ such that the set $\{t \in [0, 1] : \|f(t)\| \geq \varepsilon\}$ has Lebesgue measure zero, and we denote by $\|f\|_\infty$ the infimum of all such numbers $\varepsilon > 0$. By $L_\infty([0, 1], X)$ we will denote the space of all (classes of equivalence of) essentially bounded functions $f : [0, 1] \to X$, equipped with the complete norm $\|\cdot\|_\infty$.

Recall that if τ is an infinite cardinal then the *cofinality* of τ, denoted by $\mathrm{cf}(\tau)$, is the least cardinal α such that there exists a family of ordinals $\{\beta_j : j \in \alpha\}$ satisfying $\beta_j < \tau$ for all $j \in \alpha$, and $\sup\{\beta_j : j \in \alpha\} = \tau$. A cardinal τ is said to be *regular* when $\mathrm{cf}(\tau) = \tau$; otherwise, it is said to be *singular*.

Many papers in the history of the geometry of Banach spaces have been devoted to establishing results about when certain Banach spaces contain complemented copies of c_0 or $c_0(\tau)$ for uncountable cardinals τ; see, for example, [Amir and Lindenstrauss 1968; Argyros et al. 2002; Cembranos 1984; Cembranos and Mendoza 1997; Emmanuele 1988; Sobczyk 1941; Zippin 1977]. The starting points of our research are three of these results related to the space c_0, i.e., Theorems 1, 2 and 3 below.

We begin by recalling the following immediate consequence of the classical Cembranos–Freniche theorem [Cembranos 1984, Main theorem; Freniche 1984, Corollary 2.5].

Theorem 1. *For each* $p \in [1, \infty)$,

$$c_0 \overset{c}{\hookrightarrow} L_p[0, 1]\widehat{\otimes}_\varepsilon \ell_\infty.$$

However, $c_0 \overset{c}{\not\hookrightarrow} \ell_\infty$ (see, e.g., [Diestel and Uhl 1977, Corollary 11, p. 156]).

On the other hand, Oja proved the following stability property.

Theorem 2 [Oja 1991, Theorem 3b]. *If X is a Banach space and $p \in (1, \infty)$, then*

$$c_0 \overset{c}{\hookrightarrow} L_p[0, 1]\widehat{\otimes}_\pi X \Longrightarrow c_0 \overset{c}{\hookrightarrow} X.$$

Observe that Theorem 2 does not hold for $p = 1$. Indeed, $L_1([0, 1], X)$ is linearly isometric to $L_1[0,1]\hat{\otimes}_\pi X$ [Ryan 2002, Example 2.19, p. 29] and Emmanuele obtained the following result.

Theorem 3 [Emmanuele 1988, Main theorem]. *If X is a Banach space and $p\in[1,\infty)$, then*

$$c_0 \hookrightarrow X \Longrightarrow c_0 \overset{c}{\hookrightarrow} L_p([0,1],X).$$

So, in particular, $L_p([0,1],\ell_\infty)$ contains a complemented copy of c_0, but once again $c_0 \overset{c}{\not\hookrightarrow} \ell_\infty$.

We recall also that, denoting by $\|\cdot\|_{\Delta_p}$ the natural tensor norm induced on $L_p[0,1]\otimes X$ by $L_p([0,1],X)$ and by $L_p[0,1]\widehat{\otimes}_{\Delta_p}X$ the completion of $L_p[0,1]\otimes X$ with this norm, the space $L_p([0,1],X)$ is linearly isometric to $L_p[0,1]\widehat{\otimes}_{\Delta_p}X$ [Defant and Floret 1993, Chapters 7.1 and 7.2].

Thus, we are naturally led to the following problem.

Problem 4. For X a Banach space, $p\in[1,\infty)$, and τ an infinite cardinal, we want to know under which conditions

$$c_0(\tau) \overset{c}{\hookrightarrow} L_p[0,1]\widehat{\otimes}_\alpha X \Longrightarrow c_0(\tau) \overset{c}{\hookrightarrow} X$$

holds, where α denotes either the projective, injective or natural norm.

This problem becomes more interesting if we keep in mind that, in general, it is not so simple to determine whether the tensor products of E and X contain complemented copies of a certain space F, even when E contains no complemented copies of F. Indeed, there are a number of elementary questions about this topic that remain unanswered. For instance, it is not known whether $l_\infty\widehat{\otimes}_\pi l_\infty$ contains a complemented copy of c_0 or not [Cabello Sánchez et al. 2006, Remark 3].

In the present paper, we will prove that for every Banach space X, $p\in(1,\infty)$ and an infinite cardinal τ,

$$c_0(\tau) \overset{c}{\hookrightarrow} L_p[0,1]\widehat{\otimes}_\pi X \Longrightarrow c_0(\tau) \overset{c}{\hookrightarrow} X.$$

Additionally, if τ has uncountable cofinality, then for every $p\in[1,\infty)$,

$$c_0(\tau) \overset{c}{\hookrightarrow} L_p[0,1]\widehat{\otimes}_\varepsilon X \Longrightarrow c_0(\tau) \overset{c}{\hookrightarrow} X$$

and

$$c_0(\tau) \overset{c}{\hookrightarrow} L_p([0,1],X) \Longrightarrow c_0(\tau) \overset{c}{\hookrightarrow} X.$$

This paper is organized as follows. We study complemented copies of $c_0(\tau)$ in the injective (Section 2), projective (Section 3) and natural (Section 4) tensor products with $L_p[0,1]$.

2. Complemented copies of $c_0(\tau)$ in $X\hat{\otimes}_\varepsilon Y$ spaces

The goal of this section is to prove Theorem 7. We recall that given Banach spaces X and Y, the operator $S : X\hat{\otimes}_\varepsilon Y \to \mathcal{K}(X^*, Y)$ satisfying

$$S(v)(x^*) = \sum_{i=1}^{j} x^*(a_i)b_i$$

for every $x^* \in X^*$ and $v = \sum_{i=1}^{j} a_i \otimes b_i \in X \otimes Y$, is a linear isometry onto its image.

We will need the following key lemma.

Lemma 5. *Let X and Y be Banach spaces. Suppose that X has the bounded approximation property. Then there exist sets $A \subset X$ and $B \subset X^*$ such that* $\max(|A|, |B|) \leq \operatorname{dens}(X)$ *and for every $u \in X\hat{\otimes}_\varepsilon Y$ and $\delta > 0$ there exist $x_1, \dots, x_m \in A$ and $\varphi_1, \dots, \varphi_m \in B$ satisfying*

$$\left\| u - \sum_{n=1}^{m} x_n \otimes S(u)(\varphi_n) \right\|_\varepsilon < \delta.$$

Proof. By hypothesis, there exists $\lambda \geq 1$ such that for every finite-dimensional subspace Z of X there exists a finite rank operator T on X such that $\|T\| \leq \lambda$ and $T(x) = x$ for all $x \in Z$ [Casazza 2001, Theorem 3.3.(3), p. 288].

Let D be a dense subset of X with $|D| = \operatorname{dens}(X)$ and let $\mathcal{F}$ be the family of all finite, nonempty subsets of D. For each $F \in \mathcal{F}$, fix a finite rank operator T_F on X such that $\|T_F\| \leq \lambda$ and $T_F(d) = d$ for all $d \in F$. Let m_F be the dimension of $T_F(X)$, $\{x_1^F, \dots, x_{m_F}^F\}$ be a basis of $T_F(X)$ and $\varphi_1^F, \dots, \varphi_{m_F}^F \in X^*$ such that

$$T_F(x) = \sum_{n=1}^{m_F} \varphi_n^F(x) x_n^F,$$

for every $x \in X$. Define

$$A = \bigcup_{F\in\mathcal{F}} \{x_1^F, \dots, x_{m_F}^F\} \quad \text{and} \quad B = \bigcup_{F\in\mathcal{F}} \{\varphi_1^F, \dots, \varphi_{m_F}^F\}.$$

We claim that A and B have the desired properties. Indeed, notice that

$$|A| \leq |\mathcal{F}| \sup_{F\in\mathcal{F}} |\{x_1^F, \dots, x_{m_F}^F\}| \leq \max(|D|, \aleph_0) = |D|$$

and similarly $|B| \leq |D|$.

Next, let $u \in X\hat{\otimes}_\varepsilon Y$ and $\delta > 0$ be given. There exists $v = \sum_{j=1}^{k} d_j \otimes y_j \in X \otimes Y$ such that $d_1, \dots, d_k \in D$, $d_i \neq d_j$ if $i \neq j$, and

$$\|u - v\|_\varepsilon < \frac{\delta}{\lambda + 1}.$$

Writing $G = \{d_1, \dots, d_k\}$, we see that

$$\sum_{n=1}^{m_G} x_n^G \otimes S(v)(\varphi_n^G) = \sum_{j=1}^{k}\left(\sum_{n=1}^{m_G} \varphi_n^G(d_j)x_n^G\right) \otimes y_j = \sum_{j=1}^{k} T_G(d_j) \otimes y_j = v.$$

Furthermore, since

$$\left\|\sum_{n=1}^{m_G} x_n^G \otimes \varphi_n^G\right\|_\varepsilon = \sup_{x\in B_X}\left\|\sum_{n=1}^{m_G} \varphi_n^G(x)x_n^G\right\| = \|T_G\| \le \lambda,$$

we obtain

$$\begin{aligned}\left\|\sum_{n=1}^{m_G} x_n^G \otimes S(u-v)(\varphi_n^G)\right\|_\varepsilon &= \sup_{x^*\in B_{X^*}}\left\|\sum_{n=1}^{m_G} x^*(x_n^G)S(u-v)(\varphi_n^G)\right\| \\ &\le \|u-v\|_\varepsilon \sup_{x^*\in B_{X^*}}\left\|\sum_{n=1}^{m_G} x^*(x_n^G)(\varphi_n^G)\right\| \\ &< \frac{\delta}{\lambda+1}\left\|\sum_{n=1}^{m_G} x_n^G \otimes \varphi_n^G\right\|_\varepsilon \le \frac{\lambda\delta}{\lambda+1}.\end{aligned}$$

Thus,

$$\left\|u - \sum_{n=1}^{m_G} x_n^G \otimes S(u)(\varphi_n^G)\right\|_\varepsilon < \delta$$

and we are done. □

The following result [Galego and Cortes 2017] will also be used frequently throughout this work.

Theorem 6 [Galego and Cortes 2017, Theorem 2.4]. *Let X be a Banach space and τ be an infinite cardinal. The following are equivalent*:

(1) *X contains a complemented copy of $c_0(\tau)$.*

(2) *There exist a family $(x_j)_{j\in\tau}$ equivalent to the unit-vector basis of $c_0(\tau)$ in X and a weak*-null family $(x_j^*)_{j\in\tau}$ in X^* such that, for each $j, k \in \tau$,*

$$x_j^*(x_k) = \delta_{jk}.$$

(3) *There exist a family $(x_j)_{j\in\tau}$ equivalent to the unit-vector basis of $c_0(\tau)$ in X and a weak*-null family $(x_j^*)_{j\in\tau}$ in X^* such that*

$$\inf_{j\in\tau} |x_j^*(x_j)| > 0.$$

Theorem 7. *Let X and Y be Banach spaces and τ be an infinite cardinal. If X has the bounded approximation property and* $\operatorname{cf}(\tau) > \operatorname{dens}(X)$, *then*

$$c_0(\tau) \overset{c}{\hookrightarrow} X \hat{\otimes}_\varepsilon Y \Longrightarrow c_0(\tau) \overset{c}{\hookrightarrow} Y.$$

Proof. Let $A \subset X$ and $B \subset X^*$ be the sets provided by Lemma 5. By Theorem 6, there exist families $(u_i)_{i\in\tau}$ in $X\hat{\otimes}_\varepsilon Y$ and $(\psi_i)_{i\in\tau}$ in $(X\hat{\otimes}_\varepsilon Y)^*$ such that $(u_i)_{i\in\tau}$ is equivalent to the usual unit-vector basis of $c_0(\tau)$, $(\psi_i)_{i\in\tau}$ is weak*-null and $\psi_i(u_j) = \delta_{ij}$ for each $i, j \in \tau$. Let $s = \sup_{i\in\tau} \|\psi_i\| < \infty$.

For each $i \in \tau$ there exist $x^i_1, \dots, x^i_{m_i} \in A$ and $\varphi^i_1, \dots, \varphi^i_{m_i} \in B$ such that

$$\left\| u_i - \sum_{n=1}^{m_i} x^i_n \otimes S(u_i)(\varphi^i_n) \right\|_\varepsilon < \frac{1}{2s}$$

and hence

$$\frac{1}{2} < \sum_{n=1}^{m_i} |\psi_i(x^i_n \otimes S(u_i)(\varphi^i_n))|.$$

Put $\mathcal{M} = \{m_i : i \in \tau\}$ and for each $m \in \mathcal{M}$ define $\alpha_m = \{i \in \tau : m_i = m\}$. Since $\mathcal{M}$ is countable and τ has uncountable cofinality, there exists $M \in \mathcal{M}$ such that $|\alpha_M| = \tau$. Setting $\tau_1 = \alpha_M$, we have

$$\frac{1}{2} < \sum_{n=1}^{M} |\psi_i(x^i_n \otimes S(u_i)(\varphi^i_n))| \quad \text{for all } i \in \tau_1.$$

Next, for each $i \in \tau_1$ there exists $1 \le n_i \le M$ satisfying

$$\frac{1}{2M} < |\psi_i(x^i_{n_i} \otimes S(u_i)(\varphi^i_{n_i}))|.$$

Let $\mathcal{N} = \{n_i : i \in \tau_1\}$ and for each $n \in \mathcal{N}$ consider $\beta_n = \{i \in \tau_1 : n_i = n\}$. Since $\mathcal{N}$ is finite, there exists $N \in \mathcal{N}$ such that $|\beta_N| = \tau$. Setting $\tau_2 = \beta_N$, we obtain

$$\frac{1}{2M} < |\psi_i(x^i_N \otimes S(u_i)(\varphi^i_N))| \quad \text{for all } i \in \tau_2.$$

Now let $\mathcal{A} = \{x^i_N : i \in \tau_2\}$ and for each $a \in \mathcal{A}$ put $\gamma_a = \{i \in \tau_2 : x^i_N = a\}$. Since $\operatorname{cf}(\tau) > \operatorname{dens}(X) \ge |\mathcal{A}|$, there exists $x_0 \in \mathcal{A}$ such that $|\gamma_{x_0}| = \tau$. Setting $\tau_3 = \gamma_{x_0}$, we get

$$\frac{1}{2M} < |\psi_i(x_0 \otimes S(u_i)(\varphi^i_N))| \quad \text{for all } i \in \tau_3.$$

Finally, let $\mathcal{B} = \{\varphi^i_N : i \in \tau_3\}$, and for each $\varphi \in \mathcal{B}$ put $\lambda_\varphi = \{i \in \tau_3 : \varphi^i_N = \varphi\}$. Since $\operatorname{cf}(\tau) > \operatorname{dens}(X) \ge |\mathcal{B}|$, there exists $\varphi_0 \in \mathcal{B}$ such that $|\lambda_{\varphi_0}| = \tau$. Setting $\tau_4 = \lambda_{\varphi_0}$, we obtain

$$\frac{1}{2M} < |\psi_i(x_0 \otimes S(u_i)(\varphi_0))| \quad \text{for all } i \in \tau_4. \tag{2-1}$$

For each $i \in \tau_4$, write $y_i = S(u_i)(\varphi_0) \in Y$ and consider the linear functional $y_i^* \in Y^*$ defined by $y_i^*(y) = \psi_i(x_0 \otimes y)$, for every $y \in Y$. By (2-1), we have

$$\frac{1}{2M} < |y_i^*(y_i)| \le \|\psi_i\| \|x_0\| \|y_i\| \le s\|x_0\| \|y_i\| \quad \text{for all } i \in \tau_4,$$

and therefore

$$\frac{1}{2Ms\|x_0\|} < \|y_i\| \quad \text{for all } i \in \tau_4. \tag{2-2}$$

Denote by $(e_i)_{i\in\tau}$ the unit-vector basis of $c_0(\tau)$ and let $T : c_0(\tau) \to X\hat{\otimes}_\varepsilon Y$ be an isomorphism from $c_0(\tau)$ onto its image such that $T(e_i) = u_i$ for each $i \in \tau$. Consider $P : X\hat{\otimes}_\varepsilon Y \to Y$ the linear operator defined by $P(u) = S(u)(\varphi_0)$ for every $u \in X\hat{\otimes}_\varepsilon Y$. The inequality (2-2) then yields

$$\|(P \circ T)(e_i)\| = \|y_i\| \ge \frac{1}{2Ms\|x_0\|} > 0 \quad \text{for all } i \in \tau_4$$

and thus, by [Rosenthal 1970, remark following Theorem 3.4], there exists $\tau_5 \subset \tau_4$ such that $|\tau_5| = \tau$ and $P \circ T_{|c_0(\tau_5)}$ is an isomophism onto its image. This shows that $(y_i)_{i\in\tau_5} = (P(T(e_i))_{i\in\tau_5}$ is equivalent to the unit-vector basis of $c_0(\tau_5)$ in Y. Notice also that

$$(y_i^*(y))_{i\in\tau_5} = (\psi_i(x_0 \otimes y))_{i\in\tau_5} \in c_0(\tau_5) \quad \text{for all } y \in Y,$$

since $(\psi_i)_{i\in\tau}$ is weak*-null by hypothesis. Thus, $(y_i^*)_{i\in\tau_5}$ is weak*-null in Y^*. Combining these facts with (2-1), an appeal to Theorem 6 yields a complemented copy of $c_0(\tau)$ in Y. □

Note that according to Theorem 1, the above result is optimal. Moreover, Theorem 7 does not hold for cardinals with uncountable cofinality equal to the density of X. Indeed, by [Galego and Hagler 2012, Theorem 4.5] it follows that $c_0(\tau) \overset{c}{\hookrightarrow} \ell_1(\tau)\hat{\otimes}_\varepsilon \ell_\infty(\tau)$, however according to [Diestel and Uhl 1977, Corollary 11, p. 156], $c_0(\tau) \overset{c}{\not\hookrightarrow} \ell_\infty(\tau)$.

As a direct application of Theorem 7, we have:

Corollary 8. *Let X be a Banach space, $p \in [1, \infty)$ and τ an infinite cardinal with* $\mathrm{cf}(\tau) > \aleph_0$. *Then*

$$c_0(\tau) \overset{c}{\hookrightarrow} L_p[0,1]\hat{\otimes}_\varepsilon X \Longrightarrow c_0(\tau) \overset{c}{\hookrightarrow} X.$$

3. Complemented copies of $c_0(\tau)$ in $L_p[0,1]\hat{\otimes}_\pi X$ spaces

We will use a convenient characterization of $L_p[0,1]\hat{\otimes}_\pi X$ as a sequence space.

3.1. ***The spaces $L_p^{\mathrm{weak}}(X)$ and $L_p\langle X\rangle$.*** We will denote by $(\chi_n)_{n\geq 1}$ the Haar system, that is, the sequence of functions defined on [0, 1] by $\chi_1(t) = 1$, for every $t \in [0, 1]$, and

$$\chi_{2^k+j}(t) = \begin{cases} 1 & \text{if } t \in \left[\frac{2j-2}{2^{k+1}}, \frac{2j-1}{2^{k+1}}\right), \\ -1 & \text{if } t \in \left[\frac{2j-1}{2^{k+1}}, \frac{2j}{2^{k+1}}\right), \\ 0 & \text{otherwise,} \end{cases}$$

for each $k \geq 0$ and $1 \leq j \leq 2^k$. It is well known (see [Lindenstrauss and Tzafriri 1977, p. 19; 1979, p. 155]) that the Haar system is an unconditional basis of $L_p[0, 1]$, $p \in (1, \infty)$, and we will denote its unconditional basis constant by K_p. Following [Bu 2002; Dowling 2004], we renorm $L_p[0, 1]$ by

$$\|f\|_p^{\mathrm{new}} = \sup\left\{\left\|\sum_{n=1}^{\infty} \theta_n \alpha_n \chi_n\right\|_p : \theta_n = \pm 1, n \geq 1\right\}$$

for each $f = \sum_{n=1}^{\infty} \alpha_n \chi_n \in L_p[0, 1]$. Then

$$\|\cdot\|_p \leq \|\cdot\|_p^{\mathrm{new}} \leq K_p \|\cdot\|_p$$

and $(\chi_n)_{n\geq 1}$ is a monotone, unconditional basis with respect to $\|\cdot\|_p^{\mathrm{new}}$. We will use $L_p^{\mathrm{new}}[0, 1]$ to denote $L_p[0, 1]$ equipped with the norm $\|\cdot\|_p^{\mathrm{new}}$.

Now, for each $n \geq 1$ let

$$e_n^p = \frac{\chi_n}{\|\chi_n\|_p^{\mathrm{new}}}.$$

The sequence $(e_n^p)_{n\geq 1}$ is a normalized, unconditional basis of $L_p^{\mathrm{new}}[0, 1]$ whose unconditional basis constant is 1. Further, by [Lindenstrauss and Tzafriri 1977, p. 19], $(e_n^p)_{n\geq 1}$ is also a boundedly complete basis.

Given X a Banach space and $p, q \in (1, \infty)$ satisfying $1/p + 1/q = 1$, we denote by $L_p^{\mathrm{weak}}(X)$ the space

$$\left\{(x_n)_{n\geq 1} \in X^{\mathbb{N}} : \sum_{n=1}^{\infty} x^*(x_n) e_n^p \text{ converges in } L_p^{\mathrm{new}}[0, 1] \text{ for each } x^* \in X^*\right\}$$

equipped with the norm

$$\|\bar{x}\|_p^{\mathrm{weak}} = \sup\left\{\left\|\sum_{n=1}^{\infty} x^*(x_n) e_n^p\right\|_p^{\mathrm{new}} : x^* \in B_{X^*}\right\},$$

and by $L_p\langle X\rangle$ the space

$$\left\{(x_n)_{n\geq 1} \in X^{\mathbb{N}} : \sum_{n=1}^{\infty} |x_n^*(x_n)| < \infty \text{ for each } (x_n^*)_{n\geq 1} \in L_q^{\mathrm{weak}}(X^*)\right\}$$

with the norm

$$\|\bar{x}\|_{L_p\langle X\rangle} = \sup\left\{\sum_{n=1}^{\infty}|x_n^*(x_n)| : (x_n^*)_{n\geq 1} \in B_{L_q^{\text{weak}}(X^*)}\right\},$$

where $\bar{x} = (x_n)_{n\geq 1}$. With their own respective norms, $L_p^{\text{weak}}(X)$ and $L_p\langle X\rangle$ are Banach spaces [Bu 2002].

For each $n \geq 1$, we will denote by

$$I_n : X \to X^{\mathbb{N}}$$

the natural inclusion

$$I_n(x) = (\delta_{mn}x)_{m\geq 1} \quad \text{for all } x \in X.$$

It is easy to see that $\|I_n(x)\|_p^{\text{weak}} = \|x\|$ and furthermore, by [Lindenstrauss and Tzafriri 1977, Proposition 1.c.7], we know $\|I_n(x)\|_{L_p\langle X\rangle} \leq 2\|x\|$, for every $x \in X$.

We shall consider also the following closed subspace of $L_p^{\text{weak}}(X)$:

$$F_p(X) = \left\{\bar{x} = (x_n)_{n\geq 1} \in L_p^{\text{weak}}(X) : \left\|\bar{x} - \sum_{n=1}^{m} I_n(x_n)\right\|_p^{\text{weak}} \to 0\right\}.$$

Next, we recall some results obtained in [Bu 2002].

Theorem 9 [Bu 2002, Theorem 2.4]. *Given X a Banach space, $p \in (1,\infty)$ and $\bar{x} = (x_n)_{n\geq 1} \in L_p\langle X\rangle$, the series $\sum_{n=1}^{\infty} I_n(x_n)$ converges to $\bar{x}$ in $L_p\langle X\rangle$.*

The next result gives a sequential representation of $L_p[0,1]\hat{\otimes}_\pi X$.

Theorem 10 [Bu 2002, Theorem 3.4]. *Let X be a Banach space and $p \in (1,\infty)$. The function $\Psi : L_p\langle X\rangle \to L_p[0,1]\hat{\otimes}_\pi X$ defined by*

$$\Psi(\bar{x}) = \sum_{n=1}^{\infty} e_n^p \otimes x_n$$

for each $\bar{x} = (x_n)_{n\geq 1} \in L_p\langle X\rangle$ is an isomorphism onto $L_p[0,1]\hat{\otimes}_\pi X$.

Theorem 11. *Let X be a Banach space and $p, q \in (1,\infty)$ such that $1/p + 1/q = 1$. Then $L_q^{\text{weak}}(X)$ is isomorphic to $\mathcal{L}(L_p[0,1], X)$ and its subspace $F_q(X)$ is isomorphic to $\mathcal{K}(L_p[0,1], X)$.*

Proof. Let $(e_n^*)_{n\geq 1}$ be the sequence of coordinate functionals in $L_p[0,1]^*$ with respect to the basis $(e_n^p)_{n\geq 1}$. It is easy to check that the usual isometry from $L_p[0,1]^*$ onto $L_q[0,1]$ associates the functional e_n^* to e_n^q.

Fix $\bar{x} = (x_n)_{n\geq 1} \in L_q^{\text{weak}}(X)$ and $f = \sum_{n=1}^{\infty} \alpha_n e_n^p \in L_p[0,1]$. We claim that the series $\sum_{n=1}^{\infty} \alpha_n x_n$ converges in X. Indeed, given $k \geq j \geq 1$ we have

$$\begin{aligned}
\Big\| \sum_{n=j}^{k} \alpha_n x_n \Big\| &= \Big\| \sum_{n=j}^{k} e_n^*(f) x_n \Big\| = \sup_{x^* \in B_{X^*}} \Big| \sum_{n=j}^{k} e_n^*(f) x^*(x_n) \Big| \\
&= \sup_{x^* \in B_{X^*}} \Big| \Big(\sum_{n=j}^{k} x^*(x_n) e_n^* \Big) \Big(\sum_{m=j}^{k} e_m^*(f) e_m^p \Big) \Big| \\
&\leq \sup_{x^* \in B_{X^*}} \Big\| \sum_{n=j}^{k} x^*(x_n) e_n^* \Big\| \Big\| \sum_{m=j}^{k} e_m^*(f) e_m^p \Big\| \\
&= \sup_{x^* \in B_{X^*}} \Big\| \sum_{n=j}^{k} x^*(x_n) e_n^q \Big\|_q \Big\| \sum_{m=j}^{k} e_m^*(f) e_m^p \Big\| \leq \|\bar{x}\|_q^{\mathrm{weak}} \Big\| \sum_{m=j}^{k} e_m^*(f) e_m^p \Big\|
\end{aligned}$$

and therefore the partial sums of the series $\sum_{n=1}^{\infty} \alpha_n x_n$ form a Cauchy sequence in X, which establishes our claim.

This proves that $\mathcal{I} : L_q^{\mathrm{weak}}(X) \to \mathcal{L}(L_p[0,1], X)$ given by

$$\mathcal{I}(\bar{x})(f) = \sum_{n=1}^{\infty} \alpha_n x_n$$

for each $\bar{x} = (x_n)_{n \geq 1} \in L_q^{\mathrm{weak}}(X)$ and $f = \sum_{n=1}^{\infty} \alpha_n e_n^p \in L_p[0,1]$ is a well-defined linear operator satisfying $\|\mathcal{I}(\bar{x})\| \leq \|\bar{x}\|_q^{\mathrm{weak}}$.

Let us show now that $\mathcal{I}$ is an isomorphism onto $\mathcal{L}(L_p[0,1], X)$. Fix $S \in \mathcal{L}(L_p[0,1], X)$ and consider $\bar{y} = (S(e_n^p))_{n \geq 1}$. We claim that $\bar{y} \in L_q^{\mathrm{weak}}$. Indeed, for each $m \geq 1$ and $x^* \in B_{X^*}$ we have

$$\begin{aligned}
&\Big\| \sum_{n=1}^{m} x^*(S(e_n^p)) e_n^q \Big\|_q^{\mathrm{new}} \\
&\quad = \sup_{\theta_n = \pm 1} \Big\| \sum_{n=1}^{m} \theta_n x^*(S(e_n^p)) e_n^q \Big\|_q = \sup_{\theta_n = \pm 1} \sup_{g \in B_{L_p[0,1]}} \Big| x^* \Big(\sum_{n=1}^{m} \theta_n e_n^*(g) S(e_n^p) \Big) \Big| \\
&\quad \leq \sup_{\theta_n = \pm 1} \sup_{g \in B_{L_p[0,1]}} \Big\| S \Big(\sum_{n=1}^{m} \theta_n e_n^*(g) e_n^p \Big) \Big\| \leq \|S\| \sup_{\theta_n = \pm 1} \sup_{g \in B_{L_p[0,1]}} \Big\| \sum_{n=1}^{m} \theta_n e_n^*(g) e_n^p \Big\|_p \\
&\quad = \|S\| \sup_{g \in B_{L_p[0,1]}} \Big\| \sum_{n=1}^{m} e_n^*(g) e_n^p \Big\|_p^{\mathrm{new}} \leq \|S\| \sup_{g \in B_{L_p[0,1]}} \Big\| \sum_{n=1}^{\infty} e_n^*(g) e_n^p \Big\|_p^{\mathrm{new}} \\
&\quad \leq K_p \|S\| \sup_{g \in B_{L_p[0,1]}} \Big\| \sum_{n=1}^{\infty} e_n^*(g) e_n^p \Big\|_p = K_p \|S\|.
\end{aligned}$$

Since $(e_n^q)_{n\geq 1}$ is a boundedly complete basis, the claim is established. This shows that $\mathcal{I}' : \mathcal{L}(L_p[0,1], X) \to L_q^{\text{weak}}(X)$ defined by $\mathcal{I}'(S) = (S(e_n^p))_{n\geq 1}$ is a bounded linear operator with $\|\mathcal{I}'\| \leq K_p$. Furthermore, it is easy to see that $\mathcal{I}'$ is the inverse of $\mathcal{I}$. Thus, $\mathcal{I}$ is an isomorphism onto $\mathcal{L}(L_p[0,1], X)$.

Next we will show that $\mathcal{I}$ maps $F_q(X)$ onto $\mathcal{K}(L_p[0,1], X)$. It is clear that $\mathcal{I}(F_q(X))$ is subset of $\mathcal{K}(L_p[0,1], X)$. Next, fix $T \in \mathcal{K}(L_p[0,1], X)$. Since $\mathcal{I}$ is onto $\mathcal{L}(L_p[0,1], X)$, there exists a unique $\bar{y} = (y_n)_{n\geq 1} \in L_q^{\text{weak}}(X)$ such that $\mathcal{I}(\bar{y}) = T$. We will show that $\bar{y} \in F_q(X)$. Fix $\varepsilon > 0$ and denote by $(P_n)_{n\geq 1}$ the sequence of projections associated to the basis $(e_n^p)_n$. Since $(e_n^*)_{n\geq 1}$ is a Schauder basis of $L_p[0,1]^*$ and T is compact, the sequence $(P_n^*)_{n\geq 1}$ converges uniformly to the identity operator on the compact set $T^*(B_{X^*})$. Hence, there exists $N \geq 1$ such that $\|P_m^*(T^*(x^*)) - T^*(x^*)\| < \varepsilon/K_p$ for every $x^* \in B_{X^*}$ and $m \geq N$, and thus $\|T \circ P_m - T\| \leq \varepsilon/K_p$ for every $m \geq N$. It is easy to see that

$$\mathcal{I}\biggl(\sum_{n=1}^{m} I_n(y_n)\biggr) = T \circ P_m$$

for every $m \geq 1$. Therefore we have

$$\biggl\|\bar{y} - \sum_{n=1}^{m} I_n(y_n)\biggr\|_q^{\text{weak}} \leq \|\mathcal{I}^{-1}\| \|T - T \circ P_m\| < \varepsilon$$

for every $m \geq N$, and thus $\bar{y} \in F_q(X)$. The proof is complete. □

3.2. ***The duals of the spaces $L_p\langle X\rangle$ and $F_q(X)$.*** It is well known that $L_p\langle X\rangle^*$ is linearly isomorphic to $\mathcal{L}(L_p[0,1], X^*)$ [Ryan 2002, Theorem 2.9] and that $F_q(X)^*$ is linearly isomorphic to $L_p[0,1]\widehat{\otimes}_\pi X^*$ [Ryan 2002, Theorem 5.33].

This subsection will be devoted to obtaining convenient characterizations of the duals of the spaces $F_q(X)$ and $L_p\langle X\rangle$.

Proposition 12. *Given X a Banach space, $p \in (1,\infty)$, $\bar{x} = (x_n)_{n\geq 1} \in L_p\langle X\rangle$ and $\varphi \in L_p\langle X\rangle^*$, the series $\sum_{n=1}^\infty (\varphi \circ I_n)(x_n)$ converges absolutely.*

Proof. For each $n \geq 1$, let $\theta_n = \operatorname{sign}(\varphi \circ I_n)(x_n)$. Then $\bar{y} = (\theta_n x_n)_{n\geq 1} \in L_p\langle X\rangle$ and by Theorem 9 we have

$$\sum_{n=1}^{\infty} |(\varphi \circ I_n)(x_n)| = \sum_{n=1}^{\infty} (\varphi \circ I_n)(\theta_n x_n) = \varphi(\bar{y}),$$

as desired. □

Similarly to the previous proposition, we have:

Proposition 13. *Given X a Banach space, $p \in (1,\infty)$, $\bar{x} = (x_n)_{n\geq 1} \in F_p(X)$ and $\varphi \in F_p(X)^*$, the series $\sum_{n=1}^\infty (\varphi \circ I_n)(x_n)$ converges absolutely.*

Proof. For each $n \geq 1$, let $\theta_n = \operatorname{sign}(\varphi \circ I_n)(x_n)$. Since $(e_n^p)_{n\geq 1}$ is an unconditional basis with unconditional constant equal to 1, it follows that the series $\sum_{n=1}^{\infty} \theta_n x^*(x_n) e_n^p$ converges in $L_p^{\text{new}}[0,1]$ for every $x^* \in X^*$. Moreover, for every $k \geq 1$ and $x^* \in X^*$ we have

$$\left\| \sum_{n=k}^{\infty} \theta_n x^*(x_n) e_n^p \right\|_p^{\text{new}} = \left\| \sum_{n=k}^{\infty} x^*(x_n) e_n^p \right\|_p^{\text{new}}$$

and so $(\theta_n x_n)_{n\geq 1} \in F_p(X)$. Thus, $\sum_{n=1}^{\infty} \theta_n (\varphi \circ I_n)(x_n)$ converges. □

Proposition 14. *Let X be a Banach space and $p, q \in (1,\infty)$ such that $1/p + 1/q = 1$. A sequence $\bar{x}^* = (x_n^*)_{n\geq 1}$ of elements of X^* belongs to $L_p\langle X^*\rangle$ if, and only if, the series $\sum_{n=1}^{\infty} x_n^*(x_n)$ converges absolutely for each $\bar{x} = (x_n)_{n\geq 1} \in F_q(X)$. Furthermore, in this case one has*

$$\|\bar{x}^*\|_{L_p\langle X\rangle} \leq \sup\left\{ \sum_{n=1}^{\infty} |x_n^*(x_n)| : \bar{x} = (x_n)_{n\geq 1} \in B_{F_q(X)} \right\} < \infty.$$

Proof. Let us show the nontrivial implication. Let $\bar{x}^* = (x_n^*)_{n\geq 1}$ be a sequence of elements of X^* such that the series $\sum_{n=1}^{\infty} x_n^*(x_n)$ converges absolutely for each $\bar{x} = (x_n)_{n\geq 1} \in F_q(X)$. We claim that

$$\mathcal{S}(\bar{x}^*) = \sup\left\{ \sum_{n=1}^{\infty} |x_n^*(x_n)| : \bar{x} = (x_n)_{n\geq 1} \in B_{F_q(X)} \right\} < \infty.$$

Indeed, for each $m \geq 1$, consider the set

$$U_m = \left\{ \bar{x} = (x_n)_{n\geq 1} \in B_{F_q(X)} : \sum_{n\geq 1} |x_n^*(x_n)| \leq m \right\}.$$

It is easy to check that U_m is a closed, absolutely convex subset of $B_{F_q(X)}$. Since $B_{F_q(X)} = \bigcup_{m\geq 1} U_m$ has nonempty interior, by Baire's theorem there exists $M \geq 1$ such that U_M has nonempty interior. The absolute convexity of U_M implies that 0 is an interior point of U_M, that is, there exists $r > 0$ satisfying

$$\{\bar{x} = (x_n)_{n\geq 1} \in B_{F_q(X)} : \|\bar{x}\|_q^{\text{weak}} \leq r\} \subset U_M.$$

This proves that $\mathcal{S}(\bar{x}^*) \leq M/r$ and our claim is established.

Next, let us show that $\bar{x}^* = (x_n^*)_{n\geq 1} \in L_p\langle X^*\rangle$. Fix $\bar{x}^{**} = (x_n^{**})_{n\geq 1} \in L_q^{\text{weak}}(X^{**})$, $m \geq 1$ and $\varepsilon > 0$. Put $Y = \operatorname{span}\{x_1^{**}, \ldots, x_m^{**}\}$. By the principle of local reflexivity [Martínez-Abejón 1999, Theorem 2], there exists a linear operator $T : Y \to X$ satisfying $\|T\| \leq 1 + \varepsilon$ and $x_n^*(T(x_n^{**})) = x_n^{**}(x_n^*)$ for each $1 \leq n \leq m$. Put $\bar{y} = (y_n)_{n\geq 1} \in F_q(X)$, where $y_n = T(x_n^{**})$, if $1 \leq n \leq m$, and $y_n = 0$ otherwise.

Since $(e_n^q)_{n\geq 1}$ is an unconditional basis, by [Lindenstrauss and Tzafriri 1977, p. 18] we have

$$\begin{aligned}\|\bar{y}\|_q^{\text{weak}} &= \sup_{x^*\in B_{X^*}} \Big\|\sum_{n=1}^{m}(x^*\circ T)(x_n^{**})e_n^q\Big\|_q^{\text{new}} \\ &\leq (1+\varepsilon)\sup_{\varphi\in B_{X^{***}}}\Big\|\sum_{n=1}^{m}\varphi(x_n^{**})e_n^q\Big\|_q^{\text{new}} \\ &\leq (1+\varepsilon)\sup_{\varphi\in B_{X^{***}}}\Big\|\sum_{n=1}^{\infty}\varphi(x_n^{**})e_n^q\Big\|_q^{\text{new}} = (1+\varepsilon)\|\bar{x}^{**}\|_q^{\text{weak}}\end{aligned}$$

and hence

$$\sum_{n=1}^{m}|x_n^{**}(x_n^*)| \leq \mathcal{S}(\bar{x}^*)\|\bar{y}\|_q^{\text{weak}} \leq (1+\varepsilon)\mathcal{S}(\bar{x}^*)\|\bar{x}^{**}\|_q^{\text{weak}}.$$

Since $\varepsilon > 0$ was arbitrary, we obtain

$$\sum_{n=1}^{m}|x_n^{**}(x_n^*)| \leq \mathcal{S}(\bar{x}^*)\|\bar{x}^{**}\|_q^{\text{weak}}$$

for each $m \geq 1$, which in turn implies

$$\sum_{n=1}^{\infty}|x_n^{**}(x_n^*)| \leq \mathcal{S}(\bar{x}^*)\|\bar{x}^{**}\|_q^{\text{weak}}.$$

Thus, $\bar{x}^* \in L_p\langle X^*\rangle$ and $\|\bar{x}^*\|_{L_p\langle X\rangle} \leq \mathcal{S}(\bar{x}^*)$, as desired. □

Theorem 15. *Let X be a Banach space and $p, q \in (1,\infty)$ such that $1/p+1/q=1$. The function $\mathcal{H} : F_q(X)^* \to L_p\langle X^*\rangle$ defined by*

$$\mathcal{H}(\varphi) = (\varphi\circ I_n)_{n\geq 1}$$

for each $\varphi \in F_q(X)^*$ is a linear isometry onto $L_p\langle X^*\rangle$.

Proof. Given $\varphi \in F_q(X)^*$, Propositions 13 and 14 imply that $(\varphi\circ I_n)_{n\geq 1} \in L_p\langle X^*\rangle$. Thus, $\mathcal{H}$ is well defined. It is clear that $\mathcal{H}$ is linear.

By Proposition 13, we have

$$\begin{aligned}\|\mathcal{H}(\varphi)\|_{L_p\langle X^*\rangle} &\leq \sup\Big\{\sum_{n=1}^{\infty}|(\varphi\circ I_n)(x_n)| : \bar{x} = (x_n)_{n\geq 1} \in B_{F_q(X)}\Big\} \\ &= \sup\Big\{\Big|\sum_{n=1}^{\infty}(\varphi\circ I_n)(x_n)\Big| : \bar{x} = (x_n)_{n\geq 1} \in B_{F_q(X)}\Big\} \\ &= \sup\{|\varphi(\bar{x})| : \bar{x} = (x_n)_{n\geq 1} \in B_{F_q(X)}\} = \|\varphi\|,\end{aligned}$$

where the first equality follows immediately from the proof of Proposition 13. On the other hand,

$$\begin{aligned}\|\mathcal{H}(\varphi)\|_{L_p\langle X^*\rangle} &= \sup\left\{\sum_{n=1}^{\infty}|x_n^{**}(\varphi\circ I_n)| : \bar{x}^{**} = (x_n^{**})_{n\geq 1} \in B_{L_q^{\text{weak}}(X^{**})}\right\}\\ &\geq \sup\left\{\sum_{n=1}^{\infty}|(\varphi\circ I_n)(x_n)| : \bar{x} = (x_n)_{n\geq 1} \in B_{F_q(X)}\right\} = \|\varphi\|.\end{aligned}$$

This shows that $\mathcal{H}$ is an isometry onto its image.

Finally, given $\bar{x}^* = (x_n^*)_{n\geq 1} \in L_p\langle X^*\rangle$, the function $\psi : F_q(X) \to \mathbb{R}$ defined by $\psi(\bar{x}) = \sum_{n=1}^{\infty} x_n^*(x_n)$ for each $\bar{x} = (x_n)_{n\geq 1} \in F_q(X)$ is a linear functional on $F_q(X)$ and it is clear that $\mathcal{H}(\psi) = \bar{x}^*$. This completes the proof. □

Next, we establish an isomorphism from $L_p\langle X\rangle^*$ onto $L_q^{\text{weak}}(X^*)$.

Theorem 16. *Let X be a Banach space and $p, q \in (1, \infty)$ such that $1/p + 1/q = 1$. The function $\mathcal{J} : L_q^{\text{weak}}(X^*) \to L_p\langle X\rangle^*$ given by*

$$\mathcal{J}(\bar{x}^*)(\bar{x}) = \sum_{n=1}^{\infty} x_n^*(x_n)$$

for each $\bar{x}^* = (x_n)_{n\geq 1} \in L_q^{\text{weak}}(X^*)$ and $\bar{x} = (x_n)_{n\geq 1} \in L_p\langle X\rangle$ is an isomorphism onto $L_p\langle X\rangle^*$.

Proof. Let $\Psi : L_p\langle X\rangle \to L_p[0,1]\hat{\otimes}_\pi X$ be the isomorphism defined in Theorem 10, $\mathcal{I} : L_q^{\text{weak}}(X^*) \to \mathcal{L}(L_p[0,1], X^*)$ be the isomorphism defined in Theorem 11, and consider $\Phi : \mathcal{L}(L_p[0,1], X^*) \to (L_p[0,1]\hat{\otimes}_\pi X)^*$ the canonical linear isometry [Ryan 2002, p. 24]. Given $\bar{x}^* = (x_n)_{n\geq 1} \in L_q^{\text{weak}}(X^*)$ and $\bar{x} = (x_n)_{n\geq 1} \in L_p\langle X\rangle$, we have

$$\begin{aligned}(\Psi^*\circ\Phi\circ\mathcal{I})(\bar{x}^*)(\bar{x}) = (\Phi\circ\mathcal{I})(\bar{x}^*)(\Psi(\bar{x})) &= \sum_{n=1}^{\infty}(\Phi\circ\mathcal{I})(\bar{x}^*)(e_n^p\otimes x_n)\\ &= \sum_{n=1}^{\infty}\mathcal{I}(\bar{x}^*)(e_n^p)(x_n) = \mathcal{J}(\bar{x}^*)(\bar{x}) = \sum_{n=1}^{\infty} x_n^*(x_n)\end{aligned}$$

and therefore $\mathcal{J} = \Psi^*\circ\Phi\circ\mathcal{I}$. The proof is complete. □

3.3. *Complemented copies of $c_0(\tau)$ in $L_p\langle X\rangle$ spaces.* The next lemma will play a crucial role in the proof of Theorem 18.

Lemma 17. *Let X be a Banach space, τ be an infinite cardinal and $p, q \in (1, \infty)$ such that $1/p + 1/q = 1$. Suppose that $(\bar{x}_i)_{i\in\tau} = ((x_n^i)_{n\geq 1})_{i\in\tau}$ is a family equivalent to the canonical basis of $c_0(\tau)$ in $L_p\langle X\rangle$ and let $(\varphi_i)_{i\in\tau}$ be a bounded family in*

$L_p\langle X\rangle^*$. *Then for each* $\varepsilon > 0$ *there exists* $M \geq 0$ *such that*

$$\left|\sum_{n=M+1}^{\infty} (\varphi_i \circ I_n)(x_n^i)\right| < \varepsilon, \quad \textit{for all } i \in \tau.$$

Proof. We recall that the series $\sum_{n=1}^{\infty} (\varphi_i \circ I_n)(x_n^i)$ converges absolutely for each $i \in \tau$, by Proposition 12. Let $s = \sup_{i\in\tau} \|\psi_i\| < \infty$.

Suppose the thesis does not hold. Then there exists $\varepsilon > 0$ such that, for each $m \geq 0$, there exists $i \in \tau$ satisfying

$$\left|\sum_{n=m+1}^{\infty} (\varphi_i \circ I_n)(x_n^i)\right| \geq \varepsilon.$$

We proceed by induction. For $M_0 = 0$, there exists $i_1 \in \tau$ such that

$$\left|\sum_{n=1}^{\infty} (\varphi_{i_1} \circ I_n)(x_n^{i_1})\right| \geq \varepsilon.$$

The absolute convergence of $\sum_{n=1}^{\infty} (\varphi_{i_1} \circ I_n)(x_n^{i_1})$ yields $M_1 \geq 1$ such that

$$\sum_{n=M_1+1}^{\infty} |(\varphi_{i_1} \circ I_n)(x_n^{i_1})| < \frac{\varepsilon}{2}.$$

Thus we have

$$\left|\sum_{n=1}^{M_1} (\varphi_{i_1} \circ I_n)(x_n^{i_1})\right| > \frac{\varepsilon}{2}.$$

Suppose we have obtained, for some $k \geq 1$, strictly increasing natural numbers $0 = M_0 < M_1 < \cdots < M_k$ and distinct $i_1, \ldots, i_k \in \tau$ satisfying

$$\left|\sum_{n=N_j}^{M_j} (\varphi_{i_j} \circ I_n)(x_n^{i_j})\right| > \frac{\varepsilon}{2} > \sum_{n=M_j+1}^{\infty} |(\varphi_{i_j} \circ I_n)(x_n^{i_j})|, \tag{3-1}$$

where $N_j = M_{j-1} + 1$, for each $1 \leq j \leq k$. By hypothesis, there exists $i_{k+1} \in \tau$ such that

$$\left|\sum_{n=M_k+1}^{\infty} (\varphi_{i_{k+1}} \circ I_n)(x_n^{i_{k+1}})\right| \geq \varepsilon.$$

The absolute convergence of $\sum_{n=1}^{\infty} (\varphi_{i_{k+1}} \circ I_n)(x_n^{i_{k+1}})$ yields $M_{k+1} \geq M_k + 1$ such that

$$\sum_{n=M_{k+1}}^{\infty} |(\varphi_{i_{k+1}} \circ I_n)(x_n^{i_{k+1}})| < \frac{\varepsilon}{2}.$$

Thus we have

$$\left|\sum_{n=M_k+1}^{M_{k+1}} (\varphi_{i_{k+1}} \circ I_n)(x_n^{i_{k+1}})\right| > \frac{\varepsilon}{2}.$$

The above inequality and (3-1) imply that $i_{k+1} \notin \{i_1, \ldots, i_k\}$.

For each $j \geq 1$, consider $\bar{x}_j^* = (x_{j,n}^*)_{n\geq 1} \in F_q(X^*)$, where

$$x_{j,n}^* = \begin{cases} \varphi_{i_j} \circ I_n, & \text{if } N_j \leq n \leq M_j, \\ 0, & \text{otherwise.} \end{cases}$$

We claim that $(\bar{x}_j^*)_{j\geq 1}$ is weakly-null in $F_q(X^*)$. Indeed, fix $\psi \in F_q(X)^*$ and $\delta > 0$. Let $\mathcal{J}$ be the isomorphism defined in Theorem 16. By Theorem 15, the sequence $(\psi \circ J_n)_{n\geq 1}$ belongs to $L_p\langle X^*\rangle$, where $J_n : X^* \to (X^*)^{\mathbb{N}}$ is the usual inclusion. By Theorem 9, there exists $N \geq 1$ such that

$$\left\|\sum_{n=m}^{\infty} K_n(\psi \circ J_n)\right\|_{L_p\langle X^*\rangle} < \frac{\delta}{s\|\mathcal{J}^{-1}\|}$$

for each $m \geq N$, where $K_n : X^{**} \to (X^{**})^{\mathbb{N}}$ is the usual inclusion. Since the sequence $(N_j)_{j\geq 1}$ is strictly increasing, there exists $J \geq 1$ such that $N_j \geq N$, for all $j \geq J$. Thus we have

$$\begin{aligned} |\psi(\bar{x}_j^*)| = \left|\sum_{n=N_j}^{M_j} (\psi \circ J_n)(x_{j,n}^*)\right| &\leq \|\bar{x}_j^*\|_q^{\text{weak}} \left\|\sum_{n=N_j}^{M_j} K_n(\psi \circ J_n)\right\|_{L_p\langle X^*\rangle} \\ &\leq \|(\varphi_{i_j} \circ I_n)_{n\geq 1}\|_q^{\text{weak}} \frac{\delta}{s\|\mathcal{J}^{-1}\|} = \|\mathcal{J}^{-1}(\varphi_{i_j})\|_q^{\text{weak}} \frac{\delta}{s\|\mathcal{J}^{-1}\|} \leq \delta \end{aligned}$$

for all $j \geq J$. This establishes the claim.

Now, let $\theta_j = \mathcal{J}(\bar{x}_j^*) \in L_p\langle X\rangle^*$ for each $j \geq 1$. By our claim, $(\theta_j)_{j\geq 1}$ is weakly-null. On the other hand, by (3-1) we have

$$|\theta_j(\overline{x_{i_j}})| = \left|\sum_{n=N_j}^{M_j} (\varphi_{i_j} \circ I_n)(x_n^{i_j})\right| > \frac{\varepsilon}{2} \quad \text{for all } j \geq 1.$$

This contradicts the Dunford–Pettis property of c_0 [Fabian et al. 2010, p. 596], and we are done. □

Theorem 18. *Given X a Banach space, $p \in (1, \infty)$ and τ an infinite cardinal, we have*

$$c_0(\tau) \overset{c}{\hookrightarrow} L_p[0,1]\hat{\otimes}_\pi X \Longrightarrow c_0(\tau) \overset{c}{\hookrightarrow} X.$$

Proof. By Theorems 6 and 10, there exist families $(\overline{x_i})_{i\in\tau} = ((x_n^i)_{n\geq 1})_{i\in\tau}$ in $L_p\langle X\rangle$ and $(\psi_i)_{i\in\tau}$ in $L_p\langle X\rangle^*$ such that $(\overline{x_i})_{i\in\tau}$ is equivalent to the usual unit-vector basis of $c_0(\tau)$, $(\psi_i)_{i\in\tau}$ is weak*-null and $\psi_i(\overline{x_j}) = \delta_{ij}$ for each $i, j \in \tau$. Let $s = \sup_{i\in\tau} \|\psi_i\| < \infty$.

An appeal to Lemma 17 yields $M \geq 0$ such that

$$\left|\sum_{n=M+1}^{\infty} (\varphi_i \circ I_n)(x_n^i)\right| < \frac{1}{2} \quad \text{for all } i \in \tau.$$

Since $1 = \psi_i(\overline{x_i}) = \sum_{n=1}^{\infty} (\varphi_i \circ I_n)(x_n^i)$, we have $M \geq 1$ and

$$\frac{1}{2} < \sum_{n=1}^{M} |(\psi_i \circ I_n)(x_n^i)| \quad \text{for all } i \in \tau.$$

Next, for each $i \in \tau$ there exists $1 \leq n_i \leq M$ satisfying

$$\frac{1}{2M} < |(\psi_i \circ I_{n_i})(x_{n_i}^i)|.$$

Let $\mathcal{N} = \{n_i : i \in \tau\}$ and for each $n \in \mathcal{N}$ consider $\alpha_n = \{i \in \tau : n_i = n\}$. Since $\mathcal{N}$ is finite, there exists $N \in \mathcal{N}$ such that $|\alpha_N| = \tau$. Setting $\tau_1 = \alpha_N$, we obtain

$$\frac{1}{2M} < |(\psi_i \circ I_N)(x_N^i)| \quad \text{for all } i \in \tau_1. \tag{3-2}$$

For each $i \in \tau_1$, define $x_i = x_N^i \in X$ and $x_i^* = \psi_i \circ I_N \in X^*$. By (3-2), we have

$$\frac{1}{2M} < |x_i^*(x_i)| \leq \|\psi_i\| \|I_N\| \|x_i\| \leq s \|I_N\| \|y_i\| \quad \text{for all } i \in \tau_1,$$

and therefore

$$\frac{1}{2Ms\|I_N\|} < \|x_i\| \quad \text{for all } i \in \tau_1. \tag{3-3}$$

Next, let $(e_i)_{i\in\tau}$ denote the unit-vector basis of $c_0(\tau)$. By hypothesis, there exists $T : c_0(\tau) \to L_p\langle X\rangle$ an isomorphism from $c_0(\tau)$ onto its image such that $T(e_i) = \overline{x_i}$ for each $i \in \tau$. By (3-3), we have

$$\|(P_N \circ T)(e_i)\| = \|x_i\| \geq \frac{1}{2Ms\|I_N\|} > 0 \quad \text{for all } i \in \tau_1.$$

Therefore, by [Rosenthal 1970, remark following Theorem 3.4], there exists $\tau_2 \subset \tau_1$ such that $|\tau_2| = \tau$ and $P_N \circ T_{|c_0(\tau_2)}$ is an isomophism onto its image; hence, $(x_i)_{i\in\tau_2} = (P_N(T(e_i))_{i\in\tau_2}$ is equivalent to the unit-vector basis of $c_0(\tau_2)$.

Finally, given $x \in X$, observe that

$$(x_i^*(x))_{i\in\tau_2} = (\psi_i(I_N(x)))_{i\in\tau_2} \in c_0(\tau_2),$$

since $(\psi_i)_{i\in\tau}$ is weak*-null by hypothesis. This shows that $(x_i^*)_{i\in\tau_2}$ is weak*-null in X^*.

Combining these facts with (3-2), an appeal to Theorem 6 yields a complemented copy of $c_0(\tau)$ in X. □

4. Complemented copies of $c_0(\tau)$ in $L_p([0,1],X)$ spaces

Let $\rho : L_p[0,1]\widehat{\otimes}_{\Delta_p} X \to L_p([0,1],X)$ be the unique linear extension of the natural mapping $g\otimes x \mapsto g(\,\cdot\,)x$, where $g\in L_p[0,1]$ and $x\in X$. By [Defant and Floret 1993, Chapters 7.1 and 7.2], ρ is a linear isometry from $L_p[0,1]\widehat{\otimes}_{\Delta_p} X$ onto $L_p([0,1],X)$.

For every integer m and $u\in L_p[0,1]$, we define

$$\sigma_m(u)=\sum_{n=1}^{m} c_n\chi_n(\cdot)\int_0^1 \chi_n(s)u(s)\,ds,$$

where $c_1=1$ and $c_{2^k+j}=2^k$ for each $k\ge 0$ and $1\le j\le 2^k$.

We define also the function H_m on $[0,1]\times[0,1]$ by

$$H_m(t,s)=\sum_{n=1}^{m} c_n\chi_n(t)\chi_n(s).$$

For every integer $k\ge 1$ we denote

$$I_{k,l}=\begin{cases}\left[\frac{l-1}{2^k},\frac{l}{2^k}\right) & \text{if } 1\le l\le 2^k-1,\\ \left[1-\frac{1}{2^k},1\right] & \text{if } l=2^k.\end{cases}$$

We also write $I_{0,1}=[0,1]$ and $C_{k,l}=I_{k,l}\times I_{k,l}$.

It is easy to check by induction that for each $k\ge 0,\ 1\le l\le 2^k$ and $m=2^k+l$ we have

$$H_m=2^{k+1}\sum_{i=1}^{2l} 1_{C_{k+1,i}}+2^k\sum_{i=l+1}^{2^k} 1_{C_{k,i}},$$

[Novikov and Semenov 1997, p. 17], where 1_A denotes the characteristic function of $A\subset[0,1]$, and thus H_m is a positive function on $[0,1]\times[0,1]$. Since one has

$$\sigma_m(g)=\int_0^1 H_m(\,\cdot\,,s)g(s)\,ds$$

for each $g\in L_p[0,1]$, we conclude that σ_m is a positive operator on $L_p[0,1]$. Furthermore, $\|\sigma_m\|=1$ and

$$\lim_{m\to\infty}\|\sigma_m(g)-g\|_p=0 \tag{4-1}$$

for each $f\in L_p[0,1]$, by [Lindenstrauss and Tzafriri 1977, p. 3] or [Singer 1970, Example 2.3, p. 13].

Lemma 19. *Given X a Banach space, $p \in [1,\infty)$ and $f \in L_p([0,1],X)$, the series*

$$\sum_{n=1}^{\infty} c_n \chi_n(\,\cdot\,) \int_0^1 \chi_n(s) f(s)\,ds$$

converges to f in $L_p([0,1],X)$, where $c_1 = 1$ and $c_{2^k+j} = 2^k$ for each $k \geq 0$ and $1 \leq j \leq 2^k$.

Proof. The natural tensor norm $\|\cdot\|_{\Delta_p}$ is not an uniform cross norm, nevertheless the operator $s_m = \sigma_m \otimes I_X$ is bounded and $\|s_m\| = 1$ by [Defant and Floret 1993, Chapter 7.2]. By (4-1), we have

$$\lim_{m\to\infty} \|s_m(g\otimes x) - g\otimes x\|_{\Delta_p} = 0$$

and hence

$$\lim_{m\to\infty} \|s_m(u) - u\|_{\Delta_p} = 0$$

for every $u \in L_p[0,1]\widehat{\otimes}_{\Delta_p} X$. The result then follows from the fact that ρ is a linear isometry onto $L_p([0,1],X)$. □

We are now ready to prove the main result of this section.

Theorem 20. *Let X be a Banach space, τ be an infinite cardinal and $p \in [1,\infty)$. If $\operatorname{cf}(\tau) > \aleph_0$, then*

$$c_0(\tau) \overset{c}{\hookrightarrow} L_p([0,1],X) \Longrightarrow c_0(\tau) \overset{c}{\hookrightarrow} X.$$

Proof. By Theorem 6, there exist families $(f_i)_{i\in\tau}$ in $L_p([0,1],X)$ and $(\psi_i)_{i\in\tau}$ in $L_p([0,1],X)^*$ such that $(f_i)_{i\in\tau}$ is equivalent to the usual unit-vector basis of $c_0(\tau)$, $(\psi_i)_{i\in\tau}$ is weak*-null and $\psi(f_j) = \delta_{ij}$, for each $i, j \in \tau$. Let $s = \sup_{i\in\tau} \|\psi_i\| < \infty$.

By Lemma 19, for each $i \in \tau$ we have

$$1 = |\psi_i(f_i)| \leq \sum_{n=1}^{\infty} c_n |\psi_i(\chi_n(\,\cdot\,)x_n^i)|,$$

where $x_n^i = \int_0^1 \chi_n(t) f_i(t)\,dt$, and thus there exists $m_i \geq 1$ such that

$$\frac{1}{2} < \sum_{n=1}^{m_i} c_n |\psi_i(\chi_n(\,\cdot\,)x_n^i)|.$$

Put $\mathcal{M} = \{m_i : i \in \tau\}$ and for each $m \in \mathcal{M}$ define $\alpha_m = \{i \in \tau : m_i = m\}$. Since $\mathcal{M}$ is countable and τ has uncountable cofinality, there exists $M \in \mathcal{M}$ such that $|\alpha_M| = \tau$. Setting $\tau_1 = \alpha_M$, we have

$$\frac{1}{2} < \sum_{n=1}^{M} c_n |\psi_i(\chi_n(\,\cdot\,)x_n^i)| \quad \text{for all } i \in \tau_1.$$

Next, for each $i \in \tau_1$ there exists $1 \leq n_i \leq M$ satisfying

$$\frac{1}{2M} < c_{n_i} |\psi_i(\chi_{n_i}(\,\cdot\,)x^i_{n_i})|.$$

Let $\mathcal{N} = \{n_i : i \in \tau_1\}$ and for each $n \in \mathcal{N}$ consider $\beta_n = \{i \in \tau_1 : n_i = n\}$. Since $\mathcal{N}$ is finite, there exists $N \in \mathcal{N}$ such that $|\beta_N| = \tau$. Setting $\tau_2 = \beta_N$, we obtain

$$\frac{1}{2Mc_N} < |\psi_i(\chi_N(\,\cdot\,)x^i_N)| \quad \text{for all } i \in \tau_2. \tag{4-2}$$

For each $i \in \tau_2$, write $x_i = x^i_N$ and consider the linear functional $x^*_i \in X^*$ defined by

$$x^*_i(x) = \psi_i(\chi_N(\,\cdot\,)(x)) \quad \text{for all } x \in X.$$

By (4-2), we obtain

$$\frac{1}{2Mc_N} < |x^*_i(x_i)| \leq \|\psi_i\| \|\chi_N(\,\cdot\,)x_i\|_p \leq \delta \|\chi_N\|_p \|x_i\| \quad \text{for all } i \in \tau_2,$$

and therefore

$$\Delta < \|x_i\| \quad \text{for all } i \in \tau_2, \tag{4-3}$$

where $\Delta = (2Msc_N\|\chi_N\|_p)^{-1}$.

Next, let $(e_i)_{i\in\tau}$ be the unit-vector basis of $c_0(\tau)$ and $T : c_0(\tau) \to L_p([0, 1], X)$ be an isomorphism from $c_0(\tau)$ onto its image such that $T(e_i) = f_i$ for each $i \in \tau$. Consider $P : L_p([0, 1], X) \to X$ the linear operator defined by

$$P(f) = \int_0^1 \chi_N(t) f(t)\, dt \quad \text{for all } f \in L_p([0, 1], X).$$

By (4-3), we have

$$\|(P \circ T)(e_i)\| = \|x_i\| \geq \Delta > 0 \quad \text{for all } i \in \tau_2.$$

Therefore, by [Rosenthal 1970, remark following Theorem 3.4], there exists $\tau_3 \subset \tau_2$ such that $|\tau_3| = \tau$ and $P \circ T_{|c_0(\tau_3)}$ is an isomorphism onto its image; hence,

$$(x_i)_{i\in\tau_3} = (P(T(e_i))_{i\in\tau_3}$$

is equivalent to the unit-vector basis of $c_0(\tau_3)$.

Finally, given $x \in X$, observe that

$$(x^*_i(x))_{i\in\tau_3} = (\psi_i(\chi_N(\,\cdot\,)(x)))_{i\in\tau_3} \in c_0(\tau_3),$$

since $(\psi_i)_{i\in\tau}$ is weak*-null by hypothesis. This proves that $(x^*_i)_{i\in\tau_3}$ is weak*-null in X^*.

Combining these facts with (2-1), an appeal to Theorem 6 yields a complemented copy of $c_0(\tau)$ in X. □

We do not know if the statement of Theorem 20 remains true in the case $p = \infty$.

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