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LOXODROMICS FOR THE CYCLIC SPLITTING COMPLEX AND THEIR CENTRALIZERS

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We show that an outer automorphism acts loxodromically on the cyclic splitting complex if and only if it has a filling lamination and no generic leaf of the lamination is carried by a vertex group of a cyclic splitting. This is the analog for the cyclic splitting complex of Handel–Mosher's theorem on loxodromics for the free splitting complex. We also show that such outer automorphisms have virtually cyclic centralizers.

1. Introduction

The study of the mapping class group of a closed orientable surface S has benefited greatly from its action on the curve complex, $\mathcal{C}(S)$, which was shown to be hyperbolic in [Masur and Minsky 1999]. Curve complexes have been used for bounded cohomology of subgroups of mapping class groups, rigidity results, and myriad other applications.

The outer automorphism group of a finite rank free group $\mathbb{F}$, denoted by $\mathrm{Out}(\mathbb{F})$, is defined as the quotient of $\mathrm{Aut}(\mathbb{F})$ by the inner automorphisms, those which arise from conjugation by a fixed element. Much of the study of $\mathrm{Out}(\mathbb{F})$ draws parallels with the study of mapping class groups. This analogy, however, is far from perfect; there are several $\mathrm{Out}(\mathbb{F})$-complexes that act as analogs for the curve complex. Among them are the free splitting complex $\mathcal{FS}$, the cyclic splitting complex $\mathcal{FZ}$, and the free factor complex $\mathcal{FF}$, all of which have been shown to be hyperbolic [Handel and Mosher 2013a; Mann 2014; Bestvina and Feighn 2014]. Just as curve complexes have yielded useful information about mapping class groups, so too have these complexes furthered our understanding of $\mathrm{Out}(\mathbb{F})$.

The three hyperbolic $\mathrm{Out}(\mathbb{F})$-complexes mentioned above are related via coarse Lipschitz maps, $\mathcal{FS} \to \mathcal{FZ} \to \mathcal{FF}$. The loxodromics for $\mathcal{FF}$ have been identified

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with the set of fully irreducible outer automorphisms [Bestvina and Feighn 2014]. Handel and Mosher [2014] proved that an outer automorphism, ϕ, acts loxodromically on $\mathcal{FS}$ precisely when ϕ has a *filling lamination*, that is, some element of the finite set of laminations associated to ϕ (see [Bestvina et al. 2000]) is not carried by a vertex group of any free splitting. In this paper, we focus our attention on the isometry type of outer automorphisms, considered as elements of $\mathrm{Isom}(\mathcal{FZ})$.

A $\mathcal{Z}$-splitting of $\mathbb{F}$ is a splitting in which edge stabilizers are either trivial or cyclic. The cyclic splitting complex $\mathcal{FZ}$, introduced in [Mann 2014], is defined as follows (see Section 2L): vertices are one-edge $\mathcal{Z}$-splittings of $\mathbb{F}$ and k-simplices correspond to collections of $k+1$ vertices which are compatible with a common $k+1$-edge $\mathcal{Z}$-splitting. In this paper, we determine precisely which outer automorphisms act loxodromically on $\mathcal{FZ}$. Closely related to $\mathcal{Z}$-splittings are the maximally-cyclic splittings, called $\mathcal{Z}^{\max}$-splittings, in which the edge groups are required to be trivial or maximal cyclic (i.e., not contained in a larger cyclic subgroup). The results of this paper also apply to the maximally-cyclic splitting complex $\mathcal{FZ}^{\max}$ which is defined exactly as $\mathcal{FZ}$ except that splittings are required to be in the class $\mathcal{Z}^{\max}$. We will use the notation $\mathcal{FZ}^{(\max)}$ to mean either $\mathcal{FZ}$ or $\mathcal{FZ}^{\max}$.

In [Bestvina et al. 2000], the authors associate to each $\phi \in \mathrm{Out}(\mathbb{F})$ a finite set of attracting laminations, denoted by $\mathcal{L}(\phi)$. We say that a lamination $\Lambda \in \mathcal{L}(\phi)$ is $\mathcal{Z}^{(\max)}$*-filling* if no generic leaf (see Section 2N for definitions) of Λ is carried by a vertex group of a one-edge $\mathcal{Z}^{(\max)}$-splitting; we say that ϕ has a $\mathcal{Z}^{(\max)}$-filling lamination if some element of $\mathcal{L}(\phi)$ is $\mathcal{Z}^{(\max)}$-filling. We prove

Theorem 1.1. *For a free group of rank at least* 3, *an outer automorphism ϕ acts loxodromically on $\mathcal{FZ}^{(\max)}$ if and only if it has a $\mathcal{Z}^{(\max)}$-filling lamination. Furthermore*, *if ϕ has a filling lamination which is not $\mathcal{Z}^{(\max)}$-filling*, *then a power of ϕ fixes a point in $\mathcal{FZ}^{(\max)}$.*

Horbez and Wade [2015] showed that every isometry of $\mathcal{FZ}^{(\max)}$ is induced by an outer automorphism. Combining their result with [Handel and Mosher 2014, Theorem 1.1] and Theorem 1.1, this amounts to a classification of the isometries of $\mathcal{FZ}^{(\max)}$.

Corollary 1.2 (classification of isometries). *For all $\phi \in \mathrm{Isom}(\mathcal{FZ}^{(\max)})$ we have that*:

(1) *The action of ϕ on $\mathcal{FZ}^{(\max)}$ is loxodromic if and only if some element of $\mathcal{L}(\phi)$ is $\mathcal{Z}^{(\max)}$-filling.*

(2) *If the action of ϕ on $\mathcal{FZ}^{(\max)}$ is not loxodromic*, *then it has bounded orbits* (*there are no parabolic isometries*).

The proof of Theorem 1.1 relies on the description of the boundary of $\mathcal{FZ}^{(\max)}$ due to Horbez [2016]; points in the boundary of $\mathcal{FZ}^{(\max)}$ are equivalence classes of

$\mathcal{Z}^{(\max)}$-averse trees. The proof is carried out as follows. In Section 3, we extend the theory of folding paths to the boundary of Culler and Vogtmann's outer space, $\mathbb{P}\mathcal{O}$, defining a folding path guided by ϕ which is entirely contained in $\partial\mathbb{P}\mathcal{O}$. In Section 4, we show that the limit of the folding path thus constructed is $\mathcal{Z}^{(\max)}$-averse. In Section 5, we show that an outer automorphism with a filling but not a $\mathcal{Z}^{(\max)}$-filling lamination fixes (up to taking a power) a point in $\mathcal{FZ}^{(\max)}$ and conclude with a proof of Theorem 1.1.

The remainder of the paper is devoted to a study of the centralizers of automorphisms with filling laminations. We prove the following result:

Theorem 1.3. *If an outer automorphism ϕ has a $\mathcal{Z}$-filling lamination, then its centralizer in* $\mathrm{Out}(\mathbb{F})$ *is virtually cyclic. Conversely, if ϕ has a filling but not a $\mathcal{Z}$-filling lamination, then the centralizer of some power of ϕ in* $\mathrm{Out}(\mathbb{F})$ *is not virtually cyclic.*

The key tools used to prove Theorem 1.3 are the completely split train tracks introduced in [Feighn and Handel 2011] and the disintegration theory for outer automorphisms developed in [Feighn and Handel 2009]. We first show (Proposition 7.3) that the disintegration of any outer automorphism ϕ, that has a $\mathcal{Z}$-filling lamination, is virtually cyclic. Then we show that Proposition 7.3 implies the centralizer of ϕ is also virtually cyclic. Conversely, in Proposition 7.11, we show that if ϕ has a filling lamination that is not $\mathcal{Z}$-filling, then ϕ commutes with an appropriately chosen partial conjugation.

The method used to prove Theorem 1.3 provides an alternate (and simple) proof of the well-known fact, due to Bestvina, Feighn and Handel, that centralizers of fully irreducible outer automorphisms are virtually cyclic. In [Bestvina et al. 2000], the stretch factor homomorphism is used to show that the stabilizer of the lamination of a fully irreducible outer automorphism is virtually cyclic, which implies that the centralizer is also virtually cyclic. In general, little is known about the centralizers of outer automorphisms. Rodenhausen and Wade [2015] described an algorithm to find a presentation of the centralizer of a Dehn Twist automorphism. Feighn and Handel [2009] showed that the disintegration of an outer automorphism $\mathcal{D}(\phi)$ is contained in the weak center of the centralizer of ϕ. Recently, Algom-Kfir and Pfaff [2017] showed that centralizers of fully irreducible outer automorphisms with lone axes are isomorphic to $\mathbb{Z}$. We also mention a result of Kapovich and Lustig [2011]: automorphisms whose limiting trees are free have virtually cyclic centralizers.

The main motivation for examining the centralizers of loxodromic elements of $\mathcal{FZ}$ (and $\mathcal{FS}$) is to understand which automorphisms have the potential to be WPD elements for the action of $\mathrm{Out}(\mathbb{F})$ on these complexes.

Corollary 1.4. *Any outer automorphism that is loxodromic for the action of* $\mathrm{Out}(\mathbb{F})$ *on $\mathcal{FS}$ but elliptic for the action on $\mathcal{FZ}$ is not a WPD element for the action on $\mathcal{FS}$.*

The result that centralizers of loxodromic elements of $\mathcal{FZ}$ are virtually cyclic is a promising sign for the following conjecture:

Conjecture 1.5. The action of $\mathrm{Out}(\mathbb{F})$ on $\mathcal{FZ}$ is a WPD action. That is, every loxodromic element for the action satisfies WPD.

2. Preliminaries

Before proceeding, we fix a free group $\mathbb{F}$ of rank ≥ 3.

2A. ***Isometries of metric spaces.*** Let X be a Gromov hyperbolic metric space. We say that an infinite order isometry g of X is *loxodromic* if it acts with positive translation length on X: $\lim_{N\to\infty}(d(x, g^N(x))/N) > 0$ for some $x \in X$. Every loxodromic element has exactly two limit points in the Gromov boundary of X.

Given a group G acting by isometries on the hyperbolic space X, we denote by $\Lambda_X G$ the limit set of G in $\partial_\infty X$, which is defined as the intersection of $\partial_\infty X$ with the closure of the orbit of any point in X under the G-action. The following theorem, essentially due to Gromov, and formulated here for the case that G is cyclic, gives a classification of isometry groups of (possibly nonproper) Gromov hyperbolic spaces. A sketch of a proof can be found in [Caprace et al. 2015, Proposition 3.1].

Theorem 2.1 [Gromov 1987, Section 8.2]. *Let X be a hyperbolic geodesic metric space, and let G be a cyclic group acting by isometries on X. Then G is either*

- *bounded, i.e., all G-orbits in X are bounded; in this case $\Lambda_X G = \varnothing$, or*
- *horocyclic, i.e., G is not bounded and contains no loxodromic element; in this case $\Lambda_X G$ is reduced to one point, or*
- *lineal, i.e., G contains a loxodromic element, and any two loxodromic elements have the same fixed points in $\partial_\infty X$; in this case $\Lambda_X G$ consists of these two points.*

2B. ***Outer space and its compactification.*** Culler–Vogtmann *outer space*, $\mathbb{P}\mathcal{O}$, is defined in [Culler and Vogtmann 1986] as the space of simplicial, free, and minimal isometric actions of $\mathbb{F}$ on simplicial metric trees up to $\mathbb{F}$-equivariant homothety. We denote by $\mathcal{O}$ the *unprojectivized outer space*, in which the trees are considered up to isometry, rather than homothety. Each of these spaces is equipped with a natural (right) action of $\mathrm{Out}(\mathbb{F})$.

An $\mathbb{F}$-tree is an $\mathbb{R}$-tree with an isometric action of $\mathbb{F}$. An $\mathbb{F}$-tree is called *very small* if the action is minimal, arc stabilizers are either trivial or maximal cyclic, and tripod stabilizers are trivial. Outer space can be mapped into $\mathbb{R}^{\mathbb{F}}$ by the map $T \mapsto (\|g\|_T)_{g\in\mathbb{F}}$, where $\|g\|_T$ denotes the translation length of g in T. This was shown in [Culler and Morgan 1987] to be a continuous injection. The closure of the image of $\mathbb{P}\mathcal{O}$ under this embedding is compact and was identified in [Bestvina and

Feighn 1992] and [Cohen and Lustig 1995] with the space of very small $\mathbb{F}$-trees. We denote by $\overline{\mathbb{P}\mathcal{O}}$ the closure of outer space in $\mathbb{P}\mathbb{R}^{\mathbb{F}}$ and by $\partial\mathbb{P}\mathcal{O}$ its boundary. We will denote the preimage of $\overline{\mathbb{P}\mathcal{O}}$ in $\mathbb{R}^{\mathbb{F}}$ by $\overline{\mathcal{O}}$.

2C. ***Free factor system.*** A free factor system of $\mathbb{F}$ is a finite collection of conjugacy classes of proper free factors of $\mathbb{F}$ of the form $\mathcal{A} = \{[A_1], \dots, [A_k]\}$, where $k \geq 0$ and $[\cdot]$ denotes the conjugacy class of a subgroup, such that there exists a free factorization $\mathbb{F} = A_1 * \cdots * A_k * F_N$. We refer to the free factor F_N as the *cofactor* of $\mathcal{A}$, keeping in mind that it is not unique, even up to conjugacy.

The main geometric example of a free factor system is as follows: suppose G is a marked graph and K is a subgraph whose noncontractible connected components are denoted $C_1, \dots, C_k$. Let $[A_i]$ be the conjugacy class of a free factor of $\mathbb{F}$ determined by $\pi_1(C_i)$. Then $\mathcal{A} = \{[A_1], \dots, [A_k]\}$ is a free factor system. We say $\mathcal{A}$ is *realized by* K and we denote it by $\mathcal{F}(K)$.

2D. ***Marked graphs.*** We recall some basic definitions from [Bestvina and Handel 1992]. Identify $\mathbb{F}$ with $\pi_1(\mathcal{R}, *)$ where $\mathcal{R}$ is a rose with $\mathfrak{n}$ petals, $\mathfrak{n}$ being the rank of $\mathbb{F}$. A *marked graph* G is a graph of rank $\mathfrak{n}$, all of whose vertices have valence at least three, equipped with a homotopy equivalence $m : \mathcal{R} \to G$ called a *marking*. The marking determines an identification of $\mathbb{F}$ with $\pi_1(G, m(*))$. A homotopy equivalence $f : G \to G$ induces an outer automorphism of $\pi_1(G)$ and hence an element ϕ of $\mathrm{Out}(\mathbb{F})$. If f sends vertices to vertices and the restriction of f to edges is an immersion then we say that f is a *topological representative* of ϕ.

2E. ***Paths, circuits, and tightening.*** Let Γ be either a marked graph or an $\mathbb{F}$-tree. A *path* in Γ is either an isometric immersion of a (possibly infinite) closed interval $\sigma : I \to \Gamma$ or a constant map $\sigma : I \to \Gamma$. If σ is a constant map, the path will be called *trivial*. If I is finite, then any map $\sigma : I \to \Gamma$ is homotopic rel endpoints to a unique path $[\sigma]$. We say that $[\sigma]$ is obtained by *tightening* σ. If $f : \Gamma \to \Gamma$ is continuous and σ is a path in Γ, we define $f_\#(\sigma)$ as $[f(\sigma)]$. If the domain of σ is finite and Γ is either a graph or a simplicial tree, then the image has a natural decomposition into edges $E_1 E_2 \cdots E_k$ called the *edge path associated to* σ. If Γ is a tree, we may use $[x, x']$ to denote the unique geodesic path connecting x and x'.

A *circuit* is an immersion $\sigma : S^1 \to \Gamma$. For any path or circuit, let $\overline{\sigma}$ be σ with its orientation reversed. A decomposition of a path or circuit into subpaths is a *splitting* for $f : \Gamma \to \Gamma$ and is denoted $\sigma = \cdots \sigma_1 \cdot \sigma_2 \cdots$ if $f_\#^k(\sigma) = \cdots f_\#^k(\sigma_1) f_\#^k(\sigma_2) \cdots$ for all $k \geq 1$.

2F. ***Turns, directions and train track structure.*** Let Γ be an $\mathbb{F}$-tree. A direction d based at $p \in \Gamma$ is a component of $\Gamma - \{p\}$. A *turn* is an unordered pair of directions based at the same point. In the case that Γ is a simplicial tree, and p is a vertex, we identify directions at p with edges emanating from p. An *illegal turn structure* on Γ is an equivalence relation on the set of directions at each point $p \in \Gamma$. The classes of this relation are called *gates*. A turn (d, d') is *legal* if d and d' do not belong to the same gate. If in addition there are at least two gates at every vertex of Γ, then the illegal turn structure is called a *train track structure*. A path is legal if it only crosses legal turns.

2G. *Optimal morphism.* Given two $\mathbb{F}$-trees Γ and Γ', an $\mathbb{F}$-equivariant map $f : \Gamma \to \Gamma'$ is called a *morphism* if every segment of Γ can be subdivided into finitely many subintervals onto which f restricts to an isometric embedding. A morphism between $\mathbb{F}$-trees induces an illegal turn structure on the domain Γ as follows: for every $x \in \Gamma$, the map f determines a map $Df_x : D_x \to D_{f(x)}$, on the set of directions D_x at x. For $d, d' \in D_x$, we then declare $d \sim d'$ if $D(f^k)(d) = D(f^k)(d')$ for some $k \geq 0$. A morphism is called *optimal* if there are at least two gates at each point of Γ. A morphism f that induces a train track structure is an optimal morphism.

The map f is called *alignment preserving* (or a *collapse map*) if the f-image of every segment in Γ is a segment in Γ'.

2H. *Train track maps.* An optimal morphism is called a *train track map* if $f : \Gamma \to \Gamma'$ is an embedding on each edge and maps legal turns to legal turns. In particular, legal paths map to legal paths. Note that usually the term *train track map* is used for self maps, but Bestvina and Feighn [2014] defined it for a map between different $\mathbb{F}$-trees, each equipped with its own abstract train track structure.

The terminology can also be extended to graphs by passing to their universal covers. For more details on train track maps, the reader is referred to [Bestvina and Feighn 2014; Bestvina and Handel 1992].

2I. *Relative train track maps and CTs.* A *filtration* for a topological representative $f : G \to G$ of an outer automorphism ϕ, where G is a marked graph, is an increasing sequence of f-invariant subgraphs $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_M = G$. We let $H_i = \overline{G_i \setminus G_{i-1}}$ and call H_i the *i-th stratum*. A turn with one edge in H_i and the other in G_{i-1} is called *mixed* while a turn with both edges in H_i is called a *turn in* H_i. If $\sigma \subset G_i$ does not contain any illegal turns in H_i, then we say σ is *i-legal*.

We denote by M_i the submatrix of the transition matrix for f obtained by deleting all rows and columns except those labeled by edges in H_i. For the topological representatives that will be of interest to us, the transition matrices M_i will come in three flavors: M_i may be a zero matrix, it may be the 1×1 identity matrix, or it may be an irreducible matrix with Perron–Frobenius eigenvalue $\lambda_i > 1$. We will call H_i a *zero* (Z), *nonexponentially growing* (NEG), or *exponentially growing* (EG) stratum, respectively. Any stratum which is not a zero stratum is called an *irreducible stratum*.

Definition 2.2 [Bestvina and Handel 1992]. We say that $f : G \to G$ is a *relative train track map* representing $\phi \in \mathrm{Out}(F_n)$ if for every exponentially growing stratum H_r, the following hold:

(RTT i) Df maps the set of oriented edges in H_r to itself; in particular all mixed turns are legal.

(RTT ii) If $\sigma \subset G_{r-1}$ is a nontrivial path with endpoints in $H_r \cap G_{r-1}$, then so is $f_\#(\sigma)$.

(RTT iii) If $\sigma \subset G_r$ is r-legal, then $f_\#(\sigma)$ is r-legal.

Suppose that $u < r$, that H_u is irreducible, H_r is EG and each component of G_r is noncontractible, and that for each $u < i < r$, H_i is a zero stratum which is a component of G_{r-1} and each vertex of H_i has valence at least two in G_r. Then we say that H_i is *enveloped by* H_r and we define $H_r^z = \bigcup_{k=u+1}^{r} H_k$.

A path or circuit σ in a representative $f : G \to G$ is called a *periodic Nielsen path* if $f_\#^k(\sigma) = \sigma$ for some $k \geq 1$. If $k = 1$, then σ is a *Nielsen path*. A Nielsen path is *indivisible*, denoted INP, if it cannot be written as a concatenation of nontrivial Nielsen paths. If w is a closed root-free Nielsen path and E_i is an edge such that $f(E_i) = E_i w^{d_i}$, then we say E_i *is a linear edge* and we call w the *axis* of E. If E_i, E_j are distinct linear edges with the same axis such that $d_i \neq d_j$ and $d_i, d_j > 0$, then we call a path of the form $E_i w^* \bar{E}_j$ an *exceptional path*. We say that x and y are *Nielsen equivalent* if there is a Nielsen path σ in G whose endpoints are x and y. We say that a periodic point $x \in G$ is *principal* if neither of the following conditions hold:

- x is an endpoint of a nontrivial periodic Nielsen path and there are exactly two periodic directions at x, both of which are contained in the same EG stratum.
- x is contained in a component C of periodic points that is topologically a circle and each point in C has exactly two periodic directions.

A relative train track map f is called *rotationless* if each principal periodic vertex is fixed and if each periodic direction based at a principal vertex is fixed.

For an EG stratum, H_r, we call a nontrivial path $\sigma \subset G_{r-1}$ with endpoints in $H_r \cap G_{r-1}$ a *connecting path for* H_r. Let E be an edge in an irreducible stratum, H_r, and let σ be a maximal subpath of $f_\#^k(E)$ in a zero stratum for some $k \geq 1$. Then we say that σ is *taken*. A nontrivial path or circuit σ is called *completely split* if it has a splitting $\sigma = \tau_1 \cdot \tau_2 \cdots \tau_k$ where each of the τ_i's is a single edge in an irreducible stratum, an indivisible Nielsen path, an exceptional path, or a connecting path in a zero stratum which is both maximal and taken. We say that a relative train track map is *completely split* if $f(E)$ is completely split for every edge E in an irreducible stratum *and* if for every taken connecting path σ in a zero stratum, $f_\#(\sigma)$ is completely split.

The following theorem/definition is the main existence result for CTs:

Theorem 2.3 [Feighn and Handel 2011, Theorem 4.28; 2009, Corollary 3.5]. *There exists $k > 0$ depending only on $\mathtt{n}$, so that given any $\phi \in \mathrm{Out}(\mathbb{F})$ and any nested sequence of ϕ^k-invariant free factor systems, there is a **completely split improved relative train track map*** (CT *for short*) *$f : G \to G$ representing ϕ^k such that each free factor system is realized by some filtration element. The map f satisfies the following properties*:

- (*rotationless*) *$f : G \to G$ is rotationless.*
- (*completely split*) *$f : G \to G$ is completely split.*
- (*filtration*) *$\mathcal{F}$ is reduced. The core of each filtration element is a filtration element.*
- (*vertices*) *The endpoints of all indivisible periodic* (*necessarily fixed*) *Nielsen paths are* (*necessarily principal*) *vertices. The terminal endpoint of each nonfixed NEG edge is principal* (*and hence fixed*).
- (*periodic edges*) *Each periodic edge is fixed and each endpoint of a fixed edge is principal. If the unique edge E_r in a fixed stratum H_r is not a loop then G_{r-1} is a core graph and both ends of E_r are contained in G_{r-1}.*
- (*zero strata*) *If H_i is a zero stratum, then H_i is enveloped by an EG stratum H_r, each edge in H_i is r-taken and each vertex in H_i is contained in H_r and has link contained in $H_i \cup H_r$.*
- (*linear edges*) *For each linear E_i there is a closed root-free Nielsen path w_i such that $f(E_i) = E_i w_i^{d_i}$ for some $d_i \neq 0$. If E_i and E_j are distinct linear edges with the same axes then $w_i = w_j$ and $d_i \neq d_j$.*
- (*NEG Nielsen paths*) *If the highest edges in an indivisible Nielsen path σ belong to an NEG stratum then there is a linear edge E_i with w_i as in* (*linear edges*) *and there exists $k \neq 0$ such that $\sigma = E_i w_i^k \overline{E}_i$. Moreover, if ϕ is rotationless in the sense of* [*Feighn and Handel 2011*], *then we may take $k = 1$.*

It follows directly from the definitions that, for completely split paths and circuits, all cancellation under iteration of $f_\#$ is confined to the individual terms of the splitting. Moreover, $f_\#(\sigma)$ has a complete splitting which refines that of σ. Finally, just as with improved relative train track maps introduced in [Bestvina et al. 2000], every circuit or path with endpoints at vertices eventually is completely split [Feighn and Handel 2011, Lemma 4.25]. The reader is directed to [Feighn and Handel 2011, §4] for many useful properties of CTs that we will use frequently in the sequel, often without a specific reference.

2J. *Bounded backtracking* (*BBT*). Let $f : T \to T'$ be a continuous map between two $\mathbb{R}$-trees T and T'. We say that f has bounded backtracking if the f image of any path $[p, q]$ is contained in a C-neighborhood of $[f(p), f(q)]$. The smallest such C is called the *bounded backtracking constant* of f, denoted $\mathrm{BBT}(f)$.

2K. *Folding paths.* Given simplicial $\mathbb{F}$-trees T and T' and an optimal morphism $f : T \to T'$ Guirardel and Levitt [2007b, Section 3] construct a *canonical optimal folding path* $(T_t)_{t\in\mathbb{R}^+}$ guided by f. The tree T_t is constructed as follows. Given $a, b \in T$ with $f(a) = f(b)$, the *identification time* of a and b is defined as $\tau(a, b) = \sup_{x\in[a,b]} d_{T'}(f(x), f(a))$. Define $L := \frac{1}{2}\,\mathrm{BBT}(f)$. For each $t \in [0, L]$, one defines an equivalence relation $\sim_t$ by $a \sim_t b$ if $f(a) = f(b)$ and $\tau(a, b) < t$. The tree T_t is then a quotient of T by the equivalence relation $\sim_t$. Guirardel and Levitt prove that for each $t \in [0, L]$, T_t is an $\mathbb{R}$-tree. The collection of trees $(T_t)_{t\in[0,L]}$ comes equipped with $\mathbb{F}$-equivariant morphisms $f_{s,t} : T_t \to T_s$ for all $t < s$ and these maps satisfy the semiflow property: for all $r < s < t$, we have $f_{t,s} \circ f_{s,r} = f_{t,r}$. Moreover $T_L = T'$ and $f_{L,0} = f$. The trees $(T_t)_{t\in[0,L]}$ and the maps $(f_{s,t} : T_t \to T_s)_{t<s\in[0,L]}$ are called the *connection data* for the folding path.

2L. *The $\mathcal{Z}$-splitting complex.* Let $\mathcal{Z}$ be the collection of subgroups of $\mathbb{F}$ that are either trivial or cyclic. We denote by $\mathcal{Z}^{\max}$ the collection of elements of $\mathcal{Z}$ which are either trivial or closed under taking roots. We use the notation $\mathcal{Z}^{(\max)}$ to mean either $\mathcal{Z}$ or $\mathcal{Z}^{\max}$. A $\mathcal{Z}^{(\max)}$*-splitting* is a minimal, simplicial $\mathbb{F}$-tree whose edge stabilizers belong to the set $\mathcal{Z}^{(\max)}$; it is a *one-edge splitting* if there is one $\mathbb{F}$ orbit of edges. A *cyclic splitting* (resp. maximally-cyclic splitting) is a one-edge $\mathcal{Z}$-splitting (resp. $\mathcal{Z}^{\max}$-splitting) whose edge stabilizer is infinite cyclic. Two $\mathcal{Z}^{(\max)}$-splittings are *equivalent* if the corresponding Bass–Serre trees are $\mathbb{F}$-equivariantly homeomorphic. We will often blur the distinction between a splitting and its Bass–Serre tree.

If S is a one-edge free splitting (resp. $\mathcal{Z}^{(\max)}$-splitting) and v is a vertex in the Bass–Serre tree, then $\mathrm{Stab}(v)$ will be called a *vertex group* of S. Vertex groups of free splittings are free factors.

Given two $\mathcal{Z}^{(\max)}$-splittings $\overline{T}$ and T, we say that $\overline{T}$ is a *refinement* of T if there is a collapse map from $\overline{T}$ to T. Two $\mathcal{Z}^{(\max)}$-splittings T and T' are *compatible* if they have a common refinement, i.e., if there exists a tree that collapses onto both T and T'. A tree T is $\mathcal{Z}^{(\max)}$*-incompatible* if the set of $\mathcal{Z}^{(\max)}$-splittings compatible with T is empty. The (maximally-) cyclic splitting complex $\mathcal{FZ}^{(\max)}$ is the simplicial complex whose vertices are equivalence classes of one-edge $\mathcal{Z}^{(\max)}$-splittings and whose k-simplices are collections of $k+1$ pairwise compatible one-edge $\mathcal{Z}^{(\max)}$-splittings. Mann [2014] showed that $\mathcal{FZ}$ is δ-hyperbolic. More recently, Horbez [2016] used the same argument to prove that $\mathcal{FZ}^{\max}$ is δ-hyperbolic.

The results of Shenitzer [1955], Stallings [1991] and Swarup [1986] imply that every one-edge cyclic splitting of $\mathbb{F}$ is obtained from a one-edge free splitting of $\mathbb{F}$ by the "edge folding" process described as follows. Let T be a free splitting of $\mathbb{F}$, let v be a vertex of T and let G_v be its stabilizer. Consider $w \in G_v$ and $\langle w\rangle$, the cyclic group generated by w. Construct a new $\mathbb{F}$-tree T' by first choosing an edge e incident at v, then, for every $\gamma \in \mathbb{F}$, identifying γe with its orbit under $\langle \gamma w \gamma^{-1} \rangle \subseteq G_{\gamma v}$. The tree T' has an edge with stabilizer equal to $\langle w \rangle$. We say T' is obtained from T by an equivariant *edge fold*, or to be more specific, we sometimes say that T' is obtained from T by performing the *edge fold corresponding to* $\langle w \rangle$.

2M. ***$\mathcal{Z}$-averse trees and boundary of $\mathcal{FZ}$.*** A tree T in $\overline{\mathcal{O}}$ is called $\mathcal{Z}^{(\max)}$-averse [Horbez 2016, Definition 4.2] if there is no finite chain of compatibility between T and a $\mathcal{Z}^{(\max)}$-splitting: i.e., if there is no finite set of trees $(T = T_0, T_1, \dots, T_k = T')$ in $\overline{\mathcal{O}}$ such that T' is a $\mathcal{Z}^{(\max)}$-splitting and for each $i \in \{0, \dots, k-1\}$, the trees T_i and T_{i+1} are compatible. Two $\mathcal{Z}^{(\max)}$-averse trees, T, T', are called *equivalent* if there is a finite chain of compatible trees in $\overline{\mathcal{O}}$ relating T to T' as above. The reader will note that the notions of $\mathcal{Z}^{(\max)}$-compatibility and $\mathcal{Z}^{(\max)}$-aversity are independent of the homothety class of T; in particular, it makes sense to say that a tree in $\overline{\mathbb{P}\mathcal{O}}$ is $\mathcal{Z}$-averse, or that two trees in $\overline{\mathbb{P}\mathcal{O}}$ are equivalent. We denote by $\mathcal{X}^{(\max)}$ (resp. $\mathbb{P}\mathcal{X}^{(\max)}$) the subspace of $\overline{\mathcal{O}}$ (resp. $\overline{\mathbb{P}\mathcal{O}}$) consisting of $\mathcal{Z}^{(\max)}$-averse trees.

There is a natural map from a subset of $\partial\mathbb{P}\mathcal{O}$ to the Gromov boundary of $\mathcal{FZ}^{(\max)}$ relating the geometries at infinity of these two spaces, which we now describe. There is a map $\psi^{(\max)} : \mathbb{P}\mathcal{O} \to \mathcal{FZ}^{(\max)}$, which extends to the set of simplicial trees in $\overline{\mathcal{O}}$ with trivial edge stabilizers, defined by choosing a one-edge collapse of every simplicial tree in $\mathbb{P}\mathcal{O}$. This map is not quite $\mathrm{Out}(\mathbb{F})$-equivariant because we must make choices, however differing choices change distances by at most 2. The following theorem due to Horbez describes the boundary of the free splitting complex.

Theorem 2.4 [Horbez 2016, Theorem 0.1]. *There is a unique* $\mathrm{Out}(\mathbb{F})$*-equivariant homeomorphism*

$$\partial\psi^{(\max)} : \mathcal{X}^{(\max)}/\sim \quad \longrightarrow \quad \partial_\infty \mathcal{FZ}^{(\max)}$$

so that for all $T \in \mathcal{X}^{(\max)}$ and all sequences $(T_n) \in \mathcal{O}^{\mathbb{N}}$ converging to T, the sequence $(\psi^{(\max)}(T_n))_{n\in\mathbb{N}}$ converges to $\psi(T)$.

Given a tree $T \in \overline{\mathcal{O}}$, a $\mathcal{Z}^{(\max)}$-splitting S is called a *reducing splitting* for T, if S is compatible with some $T' \in \overline{\mathcal{O}}$, which is itself compatible with T.

2N. ***Lines and laminations.*** We briefly recall some definitions, but the reader is directed to [Bestvina et al. 2000] for details. The *space of abstract lines*, $\widetilde{\mathcal{B}} = (\partial\mathbb{F} \times \partial\mathbb{F} - \Delta)/\mathbb{Z}_2$, is the set of unordered distinct pairs of points in the boundary of $\mathbb{F}$ and is equipped with the natural (subspace/product/quotient) topology. The quotient of $\widetilde{\mathcal{B}}$ by the natural $\mathbb{F}$ action is *the space of lines in* $\mathcal{R}$ and is called $\mathcal{B}$. It is endowed with the quotient topology, which satisfies none of the separation axioms. Points in $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ will be called lines.

A closed subset Λ of $\mathcal{B}$ is an *attracting lamination for ϕ* if it is the closure of a single line β that is *bireccurrent* (every finite subpath σ of β occurs infinitely many times as an unoriented subpath of each end of β), has an *attracting neighborhood* (there is some open $U \ni \beta$ so that $\phi^k(\gamma) \to \beta$ for all $\gamma \in U$), and is not carried by a rank one ϕ-periodic free factor. The lines in Λ satisfying the above properties are called the *generic leaves* of Λ.

A subgroup A of $\mathbb{F}$ determines a subset of the boundary of $\mathbb{F}$ called $\partial A \subset \partial \mathbb{F}$. We say that A *carries* a line β if there is some lift $\widetilde{\beta}$ whose endpoints are in ∂A. We then say that the A *carries* the lamination Λ if A carries some (any) generic leaf of Λ. A lamination Λ is said to be *filling* (resp. $\mathcal{Z}^{(\max)}$*-filling*) if Λ is not carried by any vertex group of any free splitting (resp. $\mathcal{Z}^{(\max)}$-splitting).

Let $\pi_A : G_A \to \mathcal{R}$ be the immersion from the core of the cover of $\mathcal{R}$ corresponding to the subgroup A and let β be a line. Then clearly β is carried by A if and only if there exist immersions $\rho_A : \mathbb{R} \to G_A$ and $\rho : \mathbb{R} \to \mathcal{R}$ such that $\rho = \pi_A \rho_A$. If we further assume that A is finitely generated, it's easy to see that β is carried by A if and only if every finite subsegment of β can be immersed into G_A.

3. Folding in the boundary of outer space

Throughout this section, ϕ will be an outer automorphism with a $\mathcal{Z}^{(\max)}$-filling lamination Λ^+_ϕ. Our first goal is to extract from ϕ a folding path converging to a tree in $\partial\mathbb{P}\mathcal{O}$ which "witnesses" the lamination Λ^+_ϕ. The automorphism ϕ is fully irreducible relative to some maximal ϕ-invariant free factor system $\mathcal{A}$. Since ϕ has a filling lamination, $\mathcal{A}$ is not an exceptional free factor system, that is, it is not of the form $\{A\}$ or $\{A_1, A_2\}$, where $\mathbb{F} = A * \mathbb{Z}$ or $\mathbb{F} = A_1 * A_2$. Let $f : T \to T$ be the universal cover of a relative train track representative of ϕ realizing the invariant free factor system $\mathcal{A}$. Let $G = T/\mathbb{F}$ be the quotient graph, which comes with a filtration

$$\varnothing = G_0 \subset G_1 \subset \cdots \subset G_r = G$$

such that $\mathcal{F}(G_{r-1}) = \mathcal{A}$ and H_r is an EG stratum with Perron–Frobenius eigenvalue λ_ϕ. Let T_r (resp. T_{r-1}) denote the full preimage of H_r (resp. G_{r-1}) under the quotient map $T \to G$. We endow G (and hence T) with a metric by declaring all edges to have length 1. We will henceforth consider T as a point in unprojectivized outer space $\mathcal{O}$, whereby f may be thought of as an $\mathbb{F}$-equivariant map $T \to T \cdot \phi$.

Let T'_0 be the tree obtained from T by equivariantly collapsing the $\mathcal{A}$-minimal subtree. Our present aim is to construct a folding path ending at $T^+_\phi := \lim_{n\to\infty} T'_0\phi^n/\lambda^n_\phi$. To accomplish this, we will construct simplicial trees T_0, T_1 and define an optimal morphism $f_0 : T_0 \to T_1$. From this we will obtain a periodic canonical optimal folding path $(f_t)_{t\in[0,L]}$ which will end at T^+_ϕ. It is worth noting that the natural map $f'_0 : T'_0 \to T'_0\phi$ induced by f is neither optimal nor a morphism as there may be nondegenerate intervals which are mapped to points.

We remark that existence of an optimal morphism which is a train track map representing a relative fully irreducible outer automorphism is a special case of the results of [Francaviglia and Martino 2015] and [Meinert 2015], for free products and deformation spaces, respectively. Francaviglia and Martino [2015] developed metric theory for relative outer space for free products which is used to show the existence of optimal maps. This requires a considerable amount of work due to lack of applicability of the Arzela–Ascoli theorem in this setting. In what follows, we provide a shorter proof of existence of a train track map representing ϕ in the context of free groups.

Constructing T_0. The following is based on the construction in the proof of [Bestvina and Handel 1992, Lemma 5.10]. Define a measure μ on T with support contained in the set $\{x \in T_r : f^k(x) \in T_r \text{ for all } k \geq 0\}$ as follows: choose a Perron–Frobenius eigenvector $\vec{v}$ corresponding to the PF eigenvalue λ_ϕ. For an edge e in T_r, let $\mu(e) = v_e$, where v_e is the component of $\vec{v}$ corresponding to e. Define $\mu(e) = 0$ for all edges $e \in T_{r-1}$. Let V be the set of vertices of T and let $V_m := \{x \in T : f^m(x) \in V\}$. Subdividing T at V_m divides each edge into segments that map to edge paths under f^m. If a is such a segment then define $\mu(a) = \mu(f^m(a))/\lambda_\phi^m$. The definition of μ together with the fact that relative train track maps take r-legal paths to r-legal paths implies:

Lemma 3.1. *If $[x, y]$ is an r-legal path in T, then $\mu(f_\#([x, y])) = \lambda_\phi \mu([x, y])$. If $[x, y]$ contains an initial or terminal segment of some edge in T_r, then $\mu([x, y]) > 0$.*

The measure μ defines a pseudometric d_μ on T. Collapsing the sets of μ-measure zero to make d_μ into a metric, we obtain a tree T_0. Let $p : T \to T_0$ be the collapse map.

Lemma 3.2. *T_0 is simplicial.*

Proof. We will show that the $\mathbb{F}$-orbit of any point in T_0 must be discrete. Let $x \in T_0$ and choose a point $\tilde{x} \in p^{-1}(x)$. The $\mathbb{F}$-orbit of $\tilde{x}$ in T is discrete, and to understand the orbit of x, we need only understand $\mu([\tilde{x}, g\tilde{x}])$ for $g \in \mathbb{F}$. If $[\tilde{x}, g\tilde{x}]$ contains no edges in T_r, then $\mu([\tilde{x}, g\tilde{x}]) = 0$, in which case $g \in \mathrm{Stab}(x)$. Otherwise, the segment contains an edge in T_r, and hence has positive μ-measure. Since there are only finitely many $\mathbb{F}$-orbits of edges in T_r, there is a lower bound on the μ-measure of $[\tilde{x}, g\tilde{x}]$. Hence, there is a lower bound on $d_{T_0}(x, gx)$. This concludes the proof. □

The trees T_0 and T_0' are $\mathbb{F}$-equivariantly homeomorphic. The problem with T_0' is that the "obvious" map $f_0' : T_0' \to T_0' \phi$ sends nondegenerate segments to points and, because of that, is not useful for making a folding path. The map f_0 defined in the sequel is an improvement because it can be used to construct a folding path.

Defining $f_0 : T_0 \to T_1$. Let T_1 be the tree $\lambda_\phi^{-1} T_0 \cdot \phi$: the leading coefficient indicates that the metric has been scaled by λ_ϕ^{-1}. The relative train track map $f : T \to T \cdot \phi$

naturally induces a map $f_0 : T_0 \to T_1$. For each $x \in T_0$, its pre-image $p^{-1}(x)$ is a connected subtree of T with μ-measure zero. The definition of μ guarantees that the f-image of this set is also connected and has μ-measure zero. Therefore $p \circ f \circ p^{-1}(x)$ is a single point in $T_0 \cdot \phi$, which is identified with T_1 and we define $f_0 := p \circ f \circ p^{-1}$.

Lemma 3.3. *The map f_0 is an optimal morphism.*

Proof. We first show that f_0 is a morphism, which will follow from the definition of μ and properties of relative train track maps. Given a nondegenerate segment $[x, x']$ in T_0, choose $\tilde{x} \in p^{-1}(x)$ and $\tilde{x}' \in p^{-1}(x')$. The intersection of $[\tilde{x}, \tilde{x}']$ with the vertices of T is a finite set $\{\tilde{x}_1, \dots, \tilde{x}_{k-1}\}$. Let $\tilde{x}_0 := \tilde{x}$ and $\tilde{x}_k := \tilde{x}'$. Taking the p-image of $\tilde{x}_i$ for $i \in \{0, \dots, k\}$ yields a subdivision of $[x, x']$ into finitely many subsegments $[x_i, x_{i+1}]$, some of which may be degenerate. We will ignore the degenerate subdivisions: they occur as the projections of edges in T_{r-1} (all of which have μ-measure zero).

We claim that f_0 is an isometry in restriction to each of these subsegments. Indeed, let $e = [\tilde{x}_i, \tilde{x}_{i+1}]$ be an edge in T. Assume without loss of generality that $x_i \neq x_{i+1}$ so that $\mu(e) \neq 0$ and e is therefore an edge in T_r. It is an immediate consequence of Lemma 3.1 that for each $y \in e$, we have $\mu([f(\tilde{x}_i), f(y)]) = \lambda_\phi \mu([\tilde{x}_i, y])$ and hence f_0 is an isometry in restriction to $[x_i, x_{i+1}]$.

We now address the optimality of f_0. There are three types of points to consider: points in the interior of an edge, vertices with trivial stabilizer, and vertices with nontrivial stabilizer. We have already established that f_0 is an isometry in restriction to edges, so there are two gates at each $x \in T_0$ contained in the interior of an edge. If $x \in T_0$ is a vertex with trivial stabilizer, then $p^{-1}(x)$ is a vertex (Lemma 3.1) contained in $T_r \setminus T_{r-1}$. As f is a relative train track map, there are at least two gates at $p^{-1}(x)$ and each is necessarily contained in T_r. A short path in T containing $p^{-1}(x)$ entering through the first gate and leaving through the second will be legal. Lemma 3.1 gives that f_0 is an isometry in restriction to such a path, so there are at least two gates at x.

Now let $x \in T_0$ be a vertex with nontrivial stabilizer. Then $p^{-1}(x)$ is a subtree which is the inverse image of a component of G_{r-1} under the quotient map $T \to G$. Let $\tilde{x}, \tilde{x}' \in p^{-1}(x)$ be distinct vertices in $T_r \cap T_{r-1}$ and let d (resp. d') be a direction based at $\tilde{x}$ (resp. $\tilde{x}'$) corresponding to an edge e (resp. e') in T_r. Lemma 3.1 provides that d and d' determine distinct directions at x. As mixed turns are legal, the path $\overline{e} \cup [\tilde{x}, \tilde{x}'] \cup e'$ in T is r-legal. A final application of Lemma 3.1 gives that the restriction of f_0 to the p-image of this path is an isometry, and hence that there are at least two directions at x. □

The reader will note that we have proved the following:

Lemma 3.4. *The map f_0 is a train track map.*

As T_0 and T_0' are $\mathbb{F}$-equivariantly homeomorphic, there is a bijection between ($\mathbb{F}$-orbits of) edges of each. It is easily verified that the transition matrix of f_0 and that of f are equal. In particular, we will speak of edges, transition matrices, PF eigenvalues, and related notions for $f_0 : T_0 \to T_1$, without reference to this bijection.

Next, we use f_0 to construct a folding path starting at $S_0 := T_0$. This folding path will converge in $\partial\mathbb{P}\mathcal{O}$ to a tree S_L. We then prove that S_L is in fact the tree T_ϕ^+ as defined above.

Folding T_0. Applying the canonical folding path construction, we obtain a folding path $(S_t)_{t\in[0,L_1]}$ guided by $f_0 : T_0 \to T_1$ which begins at $T_0 = S_0$ and ends at $T_1 = S_{L_1}$, where $L_1 = \frac{1}{2}\operatorname{BBT}(f_0)$. Adapting a construction of Handel and Mosher [2011, Section 7.1], we now extend this to a *periodic fold path guided by* f_0. For each $i \in \mathbb{N}$, let $T_i = \lambda_\phi^{-i} T_0 \cdot \phi^i$, whence we have optimal morphisms $f_i : T_i \to T_{i+1}$ satisfying $\operatorname{BBT}(f_i) = \lambda_\phi^{-i}\operatorname{BBT}(f_0)$. For each i, inductively define $L_i := L_{i-1} + \frac{1}{2}\operatorname{BBT}(f_{i-1})$ and extend the folding path (which has so far been defined on $[0, L_{i-1}]$) using f_{i-1} to a folding path $(S_t)_{t\in[0,L_i]}$. Define $L := \lim_{i\to\infty} L_i$, which is finite as $\operatorname{BBT}(f_i)$ is a geometric sequence. We have thus defined the trees $(S_t)_{t\in[0,L)}$.

The notation here is less than ideal. In the above, $(T_i)_{i\in\mathbb{N}}$ is used for the trees $\lambda_\phi^{-i} T_0 \cdot \phi^i$, while $(S_t)_{t\in[0,L)}$ denotes a continuous folding path which is folded at constant speed. The reason for the differing names (S and T) is simply that the parameterizations differ; in particular $S_{L_i} = T_i$.

We now describe the maps $f_{t,s}$ for $s, t \in [0, L)$ with $s < t$. Indeed, given s, t, there is a natural choice of a map $f_{t,s} : S_s \to S_t$. Suppose $s \in [L_i, L_{i+1})$ and $t \in [L_j, L_{j+1})$. Then

$$f_{t,s} := f_{t,L_j} \circ f_{j-1} \circ f_{j-2} \circ \cdots \circ f_{i+1} \circ f_{L_{i+1},s}.$$

The semiflow property for the connection data follows from the definitions. Though our setting differs slightly from that of [Bestvina and Feighn 2014], Proposition 2.2 (5) therein can still be applied to give that each tree S_t has a well defined train track structure.

Along with the connection data, the fold path $(S_t)_{t\in[0,L)}$ forms a directed system in the category of $\mathbb{F}$-equivariant metric spaces and distance nonincreasing maps. As direct limits exist in this category, let $S_L := \varinjlim S_t$ and let $f_{L,t}$ be the direct limit maps. The proof of the following proposition is contained in Section 7.3 of [Handel and Mosher 2011], though it is not stated in this way. While Handel and Mosher deal with trees in $\mathcal{O}$ rather than $\partial\mathbb{P}\mathcal{O}$, the reader will easily verify that their proof goes through directly in our setting.

Proposition 3.5 [Handel and Mosher 2011]. *S_L is a non-trivial, minimal, $\mathbb{R}$-tree. Moreover S_t converges to S_L in the length function topology.*

We have described two trees in the boundary of outer space: $T_\phi^+ = \lim_{n\to\infty} T_0'\phi^n$ and S_L. We observe that both S_0 and T_0' are points in the relative outer space $\mathcal{O}(\mathbb{F}, \mathcal{A})$, which inherits the subspace topology from $\overline{\mathbb{P}\mathcal{O}}$. Moreover, ϕ is fully irreducible relative to $\mathcal{A}$, and as such, it acts with north-south dynamics on $\overline{\mathbb{P}\mathcal{O}(\mathbb{F},\mathcal{A})}$ [Gupta 2018]. Recall that for each $i \in \mathbb{N}$, $S_{L_i} = \lambda_\phi^{-i} S_0 \cdot \phi^i$, and that $L_i \to L$. As S_L is the limit of the fold path $(S_t)_{t\in[0,L)}$, we conclude:

Lemma 3.6. $S_L = T_\phi^+$.

We conclude this section with a lemma.

Lemma 3.7. *For all $t \in [0, L)$, the tree S_t is simplicial.*

Proof. Let $t \in [0, L)$. If $t = 0$, Lemma 3.2 provides that S_0 is simplicial. Since $S_{L_i} = \lambda_\phi^{-i} S_0 \cdot \phi^i$, the lemma holds when $t = L_i$ for some $i \in \mathbb{N}$. The other possibility is that $t \in (L_i, L_{i+1})$ for some i. Since both S_{L_i} and $S_{L_{i+1}}$ have trivial edge stabilizers, Proposition 1.1 of [Horbez 2016] applies to the folding path guided by f_i and allows one to conclude that all trees S_t, $t \in [L_i, L_{i+1}]$ are simplicial, as desired. □

4. The stable tree is $\mathcal{Z}^{(\mathbf{max})}$-averse

Our present aim is to understand T_ϕ^+; we would like to show that it is $\mathcal{Z}^{(\max)}$-averse. In this section, we will use the leaves of the topmost lamination Λ_ϕ^+ to construct a transverse covering of T_ϕ^+, and then use the transverse covering to achieve our goal.

Definition 4.1. Let G be a group and T be an $\mathbb{R}$-tree equipped with an action of G by isometries; and let $K \subseteq T$ be a subtree. We say that the action $G \curvearrowright T$ is *supported on K* if for any finite arc $J \subseteq T$, there are $g_1, \ldots, g_r \in G$ such that $I \subseteq g_1 K \cup \cdots \cup g_r K$.

Let I_0 be a segment of a leaf of the lamination Λ_ϕ^+ in S_0. Define the arc I_t in S_t by $I_t := f_{t,0}(I_0)$. We will denote I_L simply by I and we will call any segment in T_ϕ^+ obtained in this way *a segment of a leaf of* Λ_ϕ^+.

Lemma 4.2. *The action $\mathbb{F} \curvearrowright T_\phi^+$ is supported on I.*

Proof. Let $I = [x, y]$ and let $J = [x', y']$ be a nondegenerate arc in T_ϕ^+. The construction in Section 3 provides an optimal folding path $(S_t)_{t\in[0,L]}$, and optimal morphisms $f_{s,t} : S_t \to S_s$ for all $s, t \in [0, L]$ with $s > t$ which satisfy the semiflow property. It follows easily from the definitions that for a folding path (S_t) and any z in $S_L = T_\phi^+$, the set $f_{L,0}^{-1}(z)$ is a discrete set of points in S_0. Let $x_0' \in f_{L,0}^{-1}(x')$ and $y_0' \in f_{L,0}^{-1}(y')$ be points in S_0 chosen so that (x_0', y_0') contains no points in $f_{L,0}^{-1}(x') \cup f_{L,0}^{-1}(y')$ and define $J_0 = [x_0', y_0']$. Since I_0 is legal, it is never folded under the maps $f_{t,0}$, so the corresponding property already holds for I_0. Define the arc J_t in T_t by $J_t := [f_{t,0}(J_0)]$. The definitions of I_0 and J_0 guarantee that $[f_{L,0}(I_0)] = I$ and similarly for J_0. The semiflow property of the maps $f_{s,t}$ gives that for all $s, t \in [0, L]$ with $s > t$, we have $[f_{s,t}(I_t)] = I_s$ (resp. $[f_{s,t}(J_t)] = J_s$).

Since I_0 is a leaf segment and therefore legal with respect to the train track structure on S_0, it is never folded under the maps $f_{t,0}$. In particular, the length of I_t is constant in t. The maximum length of any edge in S_t tends to 0 as $t \to L$ because edge lengths can only decrease along the fold path and the metric in S_{L_i} has been scaled by λ_ϕ^{-i}. Thus, for sufficiently large t, I_t crosses an entire edge of S_t. Irreducibility of the transition matrix for f_0 implies that by further enlarging t, we may assume that I_t crosses an edge from every $\mathbb{F}$-orbit of edges in S_t.

We are now ready to complete the proof. Indeed, write J_t as an edge path $J_t = e_0 e_1 \cdots e_k$ in S_t (the first and last edges may be partial edges). Since I_t crosses every $\mathbb{F}$-orbit of edges in S_t, there exist $g_0, \ldots, g_k \in \mathbb{F}$ so that for all j, $g_j I_t$ crosses the edge e_j. Now we simply use $\mathbb{F}$-equivariance of the maps $f_{L,t}$ to conclude that

$$f_{L,t}(J_t) \subseteq g_0 f_{L,t}(I_t) \cup g_1 f_{L,t}(I_t) \cup \cdots \cup g_k f_{L,t}(I_t)$$

As I_t is legal, $f_{L,t}(I_t) = I$. While J_t is not necessarily legal, it's still true that $J = [f_{L,t}(J_t)] \subseteq f_{L,t}(J_t)$, completing the proof. □

4A. ***Mixing and indecomposable trees.*** A tree $T \in \overline{\mathbb{P}\mathcal{O}}$ is *mixing* if for all finite subarcs $I, J \subset T$, there exist $g_0, \ldots, g_k \in \mathbb{F}$ such that $J \subseteq g_0 I \cup g_1 I \cup \cdots \cup g_k I$ and $g_j I \cap g_{j+1} I \neq \varnothing$ for all $j \in \{0, \ldots, k-1\}$. A tree $T \in \overline{\mathbb{P}\mathcal{O}}$ is called *indecomposable* [Guirardel 2008] if it is mixing and the g_j's can be chosen so that $g_j I \cap g_{j+1} I$ is a nondegenerate arc for each $j \in \{0, \ldots, k-1\}$.

Lemma 4.3. *T_ϕ^+ is mixing.*

Proof. The proof is similar to that of Lemma 4.2, so we will retain our notation from that proof. Indeed, it's clearly enough to show that every arc J can be covered by finitely many translates with nonempty overlap of the fixed arc I and conversely that I can be covered similarly by translates of J. Recall the cover of J by translates of I constructed in proof of Lemma 4.2. Since consecutive edges in the edge path of $J_t = e_0 \cdots e_k$ intersect in a point, it follows that $g_j I_t \cap g_{j+1} I_t \neq \varnothing$ for all $j \in \{0, \ldots, k-1\}$. Again, this behavior persists in the limit.

Conversely, to see that I can be covered by translates of J we use essentially the same argument as before, only now there is a slight difficulty in producing an edge in some J_t that isometrically embeds in the limit. Now J_t may have illegal turns, so we write J_t as a concatenation of maximal legal subpaths, $J_t = J_t^0 J_t^1 \cdots J_t^k$. Now $f_{L,t}(J_t)$ is a concatenation of the $f_{L,t}$-images of J_t^i, which are themselves segments in S_L. Thus, the tightened image $J = [f_{L,t}(J_t)]$ is contained in the union $f_{L,t}(J_t^0) \cup \cdots \cup f_{L,t}(J_t^k)$. Now choose an $i \in \{0, \ldots, k\}$ so that $J \cap f_{L,t}(J_t^i)$ is a nondegenerate subsegment of J and replace J by the subsegment $J' = J \cap f_{L,t}(J_t^i)$. The proof of Lemma 4.2 can now be applied to J', allowing us to conclude that I can be covered by finitely many translates J' with nonempty overlaps. As J' is a subsegment of J, the same finite set of group elements witnesses the fact that I can be covered by finitely many translates J with nonempty overlaps. □

4B. ***Transverse families and transverse coverings.*** A subtree Y of a tree T is called *closed* [Guirardel 2004, Definition 2.4] if $Y \cap \sigma$ is either empty or a path in T for all paths $\sigma \subset T$; recall that paths are defined on closed intervals. A *transverse family* [Guirardel 2004, Definition 4.6] of an $\mathbb{R}$-tree T is a family $\mathcal{Y}$ of nondegenerate closed subtrees of T such that any two distinct subtrees in $\mathcal{Y}$ intersect in at most one point. If every path in T is covered by finitely many subtrees in $\mathcal{Y}$, then the transverse family is called a *transverse covering*.

The idea of the following definition is to start with an interval and "fill it out" into an entire subtree by translating it around, always requiring that overlaps are nondegenerate.

Definition 4.4 (the transverse family generated by Λ_ϕ^+). Let $I = [x, y]$ be a segment of a leaf of Λ_ϕ^+ in T_ϕ^+. Define Y_I as the union of all arcs J such that there exists $g_0, \dots, g_k \in \mathbb{F}$ satisfying:

- $J \subseteq g_0 I \cup \cdots \cup g_k I$.
- $g_j I \cap g_{j+1} I$ is a nondegenerate segment for each $i \in \{0, \dots, k-1\}$.
- $g_0 I \cap I$ is a nondegenerate segment.

It's immediate that the collection $\mathcal{Y} = \{gY_I\}_{g \in \mathbb{F}}$ is a transverse family in T_ϕ^+ since, by definition, distinct $\mathbb{F}$-translates of Y_I intersect in a point or not at all. This construction is essentially due to Guirardel–Levitt.

Lemma 4.5. *With notation as above, Y_I is indecomposable with respect to the* $\mathrm{Stab}(Y_I)$ *action. Moreover, $\mathcal{Y} = \{gY_I\}_{g \in \mathbb{F}}$ is a transverse covering of T_ϕ^+.*

Proof. We first show that Y_I is indecomposable. The proof is similar to that of Lemmas 4.2 and 4.3, so we will retain our notation from those proofs. As before, it is enough to show that every arc $J \subseteq Y_I$ can be covered by finitely many translates with nondegenerate overlap of the fixed arc I, and conversely that I can be covered by finitely many translates of J with nondegenerate overlap. The definition of Y_I guarantees that J can be covered by finitely many translates of I, so we are left to show the converse.

First, replace J by an appropriately chosen subinterval exactly as in the proof of Lemma 4.3. Now we run the proof of Lemma 4.2 with a minor modification. For $t \in [0, L)$, let J_t and I_t be as in that proof. This time, choose t large enough so that I_t crosses every $\mathbb{F}$-orbit of turns taken by a leaf of Λ_ϕ^+. By further enlarging t if necessary, we may arrange that J_t also crosses every turn taken by a leaf. Write I_t as an edge path $I_t = e_0 e_1 \cdots e_k$ in S_t, where the first and last edges may be partial edges. Since J_t crosses every $\mathbb{F}$-orbit of turns taken by a leaf in S_t, there exist

$g_0, \dots, g_k \in \mathbb{F}$ so that for all $j \in \{0, \dots, k-1\}$, $g_j J_t$ crosses the edge path $e_j e_{j+1}$. Now we conclude exactly as before, using $\mathbb{F}$-equivariance of the maps $f_{L,t}$ to see that

$$f_{L,t}(I_t) \subseteq g_0 f_{L,t}(J_t) \cup g_1 f_{L,t}(J_t) \cup \cdots \cup g_k f_{L,t}(J_t)$$

Since both I_t and J_t are legal, this set containment (and nondegeneracy of the overlaps) is unaffected by tightening and the proof is complete.

To see that $\mathcal{Y}$ is a transverse covering we again reference the proof of Lemma 4.2, which shows that every path in T_ϕ^+ can be covered by finitely many trees in $\mathcal{Y}$. □

Lemma 4.6. *Let β be a generic leaf of Λ_ϕ^+ and let J be a finite subsegment of a realization of β in T_ϕ^+. Then there exists $g \in \mathbb{F}$ which is contained in a conjugate of* $\mathrm{Stab}(Y_I)$ *and whose axis, A_g, in T_ϕ^+ contains the segment J.*

Proof. We retain our notation from above, so that J_t is a segment in S_t which maps to J under $f_{L,t}$. We will denote the realization of β in S_t by β_t. Choose t large enough so that J_t crosses every turn taken by β_t, then lengthen J_t by following the leaf to arrange that both endpoints of J_t are vertices in the same $\mathbb{F}$-orbit. Write J_t as an edge path $J_t = e_0 e_1 \cdots e_k$. If necessary, further lengthen J_t (again following β_t) to arrange that the turn $\{e_0, \overline{e_k}\}$ is taken by a leaf. Let x_t (resp. y_t) be the initial (resp. terminal) endpoint of J_t.

Now let $g \in \mathbb{F}$ be a group element taking x_t to y_t. After postcomposing with an element of $\mathrm{Stab}(y_t)$ if necessary, we may assume that the turn $\{\overline{e_k}, g(e_0)\}$ is taken by a generic leaf of Λ_ϕ^+. We claim that the axis of g in S_t crosses J_t. Indeed, to get from x_t to y_t, one traverses the edge path $e_0 e_1 \cdots e_k$. Thus, to get from $y_t = g \cdot x_t$ to $g \cdot y_t = g^2 \cdot x_t$, one traverses the same (up to $\mathbb{F}$-orbit) edge path. As $e_0 \neq \overline{e_k}$ and S_t is a tree, we have that $d(x_t, g^2 \cdot x_t) = 2d(x_t, g \cdot x_t)$. It is an elementary exercise to show that this is equivalent to x being on the axis of g. Both β_t and the axis of g are legal, so the restriction of $f_{L,t}$ to each is an immersion. Thus, we can push this picture forward to the limit using $f_{L,t}$ to reach the desired conclusion.

We've seen that any realization of β in T_ϕ^+ is contained in a single $\mathbb{F}$-translate of Y_I. As we have arranged that every turn taken by the axis of g in S_t is also taken by a leaf, the argument given in the proof of Lemma 4.5 allows us to conclude that A_g is contained in a single $\mathbb{F}$-translate of Y_I. Thus g is contained in a conjugate of $\mathrm{Stab}(Y_I)$, as desired. □

For convenience of the reader, we recall two essential facts:

Proposition 4.7 [Horbez 2016, Propositions 4.27, 4.3]. *If $T \in \overline{\mathcal{O}}$ is mixing, then T is $\mathcal{Z}^{(\max)}$-averse if and only if T is $\mathcal{Z}^{(\max)}$-incompatible.*

Lemma 4.8 [Guirardel 2008, Lemma 1.18]. *Let $T \in \overline{\mathcal{O}}$ be compatible with a $\mathcal{Z}^{(\max)}$-splitting, S. Let $H \subseteq \mathbb{F}$ be a subgroup, such that the H-minimal subtree T_H of T is indecomposable. Then H is elliptic in S.*

Proposition 4.9. *T_ϕ^+ is $\mathcal{Z}^{(\max)}$-averse.*

Proof. We assume that T_ϕ^+ is not $\mathcal{Z}^{(\max)}$-averse and argue towards a contradiction. Indeed, as T_ϕ^+ is mixing, Proposition 4.7 implies that it is compatible with a $\mathcal{Z}^{(\max)}$-splitting S. Now let $H = \mathrm{Stab}(Y_I)$. If $Y_I = T_\phi^+$, then $H = \mathbb{F}$ and Lemma 4.8 gives that $\mathbb{F}$ is elliptic in S, a contradiction as S is a nontrivial minimal splitting.

The other possibility is that Y_I is a proper subtree in T_ϕ^+, and in this situation we argue that Λ_ϕ^+ is carried by a vertex group of S. As above, we apply Lemma 4.8 to conclude that $H = \mathrm{Stab}(Y_I)$ is carried by a vertex group A of the splitting S. We have a tower of covers corresponding to subgroups as follows (we temporarily blur the distinction between $\mathbb{F}$ and the universal cover of $\mathcal{R}$):

$$\mathbb{F} \xrightarrow{\pi_{H,\mathbb{F}}} X_H \xrightarrow{\pi_{A,H}} X_A \xrightarrow{\pi_{\mathcal{R},A}} \mathcal{R}$$

We denote by G_A and G_H the core of the corresponding covers.

Let β be a generic leaf of Λ_ϕ^+. Even though H may not be finitely generated, we claim it is enough to show that every finite subsegment of β can be immersed into G_H. Indeed, by postcomposing these immersions with $\pi_{A,H}$ (also an immersion), we see that every finite subpath of β can then be immersed into G_A. Since A is finitely generated, we conclude that β can be immersed into G_A, and therefore that Λ_ϕ^+ is carried by a vertex group of the cyclic splitting S.

Let $h : \mathbb{F} \to T_\phi^+$ be an $\mathbb{F}$-equivariant map which is linear on edges and Lipschitz (it's easy to see that such maps exist). Lemma 3.1 of [Bestvina et al. 1997] gives that $\mathrm{BBT}(h)$ is finite. Color the line β_L in T_ϕ^+ red and let $\beta_{\mathbb{F}}$ be the realization of β in $\mathbb{F}$. Pull back the coloring via h to $\beta_{\mathbb{F}}$ as follows (keeping in mind the bounded cancellation): if $x \in \beta_{\mathbb{F}}$ is such that $h(x)$ is red, then color x red, otherwise do not color x. It's clear that both ends of $\beta_{\mathbb{F}}$ have red segments.

Let $J_{\mathbb{F}}$ be a subsegment of $\beta_{\mathbb{F}}$. Extend $J_{\mathbb{F}}$ along $\beta_{\mathbb{F}}$ if necessary to ensure that both endpoints of $J_{\mathbb{F}}$ are red. Define $J = h_\#(J_{\mathbb{F}})$. The fact that the endpoints of $J_{\mathbb{F}}$ are red ensures that J is a subsegment of β_L. Apply Lemma 4.6 to obtain an element $g \in H$ whose axis contains J. Color the axis of g in T_ϕ^+ blue. Pull back this coloring to the axis of g in $\mathbb{F}$ exactly as above. Equivariance of h, coupled with the fact that g is not elliptic in $\mathbb{F}$ or T_ϕ^+, implies that every subray of the axis of g in $\mathbb{F}$ contains blue points. In particular, there are blue points on either side of $J_{\mathbb{F}}$. Thus the axis of g in $\mathbb{F}$ contains the prescribed segment $J_{\mathbb{F}}$. It's now evident that $J_{\mathbb{F}}$ is contained in the H-minimal subtree of $\mathbb{F}$. This implies that $\pi_{H,\mathbb{F}}(J_{\mathbb{F}})$ is contained in the core G_H of the cover, completing the proof. □

5. Filling but not $\mathcal{Z}^{(\max)}$-filling laminations

In this section, we study filling laminations which are not $\mathcal{Z}^{(\max)}$-filling. We then use this understanding to establish the following proposition, which is a restatement of the second claim in Theorem 1.1. This section concludes with a proof of the first statement in Theorem 1.1.

Proposition 5.1. *Let ϕ be an automorphism with a filling lamination Λ^+_ϕ that is not $\mathcal{Z}^{(\max)}$-filling, so that Λ^+_ϕ is carried by a vertex group of a (maximally-) cyclic splitting S. Then there is a (maximally-) cyclic splitting S' that is fixed by a power of ϕ.*

The splitting S' is canonical in the sense that the vertex group which carries Λ^+_ϕ is as small as possible. The proof of Proposition 5.1 will require an excursion into the theory of JSJ-decompositions; the reader is referred to [Fujiwara and Papasoglu 2006] for details about JSJ theory.

We say a lamination is *elliptic* in an $\mathbb{F}$-tree T if it is is carried by a vertex stabilizer of T. Let $\mathfrak{S}$ be the set of all one-edge $\mathcal{Z}^{(\max)}$-splittings in which the lamination Λ^+_ϕ is elliptic. Since Λ^+_ϕ is filling, the set $\mathfrak{S}$ does not contain any free splittings.

Definition 5.2 (types of pairs of splittings [Rips and Sela 1997]). Let $S = A *_C B$ (or $A*_C$) and $S' = A' *_{C'} B'$ (or $A'*_{C'}$) be one-edge cyclic splittings with corresponding Bass–Serre trees T and T'. We say S is *hyperbolic* with respect to S' if there is an element $c \in C$ that acts hyperbolically on T'. We say S is *elliptic* with respect to S' if C fixes a point of T'. We say this pair is *hyperbolic-hyperbolic* if each splitting is hyperbolic with respect to the other. We define elliptic-elliptic, hyperbolic-elliptic and elliptic-hyperbolic splittings similarly.

Lemma 5.3. *With notation as above, suppose that $S, S' \in \mathfrak{S}$, and assume without loss that Λ^+_ϕ is carried by the vertex groups A and A'. Then Λ^+_ϕ is elliptic in the minimal subtree of A in T', denoted T'_A, and in the minimal subtree of A' in T, denoted $T_{A'}$.*

Proof. Since A and A' both carry Λ^+_ϕ, their intersection $A \cap A'$ also carries Λ^+_ϕ. The vertex stabilizers of $T_{A'}$ are precisely the intersection of vertex stabilizers of T with A', namely the conjugates of $A \cap A'$. Thus Λ^+_ϕ is carried by a vertex group of $T_{A'}$. □

Lemma 5.4. *With notation as above, suppose that S, S' are one-edge $\mathcal{Z}^{(\max)}$-splittings in $\mathfrak{S}$. Then S and S' are either hyperbolic-hyperbolic or elliptic-elliptic.*

Proof. The following is based on the proof of [Fujiwara and Papasoglu 2006, Proposition 2.2]. We will address the case that both the splittings are free products with amalgamations; when one or both are HNN extensions, the proof is similar. Toward a contradiction, suppose some element of C acts hyperbolically in T' and that C' is elliptic in T. Without loss of generality, we may assume that C' fixes the vertex stabilized by A in T. Suppose first that both A' and B' fix vertices in T. The two subgroups cannot fix the same vertex because they generate $\mathbb{F}$. On the other hand, if the vertices are distinct, then C' fixes an edge in T. Hence C' must be a finite index subgroup of C, in contradiction to the assumption that C is hyperbolic in T'. Thus, one of A' or B' does not fix a vertex in T.

Assume without loss that A' does not fix a vertex of T. The minimal subtree of A' in T, denoted $T_{A'}$, gives a minimal splitting of A' over an infinite index subgroup of C (i.e., a free splitting). Indeed, were A' to split over a finite index subgroup C_1 of C, then C_1 would be elliptic in T' contradicting our assumption that C is hyperbolic in T'. As C' is elliptic in T, it is also elliptic in $T_{A'}$. Now blow up the vertex stabilized by A' in T' to the free splitting of A' just obtained, and then collapse the edge stabilized by C' to get a free splitting T_0 of $\mathbb{F}$. Then B' is still elliptic in T_0. If Λ_ϕ^+ is carried by B', then Λ_ϕ^+ is elliptic in the free splitting T_0, which is a contradiction. If Λ_ϕ^+ is carried by A', then Lemma 5.3 implies that Λ_ϕ^+ is elliptic in $T_{A'}$. Thus Λ_ϕ^+ is also elliptic in the free splitting T_0, again a contradiction. □

In [Fujiwara and Papasoglu 2006], the existence of JSJ decompositions for splittings with slender edge groups ([loc. cit., Theorem 5.13]) is established via an iterative process: one starts with a pair of splittings, and produces a new splitting which is a common refinement (in the case of an elliptic-elliptic pair) [loc. cit., Proposition 5.10], or an enclosing subgroup [loc. cit., Definition 4.5] (in the case of a hyperbolic-hyperbolic pair) [loc. cit., Proposition 5.8]. One then repeats this process for all the splittings under consideration, and uses an accessibility result due to Bestvina and Feighn [1991] to conclude that the process stops after finitely many iterations. In order to use techniques of Fujiwara and Papasoglu, we need only ensure that if two splittings belong to the set $\mathfrak{S}$, then the splittings created in this process also belong to $\mathfrak{S}$. By examining the construction of an enclosing subgroup for a pair of hyperbolic-hyperbolic splittings [Fujiwara and Papasoglu 2006, Proposition 4.7] and using Lemma 5.3, we see that the enclosing graph decomposition of $\mathbb{F}$ for this pair of splittings indeed belongs to $\mathfrak{S}$. Similarly, Lemma 5.3 implies that the refinement of two elliptic-elliptic splittings that are contained in $\mathfrak{S}$ is itself contained in $\mathfrak{S}$. This discussion implies that JSJ decompositions exist for cyclic splittings of $\mathbb{F}$ in which Λ_ϕ^+ is elliptic.

We conclude our foray into JSJ decompositions by using the theory of deformation spaces [Forester 2002; Guirardel and Levitt 2007a] to show that the set of JSJ splittings of $\mathbb{F}$ in which Λ_ϕ^+ is elliptic is finite. By passing to a power, we will then obtain a ϕ-invariant splitting in $\mathfrak{S}$.

Definition 5.5 (slide moves [Guirardel and Levitt 2007a, Section 7]). Let $e = vw$ and $f = vu$ be adjacent edges in an $\mathbb{F}$-tree T such that the edge stabilizer of f, denoted G_f, is contained in G_e. Assume that e and f are not in the same orbit as nonoriented edges. Define a new tree T' with the same vertex set as T and replace f by an edge $f' = wu$ equivariantly. Then we say f *slides* across e. Often, a slide move is described on the quotient of T by $\mathbb{F}$.

Definition 5.6 [Guirardel and Levitt 2007a; Forester 2002]. The *deformation space* $\mathcal{D}$ containing a tree T is the set of all trees T' such that there are equivariant maps from T to T' and from T' to T, up to equivariant isometry.

Definition 5.7 [Forester 2002]. A tree T is reduced if no inclusion of an edge group into either of its vertex groups is an isomorphism.

Theorem 5.8 [Guirardel and Levitt 2007a, Theorem 7.2]. *If $\mathcal{D}$ is a nonascending deformation space, then any two reduced simplicial trees $T, T' \in \mathcal{D}$ may be connected by a finite sequence of slides.*

Deformation spaces consisting of trees such that no edge stabilizer properly contains a conjugate of itself are examples of nonascending deformation spaces [Guirardel and Levitt 2007a, Section 7]. We are only interested in such deformation spaces here.

Lemma 5.9. *Given a reduced cyclic splitting S, there are only finitely many slide moves that can be performed on S. Moreover, any sequence of slide moves starting at S has bounded length.*

Proof. The first statement follows from the fact that S has finitely many orbits of edges. For the second statement, first suppose that the splitting $S/\mathbb{F}$ does not have any loops or circuits. Then it is clear that only finitely many slide moves can be performed on S. If S has a loop, then we can slide an edge f along the loop e only once. Indeed, we have $G_f \subseteq G_e$ and after sliding we have $G_{f'} \subseteq tG_et^{-1}$, where t is the stable letter corresponding to the loop. Since $G_e \cong \mathbb{Z}$ and $G_e \cap tG_et^{-1} = 1$, $G_{f'} \nsubseteq G_e$ which prevents sliding of f' over e. The proof in the case of a circuit is similar. □

Proof of Proposition 5.1. By assumption, there exists a one-edge cyclic splitting S such that Λ^+_ϕ is elliptic in S. The existence of JSJ decomposition for splittings in $\mathfrak{S}$ implies that the deformation space $\mathcal{D}$ for cyclic splittings in $\mathfrak{S}$ is nonempty. Since the edge stabilizer of the trees in $\mathcal{D}$ is $\mathbb{Z}$, the space $\mathcal{D}$ is nonascending. Theorem 5.8 and Lemma 5.9 together imply that the set of reduced trees in $\mathcal{D}$ is finite. As the set of reduced trees in $\mathcal{D}$ is ϕ-invariant, passing to a power yields a reduced cyclic splitting S' in $\mathcal{D}$ which is fixed by ϕ^k. The same argument works if S is a maximally-cyclic splitting. □

Proof of Theorem 1.1 (loxodromic). Suppose that ϕ has a $\mathcal{Z}^{(\max)}$-filling lamination, whereby ϕ^{-1} does as well. Applying Proposition 4.9 we conclude that both T^+_ϕ and T^-_ϕ are $\mathcal{Z}^{(\max)}$-averse. We now argue that these trees determine distinct points in $\mathcal{X}^{(\max)}$. We denote the dual lamination of a tree T by $L(T)$ [Coulbois et al. 2008]. Since the attracting lamination Λ^+_ϕ and the repelling lamination Λ^-_ϕ are different, and $\Lambda^\mp_\phi \subseteq L(T^\pm_\phi)$ and $\Lambda^\pm_\phi \nsubseteq L(T^\pm_\phi)$, we have that T^+_ϕ and T^-_ϕ are distinct points in $\overline{\mathcal{O}}$. Both trees are mixing (Lemma 4.3), but [Horbez 2016, Proposition 4.3] provides that if two mixing trees in $\overline{\mathcal{O}}$ are equivalent (i.e., determine the same point in $\mathcal{X}^{(\max)}$), then each must collapse onto the other. If there a collapse map from $T \to T'$, then $L(T) \subseteq L(T')$. So if T_ϕ^+ and T_ϕ^- were equivalent, then their dual laminations would be equal, a contradiction.

We now argue that the limit set of $\langle\phi\rangle$ acting on $\mathcal{FZ}^{(\max)}$ consists of two points. There is a minor complication arising from the fact that the folding path constructed in Section 3 consisted entirely of trees in the boundary of outer space, but Theorem 2.4 applies only to sequences in the interior. Indeed, recall from Section 3 that T denotes the universal cover of a relative train track map representing ϕ and that T_0 was obtained from T by first collapsing the $\mathbb{F}$-translates of the $\mathcal{A}$-minimal subtree in T, then further collapsing according to a measure μ. Finally, recall (Proposition 4.9) that the sequence $T_i = \lambda_\phi^{-i} T_0 \phi^i$ where $i \in \mathbb{N}$ converges to T_ϕ^+, which is $\mathcal{Z}^{(\max)}$-averse. Let $R_i = T\phi^i$ and let $R_\infty = \lim_{i\to\infty} R_i$. For all $i \in \mathbb{N}$, R_i collapses onto T_i, so R_i and T_i are compatible. That compatibility passes to the limit follows from [Guirardel and Levitt 2017, Corollary A.12], so R_∞ is compatible with T_ϕ^+ and is therefore $\mathcal{Z}$-averse. Applying Theorem 2.4 to the sequence $\{R_i\}_{i\in\mathbb{N}}$, we conclude that the image sequence $\psi(R_i)$ converges to $[T_\phi^+] \in \mathcal{X}^{(\max)}$. Finally, since the set of reducing splittings for a free simplicial $\mathbb{F}$-tree is bounded, if S is any $\mathcal{Z}^{(\max)}$-splitting we have that $S\phi^i$ converges to $[T_\phi^+]$, with a similar statement holding for iterates of ϕ^{-1}. Thus, $\Lambda_{\mathcal{FZ}}\langle\phi\rangle$ consists of exactly two points and ϕ therefore acts loxodromically on $\mathcal{FZ}^{(\max)}$.

We now prove the converse: if ϕ acts loxodromically on $\mathcal{FZ}^{(\max)}$, then ϕ has a $\mathcal{Z}^{(\max)}$-filling lamination. Indeed, if ϕ acts loxodromically on $\mathcal{FZ}^{(\max)}$, then ϕ necessarily acts loxodromically on $\mathcal{FS}$, and thus has a filling lamination Λ_ϕ^+. If the lamination is not $\mathcal{Z}^{(\max)}$-filling, then Proposition 5.1 implies that a power of ϕ fixes a point in $\mathcal{FZ}^{(\max)}$, contradicting our assumption on ϕ. Thus, Λ_ϕ^+ is $\mathcal{Z}^{(\max)}$-filling. □

6. Examples

This section will provide several examples exhibiting the range of behaviors of outer automorphisms acting on $\mathcal{FZ}$. We begin with an automorphism that acts loxodromically on $\mathcal{FZ}$.

Example 6.1 (loxodromic element). Let ϕ be a rotationless automorphism with a CT representative $f : G \to G$ satisfying the following properties:

- f has exactly two strata, each of which is EG and nongeometric.
- The lamination corresponding to the top stratum of f is filling.

An explicit example satisfying these properties can be constructed using the sage-train-tracks package written by T. Coulbois [2015]. The fact that the top lamination is filling guarantees that ϕ acts loxodromically on $\mathcal{FS}$. As both strata are nongeometric,

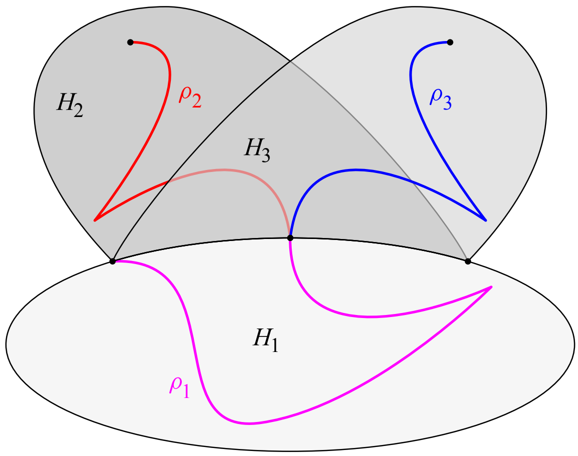

Figure 1. A CT representative for the automorphism constructed in Example 6.2, which acts with bounded orbits but no fixed point.

[Handel and Mosher 2013b, Fact 1.42(1a)] guarantees that ϕ does not fix the conjugacy class of any element of $\mathbb{F}$, and therefore cannot possibly fix a cyclic splitting. Corollary 1.2 implies that ϕ acts loxodromically.

Example 6.2 (bounded orbit without fixed point). By building on Example 6.1 and [Handel and Mosher 2014, Example 4.2], we can construct an automorphism ψ which acts on $\mathcal{FZ}$ with bounded orbits but without a fixed point. Let ψ be a three stratum automorphism obtained from f by creating a duplicate of H_2. Explicitly, ψ has a CT representative $f' : G' \to G'$ defined as follows. The graph G' is obtained by taking two copies of G and identifying them along G_1. Each edge E of G' is naturally identified with an edge of G, and $f'(E)$ is defined via this identification. Moreover, the marking of G naturally gives a marking of G' (by a larger free group). That f' is a CT is evident from the fact that f is a CT.

There are three laminations in $\mathcal{L}(\psi)$, and it's evident that none are filling. Since the top lamination in $\mathcal{L}(\phi)$ (where ϕ is as in Example 6.1) is filling, we know that $\mathcal{L}(\psi)$ must fill. Thus, ψ acts on $\mathcal{FS}$ with bounded orbits. As before, [Handel and Mosher 2013b, Fact 1.42 (1a)] implies that ψ doesn't fix the conjugacy class of any element of $\mathbb{F}$: while each stratum may have an INP, ρ_i, none of these INPs can be closed loops, nor can they be concatenated to form a closed loop. Thus, ψ does not fix any one-edge cyclic splitting and therefore must act on $\mathcal{FZ}$ with bounded orbits, but no fixed point. See Figure 1 for a pictorial representation of ψ. The INPs ρ_2 and ρ_3 must each have at least one endpoint which is not in H_1.

Example 6.3 (loxodromic element). Consider the outer automorphism $\phi : F_4 \to F_4$ given by

$$\phi(a) = ab, \quad \phi(b) = bcab, \quad \phi(c) = d, \quad \phi(d) = cd.$$

In [Reynolds 2012], it is shown that the stable tree for ϕ is indecomposable and hence $\mathcal{Z}$-averse. Therefore ϕ acts loxodromically on $\mathcal{FZ}$.

Example 6.4 (fixed point). Let $\Sigma_{2,1}$ be the surface of genus two with one puncture. Consider the free homotopy class of a simple separating curve which divides $\Sigma_{2,1}$ into two subsurfaces: a once punctured torus and a twice punctured torus. Placing a pseudo-Anosov on each of these subsurfaces and taking the outer automorphism induced by this mapping class yields an element of $\mathrm{Out}(\mathbb{F})$ that acts loxodromically on $\mathcal{FS}$, but fixes a point in $\mathcal{FZ}$. A similar example using nonseparating simple curve can be found in the proof of [Mann 2014, Proposition 3].

7. Virtually cyclic centralizers

In this section, we investigate centralizers of automorphisms acting loxodromically on $\mathcal{FZ}$. To do this, we use the machinery of completely split train tracks, and the "disintegration" procedure of [Feighn and Handel 2009], which takes a rotationless outer automorphism and returns an abelian subgroup of $\mathrm{Out}(\mathbb{F})$. The main result is:

Theorem 1.3. *An outer automorphism with a filling lamination has a virtually cyclic centralizer in* $\mathrm{Out}(\mathbb{F})$ *if and only if the lamination is* $\mathcal{Z}$*-filling.*

We begin with a terse review of disintegration for outer automorphisms.

7A. ***Disintegration and rotationless abelian subgroups in* Out($\mathbb{F}$).** Given a mapping class f in Thurston normal form, there is a straightforward way of making a subgroup of the mapping class group, called *the disintegration of f*, by "doing one piece at a time." The subgroup is easily seen to be abelian as each pair of generators can be realized as homeomorphisms with disjoint supports. The process of disintegration in $\mathrm{Out}(\mathbb{F})$ is analogous, but more difficult.

The reader is warned that we will only review those ingredients from [Feighn and Handel 2009] that will be used directly; the reader is directed there, specifically to Section 6, for complete details. Given a rotationless outer automorphism ϕ, one can form an abelian subgroup called $\mathcal{D}(\phi)$. The process of disintegrating ϕ begins by creating a finite graph, B, which records the interactions between different strata in a CT representing ϕ. As a first approximation, the components of B correspond to generators of $\mathcal{D}(\phi)$. However, there may be additional relations between strata that are unseen by B, so the number of components of B only gives an upper bound to the rank of $\mathcal{D}(\phi)$.

Let $f : G \to G$ be a CT representing the rotationless outer automorphism ϕ. While the construction of $\mathcal{D}(\phi)$ does depend on f, using different representatives will produce subgroups that are commensurable.

Let E_i, E_j be distinct linear edges in G with the same axis w so that $f(E_i) = E_i w^{d_i}$ and $f(E_j) = E_j w^{d_j}$ for integers $d_i \neq d_j$. Recall that if $d_i, d_j > 0$, then any path of the form $E_i w^* \overline{E}_j$ called an *exceptional path*. In the same scenario, if d_i and d_j have different signs, we call such a path a *quasi-exceptional path*. It would be instructive for the reader to compute the f-image of some exceptional and quasi-exceptional paths. We will need to consider a weakening of the complete splitting of paths and circuits in f. The *quasi-exceptional splitting* of a completely split path or circuit σ is the coarsening of the complete splitting obtained by considering each quasi-exceptional subpath to be a single element.

Definition 7.1. Define a finite directed graph B as follows. There is one vertex v_i^B for each nonfixed irreducible stratum H_i. If H_i is NEG, then a v_i^B-path is defined as the unique edge in H_i; if H_i is EG, then a v_i^B-path is either an edge in H_i or a taken connecting path in a zero stratum contained in H_i^z. There is a directed edge from v_i^B to v_j^B if there exists a v_i^B-path κ_i such that some term in the QE-splitting of $f_\#(\kappa_i)$ is an edge in H_j. The components of B are labeled $B_1, \dots, B_K$. For each B_s, define X_s to be the minimal subgraph of G that contains H_i for each NEG stratum with $v_i^B \in B_s$ and contains H_i^z for each EG stratum with $v_i^B \in B_s$. We say that $X_1, \dots, X_K$ are the *almost invariant subgraphs* associated to $f : G \to G$.

The reader should note that the number of components of B is left unchanged if an iterate of $f_\#$ is used in the definition, rather than $f_\#$ itself. In the sequel, we will frequently make statements about B using an iterate of $f_\#$.

For each K-tuple $\vec{a} = (a_1, \dots, a_K)$ of nonnegative integers, define

$$f_{\vec{a}}(E) = \begin{cases} f_\#^{a_i}(E) & \text{if } E \in X_i, \\ E & \text{if } E \text{ is fixed by } f. \end{cases}$$

It turns out that $f_{\vec{a}}$ is always a homotopy equivalence of G [Feighn and Handel 2009, Lemma 6.7], but in general $\langle f_{\vec{a}} \mid \vec{a} \text{ is a nonnegative tuple} \rangle$ is not abelian. To obtain an abelian subgroup, one has to pass to a certain subset of tuples which take into account interactions between the almost invariant subgraphs that are unseen by B. The reader is referred to [loc. cit., Example 6.9] for an example.

Definition 7.2. A K-tuple $(a_1, \dots, a_K)$ is called *admissible* if, for all axes μ, if

- X_s contains a linear edge E_i with axis μ and exponent d_i,
- X_t contains a linear edge E_j with axis μ and exponent d_j,
- there is a vertex v^B of B and a v^B-path $\kappa \subseteq X_r$ such that some element in the quasi-exceptional family $E_i \overline{E}_j$ is a term in the QE-splitting of $f_\#(\kappa)$,

then $a_r(d_i - d_j) = a_s d_i - a_t d_j$.

The disintegration of ϕ is then defined as

$$\mathcal{D}(\phi) = \langle f_{\vec{a}} \mid \vec{a} \text{ is admissible} \rangle,$$

which is abelian by [loc. cit., Corollary 6.16].

We now recall some useful facts concerning abelian subgroups of Out($\mathbb{F}$), which were studied in [Feighn and Handel 2009].

If an abelian subgroup H is generated by rotationless automorphisms, then all elements of H are rotationless [loc. cit., Corollary 3.13]. In this case, H is said to be rotationless. Rotationless abelian subgroups of Out($\mathbb{F}$) have finitely many attracting laminations ([loc. cit., Lemma 4.4]), i.e., if H is abelian and $\mathcal{L}(H) := \bigcup_{\phi \in H} \mathcal{L}(\phi)$, then $|\mathcal{L}(H)| < \infty$.

Feighn and Handel [2009] associated to each rotationless abelian subgroup of Out($\mathbb{F}$) a finite collection of (nontrivial) homomorphisms to $\mathbb{Z}$. Combining these, one obtains a homomorphism $\Omega : H \to \mathbb{Z}^N$ that is injective [Feighn and Handel 2009, Lemma 4.6]. An element $\psi \in H$ is said to be *generic* if all coordinates of $\Omega(\psi)$ are nonzero. For the purposes of this section, we require only two facts concerning Ω. First, some of the coordinates of Ω correspond to elements in the finite set $\mathcal{L}(H)$ (there are other coordinates, which we will not need). Second is the fact that the coordinate of $\Omega(\psi)$ corresponding to $\Lambda \in \mathcal{L}(H)$ is positive if and only if $\Lambda \in \mathcal{L}(\psi)$.

7B. ***From disintegrations to centralizers.*** In this subsection, we explain how to deduce Theorem 1.3 from the following proposition concerning the disintegration of elements acting loxodromically on $\mathcal{FZ}$. The proof of Proposition 7.3 is postponed until the next subsection.

Proposition 7.3. *If ϕ is rotationless and has a $\mathcal{Z}$-filling lamination, then $\mathcal{D}(\phi)$ is virtually cyclic.*

Proof of Theorem 1.3. Suppose $\psi \in C(\phi)$ has infinite order and assume that $\langle \phi, \psi \rangle \simeq \mathbb{Z}^2$. If no such element exists, then $C(\phi)$ is virtually cyclic, as there is a bound on the order of a finite subgroup of Out($\mathbb{F}$) [Culler 1984]. Now let H_R be the finite index subgroup of $\langle \phi, \psi \rangle$ consisting of rotationless elements [Feighn and Handel 2009, Corollary 3.14] and let ψ' be a generic element of this subgroup. If the coordinate of $\Omega(\psi')$ corresponding to the $\mathcal{Z}$-filling lamination Λ_ϕ^+ is negative, then replace ψ' by $(\psi')^{-1}$, which is also generic. Since $\Lambda_\phi^+ \in \mathcal{L}(\psi')$ is $\mathcal{Z}$-filling, Theorem 1.1 implies that ψ' acts loxodromically on $\mathcal{FZ}$. Since ψ' is generic in H_R, [Feighn and Handel 2009, Theorem 7.2] says that $\mathcal{D}(\psi') \cap \langle \phi, \psi \rangle$ has finite index in $\langle \phi, \psi \rangle$. This contradicts Proposition 7.3, which says that the disintegration of ψ' is virtually cyclic. □

7C. ***The proof of Proposition 7.3.*** The idea of the proof is as follows. We noted above that the number of components in B only gives an upper bound to the rank of $\mathcal{D}(\phi)$; it may happen that there are interactions between the strata of f that are unseen by B (Definition 7.2). We will obtain precise information about the structure of B; it consists of one main component (B_1), and several components consisting of a single point $(B_2, \dots, B_K)$. We will then show that the admissibility condition provides sufficiently many constraints so that choosing a_1 determines $a_2, \dots, a_K$. Thus, the set of admissible tuples consists of a line in $\mathbb{Z}^K$.

Let $f : G \to G$ be a CT representing ϕ with filtration $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_M = G$. Let $\Lambda^+_\phi \in \mathcal{L}(\phi)$ be $\mathcal{Z}$-filling and let $\ell \in \Lambda^+_\phi$ be a generic leaf. As Λ^+_ϕ is filling, the corresponding EG stratum is necessarily the top stratum, H_M. We will understand the graph B by studying the realization of ℓ in G. The results of [Bestvina et al. 2000, §3.1], together with Lemma 4.25 of [Feighn and Handel 2011] give that the realization of ℓ in G is completely split, and this splitting is unique. Thus, we may consider the QE-splitting of ℓ.

We begin with a lemma that allows the structure of INPs and quasi-exceptional paths to be understood inductively.

Lemma 7.4. *Let H_r be a nonfixed irreducible stratum and let ρ be a path of height $s \geq r$ which is either an INP or a quasi-exceptional path. Assume further that ρ intersects H_r nontrivially. Then one of the following holds*:

- *H_r and H_s are NEG linear strata with the same axis, each consisting of a single edge E_r (resp. E_s), and $\rho = E_s w^k \overline{E}_r$, for some $k \in \mathbb{Z}$, where w is a closed, root-free Nielsen path of height $< s$.*
- *ρ can be written as a concatenation $\rho = \beta_0 \rho_1 \beta_1 \rho_2 \beta_2 \cdots \rho_j \beta_j$, where each ρ_i is an INP of height r and each β_i is a path contained in $G - \operatorname{int}(H_r)$ (some of the β_i's may be trivial).*

Proof. The proof proceeds by strong induction on the height s of the path ρ. In the base case, $s = r$, and ρ is either an INP of height r or a quasi-exceptional path of the form described. The inductive step breaks into cases according whether H_s is an EG stratum, or an NEG stratum.

If H_s is an EG stratum, then ρ must be an INP, as there are no exceptional paths of EG height. In this case, [Feighn and Handel 2011, Lemma 4.24 (2)] provides a decomposition of ρ into subpaths of height s and maximal subpaths of height $< s$, and each of the subpaths of height $< s$ is a Nielsen path. The inductive hypothesis then guarantees that each of these Nielsen paths has the desired form. By breaking apart and combining these terms appropriately, we conclude that ρ does as well.

Suppose now that H_s is an NEG stratum and let E_s be the unique edge in H_s. Using (NEG Nielsen paths), we see that E_s must be a linear edge, and therefore that ρ is either $E_s w^k \overline{E_s}$ or $E_s w^k \overline{E'}$, where E' is another linear edge with the same axis and w is a closed root free Nielsen path of height $< s$. If H_r is NEG linear, and $E' = E_r$, then the first conclusion holds. Otherwise, we may apply the inductive hypothesis to w to obtain a decomposition as desired. This completes the proof. □

We now begin our study of the graph B. We call the component of B containing v^B_M, the vertex corresponding to the topmost stratum of f, the *main component*.

Lemma 7.5. *All nonlinear NEG strata are in the main component of B.*

Proof. Let H_r be a nonlinear NEG stratum, with single edge E_r. It is enough to show that the single edge E_r occurs as a term in the QE-splitting of ℓ (henceforth, we will say that E_r is a *QE-splitting unit* in ℓ), as this implies that there is an edge in B connecting v_M^B to v_r^B. As ℓ is filling, we know that its realization in G must cross E_r. If the corresponding QE-splitting unit of ℓ is the single edge E, then we are done. The only other possibility is that the QE-splitting unit is an INP or a quasi-exceptional path of some height $s \geq r$. An application of Lemma 7.4 shows that this is impossible, as it would imply the existence of an INP of height r or a quasi-exceptional path of the form $E_r w^* \overline{E}'$, contradicting (NEG Nielsen paths). □

Lemma 7.6. *All EG strata are in the main component of B.*

Proof. Let H_r be an EG stratum. As before, it is enough to show that some (every) edge of H_r occurs as a QE-splitting unit of ℓ. There are three types of QE-splitting units that can cross H_r: a single edge in H_r, an INP of height $\geq r$, or a quasi-exceptional path. In the first case, we are done, so suppose that every time ℓ crosses H_r, the corresponding QE-splitting unit is an INP or a quasi-exceptional path. We now argue that this situation leads to a contradiction.

We may write ℓ as a concatenation $\ell = \cdots \gamma_1\sigma_1\gamma_2\sigma_2 \cdots$, where each σ_i is a QE-splitting unit of ℓ which intersects $\operatorname{int}(H_r)$, and each γ_i is a maximal concatenation of QE-splitting units of ℓ which do not intersect $\operatorname{int}(H_r)$ (some γ_i's may be trivial). By assumption, each σ_i is an INP or a QEP. Applying Lemma 7.4 to each of the σ_i's, then combining and breaking apart the terms appropriately, we see that ℓ can be written as a concatenation $\ell = \cdots \gamma_1\rho_1\gamma_2\rho_2 \cdots$ where each ρ_i is the unique INP of height r or its inverse. Call this INP ρ.

We will now use the information we have about ℓ to find a $\mathcal{Z}$-splitting in which ℓ is carried by a vertex group. The existence of such a splitting will contradict our assumption that ℓ is a generic leaf of the $\mathcal{Z}$-filling lamination Λ_ϕ^+.

We now modify G to produce a 2-complex, G'', whose fundamental group is identified with $\mathbb{F}$. First assume H_r is nongeometric, so that ρ has distinct endpoints, v_0 and v_1. Let G' be the graph obtained from G by replacing each vertex v_i for $i \in \{0, 1\}$ with two vertices, v_i^u and v_i^d (u and d stand for "up" and "down"), which are to be connected by an edge E_i. For each edge E of G incident to v_i, connect it in G' to the new vertices as follows: if $E \in H_r$, then E is connected to v_i^d, and if $E \notin H_r$, then E is connected to v_i^u. G' deformation retracts onto G by collapsing the new edges, and this retraction identifies $\pi_1(G')$ with $\mathbb{F}$ via the marking of G. Let $R = [0, 1] \times [0, 1]$ be a rectangle and define G'' by gluing $\{i\} \times [0, 1]$ homeomorphically onto E_i for $i \in \{0, 1\}$, then gluing $[0, 1] \times \{0\}$ homeomorphically to the INP ρ. As only three sides of the rectangle have been glued, G'' deformation retracts onto G', and its fundamental group is again identified with $\mathbb{F}$.

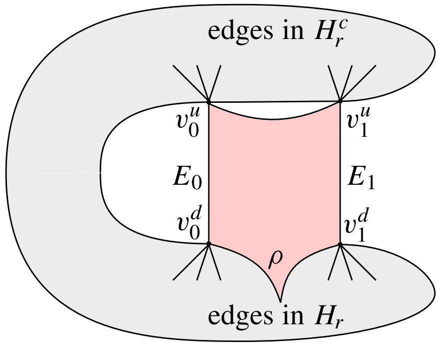

Figure 2. G'' when H_r is a nongeometric EG stratum.

The construction of G'' differs only slightly if H_r is geometric. In this case, ρ is a closed loop based at v_0 and we blow up v_0 to two vertices, v_0^u and v_0^d, that are connected by an edge E_0. Instead of gluing in a rectangle, we glue in a cylinder $R = S^1 \times [0, 1]$; $\{p\} \times [0, 1]$ is glued homeomorphically to E_0, where p is a point in S^1, and $S^1 \times \{0\}$ is glued homeomorphically to ρ.

Recall that in G, the leaf ℓ can be written as a concatenation $\ell = \cdots \gamma_1 \rho_1 \gamma_2 \rho_2 \cdots$, where each ρ_i is either ρ or $\bar{\rho}$. Thus we can realize ℓ in G' as $\ell = \cdots \gamma_1 \rho_1' \gamma_2 \rho_2' \cdots$, where each ρ_i' is either $E_0 \rho \bar{E}_1$ or $E_1 \bar{\rho} \bar{E}_0$. In G'', each ρ_i' is homotopic rel endpoints to a path that travels along the top of R, rather than down-across-and-up. Thus, after performing a (proper!) homotopy to the image of ℓ, we can arrange that it never intersects the interior of R, nor the vertical sides of R. Cutting R along its centerline yields a $\mathcal{Z}$-splitting S of $\mathbb{F}$, and ℓ is carried by a vertex group of this splitting. If H_r is nongeometric, then S is a free splitting and if H_r is geometric, then S is a cyclic splitting. In either case, so long as S is nontrivial, we have contradicted our assumption that the lamination is $\mathcal{Z}$-filling. □

Claim 7.7. *The splitting S is nontrivial.*

Proof of Claim 7.7. We first handle the case that H_r is geometric. We have described a one-edge cyclic splitting S which was obtained as follows: cut G' along the edge E_0, that is, collapse $G' - E_0$ to get a free splitting of $\mathbb{F}$, then perform the edge fold corresponding to $\langle w \rangle$ (see Section 2L for definition), where w is the conjugacy class of the INP ρ. If $G' - E_0$ is connected, then the free splitting is an HNN extension, and there is no danger of S being trivial as $\mathrm{rk}(\mathbb{F}) \geq 3$. On the other hand, if $G' - E_0$ is disconnected, then let $G^{d'}$ and $G^{u'}$ be the components of $G' - E_0$ containing v_0^d and v_0^u respectively. The free splitting which is folded to get S is precisely $\pi_1(G^{d'}) * \pi_1(G^{u'})$. In this case, $G^{d'}$ is necessarily a component of G_r and [Feighn and Handel 2011, Proposition 2.20 (2)] together with (filtration) imply that this component is a core graph. As H_r is EG, the rank of $\pi_1(G^{d'})$ is at least two and the splitting S is therefore nontrivial. To see that $\mathrm{rk}(\pi_1(G^{u'})) \geq 1$, we need only recall that ℓ is not periodic and is carried by $\pi_1(G^{c'}) * \langle w \rangle$.

In the case that H_r is nongeometric, the splitting obtained above is a free splitting. If $G' - \{E_0, E_1\}$ is connected, then the free splitting is an HNN extension, and as before S is nontrivial. If $G' - \{E_0, E_1\}$ is disconnected, then the component containing v_0^d (and by necessity v_1^d), denoted $G^{d'}$, corresponds to a vertex group of S. By the same reasoning as in the previous case, we get that $\pi_1(G^{d'})$ is nontrivial. As before, the other vertex group of S carries the leaf ℓ and hence S is a nontrivial free splitting. □

Remark 7.8. We would like the reader to note that the above proof actually gives restrictions on the way two EG strata in a CT can interact. For example, suppose that ϕ is represented by a CT, $f : G \to G$, with exactly two strata, both of which are EG. Assume further that H_1 is nongeometric and has an INP. A priori, there are three ways that H_2 can interact with H_1: (1) there is some edge E in H_2 such that $f_\#(E)$ contains an edge from H_1 as a splitting unit, (2) the $f_\#$ image of each edge in H_2 is entirely contained in H_2, or (3) whenever E is an edge from H_2 and $f_\#(E)$ crosses H_1, the corresponding splitting unit is the INP of height 1. In the first case, $\Lambda_2 \supset \Lambda_1$. In the second case, we may think of the strata as being side-by-side, rather than H_2 being stacked on top of H_1. The proof of Lemma 7.6 implies that the third possibility never happens. Indeed, the proof provides a free splitting which is ϕ-invariant and the vertex groups of this splitting form a free factor system which lies strictly between the free factor systems $\pi_1(G_1)$ and $\pi_1(G_2)$. This contradicts (filtration) in the definition of a CT, which states that the filtration $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_M = G$ must be reduced.

Before we address the NEG linear strata and conclude the proof of Proposition 7.3, we present a final lemma concerning the structure of B.

Lemma 7.9. *Assume H_r is a linear NEG stratum consisting of an edge E_r. If v_r^B is not in the main component of B, then the component of B containing v_r^B is a single point.*

Proof. This follows directly from the definition of B, together with Lemmas 7.5 and 7.6. If H_r is a linear NEG stratum, then the definition of B implies that v_r^B has no outgoing edges. For any edge in B whose terminal vertex is v_r^B, its initial vertex necessarily corresponds to a nonlinear NEG stratum or an EG stratum, and hence is in the main component of B. □

When dealing with an NEG linear stratum, we would like to carry out a similar strategy to the EG case: blow up the terminal vertex, v_0, to an edge and glue in a cylinder, thereby producing a cyclic splitting in which ℓ is carried by a vertex group. The main difficulty in implementing this comes from other linear edges with the same axis; for each such edge, one has to decide whether to glue it in G' to v_0^d or v_0^u.

Let μ be an axis with corresponding unoriented root-free conjugacy class w. Let $\mathcal{E}_\mu$ be the set of linear edges in G with axis μ. Define a relation on $\mathcal{E}_\mu$ by declaring $E \sim_R E'$ if the quasi-exceptional path $Ew^*\bar{E}'$ is a QE-splitting unit in ℓ or if both E and E' are QE-splitting units in ℓ. Then let $\sim$ be the equivalence relation generated by $\sim_R$. Note that all edges in $\mathcal{E}_\mu$ which occur as QE-splitting units in ℓ are in the same equivalence class.

As mentioned above, the difficulty in adapting the strategy used for EG stratum to the present situation lies in deciding where to glue edges (top or bottom) in G'. The existence of multiple classes in the equivalence relation $\sim$ will provide instructions for how to glue edges from $\mathcal{E}_\mu$ in G' so that the leaf never crosses the cylinder in G''.

Lemma 7.10. *There is only one equivalence class of* $\sim$. *Moreover, at least one edge in* $\mathcal{E}_\mu$ *occurs as a term in the QE-splitting of* ℓ.

Proof. Suppose for a contradiction that there is more than one equivalence class of $\sim$ and let $[E]$ be an equivalence class for which no edge in $[E]$ is a QE-splitting unit in ℓ. Now build G' as in the proof of Lemma 7.6. Let v_0 be the terminal vertex of the edges in $\mathcal{E}_\mu$ (they all have the same terminal vertex), and define G' by blowing up v_0 into two vertices, v_0^u and v_0^d, which are connected by an edge E_0. The terminal vertex of each edge of $[E]$ is to be glued in G' to v_0^u, while all other edges in G that are incident to v_0 are glued to v_0^d. Define G'' as before, gluing the bottom of a cylinder R along the closed loop w, and gluing the vertical interval above v_0 homeomorphically to the edge E_0.

The definition of $\sim$ guarantees that ℓ is carried by a vertex group of the cyclic splitting determined by cutting along the centerline of R. Indeed, whenever ℓ crosses an edge from $[E]$, the corresponding QE-splitting unit is either an INP or a quasi-exceptional path $E'w^*\bar{E}''$, where $E', E'' \in [E]$. Repeatedly applying Lemma 7.4 to each of these terms, then rearranging and combining terms appropriately, we see that ℓ can be written in G as a concatenation $\ell = \cdots \gamma_1\rho_1\gamma_2\rho_2 \cdots$ where each ρ_i is either $E'w^*\overline{E'}$ or $E'w^*\overline{E''}$ with $E', E'' \in [E]$. Thus we can realize ℓ in G' as $\ell = \cdots \gamma_1\rho_1'\gamma_2\rho_2' \cdots$, where each ρ_i' is $E'E_0w^*\bar{E}_0\overline{E'}$ or $E'E_0w^*\bar{E}_0\overline{E''}$. In G'', each ρ_i' is homotopic rel endpoints to a path that travels along the top of R, rather than down-across-and-up. Thus, we have again produced a cyclic splitting in which ℓ is carried by a vertex group.

We now argue that the splitting is nontrivial. There is a free splitting S which comes from cutting the edge E_0 in G', which cannot be a self loop. The cyclic splitting of interest S' is obtained from S by performing the edge fold corresponding to w. If $G' - E_0$ is connected, then S' is an HNN extension with edge group $\langle [w] \rangle$. As $\mathrm{rk}(\mathbb{F}) \geq 3$, the vertex group has rank at least two and we are done. Now suppose E_0 is separating so that $G' - E_0$ consists of two components. Let G'^u be the component containing the vertex v_0^u and let G'^d be the other component. The vertex groups of the splitting S' are $\pi_1(G^d)$ and $\pi_1(G^u) * \langle [w] \rangle$. The fact that v is a principal vertex guarantees that $\pi_1(G^d) \ncong \mathbb{Z}$, and the fact that G is a finite graph without valence one vertices ensures that $\pi_1(G^u)$ is nontrivial.

The proof of the second statement is exactly the same as that of the first. □

Finally, we finish the proof of Proposition 7.3. As before, B_1 is the main component of B, with corresponding almost invariant subgraph X_1. All other components $B_2, \dots, B_K$ are single points, and each almost invariant subgraph X_i consist of a single linear edge. Let $(a_1, \dots, a_K)$ be a K-tuple and suppose that a_1 has been chosen. We claim that imposing the admissibility condition determines all other a_i's.

Suppose first that E_i, E_j are linear edges with the same axis, μ, such that $E_i \in X_1$, $E_j \in X_k$, and $E_i \sim_R E_j$. Let d_i and d_j be the exponents of E_i and E_j respectively. Applying the definition of admissibility with $s = r = 1$, $t = k$, and κ a v^B path such that $f_\#(\kappa)$ contains a quasi-exceptional path of the form $E_i w^* \overline{E}_j$ in its QE-splitting (such a κ must exist as a quasi-exceptional path of this type occurs in the QE-splitting of ℓ), we obtain the relation $a_1(d_i - d_j) = a_1 d_i - a_k d_j$. Thus a_k is determined by a_1.

Now suppose E_i and E_j are as above, but rather than being related by $\sim_R$, we only have that $E_i \sim E_j$. There is a finite chain of $\sim_R$-relations to get from E_i to E_j. At each stage in this chain, the definition of admissibility (applied with $r = 1$ and κ chosen appropriately) will impose a relation that determines the next coordinate from the previous ones. Ultimately, this determines a_k.

We have thus shown that an admissible tuple is completely determined by choosing a_1, and therefore that the set of admissible tuples forms a line in $\mathbb{Z}^K$. Therefore $\mathcal{D}(\phi)$ is virtually cyclic.

7D. *A converse to Proposition 7.3.*

Proposition 7.11. *If ϕ has a filling lamination which is not $\mathcal{Z}$-filling, then the centralizer of some power of ϕ in* Out($\mathbb{F}$) *is not virtually cyclic.*

Proof. Since ϕ has a filling lamination which is not $\mathcal{Z}$-filling, it follows by Proposition 5.1 that for some k, ϕ^k fixes a one-edge cyclic splitting S.

Suppose $S/\mathbb{F}$ is a free product with amalgamation with vertex stabilizers $\langle A, w \rangle$ and B and edge group $\langle w \rangle \subset B$. Consider the Dehn twist D_w given by S as follows: D_w acts as identity on B and conjugation by w on A. The automorphism D_w has infinite order. We claim that D_w and ϕ^k commute. Indeed, consider a generating set $\{a_1, \dots, a_k, b_1, \dots b_m\}$ for $\mathbb{F}$ such that the a_i's generate A and the b_i's generate B. Choose a representative Φ of ϕ such that $\Phi^k(B) = B$ and $\Phi^k(\langle A, w \rangle) = \langle A, w \rangle^b$ for some element $b \in B$. Since D_w is identity on B and $\Phi^k(B) = B$, we have $\Phi^k(D_w(b_i)) = D_w(\Phi^k(b_i))$ for all generators b_i. Since $D_w(a_i) = w a_i \overline{w}$,

$\Phi^k(w) = w$ and $\Phi^k(\langle A, w \rangle) = \langle A, w \rangle^b$, we have $D_w(\Phi^k(a_i)) = \Phi^k(D_w(a_i))$ for all generators a_i. Thus D_w and ϕ^k commute.

We now address the case that $S/\mathbb{F}$ is an HNN extension. Assume $S/\mathbb{F}$ has stable letter t, edge group $\langle w \rangle$ and vertex group $\langle A, \bar{t}wt \rangle$. Since the cyclic splitting S is obtained from a free HNN extension, with vertex group A and stable letter t, by an edge fold, we have that a basis of $\mathbb{F}$ is given by $\{a_1, a_2, \ldots, a_k, t\}$, where the a_i's generate A. Consider the Dehn twist D_w determined by S such that D_w is identity on A and sends t to wt. The automorphism D_w has infinite order. Choose a representative Φ of ϕ such that $\langle A, \bar{t}wt \rangle$ is Φ^k-invariant. Then for every generator a_i, $\Phi^k(a_i)$ is a word in the a_i's and $\bar{t}wt$. Since D_w is identity on A and fixes $\bar{t}wt$, we get $\Phi^k(D_w(a_i)) = D_w(\Phi^k(a_i))$. Again, since $\langle A, \bar{t}wt \rangle$ is Φ^k-invariant, $\Phi^k(t)$ is equal to $w^m t\alpha$, where α is some word in $\langle A, \bar{t}wt \rangle$ and $m \in \mathbb{Z}$. On one hand, $\Phi^k(D_w(t)) = \Phi^k(wt) = \Phi^k(w)\Phi^k(t) = ww^m t\alpha$ and on the other hand, $D_w(\Phi^k(t)) = D_w(w^m t\alpha) = w^m D_w(t) D_w(\alpha) = w^m wt\alpha$. Thus D_w and ϕ^k commute.

Thus when ϕ^k fixes a cyclic splitting, then an infinite order element other than a power of ϕ^k exists in the centralizer of ϕ^k. □

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RADHIKA GUPTA
DEPARTMENT OF MATHEMATICS
TECHNION – ISRAEL INSTITUTE OF TECHNOLOGY
HAIFA
ISRAEL

radhikagup@technion.ac.il

DERRICK WIGGLESWORTH
DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF ARKANSAS
FAYETTEVILLE, AR
UNITED STATES

derrick.wigglesworth@gmail.com

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