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LIE 2-ALGEBROIDS AND MATCHED PAIRS OF 2-REPRESENTATIONS: A GEOMETRIC APPROACH

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Li-Bland's correspondence between linear Courant algebroids and Lie 2-algebroids is explained at the level of linear and core sections versus graded functions, and shown to be an equivalence of categories. More precisely, decomposed VB-Courant algebroids are shown to be equivalent to split Lie 2-algebroids in the same manner as decomposed VB-algebroids are equivalent to 2-term representations up to homotopy (Gracia-Saz and Mehta). Several special cases are discussed, yielding new examples of split Lie 2-algebroids.

We prove that the bicrossproduct of a matched pair of 2-representations is a split Lie 2-algebroid and we explain this result geometrically, as a consequence of the equivalence of VB-Courant algebroids and Lie 2-algebroids. This explains in particular how the two notions of the "double" of a matched pair of representations are geometrically related. In the same manner, we explain the geometric link between the two notions of the double of a Lie bialgebroid.

1. Introduction

Lie bialgebroids and matched pairs of Lie algebroids. A matched pair of Lie algebroids is a pair of Lie algebroids A and B over a smooth manifold M, together

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with a representation of A on B and a representation of B on A,[1] satisfying some compatibility conditions, which can be interpreted in two manners: first the direct sum $A \oplus B$ carries a Lie algebroid structure over M, such that A and B are Lie subalgebroids and such that the representations give "mixed" brackets

$$[(a,0),(0,b)] = (-\nabla_b a, \nabla_a b)$$

for all $a \in \Gamma(A)$ and $b \in \Gamma(B)$. The direct sum $A \oplus B$ with this Lie algebroid structure is called here the *bicrossproduct of the matched pair*. Note that conversely, any Lie algebroid with two transverse and complementary subalgebroids defines a matched pair of Lie algebroids [Mokri 1997].

Alternatively, the fibre product $A \times_M B$, which has a double vector bundle structure with sides A and B and with trivial core, is as follows a double Lie algebroid: for $a \in \Gamma(A)$, we write $a^l : B \to A \times_M B$, $b_m \mapsto (a(m), b_m)$ for the linear section of $A \times_M B \to B$, and similarly, a section $b \in \Gamma(B)$ defines a linear section $b^l \in \Gamma_A(A \times_M B)$. The Lie algebroid structure on $A \times_M B \to B$ is defined by

$$[a_1^l, a_2^l] = [a_1, a_2]^l \quad \text{and} \quad \rho(a^l) = \widehat{\nabla}_a \in \mathfrak{X}^l(B)$$

for $a, a_1, a_2 \in \Gamma(A)$, where we denote by $\widehat{D} \in \mathfrak{X}(B)$ the linear vector field defined by a derivation D on B. The Lie algebroid structure on $A \times_M B \to A$ is defined accordingly by the Lie bracket on sections of B and the B-connection on A. The double Lie algebroid $A \times_M B$ is then called the *double of the matched pair*. Note that conversely, any double Lie algebroid with trivial core is the fibre product of two vector bundles and defines a matched pair of Lie algebroids [Mackenzie 2011].

These two constructions encoding the compatibility conditions for a matched pair of representations seem at first sight only related by the fact that they both encode matched pairs. A similar phenomenon can be observed with the notion of Lie bialgebroid: A Lie bialgebroid is a pair of Lie algebroids $A, A^* \to M$ in duality, satisfying some compatibility conditions, which can be described in two manners. First, the direct sum $A \oplus A^* \to M$ inherits a Courant algebroid structure with the two Lie algebroids A and A^* as transverse Dirac structures, and mixed brackets given by

$$[\![(a,0),(0,\alpha)]\!] = (-\boldsymbol{i}_\alpha \boldsymbol{d}_{A^*} a, \pounds_a \alpha)$$

for all $a \in \Gamma(A)$ and $\alpha \in \Gamma(A^*)$. Alternatively, the cotangent bundle T^*A, a double vector bundle with sides A and A^* and core T^*M, which is isomorphic as a double vector bundle to T^*A^*, carries two linear Lie algebroid structures. The first, on $T^*A \to A$, is the cotangent Lie algebroid induced by the linear Poisson

[1] For the sake of simplicity, we write $\nabla : \Gamma(A) \times \Gamma(B) \to \Gamma(B)$ and $\nabla : \Gamma(B) \times \Gamma(A) \to \Gamma(A)$ for the two flat connections. It is clear from the indexes which connection is meant.

structure defined on A by the Lie algebroid structure on A^*. The second, on $T^*A \simeq T^*A^* \to A^*$, is defined in the same manner by the Lie algebroid structure on A. The compatibility conditions for A and A^* to build a Lie bialgebroid are equivalent to the double Lie algebroid condition for (T^*A, A, A^*, M) [Mackenzie 2011; Gracia-Saz et al. 2018]. Again, the cotangent double of the Lie algebroid and the bicrossproduct Courant algebroid seem only related by the fact that they are two elegant ways of encoding the Lie bialgebroid conditions.

One feature of this paper is the explanation of the deeper, more intrinsic relation between the bicrossproduct of a matched pair of Lie algebroids and its double on the one hand, and between the bicrossproduct of a Lie bialgebroid and its cotangent double on the other hand. In both cases, the bicrossproduct can be understood as a purely algebraic construction, which is *geometrised* by the corresponding double Lie algebroid. More generally, we explain how the matched pair of two 2-term representations up to homotopy [Gracia-Saz et al. 2018] defines a bicrossproduct split Lie 2-algebroid, and we relate the latter to the decomposed double Lie algebroid found in [Gracia-Saz et al. 2018] to be equivalent to the matched pair of 2-representations.

These three classes of examples of bicrossproduct constructions versus double Lie algebroid constructions are described here as three special cases of the equivalence between the category of VB-Courant algebroid, and the category of Lie 2-algebroids [Li-Bland 2012].

The equivalence of VB-Courant algebroids with Lie 2-algebroids. Let us be a little more precise. Supermanifolds were introduced in the 1970s by physicists, as a formalism to describe supersymmetric field theories, and have been extensively studied since then (see, e.g., [Sardanashvily 2009; Varadarajan 2004]). A supermanifold is a smooth manifold the algebra of functions of which is enriched by anticommuting coordinates. Supermanifolds with an additional $\mathbb{Z}$-grading have been used since the late 1990s among others in relation with Poisson geometry and Lie and Courant algebroids [Ševera 2005; Roytenberg 2002; Voronov 2002].

An equivalence between Courant algebroids and $\mathbb{N}$-manifolds of degree 2 endowed with a symplectic structure and a compatible homological vector field [Roytenberg 2002] is at the heart of the current interest in $\mathbb{N}$-graded manifolds in Poisson geometry, as this algebraic description of Courant algebroids leads to possible paths to their integration [Ševera 2005; Li-Bland and Ševera 2012; Mehta and Tang 2011]. In [Jotz Lean 2018b] we showed how the category of $\mathbb{N}$-manifolds of degree 2 is equivalent to a category of double vector bundles endowed with a linear involution. The latter involutive double vector bundles are dual to double vector bundles endowed with a linear metric. In this paper we extend this correspondence to an equivalence between the category of $\mathbb{N}$-manifolds of degree 2 endowed with a homological vector field and a category of VB-Courant algebroids, i.e., metric double vector bundles endowed with a linear Courant algebroid structure. We recover in this manner Li-Bland's one-to-one correspondence between Lie 2-algebroids and VB-Courant algebroids [2012], which we better formulate as an equivalence of categories.

Li-Bland's construction of a VB-Courant algebroid from a given Lie 2-algebroid relies on the equivalence of symplectic Lie 2-algebroids with Courant algebroids [Roytenberg 2002]: given a Lie 2-algebroid, its cotangent space is a symplectic Lie 2-algebroid, which corresponds hence to a Courant algebroid. The linear property of the cotangent space induces an additional vector bundle structure on the obtained Courant algebroid, a linear structure which turns out to be compatible with the pairing, the anchor and the bracket. While this method is nice and very simple, it is not constructive in the sense that the sheaf of graded functions on the Lie 2-algebroid are not described as a sheaf of special sections of the corresponding VB-Courant algebroid. Further, the exact correspondences of the degree 2 structure with the linear pairing (that we describe in [Jotz Lean 2018b]) and of the homological vector field with the linear anchor and bracket cannot be read directly from Li-Bland's proof.

We remedy this and provide a new formulation of Li-Bland's equivalence that *does not* use Roytenberg's description [2002] of Courant algebroids via symplectic Lie 2-algebroids. Since we explain precisely how functions of degree 0, 1 and 2 on the Lie 2-algebroid side correspond to special functions and sections of the corresponding VB-Courant algebroid, the result presented here is in our opinion more convenient to work with when looking at concrete examples.

Original motivation. Let us explain in more detail our methodology and our original motivation. A VB-Lie algebroid is a double vector bundle $(D; A, B; M)$ with one side $D \to B$ endowed with a Lie algebroid bracket and an anchor that are *linear* over a Lie algebroid structure on $A \to M$. Gracia-Saz and Mehta [2010] prove that linear decompositions of VB-algebroids are equivalent to super-representations, or in other words, to 2-representations.

The definition of a VB-Courant algebroid is very similar to the one of a VB-algebroid. The Courant bracket, the anchor and the nondegenerate pairing all have to be linear. In [Jotz Lean 2018a] we prove that the standard Courant algebroid over a vector bundle can be decomposed into a connection, a Dorfman connection, a curvature term and a vector bundle map, in a manner that resembles very much the main result in [Gracia-Saz and Mehta 2010]. In other words, as linear splittings of the tangent space TE of a vector bundle E are equivalent to linear connections on the vector bundle, linear splittings of the Pontryagin bundle $TE \oplus T^*E$ over E are equivalent to a certain class of Dorfman connections [Jotz Lean 2018a]. Further, as the Lie algebroid structure on $TE \to E$ can be described in a splitting in terms of the corresponding connection, the Courant algebroid structure on $TE \oplus T^*E \to E$

is completely encoded in a splitting by the corresponding Dorfman connection [Jotz Lean 2018a].

Our original goal in this project was to show that the work done in [Jotz Lean 2018a] is in fact a very special case of a general result on linear splittings of VB-Courant algebroids, in the spirit of Gracia-Saz and Mehta's work [2010]. Along the way, we proved the equivalence of [2]-manifolds with metric double vector bundles [Jotz Lean 2018b]. This paper builds upon that equivalence and proves that a linear Lagrangian splitting of a VB-Courant algebroid decomposes the VB-Courant algebroid structure in the components of a split Lie 2-algebroid.

Note that our correspondence of decomposed VB-Courant algebroids with split Lie 2-algebroids is also described (with slightly different conventions) in the independent work of del Carpio-Marek [2015].

While the methods used in [Jotz Lean 2018b; Li-Bland 2012] do not use splittings of the [2]-manifolds and metric double vector bundles, it appears here more natural to us to work with split objects. First, the equivalence of the underlying [2]-manifolds with metric double vector bundles was already established and it is now much more convenient to work in splittings versus Lagrangian double vector bundle charts — the definition of the homological vector field that corresponds to a linear Courant algebroid structure is easily done in splittings (see Section 3B), but we did not find a good coordinate free definition of it using the techniques given by [Jotz Lean 2018b]. Second, working with splittings is necessary in order to exhibit the similarity with Gracia-Saz and Mehta's techniques [2010], which is one of our main goals. Finally, as explained below, the construction of the bicrossproduct of a matched pair of 2-representations is an algebraic description of the construction of a *decomposed* VB-Courant algebroid from a *decomposed* double Lie algebroid, just as 2-representations are equivalent to *decomposed* VB-Lie algebroids.

Application: the bicrossproduct of a matched pair of 2-representations. The equivalence of matched pairs of 2-representations with a certain class of split Lie 2-algebroids appears as a natural class of examples of our correspondence of decomposed VB-Courant algebroids with split Lie 2-algebroids. A double vector bundle $(D; A, B; M)$ with core C and two linear Lie algebroid structures on $D \to A$ and $D \to B$ is a double Lie algebroid if and only if the pair of duals $(D^*_A; D^*_B)$ is a VB-Lie bialgebroid over C^*. Equivalently, $D^*_A \oplus_{C^*} D^*_B$ is a VB-Courant algebroid over C^*, with side $A \oplus B$ and core $B^* \oplus A^*$, and with two transverse Dirac structures D^*_A and D^*_B. A decomposition of D defines on the one hand a matched pair of 2-representations [Gracia-Saz et al. 2018], and on the other hand a Lagrangian decomposition of $D^*_A \oplus_{C^*} D^*_B$, hence a split Lie 2-algebroid. Once this geometric correspondence has been found, it is straightforward to construct algebraically the split Lie 2-algebroid from the matched pair, and vice versa.

Outline, main results and applications. This paper is organised as follows.

Section 2: We describe the main result in [Jotz Lean 2018b] — the equivalence of [2]-manifolds with metric double vector bundles — and we recall the background on double Lie algebroids and matched pairs of representations up to homotopy that will be necessary for our main application on the bicrossproduct of a matched pair of 2-representations.

Section 3: We start by recalling necessary background on Courant algebroids, Dirac structures and Dorfman connections. Then we formulate in our own manner Sheng and Zhu's definition [2017] of split Lie 2-algebroids. We write in coordinates the homological vector field corresponding to a split Lie 2-algebroid, showing where the components of the split Lie 2-algebroid appear. In Section 3D, we give several classes of examples of split Lie 2-algebroids, introducing in particular the standard split Lie 2-algebroids defined by a vector bundle. Finally we describe morphisms of split Lie 2-algebroids.

Section 4: We give the definition of VB-Courant algebroids [Li-Bland 2012] and we relate split Lie 2-algebroids with Lagrangian splittings of VB-Courant algebroids, in the spirit of Gracia-Saz and Mehta's description of split VB-algebroids via 2-term representations up to homotopy [2010]. Then we describe the VB-Courant algebroids corresponding to the examples of split Lie 2-algebroids found in the preceding section, and we prove that the equivalence of categories established in [Jotz Lean 2018b] induces an equivalence of the category of VB-Courant algebroids with the category of Lie 2-algebroids.

Section 5: We construct the bicrossproduct of a matched pair of 2-representations and prove that it is a split Lie 2-algebroid. We then explain geometrically this result by studying VB-bialgebroids and double Lie algebroids.

Appendix: We give the proof of our main theorem, describing decomposed VB-Courant algebroids via split Lie 2-algebroids.

Prerequisites, notation and conventions. We write $p_M : TM \to M$, $q_E : E \to M$ for vector bundle maps. For a vector bundle $Q \to M$ we often identify without further mention the vector bundle $(Q^*)^*$ with Q via the canonical isomorphism. We write $\langle \cdot\,, \cdot \rangle$ for the canonical pairing of a vector bundle with its dual; i.e., $\langle a_m, \alpha_m \rangle = \alpha_m(a_m)$ for $a_m \in A$ and $\alpha_m \in A^*$. We use several different pairings; in general, which pairing is used is clear from its arguments. Given a section ε of E^*, we always write $\ell_\varepsilon : E \to \mathbb{R}$ for the linear function associated to it, i.e., the function defined by $e_m \mapsto \langle \varepsilon(m), e_m \rangle$ for all $e_m \in E$.

Let M be a smooth manifold. We denote by $\mathfrak{X}(M)$ and $\Omega^1(M)$ the sheaves of local smooth sections of the tangent and the cotangent bundle, respectively. For an arbitrary vector bundle $E \to M$, the sheaf of local sections of E will be written $\Gamma(E)$. Let $f : M \to N$ be a smooth map between two smooth manifolds M

and N. Then two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are said to be *f-related* if $Tf \circ X = Y \circ f$ on $\mathrm{Dom}(X) \cap f^{-1}(\mathrm{Dom}(Y))$. We write then $X \sim_f Y$. In the same manner, if $\phi : A \to B$ is a vector bundle morphism over $\phi_0 : M \to N$, then a section $a \in \Gamma_M(A)$ is ϕ-related to $b \in \Gamma_N(B)$ if $\phi(a(m)) = b(\phi_0(m))$ for all $m \in M$. We write then $a \sim_\phi b$. The dual of the morphism ϕ is in general not a morphism of vector bundles, but a relation $R_{\phi^*} \subseteq A^* \times B^*$ defined as

$$R_{\phi^*} = \{(\phi_m^* \beta_{\phi_0(m)}, \beta_{\phi_0(m)}) \mid m \in M, \beta_{\phi_0(m)} \in B^*_{\phi_0(m)}\},$$

where $\phi_m : A_m \to B_{\phi_0(m)}$ is the morphism of vector spaces.

We will say 2-representations for 2-term representations up to homotopy. We write "[n]-manifold" for "$\mathbb{N}$-manifolds of degree n". We refer the reader to [Jotz Lean 2018b; Bonavolontà and Poncin 2013] for a quick review of split $\mathbb{N}$-manifolds, and for our notation convention. Let E_1 and E_2 be smooth vector bundles of finite ranks r_1, r_2 over M. The [2]-manifold $E_1[-1] \oplus E_2[-2]$ has local basis sections of ${E_i}^*$ as local generators of degree i, for $i = 1, 2$, and so dimension $(p; r_1, r_2)$. A [2]-manifold $\mathcal{M} = E_1[-1] \oplus E_2[-2]$ defined in this manner by a graded vector bundle is called a *split* [2]-*manifold*. In other words, we have

$$C^\infty(\mathcal{M})^0 = C^\infty(M), \quad C^\infty(\mathcal{M})^1 = \Gamma(E_1^*) \quad \text{and} \quad C^\infty(\mathcal{M})^2 = \Gamma(E_2^* \oplus \wedge^2 E_1^*).$$

Let $\mathcal{N} := F_1[-1] \oplus F_2[-2]$ be a second [2]-manifold over a base N. A morphism $\mu : F_1[-1] \oplus F_2[-2] \to E_1[-1] \oplus E_2[-2]$ of split [2]-manifolds over the bases N and M, respectively, consists of a smooth map $\mu_0 : N \to M$, three vector bundle morphisms $\mu_1 : F_1 \to E_1$, $\mu_2 : F_2 \to E_2$ and $\mu_{12} : \wedge^2 F_1 \to E_2$ over μ_0. The morphism $\mu^\star : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{N})$ sends a degree 1 function $\xi \in \Gamma(E_1^*)$ to ${\mu_1}^\star \xi \in \Gamma(F_1^*)$, defined by $\langle \mu_1^\star \xi, f_m \rangle = \langle \xi(\mu_0(m)), \mu_1(f_m) \rangle$ for all $f_m \in F_1(m)$. The morphism $\mu^\star$ sends a degree 2 function $\xi \in \Gamma(E_2^*)$ to ${\mu_2}^\star \xi + \mu_{12}^\star \xi \in \Gamma({F_2}^* \oplus \wedge^2 {F_1}^*)$.

2. Preliminaries

We refer to Section 2.2 of [Jotz Lean 2018b] for the definition of a double vector bundle, and for the necessary background on their linear and core sections, and on their linear splittings and dualisations. Sections 2.3–2.5 of [Jotz Lean 2018b] recall the definition of a VB-algebroid, and also the equivalence of 2-term representations up to homotopy (called here "2-representations" for short) with linear decompositions of VB-algebroids [Gracia-Saz and Mehta 2010]. The notation that we use here is the same as in [Jotz Lean 2018b].

In this section we recall the correspondence of decompositions of double Lie algebroids with matched pairs of 2-representations. Then we summarise the correspondence found in [Jotz Lean 2018b] between double vector bundles endowed with a linear metric, and $\mathbb{N}$-manifolds of degree 2.

2A. ***Double Lie algebroids and matched pairs of 2-representations.*** If $(D, A; B, M)$ is a VB-algebroid with Lie algebroid structures on $D \to B$ and $A \to M$, then the dual vector bundle $D^*_B \to B$ has a *Lie–Poisson structure* (a linear Poisson structure), and the structure on D^*_B is also Lie–Poisson with respect to $D^*_B \to C^*$ [Mackenzie 2011, 3.4]. Dualising this bundle gives a Lie algebroid structure on $(D^*_B)^*_{C^*} \to C^*$. This equips the double vector bundle $((D^*_B)^*_{C^*}; C^*, A; M)$ with a VB-algebroid structure. Using the isomorphism defined by $-\langle\cdot\,,\cdot\rangle$, (see [Mackenzie 2005] and [Jotz Lean 2018b, §2.2.4] for a summary and our sign convention), the double vector bundle $(D^*_A \to C^*; A \to M)$ is also a VB-algebroid. In the same manner, if $(D \to A, B \to M)$ is a VB-algebroid then we use $\langle\cdot\,,\cdot\rangle$ to get a VB-algebroid structure on $(D^*_B \to C^*; B \to M)$.

Let $\Sigma : A \times_M B \to D$ be a linear splitting of D and denote by $(\nabla^B, \nabla^C, R_A)$ the induced 2-representation of the Lie algebroid A on $\partial_B : C \to B$ (see [Gracia-Saz and Mehta 2010]; this is also recalled in Section 2.5 of [Jotz Lean 2018b]). The linear splitting Σ induces a linear splitting $\Sigma^\star : A \times_M C^* \to D^*_A$ of D^*_A. The 2-representation of A that is associated to this splitting is then $(\nabla^{C^*}, \nabla^{B^*}, -R^*_A)$ on the complex $\partial^*_B : B^* \to C^*$. This is proved in the appendix of [Drummond et al. 2015].

A *double Lie algebroid* [Mackenzie 2011] is a double vector bundle $(D; A, B; M)$ with core C, and with Lie algebroid structures on each of $A \to M$, $B \to M$, $D \to A$ and $D \to B$ such that each pair of parallel Lie algebroids gives D the structure of a VB-algebroid, and such that the pair (D^*_A, D^*_B) with the induced Lie algebroid structures on base C^* and the pairing $\langle\cdot\,,\cdot\rangle$, is a Lie bialgebroid.

Consider a double vector bundle $(D; A, B; M)$ with core C and a VB-Lie algebroid structure on each of its sides. After a choice of splitting $\Sigma : A \times_M B \to D$, the Lie algebroid structures on the two sides of D are described by two 2-representations [Gracia-Saz and Mehta 2010]. We prove in [Gracia-Saz et al. 2018] that (D^*_A, D^*_B) is a Lie bialgebroid over C^* if and only if, for any splitting of D, the two induced 2-representations form a matched pair as in the following definition [Gracia-Saz et al. 2018].

Definition 2.1. Let $(A \to M, \rho_A, [\cdot\,,\cdot])$ and $(B \to M, \rho_B, [\cdot\,,\cdot])$ be two Lie algebroids and assume that A acts on $\partial_B : C \to B$ up to homotopy via $(\nabla^B, \nabla^C, R_A)$ and B acts on $\partial_A : C \to A$ up to homotopy via $(\nabla^A, \nabla^C, R_B)$.[2] Then we say that the two representations up to homotopy form a matched pair if

(M1) $\nabla_{\partial_A c_1} c_2 - \nabla_{\partial_B c_2} c_1 = -(\nabla_{\partial_A c_2} c_1 - \nabla_{\partial_B c_1} c_2)$,

(M2) $[a, \partial_A c] = \partial_A(\nabla_a c) - \nabla_{\partial_B c} a$,

[2] For the sake of simplicity, we write in this definition ∇ for all the four connections. It will always be clear from the indexes which connection is meant. We write ∇^A for the A-connection induced by ∇^{AB} and ∇^{AC} on $\wedge^2 B^* \otimes C$ and ∇^B for the B-connection induced on $\wedge^2 A^* \otimes C$.

(M3) $[b, \partial_B c] = \partial_B(\nabla_b c) - \nabla_{\partial_A c} b$,

(M4) $\nabla_b \nabla_a c - \nabla_a \nabla_b c - \nabla_{\nabla_b a} c + \nabla_{\nabla_a b} c = R_B(b, \partial_B c)a - R_A(a, \partial_A c)b$,

(M5) $\partial_A(R_A(a_1, a_2)b)$
$$= -\nabla_b[a_1, a_2] + [\nabla_b a_1, a_2] + [a_1, \nabla_b a_2] + \nabla_{\nabla_{a_2} b} a_1 - \nabla_{\nabla_{a_1} b} a_2,$$

(M6) $\partial_B(R_B(b_1, b_2)a)$
$$= -\nabla_a[b_1, b_2] + [\nabla_a b_1, b_2] + [b_1, \nabla_a b_2] + \nabla_{\nabla_{b_2} a} b_1 - \nabla_{\nabla_{b_1} a} b_2,$$

for all $a, a_1, a_2 \in \Gamma(A)$, $b, b_1, b_2 \in \Gamma(B)$ and $c, c_1, c_2 \in \Gamma(C)$, and

(M7) $\boldsymbol{d}_{\nabla^A} R_B = \boldsymbol{d}_{\nabla^B} R_A \in \Omega^2(A, \wedge^2 B^* \otimes C) = \Omega^2(B, \wedge^2 A^* \otimes C)$, where R_B is seen as an element of $\Omega^1(A, \wedge^2 B^* \otimes C)$ and R_A as an element of $\Omega^1(B, \wedge^2 A^* \otimes C)$.

2B. ***The equivalence of* [2]*-manifolds with metric double vector bundles.*** We quickly recall in this section the main result in [Jotz Lean 2018b].

A *metric double vector bundle* is a double vector bundle $(\mathbb{E}, Q; B, M)$ with core Q^*, equipped with a *linear symmetric nondegenerate pairing* $\mathbb{E} \times_B \mathbb{E} \to \mathbb{R}$, i.e., such that

(1) $\langle \tau_1^\dagger, \tau_2^\dagger \rangle = 0$ for $\tau_1, \tau_2 \in \Gamma(Q^*)$,

(2) $\langle \chi, \tau^\dagger \rangle = q_B^* \langle q, \tau \rangle$ for $\chi \in \Gamma_B^l(\mathbb{E})$ linear over $q \in \Gamma(Q)$, and $\tau \in \Gamma(Q^*)$ and

(3) $\langle \chi_1, \chi_2 \rangle$ is a linear function on B for $\chi_1, \chi_2 \in \Gamma_B^l(\mathbb{E})$.

Note that the *opposite* $(\overline{\mathbb{E}}; Q; B, M)$ of a metric double vector bundle $(\mathbb{E}; B; Q, M)$ is the metric double vector bundle with $\langle \cdot, \cdot \rangle_{\overline{\mathbb{E}}} = -\langle \cdot, \cdot \rangle_{\mathbb{E}}$.

A linear splitting $\Sigma : Q \times_M B \to \mathbb{E}$ is said to be *Lagrangian* if its image is maximal isotropic in $\mathbb{E} \to B$. The corresponding horizontal lifts $\sigma_Q : \Gamma(Q) \to \Gamma_B^l(\mathbb{E})$ and $\sigma_B : \Gamma(B) \to \Gamma_Q^l(\mathbb{E})$ are then also said to be *Lagrangian*. By definition, a horizontal lift $\sigma_Q : \Gamma(Q) \to \Gamma_B^l(\mathbb{E})$ is Lagrangian if and only if $\langle \sigma_Q(q_1), \sigma_Q(q_2) \rangle = 0$ for all $q_1, q_2 \in \Gamma(Q)$. Showing the existence of a Lagrangian splitting of $\mathbb{E}$ is relatively easy [Jotz Lean 2018b]. Further, if Σ^1 and $\Sigma^2 : Q \times_M B \to \mathbb{E}$ are Lagrangian, then the change of splitting $\phi_{12} \in \Gamma(Q^* \otimes Q^* \otimes B^*)$ defined by $\Sigma^2(q, b) = \Sigma^1(q, b) + \widetilde{\phi(q, b)}$ for all $(q, b) \in Q \times_M B$, is a section of $Q^* \wedge Q^* \otimes B^*$.

Example 2.2. Let $E \to M$ be a vector bundle endowed with a symmetric nondegenerate pairing $\langle \cdot, \cdot \rangle : E \times_M E \to \mathbb{R}$ (a *metric vector bundle*). Then $E \simeq E^*$ and the tangent double is a metric double vector bundle $(TE, E; TM, M)$ with pairing $TE \times_{TM} TE \to \mathbb{R}$ the tangent of the pairing $E \times_M E \to \mathbb{R}$. In particular, we have $\langle Te_1, Te_2 \rangle_{TE} = \ell_{\boldsymbol{d}\langle e_1, e_2 \rangle}$, $\langle Te_1, e_2^\dagger \rangle_{TE} = p_M^* \langle e_1, e_2 \rangle$ and $\langle e_1^\dagger, e_2^\dagger \rangle_{TE} = 0$ for $e_1, e_2 \in \Gamma(E)$.

Recall from [Jotz Lean 2018b, Example 3.11] that linear splittings of TE are equivalent to linear connections $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$. The Lagrangian splittings of TE are exactly the linear splittings that correspond to *metric* connections, i.e., linear connections $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ that preserve the metric: $\langle \nabla_{\cdot} e_1, e_2 \rangle + \langle e_1, \nabla_{\cdot} e_2 \rangle = \boldsymbol{d} \langle e_1, e_2 \rangle$ for $e_1, e_2 \in \Gamma(E)$.

Let $(\mathbb{E}, B; Q, M)$ be a metric double vector bundle. Define $\mathcal{C}(\mathbb{E}) \subseteq \Gamma^l_Q(\mathbb{E})$ as the $C^\infty(M)$-submodule of linear sections with isotropic image in $\mathbb{E}$. After the choice of a Lagrangian splitting $\Sigma : Q \times_M B \to \mathbb{E}$, $\mathcal{C}(\mathbb{E})$ can be written $\mathcal{C}(\mathbb{E}) := \sigma_B(\Gamma(B)) + \{\tilde{\omega} \mid \omega \in \Gamma(Q^* \wedge Q^*)\}$. This shows that $\mathcal{C}(\mathbb{E})$ together with $\Gamma^c_Q(\mathbb{E}) \simeq \Gamma(Q^*)$ span $\mathbb{E}$ as a vector bundle over Q.

An *involutive double vector bundle* is a double vector bundle (D, Q, Q, M) with core B^* equipped with a morphism $\mathcal{I} : D \to D$ of double vector bundles satisfying $\mathcal{I}^2 = \mathrm{Id}_D$ and $\pi_1 \circ \mathcal{I} = \pi_2$, $\pi_2 \circ \mathcal{I} = \pi_1$, where $\pi_1, \pi_2 : D \to Q$ are the two side projections, and with core morphism $-\mathrm{Id}_{B^*} : B^* \to B^*$. A *morphism* $\Omega : D_1 \to D_2$ of *involutive double vector bundles* is a morphism of double vector bundles such that $\Omega \circ \mathcal{I}_1 = \mathcal{I}_2 \circ \Omega$. [Jotz Lean 2018b, Proposition 3.15] proves a duality of involutive double vector bundles with metric double vector bundles: the dual $(D^*_{\pi_1}; Q, B; M)$ with core Q^* carries an induced linear metric. Conversely, given a metric double vector bundle $(\mathbb{E}; Q, B; M)$ with core Q^*, the dual $(\mathbb{E}^*_Q; Q, Q; M)$ with core B^* carries an induced involution as above. We define morphisms of metric double vector bundles as the dual morphisms to morphisms of involutive double vector bundles. A *morphism* $\Omega : \mathbb{F} \to \mathbb{E}$ of *metric double vector bundles* is hence a relation $\Omega \subseteq \overline{\mathbb{F}} \times \mathbb{E}$ that is the dual of a morphism of involutive double vector bundles $\omega : \mathbb{F}^*_P \to \mathbb{E}^*_Q$.

$$
\begin{tikzcd}
\mathbb{F}^*_P \arrow[rr, "\omega"] \arrow[dr] \arrow[dd] & & \mathbb{E}^*_Q \arrow[dr] \arrow[dd] & \\
& P \arrow[rr, "\omega_P"] \arrow[dd] & & Q \arrow[dd] \\
P^{**} \arrow[rr] \arrow[dr] & & Q^{**} \arrow[dr] & \\
& N \arrow[rr, "\omega_0"] & & M
\end{tikzcd}
$$

Note that the dual of Ω is compatible with the involutions if and only if Ω is an isotropic subspace of $\overline{\mathbb{F}} \times \mathbb{E}$. Equivalently [Jotz Lean 2018b], one can define a morphism $\Omega : \mathbb{F} \to \mathbb{E}$ of metric double vector bundles as a pair of maps $\omega^\star : \mathcal{C}(\mathbb{E}) \to \mathcal{C}(\mathbb{F})$ and $\omega_P^\star : \Gamma(Q^*) \to \Gamma(P^*)$ together with a smooth map $\omega_0 : N \to M$ such that

(1) $\omega^\star(\widetilde{\tau_1 \wedge \tau_2}) = \widetilde{\omega_P^\star \tau_1 \wedge \omega_P^\star \tau_2}$,

(2) $\omega^\star(q_Q^* f \cdot \chi) = q_P^*(\omega_0^* f) \cdot \omega^\star(\chi)$ and

(3) $\omega_P^\star(f \cdot \tau) = \omega_0^* f \cdot \omega_P^\star \tau$

for all $\tau, \tau_1, \tau_2 \in \Gamma(Q^*)$, $f \in C^\infty(M)$ and $\chi \in \mathcal{C}(\mathbb{E})$. We write MDVB for the obtained category of metric double vector bundles. The following theorem is proved in [Jotz Lean 2018b] and independently in [del Carpio-Marek 2015].

Theorem 2.3 [Jotz Lean 2018b]. *There is a (covariant) equivalence between the category of* [2]*-manifolds and the category of involutive double vector bundles.*

Combining the obtained equivalence with the (contravariant) dualisation equivalence of IDVB with MDVB yields a (contravariant) equivalence between the category of metric double vector bundles with the morphisms defined above and the category of [2]-manifolds. This equivalence establishes in particular an equivalence between split [2]-manifold $\mathcal{M} = Q[-1] \oplus B^*[-2]$ and the decomposed metric double vector bundle $(Q \times_M B \times_M Q^*, B, Q, M)$ with the obvious linear metric over B. More precisely, the obtained functor from [2]-manifolds to metric double vector bundles sends by construction a split [2]-manifold to a decomposed metric double vector bundle. Conversely, the functor from metric double vector bundles to [2]-manifolds sends decomposed metric double vector bundles to split [2]-manifolds.

We quickly describe the functors between the two categories. To construct the geometrisation functor $\mathcal{G} : [2]\text{–Man} \to \text{MDVB}$, take a [2]-manifold and consider its local trivialisations. Changes of local trivialisation define a set of cocycle conditions, that correspond exactly to cocycle conditions for a double vector bundle atlas. The local trivialisations can hence be collated to a double vector bundle, which naturally inherits a linear pairing. See [Jotz Lean 2018b] for more details, and remark that this construction is as simple as the construction of a vector bundle from a locally free and finitely generated sheaf of $C^\infty(M)$-modules.

Conversely, the algebraisation functor $\mathcal{A}$ sends a metric double vector bundle $\mathbb{E}$ to the [2]-manifold defined as follows: the functions of degree 1 are the sections of $\Gamma_Q^c(\mathbb{E}) \simeq \Gamma(Q^*)$, and the functions of degree 2 are the elements of $\mathcal{C}(\mathbb{E})$. The multiplication of two core sections $\tau_1, \tau_2 \in \Gamma(Q^*)$ is the core-linear section $\widetilde{\tau_1 \wedge \tau_2} \in \mathcal{C}(\mathbb{E})$.

Note that while that equivalence can be seen as the special case of trivial homological vector field versus trivial bracket and anchor of Li-Bland's bijection of Lie 2-algebroids with VB-Courant algebroids [Li-Bland 2012], this corollary is not given there and only a very careful study of Li-Bland's proof, which would amount to the work done in [Jotz Lean 2018b] would yield it.

3. Split Lie 2-algebroids

In this section we recall the notions of Courant algebroids, Dirac structures, dull algebroids, Dorfman connections and (split) Lie 2-algebroids.

3A. ***Courant algebroids and Dorfman connections.*** We introduce in this section a generalisation of the notion of Courant algebroid, namely the one of *degenerate Courant algebroid with pairing in a vector bundle*. Later we will see that the fat bundle associated to a VB-Courant algebroid carries a natural Courant algebroid structure with pairing in the dual of the base.

An anchored vector bundle is a vector bundle $Q \to M$ endowed with a vector bundle morphism $\rho_Q : Q \to TM$ over the identity. Consider an anchored vector bundle $(\mathsf{E} \to M, \rho)$ and a vector bundle V over the same base M together with a morphism $\tilde{\rho} : \mathsf{E} \to \operatorname{Der}(V)$, such that the symbol of $\tilde{\rho}(e)$ is $\rho(e) \in \mathfrak{X}(M)$ for all $e \in \Gamma(\mathsf{E})$. Assume that E is paired with itself via a nondegenerate pairing $\langle \cdot, \cdot \rangle : \mathsf{E} \times_M \mathsf{E} \to V$ with values in V. Define $\mathcal{D} : \Gamma(V) \to \Gamma(\mathsf{E})$ by $\langle \mathcal{D}v, e \rangle = \tilde{\rho}(e)(v)$ for all $v \in \Gamma(V)$. Then $\mathsf{E} \to M$ is a *Courant algebroid with pairing in* V if E is in addition equipped with an $\mathbb{R}$-bilinear bracket $[\![\cdot, \cdot]\!]$ on the smooth sections $\Gamma(\mathsf{E})$ such that

(CA1) $[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!]$,

(CA2) $\tilde{\rho}(e_1)\langle e_2, e_3 \rangle = \langle [\![e_1, e_2]\!], e_3 \rangle + \langle e_2, [\![e_1, e_3]\!] \rangle$,

(CA3) $[\![e_1, e_2]\!] + [\![e_2, e_1]\!] = \mathcal{D}\langle e_1, e_2 \rangle$,

(CA4) $\tilde{\rho}[\![e_1, e_2]\!] = [\tilde{\rho}(e_1), \tilde{\rho}(e_2)]$

for all $e_1, e_2, e_3 \in \Gamma(\mathsf{E})$ and $f \in C^\infty(M)$. Equation (CA2) implies $[\![e_1, fe_2]\!] = f[\![e_1, e_2]\!] + (\rho(e_1)f)e_2$ for all $e_1, e_2 \in \Gamma(\mathsf{E})$ and $f \in C^\infty(M)$. If $V = \mathbb{R} \times M \to M$ is in addition the trivial bundle, then $\mathcal{D} = \rho^* \circ \boldsymbol{d} : C^\infty(M) \to \Gamma(\mathsf{E})$, where E is identified with E^* via the pairing. The quadruple $(\mathsf{E} \to M, \rho, \langle \cdot, \cdot \rangle, [\![\cdot, \cdot]\!])$ is then a *Courant algebroid* [Liu et al. 1997; Roytenberg 1999] and (CA4) follows then from (CA1), (CA2) and (CA3) (see [Uchino 2002] and also [Jotz Lean 2018a] for a quicker proof).

Note that Courant algebroids with a pairing in a vector bundle E were defined in [Chen et al. 2010] and called *E-Courant algebroids*. It is easy to check that Li-Bland's *AV-Courant algebroids* [2011] yield a special class of degenerate Courant algebroids with pairing in V. The examples of Courant algebroids with pairing in a vector bundle that we will get in Theorem 4.2 are *not* AV-Courant algebroids, so the two notions are distinct.

In our study of VB-Courant algebroids, we will need the following two lemmas.

Lemma 3.1 [Roytenberg 2002]. *Let* $(\mathsf{E} \to M, \rho, \langle \cdot, \cdot \rangle, [\![\cdot, \cdot]\!])$ *be a Courant algebroid. For all* $\theta \in \Omega^1(M)$ *and* $e \in \Gamma(\mathsf{E})$, *we have*

$$[\![e, \rho^*\theta]\!] = \rho^*(\pounds_{\rho(e)}\theta), \qquad [\![\rho^*\theta, e]\!] = -\rho^*(\boldsymbol{i}_{\rho(e)}\boldsymbol{d}\theta)$$

and so $\rho(\rho^*\theta) = 0$, *which implies* $\rho \circ \mathcal{D} = 0$.

Lemma 3.2 [Li-Bland 2012]. *Let* $\mathsf{E} \to M$ *be a vector bundle*, $\rho : \mathsf{E} \to TM$ *be a bundle map*, $\langle \cdot, \cdot \rangle$ *be a nondegenerate pairing on* E, *and let* $\mathcal{S} \subseteq \Gamma(\mathsf{E})$ *be a subspace of sections which generates* $\Gamma(\mathsf{E})$ *as a* $C^\infty(M)$*-module. Suppose that* $[\![\cdot, \cdot]\!] : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ *is a bracket satisfying*

(1) $[\![s_1, [\![s_2, s_3]\!]]\!] = [\![[\![s_1, s_2]\!], s_3]\!] + [\![s_2, [\![s_1, s_3]\!]]\!]$,

(2) $\rho(s_1)\langle s_2, s_3\rangle = \langle [\![s_1, s_2]\!], s_3\rangle + \langle s_2, [\![s_1, s_3]\!]\rangle$,

(3) $[\![s_1, s_2]\!] + [\![s_2, s_1]\!] = \rho^* \boldsymbol{d}\langle s_1, s_2\rangle$,

(4) $\rho[\![s_1, s_2]\!] = [\rho(s_1), \rho(s_2)]$

for any $s_i \in \mathcal{S}$, and that $\rho \circ \rho^* = 0$. Then there is a unique extension of $[\![\,\cdot\,,\cdot\,]\!]$ to a bracket on all of $\Gamma(\mathsf{E})$ such that $(\mathsf{E}, \rho, \langle\,\cdot\,,\cdot\,\rangle, [\![\,\cdot\,,\cdot\,]\!])$ is a Courant algebroid.

A *Dirac structure with support* [Alekseev and Xu 2001] in a Courant algebroid $\mathsf{E} \to M$ is a subbundle $D \to S$ over a submanifold S of M, such that $D(s)$ is maximal isotropic in $\mathsf{E}(s)$ for all $s \in S$ and

$$e_1|_S \in \Gamma_S(D),\ e_2|_S \in \Gamma_S(D) \quad \Rightarrow \quad [\![e_1, e_2]\!]|_S \in \Gamma_S(D)$$

for all $e_1, e_2 \in \Gamma(\mathsf{E})$. We leave to the reader the proof of the following lemma.

Lemma 3.3. *Let $\mathsf{E} \to M$ be a Courant algebroid and $D \to S$ a subbundle, with S a submanifold of M. Assume that $D \to S$ is spanned by the restrictions to S of a family $\mathcal{S} \subseteq \Gamma(\mathsf{E})$ of sections of E. Then D is a Dirac structure with support S if and only if*

(1) *$\rho_{\mathsf{E}}(e)(s) \in T_s S$ for all $e \in \mathcal{S}$ and $s \in S$,*

(2) *D_s is Lagrangian in $\mathbb{E}_s$ for all $s \in S$ and*

(3) *$[\![e_1, e_2]\!]|_S \in \Gamma_S(D)$ for all $e_1, e_2 \in \mathcal{S}$.*

Next we recall the notion of Dorfman connection [Jotz Lean 2018a]. Let $(Q \to M, \rho_Q)$ be an anchored vector bundle and let B be a vector bundle over M with a fibrewise pairing $\langle\cdot,\cdot\rangle : Q \times_M B \to \mathbb{R}$ and an $\mathbb{R}$-linear map $\delta : C^\infty(M) \to \Gamma(B)$ with $\delta(f \cdot g) = f \cdot \delta g + g \cdot \delta f$ for all $f, g \in C^\infty(M)$. A *Dorfman (Q-)connection on B* is an $\mathbb{R}$-linear map $\Delta : \Gamma(Q) \to \Gamma(\mathrm{Der}(B))$ such that

(1) Δ_q is a derivation over $\rho_Q(q) \in \mathfrak{X}(M)$,

(2) $\Delta_{fq} b = f \Delta_q b + \langle q, b\rangle \cdot \delta f$ and

(3) $\Delta_q \delta f = \delta(\rho_Q(q) f)$

for all $f \in C^\infty(M)$, $q, q' \in \Gamma(Q)$, $b \in \Gamma(B)$. The equality $\langle q, \delta f\rangle = \rho_Q(q)(f)$ follows from (2) and (3) for $q \in \Gamma(Q)$ and $f \in C^\infty(M)$.

For instance, if $B = Q^*$, the pairing is the canonical one and $\delta = \rho_Q^* \boldsymbol{d}$, we get a Q-Dorfman connection on Q^*. The map $[\![\,\cdot\,,\cdot\,]\!]_\Delta = \Delta^* : \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q)$ that is dual to Δ in the sense of dual derivations, i.e.,

$$\langle \Delta^*_{q_1} q_2, \tau\rangle = \rho_Q(q_1)\langle q_2, \tau\rangle - \langle q_2, \Delta_{q_1}\tau\rangle$$

for all $q_1, q_2 \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$, is then a *dull bracket* on $\Gamma(Q)$ in the following sense. A *dull algebroid* is an anchored vector bundle $(Q \to M, \rho_Q)$ with a bracket $[\![\cdot\,,\cdot]\!]$ on $\Gamma(Q)$ such that

$$\rho_Q[\![q_1, q_2]\!] = [\rho_Q(q_1), \rho_Q(q_2)] \tag{1}$$

and (the Leibniz identity)

$$[\![f_1 q_1, f_2 q_2]\!] = f_1 f_2 [\![q_1, q_2]\!] + f_1 \rho_Q(q_1)(f_2) q_2 - f_2 \rho_Q(q_2)(f_1) q_1$$

for all $f_1, f_2 \in C^\infty(M)$, $q_1, q_2 \in \Gamma(Q)$. In other words, a dull algebroid is a *Lie algebroid* if its bracket is in addition skew-symmetric and satisfies the Jacobi identity. Note that a dull bracket can easily be skew-symmetrised.

If $Q \to M$ is endowed with a dull algebroid structure, the *curvature* of a Dorfman connection $\Delta : \Gamma(Q) \times \Gamma(B) \to \Gamma(B)$ is the map $R_\Delta : \Gamma(Q) \times \Gamma(Q) \to \Gamma(\mathrm{End}(B))$ defined on $q, q' \in \Gamma(Q)$ by $R_\Delta(q, q') := \Delta_q \Delta_{q'} - \Delta_{q'} \Delta_q - \Delta_{[\![q, q']\!]}$. As always, Δ is said to be *flat* if R_Δ vanishes.

If the dull bracket on Q is skew-symmetric, $B = Q^*$ and Δ is the Dorfman connection that is dual to the bracket, then $R_\Delta \in \Omega^2(Q, \mathrm{End}(Q^*))$. The curvature satisfies then also

$$\langle \tau, \mathrm{Jac}_{[\![\cdot,\cdot]\!]}(q_1, q_2, q_3)\rangle = \langle R_\Delta(q_1, q_2)\tau, q_3\rangle \tag{2}$$

for $q_1, q_2, q_3 \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$, where

$$\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(q_1, q_2, q_3) = [\![[\![q_1, q_2]\!]_\Delta, q_3]\!] + [\![q_2, [\![q_1, q_3]\!]]\!] - [\![q_1, [\![q_2, q_3]\!]]\!]$$

is the Jacobiator of $[\![\cdot\,,\cdot]\!]$. Hence, the Dorfman connection is flat if and only if the corresponding dull bracket satisfies the Jacobi identity in Leibniz form.

3B. *Split Lie 2-algebroids.* A *homological* vector field χ on an $[n]$-manifold $\mathcal{M}$ is a derivation of degree 1 of $C^\infty(\mathcal{M})$ such that $\mathcal{Q}^2 = \frac{1}{2}[\mathcal{Q}, \mathcal{Q}]$ vanishes. A homological vector field on a [1]-manifold $\mathcal{M} = E[-1]$ is the de Rham differential $\boldsymbol{d}_E$ associated to a Lie algebroid structure on E [Vaintrob 1997]. A *Lie n-algebroid* is an $[n]$-manifold endowed with a homological vector field (an $\mathbb{N}\mathcal{Q}$*-manifold* of degree n).

A *split Lie n-algebroid* is a split $[n]$-manifold endowed with a homological vector field. Split Lie n-algebroids were studied by Sheng and Zhu [2017] and described as vector bundles endowed with a bracket that satisfies the Jacobi identity up to some correction terms; see also [Bonavolontà and Poncin 2013]. Our definition of a split Lie 2-algebroid turns out to be a Lie algebroid version of Baez and Crans' definition of a Lie 2-algebra [2004].

Definition 3.4. A split Lie 2-algebroid $B^* \to Q$ is the pair of an anchored vector bundle[3] $(Q \to M, \rho_Q)$ and a vector bundle $B \to M$, together with a vector bundle

[3] The names that we choose for the vector bundles will become natural in a moment.

map $l : B^* \to Q$, a skew-symmetric dull bracket[4] $[\![\cdot,\cdot]\!] : \Gamma(Q)\times\Gamma(Q)\to\Gamma(Q)$, a linear connection $\nabla : \Gamma(Q)\times\Gamma(B)\to\Gamma(B)$ and a vector bundle valued 3-form $\omega\in\Omega^3(Q, B^*)$, such that

(i) $\nabla^*_{l(\beta_1)}\beta_2+\nabla^*_{l(\beta_2)}\beta_1=0$ for all $\beta_1,\beta_2\in\Gamma(B^*)$,

(ii) $[\![q, l(\beta)]\!]=l(\nabla^*_q\beta)$ for $q\in\Gamma(Q)$ and $\beta\in\Gamma(B^*)$,

(iii) $\mathrm{Jac}_{[\![\cdot,\cdot]\!]}=l\circ\omega\in\Omega^3(Q,Q)$,

(iv) $R_\nabla(q_1,q_2)b=l^*\langle \boldsymbol{i}_{q_2}\boldsymbol{i}_{q_1}\omega, b\rangle$ for $q_1,q_2\in\Gamma(Q)$ and $b\in\Gamma(B)$, and

(v) $\boldsymbol{d}_{\nabla^*}\omega=0$.

From (iii) follows the identity $\rho_Q\circ l=0$. In the following, we will also work with $\partial_B := l^* : Q^*\to B$, with the Dorfman connection $\Delta : \Gamma(Q)\times\Gamma(Q^*)\to\Gamma(Q^*)$ that is dual to $[\![\cdot,\cdot]\!]$, and with $R_\omega\in\Omega^2(Q,\mathrm{Hom}(B,Q^*))$ which is defined by $R_\omega(q_1,q_2)b=\langle \boldsymbol{i}_{q_2}\boldsymbol{i}_{q_1}\omega, b\rangle$. Then (ii) is equivalent to $\partial_B\circ\Delta_q=\nabla_q\circ\partial_B$, (iii) is $R_\omega(q_1,q_2)\circ\partial_B=R_\Delta(q_1,q_2)$ for $q,q_1,q_2\in\Gamma(Q)$, and (iv) is $R_\nabla(q_1,q_2)=\partial_B\circ R_\omega(q_1,q_2)$ for all $q_1,q_2\in\Gamma(Q)$.

3C. ***Split Lie-2-algebroids as split* [2]*Q-manifolds.*** Before we go on with the study of examples, we briefly describe how to construct from the objects in Definition 3.4 the corresponding homological vector fields on split [2]-manifolds. Note that local descriptions of homological vector fields are also given in [Sheng and Zhu 2017] and [Bonavolontà and Poncin 2013].

Consider a split [2]-manifold $\mathcal{M}=Q[-1]\oplus B^*[-2]$. Assume that Q is endowed with an anchor ρ_Q and a skew-symmetric dull bracket $[\![\cdot,\cdot]\!]$, that it acts on B via a linear connection $\nabla : \Gamma(Q)\times\Gamma(B)\to\Gamma(B)$, that ω is an element of $\Omega^3(Q, B^*)$ and that $\partial_B : Q^*\to B$ is a vector bundle morphism. Define a vector field $\mathcal{Q}$ of degree 1 on $\mathcal{M}$ by the formulas

$$\mathcal{Q}(f)=\rho_Q^*\boldsymbol{d}f\in\Gamma(Q^*)$$

for $f\in C^\infty(M)$,

$$\mathcal{Q}(\tau)=\boldsymbol{d}_Q\tau+\partial_B\tau\in\Omega^2(Q)\oplus\Gamma(B)$$

for $\tau\in\Gamma(Q^*)$ and

$$\mathcal{Q}(b)=\boldsymbol{d}_\nabla b-\langle\omega, b\rangle\in\Omega^1(Q,B)\oplus\Omega^3(Q)$$

for $b\in\Gamma(B)$. Conversely, a relatively easy degree count and study of the graded Leibniz identity for an arbitrary vector field of degree 1 on $\mathcal{M}=Q[-1]\oplus B^*[-2]$

[4]To get the definition in [Sheng and Zhu 2017], set $l_1 := -l$, $l_3 := \omega$ and consider the skew symmetric bracket $l_2 : \Gamma(Q\oplus B^*)\times\Gamma(Q\oplus B^*)\to\Gamma(Q\oplus B^*)$, $l_2((q_1,\beta_1),(q_2,\beta_2)) = ([\![q_1,q_2]\!], \nabla^*_{q_1}\beta_2-\nabla^*_{q_2}\beta_1)$ for $q_1,q_2\in\Gamma(Q)$ and $\beta_1,\beta_2\in\Gamma(B^*)$. Note that this bracket satisfies a Leibniz identity with anchor $\rho_Q\circ\mathrm{pr}_Q : Q\oplus B^*\to TM$ and that the Jacobiator of this bracket is given by $\mathrm{Jac}_{l_2}((q_1,\beta_1),(q_2,\beta_2),(q_3,\beta_3))=(-l(\omega(q_1,q_2,q_3)),\omega(q_1,q_2,l(\beta_3))+\text{c.p.}$

shows that it must be given as above, defining therefore an anchor ρ_Q, and the structure objects $[\![\cdot\,,\cdot]\!]$, ∇, ω and ∂_B.

We show that $\mathcal{Q}^2 = 0$ if and only if $(\partial_B^* : B^* \to Q, [\![\cdot\,,\cdot]\!], \nabla, \omega)$ is a split Lie 2-algebroid anchored by ρ_Q. For $f \in C^\infty(M)$ we have

$$\mathcal{Q}^2(f) = \boldsymbol{d}_Q(\rho_Q^* \boldsymbol{d} f) + \partial_B(\rho_Q^* \boldsymbol{d} f) \in \Omega^2(Q) \oplus \Gamma(B).$$

Hence $\mathcal{Q}^2(f) = 0$ for all $f \in C^\infty(M)$ if and only if $\partial_B \circ \rho_Q^* = 0$ and $\rho_Q[\![q_1, q_2]\!]_\Delta = [\rho_Q(q_1), \rho_Q(q_2)]$ for all $q_1, q_2 \in \Gamma(Q)$. Now we assume that these two conditions are satisfied. For $\tau \in \Gamma(Q^*)$ we have

$$\mathcal{Q}^2(\tau) = (\boldsymbol{d}_Q^2 \tau - \langle \omega, \partial_B \tau\rangle) + (\partial_B \boldsymbol{d}_Q \tau + \boldsymbol{d}_\nabla(\partial_B \tau)) \in \Omega^3(Q) \oplus \Omega^1(Q, B),$$

where $\partial_B : \Omega^k(Q) \to \Omega^{k-1}(Q, B)$ is the vector bundle morphism defined by

$$\partial_B(\tau_1 \wedge \cdots \wedge \tau_k) = \sum_{i=1}^{k} (-1)^{i+1} \tau_1 \wedge \cdots \wedge \hat{i} \wedge \cdots \tau_k \wedge \partial_B \tau_i$$

for all $\tau_1, \tau_2 \in \Gamma(Q^*)$. We find $\boldsymbol{d}_Q^2 \tau(q_1, q_2, q_3) = \langle \mathrm{Jac}_{[\![\cdot,\cdot]\!]}(q_1, q_2, q_3), \tau\rangle$ and $(\partial_B \boldsymbol{d}_Q \tau)(q, \beta) = -\langle \partial_B \Delta_q \tau, \beta\rangle$, and so $\mathcal{Q}^2(\tau) = 0$ for all $\tau \in \Gamma(Q^*)$ if and only if $\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(q_1, q_2, q_3) = \partial_B^* \omega(q_1, q_2, q_3)$ for all $q_1, q_2, q_3 \in \Gamma(Q)$ and $\partial_B \Delta_q \tau = \nabla_q(\partial_B \tau)$ for all $q \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$.

Finally, we find for $b \in \Gamma(B)$:

$$\mathcal{Q}^2(b) = \mathcal{Q}(\boldsymbol{d}_\nabla b) - \boldsymbol{d}_Q\langle \omega, b\rangle - \partial_B\langle \omega, b\rangle.$$

The term $\partial_B\langle \omega, b\rangle$ is an element of $\Omega^2(Q, B)$ and the term $\boldsymbol{d}_Q\langle \omega, b\rangle$ is an element of $\Omega^4(Q)$. A computation yields that the $\Omega^4(Q)$-term of $\mathcal{Q}(\boldsymbol{d}_\nabla b)$ is $-\langle \omega, \boldsymbol{d}_\nabla b\rangle$, which is defined by

$$\langle \omega, \boldsymbol{d}_\nabla b\rangle(q_1, q_2, q_3, q_4) = \sum_{\sigma \in Z_4} (-1)^\sigma \langle \omega(q_{\sigma(1)}, q_{\sigma(2)}, q_{\sigma(3)}), \nabla_{q_{\sigma(4)}} b\rangle,$$

where Z_4 is the group of cyclic permutations of $\{1, 2, 3, 4\}$. The $\Omega^2(Q, B)$-term is $R_\nabla(\cdot\,,\cdot)b$ and the $\Gamma(S^2 B)$-term is $\nabla_{\partial_B^*} b$ defined by $(\nabla_{\partial_B^*} b)(\beta_1, \beta_2) = \langle \nabla_{\partial_B^* \beta_1} b, \beta_2\rangle + \langle \nabla_{\partial_B^* \beta_2} b, \beta_1\rangle$ for all $\beta_1, \beta_2 \in \Gamma(B^*)$. Hence $\mathcal{Q}^2(b) = 0$ if and only if $\boldsymbol{d}_Q\langle \omega, b\rangle + \langle \omega, \boldsymbol{d}_\nabla b\rangle = 0$, which is equivalent to $\boldsymbol{d}_{\nabla^*} \omega = 0$; $\nabla_{\partial_B^*} b = 0$, which is equivalent to

$$\nabla^*_{\partial_B^* \beta_1} \beta_2 + \nabla^*_{\partial_B^* \beta_2} \beta_1 = 0$$

for all $\beta_1, \beta_2 \in \Gamma(B^*)$; and

$$R_\nabla(\cdot\,,\cdot)b = \partial_B\langle \omega, b\rangle,$$

which is equivalent to $R_{\nabla^*}(q_1, q_2)\beta = \omega(q_1, q_2, \partial_B^* \beta)$ for all $q_1, q_2 \in \Gamma(Q)$ and $\beta \in \Gamma(B^*)$.

3D. ***Examples of split Lie 2-algebroids.*** We describe here four classes of examples of split Lie 2-algebroids. Later we will discuss their geometric meanings. We do not verify in detail the axioms of split Lie 2-algebroids. The computations in order to do this for Examples 3D2 and 3D3 are long, but straightforward. Note that, alternatively, the next section will provide a geometric proof of the fact that the following objects are split Lie 2-algebroids, since we will find them to be equivalent to special classes of VB-Courant algebroids. Note finally that a fifth important class of examples is discussed in Section 5.

3D1. *Lie algebroid representations.* Let $(Q \to M, \rho, [\cdot\,,\cdot])$ be a Lie algebroid and $\nabla : \Gamma(Q) \times \Gamma(B) \to \Gamma(B)$ a representation of Q on a vector bundle B. Then $(0 : B^* \to Q, [\cdot\,,\cdot], \nabla, 0)$ is a split Lie 2-algebroid. It is a semidirect extension of the Lie algebroid Q (and a special case of the bicrossproduct Lie 2-algebroids defined in Section 5A): the corresponding bracket l_2 is given by $l_2(q_1+\beta_1, q_2+\beta_2) = [q_1, q_2] + (\nabla^*_{q_1}\beta_2 - \nabla^*_{q_2}\beta_1)$ for $q_1, q_2 \in \Gamma(Q)$ and $\beta_1, \beta_2 \in \Gamma(B^*)$. Hence $(Q \oplus B^* \to M,\ \rho = \rho_Q \circ \mathrm{pr}_Q, l_2)$ is simply a Lie algebroid.

3D2. *Standard split Lie 2-algebroids.* Let $E \to M$ be a vector bundle, set

$$\partial_E = \mathrm{pr}_E : E \oplus T^*M \to E,$$

consider a skew-symmetric dull bracket $[\![\cdot\,,\cdot]\!]$ on $\Gamma(TM \oplus E^*)$, with $TM \oplus E^*$ anchored by pr_{TM}, and let

$$\Delta : \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \to \Gamma(E \oplus T^*M)$$

be the dual Dorfman connection. This defines as follows a split Lie 2-algebroid structure on the vector bundles $(TM \oplus E^*, \mathrm{pr}_{TM})$ and E^*.

Let $\nabla : \Gamma(TM \oplus E^*) \times \Gamma(E) \to \Gamma(E)$ be the ordinary linear connection[5] defined by $\nabla = \mathrm{pr}_E \circ \Delta \circ \iota_E$. The vector bundle map $l = \mathrm{pr}_E^* : E^* \to TM \oplus E^*$ is just the canonical inclusion. Define ω by $\omega(v_1, v_2, v_3) = \mathrm{Jac}_{[\![\cdot,\cdot]\!]}(v_1, v_2, v_3)$. Note that since $TM \oplus E^*$ is anchored by pr_{TM}, the tangent part of the dull bracket must just be the Lie bracket of vector fields. The Jacobiator $\mathrm{Jac}_{[\![\cdot,\cdot]\!]}$ can hence be seen as an element of $\Omega^3(TM \oplus E^*, E^*)$.

A straightforward verification of the axioms shows that l, $[\![\cdot\,,\cdot]\!]$, ∇^*, ω define a split Lie 2-algebroid. For reasons that will become clearer in Section 4D1, we call *standard* this type of split Lie 2-algebroid.

3D3. *Adjoint split Lie 2-algebroids.* The *adjoint* split Lie 2-algebroids can be described as follows. Let $\mathsf{E} \to M$ be a Courant algebroid with anchor ρ_{E} and bracket $[\![\cdot\,,\cdot]\!]$ and choose a metric linear connection $\nabla : \mathfrak{X}(M) \times \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$,

[5] To see that $\nabla = \mathrm{pr}_E \circ \Delta \circ \iota_E$ is an ordinary connection, recall that since $TM \oplus E^*$ is anchored by pr_{TM}, the map $\boldsymbol{d}_{E\oplus T^*M} = \mathrm{pr}_{TM}^* \boldsymbol{d} : C^\infty(M) \to \Gamma(E \oplus T^*M)$ sends $f \to (0, \boldsymbol{d} f)$.

i.e., a linear connection that preserves the pairing. Set $\partial_{TM} = \rho_{\mathsf{E}} : \mathsf{E} \to TM$ and identify E with its dual via the pairing. The map $\Delta : \Gamma(\mathsf{E}) \times \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$,

$$\Delta_e e' = [\![e, e']\!] + \nabla_{\rho(e')} e$$

is a Dorfman connection, which we call the *basic Dorfman connection associated to* ∇. The dual skew-symmetric(!) dull bracket is given by

$$[\![e, e']\!]_\Delta = [\![e, e']\!] - \rho^* \langle \nabla_\cdot e, e' \rangle$$

for all $e, e' \in \Gamma(\mathsf{E})$. The map

$$\nabla^{\mathrm{bas}} : \Gamma(\mathsf{E}) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \quad \nabla^{\mathrm{bas}}_e X = [\rho(e), X] + \rho(\nabla_X e)$$

is a linear connection, the *basic connection associated to* ∇.

We now define the *basic curvature* $R^{\mathrm{bas}}_\Delta \in \Omega^2(\mathsf{E}, \mathrm{Hom}(TM, \mathsf{E}))$ by[6]

$$\begin{aligned} R^{\mathrm{bas}}_\Delta(e_1, e_2)X = -\nabla_X [\![e_1, e_2]\!] + [\![\nabla_X e_1, e_2]\!] + [\![e_1, \nabla_X e_2]\!] \\ + \nabla_{\nabla^{\mathrm{bas}}_{e_2} X} e_1 - \nabla_{\nabla^{\mathrm{bas}}_{e_1} X} e_2 - \beta^{-1} \langle \nabla_{\nabla^{\mathrm{bas}}_\cdot X} e_1, e_2 \rangle \end{aligned} \tag{3}$$

for all $e_1, e_2 \in \Gamma(\mathsf{E})$ and $X \in \mathfrak{X}(M)$. Note the similarity of these constructions with the one of the adjoint representation up to homotopy (see [Gracia-Saz and Mehta 2010]). The meaning of this similarity will become clear in Section 4D3. The map l is $\rho^*_{\mathsf{E}} : T^*M \to \mathsf{E}$ and the form $\omega \in \Omega^3(\mathsf{E}, T^*M)$ is given by $\omega(e_1, e_2, e_3) = \langle R^{\mathrm{bas}}_\Delta(e_1, e_2), e_3 \rangle$. Note that it corresponds to the tensor Ψ defined in [Li-Bland 2012, Definition 4.1.2] (the right-hand side of (3)). The adjoint split Lie 2-algebroids are exactly the *split symplectic Lie 2-algebroids*, and correspond hence to splittings of the tangent doubles of Courant algebroids [Jotz Lean 2018b].

3D4. *Split Lie 2-algebroid defined by a 2-representation.* Let $(\partial_B : C \to B, \nabla, \nabla, R)$ be a representation up to homotopy of a Lie algebroid A on $B \oplus C$. We anchor $A \oplus C^*$ by $\rho_A \circ \mathrm{pr}_A$ and define $\Delta : \Gamma(A \oplus C^*) \times \Gamma(C \oplus A^*) \to \Gamma(C \oplus A^*)$ by

$$\Delta_{(a,\gamma)}(c, \alpha) = (\nabla_a c, \pounds_a \alpha + \langle \nabla^*_\cdot \gamma, c \rangle),$$

and $\nabla : \Gamma(A \oplus C^*) \times \Gamma(B) \to \Gamma(B)$ by $\nabla_{(a,\gamma)} b = \nabla_a b$. The vector bundle map l is here $l = \iota_{C^*} \circ \partial_B^*$, where $\iota_{C^*} : C^* \to A \oplus C^*$ is the canonical inclusion, and the dull bracket that is dual to Δ is given by

$$[\![(a_1, \gamma_1), (a_2, \gamma_2)]\!] = ([a_1, a_2], \nabla^*_{a_1} \gamma_2 - \nabla^*_{a_2} \gamma_1)$$

[6] We have then $R^{\mathrm{bas}}_\Delta(e_1, e_2)X = -\nabla_X [\![e_1, e_2]\!]_\Delta + [\![\nabla_X e_1, e_2]\!]_\Delta + [\![e_1, \nabla_X e_2]\!]_\Delta + \nabla_{\nabla^{\mathrm{bas}}_{e_2} X} e_1 - \nabla_{\nabla^{\mathrm{bas}}_{e_1} X} e_2 - \beta^{-1} \rho^* \langle R_\nabla(X, \cdot) e_1, e_2 \rangle$. Using $-R^*_\nabla = R_{\nabla^*} = R_\nabla$ (where we identify E with its dual using $\langle \cdot, \cdot \rangle$), the identity $R^{\mathrm{bas}}_\Delta(e_1, e_2) = -R^{\mathrm{bas}}_\Delta(e_2, e_1)$ is then immediate.

for $a_1, a_2 \in \Gamma(A)$, $\gamma_1, \gamma_2 \in \Gamma(C^*)$. The tensor ω is given by

$$\omega((a_1,\gamma_1),(a_2,\gamma_2),(a_3,\gamma_3)) = \langle R(a_1,a_2),\gamma_3\rangle + \text{c.p.}$$

Note that if we work with the dual A-representation up to homotopy $(\partial_B^* : B^* \to C^*, \nabla^*, \nabla^*, -R^*)$, then we get the Lie 2-algebroid defined in [Sheng and Zhu 2017, Proposition 3.5] as the semidirect product of a 2-representation and a Lie algebroid. This is then also a special case of the bicrossproduct of a matched pair of 2-representations (see Section 5A). Later we will explain why the choice that we make here is more natural.

3E. ***Morphisms of (split) Lie 2-algebroids.*** In this section we quickly discuss morphisms of split Lie 2-algebroids; see also [Bonavolontà and Poncin 2013].

A morphism $\mu : (\mathcal{M}_1, \mathcal{Q}_1) \to (\mathcal{M}_2, \mathcal{Q}_2)$ of Lie 2-algebroids is a morphism $\mu : \mathcal{M}_1 \to \mathcal{M}_2$ of the underlying [2]-manifolds, such that

$$\mu^\star \circ \mathcal{Q}_2 = \mathcal{Q}_1 \circ \mu^\star : C^\infty(\mathcal{M}_2) \to C^\infty(\mathcal{M}_1). \tag{4}$$

Assume that the two [2]-manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ are split [2]-manifolds $\mathcal{M}_1 = Q_1[-1] \oplus B_1^*[-2]$ and $\mathcal{M}_2 = Q_2[-1] \oplus B_2^*[-2]$. Then the homological vector fields $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are defined as in Section 3C with two split Lie 2-algebroids; $(\rho_1 : Q_1 \to TM_1, \partial_1 : Q_1^* \to B_1, [\![\cdot,\cdot]\!]_1, \nabla^1, \omega_1)$ and $(\rho_2 : Q_2 \to TM_2, \partial_2 : Q_2^* \to B_2, [\![\cdot,\cdot]\!]_2, \nabla^2, \omega_2)$. Further, the morphism $\mu^\star : C^\infty(\mathcal{M}_2) \to C^\infty(\mathcal{M}_1)$ over $\mu_0^* : C^\infty(M_2) \to C^\infty(M_1)$ decomposes as $\mu_Q : Q_1 \to Q_2$, $\mu_B : B_1^* \to B_2^*$ and $\mu_{12} : \wedge^2 Q_1 \to B_2^*$, all morphisms over $\mu_0 : M_1 \to M_2$. We study (4) in these decompositions.

(1) The condition $\mu^\star(\mathcal{Q}_2(f)) = \mathcal{Q}_1(\mu^\star(f))$ for all $f \in C^\infty(M_2)$ is $\mu_Q^\star(\rho_2^* \boldsymbol{d} f) = \rho_1^* \boldsymbol{d}(\mu_0^* f)$ for all $f \in C^\infty(M_2)$, which is equivalent to

$$T_m\mu_0(\rho_1(q_m)) = \rho_2(\mu_Q(q_m))$$

for all $q_m \in Q_1$. In other words $\mu_Q : Q_1 \to Q_2$ over $\mu_0 : M_1 \to M_2$ is compatible with the anchors $\rho_1 : Q_1 \to TM_1$ and $\rho_2 : Q_2 \to TM_2$.

(2) The condition $\mu^\star(\mathcal{Q}_2(\tau)) = \mathcal{Q}_1(\mu^\star(\tau))$ for all $\tau \in \Gamma(Q_2^*)$ reads

$$\mu^\star(\boldsymbol{d}_2\tau + \partial_2\tau) = \partial_1(\mu_Q^\star\tau) + \boldsymbol{d}_1(\mu_Q^\star\tau)$$

for all $\tau \in \Gamma(Q_2^*)$. The left-hand side is

$$\underbrace{\mu_Q^\star(\boldsymbol{d}_2\tau) + \mu_{12}^\star(\partial_2\tau)}_{\in\Omega^2(Q_1)} + \underbrace{\mu_B^\star(\partial_2\tau)}_{\in\Gamma(B_1)}$$

and the right-hand side is

$$\partial_1(\mu_Q^\star\tau) + \boldsymbol{d}_1(\mu_Q^\star\tau) \in \Gamma(B_1) \oplus \Omega^2(Q_1).$$

Hence, $\mu^\star \circ \mathcal{Q}_2 = \mathcal{Q}_1 \circ \mu^\star$ on degree 1 functions if and only if $\mu_Q \circ \partial_1^* = \partial_2^* \circ \mu_B$ and $\mu_Q^\star(\boldsymbol{d}_2\tau) + \mu_{12}^\star(\partial_2\tau) = \boldsymbol{d}_1(\mu_Q^\star\tau)$ for all $\tau \in \Gamma(Q_2^*)$.

(3) Finally we find that $\mu^\star(\mathcal{Q}_2(b)) = \mathcal{Q}_1(\mu^\star(b))$ for all $b \in \Gamma(B_2)$ if and only if

$$\mu^\star(\boldsymbol{d}_{\nabla^2} b) = \boldsymbol{d}_{\nabla^1}(\mu_B^\star(b)) + \partial_1(\mu_{12}^\star(b)) \in \Omega^1(Q_1, B_1)$$

for all $b \in \Gamma(B_2)$ and

$$\mu_Q^\star\omega_2 = \mu_B \circ \omega_1 - \boldsymbol{d}_{\mu_0^\star\nabla^2}\mu_{12} \in \Omega^3(Q_1, \mu_0^* B_2^*).$$

In the equalities above we have used the following constructions. The form $\mu^\star(\boldsymbol{d}_{\nabla^2} b) \in \Omega^1(Q_1, B_1)$ is defined by

$$(\mu^\star(\boldsymbol{d}_{\nabla^2} b))(q_m) = \mu_{B_m}^*(\nabla^2_{\mu_Q(q_m)} b) \in B_1(m)$$

for all $q_m \in Q_1$. Recall that μ_{12} can be seen as an element of $\Omega^2(Q_1, \mu_0^* B_2^*)$. The tensors $\mu_Q^\star\omega_2 \in \Omega^2(Q_1, \mu_0^* B_2^*)$ and $\mu_B \circ \omega_1 \in \Omega^2(Q_1, \mu_0^* B_2^*)$ can be defined as follows:

$$(\mu_Q^\star\omega_2)(q_1(m), q_2(m), q_3(m)) = \omega_2(\mu_Q(q_1(m)), \mu_Q(q_2(m)), \mu_Q(q_3(m)))$$

in $B_2^*(\mu_0(m))$, and

$$(\mu_B \circ \omega_1)(q_1(m), q_2(m), q_3(m)) = \mu_B(\omega_1)(q_1(m), q_2(m), q_3(m))) \in B_2^*(\mu_0(m))$$

for all $q_1, q_2, q_3 \in \Gamma(Q_1)$. The linear connection

$$\mu_Q^\star\nabla^2 : \Gamma(Q_1) \times \Gamma(\mu_0^* B_2^*) \to \Gamma(\mu_0^* B_2^*)$$

is defined by

$$(\mu_Q^\star\nabla^2)_q(\mu_0^!\beta)(m) = {\nabla^2}^*_{\mu_Q(q(m))}\beta \in B_2^*(\mu_0(m))$$

for all $q \in \Gamma(Q_1)$ and $\beta \in \Gamma(B_2^*)$.

We call a triple (μ_Q, μ_B, μ_{12}) over μ_0 as above a *morphism of split Lie 2-algebroids*. In particular, if $\mathcal{M}_1 = \mathcal{M}_2$, $\mu_0 = \mathrm{Id}_M : M \to M$, $\mu_Q = \mathrm{Id}_Q : Q \to Q$ and $\mu_B = \mathrm{Id}_{B^*} : B^* \to B^*$, then $\mu_{12} \in \Omega^2(Q, B^*)$ is just a change of splitting. The five conditions above simplify to the following:

(1) The dull brackets are related by $[\![q, q']\!]_2 = [\![q, q']\!]_1 + \partial_B^*\mu_{12}(q, q')$.

(2) The connections are related by $\nabla_q^2 b = \nabla_q^1 b - \partial_B\langle\mu_{12}(q, \cdot), b\rangle$.

(3) The curvature terms are related by $\omega_1 - \omega_2 = \boldsymbol{d}_{1,\nabla^2}\mu_{12}$.

The operator $\boldsymbol{d}_{1,\nabla^2} : \Omega^\bullet(Q, B^*) \to \Omega^{\bullet+1}(Q, B^*)$ is defined by the dull bracket $[\![\cdot\,,\cdot]\!]_1$ and the connection ${\nabla^2}^*$.

4. VB-Courant algebroids and Lie 2-algebroids

In this section we describe and prove in detail the equivalence between VB-Courant algebroids and Lie 2-algebroids. In short, a homological vector field on a [2]-manifold defines an anchor and a Courant bracket on the corresponding metric double vector bundle. This Courant bracket and this anchor are automatically compatible with the metric and define so a linear Courant algebroid structure on the double vector bundle. Note that a correspondence of Lie 2-algebroids and VB-Courant algebroids has already been discussed by Li-Bland [2012]. Our goal is to make this result constructive by deducing it from the results in [Jotz Lean 2018b] and presenting it as the counterpart of the main result in [Gracia-Saz and Mehta 2010], and to illustrate it with several (partly new) examples.

4A. ***Definition and observations.*** We will work with the following definition of a VB-Courant algebroid, which is due to Li-Bland [2012].

Definition 4.1. A VB-Courant algebroid is a metric double vector bundle

$$\begin{tikzcd} \mathbb{E}_{\pi_Q} \arrow[r, "\pi_B"] \arrow[d] & B \arrow[d, "q_B"] \\ Q_{q_Q} \arrow[r] & M \end{tikzcd}$$

with core Q^* such that $\mathbb{E} \to B$ is a Courant algebroid and the following conditions are satisfied.

(1) The anchor map $\Theta : \mathbb{E} \to TB$ is linear. That is,

$$\begin{tikzcd} \mathbb{E}_{\pi_Q} \arrow[rr, "\pi_B"] \arrow[dd] & & B \arrow[dd, "q_B"] \\ & C \arrow[dr] & \\ Q_{q_Q} \arrow[rr] & & M \end{tikzcd} \xrightarrow{\Theta} \begin{tikzcd} TB_{Tq_B} \arrow[rr, "p_B"] \arrow[dd] & & B \arrow[dd, "q_B"] \\ & B \arrow[dr] & \\ TM_{p_M} \arrow[rr] & & M \end{tikzcd} \tag{5}$$

is a morphism of double vector bundles.

(2) The Courant bracket is linear. That is,

$$[\![\Gamma^l_B(\mathbb{E}), \Gamma^l_B(\mathbb{E})]\!] \subseteq \Gamma^l_B(\mathbb{E}), \quad [\![\Gamma^l_B(\mathbb{E}), \Gamma^c_B(\mathbb{E})]\!] \subseteq \Gamma^c_B(\mathbb{E}), \quad [\![\Gamma^c_B(\mathbb{E}), \Gamma^c_B(\mathbb{E})]\!] = 0.$$

We make the following observations. Let $\rho_Q : Q \to TM$ be the side map of the anchor, i.e., if $\pi_Q(\chi) = q$ for $\chi \in \mathbb{E}$, then $Tq_B(\Theta(\chi)) = \rho_Q(q)$. In other words, if $\chi \in \Gamma^l_B(\mathbb{E})$ is linear over $q \in \Gamma(Q)$ then $\Theta(\chi)$ is linear over $\rho_Q(q)$. Let $\partial_B : Q^* \to B$ be the core map defined by the anchor Θ as

$$\Theta(\sigma^\dagger) = (\partial_B \sigma)^\uparrow \tag{6}$$

for all $\sigma \in \Gamma(Q^*)$. (∂_B is a morphism of vector bundles.) In the following, we call ρ_Q the *side-anchor* and ∂_B the *core-anchor*. The operator $\mathcal{D} = \Theta^* \boldsymbol{d} : C^\infty(B) \to \Gamma_B(\mathbb{E})$ satisfies $\mathcal{D}(q_B^* f) = (\rho_Q^* \boldsymbol{d} f)^\dagger$ for all $f \in C^\infty(M)$ and Lemma 3.1 yields immediately

$$\partial_B \circ \rho_Q^* = 0, \qquad \text{which is equivalent to } \ \rho_Q \circ \partial_B^* = 0. \tag{7}$$

Recall that if $\chi \in \Gamma^l_B(\mathbb{E})$ is linear over $q \in \Gamma(Q)$, then $\langle \chi, \tau^\dagger \rangle = q_B^* \langle q, \tau \rangle$ for all $\tau \in \Gamma(Q^*)$.

4B. ***The fat Courant algebroid.*** Here we denote by $\widehat{\mathbb{E}} \to M$ the fat bundle, that is the vector bundle whose sheaf of sections is the sheaf of $C^\infty(M)$-modules $\Gamma^l_B(\mathbb{E})$, the linear sections of $\mathbb{E}$ over B. Gracia-Saz and Mehta [2010] showed that if $\mathbb{E}$ is endowed with a linear Lie algebroid structure over B, then $\widehat{\mathbb{E}} \to M$ inherits a Lie algebroid structure, which is called the “fat Lie algebroid”. For completeness, we describe here quickly the counterpart of this in the case of a linear Courant algebroid structure on $\mathbb{E} \to B$.

Note that the restriction of the pairing on $\mathbb{E}$ to linear sections of $\mathbb{E}$ defines a nondegenerate pairing on $\widehat{\mathbb{E}}$ with values in B^*. Since the Courant bracket of linear sections is again linear, we get the following theorem.

Theorem 4.2. *Let $(\mathbb{E}, B, Q, M)$ be a VB-Courant algebroid. Then $\widehat{\mathbb{E}}$ is a Courant algebroid with pairing in B^*.*

Note that in [Jotz Lean and Kirchhoff-Lukat 2018] we explain how the Courant algebroid with pairing in E^* that is obtained from the VB-Courant algebroid $TE \oplus T^*E$, for a vector bundle E, is equivalent to the omni-Lie algebroids described in [Chen and Liu 2010; Chen et al. 2011].

We will come back in Corollary 4.8 to the structure found in Theorem 4.2. Recall that for $\phi \in \Gamma(\operatorname{Hom}(B, Q^*))$, the core-linear section $\widetilde{\phi}$ of $\mathbb{E} \to B$ is defined by $\widetilde{\phi}(b_m) = 0_{b_m} +_B \overline{\phi(b_m)}$. Note that $\widehat{\mathbb{E}}$ is also naturally paired with Q^* via $\langle \chi(m), \sigma(m) \rangle = \langle \pi_Q(\chi(m)), \sigma(m) \rangle$ for all $\chi \in \Gamma^l_B(\mathbb{E}) = \Gamma(\widehat{\mathbb{E}})$ and $\sigma \in \Gamma(Q^*)$. This pairing is degenerate since it restricts to 0 on $\operatorname{Hom}(B, Q^*) \times_M Q^*$. The following proposition can easily be proved.

Proposition 4.3. (1) *The map $\Delta : \Gamma(\widehat{\mathbb{E}}) \times \Gamma(Q^*) \to \Gamma(Q^*)$ defined by $(\Delta_\chi \tau)^\dagger = [\![\chi, \tau^\dagger]\!]$ is a flat Dorfman connection, where $\widehat{\mathbb{E}}$ is endowed with the anchor $\rho_Q \circ \pi_Q$ and paired with Q^* as above. The map $\delta : C^\infty(M) \to \Gamma(Q^*)$ sends f to $\rho^* \boldsymbol{d} f$.*

(2) *The map* $\nabla : \Gamma(\widehat{\mathbb{E}}) \times \Gamma(B) \to \Gamma(B)$ *defined by* $\Theta(\chi) = \widehat{\nabla}_\chi \in \mathfrak{X}^l(B)$ *is a flat connection.*

The maps Δ *and* ∇ *satisfy*

$$\partial_B \circ \Delta = \nabla \circ \partial_B \quad \text{and} \quad [\![\chi, \widetilde{\phi}]\!]_{\widehat{\mathbb{E}}} = \widetilde{\Delta_\chi \circ \phi - \phi \circ \nabla_\chi}$$

for $\chi \in \Gamma(\widehat{\mathbb{E}})$ *and* $\phi \in \Gamma(\mathrm{Hom}(B, Q^*))$.

Proof. (1) and (2) are easy to prove. For the first equation, choose $\chi \in \Gamma^l_B(\mathbb{E})$ and $\tau \in \Gamma(Q^*)$. Then

$$(\partial_B \circ \Delta_\chi \tau)^\uparrow = \Theta(\Delta_\chi \tau^\dagger) = \Theta([\![\chi, \tau^\dagger]\!]) = [\Theta(\chi), (\partial_B \tau)^\uparrow] = (\nabla_\chi(\partial_B \tau))^\uparrow.$$

The second equation is easy to check by writing $\widetilde{\phi} = \sum_{i=1}^n \ell_{\beta_i} \cdot \tau_i^\dagger$ with $\beta_i \in \Gamma(B^*)$ and $\tau_i \in \Gamma(Q^*)$. □

Lemma 4.4. *For* $\phi, \psi \in \Gamma(\mathrm{Hom}(B, Q^*))$ *and* $\tau \in \Gamma(Q^*)$, *we have*

(1) $[\![\tau^\dagger, \widetilde{\phi}]\!] = (\phi(\partial_B \tau))^\dagger = -[\![\widetilde{\phi}, \tau^\dagger]\!]$ *and*

(2) $[\![\widetilde{\phi}, \widetilde{\psi}]\!] = \widetilde{\psi \circ \partial_B \circ \phi - \phi \circ \partial_B \circ \psi}$.

Remark 4.5. Note that (2) is the bracket of the induced Lie algebra bundle structure induced on $\mathrm{Hom}(B, Q^*)$ by ∂_B.

Proof of Lemma 4.4. We write $\phi = \sum_{i=1}^n \beta_i \otimes \tau_i$ and $\psi = \sum_{j=1}^n \beta'_j \otimes \tau_j$ with $\beta_1, \ldots, \beta_n, \beta'_1, \ldots, \beta'_n \in \Gamma(B^*)$ and $\tau_1, \ldots, \tau_n \in \Gamma(Q^*)$. Hence, we have $\widetilde{\phi} = \sum_{i=1}^n \ell_{\beta_i} \tau_i^\dagger$ and $\widetilde{\psi} = \sum_{j=1}^n \ell_{\beta'_j} \tau_j^\dagger$. First we compute

$$\left[\!\!\left[\tau^\dagger, \sum_{i=1}^n \ell_{\beta_i} \tau_i^\dagger\right]\!\!\right] = \sum_{i=1}^n (\partial_B \tau)^\uparrow (\ell_{\beta_i}) \tau_i^\dagger = \sum_{i=1}^n q_B^* \langle \partial_B \tau, \beta_i \rangle \tau_i^\dagger = \left(\sum_{i=1}^n \langle \partial_B \tau, \beta_i \rangle \tau_i\right)^\dagger$$

and we get (1). Since $\langle \tau^\dagger, \widetilde{\phi} \rangle = 0$, the second equality follows. Then we have

$$\begin{aligned}\left[\!\!\left[\sum_{i=1}^n \ell_{\beta_i} \tau_i^\dagger, \sum_{j=1}^n \ell_{\beta'_j} \tau_j^\dagger\right]\!\!\right] &= \sum_{i=1}^n \sum_{j=1}^n \ell_{\beta_i} (\partial_B \tau_i)^\uparrow (\ell_{\beta'_j}) \tau_j^\dagger - \ell_{\beta'_j} (\partial_B \tau_j)^\uparrow (\ell_{\beta_i}) \tau_i^\dagger \\ &= \left(\sum_{i=1}^n \sum_{j=1}^n \langle \partial_B \tau_i, \beta'_j \rangle \cdot \beta_i \cdot \tau_j - \langle \partial_B \tau_j, \beta_i \rangle \cdot \beta'_j \cdot \tau_i\right)^\dagger,\end{aligned}$$

which leads to (2). □

4C. ***Lagrangian decompositions of VB-Courant algebroids.*** In this section, we study in detail the structure of VB-Courant algebroids, using Lagrangian decompositions of the underlying metric double vector bundle. Our goal is the following theorem. Note the similarity of this result with Gracia-Saz and Mehta's theorem [2010] in the VB-algebroid case.

Theorem 4.6. *Let* $(\mathbb{E}; Q, B; M)$ *be a VB-Courant algebroid and choose a Lagrangian splitting* $\Sigma : Q \times_M B \to \mathbb{E}$. *Then there is a split Lie* 2*-algebroid structure* $(\rho_Q, l = \partial_B^*, [\![\cdot\,,\cdot]\!], \nabla, \omega)$ *on* $Q \oplus B^*$ *such that*

$$\Theta(\sigma_Q(q)) = \widehat{\nabla}_q \in \mathfrak{X}(B), \quad [\![\sigma_Q(q), \tau^\dagger]\!] = (\Delta_q \tau)^\dagger \tag{8}$$
$$[\![\sigma_Q(q_1), \sigma_Q(q_2)]\!] = \sigma_Q[\![q_1, q_2]\!] - \widetilde{R_\omega(q_1, q_2)},$$

for all $q, q_1, q_2 \in \Gamma(Q)$ *and* $\tau \in \Gamma(Q^*)$, *where* $\Delta : \Gamma(Q) \times \Gamma(Q^*) \to \Gamma(Q^*)$ *is the Dorfman connection that is dual to the dull bracket.*

Conversely, a Lagrangian splitting $\Sigma : Q \times B^* \to \mathbb{E}$ *of the metric double vector bundle* $\mathbb{E}$ *together with a split Lie* 2*-algebroid on* $Q \oplus B^*$ *define by* (8) *a linear Courant algebroid structure on* $\mathbb{E}$.

First we will construct the objects $[\![\cdot\,,\cdot]\!]_\Delta, \Delta, \nabla, R$ as in the theorem, and then we will prove in the Appendix that they satisfy the axioms of a split Lie 2-algebroid.

4C1. *Construction of the split Lie 2-algebroid.* First recall that, by definition, the Courant bracket of two linear sections of $\mathbb{E} \to B$ is again linear. Hence, we can denote by $[\![q_1, q_2]\!]$ the section of Q such that

$$\pi_Q \circ [\![\sigma_Q(q_1), \sigma_Q(q_2)]\!] = [\![q_1, q_2]\!] \circ q_B. \tag{9}$$

Since for each $q \in \Gamma(Q)$, the anchor $\Theta(\sigma_Q(q))$ is a linear vector field on B over $\rho_Q(q) \in \mathfrak{X}(M)$, there exists a derivation $\nabla_q : \Gamma(B) \to \Gamma(B)$ over $\rho_Q(q)$ such that $\Theta(\sigma_Q(q)) = \widehat{\nabla}_q \in \mathfrak{X}^l(B)$. This defines a linear Q-connection $\nabla : \Gamma(Q) \times \Gamma(B) \to \Gamma(B)$. For $q \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$, the bracket $[\![\sigma_Q(q), \tau^\dagger]\!]$ is a core section. It is easy to check that the map $\Delta : \Gamma(Q) \times \Gamma(Q^*) \to \Gamma(Q^*)$ defined by

$$[\![\sigma_Q(q), \tau^\dagger]\!] = (\Delta_q \tau)^\dagger$$

is a Dorfman connection.[7]

The difference of the two linear sections $[\![\sigma_Q(q_1), \sigma_Q(q_2)]\!] - \sigma_Q([\![q_1, q_2]\!]_\sigma)$ is again a linear section, which projects to 0 under π_Q. Hence, there exists a vector bundle morphism $R(q_1, q_2) : B \to Q^*$ such that $\sigma_Q([\![q_1, q_2]\!]_\sigma) - [\![\sigma_Q(q_1), \sigma_Q(q_2)]\!] = \widetilde{R(q_1, q_2)}$. This defines a map $R : \Gamma(Q) \times \Gamma(Q) \to \Gamma(\mathrm{Hom}(B, Q^*))$. We show in the Appendix that R defines a 3-form $\omega \in \Omega^3(Q, B^*)$ by $R = R_\omega$, that $(l = \partial_B^*, [\![\cdot\,,\cdot]\!], \nabla, \omega)$ is a split Lie 2-algebroid, and that $[\![\cdot\,,\cdot]\!]$ is dual to Δ.

Conversely, choose a Lagrangian splitting $\Sigma : Q \times_M B$ of a metric double vector bundle $(\mathbb{E}, Q; B, M)$ with core Q^* and let $\mathcal{S} \subseteq \Gamma_B(\mathbb{E})$ be the subset

$$\{\tau^\dagger \mid \tau \in \Gamma(Q^*)\} \cup \{\sigma_Q(q) \mid q \in \Gamma(Q)\} \subseteq \Gamma(\mathbb{E}).$$

[7]Note that condition $(C3)$ then implies that $[\![\tau^\dagger, \sigma_Q(q)]\!] = (-\Delta_q \tau + \rho_Q^* \boldsymbol{d} \langle \tau, q\rangle)^\dagger$.

Choose a split Lie 2-algebroid $(l, [\![\cdot\,,\cdot]\!], \nabla, \omega)$ on $Q \oplus B^*$ with an anchor ρ_Q on Q. Consider the Dorfman connection Δ that is dual to the dull bracket. Define then a vector bundle map $\Theta : \mathbb{E} \to TB$ over the identity on B by $\Theta(\sigma_Q(q)) = \widehat{\nabla}_q$ and $\Theta(\tau^\dagger) = (l^*\tau)^\dagger$ and a bracket $[\![\cdot\,,\cdot]\!]$ on $\mathcal{S}$ by

$$[\![\sigma_Q(q_1), \sigma_Q(q_2)]\!] = \sigma_Q[\![q_1, q_2]\!] - \widetilde{R_\omega(q_1, q_2)}, \qquad [\![\sigma_Q(q), \tau^\dagger]\!] = (\Delta_q \tau)^\dagger,$$
$$[\![\tau^\dagger, \sigma_Q(q)]\!] = (-\Delta_q \tau + \rho_Q^* \boldsymbol{d}\langle \tau, q\rangle)^\dagger, \qquad [\![\tau_1^\dagger, \tau_2^\dagger]\!] = 0.$$

We show in the Appendix that this bracket, the pairing and the anchor satisfy the conditions of Lemma 3.2, and so $(\mathbb{E}, B; Q, M)$ with this structure is a VB-Courant algebroid.

4C2. *Change of Lagrangian decomposition.* Next we study how the split Lie 2-algebroid $(\partial_B^* : B^* \to Q, \nabla, [\![\cdot\,,\cdot]\!], \omega)$ associated to a Lagrangian decomposition of a VB-Courant algebroid changes when the Lagrangian decomposition changes.

The proof of the following proposition is straightforward and left to the reader. Compare this result with the equations at the end of Section 3E, that describe a change of splittings of a Lie 2-algebroid.

Proposition 4.7. *Let $\Sigma^1, \Sigma^2 : B \times_M Q \to \mathbb{E}$ be two Lagrangian splittings and let $\phi \in \Gamma(Q^* \otimes Q^* \otimes B^*)$ be the change of lift.*

(1) *The Dorfman connections are related by $\Delta_q^2 \tau = \Delta_q^1 \tau - \phi(q)(\partial_B \tau)$.*
(2) *The dull brackets are consequently related by $[\![q, q']\!]_2 = [\![q, q']\!]_1 + \partial_B^* \phi(q)^*(q')$.*
(3) *The connections are related by $\nabla_q^2 = \nabla_q^1 - \partial_B \circ \phi(q)$.*
(4) *The curvature terms are related by $\omega_1 - \omega_2 = \boldsymbol{d}_{\nabla^{2*}} \phi$, where the operator $\boldsymbol{d}_{\nabla^{2*}}$ is defined with the dull bracket $[\![\cdot\,,\cdot]\!]_1$ on $\Gamma(Q)$.*

As an application, we get the following corollary of Theorems 4.2 and 4.6. Given $\Delta : \Gamma(Q) \times \Gamma(Q^*) \to \Gamma(Q^*)$ and $\nabla : \Gamma(Q) \times \Gamma(B) \to \Gamma(B)$, we define the derivations $\Diamond : \Gamma(Q) \times \Gamma(\mathrm{Hom}(B, Q^*)) \to \Gamma(\mathrm{Hom}(B, Q^*))$ by $(\Diamond_q \phi)(b) = \Delta_q(\phi(b)) - \phi(\nabla_q b)$.

Corollary 4.8. *Let $(Q \oplus B^* \to M, \rho_Q, \partial_B^*, [\![\cdot\,,\cdot]\!], \nabla, \omega)$ be a split Lie 2-algebroid. Then the vector bundle $\mathsf{E} := Q \oplus \mathrm{Hom}(B, Q^*)$ is a Courant algebroid with pairing in B^* given by $\langle (q_1, \phi_1), (q_2, \phi_2)\rangle = \phi_1^*(q_2) + \phi_2^*(q_1)$, with the anchor $\tilde\rho : \mathsf{E} \to \widehat{\mathrm{Der}(B)}$, $\tilde\rho(q, \phi)^* = \nabla_q^* + \phi^* \circ \partial_B^*$ over $\rho(q)$ and the bracket given by*

$$[\![(q_1, \phi_1), (q_2, \phi_2)]\!] = \big([\![q_1, q_2]\!]_\Delta + \partial_B(\phi_1^*(q_2)), \Diamond_{q_1}\phi_2 - \Diamond_{q_2}\phi_1 + \nabla_\cdot^*(\phi_1^*(q_2)) \\ + \phi_2 \circ \partial_B \circ \phi_1 - \phi_1 \circ \partial_B \circ \phi_2 + R_\omega(q_1, q_2)\big).$$

The map $\mathcal{D} : \Gamma(B^*) \to \Gamma(\mathsf{E})$ sends q to $(\partial_B^* q, \nabla_\cdot^* q)$. The bracket does not depend on the choice of splitting.

4D. ***Examples of VB-Courant algebroids and of the corresponding split Lie 2-algebroids.*** We give here some examples of VB-Courant algebroids, and we compute the corresponding classes of split Lie 2-algebroids. We find the split Lie 2-algebroids described in Section 3D. In each of the examples below, it is easy to check that the Courant algebroid structure is linear. Hence, it is easy to check geometrically that the objects described in 3D are indeed split Lie 2-algebroids.

4D1. *The standard Courant algebroid over a vector bundle.* We have discussed this example in great detail in [Jotz Lean 2018a], but not in the language of split Lie 2-algebroids. Note further that, in [Jotz Lean 2018a], we worked with general, not necessarily Lagrangian, linear splittings.

Let $q_E : E \to M$ be a vector bundle and consider the VB-Courant algebroid

$$\begin{array}{ccc} TE \oplus T^*E_{\pi_E} & \xrightarrow{\Phi_E := (q_{E*}, r_E)} & TM \oplus E^* \\ \downarrow & & \downarrow \\ E_{q_E} & \longrightarrow & M \end{array}$$

with base E and side $TM \oplus E^* \to M$, and with core $E \oplus T^*M \to M$, or in other words the standard (VB-)Courant algebroid over a vector bundle $q_E : E \to M$. Recall that $TE \oplus T^*E$ has a natural linear metric (see [Jotz Lean 2018a]). Linear splittings of $TE \oplus T^*E$ are in bijection with dull brackets on sections of $TM \oplus E^*$ [Jotz Lean 2018a], and so also with Dorfman connections $\Delta : \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \to \Gamma(E \oplus T^*M)$, and Lagrangian splittings of $TE \oplus T^*E$ are in bijection with skew-symmetric dull brackets on sections of $TM \oplus E^*$ [Jotz Lean 2018b].

The anchor $\Theta = \mathrm{pr}_{TE} : TE \oplus T^*E \to TE$ restricts to the map $\partial_E = \mathrm{pr}_E : E \oplus T^*M \to E$ on the cores, and defines an anchor

$$\rho_{TM \oplus E^*} = \mathrm{pr}_{TM} : TM \oplus E^* \to TM$$

on the side. In other words, the anchor of $(e, \theta)^\dagger$ is $e^\uparrow \in \mathfrak{X}^c(E)$ and if $\widetilde{(X, \varepsilon)}$ is a linear section of $TE \oplus T^*E \to E$ over $(X, \varepsilon) \in \Gamma(TM \oplus E^*)$, the anchor $\Theta(\widetilde{(X, \varepsilon)}) \in \mathfrak{X}^l(E)$ is linear over X. Let $\iota_E : E \to E \oplus T^*M$ be the canonical inclusion. In [Jotz Lean 2018a] we proved that for $q, q_1, q_2 \in \Gamma(TM \oplus E^*)$ and $\tau, \tau_1, \tau_2 \in \Gamma(E \oplus T^*M)$, the Courant–Dorfman bracket on sections of $TE \oplus T^*E \to E$ is given by

(1) $[\![\sigma(q), \tau^\dagger]\!] = (\Delta_q \tau)^\dagger$,

(2) $[\![\sigma(q_1), \sigma(q_2)]\!] = \sigma([\![q_1, q_2]\!]_\Delta) - \widetilde{R_\Delta(q_1, q_2) \circ \iota_E}$,

and that the anchor ρ is described by $\Theta(\sigma(q)) = \widehat{\nabla}_q^* \in \mathfrak{X}(E)$, where

$$\nabla : \Gamma(TM \oplus E^*) \times \Gamma(E) \to \Gamma(E)$$

is defined by $\nabla_q = \mathrm{pr}_E \circ \Delta_q \circ \iota_E$ for all $q \in \Gamma(TM \oplus E^*)$.

Hence, if we choose a Lagrangian splitting of $TE \oplus T^*E$, we find the split Lie 2-algebroid of Section 3D2.

4D2. *The VB-Courant algebroid defined by a VB-Lie algebroid.* More generally, let

$$\begin{array}{ccc} D_{\pi_A} & \xrightarrow{\pi_B} & B \\ \downarrow & & \downarrow{\scriptstyle q_B} \\ A_{q_A} & \longrightarrow & M \end{array}$$

(with core C) be endowed with a VB-Lie algebroid structure ($D \to B, A \to M$). Then the pair (D, D^*_B) of vector bundles over B is a Lie bialgebroid, with D^*_B endowed with the trivial Lie algebroid structure. We get a linear Courant algebroid $D \oplus_B (D^*_B)$ over B with side $A \oplus C^*$,

$$\begin{array}{ccc} D \oplus_B (D^*_B) & \longrightarrow & B \\ \downarrow & & \downarrow \\ A \oplus C^* & \longrightarrow & M \end{array}$$

and core $C \oplus A^*$. We check that the Courant algebroid structure is linear. Let $\Sigma : A \times_M B \to D$ be a linear splitting of D. Recall that we can define a linear splitting of D^*_B by $\Sigma^\star : B \times_M C^* \to D^*_B$, $\langle \Sigma^\star(b_m, \gamma_m), \Sigma(a_m, b_m)\rangle = 0$ and $\langle \Sigma^\star(b_m, \gamma_m), c^\dagger(b_m)\rangle = \langle \gamma_m, c(m)\rangle$ for all $b_m \in B$, $a_m \in A$, $\gamma_m \in C^*$ and $c \in \Gamma(C)$. The linear splitting $\widetilde{\Sigma} : B \times_M (A \oplus C^*) \to D \oplus_B (D^*_B)$, $\widetilde{\Sigma}(b_m, (a_m, \gamma_m)) = (\Sigma(a_m, b_m), \Sigma^\star(b_m, \gamma_m))$ is then a Lagrangian splitting. A computation shows that the Courant bracket on $\Gamma_B(D \oplus_B (D^*_B))$ is given by

$$\begin{aligned}
&[\![\tilde{\sigma}_{A\oplus C^*}(a_1, \gamma_1), \tilde{\sigma}_{A\oplus C^*}(a_2, \gamma_2)]\!] \\
&\qquad = ([\sigma_A(a_1), \sigma_A(a_2)], \pounds_{\sigma_A(a_1)}\sigma^\star_{C^*}(\gamma_2) - \boldsymbol{i}_{\sigma_A(a_2)}\boldsymbol{d}\sigma^\star_{C^*}(\gamma_1)) \\
&\qquad = \big(\sigma_A[a_1, a_2] - \widetilde{R(a_1, a_2)}, \sigma^\star_{C^*}(\nabla^*_{a_1}\gamma_2 - \nabla^*_{a_2}\gamma_1 \\
&\qquad\qquad\qquad - \widetilde{\langle \gamma_2, R(a_1, \cdot)\rangle} + \widetilde{\langle \gamma_1, R(a_2, \cdot)\rangle})\big),
\end{aligned}$$

$$[\![\tilde{\sigma}_{A\oplus C^*}(a, \gamma), (c, \alpha)^\dagger]\!] = (\nabla_a c^\dagger, (\pounds_a \alpha + \langle \nabla^*_\cdot \gamma, c\rangle)^\dagger),$$

$$[\![(c_1, \alpha_1)^\dagger, (c_2, \alpha_2)^\dagger]\!] = 0,$$

and the anchor of $D \oplus_B (D^*_B)$ is defined by

$$\Theta(\tilde{\sigma}_{A\oplus C^*}(a, \gamma)) = \Theta(\sigma_A(a)) = \widehat{\nabla}_a \in \mathfrak{X}^l(B), \quad \Theta((c, \alpha)^\dagger) = (\partial_B c)^\uparrow \in \mathfrak{X}^c(B),$$

where $(\partial_B : C \to B, \nabla : \Gamma(A) \times \Gamma(B) \to \Gamma(B), \nabla : \Gamma(A) \times \Gamma(C) \to \Gamma(C), R)$ is the 2-representation of A associated to the splitting $\Sigma : A \times_M B \to D$ of the VB-algebroid $(D \to B, A \to M)$. Hence, we have found the split Lie 2-algebroid described in Section 3D4.

4D3. *The tangent Courant algebroid.* We consider here a Courant algebroid

$$(\mathsf{E}, \rho_{\mathsf{E}}, [\![\cdot\,,\cdot]\!], \langle\cdot\,,\cdot\rangle).$$

In this example, E will always be anchored by the Courant algebroid anchor map ρ_{E} and paired with itself by $\langle\cdot\,,\cdot\rangle$ and $\mathcal{D} = \boldsymbol{\beta}^{-1} \circ \rho_{\mathsf{E}}^* \circ \boldsymbol{d} : C^\infty(M) \to \Gamma(\mathsf{E})$. Note that $[\![\cdot\,,\cdot]\!]$ is not a dull bracket.

We show that, after the choice of a metric connection on E and so of a Lagrangian splitting $\Sigma^\nabla : TM \times_M \mathsf{E} \to T\mathsf{E}$ (see Example 2.2), the VB-Courant algebroid structure on $(T\mathsf{E} \to TM, \mathsf{E} \to M)$ is equivalent to the split Lie 2-algebroid defined by ∇ as in Section 3D3.

Theorem 4.9. *Choose a linear connection $\nabla : \mathfrak{X}(M) \times \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ that preserves the pairing on E. The Courant algebroid structure on $T\mathsf{E} \to TM$ can be described as follows, for all $e, e_1, e_2 \in \Gamma(\mathsf{E})$:*

(1) *The pairing is given by*

$$\langle e_1^\dagger, e_2^\dagger\rangle = 0, \quad \langle \sigma_{\mathsf{E}}^\nabla(e_1), e_2^\dagger\rangle = p_M^*\langle e_1, e_2\rangle, \quad \textit{and} \quad \langle \sigma_{\mathsf{E}}^\nabla(e_1), \sigma_{\mathsf{E}}^\nabla(e_2)\rangle = 0.$$

(2) *The anchor is given by $\Theta(\sigma_{\mathsf{E}}^\nabla(e)) = \widehat{\nabla}_e^{\mathrm{bas}}$ and $\Theta(e^\dagger) = (\rho_{\mathsf{E}}(e))^\uparrow$.*

(3) *The bracket is given by*

$$[\![e_1^\dagger, e_2^\dagger]\!] = 0, \qquad [\![\sigma_{\mathsf{E}}^\nabla(e_1), e_2^\dagger]\!] = (\Delta_{e_1} e_2)^\dagger$$

and

$$[\![\sigma_{\mathsf{E}}^\nabla(e_1), \sigma_{\mathsf{E}}^\nabla(e_2)]\!] = \sigma_{\mathsf{E}}^\nabla([\![e_1, e_2]\!]_\Delta) - \widetilde{R_\Delta^{\mathrm{bas}}(e_1, e_2)}$$

Proof. We use the characterisation of the tangent Courant algebroid in [Boumaiza and Zaalani 2009] (see also [Li-Bland 2012]): the pairing has already been discussed in Example 2.2. It is given by $\langle Te_1, Te_2\rangle = \ell_{\boldsymbol{d}\langle e_1, e_2\rangle}$ and $\langle Te_1, e_2^\dagger\rangle = p_M^*\langle e_1, e_2\rangle$. The anchor is given by $\Theta(Te) = \widehat{\pounds_{\rho_{\mathsf{E}}(e)}} \in \mathfrak{X}(TM)$ and $\Theta(e^\dagger) = (\rho_{\mathsf{E}}(e))^\uparrow \in \mathfrak{X}(TM)$. The bracket is given by $[\![Te_1, Te_2]\!] = T[\![e_1, e_2]\!]$ and $[\![Te_1, e_2^\dagger]\!] = [\![e_1, e_2]\!]^\dagger$ for all $e, e_1, e_2 \in \Gamma(\mathsf{E})$.

(1) is easy to check (see Example 2.2 and [Jotz Lean 2018b]). We here check (2), i.e., that the anchor satisfies $\Theta(\sigma_{\mathsf{E}}^\nabla(e)) = \widehat{\nabla}_e^{\mathrm{bas}}$: For $\theta \in \Omega^1(M)$ and $v_m \in TM$, we have $\Theta(\sigma_{\mathsf{E}}^\nabla(e)(v_m))(\ell_\theta) = \ell_{\pounds_{\rho_{\mathsf{E}}(e)}\theta}(v_m) - \langle \theta_m, \rho_{\mathsf{E}}(\nabla_{v_m} e)\rangle = \ell_{\nabla_e^{\mathrm{bas}*}\theta}(v_m)$ and for $f \in C^\infty(M)$, we have

$$\Theta(\sigma_{\mathsf{E}}^\nabla(e))(p_M^* f) = p_M^*(\rho_{\mathsf{E}}(e) f).$$

This proves the equality.

Then we compute the brackets of our linear and core sections. Choose sections ϕ, ϕ' of $\mathrm{Hom}(TM, \mathsf{E})$. Then $[\![Te, \widetilde{\phi}]\!] = \widetilde{\pounds_e \phi}$, with $\pounds_e \phi \in \Gamma(\mathrm{Hom}(TM, \mathsf{E}))$ defined by $(\pounds_e \phi)(X) = [\![e, \phi(X)]\!] - \phi([\rho_{\mathsf{E}}(e), X])$ for all $X \in \mathfrak{X}(M)$. The equality

$[\![\widetilde{\phi}, Te]\!] = -\widetilde{\pounds_e\phi} + \mathcal{D}\ell_{\langle\phi(\cdot),e\rangle}$ follows. For $\theta \in \Omega^1(M)$, we compute $\langle \mathcal{D}\ell_\theta, e^\dagger\rangle = \Theta(e^\dagger)(\ell_\theta) = p_M^*\langle\rho_{\mathsf{E}}(e),\theta\rangle$. Thus, $\mathcal{D}\ell_\theta = T(\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\theta) + \widetilde{\psi}$ for a section $\psi \in \Gamma(\mathrm{Hom}(TM,\mathsf{E}))$ to be determined. Since $\langle\mathcal{D}\ell_\theta, Te\rangle = \Theta(Te)(\ell_\theta) = \ell_{\pounds_{\rho_{\mathsf{E}}(e)}\theta}$, the bracket $\langle T(\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\theta) + \widetilde{\psi}, Te\rangle = \ell_{\boldsymbol{d}\langle\theta,\rho_{\mathsf{E}}(e)\rangle + \langle\psi(\cdot),e\rangle}$ must equal $\ell_{\pounds_{\rho_{\mathsf{E}}(e)}\theta}$, and we find $\langle\psi(\cdot), e\rangle = \boldsymbol{i}_{\rho_{\mathsf{E}}(e)}\boldsymbol{d}\theta$. Because $e \in \Gamma(\mathsf{E})$ was arbitrary we find $\psi(X) = -\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\boldsymbol{i}_X\boldsymbol{d}\theta$ for $X \in \mathfrak{X}(M)$. We get in particular

$$[\![\widetilde{\phi}, Te]\!] = -\widetilde{\pounds_e\phi} + T(\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\langle\phi(\cdot), e\rangle) - \widetilde{\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\boldsymbol{i}_X\boldsymbol{d}\langle\phi(\cdot), e\rangle}.$$

The formula $[\![\widetilde{\phi}, \widetilde{\phi'}]\!] = \widetilde{\phi'\circ\rho_{\mathsf{E}}\circ\phi - \phi\circ\rho_{\mathsf{E}}\circ\phi'}$ can easily be checked, as well as $[\![\widetilde{\phi}, e^\dagger]\!] = -[\![e^\dagger, \widetilde{\phi}]\!] = -(\phi(\rho_{\mathsf{E}}(e)))^\dagger$. Using this, we find now easily that

$$\begin{aligned}
[\![\sigma_{\mathsf{E}}^\nabla(e_1), \sigma_{\mathsf{E}}^\nabla(e_2)]\!] &= [\![Te_1 - \widetilde{\nabla_\cdot e_1}, Te_2 - \widetilde{\nabla_\cdot e_2}]\!] \\
&= T[\![e_1,e_2]\!] - \widetilde{\pounds_{e_1}\nabla_\cdot e_2} + \widetilde{\pounds_{e_2}\nabla_\cdot e_1} - T(\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\langle\nabla_\cdot e_1, e_2\rangle) \\
&\qquad + \widetilde{\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\boldsymbol{d}\langle\nabla_\cdot e_1, e_2\rangle} + \widetilde{\nabla_{\rho_{\mathsf{E}}(\nabla_\cdot e_1)}e_2} - \widetilde{\nabla_{\rho_{\mathsf{E}}(\nabla_\cdot e_2)}e_1} \\
&= T[\![e_1,e_2]\!]_\Delta - \widetilde{\pounds_{e_1}\nabla_\cdot e_2} + \widetilde{\pounds_{e_2}\nabla_\cdot e_1} + \widetilde{\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\boldsymbol{d}\langle\nabla_\cdot e_1, e_2\rangle} \\
&\qquad + \widetilde{\nabla_{\rho_{\mathsf{E}}(\nabla_\cdot e_1)}e_2} - \widetilde{\nabla_{\rho_{\mathsf{E}}(\nabla_\cdot e_2)}e_1}.
\end{aligned}$$

Since for all $X \in \mathfrak{X}(M)$, we have

$$\begin{aligned}
&-(\pounds_{e_1}\nabla_\cdot e_2)(X) + (\pounds_{e_2}\nabla_\cdot e_1)(X) + \beta^{-1}\rho_{\mathsf{E}}^*\boldsymbol{i}_X\boldsymbol{d}\langle\nabla_\cdot e_1, e_2\rangle \\
&= -[\![e_1, \nabla_X e_2]\!] + \nabla_{[\rho_{\mathsf{E}}(e_1),X]}e_2 + [\![e_2, \nabla_X e_1]\!] - \nabla_{[\rho_{\mathsf{E}}(e_2),X]}e_1 + \beta^{-1}\rho_{\mathsf{E}}^*\boldsymbol{i}_X\boldsymbol{d}\langle\nabla_\cdot e_1, e_2\rangle \\
&= -[\![e_1, \nabla_X e_2]\!] + \nabla_{[\rho_{\mathsf{E}}(e_1),X]}e_2 - [\![\nabla_X e_1, e_2]\!] - \nabla_{[\rho_{\mathsf{E}}(e_2),X]}e_1 + \beta^{-1}\rho_{\mathsf{E}}^*\pounds_X\langle\nabla_\cdot e_1, e_2\rangle,
\end{aligned}$$

we find that $[\![\sigma_{\mathsf{E}}^\nabla(e_1), \sigma_{\mathsf{E}}^\nabla(e_2)]\!] = T[\![e_1,e_2]\!]_\Delta - \widetilde{R_\Delta^{\mathrm{bas}}(e_1,e_2)}$. Finally we compute $[\![\sigma_{\mathsf{E}}^\nabla(e_1), e_2^\dagger]\!] = [\![Te_1 - \widetilde{\nabla_\cdot e_1}, e_2^\dagger]\!] = [\![e_1,e_2]\!]^\dagger + \nabla_{\rho_{\mathsf{E}}(e_2)}e_1^\dagger = \Delta_{e_1}e_2^\dagger$. □

4E. ***Categorical equivalence of Lie 2-algebroids and VB-Courant algebroids.*** In this section we quickly describe morphisms of VB-Courant algebroids. Then we find an equivalence between the category of VB-Courant algebroids and the category of Lie 2-algebroids. Note that a bijection between VB-Courant algebroids and Lie 2-algebroids was already described in [Li-Bland 2012].

4E1. *Morphisms of VB-Courant algebroids.* Recall from Section 2B that a morphism $\Omega : \mathbb{E}_1 \to \mathbb{E}_2$ of metric double vector bundles is an isotropic relation $\Omega \subseteq \overline{\mathbb{E}}_1 \times \mathbb{E}_2$ that is the dual of a morphism $(E_1)^*_{Q_1} \to (E_2)^*_{Q_2}$. Assume that $\mathbb{E}_1$ and $\mathbb{E}_2$ have linear Courant algebroid structures. Then Ω is a morphism of VB-Courant algebroids if it is a Dirac structure (with support) in $\overline{\mathbb{E}}_1 \times \mathbb{E}_2$.

Choose two Lagrangian splittings $\Sigma^1 : Q_1 \times B_1 \to \mathbb{E}_1$ and $\Sigma^2 : Q_2 \times B_2 \to \mathbb{E}_2$.

Then there exist four structure maps

$$\omega_0 : M_1 \to M_2, \quad \omega_Q : Q_1 \to Q_2, \quad \omega_B : B_1^* \to B_2^*, \quad \omega_{12} \in \Omega^2(Q_1, \omega_0^* B_2^*)$$

that define completely Ω. More precisely, Ω is spanned over $\operatorname{Graph}(\omega_Q : Q_1 \to Q_2)$ by sections $\tilde{b} : \operatorname{Graph}(\omega_Q) \to \Omega$,

$$\tilde{b}(q_m, \omega_Q(q_m)) = (\sigma_{B_1}(\omega_B^\star b)(q_m) + \widetilde{\omega_{12}^\star(b)}(q_m), \sigma_{B_2}(b)(\omega_Q(q_m)))$$

for all $b \in \Gamma_{M_2}(B_2)$, and $\tau^\times : \operatorname{Graph}(\omega_Q) \to \Omega$,

$$\tau^\times(q_m, \omega_Q(q_m)) = ((\omega_Q^\star \tau)^\dagger(q_m), \tau^\dagger(\omega_Q(q_m)))$$

for all $\tau \in \Gamma_{M_2}(Q_2^*)$. Note that Ω projects under $\pi_{B_1} \times \pi_{B_2}$ to $R_{\omega_B^*} \subseteq B_1 \times B_2$. If $q \in \Gamma(Q_1)$ then $\omega_Q^! q \in \Gamma_{M_1}(\omega_0^* Q_2)$ can be written as $\sum_i f_i \omega_0^! q_i$ with $f_i \in C^\infty(M_1)$ and $q_i \in \Gamma_{M_2}(Q_2)$. The pair

$$(\sigma_{B_1}(\omega_B^\star b)(q_m) + \widetilde{\omega_{12}^\star(b)}(q_m),\ \sigma_{B_2}(b)(\omega_Q(q_m)))$$

can be written as

$$\left((\sigma_{Q_1}(q) + \langle \omega_{12}(q, \cdot), b(\omega_0(m))\rangle^\dagger)(\omega_B^\star b(m)),\ \sum_i f_i(m) \sigma_{Q_2}(q_i)(b(\omega_0(m)))\right).$$

Hence, Ω is spanned by the restrictions to $R_{\omega_B^*}$ of sections

$$\left(\sigma_{Q_1}(q) \circ \mathrm{pr}_1 + \langle \omega_{12}(q, \cdot), \mathrm{pr}_2\rangle^\dagger \circ \mathrm{pr}_1, \sum_i (f_i \circ q_{B_1} \circ \mathrm{pr}_1) \cdot (\sigma_{Q_2}(q_i) \circ \mathrm{pr}_2)\right) \tag{10}$$

for all $q \in \Gamma_{M_1}(Q_1)$ and

$$((\omega_Q^\star \tau)^\dagger \circ \mathrm{pr}_1, \tau^\dagger \circ \mathrm{pr}_2) \tag{11}$$

for all $\tau \in \Gamma(Q_2^*)$.

Checking all the conditions in Lemma 3.3 on the two types of sections (10) and (11) yields that $\Omega \to R_{\omega_B^*}$ is a Dirac structure with support if and only if

(1) $\omega_Q : Q_1 \to Q_2$ over $\omega_0 : M_1 \to M_2$ is compatible with the anchors $\rho_1 : Q_1 \to TM_1$ and $\rho_2 : Q_2 \to TM_2$:

$$T_m \omega_0(\rho_1(q_m)) = \rho_2(\omega_Q(q_m))$$

for all $q_m \in Q_1$,

(2) $\partial_1 \circ \omega_Q^\star = \omega_B^\star \circ \partial_2$ as maps from $\Gamma(Q_2^*)$ to $\Gamma(B_1)$, or equivalently $\omega_Q \circ \partial_1^* = \partial_2^* \circ \omega_B$,

(3) ω_Q preserves the dull brackets up to $\partial_2^* \omega_{12}$: i.e., $\omega_Q^\star(\boldsymbol{d}_2 \tau) + \omega_{12}^\star(\partial_2 \tau) = \boldsymbol{d}_1(\omega_Q^\star \tau)$ for all $\tau \in \Gamma(Q_2^*)$.

(4) ω_B and ω_Q intertwine the connections ∇^1 and ∇^2 up to $\partial_1\circ\omega_{12}$:

$$\omega_B^\star((\omega_Q^\star\nabla^2)_q b)=\nabla_q^1(\omega_B^\star(b))-\partial_1\circ\langle\omega_{12}(q,\cdot),b\rangle\in\Gamma(B_1)$$

for all $q_m\in Q_1$ and $b\in\Gamma(B^2)$, and

(5) $\omega_Q^\star\omega_{R_2}-\omega_B\circ\omega_{R_1}=-\boldsymbol{d}_{(\omega_Q^\star\nabla^2)}\omega_{12}\in\Omega^3(Q_1,\omega_0^*B_2^*)$.

We thus find that Ω is a morphism of VB-Courant algebroids if and only if it induces a morphism of split Lie 2-algebroids after any choice of Lagrangian decompositions of $\mathbb{E}_1$ and $\mathbb{E}_2$.

4E2. *Equivalence of categories.* The functors Section 2B between the category of metric double vector bundles and the category of [2]-manifolds refine to functors between the category of VB-Courant algebroids and the category of Lie [2]-algebroids.

Theorem 4.10. *The category of Lie 2-algebroids is equivalent to the category of VB-Courant algebroids.*

Proof. Let $(\mathcal{M},\mathcal{Q})$ be a Lie 2-algebroid and consider the double vector bundle $\mathbb{E}_{\mathcal{M}}$ corresponding to $\mathcal{M}$. Choose a splitting $\mathcal{M}\simeq Q[-1]\oplus B^*[-2]$ of $\mathcal{M}$ and consider the corresponding Lagrangian splitting Σ of $\mathbb{E}_{\mathcal{M}}$.

By Theorem 4.6, the split Lie 2-algebroid $(Q[-1]\oplus B^*[-2],\mathcal{Q})$ defines a VB-Courant algebroid structure on the decomposition of $\mathbb{E}_{\mathcal{M}}$ and so by isomorphism on $\mathbb{E}_{\mathcal{M}}$. Further, by Proposition 4.7, the Courant algebroid structure on $\mathbb{E}_{\mathcal{M}}$ does not depend on the choice of splitting of $\mathcal{M}$, since a different choice of splitting will induce a change of Lagrangian splitting of $\mathbb{E}_{\mathcal{M}}$. This shows that the functor $\mathcal{G}$ lifts to a functor $\mathcal{G}_Q$ from the category of Lie 2-algebroids to the category of VB-Courant algebroids.

Sections 3E and 4E1 show that morphisms of split Lie 2-algebroids are sent by $\mathcal{G}$ to morphisms of decomposed VB-Courant algebroids.

The functor $\mathcal{F}$ lifts in a similar manner to a functor $\mathcal{F}_{\mathrm{VBC}}$ from the category of VB-Courant algebroids to the category of Lie 2-algebroids. The natural transformations found in the proof of Theorem 2.3 refine to natural transformations $\mathcal{F}_{\mathrm{VBC}}\mathcal{G}_Q\simeq\mathrm{Id}$ and $\mathcal{G}_Q\mathcal{F}_{\mathrm{VBC}}\simeq\mathrm{Id}$. □

Remark 4.11. Note that we use splittings and decompositions in order to obtain this equivalence of categories, which does not involve splittings and decompositions.

First, while the linear metric of the linear VB-Courant algebroid is at the heart of the equivalence of the underlying (metric) double vector bundle $(\mathbb{E};B,Q;M)$ with the underlying [2]-manifold of the corresponding Lie 2-algebroid, the linear Courant bracket and the linear anchor do not translate to very elegant structures on the linear isotropic sections of $\mathbb{E}\to Q$ and on its core sections. Only in a decomposition, the ingredients of the linear bracket and anchor are recognised in a straightforward manner as the ingredients of a split Lie 2-algebroid.

Since our main goal was to show that, as decomposed VB-algebroids are the same as 2-representations [Gracia-Saz and Mehta 2010], decomposed VB-Courant algebroids are the same as split Lie 2-algebroids, it is natural for us to establish here our equivalence in decompositions and splittings. The main work for the "splitting free" version of the equivalence was done in [Jotz Lean 2018b]. Another approach can of course be found in [Li-Bland 2012], but the equivalence there is not really constructive, in the sense that it is difficult to even recognise the graded functions on the underlying [2]-manifold as sections of the metric double vector bundle. To our understanding, the equivalence of [2]-manifolds with metric double vector bundles is not easy to recognise in the proof of [Li-Bland 2012].

Further, our main application in Section 5 is a statement about a certain class of *decomposed* VB-Courant algebroids versus *split* Lie 2-algebroids. Similarly, in a sequel of this paper [Jotz Lean 2018c], we work exclusively with decomposed or split objects to express Li-Bland's definition of an LA-Courant algebroid [Li-Bland 2012] in a decomposition. This yields a new definition that involves the "matched pair" of a split Lie 2-algebroid with a self-dual 2-representation. This new approach is far more useful for concrete computations, since there is no need anymore to consider the tangent triple vector bundle of $\mathbb{E}$ (see [Li-Bland 2012]).

5. VB-bialgebroids and bicrossproducts of matched pairs of 2-representations

In this section we show that the bicrossproduct of a matched pair of 2-representations is a split Lie 2-algebroid and we geometrically explain this result.

5A. ***The bicrossproduct of a matched pair of 2-representations.*** We construct a split Lie 2-algebroid $(A\oplus B)\oplus C$ induced by a matched pair of 2-representations as in Definition 2.1. The vector bundle $A\oplus B\to M$ is anchored by $\rho_A\circ\mathrm{pr}_A+\rho_B\circ\mathrm{pr}_B$ and paired with $A^*\oplus B^*$ as follows:

$$\langle (a,b),(\alpha,\beta)\rangle=\alpha(a)-\beta(b)$$

for all $a\in\Gamma(A)$, $b\in\Gamma(B)$, $\alpha\in\Gamma(A^*)$ and $\beta\in\Gamma(B^*)$. The morphism $A^*\oplus B^*\to C^*$ is $\partial_A^*\circ\mathrm{pr}_{A^*}+\partial_B^*\circ\mathrm{pr}_{B^*}$. The $A\oplus B$-Dorfman connection on $A^*\oplus B^*$ is defined by

$$\Delta_{(a,b)}(\alpha,\beta)=(\nabla_b^*\alpha+\pounds_a\alpha-\langle\nabla_\cdot b,\beta\rangle,\ \nabla_a^*\beta+\pounds_b\beta-\langle\nabla_\cdot a,\alpha\rangle).$$

The dual dull bracket on $\Gamma(A\oplus B)$ is

$$[\![(a,b),(a',b')]\!]=([a,a']+\nabla_b a'-\nabla_{b'}a,\ [b,b']+\nabla_a b'-\nabla_{a'}b). \tag{12}$$

The $A\oplus B$-connection on C^* is simply given by $\nabla^*_{(a,b)}\gamma=\nabla_a^*\gamma+\nabla_b^*\gamma$ and the dual connection is $\nabla:\Gamma(A\oplus B)\times\Gamma(C)\to\Gamma(C)$,

$$\nabla_{(a,b)}c=\nabla_a c+\nabla_b c. \tag{13}$$

Finally, the form $\omega \in \Omega^3(A \oplus B, C)$ is given by

$$\omega((a_1,b_1),(a_2,b_2),(a_3,b_3)) = R(a_1,a_2)b_3 + R(a_2,a_3)b_1 + R(a_3,a_1)b_2 \\ - R(b_1,b_2)a_3 - R(b_2,b_3)a_1 - R(b_3,b_1)a_2. \tag{14}$$

The vector bundle $(A \oplus B) \oplus C \to M$ with the anchor $\rho_A \circ \mathrm{pr}_A + \rho_B \circ \mathrm{pr}_B : A \oplus B \to TM$, $l = (-\partial_A; \partial_B) : C \to A \oplus B$, ω_R and the skew-symmetric dull bracket (12) define a split Lie 2-algebroid. Moreover, we prove the following theorem:

Theorem 5.1. *The bicrossproduct of a matched pair of 2-representations is a split Lie 2-algebroid with the structure given above. Conversely if $(A \oplus B) \oplus C$ has a split Lie 2-algebroid structure such that*

(1) $[\![(a_1,0),(a_2,0)]\!] = ([a_1,a_2],0)$ *with a section* $[a_1,a_2] \in \Gamma(A)$ *for all* $a_1, a_2 \in \Gamma(A)$ *and in the same manner* $[\![(0,b_1),(0,b_2)]\!] = (0,[b_1,b_2])$ *with a section* $[b_1,b_2] \in \Gamma(B)$ *for all* $b_1, b_2 \in \Gamma(B)$, *and*
(2) $\omega((a_1,0),(a_2,0),(a_3,0)) = 0$ *and* $\omega((0,b_1),(0,b_2),(0,b_3)) = 0$ *for all* a_1, a_2 *and* a_3 *in* $\Gamma(A)$ *and* b_1, b_2 *and* b_3 *in* $\Gamma(B)$,

then A and B are Lie subalgebroids of $(A \oplus B) \oplus C$ and $(A \oplus B) \oplus C$ is the bicrossproduct of a matched pair of 2-representations of A on $B \oplus C$ and of B on $A \oplus C$. The 2-representation of A is given by

$$\begin{aligned} \partial_B(c) &= \mathrm{pr}_B(l(c)), & \nabla_a b &= \mathrm{pr}_B[\![(a,0),(0,b)]\!], \\ \nabla_a c &= \nabla_{(a,0)} c, & R_{AB}(a_1,a_2)b &= \omega(a_1,a_2,b) \end{aligned} \tag{15}$$

and the B-representation is given by

$$\begin{aligned} \partial_A(c) &= -\,\mathrm{pr}_A(l(c)), & \nabla_b a &= \mathrm{pr}_A[\![(0,b),(a,0)]\!], \\ \nabla_b c &= \nabla_{(0,b)} c, & R_{BA}(b_1,b_2)a &= -\omega(b_1,b_2,a). \end{aligned} \tag{16}$$

Proof. Assume first that $(A \oplus B) \oplus C$ is a split Lie 2-algebroid with (1) and (2). The bracket $[\cdot\,,\cdot] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ defined by $[\![(a_1,0),(a_2,0)]\!] = ([a_1,a_2],0)$ is obviously skew-symmetric and $\mathbb{R}$-bilinear. Define an anchor ρ_A on A by $\rho_A(a) = \rho_{A\oplus B}(a,0)$. Then we get immediately

$$([a_1, fa_2],0) = [\![(a_1,0), f(a_2,0)]\!] = f([a_1,a_2],0) + \rho_{A\oplus B}(a_1,0)(f)(a_2,0),$$

which shows that $[a_1, fa_2] = f[a_1,a_2] + \rho_A(a_1)(f)a_2$ for all $a_1, a_2 \in \Gamma(A)$. Further, we find

$$\begin{aligned} \mathrm{Jac}_{[\cdot,\cdot]}(a_1,a_2,a_3) &= \mathrm{pr}_A(\mathrm{Jac}_{[\![\cdot,\cdot]\!]}((a_1,0),(a_2,0),(a_3,0))) \\ &= -(\mathrm{pr}_A \circ l \circ \omega)((a_1,0),(a_2,0),(a_3,0))) = 0 \end{aligned}$$

since ω vanishes on sections of A. Hence A is a wide subalgebroid of the split Lie

2-algebroid. In a similar manner, we find a Lie algebroid structure on B. Next we prove that (15) defines a 2-representation of A. Using (ii) in Definition 3.4 we find for $a \in \Gamma(A)$ and $c \in \Gamma(C)$ that

$$\begin{aligned}\partial_B(\nabla_a c) &= (\mathrm{pr}_B \circ l)(\nabla_{(a,0)} c)\\ &\overset{\text{(ii)}}{=} \mathrm{pr}_B[\![(a,0), l(c)]\!] = \mathrm{pr}_B[\![(a,0),(0,\mathrm{pr}_B(l(c)))]\!] = \nabla_a(\partial_B c).\end{aligned}$$

In the third equation we have used condition (1) and in the last equation the definitions of ∂_B and $\nabla_a : \Gamma(B) \to \Gamma(B)$. In the following, we will write for simplicity a for $(a,0) \in \Gamma(A \oplus B)$, etc. We easily get

$$R_{AB}(a_1,a_2)\partial_B c = \omega(a_1,a_2,\mathrm{pr}_B(l(c))) = \omega(a_1,a_2,l(c)) \overset{\text{(iv)}}{=} R_\nabla(a_1,a_2)c$$

and

$$\partial_B R_{AB}(a_1,a_2)b = (\mathrm{pr}_B \circ l \circ \omega)(a_1,a_2,b) \overset{\text{(iii)}}{=} -\mathrm{pr}_B(\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(a_1,a_2,b))$$

for all $a_1,a_2 \in \Gamma(A)$, $b \in \Gamma(B)$ and $c \in \Gamma(C)$. By condition (1) and the definition of $\nabla_a : \Gamma(B) \to \Gamma(B)$, we find

$$\begin{aligned}R_\nabla(a_1,a_2)b &= \mathrm{pr}_B[\![a_1,[\![a_2,b]\!]]\!] - \mathrm{pr}_B[\![a_2[\![a_1,b]\!]]\!] - \mathrm{pr}_B[\![[\![a_1,a_2]\!],b]\!]\\ &= -\mathrm{pr}_B(\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(a_1,a_2,b)).\end{aligned}$$

Hence, $\partial_B R_{AB}(a_1,a_2)b = R_\nabla(a_1,a_2)b$. Finally, an easy computation along the same lines shows that

$$\langle(\boldsymbol{d}_{\nabla^{\mathrm{Hom}}} R_{AB})(a_1,a_2,a_3), b\rangle = (\boldsymbol{d}_\nabla \omega)(a_1,a_2,a_3,b) \tag{17}$$

for $a_1,a_2,a_3 \in \Gamma(A)$ and $b \in \Gamma(B)$. Since $\boldsymbol{d}_\nabla\omega = 0$, we find $\boldsymbol{d}_{\nabla^{\mathrm{Hom}}} R_{AB} = 0$. In a similar manner, we prove that (16) defines a 2-representation of B. Further, by construction of the 2-representations, the split Lie 2-algebroid structure on $(A \oplus B) \oplus C)$ must be defined as in (12), (13) and (14), with the anchor $\rho_A \circ \mathrm{pr}_A + \rho_B \circ \mathrm{pr}_B$ and $l = (-\partial_A, \partial_B)$. Hence, to conclude the proof, it only remains to check that the split Lie 2-algebroid conditions for these objects are equivalent to the seven conditions in Definition 2.1 for the two 2-representations.

First, we find immediately that (M1) is equivalent to (i). Then we find by construction

$$[a,\partial_A c] + \nabla_{\partial_B c} a = -[a,\mathrm{pr}_A(l(c))] + \nabla_{\mathrm{pr}_B(l(c))} a = \mathrm{pr}_A[\![l(c),a]\!] = -\mathrm{pr}_A[\![a,l(c)]\!].$$

Hence, we find that (M2) holds if and only if $\mathrm{pr}_A[\![a,l(c)]\!] = \mathrm{pr}_A \circ l(\nabla_a c)$. But since

$$\begin{aligned}[\![a,lc]\!] &= (\mathrm{pr}_A[\![a,l(c)]\!], \nabla_a \mathrm{pr}_B\, l(c)) = (\mathrm{pr}_A[\![a,l(c)]\!], \nabla_a \partial_B(c))\\ &= (\mathrm{pr}_A[\![a,l(c)]\!], \partial_B \nabla_a c) = (\mathrm{pr}_A[\![a,l(c)]\!], \mathrm{pr}_B(l(\nabla_a c))),\end{aligned}$$

we have $\mathrm{pr}_A[\![a,l(c)]\!] = \mathrm{pr}_A \circ l(\nabla_a c)$ if and only if $[\![a,lc]\!] = l(\nabla_a c)$. Hence (M2) is satisfied if and only if $[\![a, l(c)]\!] = l(\nabla_a c)$ for all $a \in \Gamma(A)$ and $c \in \Gamma(C)$. In a similar manner, we find that (M3) is equivalent to $[\![b, lc]\!] = l(\nabla_b c)$ for all $b \in \Gamma(B)$ and $c \in \Gamma(C)$. This shows that (M2) and (M3) together are equivalent to (ii).

Next, a simple computation shows that (M4) is equivalent to $R_\nabla(b,a)c = \omega(b,a,l(c))$. Since

$$R_\nabla(a,a')c = R_{AB}(a,a')\partial_B c = \omega(a,a',\mathrm{pr}_B(l(c))) = \omega(a,a',l(c))$$

and $R_\nabla(b,b')c = \omega(b,b',l(c))$, we get that (M4) is equivalent to (iv).

Two straightforward computations show that (M5) is equivalent to

$$\mathrm{pr}_A(\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(a_1,a_2,b)) = -\,\mathrm{pr}_A(l\omega(a_1,a_2,b))$$

and that (M6) is equivalent to

$$\mathrm{pr}_B(\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(b_1,b_2,a)) = -\,\mathrm{pr}_B(l\omega(b_1,b_2,a)).$$

But since $\mathrm{pr}_B(\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(a_1,a_2,b)) = -R_\nabla(a_1,a_2)b$ by construction and

$$R_\nabla(a_1,a_2)b = \partial_B R_{AB}(a_1,a_2)b = \mathrm{pr}_B(l\omega(a_1,a_2,b)),$$

we find

$$\mathrm{pr}_B(\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(a_1,a_2,b)) = -\,\mathrm{pr}_B(l\omega(a_1,a_2,b)),$$

and in a similar manner

$$\mathrm{pr}_A(\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(b_1,b_2,a)) = -\,\mathrm{pr}_A(l\omega(b_1,b_2,a)).$$

Since $\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(a_1,a_2,a_3) = 0$, $\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(b_1,b_2,b_3) = 0$, and ω vanishes on sections of A, and respectively on sections of B, we conclude that (M5) and (M6) together are equivalent to (iii).

Finally, a slightly longer, but still straightforward computation shows that

$$(\boldsymbol{d}_{\nabla^B} R_{AB})(b_1,b_2)(a_1,a_2) - (\boldsymbol{d}_{\nabla^A} R_{BA})(a_1,a_2)(b_1,b_2) = (\boldsymbol{d}_\nabla \omega)(a_1,a_2,b_1,b_2)$$

for all $a_1, a_2 \in \Gamma(A)$ and $b_1, b_2 \in \Gamma(B)$. This, (17), the corresponding identity for R_{BA}, and the vanishing of ω on sections of A, and, respectively, on sections of B, show that (M7) is equivalent to (v). □

If $C = 0$, then $R_{AB} = 0$, $R_{BA} = 0$, $\partial_A = 0$ and $\partial_B = 0$ and the matched pair of 2-representations is just a matched pair of Lie algebroids. The double is then concentrated in degree 0, with $\omega = 0$, and l_2 is the bicrossproduct Lie algebroid structure on $A \oplus B$ with anchor $\rho_A + \rho_B$ [Lu 1997; Mokri 1997]. Hence, in that case the split Lie 2-algebroid is just the bicrossproduct of a matched pair of representations and the dual (flat) Dorfman connection is the corresponding Lie derivative. The Lie 2-algebroid is in that case a genuine Lie 1-algebroid.

In the case where B has a trivial Lie algebroid structure and acts trivially up to homotopy on $\partial_A = 0 : C \to A$, the double is the semidirect product Lie 2-algebroid found in [Sheng and Zhu 2017, Proposition 3.5] (see Section 3D4).

5B. ***VB-bialgebroids and double Lie algebroids.*** Consider a double vector bundle $(D; A, B; M)$ with core C and a VB-Lie algebroid structure on each of its sides. Recall from Section 2A that $(D; A, B, M)$ is a double Lie algebroid if and only if, for any linear splitting of D, the two induced 2-representations (denoted as in Section 2A) form a matched pair [Gracia-Saz et al. 2018]. By definition of a double Lie algebroid, (D^*_A, D^*_B) is then a Lie bialgebroid over C^* [Mackenzie 2011], and so the double vector bundle

$$\begin{tikzcd} D^*_A \oplus D^*_B \arrow[r] \arrow[d] & C^* \arrow[d] \\ A \oplus B \arrow[r] & M \end{tikzcd}$$

with core $B^* \oplus A^*$ has the structure of a VB-Courant algebroid with base C^* and side $A \oplus B$. Note that we call the pair (D^*_A, D^*_B) a *VB-bialgebroid over* C^*. Conversely, a VB-Courant algebroid $(\mathbb{E}; Q, B; M)$ with two transverse VB-Dirac structures $(D_1; Q_1, B; M)$ and $(D_2; Q_2, B; M)$ defines a VB-bialgebroid (D_1, D_2) over B. It is not difficult to see that a VB-bialgebroid[8] $(D_A \to X, A \to M)$, $(D_B \to X, B \to M)$ is equivalent to a double Lie algebroid structure on

$$((D_A)^*_A; B, A; M) \simeq ((D_B)^*_B; B, A; M)$$

with core X^*.

Consider again a double Lie algebroid $(D; A, B; M)$, together with a linear splitting $\Sigma : A \times_M B \to D$. Then the "dual splittings" $\sigma^\star_A : \Gamma(A) \to \Gamma^l_{C^*}(D^*_A)$ and $\sigma^\star_B : \Gamma(B) \to \Gamma^l_{C^*}(D^*_B)$ are defined as in Section 2.2.3 in [Jotz Lean 2018b], and satisfy the equations

$$\langle \sigma^\star_A(a), \sigma^\star_B(b)\rangle = 0, \quad \langle \sigma^\star_A(a), \alpha^\dagger\rangle = -q^*_{C^*}\langle \alpha, a\rangle, \quad \langle \beta^\dagger, \sigma^\star_B(b)\rangle = q^*_{C^*}\langle \beta, b\rangle, \tag{18}$$

for all $a \in \Gamma(A)$, $b \in \Gamma(B)$, $\alpha \in \Gamma(A^*)$ and $\beta \in \Gamma(B^*)$.

Then

$$\tilde{\Sigma} : (A \oplus B) \times_M C^* \to D^*_A \oplus D^*_B,$$

defined by $\tilde{\Sigma}((a(m), b(m)), \gamma_m) = (\sigma^\star_A(a)(\gamma_m), \sigma^\star_B(b)(\gamma_m))$, is a linear Lagrangian splitting of $D^*_A \oplus D^*_B$.

Recall from Section 2A that the splitting

$$\Sigma^\star : A \times_M C^* \to D^*_A$$

[8] D_A has necessarily core B^* and D_B has core A^*.

of the VB-algebroid $(D_A^* \to C^*, A \to M)$ corresponds to the 2-representation $(\nabla^{C^*}, \nabla^{B^*}, -R^*)$ of A on the complex $\partial_B^*: B^* \to C^*$. In the same manner, the splitting $\Sigma^\star: B \times_M C^* \to D_B^*$ of the VB-algebroid $(D_B^* \to C^*, B \to M)$ corresponds to the 2-representation $(\nabla^{C^*}, \nabla^{A^*}, -R^*)$ of B on the complex $\partial_A^*: A^* \to C^*$.

We check that the split Lie 2-algebroid corresponding to the linear splitting $\tilde{\Sigma}$ of $D_A^* \oplus D_B^*$ is the bicrossproduct of the matched pair of 2-representations. The equalities in (18) imply that we have to consider $A \oplus B$ as paired with $A^* \oplus B^*$ in the nonstandard way:

$$\langle (a,b),(\alpha,\beta)\rangle = \alpha(a) - \beta(b)$$

for all $a \in \Gamma(A),\ b \in \Gamma(B),\ \alpha \in \Gamma(A^*)$ and $\beta \in \Gamma(B^*)$. The anchor of $\tilde{\sigma}(a,b) = (\sigma^\star(a), \sigma^\star(b))$ is

$$\widehat{\nabla}_a^* + \widehat{\nabla}_b^* \in \mathfrak{X}^l(C^*),$$

and the anchor of $(\alpha,\beta)^\dagger = (\beta^\dagger, \alpha^\dagger) \in \Gamma_{C^*}^c(D_A^* \oplus D_B^*)$ is

$$(\partial_B^*\beta + \partial_A^*\alpha)^\uparrow \in \mathfrak{X}^c(C^*).$$

The Courant bracket $[\![(\sigma_A^\star(a), \sigma_B^\star(b)), (\beta^\dagger, \alpha^\dagger)]\!]$ is

$$([\sigma_A^\star(a), \beta^\dagger] + \pounds_{\sigma_B^\star(b)}\beta^\dagger - \boldsymbol{i}_{\alpha^\dagger}\boldsymbol{d}_{D_B^*}\sigma_A^\star(a), [\sigma_B^\star(b), \alpha^\dagger] + \pounds_{\sigma_A^\star(a)}\alpha^\dagger - \boldsymbol{i}_{\beta^\dagger}\boldsymbol{d}_{D_A^*}\sigma_B^\star(b)),$$

where $\boldsymbol{d}_{D_A^*}: \Gamma_{C^*}\big(\bigwedge^\bullet D_B^*\big) \to \Gamma_{C^*}\big(\bigwedge^{\bullet+1} D_B^*\big)$ is defined as usual by the Lie algebroid D_A^*, and similarly for D_B^* (bear in mind that some nonstandard signs arise from the signs in (18)). The derivation $\pounds: \Gamma(D_A^*) \times \Gamma(D_B^*) \to \Gamma(D_B^*)$ is described by

$$\begin{aligned} \pounds_{\beta^\dagger}\alpha^\dagger &= 0, & \pounds_{\beta^\dagger}\sigma_B^\star(b) &= -\langle b, \nabla_\cdot^*\beta\rangle^\dagger, \\ \pounds_{\sigma_A^\star(a)}\alpha^\dagger &= \pounds_a\alpha^\dagger, & \pounds_{\sigma_A^\star(a)}\sigma_B^\star(b) &= \sigma_B^\star(\nabla_a b) + \widetilde{R(a,\cdot)b} \end{aligned}$$

in [Gracia-Saz et al. 2018, Lemma 4.8]. Similar formulae hold for

$$\pounds: \Gamma(D_B^*) \times \Gamma(D_A^*) \to \Gamma(D_A^*).$$

We get

$$[\![(\sigma_A^\star(a), \sigma_B^\star(b)), (\beta^\dagger, \alpha^\dagger)]\!] = ((\nabla_a^*\beta + \pounds_b\beta - \langle \nabla_\cdot a, \alpha\rangle)^\dagger, (\nabla_b^*\alpha + \pounds_a\alpha - \langle \nabla_\cdot b, \beta\rangle)^\dagger).$$

In the same manner, we get

$$\begin{aligned} &[\![(\sigma_A^\star(a_1), \sigma_B^\star(b_1)), (\sigma_A^\star(a_2), \sigma_B^\star(b_2))]\!] \\ &\quad = (\sigma_A^\star([a,a'] + \nabla_b a' - \nabla_{b'} a), \sigma_B^\star([b,b'] + \nabla_a b' - \nabla_{a'} b)) \\ &\qquad + \big(-\widetilde{R(a_1,a_2)} + \widetilde{R(b_1,\cdot)a_2} - \widetilde{R(b_2,\cdot)a_1}, -\widetilde{R(b_1,b_2)} + \widetilde{R(a_1,\cdot)b_2} - \widetilde{R(a_2,\cdot)b_1}\big). \end{aligned}$$

Hence we have the following result. Recall that we have found above that double Lie algebroids are equivalent to VB-Courant algebroids with two transverse VB-Dirac structures.

Theorem 5.2. *The correspondence established in Theorem 4.6, between decomposed VB-Courant algebroids and split Lie 2-algebroids, restricts to a correspondence between decomposed double Lie algebroids and split Lie 2-algebroids that are the bicrossproducts of matched pairs of 2-representations.*

In other words, decomposed VB-bialgebroids are equivalent to matched pairs of 2-representations.

Recall that if the vector bundle C is trivial, the matched pair of 2-representations is just a matched pair of the Lie algebroids A and B. The corresponding double Lie algebroid is the decomposed double Lie algebroid $(A \times_M B, A, B, M)$ found in [Mackenzie 2011]. The corresponding VB-Courant algebroid is

$$\begin{array}{ccc} A \times_M B^* \oplus A^* \times_M B & \longrightarrow & 0 \times M \\ \downarrow & & \downarrow \\ A \oplus B & \longrightarrow & M \end{array}$$

with core $B^* \oplus A^*$. In that case there is a natural Lagrangian splitting and the corresponding Lie 2-algebroid is just the bicrossproduct Lie algebroid structure defined on $A \oplus B$ by the matched pair; see also the end of Section 5. This shows that the two notions of the double of a matched pair of Lie algebroids — the bicrossproduct Lie algebroid in [Mokri 1997] and the double Lie algebroid in [Mackenzie 2011] are just the $\mathbb{N}$-geometric and the classical descriptions of the same object, and special cases of Theorem 5.2.

5C. ***Example*: *the two "doubles" of a Lie bialgebroid.*** Recall that a Lie bialgebroid (A, A^*) is a pair of Lie algebroids $(A \to M, \rho, [\cdot\,,\cdot])$ and $(A^* \to M, \rho_\star, [\cdot\,,\cdot]_\star)$ in duality such that $A \oplus A^* \to M$ with the anchor $\rho + \rho_\star$, the pairing

$$\langle (a_1, \alpha_1), (a_2, \alpha_2) \rangle = \alpha_1(a_2) + \alpha_2(a_1),$$

and the bracket

$$[\![(a_1, \alpha_1), (a_2, \alpha_2)]\!] = ([a_1, a_2] + \pounds_{\alpha_1} a_2 - \boldsymbol{i}_{\alpha_2} \boldsymbol{d}_{A^*} a_1, [\alpha_1, \alpha_2]_\star + \pounds_{a_1} \alpha_2 - \boldsymbol{i}_{a_2} \boldsymbol{d}_A \alpha_1)$$

is a Courant algebroid. Lie bialgebroids were originally defined in a different manner [Mackenzie and Xu 1994], and the definition above is at the origin of the abstract definition of Courant algebroids [Liu et al. 1997]. This Courant algebroid is sometimes called the bicrossproduct of the Lie bialgebroid, or the double of the Lie bialgebroid.

Mackenzie [2011] came up with an alternative notion of the double of a Lie bialgebroid. Given a Lie bialgebroid as above, the double vector bundle

$$\begin{array}{ccc} T^*A \simeq T^*A^* & \longrightarrow & A^* \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array}$$

is a double Lie algebroid with the following structures. The Lie algebroid structure on A defines a linear Poisson structure on A^*, and so a linear Lie algebroid structure on $T^*A^* \to A^*$. In the same manner, the Lie algebroid structure on A^* defines a linear Poisson structure on A, and so a linear Lie algebroid structure on $T^*A \to A$ (see [Gracia-Saz et al. 2018] for more details and for the matched pairs of 2-representations associated to a choice of linear splitting). The VB-Courant algebroid defined by this double Lie algebroid is $(T^*A)^*_A \oplus (T^*A)^*_{A^*}$ which is isomorphic to

$$\begin{array}{ccc} TA \oplus TA^* & \longrightarrow & TM \\ \downarrow & & \downarrow \\ A \oplus A^* & \longrightarrow & M \end{array}$$

Computations reveal that the Courant algebroid structure is just the tangent of the Courant algebroid structure on $A \oplus A^*$, and so that the two notions of the double of a Lie bialgebroid can be understood as an algebraic and a geometric interpretation of the same object.

Appendix: Proof of Theorem 4.6

Let $(\mathbb{E}; Q, B; M)$ be a VB-Courant algebroid and choose a Lagrangian splitting $\Sigma : Q \times_M B$. We prove here that the obtained split linear Courant algebroid is equivalent to a split Lie 2-algebroid. Recall the construction of the objects $\partial_B, \Delta, \nabla, [\![\cdot\,,\cdot]\!]_\sigma, R$ in Section 4C1, and recall that $\mathcal{S} \subseteq \Gamma_B(\mathbb{E})$ is the subset

$$\{\tau^\dagger \mid \tau \in \Gamma(Q^*)\} \cup \{\sigma_Q(q) \mid q \in \Gamma(Q)\} \subseteq \Gamma_B(\mathbb{E}).$$

Recall also that the tangent double $(TB \to B; TM \to M)$ has a VB-Lie algebroid structure, which is described in [Jotz Lean 2018b, Section 2.2.2]. We begin by giving two useful lemmas.

Lemma A.1. *For $\beta \in \Gamma(B^*)$, we have*

$$\mathcal{D}(\ell_\beta) = \sigma_Q(\partial_B^*\beta) + \widetilde{\nabla_\cdot^*\beta},$$

where $\nabla_\cdot^*\beta$ is seen as follows as a section of $\Gamma(\mathrm{Hom}(B, Q^*))$: $(\nabla_\cdot^*\beta)(b) = \langle \nabla_\cdot^*\beta, b\rangle \in \Gamma(Q^*)$ for all $b \in \Gamma(B)$.

Proof. For $\beta\in\Gamma(B^*)$, the section $\boldsymbol{d}\ell_\beta$ is a linear section of $T^*B\to B$. Since the anchor Θ is linear, the section $\mathcal{D}\ell_\beta=\Theta^*\boldsymbol{d}\ell_\beta$ is linear. Since for any $\tau\in\Gamma(Q^*)$,

$$\langle\mathcal{D}(\ell_\beta),\tau^\dagger\rangle=\Theta(\tau^\dagger)(\ell_\beta)=q_B^*\langle\partial_B\tau,\beta\rangle,$$

we find that $\mathcal{D}(\ell_\beta)-\sigma_Q(\partial_B^*\beta)\in\Gamma(\ker\pi_Q)$. Hence, $\mathcal{D}(\ell_\beta)-\sigma_Q(\partial_B^*\beta)$ is a core-linear section of $\mathbb{E}\to B$ and there exists a section ϕ of $\operatorname{Hom}(B,Q^*)$ such that $\mathcal{D}(\ell_\beta)-\sigma_Q(\partial_B^*\beta)=\widetilde{\phi}$. We have

$$\ell_{\langle\phi,q\rangle}=\langle\widetilde{\phi},\sigma_Q(q)\rangle=\langle\mathcal{D}(\ell_\beta)-\sigma_Q(\partial_B^*\beta),\sigma_Q(q)\rangle=\Theta(\sigma_Q(q))(\ell_\beta)=\ell_{\nabla_q^*\beta}$$

and so $\phi(b)=\langle\nabla_\cdot^*\beta,b\rangle\in\Gamma(Q^*)$ for all $b\in\Gamma(B)$. $\square$

For each $q\in\Gamma(Q)$, ∇_q, and Δ_q define a derivation $\Diamond_q$ of $\Gamma(\operatorname{Hom}(B,Q^*))$ as follows: for $\phi\in\Gamma(\operatorname{Hom}(B,Q^*))$ and $b\in\Gamma(B)$,

$$(\Diamond_q\phi)(b)=\Delta_q(\phi(b))-\phi(\nabla_q b).$$

Lemma A.2. *For $q\in\Gamma(Q)$ and $\phi\in\Gamma(\operatorname{Hom}(B,Q^*))$, we have $[\![\sigma_Q(q),\widetilde{\phi}]\!]=\widetilde{\Diamond_q\phi}$.*

Proof. The proof is an easy computation as in the proof of Lemma 4.4. $\square$

Now we can express all the conditions of Lemma 3.2 in terms of the objects $\partial_B,\Delta,\nabla,[\![\cdot\,,\cdot]\!]_\sigma, R$ found in Section 4C1.

Proposition A.3. *The anchor satisfies $\Theta\circ\Theta^*=0$ if and only if $\rho_Q\circ\partial_B^*=0$ and $\nabla^*_{\partial_B^*\beta_1}\beta_2+\nabla^*_{\partial_B^*\beta_2}\beta_1=0$ for all $\beta_1,\beta_2\in\Gamma(B^*)$.*

Proof. The composition $\Theta\circ\Theta^*$ vanishes if and only if $(\Theta\circ\Theta^*)\boldsymbol{d}F=0$ for all linear and pullback functions $F\in C^\infty(B)$. For $f\in C^\infty(M)$,

$$\Theta(\Theta^*\boldsymbol{d}(q_B^*f))=((\partial_B\circ\rho_Q^*)\boldsymbol{d}f)^\uparrow.$$

For $\beta\in\Gamma(B^*)$, we find, using Lemma A.1,

$$\Theta(\Theta^*\boldsymbol{d}\ell_\beta)=\Theta(\mathcal{D}\ell_\beta)=\Theta(\sigma_Q(\partial_B^*\beta)+\widetilde{\nabla_\cdot^*\beta})=\widehat{\nabla}_{\partial_B^*\beta}+\widetilde{\partial_B\circ\langle\nabla_\cdot^*\beta,\cdot\rangle}.$$

Here, $\partial_B\circ\langle\nabla_\cdot^*\beta,\cdot\,\rangle$ is as follows a morphism $B\to B$; $b\mapsto\partial_B(\langle\nabla_\cdot^*\beta,b\rangle)$. On a linear function $\ell_{\beta'}$, $\beta'\in\Gamma(B^*)$, we have $\Theta(\Theta^*\boldsymbol{d}\ell_\beta)(\ell_{\beta'})=\ell_{\nabla^*_{\partial_B^*\beta}\beta'}+\ell_{\nabla^*_{\partial_B^*\beta'}\beta}$. On a pullback q_B^*f, $f\in C^\infty(M)$, this is $q_B^*(\pounds_{(\rho_Q\circ\partial_B^*)(\beta)}f)$. $\square$

Proposition A.4. *The compatibility of Θ with the Courant algebroid bracket $[\![\cdot\,,\cdot]\!]$ is equivalent to*

1. $\partial_B\circ R(q_1,q_2)=R_\nabla(q_1,q_2)$,
2. $\rho_Q\circ[\![\cdot\,,\cdot]\!]_\sigma=[\cdot\,,\cdot]\circ(\rho_Q,\rho_Q)$, *or* $\Delta_q(\rho_Q^*\boldsymbol{d}f)=\rho_Q^*\boldsymbol{d}(\rho_Q(q)(f))$ *for all* $q\in\Gamma(Q)$ *and* $f\in C^\infty(M)$, *and*
3. $\partial_B\circ\Delta=\nabla\circ\partial_B$.

Proof. We have $\Theta[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!] = [\Theta(\sigma_Q(q_1)),\Theta(\sigma_Q(q_2))] = [\widehat{\nabla}_{q_1},\widehat{\nabla}_{q_2}]$ and

$$\Theta(\sigma_Q([\![q_1,q_2]\!]_\sigma) - \widetilde{R(q_1,q_2)}) = \widehat{\nabla}_{[\![q_1,q_2]\!]_\sigma} - \widetilde{\partial_B \circ R(q_1,q_2)}.$$

Applying both derivations to a pullback function $q_B^* f$ for $f \in C^\infty(M)$ yields

$$[\widehat{\nabla}_{q_1},\widehat{\nabla}_{q_2}](q_B^* f) = q_B^*([\rho_Q(q_1),\rho_Q(q_2)]f).$$

and

$$(\widehat{\nabla}_{[\![q_1,q_2]\!]_\sigma} - \widetilde{\partial_B \circ R(q_1,q_2)})(q_B^* f) = q_B^*(\rho_Q[\![q_1,q_2]\!]_\sigma(f)).$$

Applying both vector fields to a linear function $\ell_\beta \in C^\infty(B)$, $\beta \in \Gamma(B^*)$, we get

$$[\widehat{\nabla}_{q_1},\widehat{\nabla}_{q_2}](\ell_\beta) = \ell_{\nabla^*_{q_1}\nabla^*_{q_2}\beta - \nabla^*_{q_2}\nabla^*_{q_1}\beta}$$

and

$$(\widehat{\nabla}_{[\![q_1,q_2]\!]_\sigma} - \widetilde{\partial_B \circ R(q_1,q_2)})(\ell_\beta) = \ell_{\nabla^*_{[\![q_1,q_2]\!]_\sigma}\beta - R(q_1,q_2)^*\partial_B^*\beta}.$$

Since $R_{\nabla^*}(q_1,q_2) = -(R_\nabla(q_1,q_2))^*$, we find that

$$\Theta[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!] = [\Theta(\sigma_Q(q_1)),\Theta(\sigma_Q(q_2))]$$

for all $q_1,q_2 \in \Gamma(Q)$ if and only if (1) and (2) are satisfied.

In the same manner, for $q \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$, we compute

$$\Theta([\![\sigma_Q(q),\tau^\dagger]\!]) = (\partial_B \Delta_q \tau)^\uparrow$$

and

$$[\Theta(\sigma_Q(q)),\Theta(\tau^\dagger)] = [\widehat{\nabla}_q,(\partial_B\tau)^\uparrow] = (\nabla_q(\partial_B\tau))^\uparrow.$$

Thus, $\Theta([\![\sigma_Q(q),\tau^\dagger]\!]) = [\Theta(\sigma_Q(q)),\Theta(\tau^\dagger)]$ if and only if $\partial_B(\Delta_q\tau) = \nabla_q(\partial_B\tau)$. □

Proposition A.5. *The condition* (3) *of* Lemma 3.2 *is equivalent to* $R(q_1,q_2) = -R(q_2,q_1)$ *and* $[\![q_1,q_2]\!]_\sigma + [\![q_2,q_1]\!]_\sigma = 0$ *for* $q_1,q_2 \in \Gamma(Q)$.

Proof. Choose q_1,q_2 in $\Gamma(Q)$. Then we have

$$[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!] + [\![\sigma_Q(q_2),\sigma_Q(q_1)]\!] \\ = \sigma_Q([\![q_1,q_2]\!]_\sigma + [\![q_2,q_1]\!]_\sigma) - \widetilde{R(q_1,q_2)} - \widetilde{R(q_2,q_1)}.$$

By the choice of the splitting, we have $\mathcal{D}\langle\sigma_Q(q_1),\sigma_Q(q_2)\rangle = \mathcal{D}(0) = 0$. Hence, (3) of Lemma 3.2 is true on horizontal lifts of sections of Q if and only if $R(q_1,q_2) = -R(q_2,q_1)$ and $[\![q_1,q_2]\!]_\sigma + [\![q_2,q_1]\!]_\sigma = 0$ for all $q_1,q_2 \in \Gamma(Q)$. Further, we have $[\![\sigma_Q(q),\tau^\dagger]\!] = (\Delta_q\tau)^\dagger$ and $[\![\tau^\dagger,\sigma_Q(q)]\!] = (-\Delta_q\tau + \rho_Q^*\boldsymbol{d}\langle\tau,q\rangle)^\dagger$ by definition. On core sections (3) is trivially satisfied since both the pairing and the bracket of two core sections vanish. □

Proposition A.6. *The derivation formula* (2) *in Lemma 3.2 is equivalent to the following*:

(1) Δ *is dual to* $[\![\cdot,\cdot]\!]_\sigma$, *that is* $[\![\cdot,\cdot]\!]_\sigma = [\![\cdot,\cdot]\!]_\Delta$,

(2) $[\![q_1,q_2]\!]_\sigma + [\![q_2,q_1]\!]_\sigma = 0$ *for all* $q_1,q_2 \in \Gamma(Q)$, *and*

(3) $R(q_1,q_2)^*q_3 = -R(q_1,q_3)^*q_2$ *for all* $q_1,q_2,q_3 \in \Gamma(Q)$.

Proof. We compute (CA2) for linear and core sections. First of all, the equations

$$\Theta(\tau_1^\dagger)\langle\tau_2^\dagger,\tau_3^\dagger\rangle = \langle[\![\tau_1^\dagger,\tau_2^\dagger]\!],\tau_3^\dagger\rangle + \langle\tau_2^\dagger,[\![\tau_1^\dagger,\tau_3^\dagger]\!]\rangle,$$
$$\Theta(\tau_1^\dagger)\langle\tau_2^\dagger,\sigma_Q(q)\rangle = \langle[\![\tau_1^\dagger,\tau_2^\dagger]\!],\sigma_Q(q)\rangle + \langle\tau_2^\dagger,[\![\tau_1^\dagger,\sigma_Q(q)]\!]\rangle$$

and

$$\Theta(\sigma_Q(q))\langle\tau_1^\dagger,\tau_2^\dagger\rangle = \langle[\![\sigma_Q(q),\tau_1^\dagger]\!],\tau_2^\dagger\rangle + \langle\tau_1^\dagger,[\![\sigma_Q(q),\tau_2^\dagger]\!]\rangle$$

are trivially satisfied for all $\tau_1,\tau_2,\tau_3 \in \Gamma(Q^*)$ and $q \in \Gamma(Q)$. Next, for $q_1,q_2 \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$, we have:

$$\begin{aligned}\Theta(\sigma_Q(q_1))\langle\sigma_Q(q_2),\tau^\dagger\rangle - \langle[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!],\tau^\dagger\rangle - \langle\sigma_Q(q_2),[\![\sigma_Q(q_1),\tau^\dagger]\!]\rangle\\ = \widehat{\nabla}_{q_1}(q_B^*\langle q_2,\tau\rangle) - q_B^*\langle[\![q_1,q_2]\!]_\sigma,\tau\rangle - q_B^*\langle q_2,\Delta_{q_1}\tau\rangle\\ = q_B^*\big(\rho_Q(q_1)\langle q_2,\tau\rangle - \langle[\![q_1,q_2]\!]_\sigma,\tau\rangle - \langle q_2,\Delta_{q_1}\tau\rangle\big)\end{aligned}$$

Thus $\Theta(\sigma_Q(q_1))\langle\sigma_Q(q_2),\tau^\dagger\rangle = \langle[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!],\tau^\dagger\rangle + \langle\sigma_Q(q_2),[\![\sigma_Q(q_1),\tau^\dagger]\!]\rangle$ for all $q_1,q_2 \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$ if and only if Δ and $[\![\cdot,\cdot]\!]_\sigma$ are dual to each other. Using this, we compute

$$\begin{aligned}&\Theta(\tau^\dagger)\langle\sigma_Q(q_1),\sigma_Q(q_2)\rangle - \langle[\![\tau^\dagger,\sigma_Q(q_1)]\!],\sigma_Q(q_2)\rangle - \langle\sigma_Q(q_1),[\![\tau^\dagger,\sigma_Q(q_2)]\!]\rangle\\ &\quad= 0 - \langle-(\Delta_{q_1}\tau)^\dagger + (\rho_Q^*\boldsymbol{d}\langle q_1,\tau\rangle)^\dagger,\sigma_Q(q_2)\rangle - \langle\sigma_Q(q_1),-(\Delta_{q_2}\tau)^\dagger + (\rho_Q^*\boldsymbol{d}\langle q_2,\tau\rangle)^\dagger\rangle\\ &\quad= -q_B^*\langle[\![q_1,q_2]\!]_\sigma + [\![q_2,q_1]\!]_\sigma,\tau\rangle.\end{aligned}$$

Finally we have $\Theta(\sigma_Q(q_1))\langle\sigma_Q(q_2),\sigma_Q(q_3)\rangle = 0$ for all $q_1,q_2,q_3 \in \Gamma(Q)$, and $\langle[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!],\sigma_Q(q_3)\rangle = \ell_{-R(q_1,q_2)^*q_3}$. This shows that

$$\begin{aligned}&\Theta(\sigma_Q(q_1))\langle\sigma_Q(q_2),\sigma_Q(q_3)\rangle\\ &\qquad= \langle[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!],\sigma_Q(q_3)\rangle + \langle\sigma_Q(q_2),[\![\sigma_Q(q_1),\sigma_Q(q_3)]\!]\rangle\end{aligned}$$

if and only if $0 = -R(q_1,q_2)^*q_3 - R(q_1,q_3)^*q_2$. □

Proposition A.7. *Assume that* Δ *and* $[\![\cdot,\cdot]\!]_\sigma$ *are dual to each other. The Jacobi identity in Leibniz form for sections in* $\mathcal{S}$ *is equivalent to*

(1) $R(q_1,q_2)\circ\partial_B = R_\Delta(q_1,q_2)$,

$$(2)\quad R(q_1,\llbracket q_2,q_3\rrbracket_\Delta)-R(q_2,\llbracket q_1,q_3\rrbracket_\Delta)-R(\llbracket q_1,q_2\rrbracket_\Delta),q_3) \\ +\lozenge_{q_1}(R(q_2,q_3))-\lozenge_{q_2}(R(q_1,q_3))+\lozenge_{q_3}(R(q_1,q_2)) \\ =\nabla^*_\cdot(R(q_1,q_2)^*q_3)$$

for all $q_1,q_2,q_3\in\Gamma(Q)$.

If R is skew-symmetric as in (1) of Proposition A.5, then the second equation is $\boldsymbol{d}_{\nabla^*}\omega=0$ for $\omega\in\Omega^3(Q,B^*)$ defined by $\omega(q_1,q_2,q_3)=R(q_1,q_2)^*q_3$.

Proof. The Jacobi identity is trivially satisfied on core sections since the bracket of two core sections is 0. Similarly, for $\tau_1,\tau_2\in\Gamma(Q^*)$ and $q\in\Gamma(Q)$, we find $\llbracket\sigma_Q(q),\llbracket\tau_1^\dagger,\tau_2^\dagger\rrbracket\rrbracket=0$ and $\llbracket\llbracket\sigma_Q(q),\tau_1^\dagger\rrbracket,\tau_2^\dagger\rrbracket+\llbracket\tau_1^\dagger,\llbracket\sigma_Q(q),\tau_2^\dagger\rrbracket\rrbracket=0$. We have

$$\llbracket\sigma_Q(q_1),\llbracket\sigma_Q(q_2),\tau^\dagger\rrbracket\rrbracket-\llbracket\sigma_Q(q_2),\llbracket\sigma_Q(q_1),\tau^\dagger\rrbracket\rrbracket \\ =\llbracket\sigma_Q(q_1),(\Delta_{q_2}\tau)^\dagger\rrbracket-\llbracket\sigma_Q(q_2),(\Delta_{q_1}\tau)^\dagger\rrbracket \\ =(\Delta_{q_1}\Delta_{q_2}\tau)^\dagger-(\Delta_{q_2}\Delta_{q_1}\tau)^\dagger,$$

and

$$\llbracket\llbracket\sigma_Q(q_1),\sigma_Q(q_2)\rrbracket,\tau^\dagger\rrbracket=\llbracket\sigma_Q(\llbracket q_1,q_2\rrbracket_\Delta)-\widetilde{R(q_1,q_2)},\tau^\dagger\rrbracket \\ =(\Delta_{\llbracket q_1,q_2\rrbracket_\Delta}\tau)^\dagger+(R(q_1,q_2)(\partial_B\tau))^\dagger$$

by Lemma 4.4. We now choose $q_1,q_2,q_3\in\Gamma(Q)$ and compute

$$\llbracket\llbracket\sigma_Q(q_1),\sigma_Q(q_2)\rrbracket,\sigma_Q(q_3)\rrbracket \\ =\llbracket\sigma_Q(\llbracket q_1,q_2\rrbracket_\Delta)-\widetilde{R(q_1,q_2)},\sigma_Q(q_3)\rrbracket \\ =\sigma_Q(\llbracket\llbracket q_1,q_2\rrbracket_\Delta,q_3\rrbracket_\Delta)-\widetilde{R(\llbracket q_1,q_2\rrbracket_\Delta,q_3)}-\mathcal{D}\ell_{\langle R(q_1,q_2)\cdot,q_3\rangle}+\widetilde{\lozenge_{q_3}R(q_1,q_2)} \\ =\sigma_Q(\llbracket\llbracket q_1,q_2\rrbracket_\Delta,q_3\rrbracket_\Delta)-\widetilde{R(\llbracket q_1,q_2\rrbracket_\Delta,q_3)} \\ -\sigma_Q(\partial_B^*\langle R(q_1,q_2)\cdot,q_3\rangle)-\widetilde{\nabla^*_\cdot\langle R(q_1,q_2)\cdot,q_3\rangle}+\widetilde{\lozenge_{q_3}R(q_1,q_2)}$$

and

$$\llbracket\sigma_Q(q_2),\llbracket\sigma_Q(q_1),\sigma_Q(q_3)\rrbracket\rrbracket \\ =\llbracket\sigma_Q(q_2),\sigma_Q(\llbracket q_1,q_3\rrbracket_\Delta)-\widetilde{R(q_1,q_3)}\rrbracket \\ =\sigma_Q(\llbracket q_2,\llbracket q_1,q_3\rrbracket_\Delta\rrbracket_\Delta)-\widetilde{R(q_2,\llbracket q_1,q_3\rrbracket_\Delta)}-\widetilde{\lozenge_{q_2}R(q_1,q_3)}.$$

We hence find that

$$\llbracket\llbracket\sigma_Q(q_1),\sigma_Q(q_2)\rrbracket,\sigma_Q(q_3)\rrbracket+\llbracket\sigma_Q(q_2),\llbracket\sigma_Q(q_1),\sigma_Q(q_3)\rrbracket\rrbracket \\ =\llbracket\sigma_Q(q_1),\llbracket\sigma_Q(q_2),\sigma_Q(q_3)\rrbracket\rrbracket$$

if and only if

$$\llbracket\llbracket q_1,q_2\rrbracket_\Delta,q_3\rrbracket_\Delta+\llbracket q_2,\llbracket q_1,q_3\rrbracket_\Delta\rrbracket_\Delta=\llbracket q_1,\llbracket q_2,q_3\rrbracket_\Delta\rrbracket_\Delta+\partial_B^*\langle R(q_1,q_2)\cdot,q_3\rangle$$

and

$$\begin{aligned} R(\llbracket q_1, q_2\rrbracket_\Delta, q_3) + \nabla^*_\cdot \langle R(q_1,q_2)\cdot, q_3\rangle - \Diamond_{q_3} R(q_1,q_2) \\ + R(q_2, \llbracket q_1, q_3\rrbracket_\Delta) + \Diamond_{q_2} R(q_1,q_3) \\ = R(q_1, \llbracket q_2, q_3\rrbracket_\Delta) + \Diamond_{q_1} R(q_2,q_3). \end{aligned}$$

We conclude using (2) on page 156. □

A combination of Propositions A.3, A.4, A.5, A.6, A.7 and Lemma 3.2 proves Theorem 4.6.

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