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HARISH-CHANDRA MODULES FOR DIVERGENCE ZERO VECTOR FIELDS ON A TORUS

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The Lie algebra of divergence zero vector fields on a torus is an infinite-dimensional Lie algebra of skew derivations over the ring of Laurent polynomials. We consider the semidirect product of the Lie algebra of divergence zero vector fields on a torus with the algebra of Laurent polynomials. In this paper, we prove that a Harish-Chandra module of the universal central extension of the derived Lie subalgebra of this semidirect product is either a uniformly bounded module or a generalized highest weight module. We also classify all the generalized highest weight Harish-Chandra modules.

1. Introduction

Harish-Chandra modules, i.e., irreducible weight modules with finite-dimensional weight spaces, are no doubt one of the most important families in the study of the representation theory of infinite-dimensional Lie algebras. The classifications of Harish-Chandra modules over the Virasoro algebra ([Kaplansky and Santharoubane 1985; Mathieu 1992]), higher rank Virasoro algebras ([Su 2003; Lu and Zhao 2006]), and many other Lie algebras related to the Virasoro algebra have been achieved in [Guo et al. 2011; 2012; Lu and Zhao 2010; Liu and Jiang 2008; Mazorchuk 2000; Su 2004a; 2004b; Su et al. 2012; 2013; Wang and Tan 2007]. Let $A=\mathbb{C}[t_1^{\pm1},t_2^{\pm1}]$ be the algebra of Laurent polynomials in commuting variables and B be the set of skew derivations of A. Let L be the universal central extension of the derived Lie subalgebra of the Lie algebra $A\rtimes B$. Set $\widetilde{L}=L\oplus\mathbb{C}d_1\oplus\mathbb{C}d_2$, where d_1,d_2 are two degree derivations. In this paper, we study Harish-Chandra modules over the Lie algebra $\widetilde{L}=L\oplus\mathbb{C}d_1\oplus\mathbb{C}d_2$, this Lie algebra is a generalization of the twisted Heisenberg–Virasoro algebra from rank one to rank two (see [Xue et al. 2006; Tan et al. 2015] for details). The structure of the Lie algebra L has been studied in [Xue et al. 2006]. Recently, the connection of the Lie algebra L with the vertex algebra

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has been established in [Guo and Wang 2016] and the representation theory of the Lie algebra L has been studied in [Tan et al. 2015; Guo and Liu 2019; Billig and Talboom 2018]. However, the classification of the Harish-Chandra modules over the Lie algebra $\widetilde{L}$ is unknown. We prove that a Harish-Chandra module of $\widetilde{L}$ is either a uniformly bounded module or a generalized highest weight module, and we classify the nonzero level Harish-Chandra modules of the Lie algebra $\widetilde{L}$. Based on these results, the classification of Harish-Chandra modules of $\widetilde{L}$ reduces to the classification of uniformly bounded modules of $\widetilde{L}$. In [Guo and Liu 2019], the uniformly bounded modules satisfying the condition that the torus subalgebra acting nonzero were classified. Another reason to study the Harish-Chandra modules of the Lie algebra $\widetilde{L}$ comes from the representation theory of the nullity 2 toroidal extended affine Lie algebras (see [Chen et al. 2018]). It was proved therein that the classification of irreducible integrable modules with finite-dimensional spaces of the nullity 2 toroidal extended affine Lie algebras of type A_1 can be reduced to the classification of Harish-Chandra modules of $\widetilde{L}$. This phenomenon is similar to the fact that the classification of irreducible integrable modules of the full toroidal Lie algebra can be reduced to the classification of irreducible $(\mathrm{Der}(A_n) \ltimes A_n)$-modules (see [Eswara Rao and Jiang 2005]), where

$$A_n = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}].$$

The techniques in this paper follow from [Lin and Tan 2006; Lin and Su 2013; Lu and Zhao 2006; Su 2003]. However, we want to point out that in [Lin and Tan 2006], the construction of the generalized highest weight modules of the Virasoro-like algebra is induced from the $\mathbb{Z}$-graded irreducible modules of a Heisenberg subalgebra, while in this paper, the construction of the generalized highest weight module of the Lie algebra $\widetilde{L}$ comes from the $\mathbb{Z}$-graded irreducible module of the subalgebra $\mathcal{H}_{\boldsymbol{b}_1}$ (see the definition in Section 2), which is the twist of three Heisenberg subalgebras. So we first need to classify the $\mathbb{Z}$-graded irreducible $\mathcal{H}_{\boldsymbol{b}_1}$-modules with finite-dimensional graded spaces, which we do in Propositions 2.6 and 2.8. For the classification of generalized highest weight Harish-Chandra modules of $\widetilde{L}$, we achieve this by considering the tensor product of the highest weight modules of the Lie algebra L with a torus. Moreover, we prove that these tensor modules of $\widetilde{L}$ are completely reducible, and every generalized highest weight Harish-Chandra module of $\widetilde{L}$ is isomorphic to one of the irreducible components of these tensor modules.

The paper is organized as follows. In Section 2, we prove that a Harish-Chandra module of $\widetilde{L}$ is either a uniformly bounded module or a generalized highest weight module. In Section 3, we prove that a nonzero level Harish-Chandra module of $\widetilde{L}$ is a generalized highest weight module. Then we characterize the generalized highest weight Harish-Chandra modules with nonzero level. In Section 4, we classify the generalized highest weight Harish-Chandra modules of $\widetilde{L}$.

Throughout this paper we use $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$ to denote the sets of complex numbers, integers, nonnegative integers and positive integers respectively. All the vector spaces mentioned in this paper are over $\mathbb{C}$. As usual, if $u_1, u_2, \dots, u_k$ are elements of a certain vector space, we denote by $\langle u_1, u_2, \dots, u_k\rangle$ the linear span of the elements $u_1, u_2, \dots, u_k$ over $\mathbb{C}$. The universal enveloping algebra for a Lie algebra $\mathfrak{g}$ is denoted by $\mathcal{U}(\mathfrak{g})$ and $\mathrm{GL}_{n\times n}(\mathbb{Z})$ denotes the set of $n\times n$ invertible matrices with entries from $\mathbb{Z}$.

2. Harish-Chandra modules of $\widetilde{L}$

In this section, we first recall some basic definitions about Harish-Chandra modules of $\widetilde{L}$ and some results for Heisenberg algebras. Then we prove that a Harish-Chandra module of $\widetilde{L}$ is either a uniformly bounded module or a generalized highest weight module.

Let $\boldsymbol{e}_1 = (1, 0)$, $\boldsymbol{e}_2 = (0, 1)$, $\Gamma = \mathbb{Z}\boldsymbol{e}_1 + \mathbb{Z}\boldsymbol{e}_2$. Letting $(x_1, x_2), (y_1, y_2) \in \Gamma$, we define $(x_1, x_2) > (y_1, y_2)$ if and only if $x_1 > y_1$ and $x_2 > y_2$, and $(x_1, x_2) \geq (y_1, y_2)$ if and only if $x_1 \geq y_1$ and $x_2 \geq y_2$. For any $\boldsymbol{b}_1 = b_{11}\boldsymbol{e}_1 + b_{12}\boldsymbol{e}_2$, $\boldsymbol{b}_2 = b_{21}\boldsymbol{e}_1 + b_{22}\boldsymbol{e}_2 \in \Gamma$, we set

$$\det\begin{pmatrix}\boldsymbol{b}_1\\ \boldsymbol{b}_2\end{pmatrix} = b_{11}b_{22} - b_{12}b_{21}.$$

Now we recall the definition of the Lie algebra arising from the two-dimensional torus (also called the Heisenberg–Virasoro algebra of rank two). See [Xue et al. 2006] (cf. [Tan et al. 2015]) for details.

Definition 2.1. The *Heisenberg–Virasoro algebra of rank two* is the Lie algebra spanned by

$$\{t^{\boldsymbol{m}}, E(\boldsymbol{m}), K_i \mid \boldsymbol{m} \in \Gamma \setminus \{\boldsymbol{0}\}, i = 1, 2, 3, 4\}$$

with Lie bracket defined by

$$[t^{\boldsymbol{m}}, t^{\boldsymbol{n}}] = 0, \quad [K_i, L] = 0, \quad i = 1, 2, 3, 4,$$

$$[t^{\boldsymbol{m}}, E(\boldsymbol{n})] = \det\begin{pmatrix}\boldsymbol{n}\\ \boldsymbol{m}\end{pmatrix} t^{\boldsymbol{m}+\boldsymbol{n}} + \delta_{\boldsymbol{m}+\boldsymbol{n},0}\, h(\boldsymbol{m}),$$

$$[E(\boldsymbol{m}), E(\boldsymbol{n})] = \det\begin{pmatrix}\boldsymbol{n}\\ \boldsymbol{m}\end{pmatrix} E(\boldsymbol{m}+\boldsymbol{n}) + \delta_{\boldsymbol{m}+\boldsymbol{n},0}\, f(\boldsymbol{m}),$$

where $\boldsymbol{m} = m_1\boldsymbol{e}_1 + m_2\boldsymbol{e}_2$, $h(\boldsymbol{m}) = m_1K_1 + m_2K_2$, $f(\boldsymbol{m}) = m_1K_3 + m_2K_4$.

We denote this Lie algebra by L. Set $E(\boldsymbol{0}) = t^{\boldsymbol{0}} = 0$ for convenience. Obviously L is a $\mathbb{Z}^2$-graded Lie algebra and the subalgebra $\langle E(\boldsymbol{m}),\ K_3,\ K_4 \mid \boldsymbol{m} \in \Gamma \setminus \{\boldsymbol{0}\}\rangle$ of L is a Virasoro-like algebra. Let $\widetilde{L} = L \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$, where d_1, d_2 are two degree derivations defined by

$$[d_i, E(\boldsymbol{m})] = m_iE(\boldsymbol{m}), \quad [d_i, t^{\boldsymbol{m}}] = m_it^{\boldsymbol{m}}, \quad [d_i, K_j] = 0, \quad [d_1, d_2] = 0,$$

for $\boldsymbol{m} = m_1\boldsymbol{e}_1 + m_2\boldsymbol{e}_2 \in \Gamma$, $i = 1, 2$ and $j = 1, 2, 3, 4$. Lemma 2.2 is easy to check.

Lemma 2.2. *Let* $0 \neq \boldsymbol{b}_1 = b_{11}\boldsymbol{e}_1 + b_{12}\boldsymbol{e}_2 \in \Gamma$ *and* $\boldsymbol{b}_2 = b_{21}\boldsymbol{e}_1 + b_{22}\boldsymbol{e}_2 \in \Gamma$.

(1) *Then* $\langle E(\pm k\boldsymbol{b}_1),\ f(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle$ *and* $\langle E(k\boldsymbol{b}_1),\ t^{-k\boldsymbol{b}_1},\ h(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle$ *and* $\langle E(-k\boldsymbol{b}_1),\ t^{k\boldsymbol{b}_1},\ h(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle$ *are three Heisenberg subalgebras of* $\widetilde{L}$, *and*

(2) $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ *is a* $\mathbb{Z}$*-basis of* Γ *if and only if* $\det\binom{\boldsymbol{b}_1}{\boldsymbol{b}_2} = \pm 1$.

Now we recall some definitions related to the Harish-Chandra modules for $\widetilde{L}$. A *weight module* of $\widetilde{L}$ is a module V with weight space decomposition

$$V = \bigoplus_{\boldsymbol{\lambda} \in \mathbb{C}^6} V_{\boldsymbol{\lambda}},$$

where $V_{\boldsymbol{\lambda}} = \{v \in V \mid d_i v = \lambda_i v,\ K_j v = \lambda_{j+2} v,\ i = 1, 2,\ j = 1, 2, 3, 4\}$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_6) \in \mathbb{C}^6$. For a weight module V, we define the weight set of V by $\mathcal{P}(V) = \{\boldsymbol{\lambda} \in \mathbb{C}^6 \mid V_{\boldsymbol{\lambda}} \neq 0\}$. A weight module is said to be *quasifinite* if all weight spaces $V_{\boldsymbol{\lambda}}$ are finite-dimensional. Furthermore, if there exists a positive integer N such that $\dim V_{\boldsymbol{\lambda}} \leq N$ for all $\boldsymbol{\lambda} \in \mathbb{C}^6$, we call V a *uniformly bounded* module. An irreducible quasifinite weight module is called a *Harish-Chandra module*. Note that the centers K_1, K_2, K_3, K_4 of $\widetilde{L}$ act on an irreducible weight module V as scalars, i.e., $K_i.v = c_i v$ for certain $c_i \in \mathbb{C},\ i = 1, 2, 3, 4$, for all $v \in V$. And we call (c_1, c_2, c_3, c_4) the *level* of the module V. For simplicity of notation, we write $V_{(\lambda_1,\lambda_2)}$ instead of $V_{(\lambda_1,\ldots,\lambda_6)}$ if the module V is irreducible, i.e., the level $(c_1, \ldots, c_4)$ is fixed. One can easily see that there exist $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $\mathcal{P}(V) \subseteq (\lambda_1, \lambda_2) + \Gamma$ for an irreducible weight module V of $\widetilde{L}$. If there exists a $\mathbb{Z}$-basis $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ of Γ and $0 \neq v_{\boldsymbol{\lambda}} \in V_{\boldsymbol{\lambda}}$ such that $V = \mathcal{U}(\widetilde{L})v_{\boldsymbol{\lambda}}$ and $E(\boldsymbol{m})v_{\boldsymbol{\lambda}} = t^{\boldsymbol{m}} v_{\boldsymbol{\lambda}} = 0$, for all $\boldsymbol{m} \in \mathbb{Z}_+\boldsymbol{b}_1 + \mathbb{Z}_+\boldsymbol{b}_2$, we call V a *generalized highest weight module* with generalized highest weight $\boldsymbol{\lambda}$ corresponding to the $\mathbb{Z}$-basis B. The nonzero vector $v_{\boldsymbol{\lambda}}$ is called a *generalized highest weight vector corresponding to the* $\mathbb{Z}$*-basis* B, or simply *generalized highest weight vector*.

Let $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ be a $\mathbb{Z}$-basis of Γ and let

$$\mathcal{H}_{\boldsymbol{b}_1} = \langle E(k\boldsymbol{b}_1),\ t^{k\boldsymbol{b}_1},\ K_i \mid k \in \mathbb{Z} \setminus \{0\},\ i = 1, 2, 3, 4\rangle.$$

Denote

$$\begin{aligned}
&\widetilde{L}_0 = \mathcal{H}_{\boldsymbol{b}_1} \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2,\\
&\widetilde{L}_i = \langle E(m\boldsymbol{b}_1 + i\boldsymbol{b}_2), t^{m\boldsymbol{b}_1 + i\boldsymbol{b}_2} \mid m \in \mathbb{Z}\rangle, \quad i \neq 0,\\
&\widetilde{L}_+ = \oplus_{i>0}\widetilde{L}_i, \qquad \widetilde{L}_- = \oplus_{i<0}\widetilde{L}_i.
\end{aligned}$$

Then $\widetilde{L} = \widetilde{L}_+ \oplus \widetilde{L}_0 \oplus \widetilde{L}_-$. Let V be an irreducible weight $\widetilde{L}_0$-module. We extend V to be a $(\widetilde{L}_+ \oplus \widetilde{L}_0)$-module by defining $\widetilde{L}_+.V = 0$. Then we obtain the induced $\widetilde{L}$-module

$$\widetilde{M}(V) = \widetilde{M}(\boldsymbol{b}_1, \boldsymbol{b}_2, V) = \operatorname{Ind}_{\widetilde{L}_+ \oplus \widetilde{L}_0}^{\widetilde{L}} V = \mathcal{U}(\widetilde{L}) \otimes_{\mathcal{U}(\widetilde{L}_+ \oplus \widetilde{L}_0)} V.$$

It is clear that, as vector spaces,

$$\widetilde{M}(\boldsymbol{b}_1, \boldsymbol{b}_2, V) \simeq \mathcal{U}(\widetilde{L}_-) \otimes_{\mathbb{C}} V.$$

The $\widetilde{L}$-module $\widetilde{M}(\boldsymbol{b}_1, \boldsymbol{b}_2, V)$ has a unique maximal submodule $J(\boldsymbol{b}_1, \boldsymbol{b}_2, V)$ trivially intersecting with V. Then we obtain the unique irreducible quotient module

$$M(V) = M(\boldsymbol{b}_1, \boldsymbol{b}_2, V) = \widetilde{M}(\boldsymbol{b}_1, \boldsymbol{b}_2, V)/J(\boldsymbol{b}_1, \boldsymbol{b}_2, V).$$

It is clear that $M(V)$ is uniquely determined by the $\mathbb{Z}$-basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ of Γ and the $\widetilde{L}_0$-module V.

Remark 2.3. The irreducible $\widetilde{L}$-module $M(\boldsymbol{b}_1, \boldsymbol{b}_2, V)$ constructed above is a generalized highest weight module corresponding to the $\mathbb{Z}$-basis $\{\boldsymbol{b}_1 + \boldsymbol{b}_2, \boldsymbol{b}_1 + 2\boldsymbol{b}_2\}$ of Γ.

We recall some results about the $\mathbb{Z}$-graded module for Heisenberg Lie algebras.

For any $0 \neq \boldsymbol{b}_1 \in \Gamma$, denote the subalgebra $\langle E(\pm k\boldsymbol{b}_1),\ f(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle$ of $\widetilde{L}$ by $E_{\boldsymbol{b}_1}$. For any $E_{\boldsymbol{b}_1}$-module V, if the eigenvalue of $f(\boldsymbol{b}_1)$ is a scalar then we call it the *level* of V. Let

$$E^{\pm}_{\boldsymbol{b}_1} = \langle E(k\boldsymbol{b}_1) \mid \pm k \in \mathbb{N}\rangle.$$

For $0 \neq a \in \mathbb{C}$, let $\mathbb{C}v_a$ be a one-dimensional $(E^{\varepsilon}_{\boldsymbol{b}_1} \oplus \mathbb{C}f(\boldsymbol{b}_1))$-module such that $E^{\varepsilon}_{\boldsymbol{b}_1}.v_a = 0,\ f(\boldsymbol{b}_1).v_a = av_a,\ \varepsilon \in \{+, -\}$. Consider the induced $E_{\boldsymbol{b}_1}$-module

$$M^{\varepsilon}(a) = \mathcal{U}(E_{\boldsymbol{b}_1}) \otimes_{\mathcal{U}(E^{\varepsilon}_{\boldsymbol{b}_1} \oplus \mathbb{C}f(\boldsymbol{b}_1))} \mathbb{C}v_a$$

associated with a and ε (a is the level of $M^{\varepsilon}(a)$). Then the $E_{\boldsymbol{b}_1}$-module $M^{\varepsilon}(a)$ is irreducible.

The following result is due to Propositions 4.3(i) and 4.5 in [Futorny 1997].

Theorem 2.4. *If $V = \bigoplus_{i \in \mathbb{Z}} V_i$ is a $\mathbb{Z}$-graded $E_{\boldsymbol{b}_1}$-module of level $0 \neq a \in \mathbb{C}$ and $\dim V_i < \infty$ for at least one $i \in \mathbb{Z}$ then*

(1) *if V is an irreducible module then $V \simeq M^{\varepsilon}(a)$ for some $\varepsilon \in \{+, -\}$;*

(2) *V is completely reducible.*

Let $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ be a $\mathbb{Z}$-basis of Γ. For a $\mathcal{H}_{\boldsymbol{b}_1}$-module V, if $f(\boldsymbol{b}_1), h(\boldsymbol{b}_1), f(\boldsymbol{b}_2), h(\boldsymbol{b}_2)$ act as scalars $c_1, c_2, c_3, c_4 \in \mathbb{C}$, then we call (c_1, c_2, c_3, c_4) the *level* of the $\mathcal{H}_{\boldsymbol{b}_1}$-module V. Furthermore if $(c_1, c_2, c_3, c_4) = (0, 0, c_3, c_4)$, we say that V is a $\mathcal{H}_{\boldsymbol{b}_1}$-module of *level zero*. Otherwise, V is nonzero level. In the following, we will discuss the irreducible $\mathcal{H}_{\boldsymbol{b}_1}$-modules. First we recall the classification of $\mathbb{Z}$-graded irreducible $\mathcal{H}_{\boldsymbol{b}_1}$-modules of level zero. Then we classify the $\mathbb{Z}$-graded $\mathcal{H}_{\boldsymbol{b}_1}$-modules of nonzero level with finite-dimensional graded subspaces.

Set $T=\mathbb{C}[t^{\pm 1}]$. Let $\rho:\mathcal{H}_{\boldsymbol{b}_1}\to\mathbb{C}$ be a linear function with $\rho(f(\boldsymbol{b}_1))=\rho(h(\boldsymbol{b}_1))=0$. We can define a $\mathcal{H}_{\boldsymbol{b}_1}$-module structure on T by

$$f(\boldsymbol{b}_1).t^n=0, \qquad E(k\boldsymbol{b}_1).t^n=\rho(E(k\boldsymbol{b}_1))t^{k+n}, \tag{2-1}$$

$$h(\boldsymbol{b}_1).t^n=0, \qquad t^{k\boldsymbol{b}_1}.t^n=\rho(t^{k\boldsymbol{b}_1})t^{k+n}, \tag{2-2}$$

$$f(\boldsymbol{b}_2).t^n=\rho(f(\boldsymbol{b}_2))t^n, \qquad h(\boldsymbol{b}_2).t^n=\rho(h(\boldsymbol{b}_2))t^n, \tag{2-3}$$

where $n\in\mathbb{Z}$, $k\in\mathbb{Z}\setminus\{0\}$. We denote

$$T_{\rho,i}(\mathcal{H}_{\boldsymbol{b}_1})=\mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1}).t^i$$

the $\mathcal{H}_{\boldsymbol{b}_1}$-submodule of T generated by t^i for $i\in\mathbb{Z}$. And we write $T_{\rho,0}(\mathcal{H}_{\boldsymbol{b}_1})$ as $T_\rho(\mathcal{H}_{\boldsymbol{b}_1})$ for short. From the definition, we see that

$$T_{\rho,i}(\mathcal{H}_{\boldsymbol{b}_1})\simeq T_{\rho,j}(\mathcal{H}_{\boldsymbol{b}_1}) \tag{2-4}$$

for $i,j\in\mathbb{Z}$ as $\mathcal{H}_{\boldsymbol{b}_1}$-modules.

Remark 2.5. For linear function $\rho:E_{\boldsymbol{b}_1}\to\mathbb{C}$ with $\rho(f(\boldsymbol{b}_1))=0$, we can define a $E_{\boldsymbol{b}_1}$-module structure on the Laurent polynomial ring T with the action given by (2-1). Similarly, let $T_{\rho,i}(E_{\boldsymbol{b}_1}):=\mathcal{U}(E_{\boldsymbol{b}_1}).t^i$ be the $E_{\boldsymbol{b}_1}$-submodule of T generated by t^i for $i\in\mathbb{Z}$. And we also write $T_{\rho,0}(E_{\boldsymbol{b}_1})$ as $T_\rho(E_{\boldsymbol{b}_1})$ for short.

Then we have the following results from Lemma 3.6 and Proposition 3.8 in [Chari 1986].

Proposition 2.6. (1) *The $\mathcal{H}_{\boldsymbol{b}_1}$-module $T_\rho(\mathcal{H}_{\boldsymbol{b}_1})$ (resp. $E_{\boldsymbol{b}_1}$-module $T_\rho(E_{\boldsymbol{b}_1})$) is irreducible if and only if $T_\rho(\mathcal{H}_{\boldsymbol{b}_1})=T_r$ (resp. $T_\rho(E_{\boldsymbol{b}_1})=T_r$) for some $r\in\mathbb{Z}_+$, where $T_0=\mathbb{C}1$ and $T_r=\mathbb{C}[t^r,t^{-r}]$ if $r\in\mathbb{N}$.*

(2) *If V is a $\mathbb{Z}$-graded irreducible $\mathcal{H}_{\boldsymbol{b}_1}$-module (resp. $E_{\boldsymbol{b}_1}$-module) of level zero, then $V\simeq T_\rho(\mathcal{H}_{\boldsymbol{b}_1})$ for some linear function $\rho:\mathcal{H}_{\boldsymbol{b}_1}\to\mathbb{C}$ with $\rho(f(\boldsymbol{b}_1))=\rho(h(\boldsymbol{b}_1))=0$ (resp. $V\simeq T_\rho(E_{\boldsymbol{b}_1})$ for some linear function $\rho:E_{\boldsymbol{b}_1}\to\mathbb{C}$ with $\rho(f(\boldsymbol{b}_1))=0$), and $T_\rho(\mathcal{H}_{\boldsymbol{b}_1})=T_r$ (resp. $T_\rho(E_{\boldsymbol{b}_1})=T_r$) for some $r\in\mathbb{Z}_+$.*

Remark 2.7. Since $\langle t^{k\boldsymbol{b}_1},E(-k\boldsymbol{b}_1),h(\boldsymbol{b}_1)\,|\,k\in\mathbb{N}\rangle$ and $\langle t^{-k\boldsymbol{b}_1},E(k\boldsymbol{b}_1),h(\boldsymbol{b}_1)\,|\,k\in\mathbb{N}\rangle$ are two Heisenberg Lie subalgebras of $\widetilde{L}$, Theorem 2.4 and Proposition 2.6 also hold for their corresponding $\mathbb{Z}$-graded irreducible modules.

For convenience, we let $\mathcal{E}_{\boldsymbol{b}_1}$ denote the set of all linear functions $\rho:\mathcal{H}_{\boldsymbol{b}_1}\to\mathbb{C}$ with $\rho(f(\boldsymbol{b}_1))=\rho(h(\boldsymbol{b}_1))=0$ such that the $\mathcal{H}_{\boldsymbol{b}_1}$-module $T_\rho(\mathcal{H}_{\boldsymbol{b}_1})$ is irreducible. Let $t_{\boldsymbol{b}_1}=\langle t^{\pm k\boldsymbol{b}_1}\mid k\in\mathbb{N}\rangle$. Note that $t_{\boldsymbol{b}_1}$ is a centerless Heisenberg subalgebra of $\widetilde{L}$. Let $T_\rho(t_{\boldsymbol{b}_1})$ be the submodule of T generated by 1, where ρ is a linear function $\rho:t_{\boldsymbol{b}_1}\to\mathbb{C}$. The structure of the $t_{\boldsymbol{b}_1}$-module T is defined in a way similar to that of $E_{\boldsymbol{b}_1}$. In the following proposition we classify the $\mathbb{Z}$-graded irreducible $\mathcal{H}_{\boldsymbol{b}_1}$-modules with nonzero level.

Proposition 2.8. *Let $V = \bigoplus_{i\in\mathbb{Z}} V_i$ be a $\mathbb{Z}$-graded irreducible $\mathcal{H}_{\boldsymbol{b}_1}$-module with $\dim V_i < \infty$ for all $i \in \mathbb{Z}$. Suppose $f(\boldsymbol{b}_1).v = c_1 v,\ h(\boldsymbol{b}_1).v = c_2 v,\ f(\boldsymbol{b}_2).v = c_3 v$ and $h(\boldsymbol{b}_2).v = c_4 v$ for $v \in V$, where $c_1, c_2, c_3, c_4 \in \mathbb{C}$ and $(c_1, c_2) \neq \mathbf{0}$.*

(1) *If $c_1 \neq 0$ and $c_2 \neq 0$, then*

$$V \simeq \mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1}) \otimes_{\mathcal{U}(\langle E(k\boldsymbol{b}_1),\ t^{k\boldsymbol{b}_1},\ f(\boldsymbol{b}_i),\ h(\boldsymbol{b}_i) | k\in\mathbb{N},\ i=1,2\rangle)} \mathbb{C}1,$$

where $\langle E(k\boldsymbol{b}_1),\ t^{k\boldsymbol{b}_1} \mid k \in \mathbb{N}\rangle.1 = 0,\ f(\boldsymbol{b}_1).1 = c_1 1,\ h(\boldsymbol{b}_1).1 = c_2 1,\ f(\boldsymbol{b}_2).1 = c_3 1$ and $h(\boldsymbol{b}_2).1 = c_4 1$ or

$$V \simeq \mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1}) \otimes_{\mathcal{U}(\langle E(-k\boldsymbol{b}_1),\ t^{-k\boldsymbol{b}_1},\ f(\boldsymbol{b}_i),\ h(\boldsymbol{b}_i) | k\in\mathbb{N},\ i=1,2\rangle)} \mathbb{C}1,$$

where $\langle E(-k\boldsymbol{b}_1),\ t^{-k\boldsymbol{b}_1} \mid k \in \mathbb{N}\rangle.1 = 0,\ f(\boldsymbol{b}_1).1 = c_1 1,\ h(\boldsymbol{b}_1).1 = c_2 1,\ f(\boldsymbol{b}_2).1 = c_3 1$ and $h(\boldsymbol{b}_2).1 = c_4 1$.

(2) *If $c_1 \neq 0$ and $c_2 = 0$, then*

$$V \simeq T_\rho(t_{\boldsymbol{b}_1}) \otimes M^\varepsilon(c_1),$$

for some linear function $\rho : t_{\boldsymbol{b}_1} \to \mathbb{C}$ such that $T_\rho(t_{\boldsymbol{b}_1}) = T_r$ for some $r \in \mathbb{Z}_+$, where $M^\varepsilon(c_1)$ is the irreducible $E_{\boldsymbol{b}_1}$-module of level c_1, $\varepsilon \in \{+, -\}$.

(3) *If $c_1 = 0$ and $c_2 \neq 0$, then*

$$V \simeq \mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1}) \otimes_{\mathcal{U}(\langle E(k\boldsymbol{b}_1),\ t^{k\boldsymbol{b}_1},\ f(\boldsymbol{b}_i),\ h(\boldsymbol{b}_i) | k\in\mathbb{N},\ i=1,2\rangle)} \mathbb{C}1,$$

where $\langle E(k\boldsymbol{b}_1),\ t^{k\boldsymbol{b}_1} \mid k \in \mathbb{N}\rangle.1 = 0,\ f(\boldsymbol{b}_1).1 = 0,\ h(\boldsymbol{b}_1).1 = c_2 1,\ f(\boldsymbol{b}_2).1 = c_3 1$ and $h(\boldsymbol{b}_2).1 = c_4 1$ or

$$V \simeq \mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1}) \otimes_{\mathcal{U}(\langle E(-k\boldsymbol{b}_1),\ t^{-k\boldsymbol{b}_1},\ f(\boldsymbol{b}_i),\ h(\boldsymbol{b}_i) | k\in\mathbb{N},\ i=1,2\rangle)} \mathbb{C}1,$$

where $\langle E(-k\boldsymbol{b}_1),\ t^{-k\boldsymbol{b}_1} \mid k \in \mathbb{N}\rangle.1 = 0,\ f(\boldsymbol{b}_1).1 = 0,\ h(\boldsymbol{b}_1).1 = c_2 1,\ f(\boldsymbol{b}_2).1 = c_3 1$ and $h(\boldsymbol{b}_2).1 = c_4 1$.

Proof. (1) If $c_1 \neq 0$ and $c_2 \neq 0$, by Theorem 2.4 we know that there exists some $0 \neq v_0 \in V_{i_0}$ for some $i_0 \in \mathbb{Z}$ such that $E(k\boldsymbol{b}_1).v_0 = 0$ for any $k \in \mathbb{N}$ or $-k \in \mathbb{N}$. Without loss of generality, we assume $k \in \mathbb{N}$; then $\mathcal{U}(\langle E(-k\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle)v_0$ is an irreducible $E_{\boldsymbol{b}_1}$-module. Let

$$W := \mathcal{U}(\langle t^{k\boldsymbol{b}_1},\ E(l\boldsymbol{b}_1),\ f(\boldsymbol{b}_1),\ h(\boldsymbol{b}_1) \mid k \in \mathbb{N},\ l \in \mathbb{Z}\setminus\{0\}\rangle)v_0 \subseteq V.$$

Note that W as a $\mathbb{Z}$-graded $\langle t^{k\boldsymbol{b}_1},\ E(-k\boldsymbol{b}_1),\ h(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle$-module is completely reducible. Then we have that

$$W = \Big(\bigoplus_{i\in I}\Big(\bigoplus_{m_i\in X_i} V^+_{i,m_i}\Big)\Big) \oplus \Big(\bigoplus_{j\in J}\Big(\bigoplus_{n_j\in Y_j} V^-_{j,n_j}\Big)\Big),$$

where

$$V^+_{i,m_i} = \mathcal{U}(\langle t^{k\boldsymbol{b}_1},\ E(-k\boldsymbol{b}_1),\ h(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle)v_{i,m_i} \simeq M^+(c_2),$$

for some $0 \neq v_{i,m_i} \in V_i \cap W$ with $t^{k\boldsymbol{b}_1}.v_{i,m_i} = 0$ for all $k \in \mathbb{N}$, $i \in I$, $m_i \in X_i$, and

$$V^-_{j,n_j} = \mathcal{U}(\langle t^{k\boldsymbol{b}_1},\ E(-k\boldsymbol{b}_1),\ h(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle)u_{j,n_j} \simeq M^-(c_2),$$

where $0 \neq u_{j,n_j} \in V_j \cap W$ with $E(-k\boldsymbol{b}_1).u_{j,n_j} = 0$ for all $k \in \mathbb{N}$, $j \in J$, $n_j \in Y_j$, $I, J, X_i, Y_j \subseteq \mathbb{Z}$. Note that I has an upper bound, J has a lower bound and all X_i, Y_j are finite sets since $\dim V_n < \infty$ for all $n \in \mathbb{Z}$. Assume $J \neq \varnothing$; then there exists some nonzero vector $w_0 \in W \cap V_i$ such that $E(-k\boldsymbol{b}_1).w_0 = 0$ for all $k \in \mathbb{N}$ and some $i \in \mathbb{Z}$. Consider $W_0 = \mathcal{U}(E_{\boldsymbol{b}_1})w_0 \subseteq W$, then

$$W_0 = \mathcal{U}(\langle E(l\boldsymbol{b}_1) \mid l \in \mathbb{N}\rangle).w_0$$

and W_0 is a free $\mathcal{U}(\langle E(l\boldsymbol{b}_1) \mid l \in \mathbb{N}\rangle)$-module. On the other hand, since $w_0 \in W$, there exists $k \in \mathbb{N}$ such that $E(k\boldsymbol{b}_1).w_0 = 0$, which is a contradiction. Thus $J = \varnothing$ and $W = \bigoplus_{i\in I}\big(\bigoplus_{m_i \in X_i} V^+_{i,m_i}\big)$. Since I has an upper bound, there exists $0 \neq u_0 \in W \cap V_{i_1}$ for some $i_1 \in \mathbb{Z}$ such that $E(k\boldsymbol{b}_1).u_0 = t^{k\boldsymbol{b}_1}.u_0 = 0$ for all $k \in \mathbb{N}$. This shows that

$$V \simeq \mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1}) \otimes_{\mathcal{U}(\langle E(k\boldsymbol{b}_1),\ t^{k\boldsymbol{b}_1},\ f(\boldsymbol{b}_i),\ h(\boldsymbol{b}_i) \mid k \in \mathbb{N},\ i=1,2\rangle)} \mathbb{C}u_0.$$

Another case is similar.

(2) If $c_1 \neq 0$ and $c_2 = 0$, we can write

$$\mathcal{H}_{\boldsymbol{b}_1} = t_{\boldsymbol{b}_1} \oplus E_{\boldsymbol{b}_1} \oplus \mathbb{C}f(\boldsymbol{b}_2) \oplus \mathbb{C}h(\boldsymbol{b}_1) \oplus \mathbb{C}h(\boldsymbol{b}_2).$$

From Theorem 2.4, Proposition 2.6 and [Li 2004, Lemma 2.7], this result follows.

(3) If $c_1 = 0$ and $c_2 \neq 0$, by Theorem 2.4, V is completely reducible when we view V as a module of the two subalgebras $\langle t^{-k\boldsymbol{b}_1},\ E(k\boldsymbol{b}_1),\ h(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle$ and $\langle t^{k\boldsymbol{b}_1},\ E(-k\boldsymbol{b}_1),\ h(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle$. We write

$$V = \Big(\bigoplus_{i\in I}\Big(\bigoplus_{m_i \in X_i} V^+_{i,m_i}\Big)\Big) \oplus \Big(\bigoplus_{j\in J}\Big(\bigoplus_{n_j \in Y_j} V^-_{j,n_j}\Big)\Big)$$

when it is viewed as the module of the Lie algebra $\langle t^{-k\boldsymbol{b}_1},\ E(k\boldsymbol{b}_1),\ h(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle$, where $I, J, X_i, Y_j \subseteq \mathbb{Z}$,

$$V^+_{i,m_i} = \mathcal{U}(\langle t^{-k\boldsymbol{b}_1},\ E(k\boldsymbol{b}_1),\ h(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle)v_{i,m_i} \simeq M^+(c_2)$$

with $E(k\boldsymbol{b}_1).v_{i,m_i} = 0$ for all $k \in \mathbb{N}$, $i \in I$, $m_i \in X_i$, $0 \neq v_{i,m_i} \in V_i$ and

$$V^-_{j,n_j} = \mathcal{U}(\langle t^{-k\boldsymbol{b}_1},\ E(k\boldsymbol{b}_1),\ h(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle)u_{j,n_j} \simeq M^-(c_2)$$

with $t^{-k\boldsymbol{b}_1}.u_{j,n_j} = 0$ for all $k \in \mathbb{N}$, $j \in J$, $n_j \in Y_j$ and $0 \neq u_{j,n_j} \in V_j$. Similarly, we write

$$V = \Big(\bigoplus_{i\in I'}\Big(\bigoplus_{p_i \in X'_i} W^+_{i,p_i}\Big)\Big) \oplus \Big(\bigoplus_{j\in J'}\Big(\bigoplus_{q_j \in Y'_j} W^-_{j,q_j}\Big)\Big)$$

when it is viewed as the module of the Lie algebra $\langle t^{k\boldsymbol{b}_1},\ E(-k\boldsymbol{b}_1),\ h(\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle$.

Note that both I and I' have upper bounds, J and J' have lower bounds and all X_i, Y_j, X_i', Y_j' are finite sets as $\dim V_n < \infty$ for all $n \in \mathbb{Z}$. If $I = \varnothing$, similar to the proof in (1), we get

$$V \simeq \mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1}) \otimes_{\mathcal{U}(\langle E(k\boldsymbol{b}_1),\ t^{k\boldsymbol{b}_1},\ f(\boldsymbol{b}_i),\ h(\boldsymbol{b}_i) | k\in\mathbb{N},\ i=1,2\rangle)} \mathbb{C}1,$$

where $\langle E(k\boldsymbol{b}_1),\ t^{k\boldsymbol{b}_1} \mid k \in \mathbb{N}\rangle.1 = 0$, $f(\boldsymbol{b}_1).1 = 0$, $h(\boldsymbol{b}_1).1 = c_2 1$, $f(\boldsymbol{b}_2).1 = c_3 1$ and $h(\boldsymbol{b}_2).1 = c_4 1$. Now suppose $I \neq \varnothing$. We can choose $0 \neq v_0 \in V^+_{i_0, m_{i_0}}$ for some $i_0 \in I$, $m_{i_0} \in X_{i_0}$ such that $E(k\boldsymbol{b}_1).v_0 = 0$ for all $k \in \mathbb{N}$. Then we have $v_0 = w_1 + w_2$, where

$$w_1 \in \Big(\bigoplus_{i\in I'}\Big(\bigoplus_{p_i \in X_i'} W^+_{i,p_i}\Big)\Big) \cap V_{i_0} \quad \text{and} \quad w_2 \in \Big(\bigoplus_{j\in J'}\Big(\bigoplus_{q_j \in Y_j'} W^-_{j,q_j}\Big)\Big) \cap V_{i_0}.$$

If $w_1 \neq 0$, we can choose large enough $k_0 \in \mathbb{N}$ such that $E(-k_0\boldsymbol{b}_1).w_2 = 0$ since J' has a lower bound. Since $\bigoplus_{i\in I'}\big(\bigoplus_{p_i\in X_i'} W^+_{i,p_i}\big)$ is a free $\langle E(-k\boldsymbol{b}_1) \mid k \in \mathbb{N}\rangle$-module, we have $0 \neq E(-k_0\boldsymbol{b}_1).w_1 = E(-k_0\boldsymbol{b}_1).v_0 \in \bigoplus_{i\in I'}\big(\bigoplus_{p_i\in X_i'} W^+_{i,p_i}\big)$ and $E(k\boldsymbol{b}_1).E(-k_0\boldsymbol{b}_1).v_0 = E(-k_0\boldsymbol{b}_1).E(k\boldsymbol{b}_1).v_0 = 0$ for all $k \in \mathbb{N}$. Now we claim that there exists $0 \neq v \in V_i$ for some $i \in \mathbb{Z}$ such that $t^{k\boldsymbol{b}_1}.v = E(k\boldsymbol{b}_1).v = 0$ for all $k \in \mathbb{N}$. In fact, if $t^{k\boldsymbol{b}_1}.(E(-k_0\boldsymbol{b}_1).v_0) = 0$ for all $k \in \mathbb{N}$, this is done by setting $v = E(-k_0\boldsymbol{b}_1).v_0$. If there exists $k_1 \in \mathbb{N}$ such that $t^{k_1\boldsymbol{b}_1}.(E(-k_0\boldsymbol{b}_1).v_0) \neq 0$, set $v_1 = t^{k_1\boldsymbol{b}_1}.(E(-k_0\boldsymbol{b}_1).v_0)$. We can repeat this process, and, since I' has an upper bound, we know that it will terminate after finitely many steps. This implies that

$$V \simeq \mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1}) \otimes_{\mathcal{U}(\langle E(k\boldsymbol{b}_1),\ t^{k\boldsymbol{b}_1},\ f(\boldsymbol{b}_i),\ h(\boldsymbol{b}_i) | k\in\mathbb{N},\ i=1,2\rangle)} \mathbb{C}1,$$

where $\langle E(k\boldsymbol{b}_1),\ t^{k\boldsymbol{b}_1} \mid k \in \mathbb{N}\rangle.1 = 0$, $f(\boldsymbol{b}_1).1 = 0$, $h(\boldsymbol{b}_1).1 = c_2 1$, $f(\boldsymbol{b}_2).1 = c_3 1$ and $h(\boldsymbol{b}_2).1 = c_4 1$. If $w_1 = 0$, i.e., $v_0 = w_2$, we know that there exists some $0 \neq u \in V_{j_0}$ for some $j_0 \in \mathbb{Z}$ such that $E(k\boldsymbol{b}_1).u = 0$ for $k \in \mathbb{Z}\setminus\{0\}$. In fact, if there exists $n_1 \in \mathbb{N}$ such that $E(-n_1\boldsymbol{b}_1)v_0 \neq 0$, set $u_1 = E(-n_1\boldsymbol{b}_1)v_0$. We also have $E(k\boldsymbol{b}_1).u_1 = 0$ for all $k \in \mathbb{N}$ since $f(\boldsymbol{b}_1).V = 0$. We can repeat this process, and, since J' has a lower bound, we know that it will terminate after finitely many steps. Then,

$$V \simeq \mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1}) \otimes_{\mathcal{U}(\langle E(m\boldsymbol{b}_1),\ f(\boldsymbol{b}_i),\ h(\boldsymbol{b}_i) | m\in\mathbb{Z}\setminus\{0\},\ i=1,2\rangle)} \mathbb{C}1,$$

where $E(m\boldsymbol{b}_1).1 = 0$ for all $m \in \mathbb{Z}\setminus\{0\}$, $f(\boldsymbol{b}_1).1 = 0$, $h(\boldsymbol{b}_1).1 = c_2 1$, $f(\boldsymbol{b}_2).1 = c_3 1$ and $h(\boldsymbol{b}_2).1 = c_4 1$. This contradicts the condition that $\dim V_i < \infty$ for all $i \in \mathbb{Z}$. Then the conclusion follows. □

Fix a $\mathbb{Z}$-basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ of Γ, $\boldsymbol{b}_1 = b_{11}\boldsymbol{e}_1 + b_{12}\boldsymbol{e}_2$, and $\lambda_1, \lambda_2 \in \mathbb{C}$. Any $\mathbb{Z}$-graded $\mathcal{H}_{\boldsymbol{b}_1}$-module $V = \oplus_{i\in\mathbb{Z}} V_i$ with fixed level can be extended to a weight module of $\widetilde{L}_0$ by defining

$$d_1 v_j = (\lambda_1 + j b_{11}) v_j, \quad d_2 v_j = (\lambda_2 + j b_{12}) v_j,$$

for $v_j \in V_j$, $j \in \mathbb{Z}$. One can easily see that the vector space V is a weight $\widetilde{L}_0$-module and $\mathcal{P}(V) \subseteq (\lambda_1, \lambda_2) + \mathbb{Z}\boldsymbol{b}_1$. For the $\mathbb{Z}$-graded irreducible $\mathcal{H}_{\boldsymbol{b}_1}$-modules given in Propositions 2.6 and 2.8, we let

$$V^+(\boldsymbol{c}) = \mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1}) \otimes_{\mathcal{U}(\langle E(k\boldsymbol{b}_1),\ t^{k\boldsymbol{b}_1},\ K_i | k\in\mathbb{N},\ i=1,2,3,4\rangle)} \mathbb{C}1,$$
$$V^-(\boldsymbol{c}) = \mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1}) \otimes_{\mathcal{U}(\langle E(-k\boldsymbol{b}_1),\ t^{-k\boldsymbol{b}_1},\ K_i | k\in\mathbb{N},\ i=1,2,3,4\rangle)} \mathbb{C}1,$$

$M_\rho^\varepsilon(\boldsymbol{c}) = T_\rho(t_{\boldsymbol{b}_1}) \otimes M^\varepsilon(c_1)$ and $T_\rho(\mathcal{H}_{\boldsymbol{b}_1})(\boldsymbol{c}) = T_\rho(\mathcal{H}_{\boldsymbol{b}_1})$. We can extend these modules to weight $\widetilde{L}_0$-modules by the above method, and then we denote the corresponding $\widetilde{L}_0$-module by $V^+(\boldsymbol{c}, \boldsymbol{\lambda})$, $V^-(\boldsymbol{c}, \boldsymbol{\lambda})$, $M_\rho^\varepsilon(\boldsymbol{c}, \boldsymbol{\lambda})$ and $T_\rho(\mathcal{H}_{\boldsymbol{b}_1})(\boldsymbol{c}, \boldsymbol{\lambda})$ respectively, where $\boldsymbol{c} = (c_1, c_2, c_3, c_4)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$, and $f(\boldsymbol{b}_1)$, $h(\boldsymbol{b}_1)$, $f(\boldsymbol{b}_2)$ and $h(\boldsymbol{b}_2)$ act as the scalars $c_1, c_2, c_3, c_4 \in \mathbb{C}$, respectively.

With this notation, the following results can be obtained from Propositions 2.6 and 2.8.

Corollary 2.9. *Let $V = \bigoplus_{i\in\mathbb{Z}} V_i$ be any irreducible weight module of $\widetilde{L}_0$ with $\dim V_i < \infty$ for all $i \in \mathbb{Z}$, and $f(\boldsymbol{b}_1).v = c_1 v$, $h(\boldsymbol{b}_1).v = c_2 v$, $f(\boldsymbol{b}_2).v = c_3 v$ and $h(\boldsymbol{b}_2).v = c_4 v$ for $v \in V$, where $V_i := V_{(\lambda_1,\lambda_2)+i\boldsymbol{b}_1}$ for some fixed $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{C}^2$.*

(1) *If $(c_1, c_2) \neq 0$, then $V \simeq V^\varepsilon(\boldsymbol{c}, \boldsymbol{\lambda})$ or $V \simeq M_\rho^\varepsilon(\boldsymbol{c}, \boldsymbol{\lambda})$ for some linear function $\rho : t_{\boldsymbol{b}_1} \to \mathbb{C}$ with $T_\rho(t_{\boldsymbol{b}_1}) = T_r$ for some $r \in \mathbb{Z}_+$ and $\varepsilon \in \{+, -\}$.*

(2) *If $(c_1, c_2) = 0$, then $V \simeq T_\rho(\mathcal{H}_{\boldsymbol{b}_1})(\boldsymbol{c}, \boldsymbol{\lambda})$ for some $\rho \in \mathcal{E}_{\boldsymbol{b}_1}$.*

The following lemma give the characterization of the irreducible weight modules of $\widetilde{L}$ with finite-dimensional weight spaces.

Lemma 2.10. *Let $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ be a $\mathbb{Z}$-basis of Γ. V is an irreducible weight module of $\widetilde{L}$ with finite-dimensional weight spaces and $f(\boldsymbol{b}_1), h(\boldsymbol{b}_1), f(\boldsymbol{b}_2), h(\boldsymbol{b}_2)$ act on V as scalars c_1, c_2, c_3, c_4 respectively. If there exist $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $V_{(\lambda_1,\lambda_2)} \neq 0$ and $\mathcal{P}(V) \cap ((\lambda_1, \lambda_2) + \mathbb{Z}\boldsymbol{b}_1 + \mathbb{N}\boldsymbol{b}_2) = \varnothing$, we have*:

(1) *If $c_1 = c_2 = 0$, $V \simeq M(\boldsymbol{b}_1, \boldsymbol{b}_2, T_\rho(\mathcal{H}_{\boldsymbol{b}_1})(\boldsymbol{c}, \boldsymbol{\lambda}))$ for some $\rho \in \mathcal{E}_{\boldsymbol{b}_1}$.*

(2) *If $c_1 \neq 0, c_2 = 0$, $V \simeq M(\boldsymbol{b}_1, \boldsymbol{b}_2, M_\rho^\varepsilon(\boldsymbol{c}, \boldsymbol{\lambda}))$ for some linear function $\rho : t_{\boldsymbol{b}_1} \to \mathbb{C}$ satisfying $T_\rho(t_{\boldsymbol{b}_1}) = T_r$ for some $r \in \mathbb{Z}_+$.*

(3) *If $c_2 \neq 0$, $V \simeq M(\boldsymbol{b}_1, \boldsymbol{b}_2, V^\varepsilon(\boldsymbol{c}, \boldsymbol{\lambda}))$, where $\varepsilon \in \{+, -\}$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$, $\boldsymbol{c} = (c_1, c_2, c_3, c_4)$.*

Proof. Let $W = \oplus_{i\in\mathbb{Z}} V_{(\lambda_1,\lambda_2)+i\boldsymbol{b}_1}$. Since $\mathcal{P}(V) \cap ((\lambda_1, \lambda_2) + \mathbb{Z}\boldsymbol{b}_1 + \mathbb{N}\boldsymbol{b}_2) = \varnothing$, we see that W is an irreducible $\widetilde{L}_0$ weight module and $\widetilde{L}_+ W = 0$. Thus by the construction of $\widetilde{M}(\boldsymbol{b}_1, \boldsymbol{b}_2, W)$ and the PBW theorem, there exists an epimorphism φ from $\widetilde{M}(\boldsymbol{b}_1, \boldsymbol{b}_2, W)$ to V such that $\varphi\,|_W = \mathrm{id}_W$. Therefore, the lemma follows from Corollary 2.9 and the irreducibility of V. □

Using the same notation as in Lemma 2.10, the following lemma shows that the cases (2) and (3) of Lemma 2.10 don't occur.

Lemma 2.11. *For any $\mathbb{Z}$-basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ of Γ, neither $M(\boldsymbol{b}_1, \boldsymbol{b}_2, M_\rho^\varepsilon(\boldsymbol{c}, \boldsymbol{\lambda}))$ nor $M(\boldsymbol{b}_1, \boldsymbol{b}_2, V^\varepsilon(\boldsymbol{c}, \boldsymbol{\lambda}))$ is a Harish-Chandra module.*

Proof. Using the notation in Lemma 2.10, for the case that $f(\boldsymbol{b}_1)$ acts as the scalar $c_1 \neq 0$, the lemma follows from Lemma 2.6 in [Lin and Tan 2006]. So we only need to consider the case where $c_1 = 0$, $c_2 \neq 0$. Without loss of generality, we may assume that there exists a weight vector $0 \neq v_0 \in V^\varepsilon(\boldsymbol{c}, \boldsymbol{\lambda})$ such that $E(k\boldsymbol{b}_1)v_0 = t^{k\boldsymbol{b}_1} v_0 = 0$ and $E(-k\boldsymbol{b}_1)v_0 \neq 0$ and $t^{-k\boldsymbol{b}_1} v_0 \neq 0$ for all $k \in \mathbb{N}$ (see Proposition 2.8(3)). For any $n \in \mathbb{N}$, we can choose $k_j \in \mathbb{Z}$, $1 \leq j \leq n$ with $0 < k_1 < k_2 < \cdots < k_n$ such that $h(-k_j\boldsymbol{b}_1 + \boldsymbol{b}_2)v_0 \neq 0$ for $1 \leq j \leq n$. We claim that

$$\{E(k_j\boldsymbol{b}_1 - \boldsymbol{b}_2)t^{-k_j\boldsymbol{b}_1} v_0 \mid 1 \leq j \leq n\} \subseteq M(\boldsymbol{b}_1, \boldsymbol{b}_2, V^\varepsilon(\boldsymbol{c}, \boldsymbol{\lambda}))_{(\lambda_1,\lambda_2)-\boldsymbol{b}_2}$$

is a set of linear independent vectors, therefore the conclusion follows. In fact, if $\sum_{j=1}^n a_j E(k_j\boldsymbol{b}_1 - \boldsymbol{b}_2)t^{-k_j\boldsymbol{b}_1} v_0 = 0$, then

$$\begin{aligned} 0 &= t^{-k_1\boldsymbol{b}_1+\boldsymbol{b}_2} \sum_{j=1}^n a_j E(k_j\boldsymbol{b}_1 - \boldsymbol{b}_2)t^{-k_j\boldsymbol{b}_1} v_0 \\ &= a_1 h(-k_1\boldsymbol{b}_1 + \boldsymbol{b}_2)t^{-k_1\boldsymbol{b}_1} v_0 + \sum_{j=2}^n a_j \det\begin{pmatrix} k_j\boldsymbol{b}_1 - \boldsymbol{b}_2 \\ -k_1\boldsymbol{b}_1 + \boldsymbol{b}_2 \end{pmatrix} t^{(k_j-k_1)\boldsymbol{b}_1} t^{-k_j\boldsymbol{b}_1} v_0. \end{aligned}$$

Since $h(-k_1\boldsymbol{b}_1 + \boldsymbol{b}_2) \neq 0$, this implies $a_1 = 0$. Similarly, we can prove $a_2 = a_3 = \cdots = a_n = 0$. Therefore the conclusion follows. □

From Lemmas 2.10 and 2.11, we have:

Proposition 2.12. *Let $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ be a $\mathbb{Z}$-basis of Γ and let V be a Harish-Chandra module of $\widetilde{L}$. If there exist $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $\mathcal{P}(V) \cap ((\lambda_1, \lambda_2) + \mathbb{Z}\boldsymbol{b}_1 + \mathbb{N}\boldsymbol{b}_2) = \varnothing$ and $V_{(\lambda_1,\lambda_2)} \neq 0$, then $V \simeq M(\boldsymbol{b}_1, \boldsymbol{b}_2, T_\rho(\mathcal{H}_{\boldsymbol{b}_1})(\boldsymbol{c}, \boldsymbol{\lambda}))$ for some $\rho \in \mathcal{E}_{\boldsymbol{b}_1}$.*

Remark 2.13. If V is a Harish-Chandra module V of $\widetilde{L}$ satisfying the conditions in Proposition 2.12, then $\boldsymbol{c} = (0, 0, c_3, c_4)$, i.e., $f(\boldsymbol{b}_1)$, $h(\boldsymbol{b}_1)$ act trivially.

As one of the main results in this paper, we prove that a Harish-Chandra module of $\widetilde{L}$ is either a generalized highest weight module or a uniformly bounded module. First, we need the following lemma.

Lemma 2.14. *An irreducible weight $\widetilde{L}$-module V is a generalized highest weight module if there is a $\mathbb{Z}$-basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ of Γ and a weight vector $v \neq 0$ such that $E(\boldsymbol{b}_1)v = E(\boldsymbol{b}_2)v = t^{\boldsymbol{b}_1} v = 0$.*

Proof. Since there is a weight vector $v \neq 0$, such that $E(\boldsymbol{b}_1)v = E(\boldsymbol{b}_2)v = t^{\boldsymbol{b}_1} v = 0$, by induction, we have

$$E(\boldsymbol{m})v = t^{\boldsymbol{m}} v = 0$$

for $\boldsymbol{m} \in \mathbb{N}\boldsymbol{b}_1 + \mathbb{N}\boldsymbol{b}_2$. Therefore, we have

$$E(\boldsymbol{m})v = t^{\boldsymbol{m}} v = 0$$

for $\boldsymbol{m} \in \mathbb{Z}_+\boldsymbol{b}_1' + \mathbb{Z}_+\boldsymbol{b}_2'$, where $\boldsymbol{b}_1' = 2\boldsymbol{b}_1 + \boldsymbol{b}_2,\ \boldsymbol{b}_2' = 3\boldsymbol{b}_1 + \boldsymbol{b}_2 \in \Gamma$. It is obvious that $\{\boldsymbol{b}_1', \boldsymbol{b}_2'\}$ is a $\mathbb{Z}$-basis of Γ. Now we get that V is a generalized highest weight module since V is irreducible. □

Proposition 2.15. *A Harish-Chandra module V of $\widetilde{L}$ is either a generalized highest weight module or a uniformly bounded module.*

Proof. Let $(\lambda_1, \lambda_2) \in \mathcal{P}(V)$ and let $V_{\boldsymbol{b}} := V_{(\lambda_1,\lambda_2)+\boldsymbol{b}}$ for $\boldsymbol{b} \in \Gamma$. Then $V = \bigoplus_{\boldsymbol{b}\in\Gamma} V_{\boldsymbol{b}}$. If V is not a generalized highest weight module, for $\boldsymbol{m} = (m_1, m_2) \in \Gamma$, consider the linear maps $E(-m_1\boldsymbol{e}_1 + \boldsymbol{e}_2) : V_{(m_1,m_2)} \to V_{(0,m_2+1)}$, $E((1 - m_1)\boldsymbol{e}_1 + \boldsymbol{e}_2) : V_{(m_1,m_2)} \to V_{(1,m_2+1)}$ and $t^{-m_1\boldsymbol{e}_1+\boldsymbol{e}_2} : V_{(m_1,m_2)} \to V_{(0,m_2+1)}$. By Lemma 2.14, we have

$$\ker E(-m_1\boldsymbol{e}_1 + \boldsymbol{e}_2) \cap \ker E((1 - m_1)\boldsymbol{e}_1 + \boldsymbol{e}_2) \cap \ker t^{-m_1\boldsymbol{e}_1+\boldsymbol{e}_2} = 0.$$

This shows that

$$\dim V_{(m_1,m_2)} \le 2 \dim V_{(0,m_2+1)} + \dim V_{(1,m_2+1)}.$$

Now we consider the linear maps $E(-\boldsymbol{e}_1 + (1 - m_2)\boldsymbol{e}_2) : V_{(0,m_2+1)} \to V_{(-1,2)}$, $E(-\boldsymbol{e}_1 - m_2\boldsymbol{e}_2) : V_{(0,m_2+1)} \to V_{(-1,1)}$ and $t^{-\boldsymbol{e}_1-m_2\boldsymbol{e}_2} : V_{(0,m_2+1)} \to V_{(-1,1)}$. By the same reasoning, we get

$$\dim V_{(0,m_2+1)} \le 2 \dim V_{(-1,1)} + \dim V_{(-1,2)}.$$

Similarly, we have

$$\dim V_{(1,m_2+1)} \le 2 \dim V_{(0,1)} + \dim V_{(0,2)}.$$

Thus, V is a uniformly bounded module. □

3. Nonzero level Harish-Chandra modules of $\widetilde{L}$

In this section, we study the nonzero level Harish-Chandra module V of $\widetilde{L}$, which satisfies $K_i.v = c_i v$ for $v \in V$, $\boldsymbol{0} \ne (c_1, c_2, c_3, c_4) \in \mathbb{C}^4$.

We denote

$$[p, q] = \{x \mid x \in \mathbb{Z}, p \le x \le q\}$$

and similarly for $(-\infty, p]$, $[q, \infty)$ and $(-\infty, +\infty)$. First, we have:

Theorem 3.1. *If V is a nonzero level Harish-Chandra module of $\widetilde{L}$, then V is a generalized highest weight module.*

Proof. Without loss of generality, we may assume the center element K_1 acts as $0 \ne c_1 \in \mathbb{C}$. Let $(\lambda_1, \lambda_2) \in \mathcal{P}(V)$. Set $W_0 := \oplus_{i\in\mathbb{Z}} V_{(\lambda_1,\lambda_2)+i\boldsymbol{e}_1} \ne 0$. From Theorem 2.4, we see that W_0 as a $\langle E(k\boldsymbol{e}_1), t^{-k\boldsymbol{e}_1}, K_1 \mid k \in \mathbb{N}\rangle$-module is completely reducible. Also from Theorem 2.4, we know that V is not a uniformly bounded module. Thus V is a generalized highest weight module. □

Corollary 3.2. *If V is a uniformly bounded Harish-Chandra module of $\widetilde{L}$, then $K_i.v = 0$ for $v \in V$, $i = 1, 2, 3, 4$.*

We assume that $V = \bigoplus_{\boldsymbol{n}\in\Gamma} V_{\boldsymbol{\lambda}+\boldsymbol{n}}$ is a nontrivial generalized highest weight Harish-Chandra $\widetilde{L}$-module with generalized highest weight $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ corresponding to a $\mathbb{Z}$-basis $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ of Γ. Without loss of generality, we assume $\boldsymbol{\lambda} = \boldsymbol{0}$.

Lemma 3.3. (1) *For any $v \in V$, there exists $p > 0$ such that $E(i\boldsymbol{b}_1 + j\boldsymbol{b}_2)v = t^{i\boldsymbol{b}_1+j\boldsymbol{b}_2}v = 0$ for all $(i, j) \geq (p, p)$.*

(2) *For any $0 \neq v \in V$, $(m_1, m_2) > \boldsymbol{0}$, we have $E(-m_1\boldsymbol{b}_1 - m_2\boldsymbol{b}_2)v \neq 0$.*

(3) *If $\boldsymbol{b} := i_1\boldsymbol{b}_1 + i_2\boldsymbol{b}_2 \in \mathcal{P}(V)$, then for any $(m_1, m_2) > \boldsymbol{0}$, there exists $m \geq 0$ such that $\{x \in \mathbb{Z} \mid \boldsymbol{b} + x\boldsymbol{a} \in \mathcal{P}(V)\} = (-\infty, m]$, where $\boldsymbol{a} = m_1\boldsymbol{b}_1 + m_2\boldsymbol{b}_2$.*

Proof. Let v_0 be the generalized highest weight vector of V corresponding to the $\mathbb{Z}$-basis B.

(1) Since $v = uv_0$ for some $u \in \mathcal{U}(\widetilde{L})$, u can be written as a linear combination of elements of the form $u_{m,n} = t^{i_1\boldsymbol{b}_1+j_1\boldsymbol{b}_2} \cdots t^{i_m\boldsymbol{b}_1+j_m\boldsymbol{b}_2} E(k_1\boldsymbol{b}_1+l_1\boldsymbol{b}_2) \cdots E(k_n\boldsymbol{b}_1+l_n\boldsymbol{b}_2)$. Without loss of generality, we may assume $u = u_{m,n}$. Take

$$p_1 = -\Sigma_{i_s<0} i_s - \Sigma_{k_t<0} k_t + 1, \quad p_2 = -\Sigma_{j_s<0} j_s - \Sigma_{l_t<0} l_t + 1.$$

Fix $m \in \mathbb{Z}_+$. By induction on n, one gets $E(i\boldsymbol{b}_1 + j\boldsymbol{b}_2)v = t^{i\boldsymbol{b}_1+j\boldsymbol{b}_2}v = 0$ for all $(i, j) \geq (p_1, p_2)$. Take $p = \max\{p_1, p_2\}$. Then the result follows.

(2) Suppose $E(-m_1\boldsymbol{b}_1 - m_2\boldsymbol{b}_2)v = 0$ for some $0 \neq v \in V$ and some $(m_1, m_2) > \boldsymbol{0}$. Let p be as in the proof of (1). Then one gets

$$E(-m_1\boldsymbol{b}_1 - m_2\boldsymbol{b}_2)v = E(\boldsymbol{b}_1 + p(m_1\boldsymbol{b}_1 + m_2\boldsymbol{b}_2))v = E(\boldsymbol{b}_2 + p(m_1\boldsymbol{b}_1 + m_2\boldsymbol{b}_2))v = 0$$
$$t^{\boldsymbol{b}_1+p(m_1\boldsymbol{b}_1+m_2\boldsymbol{b}_2)}v = t^{\boldsymbol{b}_2+p(m_1\boldsymbol{b}_1+m_2\boldsymbol{b}_2)}v = 0.$$

Note that the Lie algebra L is generated by these elements, so we have $Lv = 0$, which contradicts V being a nontrivial irreducible module.

(3) See Lemma 3.2 in [Lin and Tan 2006]. □

The following lemma follows from Lemma 3.3 and the proof is given in [Lin and Tan 2006].

Lemma 3.4. *There exists a $\mathbb{Z}$-basis $B' = \{\boldsymbol{b}'_1, \boldsymbol{b}'_2\}$ of Γ such that:*

(1) *V is a generalized highest weight module with generalized highest weight $\boldsymbol{0}$ corresponding to the $\mathbb{Z}$-basis B'.*

(2) *$\{\mathbb{Z}_+\boldsymbol{b}'_1 + \mathbb{Z}_+\boldsymbol{b}'_2\} \cap \mathcal{P}(V) = \boldsymbol{0}$.*

(3) *$\{-\mathbb{Z}_+\boldsymbol{b}'_1 - \mathbb{Z}_+\boldsymbol{b}'_2\} \subseteq \mathcal{P}(V)$.*

(4) *If $i_1\boldsymbol{b}'_1 + i_2\boldsymbol{b}'_2 \notin \mathcal{P}(V)$, then $k_1\boldsymbol{b}'_1 + k_2\boldsymbol{b}'_2 \notin \mathcal{P}(V)$ for $(k_1, k_2) \geq (i_1, i_2)$.*

(5) *If* $i_1\boldsymbol{b}'_1 + i_2\boldsymbol{b}'_2 \in \mathcal{P}(V)$, *then* $k_1\boldsymbol{b}'_1 + k_2\boldsymbol{b}'_2 \in \mathcal{P}(V)$ *for* $(k_1, k_2) \le (i_1, i_2)$.

(6) *For any* $\mathbf{0} \ne (k_1, k_2) \ge \mathbf{0}$, $(i_1, i_2) \in \Gamma$, *we have*

$$\{x \in \mathbb{Z} \mid i_1\boldsymbol{b}'_1 + i_2\boldsymbol{b}'_2 + x(k_1\boldsymbol{b}'_1 + k_2\boldsymbol{b}'_2) \in \mathcal{P}(V)\} = (-\infty, m]$$

for some $m \in \mathbb{Z}$.

From now on, we assume that V is a nontrivial generalized highest weight Harish-Chandra module with generalized highest weight $\mathbf{0}$ corresponding to the $\mathbb{Z}$-basis $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ and B satisfies the properties in Lemma 3.4. To characterize the nontrivial generalized highest weight Harish-Chandra module V of $\widetilde{L}$, we need the following lemmas due to [Lin and Tan 2006] (cf. [Lu and Zhao 2006; Su 2003]).

Lemma 3.5. *If there exist an integer* $s > 0$ *and* $(i_1, i_2), (k_1, k_2) \in \Gamma$ *such that* k_1, k_2 *are coprime*, *and*

$$\{i_1\boldsymbol{b}_1 + i_2\boldsymbol{b}_2 + x_1 s\boldsymbol{b}_1 + x_2 s\boldsymbol{b}_2 \mid (x_1, x_2) \in \Gamma, k_1x_1 + k_2x_2 = 0\} \cap \mathcal{P}(V) = \varnothing,$$

then $V \simeq M(\boldsymbol{b}'_1, \boldsymbol{b}'_2, T_\rho(\mathcal{H}_{\boldsymbol{b}'_1})(\boldsymbol{c}, \boldsymbol{\lambda}))$ *for some* $\mathbb{Z}$*-basis* $\{\boldsymbol{b}'_1, \boldsymbol{b}'_2\}$ *of* Γ *and some* $\rho \in \mathcal{E}_{\boldsymbol{b}'_1}$, *where* $f(\boldsymbol{b}'_1), h(\boldsymbol{b}'_1), f(\boldsymbol{b}'_2), h(\boldsymbol{b}'_2)$ *act as scalars* $c_1 = 0, c_2 = 0, c_3, c_4$ *respectively and* $\boldsymbol{c} = (c_1, c_2, c_3, c_4)$.

Lemma 3.6. *If there exist* (i_1, i_2), $(0, 0) \ne (k_1, k_2) \in \Gamma$ *such that*

$$\{i_1\boldsymbol{b}_1 + i_2\boldsymbol{b}_2 + x(k_1\boldsymbol{b}_1 + k_2\boldsymbol{b}_2) \mid x \in \mathbb{Z}\} \cap \mathcal{P}(V) = \varnothing,$$

then $V \simeq M(\boldsymbol{b}'_1, \boldsymbol{b}'_2, T_\rho(\mathcal{H}_{\boldsymbol{b}'_1})(\boldsymbol{c}, \boldsymbol{\lambda}))$ *for some* $\mathbb{Z}$*-basis* $\{\boldsymbol{b}'_1, \boldsymbol{b}'_2\}$ *of* Γ *and some* $\rho \in \mathcal{E}_{\boldsymbol{b}'_1}$.

Lemma 3.7. *If there exist* $(0, 0) \ne (m, n) \in \Gamma$, $(i, j) \in \Gamma$, $p, q \in \mathbb{Z}$ *such that*

$$\{x \in \mathbb{Z} \mid i\boldsymbol{b}_1 + j\boldsymbol{b}_2 + x(m\boldsymbol{b}_1 + n\boldsymbol{b}_2) \in \mathcal{P}(V)\} \supseteq (-\infty, p] \cup [q, \infty),$$

then $V \simeq M(\boldsymbol{b}'_1, \boldsymbol{b}'_2, T_\rho(\mathcal{H}_{\boldsymbol{b}'_1})(\boldsymbol{c}, \boldsymbol{\lambda}))$ *for some* $\mathbb{Z}$*-basis* $\{\boldsymbol{b}'_1, \boldsymbol{b}'_2\}$ *of* Γ *and some* $\rho \in \mathcal{E}_{\boldsymbol{b}'_1}$.

Lemma 3.8. *If there exist* $(i, j), (k, l) \in \Gamma$ *and* $x_1, x_2, x_3 \in \mathbb{Z}$ *with* $x_1 < x_2 < x_3$ *such that*

$$i\boldsymbol{b}_1 + j\boldsymbol{b}_2 + x_1(k\boldsymbol{b}_1 + l\boldsymbol{b}_2) \notin \mathcal{P}(V), \tag{3-1}$$

$$i\boldsymbol{b}_1 + j\boldsymbol{b}_2 + x_2(k\boldsymbol{b}_1 + l\boldsymbol{b}_2) \in \mathcal{P}(V), \tag{3-2}$$

$$i\boldsymbol{b}_1 + j\boldsymbol{b}_2 + x_3(k\boldsymbol{b}_1 + l\boldsymbol{b}_2) \notin \mathcal{P}(V), \tag{3-3}$$

then $V \simeq M(\boldsymbol{b}'_1, \boldsymbol{b}'_2, T_\rho(\mathcal{H}_{\boldsymbol{b}'_1})(\boldsymbol{c}, \boldsymbol{\lambda}))$ *for some* $\mathbb{Z}$*-basis* $\{\boldsymbol{b}'_1, \boldsymbol{b}'_2\}$ *of* Γ *and some* $\rho \in \mathcal{E}_{\boldsymbol{b}'_1}$.

Proof. Without loss of generality, we may assume k, l are coprime. Thus we can choose $(m, n) \in \Gamma$ with $kn - lm = 1$. Let $\boldsymbol{b}'_1 = k\boldsymbol{b}_1 + l\boldsymbol{b}_2$ and let $\boldsymbol{b}'_2 = m\boldsymbol{b}_1 + n\boldsymbol{b}_2$; then $\{\boldsymbol{b}'_1, \boldsymbol{b}'_2\}$ is a $\mathbb{Z}$-basis of Γ. Replacing x_2 by the largest $x < x_3$ with $i\boldsymbol{b}_1 + j\boldsymbol{b}_2 + x(k\boldsymbol{b}_1 + l\boldsymbol{b}_2) \in \mathcal{P}(V)$, then replacing x_3 by $x_2 + 1$ and (i, j) by $(i, j) + x_2(k, l)$, we can assume

$$x_1 < x_2 = 0 < x_3 = 1. \tag{3-4}$$

We may assume that there exists $s \in \mathbb{Z}$ with

$$i\boldsymbol{b}_1 + j\boldsymbol{b}_2 + \boldsymbol{b}'_2 + s\boldsymbol{b}'_1 = (i+m)\boldsymbol{b}_1 + (j+n)\boldsymbol{b}_2 + s(k\boldsymbol{b}_1 + l\boldsymbol{b}_2) \notin \mathcal{P}(V). \tag{3-5}$$

Otherwise, by Lemma 3.7, we are done. Thus by (3-1)–(3-5), we have

$$\begin{gathered}
E(x_1\boldsymbol{b}'_1)v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} = E(x_1(k\boldsymbol{b}_1 + l\boldsymbol{b}_2))v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} = 0,\\
t^{x_1\boldsymbol{b}'_1}v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} = t^{x_1(k\boldsymbol{b}_1+l\boldsymbol{b}_2)}v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} = 0,\\
E(\boldsymbol{b}'_1)v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} = E(k\boldsymbol{b}_1 + l\boldsymbol{b}_2)v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} = 0,\\
t^{\boldsymbol{b}'_1}v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} = t^{k\boldsymbol{b}_1+l\boldsymbol{b}_2}v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} = 0,\\
E(\boldsymbol{b}'_2 + s\boldsymbol{b}'_1)v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} = 0,\\
t^{\boldsymbol{b}'_2+s\boldsymbol{b}'_1}v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} = 0,
\end{gathered}$$

where $0 \neq v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} \in V_{i\boldsymbol{b}_1+j\boldsymbol{b}_2}$. Note that since $x_1 < 0$, we have that

$$\{E(p\boldsymbol{b}'_1 + q\boldsymbol{b}'_2),\ t^{p\boldsymbol{b}'_1+q\boldsymbol{b}'_2} \mid p \in \mathbb{Z}, q \in \mathbb{N}\}$$

belongs to the subalgebra generated by

$$\{E(x_1\boldsymbol{b}'_1),\ E(\boldsymbol{b}'_1),\ E(\boldsymbol{b}'_2 + s\boldsymbol{b}'_1),\ t^{x_1\boldsymbol{b}'_1},\ t^{\boldsymbol{b}'_1},\ t^{\boldsymbol{b}'_2+s\boldsymbol{b}'_1}\}.$$

We obtain $E(p\boldsymbol{b}'_1 + q\boldsymbol{b}'_2)v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} = t^{p\boldsymbol{b}'_1+q\boldsymbol{b}'_2}v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2} = 0$ for $p \in \mathbb{Z}, q \in \mathbb{N}$. Since $\{\boldsymbol{b}'_1, \boldsymbol{b}'_2\}$ is a $\mathbb{Z}$-basis of Γ and V is irreducible, from the PBW theorem, we have $V = \mathcal{U}(\widetilde{L})v_{i\boldsymbol{b}_1+j\boldsymbol{b}_2}$ and

$$\{i\boldsymbol{b}_1 + j\boldsymbol{b}_2 + \mathbb{Z}\boldsymbol{b}'_1 + \mathbb{N}\boldsymbol{b}'_2\} \cap \mathcal{P}(V) = \varnothing.$$

Thus the result follows from Proposition 2.12. □

Lemma 3.9. *If there exist $i > 0$, $j < 0$ and $0 \neq v_{\boldsymbol{a}} \in V_{\boldsymbol{a}}$, $\boldsymbol{a} \in \mathbb{C}^2$, $\boldsymbol{b} = m\boldsymbol{b}_1 + n\boldsymbol{b}_2 \neq \boldsymbol{0}$, such that $E(i\boldsymbol{b})v_{\boldsymbol{a}} = 0$, $E(j\boldsymbol{b})v_{\boldsymbol{a}} = 0$, then $V \simeq M(\boldsymbol{b}'_1, \boldsymbol{b}'_2, T_\rho(\mathcal{H}_{\boldsymbol{b}'_1})(\boldsymbol{c}, \boldsymbol{\lambda}))$ for some $\mathbb{Z}$-basis $\{\boldsymbol{b}'_1, \boldsymbol{b}'_2\}$ of Γ and some $\rho \in \mathcal{E}_{\boldsymbol{b}'_1}$.*

Proof. Write $(m, n) = s(m', n')$ with m', n' coprime and $s \geq 1$. Then we can choose $(m_2, n_2) \in \Gamma$ with $n'm_2 - m'n_2 = 1$. Let $\boldsymbol{b}'_1 = m'\boldsymbol{b}_1 + n'\boldsymbol{b}_2$, $\boldsymbol{b}'_2 = m_2\boldsymbol{b}_1 + n_2\boldsymbol{b}_2$; then $\{\boldsymbol{b}'_1, \boldsymbol{b}'_2\}$ is a $\mathbb{Z}$-basis of Γ. Fix any $0 \neq q \in \mathbb{Z}$.

Case 1: If $\{\boldsymbol{a} + q\boldsymbol{b}'_2 + x\boldsymbol{b}'_1 \mid x \in \mathbb{Z}\} \cap \mathcal{P}(V) = \varnothing$, then, by Lemma 3.7, we are done.

Case 2: If there exist integers $x_1 < x_2 < x_3$ with $\boldsymbol{a} + q\boldsymbol{b}'_2 + x_2\boldsymbol{b}'_1 \in \mathcal{P}(V)$ and $\boldsymbol{a} + q\boldsymbol{b}'_2 + x_i\boldsymbol{b}'_1 \notin \mathcal{P}(V)$, $i = 1, 3$, then, by Lemma 3.9, we are done.

Case 3: If there exist $m, n \in \mathbb{Z}$ with

$$(-\infty, m] \cup [n, \infty) \subseteq \{x \in \mathbb{Z} \mid \boldsymbol{a} + q\boldsymbol{b}'_2 + x\boldsymbol{b}'_1 \in \mathcal{P}(V)\},$$

then, by Lemma 3.8, we are done.

Now if the above three cases don't occur, we know that there exists some integer p_q such that $A_q := \{x \in \mathbb{Z} \mid \boldsymbol{a} + q\boldsymbol{b}'_2 + x\boldsymbol{b}'_1 \in \mathcal{P}(V)\} = (-\infty, p_q]$ or $[p_q, \infty)$. We first assume $A_q = (-\infty, p_q]$. Thus

$$E(q\boldsymbol{b}'_2 - jxs\boldsymbol{b}'_1 \pm \boldsymbol{b}'_1)v_{\boldsymbol{a}} = t^{q\boldsymbol{b}'_2 - jxs\boldsymbol{b}'_1 \pm \boldsymbol{b}'_1} v_{\boldsymbol{a}} = 0$$

for a sufficiently large integer $x > 0$. Since $E(j\boldsymbol{b})v_{\boldsymbol{a}} = E(js\boldsymbol{b}'_1)v_{\boldsymbol{a}} = 0$, we can obtain

$$E(q\boldsymbol{b}'_2 \pm \boldsymbol{b}'_1)v_{\boldsymbol{a}} = t^{q\boldsymbol{b}'_2 \pm \boldsymbol{b}'_1} v_{\boldsymbol{a}} = 0.$$

If $A_q = [p_q, \infty)$, by a similar argument, we can also obtain

$$E(q\boldsymbol{b}'_2 \pm \boldsymbol{b}'_1)v_{\boldsymbol{a}} = t^{q\boldsymbol{b}'_2 \pm \boldsymbol{b}'_1} v_{\boldsymbol{a}} = 0.$$

This implies

$$E(\pm(\boldsymbol{b}'_1 + \boldsymbol{b}'_2))v_{\boldsymbol{a}} = E(\pm(\boldsymbol{b}'_1 + 2\boldsymbol{b}'_2))v_{\boldsymbol{a}} = 0, \quad t^{\pm(\boldsymbol{b}'_1 + \boldsymbol{b}'_2)} v_{\boldsymbol{a}} = t^{\pm(\boldsymbol{b}'_1 + 2\boldsymbol{b}'_2)} v_{\boldsymbol{a}} = 0.$$

Since $\{\boldsymbol{b}'_1 + \boldsymbol{b}'_2, \boldsymbol{b}'_1 + 2\boldsymbol{b}'_2\}$ is a $\mathbb{Z}$-basis of Γ, L is generated by

$$\{E(\pm(\boldsymbol{b}'_1 + \boldsymbol{b}'_2)), E(\pm(\boldsymbol{b}'_1 + 2\boldsymbol{b}'_2)), t^{\pm(\boldsymbol{b}'_1 + \boldsymbol{b}'_2)}, t^{\pm(\boldsymbol{b}'_1 + 2\boldsymbol{b}'_2)}\}.$$

Thus $V = \mathcal{U}(\widetilde{L})v_{\boldsymbol{a}}$ is a trivial module, which is a contradiction. □

The following proposition gives the characterization of the nontrivial generalized highest weight Harish-Chandra module.

Proposition 3.10. *If V is a nontrivial generalized highest weight Harish-Chandra $\widetilde{L}$-module with generalized highest weight $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ corresponding to a $\mathbb{Z}$-basis $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ of Γ, then $V \simeq M(\boldsymbol{b}'_1, \boldsymbol{b}'_2, T_\rho(\mathcal{H}_{\boldsymbol{b}'_1})(\boldsymbol{c}, \lambda))$ for some $\mathbb{Z}$-basis $\{\boldsymbol{b}'_1, \boldsymbol{b}'_2\}$ of Γ and some $\rho \in \mathcal{E}_{\boldsymbol{b}'_1}$.*

Proof. From Lemma 3.9 and the proof of Proposition 3.9 in [Lin and Tan 2006], we can obtain our result. □

Together with Theorem 3.1 and Proposition 3.10, we have:

Theorem 3.11. *If V is a nonzero level Harish-Chandra $\widetilde{L}$-module, then*

$$V \simeq M(\boldsymbol{b}'_1, \boldsymbol{b}'_2, T_\rho(\mathcal{H}_{\boldsymbol{b}'_1})(\boldsymbol{c}, \boldsymbol{\lambda}))$$

for some $\mathbb{Z}$-basis $\{\boldsymbol{b}'_1, \boldsymbol{b}'_2\}$ of Γ and some $\rho \in \mathcal{E}_{\boldsymbol{b}'_1}$, $\boldsymbol{\lambda} \in \mathbb{C}^2$.

4. Classification of generalized highest weight Harish-Chandra $\widetilde{L}$-modules

In this section, we will provide the classification of generalized highest weight Harish-Chandra modules of $\widetilde{L}$ by using the highest weight modules of L. From Proposition 3.10, we only need to find in which case the irreducible generalized highest weight $\widetilde{L}$-module $M(\boldsymbol{b}_1, \boldsymbol{b}_2, T_\rho(\mathcal{H}_{\boldsymbol{b}_1})(\boldsymbol{c}, \boldsymbol{\lambda}))$ is a Harish-Chandra module.

First we give a triangular decomposition of L and construct a class of $\mathbb{Z}$-graded irreducible highest weight modules of L. Recall that

$$\widetilde{L}_i = \langle E(m\boldsymbol{b}_1 + i\boldsymbol{b}_2),\ t^{m\boldsymbol{b}_1+i\boldsymbol{b}_2} \mid m \in \mathbb{Z}\rangle, \quad i \in \mathbb{Z}\setminus\{0\},$$

and

$$\widetilde{L}_+ = \bigoplus_{i>0} \widetilde{L}_i, \quad \widetilde{L}_- = \bigoplus_{i<0} \widetilde{L}_i.$$

Then $L = \widetilde{L}_+ \oplus \mathcal{H}_{\boldsymbol{b}_1} \oplus \widetilde{L}_-$.

Remark 4.1. In this section, we call a L-module V a highest weight module (corresponding to the $\mathbb{Z}$-basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$) if there exists a nonzero $v \in V$ such that $V = \mathcal{U}(L)v$ and $\widetilde{L}_+.v = 0$.

For any linear function $\rho : \mathcal{H}_{\boldsymbol{b}_1} \to \mathbb{C}$ with $\rho(f(\boldsymbol{b}_1)) = \rho(h(\boldsymbol{b}_1)) = 0$, we define a one-dimensional $(\mathcal{H}_{\boldsymbol{b}_1} \oplus \widetilde{L}_+)$-module $\mathbb{C}v_0$ as follows:

$$\widetilde{L}_+.v_0 = 0, \quad x.v_0 = \rho(x)v_0, \quad x \in \mathcal{H}_{\boldsymbol{b}_1}. \tag{4-1}$$

Then we have an induced L-module

$$\overline{V}(\rho) = \mathrm{Ind}^{L}_{\mathcal{H}_{\boldsymbol{b}_1}\oplus\widetilde{L}_+} \mathbb{C}v_0 = \mathcal{U}(L) \otimes_{\mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1}\oplus\widetilde{L}_+)} \mathbb{C}v_0. \tag{4-2}$$

We see that $\overline{V}(\rho)$ is a $\mathbb{Z}$-graded module. It is clear that $\overline{V}(\rho)$ has a unique maximal $\mathbb{Z}$-graded submodule $J(\rho)$. Then we obtain a $\mathbb{Z}$-graded irreducible highest weight L-module

$$V(\rho) = \overline{V}(\rho)/J(\rho) = \oplus_{i\in\mathbb{Z}} V(\rho)_i,$$

where, for $i \in \mathbb{Z}$,

$$V(\rho)_i = \mathrm{Span}_{\mathbb{C}}\Big\{E(i_1\boldsymbol{b}_1 + j_1\boldsymbol{b}_2)E(i_2\boldsymbol{b}_1 + j_2\boldsymbol{b}_2)\cdots E(i_m\boldsymbol{b}_1 + j_m\boldsymbol{b}_2)t^{s_1\boldsymbol{b}_1+k_1\boldsymbol{b}_2}\cdots t^{s_n\boldsymbol{b}_1+k_n\boldsymbol{b}_2}v_0 \;\Big|\; m, n \in \mathbb{Z}_+,\ \sum_{p=1}^{m} j_p + \sum_{p=1}^{n} k_p = i\Big\}.$$

We call $V(\rho)_i$ for $i \in \mathbb{Z}$ the weight space of the L-module $V(\rho)$. If $\dim V(\rho)_i < \infty$, we say that the weight space $V(\rho)_i$ is finite-dimensional.

For later use, we need a conception of an exp-polynomial function. Recall from [Billig and Zhao 2004] that a function $f:\mathbb{Z}\to\mathbb{C}$ is said to be *exp-polynomial* if it can be written as a finite sum

$$f(n)=\sum c_{m,a}n^m a^n,$$

for some $c_{m,a}\in\mathbb{C}$, $m\in\mathbb{Z}_+$ and $0\neq a\in\mathbb{C}$.

The following lemma is due to [Wilson 2008].

Lemma 4.2. *A function $f:\mathbb{Z}\to\mathbb{C}$ is an* exp-polynomial function *if and only if there exist $a_0,\dots,a_n\in\mathbb{C}$ with $a_0a_n\neq 0$, such that*

$$\sum_{i=0}^{n} a_i f(m+i)=0,$$

for all $m\in\mathbb{Z}$.

Remark 4.3. In general, for fixed $a_0,\dots,a_n\in\mathbb{C}$ with $a_0a_n\neq 0$, the exp-polynomial function f satisfying $\sum_{i=0}^{n}a_i f(m+i)=0$, for all $m\in\mathbb{Z}$, is not unique.

Then we have the following result.

Proposition 4.4. *Suppose the linear function $\rho:\mathcal{H}_{\boldsymbol{b}_1}\to\mathbb{C}$ such that $\rho(f(\boldsymbol{b}_1))=\rho(h(\boldsymbol{b}_1))=0$. Then the $\mathbb{Z}$-graded L-module $V(\rho)$ has finite-dimensional weight spaces if and only if there exist two exp-polynomials $g_j:\mathbb{Z}\to\mathbb{C}$ satisfying $\sum_{i=0}^{n}a_i g_j(k+i)=0$ for $j=1,2,\ k\in\mathbb{Z},\ a_i\in\mathbb{C},\ a_0a_n\neq 0$ and*

$$g_1(0)=\det\begin{pmatrix}\boldsymbol{b}_1\\ \boldsymbol{b}_2\end{pmatrix}\rho(f(\boldsymbol{b}_2)),\qquad g_2(0)=\det\begin{pmatrix}\boldsymbol{b}_1\\ \boldsymbol{b}_2\end{pmatrix}\rho(h(\boldsymbol{b}_2)),$$

$$g_1(m)=\rho(mE(m\boldsymbol{b}_1)),\qquad g_2(m)=\rho(mt^{m\boldsymbol{b}_1}),\ m\in\mathbb{Z}\setminus\{0\}.$$

Proof. First, we define two linear maps $\phi_1,\phi_2:\mathbb{C}[t_1^{\pm1},t_2^{\pm1}]\to L$ by

$$\phi_i(t_1^{m_1}t_2^{m_2})=\begin{cases}E(m_1\boldsymbol{b}_1+m_2\boldsymbol{b}_2) & \text{if } i=1,\\ t^{m_1\boldsymbol{b}_1+m_2\boldsymbol{b}_2} & \text{if } i=2.\end{cases}$$

If $V(\rho)$ has finite-dimensional weight spaces, since $\dim V(\rho)_{-1}<\infty$ and

$$\phi_1(t_1^it_2^{-1})v_0\in V(\rho)_{-1}$$

for all $i\in\mathbb{Z}$, there exists $k\in\mathbb{Z}$ and a nonzero polynomial $P(t_1)=\sum_{i=0}^{n}a_it_1^i\in\mathbb{C}[t_1]$ with $a_0a_n\neq 0$ such that

$$\phi_1(t_2^{-1}t_1^kP(t_1))v_0=0.$$

Applying $\phi_i(t_1^st_2)$ for any $s\in\mathbb{Z},\ i=1,2$ to the above equation respectively, we get

$$\left(\sum_{i=0}^{n}a_i(k+s+i)E((k+s+i)\boldsymbol{b}_1)+\det\begin{pmatrix}\boldsymbol{b}_1\\ \boldsymbol{b}_2\end{pmatrix}a_{-k-s}f(\boldsymbol{b}_2)\right).v_0=0,\tag{4-3}$$

and

$$\left(\sum_{i=0}^{n} a_i(k+s+i)t^{(k+s+i)\boldsymbol{b}_1} + \det\begin{pmatrix}\boldsymbol{b}_1\\ \boldsymbol{b}_2\end{pmatrix} a_{-k-s}h(\boldsymbol{b}_2)\right).v_0 = 0, \tag{4-4}$$

where $a_{-k-s} = 0$ if $-k-s \notin \{0, 1, \dots, n\}$. Set $g_1 : \mathbb{Z} \to \mathbb{C}$ such that $g_1(0) = \det\binom{\boldsymbol{b}_1}{\boldsymbol{b}_2}\rho(f(\boldsymbol{b}_2))$ and $g_1(m) = \rho(mE(m\boldsymbol{b}_1))$ for $m \in \mathbb{Z} \setminus \{0\}$. Then (4-3) becomes

$$\sum_{i=0}^{n} a_i g_1(m+i) = 0, \quad \text{for all } m \in \mathbb{Z}.$$

Set $g_2 : \mathbb{Z} \to \mathbb{C}$ such that $g_2(0) = \det\binom{\boldsymbol{b}_1}{\boldsymbol{b}_2}\rho(h(\boldsymbol{b}_2))$ and $g_2(m) = \rho(mt^{m\boldsymbol{b}_1})$ for $m \in \mathbb{Z} \setminus \{0\}$. Then (4-4) becomes

$$\sum_{i=0}^{n} a_i g_2(m+i) = 0, \quad \text{for all } m \in \mathbb{Z}.$$

From Lemma 4.2, we have that g_1, g_2 are exp-polynomial functions.

Conversely, we use Theorem 1.7 in [Billig and Zhao 2004] to prove that the $\mathbb{Z}$-graded L-module $V(\rho)$ has finite-dimensional weight spaces, i.e., $\dim V(\rho)_i < \infty$ for all $i \in \mathbb{Z}$. Since $\mathbb{C}v_0$ is a one-dimensional $\mathcal{H}_{\boldsymbol{b}_1}$-module with exp-polynomial action, i.e., $\mathcal{H}_{\boldsymbol{b}_1}$ acts on $\mathbb{C}v_0$ through two exp-polynomials g_1, g_2, and $\widetilde{L}_+.v_0 = 0$, from Theorem 1.7 in [Billig and Zhao 2004], we just need to prove that L is $\mathbb{Z}$-extragraded (see Definition 1.4 of the same work). Set the index sets $X_i = \{(1, i), (2, i)\}$ for $i \in \mathbb{Z} \setminus \{0\}$ and $X_0 = \{(i, 0) \mid i = 1, 2, \dots, 6\}$. For $i \in \mathbb{Z} \setminus \{0\}$, let

$$\mathcal{L}^i_{\boldsymbol{k}}(j) = \begin{cases} E(i\mathbf{b_2} + j\mathbf{b_1}), & \boldsymbol{k} = (1, i),\ \ j \in \mathbb{Z}, \\ t^{i\mathbf{b_2}+j\mathbf{b_1}}, & \boldsymbol{k} = (2, i),\ \ j \in \mathbb{Z}, \end{cases}$$

and

$$\mathcal{L}^0_{\boldsymbol{k}}(j) = \begin{cases} jE(j\mathbf{b_1}), & \boldsymbol{k} = (1, 0),\ \ j \neq 0, \\ jt^{j\mathbf{b_1}}, & \boldsymbol{k} = (2, 0),\ \ j \neq 0, \\ K_i, & \boldsymbol{k} = (i+2, 0),\ \ i = 1, 2, 3, 4,\ \ j = 0. \end{cases}$$

Claim 1: L is a $\mathbb{Z}$-graded exp-polynomial Lie algebra (see Definition 1.2 in [Billig and Zhao 2004]).

In fact, let $L = \bigoplus_{j\in\mathbb{Z}} L(j)$, where $L(j) = \langle \mathcal{L}^i_{\boldsymbol{k}}(j) \mid i \in \mathbb{Z},\ \boldsymbol{k} \in X_i\rangle$. $[L(j_1), L(j_2)] \subseteq L(j_1 + j_2)$ for $j_1, j_2 \in \mathbb{Z}$. Thus L is $\mathbb{Z}$-graded, and it is straightforward to check that L is an exp-polynomial Lie algebra with the distinguished spanning set $\{\mathcal{L}^i_{\boldsymbol{k}}(j) \mid \boldsymbol{k} \in X_i,\ i, j \in \mathbb{Z}\}$.

Claim 2: The $\mathbb{Z}$-graded exp-polynomial Lie algebra L is $\mathbb{Z}$-extragraded.

In fact, let $L = \bigoplus_{i\in\mathbb{Z}} L^{(i)}$, where $L^{(i)} = \langle \mathcal{L}^i_{\boldsymbol{k}}(j) \mid j \in \mathbb{Z},\ \boldsymbol{k} \in X_i\rangle$. $[L^{(i_1)}, L^{(i_2)}] \subseteq L^{(i_1+i_2)}$ for $i_1, i_2 \in \mathbb{Z}$, i.e., L has another $\mathbb{Z}$-gradation. □

For linear function $\rho : \mathcal{H}_{\boldsymbol{b}_1} \to \mathbb{C}$ with $\rho(f(\boldsymbol{b}_1)) = \rho(h(\boldsymbol{b}_1)) = 0$, we say that ρ is an *exp-polynomial function over* $\mathcal{H}_{\boldsymbol{b}_1}$ if there exist $a_0, \dots, a_n \in \mathbb{C},\ a_0 a_n \neq 0$ and two exp-polynomials g_0, g_1 given by

$$g_1(0) = \det \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix} \rho(f(\boldsymbol{b}_2)), \quad g_2(0) = \det \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix} \rho(h(\boldsymbol{b}_2)),$$

and

$$g_1(m) = \rho(mE(m\boldsymbol{b}_1)), \quad g_2(m) = \rho(mt^{m\boldsymbol{b}_1})$$

for all $m \in \mathbb{Z} \setminus \{0\}$ such that $\sum_{i=0}^{n} a_i g_j(k+i) = 0$ for $j = 1, 2,\ k \in \mathbb{Z}$.

Let

$$\begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix}^{-1} = \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} \in \mathrm{GL}_{2\times 2}(\mathbb{Z}).$$

Set $\widetilde{d}_1 = p_1 d_1 + p_2 d_2,\ \widetilde{d}_2 = q_1 d_1 + q_2 d_2$. Then we have

$$[\widetilde{d}_i, E(m_1\boldsymbol{b}_1 + m_2\boldsymbol{b}_2)] = m_i E(m\boldsymbol{b}_1 + n\boldsymbol{b}_2), \quad [\widetilde{d}_i, t^{m_1\boldsymbol{b}_1 + m_2\boldsymbol{b}_2}] = m_i t^{m_1\boldsymbol{b}_1 + m_2\boldsymbol{b}_2}$$

for $i = 1, 2,\ m_1, m_2 \in \mathbb{Z}$.

Now we construct a class of $\mathbb{Z}^2$-graded irreducible generalized highest weight $\widetilde{L}$-modules by using the above $\mathbb{Z}$-graded highest weight L-module $V(\rho)$. For any linear function $\rho : \mathcal{H}_{\boldsymbol{b}_1} \to \mathbb{C}$ with $\rho(f(\boldsymbol{b}_1)) = \rho(h(\boldsymbol{b}_1)) = 0$, we set

$$\widehat{V}(\rho) = V(\rho) \otimes \mathbb{C}[t^{\pm 1}]$$

and define the actions of $\widetilde{L}$ on $\widehat{V}(\rho)$ as

$$\begin{aligned} E(m\boldsymbol{b}_1 + n\boldsymbol{b}_2).(v \otimes t^k) &= (E(m\boldsymbol{b}_1 + n\boldsymbol{b}_2).v) \otimes t^{m+k}, \\ t^{m\boldsymbol{b}_1 + n\boldsymbol{b}_2}.(v \otimes t^k) &= (t^{m\boldsymbol{b}_1 + n\boldsymbol{b}_2}.v) \otimes t^{m+k}, \\ \widetilde{d}_1.(v \otimes t^k) &= k(v \otimes t^k), \\ \widetilde{d}_2.(v \otimes t^k) &= j(v \otimes t^k), \\ K_i.(v \otimes t^k) &= (K_i.v) \otimes t^k \end{aligned}$$

for $(m, n) \in \mathbb{Z}^2 \setminus \{\boldsymbol{0}\},\ v \in V(\rho)_j,\ j \in \mathbb{Z},\ i = 1, 2, 3, 4$. It is clear that $\widehat{V}(\rho)$ is a $\mathbb{Z}^2$-graded $\widetilde{L}$-module, and

$$\widehat{V}(\rho) = \bigoplus_{m,n \in \mathbb{Z}} \widehat{V}(\rho)_{(m,n)},$$

where $\widehat{V}(\rho)_{(m,n)} = V(\rho)_m \otimes t^n$. We call $\widehat{V}(\rho)_{(m,n)},\ m, n \in \mathbb{Z}$ weight spaces of the module $\widehat{V}(\rho)$ with respect to $\widetilde{d}_1, \widetilde{d}_2$.

Let $W(i)$ be the $\widetilde{L}$-submodule of $\widehat{V}(\rho)$ generated by $v_0 \otimes t^i,\ i \in \mathbb{Z}$, where v_0 is defined in (4-1).

Lemma 4.5. *Let $\rho \in \mathcal{E}_{\boldsymbol{b}_1}$ and $W(i)$ be a $\mathbb{Z}^2$-graded irreducible $\widetilde{L}$-submodule of $\widehat{V}(\rho)$.*

(1) *If $T_\rho(\mathcal{H}_{\boldsymbol{b}_1}) = T_0$, then $\widehat{V}(\rho) = \bigoplus_{i\in\mathbb{Z}} W(i)$.*

(2) *If $T_\rho(\mathcal{H}_{\boldsymbol{b}_1}) = T_r$ for some $r \in \mathbb{N}$, then $\widehat{V}(\rho) = \bigoplus_{i=0}^{r-1} W(i)$.*

Proof. We need to notice the following two facts. First, any nonzero $\widetilde{L}$-submodule of $\widehat{V}(\rho)$ contains $v_0 \otimes t^i$ for some $i \in \mathbb{Z}$. Second, the two $\widetilde{L}$-submodules $W(m) = W(n)$ if and only if $t^{m-n} \in T_r$, where $T_r = T_\rho(\mathcal{H}_{\boldsymbol{b}_1}),\ r \in \mathbb{Z}_+$. For (1), that $W(i)$ is an $\mathbb{Z}^2$-graded irreducible $\widetilde{L}$-module follows from $V(\rho)$ being an irreducible L-module. For (2), let M be a nonzero submodule of the $\widetilde{L}$-module $W(i)$; then $v_0 \otimes t^n \in M$ for some $n \in \mathbb{Z}$. Since $\mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1})(v_0 \otimes t^i) = v_0 \otimes (T_r \cdot t^i)$ and $v_0 \otimes t^n \in \mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1})(v_0 \otimes t^i)$, we have $t^n \in T_r \cdot t^i$. This implies that $v_0 \otimes t^i \in W(n) \subseteq M$, i.e., $W(i) \subseteq M$. Thus $M = W(i)$, which shows that $W(i)$ is irreducible. □

For $\rho \in \mathcal{E}_{\boldsymbol{b}_1}$, we know that there exists a unique maximal $\mathbb{Z}^2$-graded submodule $J(i)$ of $\widehat{V}(\rho)$ which insects $W(i)$ trivially by Lemma 4.5. Then we get the $\mathbb{Z}^2$-graded irreducible $\widetilde{L}$-module

$$\widehat{V}(\rho, i) = \widehat{V}(\rho)/J(i) \simeq W(i).$$

Remark 4.6. (1) From Lemma 3.3 in [Wilson 2008], we see $\rho \in \mathcal{E}_{\boldsymbol{b}_1}$ if ρ is an exp-polynomial function over $\mathcal{H}_{\boldsymbol{b}_1}$.

(2) For $\rho \in \mathcal{E}_{\boldsymbol{b}_1}$, $W(i) \simeq W(j)$ as an $\widetilde{L}$-module up to a shift of the action of $\widetilde{d}_1$, $i, j \in \mathbb{Z}$ from (2-4) and Lemma 4.5.

Lemma 4.7. (1) *For any linear function $\rho : \mathcal{H}_{\boldsymbol{b}_1} \to \mathbb{C}$ with $\rho(f(\boldsymbol{b}_1)) = \rho(h(\boldsymbol{b}_1)) = 0$, the $\widetilde{L}$-module $\widehat{V}(\rho)$ has finite-dimensional weight spaces if and only if L-module $V(\rho)$ has finite weight spaces.*

(2) *For $\rho \in \mathcal{E}_{\boldsymbol{b}_1}$, $M(\boldsymbol{b}_1, \boldsymbol{b}_2, T_\rho(\mathcal{H}_{\boldsymbol{b}_1})(\boldsymbol{c}, \boldsymbol{\lambda})) \simeq \widehat{V}(\rho, 0)$ as an $\widetilde{L}$-module up to scalar shifts of the actions of d_1, d_2.*

Proof. (1) Since $\widehat{V}(\rho)_{(m,n)} = V(\rho)_m \otimes t^n,\ m, n \in \mathbb{Z}$, the first assertion is obvious.

(2) For any $\widetilde{L}$-module W, it is clear that we can modify the actions of d_1 and d_2. In fact, let σ be the corresponding representation of this $\widetilde{L}$-module W. Set $\pi(x) = \sigma(x)$ for $x \in L$, and $\pi(d_i) = \sigma(d_i) + a_i \,\mathrm{Id}_W$ for some fixed $a_i \in \mathbb{C},\ i = 1, 2$. Obviously, $\pi : \widetilde{L} \to \mathrm{gl}(W)$ is a representation of $\widetilde{L}$, i.e., one can define a new $\widetilde{L}$-module structure on W through shifting the actions of d_1, d_2. Note that $\mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1}).(v_0 \otimes 1) \simeq T_\rho(\mathcal{H}_{\boldsymbol{b}_1})$ for $\rho \in \mathcal{E}_{\boldsymbol{b}_1}$ and $\widetilde{L}_+.(\mathcal{U}(\mathcal{H}_{\boldsymbol{b}_1}).(v_0 \otimes 1)) = 0$. Then the result follows from the irreducibility of $\widehat{V}(\rho, 0)$. □

By Lemma 4.7, together with Proposition 4.4, we obtain the main result in this section.

Theorem 4.8. *For $\rho \in \mathcal{E}_{\boldsymbol{b}_1}$, the irreducible generalized highest weight $\widetilde{L}$-module $M(\boldsymbol{b}_1, \boldsymbol{b}_2, T_\rho(\mathcal{H}_{\boldsymbol{b}_1})(\boldsymbol{c}, \boldsymbol{\lambda}))$ is a Harish-Chandra module if and only if ρ is an exp-polynomial function over $\mathcal{H}_{\boldsymbol{b}_1}$.*

Remark 4.9. If ρ is an exp-polynomial function over $\mathcal{H}_{\boldsymbol{b}_1}$, then the generalized highest weight Harish-Chandra $M(\boldsymbol{b}_1, \boldsymbol{b}_2, T_\rho(\mathcal{H}_{\boldsymbol{b}_1})(\boldsymbol{c}, \boldsymbol{\lambda}))$ is a one-dimensional trivial module if and only if $\rho = 0$, i.e., $T_\rho(\mathcal{H}_{\boldsymbol{b}_1}) = T_0$, $\boldsymbol{c} = \boldsymbol{0}$.

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