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WEIGHTED ESTIMATES FOR ROUGH SINGULAR INTEGRALS WITH APPLICATIONS TO ANGULAR INTEGRABILITY

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We study certain singular integral operators, as well as their corresponding truncated maximal operators, along polynomial curves. Assuming that the kernels of operators are rough not only on the unit sphere but also on the radial direction, we establish certain weighted estimates for these operators. As applications, we obtain that these operators are bounded on the mixed radial-angular spaces $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)$ and on the vector-valued mixed radial-angular spaces $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n, \ell^{\tilde{p}})$. The bounds are independent of the coefficients of the polynomials in the definition of the operators. Our results we obtained improve theorems of Antonio Córdoba (2016) and Piero D'Ancona and Renato Lucà (2016).

1. Introduction

Let $\mathbb{R}^n$ be the Euclidean space of dimension n and S^{n-1} denote the unit sphere in $\mathbb{R}^n$ ($n \geq 2$) equipped with the normalized Lebesgue measure $d\sigma$. The mixed radial-angular spaces $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)$, $1 \leq p, \tilde{p} \leq \infty$, consist of all functions u satisfying $\|u\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)} < \infty$, where

$$\|u\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)} := \left(\int_0^\infty \|u(\rho\,\cdot\,)\|^p_{L^{\tilde{p}}(\mathrm{S}^{n-1})}\rho^{n-1}\,d\rho\right)^{1/p},$$
$$\|u\|_{L^\infty_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)} := \sup_{\rho>0}\|u(\rho\,\cdot\,)\|_{L^{\tilde{p}}(\mathrm{S}^{n-1})}.$$

The spaces $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)$ have the following easy properties:

(i) If $p = \tilde{p}$ and $1 \leq p \leq \infty$, then

$$\|u\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)} = \|u\|_{L^p(\mathbb{R}^n)}. \tag{1-1}$$

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(ii) If u is a radial function on $\mathbb{R}^n$ and $1 \le p \le \infty$, $1 \le \tilde{p} \le \infty$, then

$$\|u\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)} \simeq \|u\|_{L^p(\mathbb{R}^n)}.$$

(iii) If $1 \le \tilde{p}_1 \le \tilde{p}_2 \le \infty$ and $1 \le p \le \infty$, then

$$\|u\|_{L^p_{|x|}L^{\tilde{p}_1}_\theta(\mathbb{R}^n)} \le C_{n,p,\tilde{p}_1,\tilde{p}_2}\|u\|_{L^p_{|x|}L^{\tilde{p}_2}_\theta(\mathbb{R}^n)}.$$

Here the notation $A \simeq B$ means that there are positive constants C, C' such that $A \le CB$ and $B \le C'A$.

One might think that the mixed radial-angular space $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)$ is merely a formal extension of the Lebesgue space L^p, but recently it has been successfully used in studying Strichartz estimates and dispersive equations (see [Cho and Ozawa 2009; Cacciafesta and D'Ancona 2013; Fang and Wang 2011; Lucà 2014; Machihara et al. 2005; Ozawa and Rogers 2013; Sterbenz 2005; Tao 2000]). Also, it plays active roles in the theory of singular integral operator. Córdoba [2016] proved that the singular integral

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y)\frac{\Omega(y)}{|y|^n}\,dy, \tag{1-2}$$

where Ω is a homogeneous function of degree zero, is bounded on $L^p_{|x|}L^2_\theta(\mathbb{R}^n)$ for all $1 < p < \infty$, provided that $\Omega \in \mathcal{C}^1(\mathrm{S}^{n-1})$ and satisfies

$$\int_{\mathrm{S}^{n-1}} \Omega(y)\,d\sigma(y) = 0. \tag{1-3}$$

D'Ancona and Lucà [2016] then used the argument in Córdoba's Theorem 2.1 to extend the above result:

Theorem A. *Let $\Omega \in \mathcal{C}^1(\mathrm{S}^{n-1})$ satisfy* (1-3) *and $1 < p < \infty$, $1 < \tilde{p} < \infty$. Then*

$$\|T_\Omega f\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)} \le C_{\Omega,p,\tilde{p}}\|f\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)}.$$

Recently, Cacciafesta and Lucà [2016, Theorem 1.1] extended Theorem A to the weighted setting.

On the other hand, it is a long-time interesting topic to study the rough singular integral operators. Precisely, by assuming that $\Omega \in L\log L(\mathrm{S}^{n-1})$, Calderón and Zygmund [1956] proved that T_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Their proof is based on the rotation method to reduce the operator T_Ω to the directional Hilbert transform so that the well-known Riesz theorem can be applied. Fefferman [1979] considered another singular integral

$$T_{h,\Omega} f(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y)\frac{h(|y|)\Omega(y)}{|y|^n}\,dy, \tag{1-4}$$

where Ω is given as in (1-2) and $h(\cdot) \in L^\infty(\mathbb{R}_+)$ with $\mathbb{R}_+ := (0, \infty)$. Clearly, the operator T_Ω corresponds to the special case of $T_{h,\Omega}$ for $h(t) \equiv 1$. Fefferman discovered that the Calderón–Zygmund rotation method is no longer available if the operator $T_{h,\Omega}$ is also rough in the radial direction, for instance $h \in L^\infty$, so that new methods must be addressed. In his fundamental work on $T_{h,\Omega}$, Fefferman [1979] proved that $T_{h,\Omega}$ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ if $\Omega \in \mathrm{Lip}_\alpha(\mathrm{S}^{n-1})$ for some $\alpha > 0$ and $h \in L^\infty(\mathbb{R}_+)$. Afterwards, Namazi [1986] improved Fefferman's result by assuming $\Omega \in L^q(\mathrm{S}^{n-1})$ for $q > 1$ instead of $\Omega \in \mathrm{Lip}_\alpha(\mathrm{S}^{n-1})$. Subsequently, Duoandikoetxea and Rubio de Francia [1986] used the Littlewood–Paley theory to improve the above results to the case $\Omega \in L^q(\mathrm{S}^{n-1})$ for any $q > 1$ and $h \in \Delta_2(\mathbb{R}_+)$. Here $\Delta_\gamma(\mathbb{R}_+)$, $\gamma > 0$, is the set of all measurable functions h defined on $\mathbb{R}_+$ satisfying

$$\|h\|_{\Delta_\gamma(\mathbb{R}_+)} := \sup_{R>0} \left(\frac{1}{R} \int_0^R |h(t)|^\gamma \, dt \right)^{1/\gamma} < \infty.$$

In the same article, they also studied the $L^p(\mathbb{R}^n)$ boundedness for the maximal operator

$$T^*_{h,\Omega} f(x) = \sup_{\epsilon>0} \left| \int_{|y|>\epsilon} f(x-y) \frac{h(|y|)\Omega(y)}{|y|^n} \, dy \right|.$$

These results were improved and extended by many authors (see [Al-Salman and Pan 2002; Fan and Pan 1997; Liu 2014; Liu et al. 2016; Sato 2009]). It is worth remarking the following inclusion relations:

$$\mathcal{C}^1(\mathrm{S}^{n-1}) \subsetneq \mathrm{Lip}_\alpha(\mathrm{S}^{n-1}) \subsetneq L^q(\mathrm{S}^{n-1}), \tag{1-5}$$

$$L^\infty(\mathbb{R}_+) = \Delta_\infty(\mathbb{R}_+) \subsetneq \Delta_{\gamma_2}(\mathbb{R}_+) \subsetneq \Delta_{\gamma_1}(\mathbb{R}_+) \quad \text{for } 1 \le \gamma_1 < \gamma_2 < \infty. \tag{1-6}$$

In light of the above background and observation, a question that arises naturally is the following:

Question B. Are $T_{h,\Omega}$ and $T^*_{h,\Omega}$ bounded on $L^p_{|x|} L^{\tilde{p}}_\theta(\mathbb{R}^n)$ $(p \neq \tilde{p})$ if $\Omega \in L^q(\mathrm{S}^{n-1})$ and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $1 < q,\ \gamma \le \infty$?

In this paper we will give an affirmative answer to the above question by treating a family of operators that are even broader than $T_{h,\Omega}$ and $T^*_{h,\Omega}$. To be more precise, let h, Ω be given as in (1-4) and $P_N(t)$ be a real polynomial on $\mathbb{R}$ of degree N satisfying $P(0) = 0$. The corresponding singular integral operator T_{P_N} and the related maximal singular integral operator $T^*_{P_N}$ along the "polynomial curve" P_N on $\mathbb{R}^n$ are defined by

$$T_{P_N} f(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - P_N(|y|)y') \frac{h(|y|)\Omega(y)}{|y|^n} \, dy,$$

$$T^*_{P_N} f(x) = \sup_{\epsilon>0} \left| \int_{|y|>\epsilon} f(x - P_N(|y|)y') \frac{h(|y|)\Omega(y)}{|y|^n} \, dy \right|,$$

where $y' = y/|y|$ for $y \neq 0$. Clearly, $T_{P_N} = T_{h,\Omega}$ and $T^*_{P_N} = T^*_{h,\Omega}$ if $P_N(t) = t$.

In order to obtain the $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)$ boundedness of $T_{h,\Omega}$ and $T^*_{h,\Omega}$, we will establish some weighted inequalities. Our main results can be stated as follows.

Theorem 1.1. *Let $P_N(t)$ be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P_N(0) = 0$. Suppose that $\Omega \in L^q(\mathrm{S}^{n-1})$ satisfies* (1-3) *and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $1 < q, \gamma \leq \infty$:*

(i) *Let $2 \leq p < \infty$. Then for any nonnegative measurable function u on $\mathbb{R}^n$, the following inequality holds*:

$$\|T_{P_N} f\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,p,s,N} \|f\|_{L^p(\Lambda_{N,s}u)} \quad \forall s > 1. \tag{1-7}$$

(ii) *Let $1 < p < 2$ and $\{t_k\}_{k\in\mathbb{N}}$ be a sequence of positive numbers satisfying $t_1 = 2/p$ and*

$$\frac{1}{t_{k+1}} = \frac{1}{t_k} + \frac{p}{2}\Big(1 - \frac{1}{t_k}\Big).$$

Then for any nonnegative measurable function u on $\mathbb{R}^n$ and any fixed $k \in \mathbb{N}$, the following inequality holds:

$$\|T_{P_N} f\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,p,s,N,t_k} \|f\|_{L^p(\Lambda_{N,s}u)} \quad \forall s > t_k. \tag{1-8}$$

Here and throughout the paper we use $C_{h,\Omega,\alpha,\beta,\gamma,\ldots}$ to denote positive constants that depend on the functions Ω, h and parameters $\alpha, \beta, \gamma, \ldots$ appearing either in the definitions of the operators or in the statements of the theorems. In particular, they are independent of the coefficients of the polynomial P_N in the definition of T_{P_N}. In (1-8) *we also used the notation*

$$\Lambda_{N,s}u = \begin{cases} \mathrm{M}^N_s u + \mathrm{M}^2_s \widetilde{\mathrm{M}^N_s u} + H_{N,s}u & \text{if } 1 < p < 2, \\ L_{N,s}u & \text{if } 2 \leq p < \infty, \end{cases}$$

$$L_{\lambda,s}u = \sum_{i=0}^{\lambda} \mathrm{M}^{\lambda+1-i}_s M^{\tilde{\sigma}}_{i,s} \mathrm{M}_s u, \qquad H_\lambda u = \sum_{i=1}^{\lambda} \mathrm{M}^2 M^{\tilde{\sigma}}_i \mathrm{M}^{\lambda+1-i} u \quad \forall 1 \leq \lambda \leq N$$

and $M^{\tilde{\sigma}}_{\lambda,s}u = (M^{\tilde{\sigma}}_\lambda(u^s))^{1/s}$, $\mathrm{M}^k_s u = (\mathrm{M}^k u^s)^{1/s}$ for any $k \in \mathbb{N}$, $H_{\lambda,s}u = (H_\lambda u^s)^{1/s}$. Here M^k denotes the Hardy–Littlewood maximal operator M *iterated k times for all $k \in \mathbb{N}$ and $M^{\tilde{\sigma}}_\lambda$ is a maximal operator given as in the proof of Theorem 1.1.*

Theorem 1.2. *Let $P_N(t)$ be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P_N(0) = 0$. Suppose that $\Omega \in L^q(\mathrm{S}^{n-1})$ satisfies* (1-3) *and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $1 < q, \gamma \leq \infty$:*

(i) *Let $2 \leq p < \infty$. Then for any nonnegative measurable function u on $\mathbb{R}^n$, the following inequality holds*:

$$\|T^*_{P_N} f\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,p,s} \|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s u + \Theta_{N,s}\mathrm{M}^2_s u)} \quad \forall s > 1. \tag{1-9}$$

(ii) *Let* $1<p<2$ *and* $\{t_k\}_{k\in\mathbb{N}}$ *be given as in Theorem 1.1. Then for any nonnegative measurable function* u *on* $\mathbb{R}^n$ *and any fixed* $k\in\mathbb{N}$, *the following inequality holds*:

$$\|T^*_{P_N}f\|_{L^p(u)}\le C_{h,\Omega,q,\gamma,p,s,N,t_k}\|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s u+\Theta_{N,s}\mathrm{M}_s^2 u)}\quad \forall s>t_k. \tag{1-10}$$

Here

$$\Theta_{N,s}u=\begin{cases}\mathrm{M}_s^N u+\mathrm{M}_s^2\widetilde{\mathrm{M}_s^N u}+H_{N,s}u & \text{if } 1<p<2,\\ \mathrm{M}_s^N u+L_{N,s}u+I_{N,s}u+J_{N,s}u & \text{if } 2\le p<\infty,\end{cases}$$

where $L_{N,s}$ *is given as in Theorem 1.1 and*

$$I_{\lambda,s}u=\sum_{i=1}^{\lambda}\mathrm{M}_s M^{\tilde{\sigma}}_{i,s}\mathrm{M}_s^{\lambda-i}u,\quad J_{\lambda,s}u=\sum_{i=1}^{\lambda}\mathrm{M}_s^2 M^{\tilde{\sigma}}_{i-1,s}\mathrm{M}_s^{\lambda-i}u\quad \forall 1\le\lambda\le N.$$

As applications of Theorems 1.1 and 1.2, we obtain the $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)$ boundedness of the operators T_{P_N} and $T^*_{P_N}$ in following results.

Corollary 1.3. *Let* $P_N(t)$ *be a real polynomial on* $\mathbb{R}$ *of degree* N *and satisfy* $P_N(0)=0$. *Suppose that* $\Omega\in L^q(\mathrm{S}^{n-1})$ *satisfies* (1-3) *and* $h\in\Delta_\gamma(\mathbb{R}_+)$ *for some* $1<q,\ \gamma\le\infty$. *Then for* $1<p<\infty$ *and* $1<\tilde{p}<\infty$, *the following inequalities hold*:

$$\|T_{P_N}f\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)}\le C_{h,\Omega,q,\gamma,p,\tilde{p},N}\|f\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)}, \tag{1-11}$$

$$\begin{aligned}\Big\|\Big(\sum_{j\in\mathbb{Z}}|T_{P_N}f_j|^{\tilde{p}}\Big)^{1/\tilde{p}}\Big\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)}&\\ \le C_{h,\Omega,q,\gamma,p,\tilde{p},N}&\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^{\tilde{p}}\Big)^{1/\tilde{p}}\Big\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)},\end{aligned} \tag{1-12}$$

where the constants $C_{h,\Omega,q,\gamma,p,\tilde{p},N}>0$ *are independent of the coefficients of* P_N.

Corollary 1.4. *Let* $P_N(t)$ *be a real polynomial on* $\mathbb{R}$ *of degree* N *and satisfy* $P_N(0)=0$. *Suppose that* $\Omega\in L^q(\mathrm{S}^{n-1})$ *satisfies* (1-3) *and* $h\in\Delta_\gamma(\mathbb{R}_+)$ *for some* $1<q,\ \gamma\le\infty$. *Then for* $1<\tilde{p}\le p<\infty$, *the following inequalities hold*:

$$\|T^*_{P_N}f\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)}\le C_{h,\Omega,q,\gamma,p,\tilde{p},N}\|f\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)}, \tag{1-13}$$

$$\begin{aligned}\Big\|\Big(\sum_{j\in\mathbb{Z}}|T^*_{P_N}f_j|^{\tilde{p}}\Big)^{1/\tilde{p}}\Big\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)}&\\ \le C_{h,\Omega,q,\gamma,p,\tilde{p},N}&\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^{\tilde{p}}\Big)^{1/\tilde{p}}\Big\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)},\end{aligned} \tag{1-14}$$

where the constants $C_{h,\Omega,q,\gamma,p,\tilde{p},N}>0$ *are independent of the coefficients of* P_N.

Corollary 1.5. *Let $P_N(t)$ be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P_N(0) = 0$. Suppose that $\Omega \in L^q(\mathrm{S}^{n-1})$ satisfies* (1-3) *and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $1 < q,\ \gamma \le \infty$:*

(i) *If $1 < p < \infty$ and $1 < \tilde{p} < \infty$, then*

$$\left\| \left(\sum_{j\in\mathbb{Z}} |T_{P_N} f_j|^{\tilde{p}} \right)^{1/\tilde{p}} \right\|_{L^p(\mathbb{R}^n)} \le C_{h,\Omega,q,\gamma,p,\tilde{p},N} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^{\tilde{p}} \right)^{1/\tilde{p}} \right\|_{L^p(\mathbb{R}^n)}. \tag{1-15}$$

(ii) *If $1 < \tilde{p} \le p < \infty$, then*

$$\left\| \left(\sum_{j\in\mathbb{Z}} |T^*_{P_N} f_j|^{\tilde{p}} \right)^{1/\tilde{p}} \right\|_{L^p(\mathbb{R}^n)} \le C_{h,\Omega,q,\gamma,p,\tilde{p},N} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^{\tilde{p}} \right)^{1/\tilde{p}} \right\|_{L^p(\mathbb{R}^n)}. \tag{1-16}$$

The above constants $C_{h,\Omega,q,\gamma,p,\tilde{p},N} > 0$ are independent of the coefficients of P_N.

Remark 1.6. Corollary 1.3 improves and generalizes Theorem A by (1-5) and (1-6).

The rest of this paper is organized as follows. Section 2 contains a key criterion, which says that certain weighted norm inequalities for an operator will automatically imply its boundedness on the mixed radial-angular spaces, vector-valued mixed radial-angular spaces, and vector-valued inequalities. The main results of this paper will be proved in Section 3, where the proofs of Corollaries 1.3–1.5 are based on Theorems 1.1 and 1.2 and the criterion established in Section 2 (see Proposition 2.1). Finally, we will discuss several corresponding results concerning the Hardy–Littlewood maximal operator, Calderón–Zygmund operators, and the singular integral operators with Grafakos–Stefanov kernels. We would like to remark that the main idea in the proofs of our results is a combination of ideas and arguments from [Córdoba 2016; D'Ancona and Lucà 2016; Hofmann 1993; Liu 2014].

Throughout this note, for any $p \in (1, \infty)$, we let p' denote the dual exponent to p defined as $1/p + 1/p' = 1$. In what follows, for any function f, we define $\tilde{f}$ by $\tilde{f}(x) = f(-x)$. We denote by M^k the Hardy–Littlewood maximal operator M iterated k times for all $k = 1, 2, \ldots$ Specifically, $\mathrm{M}^k = \mathrm{M}$ when $k = 1$. For $s > 1$, we denote $\mathrm{M}_s u = (\mathrm{M}u^s)^{1/s}$. For $f \in L^p(u)$, we set

$$\|f\|_{L^p(u)} = \left(\int_{\mathbb{R}^n} |f(x)|^p u(x)\, dx \right)^{1/p}.$$

2. A criterion

To prove our main results, we need the following proposition, which is of interest in its own right.

Proposition 2.1. *Let $1 < p < \infty$ and $\{t_k\}_{k\in\mathbb{N}}$ be a strictly decreasing sequence of positive numbers satisfying $\lim_{k\to\infty} t_k = 1$. Assume that T is a linear or sublinear operator such that*

$$\|Tf\|_{L^p(u)} \le C_{p,s,t_k} \|f\|_{L^p(\mathcal{G}_s(u))} \quad \forall s > t_k \tag{2-1}$$

for any fixed positive integer k and any nonnegative measurable function u on $\mathbb{R}^n$, where $\mathcal{G}_s$ is a bounded operator from $L^q(\mathbb{R}^n)$ to itself for any $q \in (s,\infty)$ with $s > t_k$. Then for any $p < q < \infty$, the following inequalities hold:

$$\|Tf\|_{L^q_{|x|}L^p_\theta(\mathbb{R}^n)} \le C_{p,q}\|f\|_{L^q_{|x|}L^p_\theta(\mathbb{R}^n)}, \tag{2-2}$$

$$\bigg\|\bigg(\sum_{j\in\mathbb{Z}}|Tf_j|^p\bigg)^{1/p}\bigg\|_{L^q_{|x|}L^p_\theta(\mathbb{R}^n)} \le C_{p,q}\bigg\|\bigg(\sum_{j\in\mathbb{Z}}|f_j|^p\bigg)^{1/p}\bigg\|_{L^q_{|x|}L^p_\theta(\mathbb{R}^n)}, \tag{2-3}$$

$$\bigg\|\bigg(\sum_{j\in\mathbb{Z}}|Tf_j|^p\bigg)^{1/p}\bigg\|_{L^q(\mathbb{R}^n)} \le C_{p,q}\bigg\|\bigg(\sum_{j\in\mathbb{Z}}|f_j|^p\bigg)^{1/p}\bigg\|_{L^q(\mathbb{R}^n)}. \tag{2-4}$$

Proof. We only prove (2-2) since (2-3) and (2-4) can be proved similarly. The argument in the proof of (2-2) is essentially the same as in the proof of [D'Ancona and Lucà 2016, Theorem 2.6]. Let $1 < p < q < \infty$ and write $r = q/(q-p)$. Fix a number s in the interval $(1, r)$ and choose k_0 as the smallest integer for which we have $t_{k_0} < s$. We denote by X the set of all $g \in \mathcal{S}(\mathbb{R})$ with $\int_0^\infty g^r(\rho)\rho^{n-1}\,d\rho \le 1$. By polar coordinates, we write

$$\begin{aligned}\|Tf\|^p_{L^q_{|x|}L^p_\theta(\mathbb{R}^n)} &= \bigg(\int_0^\infty\bigg(\int_{\mathrm{S}^{n-1}}|Tf(\rho\theta)|^p d\sigma(\theta)\bigg)^{q/p}\rho^{n-1}\,d\rho\bigg)^{p/q}\\ &= \sup_{g\in X}\int_0^\infty\int_{\mathrm{S}^{n-1}}|Tf(\rho\theta)|^p g(\rho)\rho^{n-1}\,d\sigma(\theta)\,d\rho\\ &= \sup_{g\in X}\int_{\mathbb{R}^n}|Tf(x)|^p g(|x|)\,dx.\end{aligned}\tag{2-5}$$

Fix $g \in X$. Let $I(g) := \int_{\mathbb{R}^n}|Tf(x)|^p g(|x|)\,dx$ and $h(x) = g(|x|)$. Changing variables, we obtain from (2-1) and Hölder's inequality that

$$\begin{aligned}I(g) &\le C_{p,s,t_{k_0}}\int_{\mathbb{R}^n}|f(x)|^p\mathcal{G}_s(h)(x)\,dx\\ &\le C_{p,s,t_{k_0}}\int_0^\infty\int_{\mathrm{S}^{n-1}}|f(\rho\theta)|^p\,d\sigma(\theta)\,\mathcal{G}_s(g)(\rho)\rho^{n-1}\,d\rho\\ &\le C_{p,s,t_{k_0}}\bigg(\int_0^\infty\bigg(\int_{\mathrm{S}^{n-1}}|f(\rho\theta)|^p d\sigma(\theta)\bigg)^{q/p}\rho^{n-1}d\rho\bigg)^{p/q}\\ &\qquad\qquad\times\bigg(\int_0^\infty(\mathcal{G}_s(g)(\rho))^r\rho^{n-1}\,d\rho\bigg)^{1/r}\end{aligned}$$

$$\le C_{p,q}\|f\|^p_{L^q_{|x|}L^p_\theta(\mathbb{R}^n)}\|\mathcal{G}_s(h)\|_{L^r(\mathbb{R}^n)}$$
$$\le C_{p,q}\|f\|^p_{L^q_{|x|}L^p_\theta(\mathbb{R}^n)},$$

which, together with (2-5), yields (2-2). □

3. Proofs of main results

In this section we shall present the proofs of Theorems 1.1 and 1.2 and Corollaries 1.3–1.5. In what follows, we may assume, without loss of generality, that $P_N(t) = \sum_{i=1}^N a_i t^i$ with $a_i \neq 0$. We also let $P_\lambda(t) = \sum_{i=1}^\lambda a_i t^i$ for $\lambda = 1, 2, \dots, N$ and $P_0(t) = 0$.

Proof of Theorem 1.1. For $\lambda \in \{1, 2, \dots, N\}$, we define two families of measures $\{\sigma_{k,\lambda}\}_{k\in\mathbb{Z}}$ and $\{|\sigma_{k,\lambda}|\}_{k\in\mathbb{Z}}$ respectively by

$$\int_{\mathbb{R}^n} f(x)\,d\sigma_{k,\lambda}(x) = \int_{2^k<|x|\le 2^{k+1}} f(P_\lambda(|x|)x')\frac{h(|x|)\Omega(x)}{|x|^n}\,dx$$

and

$$\int_{\mathbb{R}^n} f(x)\,d|\sigma_{k,\lambda}|(x) = \int_{2^k<|x|\le 2^{k+1}} f(P_\lambda(|x|)x')\frac{|h(|x|)\Omega(x)|}{|x|^n}\,dx.$$

We also define the maximal operators M_λ^σ and $M_\lambda^{\tilde\sigma}$ respectively by

$$M_\lambda^\sigma f(x) = \sup_{k\in\mathbb{Z}} ||\sigma_{k,\lambda}| * f(x)|$$

and

$$M_\lambda^{\tilde\sigma} f(x) = \sup_{k\in\mathbb{Z}} ||\tilde\sigma_{k,\lambda}| * f(x)|,$$

where

$$\int_{\mathbb{R}^n} f(x)\,d|\tilde\sigma_{k,\lambda}|(x) = \int_{\mathbb{R}^n} f(-x)\,d|\sigma_{k,\lambda}|(x).$$

One can easily verify that

$$M_0^\sigma f(x) \le C_{h,\Omega,q,\gamma}|f(x)|, \tag{3-1}$$

$$M_\lambda^{\tilde\sigma} f(x) = M_\lambda^\sigma \tilde f(x), \tag{3-2}$$

$$T_{P_N} f(x) = \sum_{k\in\mathbb{Z}} \sigma_{k,N} * f(x). \tag{3-3}$$

Also, from [Liu 2014, Lemma 2.2] and a direct computation, one has

$$\max\bigl\{|\hat\sigma_{k,\lambda}(\xi) - \hat\sigma_{k,\lambda-1}(\xi)|, |\widehat{|\sigma_{k,\lambda}|}(\xi) - \widehat{|\sigma_{k,\lambda-1}|}(\xi)|\bigr\} \le C_{h,\Omega,q,\gamma}\min\{1, |2^{k\lambda}a_\lambda\xi|\}, \tag{3-4}$$

$$\max\bigl\{|\hat\sigma_{k,\lambda}(\xi)|, |\widehat{|\sigma_{k,\lambda}|}(\xi)|\bigr\} \le C_{h,\Omega,q,\gamma}(\min\{1, |2^{k\lambda}a_\lambda\xi|^{-1}\})^{1/(4\lambda q'\gamma')}. \tag{3-5}$$

We shall prove Theorem 1.1 by considering the following three steps:

Step 1: The bounds for M_λ^σ. We want to show that

$$\|M_\lambda^\sigma f\|_{L^p(\mathbb{R}^n)} \le C_{h,\Omega,q,\gamma}\|f\|_{L^p(\mathbb{R}^n)} \tag{3-6}$$

for all $0 \le \lambda \le N$ and $1 < p < \infty$. It is obvious that (3-6) holds for $\lambda = 0$ by (3-1). Choose a nonnegative function $\phi \in \mathscr{S}(\mathbb{R}^n)$ supported in $\{|t| \le 1\}$ satisfying $\phi(t) = 1$ when $|t| < \frac{1}{2}$. For $\lambda \in \{1, 2, \dots, N\}$, we define the family of functions $\{\psi_{k,\lambda}\}_{k\in\mathbb{Z}}$ via the Fourier transform $\widehat{\psi}_{k,\lambda}(\xi) = \phi(2^{k\lambda}|a_\lambda\xi|)$. Define the family of Borel measures $\{\omega_{k,\lambda}\}_{k\in\mathbb{Z}}$ on $\mathbb{R}^n$ by

$$\hat{\omega}_{k,\lambda}(\xi) = \widehat{|\sigma_{k,\lambda}|}(\xi) - \psi_{k,\lambda}(\xi)\widehat{|\sigma_{k,\lambda-1}|}(\xi). \tag{3-7}$$

One easily checks that (or see [Liu 2014])

$$|\hat{\omega}_{k,\lambda}(x)| \le C_{h,\Omega,q,\gamma}(\min\{1, |2^{k\lambda}a_\lambda x|, |2^{k\lambda}a_\lambda x|^{-1}\})^{1/(4\lambda q'\gamma')}, \tag{3-8}$$

$$M_\lambda^\omega f(x) \le M_\lambda^\sigma|f|(x) + \mathrm{M}M_{\lambda-1}^\sigma|f|(x), \tag{3-9}$$

$$M_\lambda^\sigma f(x) \le \mathrm{M}M_{\lambda-1}^\sigma|f|(x) + G_\lambda^\omega f(x), \tag{3-10}$$

where $M_\lambda^\omega f(x) = \sup_{k\in\mathbb{Z}}||\omega_{k,\lambda}| * f(x)|$ and $G_\lambda^\omega f(x) = \left(\sum_{k\in\mathbb{Z}}|\omega_{k,\lambda} * f(x)|^2\right)^{1/2}$. By (3-1), (3-8)–(3-10) and a standard iteration argument from [Duoandikoetxea and Rubio de Francia 1986], we can obtain (3-6) for all $1 \le \lambda \le N$. The details are omitted.

Step 2: The proof of (i) of Theorem 1.1. For $1 \le \lambda \le N$ and $s > 1$, let $\Lambda_{\lambda,s}$ be given as in Theorem 1.1. Let ϕ be given as above. We define the family of functions $\{\Phi_\lambda\}_{\lambda=1}^N$ by

$$\Phi_\lambda(\xi) = \prod_{j=\lambda}^N \phi(|2^{kj}a_j\xi|).$$

For $1 \le \lambda \le N$, define the Borel measures $\{\mu_{k,\lambda}\}_{k\in\mathbb{Z}}$ on $\mathbb{R}^n$ by

$$\hat{\mu}_{k,\lambda}(\xi) = \hat{\sigma}_{k,\lambda}(\xi)\Phi_{\lambda+1}(\xi) - \hat{\sigma}_{k,\lambda-1}(\xi)\Phi_\lambda(\xi).$$

Here we use the convention $\prod_{j\in\varnothing} a_j = 1$. One can easily check that (or see [Liu 2014])

$$\sigma_{k,N} = \sum_{\lambda=1}^N \mu_{k,\lambda}, \tag{3-11}$$

$$M_\lambda^\mu f(x) \le \mathrm{M}^{N-\lambda}M_\lambda^\sigma|f|(x) + \mathrm{M}^{N-\lambda+1}M_{\lambda-1}^\sigma|f|(x), \tag{3-12}$$

$$|\hat{\mu}_{k,\lambda}(x)| \le C_{h,\Omega,q,\gamma}(\min\{1, |2^{k\lambda}a_\lambda x|, |2^{k\lambda}a_\lambda x|^{-1}\})^{1/(4\lambda q'\gamma')}. \tag{3-13}$$

Equation (3-3) and (3-11) clearly yield that

$$T_{P_N}f(x) = \sum_{k\in\mathbb{Z}}\sum_{\lambda=1}^N \mu_{k,\lambda} * f(x) = \sum_{\lambda=1}^N\sum_{k\in\mathbb{Z}} \mu_{k,\lambda} * f(x) =: \sum_{\lambda=1}^N T_\lambda f(x). \tag{3-14}$$

Notice that $u \le \mathrm{M}_s u$, $\mathrm{M}_s u \in A_1$ (see [Coifman and Rochberg 1980]), and

$$\sum_{\lambda=1}^{N} \mathrm{M}_s M_{\lambda,s}^{\tilde{\mu}} \mathrm{M}_s u \le \sum_{\lambda=1}^{N} (\mathrm{M}_s^{N+1-\lambda} M_{\lambda,s}^{\tilde{\sigma}} \mathrm{M}_s u + \mathrm{M}_s^{N+2-\lambda} M_{\lambda-1,s}^{\tilde{\sigma}} \mathrm{M}_s u) \le L_{N,s} u$$

by (3-12). Therefore, (1-7) reduces to the following inequality:

$$\|T_\lambda f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s} \|f\|_{L^p(\Lambda_{N,s} u)} \tag{3-15}$$

for all $1 \le \lambda \le N$, $2 \le p < \infty$, $s > 1$ and any nonnegative measurable function u on $\mathbb{R}^n$.

We now prove (3-15). For $1 \le \lambda \le N$, let $\Psi_\lambda(t) \in \mathcal{C}_c^\infty\big(\big(\frac{1}{4}, 1\big)\big)$ such that $0 \le \Psi_\lambda \le 1$ and $\sum_{k\in\mathbb{Z}} (\Psi_\lambda(2^{k\lambda}|a_\lambda \xi|))^3 = 1$. Define the Fourier multiplier operators $\{S_{k,\lambda}\}_{k\in\mathbb{Z}}$ by $S_{k,\lambda} f(x) = \Theta_{k,\lambda} * f(x)$, where $\widehat{\Theta}_{k,\lambda}(\xi) = \Psi_\lambda(2^{k\lambda}|a_\lambda \xi|)$. It was shown in [Hofmann 1993] that

$$\bigg\| \bigg(\sum_{k\in\mathbb{Z}} |S_{k,\lambda} f|^2 \bigg)^{1/2} \bigg\|_{L^p(w)} \le C_{p,w,\lambda} \|f\|_{L^p(w)} \tag{3-16}$$

and

$$\bigg\| \sum_{k\in\mathbb{Z}} S_{k,\lambda} f_k \bigg\|_{L^p(w)} \le C_{p,w,\lambda} \bigg\| \bigg(\sum_{k\in\mathbb{Z}} |f_k|^2 \bigg)^{1/2} \bigg\|_{L^p(w)} \tag{3-17}$$

for all $1 < p < \infty$ and $w \in A_p$.

Write

$$\begin{aligned} T_\lambda f(x) &= \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} S_{j+k,\lambda}^3 (\mu_{k,\lambda} * f)(x) \\ &= \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} S_{j+k,\lambda}^3 (\mu_{k,\lambda} * f)(x) =: \sum_{j\in\mathbb{Z}} T_{\lambda,j} f(x). \end{aligned} \tag{3-18}$$

By (3-13) and Plancherel's theorem, it holds that

$$\int_{\mathbb{R}^n} |\mu_{k,\lambda} * S_{j+k,\lambda} w(x)|^2 \, dx \le C_{h,\Omega,q,\gamma} 2^{-|j|/(2q'\gamma')} \int_{\mathbb{R}^n} |w(x)|^2 \, dx \tag{3-19}$$

for an arbitrary function w on $\mathbb{R}^n$. Fix a nonnegative measurable function u on $\mathbb{R}^n$. It is easy to see that

$$\begin{aligned} &\int_{\mathbb{R}^n} |\mu_{k,\lambda} * S_{j+k,\lambda} w(x)|^2 u^s(x) \, dx \\ &\qquad \le \|\mu_{k,\lambda}\| \|\Theta_{j+k,\lambda}\|_{L^1(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\mu_{k,\lambda}| * |\Theta_{j+k,\lambda}| * |w|^2(x) u^s(x) \, dx \\ &\qquad \le C_{h,\Omega,q,\gamma} \int_{\mathbb{R}^n} |w(x)|^2 \mathrm{M} M_\lambda^{\tilde{\mu}} u^s(x) \, dx \end{aligned} \tag{3-20}$$

for any $s > 1$. By (3-19) and (3-20) and the interpolation of L^2-spaces with a change of measure [Bergh and Löfström 1976, Theorem 5.4.1], we obtain

$$\int_{\mathbb{R}^n} |\mu_{k,\lambda} * S_{j+k,\lambda} w(x)|^2 u(x)\,dx \leq C_{h,\Omega,q,\gamma,s} 2^{-(1-1/s)/(2q'\gamma')|j|} \int_{\mathbb{R}^n} |w(x)|^2 \mathrm{M}_s M^{\tilde{\mu}}_{\lambda,s} u(x)\,dx \tag{3-21}$$

for any $s > 1$. By (3-21) with $w = S_{j+k,\lambda} f$ and (3-16), it follows that

$$\begin{aligned}
\|T_{\lambda,j} f\|^2_{L^2(u)} &= \Big\| \sum_{k\in\mathbb{Z}} S^3_{j+k,\lambda} \mu_{k,\lambda} * f \Big\|^2_{L^2(u)} \\
&\leq C_\lambda \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} |\mu_{k,\lambda} * S^2_{j+k,\lambda} f(x)|^2 u(x)\,dx \\
&\leq C_{h,\Omega,q,\gamma,\lambda,s} 2^{-(1-1/s)/(2q'\gamma')|j|} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} |S_{j+k,\lambda} f(x)|^2 \mathrm{M}_s M^{\tilde{\mu}}_{\lambda,s} u(x)\,dx \\
&\leq C_{h,\Omega,q,\gamma,\lambda,s} 2^{-(1-1/s)/(2q'\gamma')|j|} \|f\|^2_{L^2(\mathrm{M}_s M^{\tilde{\mu}}_{\lambda,s} u)}.
\end{aligned}$$

Hence we obtain

$$\|T_{\lambda,j} f\|_{L^2(u)} \leq C_{h,\Omega,q,\gamma,\lambda,s} 2^{-(1-1/s)/(4q'\gamma')|j|} \|f\|_{L^2(\mathrm{M}_s M^{\tilde{\mu}}_{\lambda,s} u)}. \tag{3-22}$$

Next, we shall only prove

$$\|T_{\lambda,j} f\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,p,\lambda,s} \|f\|_{L^p(\mathrm{M}_s M^{\tilde{\mu}}_{\lambda,s} u)} \tag{3-23}$$

for all $2 < p < \infty$. Actually, by (3-22), (3-23), and an interpolation (see [Bergh and Löfström 1976, Corollary 5.5.4]), one has

$$\|T_{\lambda,j} f\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,p,\lambda,s} 2^{-\alpha(p,q,\gamma,s)|j|} \|f\|_{L^p(\mathrm{M}_s M^{\tilde{\mu}}_{\lambda,s} u)} \tag{3-24}$$

for $2 \leq p < \infty$ and $s > 1$. Here $\alpha(p,q,\gamma,s) > 0$ depends only on p, q, γ, and s. Equation (3-24) together with (3-18) yields (3-15).

To prove (3-23), it suffices to show that

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} |\mu_{k,\lambda} * g_k|^2 \Big)^{1/2} \Big\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,p,s} \Big\| \Big(\sum_{k\in\mathbb{Z}} |g_k|^2 \Big)^{1/2} \Big\|_{L^p(M^{\tilde{\mu}}_{\lambda,s} u)} \tag{3-25}$$

for all $2 < p < \infty$ and any $s > 1$. In fact, by (3-16), (3-17), (3-25), and the fact $M^{\tilde{\mu}}_{\lambda,s} u \leq \mathrm{M}_s M^{\tilde{\mu}}_{\lambda,s} u$, it holds that

$$\|T_{\lambda,j} f\|_{L^p(u)} = \Big\| \sum_{k\in\mathbb{Z}} S^3_{j+k,\lambda} \mu_{k,\lambda} * f \Big\|_{L^p(u)}$$

$$\begin{aligned}&\le C_{p,\lambda}\Big\|\Big(\sum_{k\in\mathbb{Z}}|\mu_{k,\lambda}*S^2_{j+k,\lambda}f|^2\Big)^{1/2}\Big\|_{L^p(u)}\\&\le C_{h,\Omega,q,\gamma,p,\lambda,s}\Big\|\Big(\sum_{k\in\mathbb{Z}}|S^2_{j+k,\lambda}f|^2\Big)^{1/2}\Big\|_{L^p(M^{\tilde\mu}_{\lambda,s}u)}\\&\le C_{h,\Omega,q,\gamma,p,\lambda,s}\|f\|_{L^p(\mathrm{M}_sM^{\tilde\mu}_{\lambda,s}u)}\end{aligned}$$

for all $2<p<\infty$ and any $s>1$. This yields (3-23).

Below we prove (3-25). Fix $2<p<\infty$. By duality we can choose a function $v\in L^{(p/2)'}(u)$ with unit norm such that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\mu_{k,\lambda}*g_k|^2\Big)^{1/2}\Big\|^2_{L^p(u)}=\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}|\mu_{k,\lambda}*g_k(x)|^2\cdot v(x)u(x)\,dx.$$

This together with the fact that $\|\mu_{k,\lambda}\|\le C_{h,\Omega,q,\gamma}$ yields that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\mu_{k,\lambda}*g_k|^2\Big)^{1/2}\Big\|^2_{L^p(u)}\le C_{h,\Omega,q,\gamma}\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}|g_k(x)|^2||\tilde\mu_{k,\lambda}|*(vu)(x)|\,dx.\tag{3-26}$$

Fix $s>1$ and let $r=\frac{1}{2}ps$. Hölder's inequality yields

$$||\tilde\mu_{k,\lambda}|*(vu)|\le(|\tilde\mu_{k,\lambda}|*u^s)^{1/r}(|\tilde\mu_{k,\lambda}|*(u^{r'/(p/2)'}v^{r'}))^{1/r'}.\tag{3-27}$$

By Hölder's inequality with exponents $\frac{1}{2}p$ and $\left(\frac{1}{2}p\right)'$ again and (3-26), (3-27), it holds that

$$\begin{aligned}&\Big\|\Big(\sum_{k\in\mathbb{Z}}|\mu_{k,\lambda}*g_k|^2\Big)^{1/2}\Big\|^2_{L^p(u)}\\&\quad\le C_{h,\Omega,q,\gamma}\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}|g_k(x)|^2(M^{\tilde\mu}_\lambda u^s)^{1/r}M^{\tilde\mu}_\lambda(u^{r'/(p/2)'}v^{r'}))^{1/r'}(x)\,dx\\&\quad\le C_{h,\Omega,q,\gamma}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|^2_{L^p(M^{\tilde\mu}_{\lambda,s}u)}\|M^{\tilde\mu}_\lambda(u^{r'/(p/2)'}v^{r'})\|^{1/r'}_{L^{(p/2)'/r'}(\mathbb{R}^n)}.\end{aligned}\tag{3-28}$$

By (3-1), (3-6), and (3-12), one has

$$\|M^{\tilde\mu}_\lambda f\|_{L^t(\mathbb{R}^n)}\le C_{h,\Omega,q,\gamma,t}\|f\|_{L^t(\mathbb{R}^n)}\quad\forall 1<t<\infty.\tag{3-29}$$

Since $\frac{1}{2}p = r/s < r$, then $\left(\frac{1}{2}p\right)' > r'$. Equation (3-29) leads to

$$\|M_\lambda^{\tilde{\mu}}(u^{r'/(p/2)'}v^{r'})\|^{1/r'}_{L^{(p/2)'/r'}(\mathbb{R}^n)} \le C_{h,\Omega,q,\gamma,p,s}\|u^{r'/(p/2)'}v^{r'}\|^{1/r'}_{L^{(p/2)'/r'}(\mathbb{R}^n)} \le C_{h,\Omega,q,\gamma,p,s}.$$

This together with (3-28) yields (3-25) and completes the proof of (i) of Theorem 1.1.

Step 3: The proof of (ii) of Theorem 1.1. For $1 \le \lambda \le N$ and $s > 1$, let $\Lambda_{\lambda,s}$, $H_{\lambda,s}$, and $\{t_k\}_{k\in\mathbb{N}}$ be given as in Theorem 1.1. To prove (1-8), it suffices to show that

$$\|T_\lambda f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s,t_k}\|f\|_{L^p(\Lambda_{N,s}u)} \quad \forall s > t_k \tag{3-30}$$

for all $1 \le \lambda \le N$, $1 < p < 2$, all $k \in \mathbb{N}$, and any nonnegative measurable function u on $\mathbb{R}^n$. Actually, (3-30) reduces to the following

$$\begin{aligned}\int_{\mathbb{R}^n} &|T_\lambda f(x)|^p u^{1/s}(x)\,dx \\ &\le C_{h,\Omega,q,\gamma,p,\lambda,s,t_k}\int_{\mathbb{R}^n}|f(x)|^p(\mathrm{M}^N u + \mathrm{M}^2\widetilde{\mathrm{M}^N u} + H_N u)^{1/s}(x)\,dx \quad \forall s > t_k\end{aligned} \tag{3-31}$$

for all $1 \le \lambda \le N$, $1 < p < 2$, all $k \in \mathbb{N}$, and any nonnegative measurable function u on $\mathbb{R}^n$. To this end, we substitute u^s for u in (3-31). With this substitution, the weight on the left becomes u and the weight on the right is not more than $\mathrm{M}_s^N u + \mathrm{M}_s^2\widetilde{\mathrm{M}_s^N u} + H_{N,s}u$.

We now prove (3-31). Fix $s > 1$ and a nonnegative measurable function u on $\mathbb{R}^n$. It follows from (3-10) that

$$\begin{aligned}\int_{\mathbb{R}^n}&(M_\lambda^\sigma f(x))^p u^{1/s}(x)\,dx \\ &\le \int_{\mathbb{R}^n}(\mathrm{M}M_{\lambda-1}^\sigma|f|(x))^p u^{1/s}(x)\,dx + \int_{\mathbb{R}^n}(G_\lambda^\omega f(x))^p u^{1/s}(x)\,dx\end{aligned} \tag{3-32}$$

for all $1 < p < 2$. Hence by the well-known Fefferman–Stein inequality for M (see (3-102) below) we have

$$\begin{aligned}\int_{\mathbb{R}^n}&(\mathrm{M}M_{\lambda-1}^\sigma|f|)(x))^p u^{1/s}(x)\,dx \\ &\le C_p\|M_{\lambda-1}^\sigma|f|\|^p_{L^p(\mathrm{M}u^{1/s})} \le C_p\|M_{\lambda-1}^\sigma|f|\|^p_{L^p((\mathrm{M}u)^{1/s})}\end{aligned} \tag{3-33}$$

for $1 < p < \infty$. Next, we consider $G_\lambda^\omega f$. By Minkowski's inequality, we obtain

$$\begin{aligned}G_\lambda^w f(x) &= \left(\sum_{k\in\mathbb{Z}}\left|\omega_{k,\lambda} * \sum_{j\in\mathbb{Z}} S^3_{j+k,\lambda}f(x)\right|^2\right)^{1/2} \\ &\le \sum_{j\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}|\omega_{k,\lambda} * S^3_{j+k,\lambda}f(x)|^2\right)^{1/2}\end{aligned}$$

$$=: \sum_{j\in\mathbb{Z}} G_{\lambda,j} f(x).$$

It follows that

$$\|G_\lambda^\omega f\|_{L^p(u^{1/s})} \le \sum_{j\in\mathbb{Z}} \|G_{\lambda,j} f\|_{L^p(u^{1/s})} \tag{3-34}$$

for all $1 < p < \infty$. It is obvious to see that

$$\|\omega_{k,\lambda} * f\|_{L^\infty(\mathbb{R}^n)} \le C_{h,\Omega,q,\gamma} \|f\|_{L^\infty(\mathbb{R}^n)}, \tag{3-35}$$

$$\|\omega_{k,\lambda} * f\|_{L^1(u)} \le C\|f\|_{L^1(M_\lambda^{\tilde\sigma} u + M_{\lambda-1}^{\tilde\sigma} \mathrm{M}u)} \le C\|f\|_{L^1(\mathrm{M}M_\lambda^{\tilde\sigma} u + \mathrm{M}M_{\lambda-1}^{\tilde\sigma} \mathrm{M}u)}. \tag{3-36}$$

Thus, by interpolating between (3-35) and (3-36), we obtain

$$\|\omega_{k,\lambda} * f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p} \|f\|_{L^p(\mathrm{M}M_\lambda^{\tilde\sigma} u + \mathrm{M}M_{\lambda-1}^{\tilde\sigma} \mathrm{M}u)} \tag{3-37}$$

for all $1 < p < 2$. It follows from (3-37) that

$$\begin{aligned} \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |\omega_{k,\lambda} * f_k(x)|^p u(x)\,dx \\ \le C_{h,\Omega,q,\gamma,p} \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |f_k(x)|^p (\mathrm{M}M_\lambda^{\tilde\sigma} u + \mathrm{M}M_{\lambda-1}^{\tilde\sigma} \mathrm{M}u)(x)\,dx \end{aligned} \tag{3-38}$$

for all $1 < p < 2$. On the other hand, we get from (3-6) and (3-9) that

$$\int_{\mathbb{R}^n} (\sup_{k\in\mathbb{Z}} |\omega_{k,\lambda} * f_k(x)|)^p\,dx \le C_{h,\Omega,q,\gamma,p} \int_{\mathbb{R}^n} (\sup_{k\in\mathbb{Z}} |f_k(x)|)^p\,dx \tag{3-39}$$

for all $1 < p < 2$. An interpolation between (3-38) and (3-39) now yields

$$\begin{aligned} \int_{\mathbb{R}^n} \Big(\sum_{k\in\mathbb{Z}} |\omega_{k,\lambda} * f_k(x)|^2 \Big)^{p/2} u^{1/t_1}(x)\,dx \\ \le C_{h,\Omega,q,\gamma,p,t_1} \int_{\mathbb{R}^n} \Big(\sum_{k\in\mathbb{Z}} |f_k(x)|^2 \Big)^{p/2} (\mathrm{M}M_\lambda^{\tilde\sigma} u + \mathrm{M}M_{\lambda-1}^{\tilde\sigma} \mathrm{M}u)^{1/t_1}(x)\,dx \end{aligned} \tag{3-40}$$

for all $1 < p < 2$, where $t_1 = 2/p$. Substitute u^{t_1} for u in (3-40), we obtain that

$$\begin{aligned} \int_{\mathbb{R}^n} \Big(\sum_{k\in\mathbb{Z}} |\omega_{k,\lambda} * f_k(x)|^2 \Big)^{p/2} u(x)\,dx \\ \le C_{h,\Omega,q,\gamma,p,t_1} \int_{\mathbb{R}^n} \Big(\sum_{k\in\mathbb{Z}} |f_k(x)|^2 \Big)^{p/2} (\mathrm{M}M_\lambda^{\tilde\sigma} u^{t_1} + \mathrm{M}M_{\lambda-1}^{\tilde\sigma} \mathrm{M}u^{t_1})^{1/t_1}(x)\,dx \\ \le C_{h,\Omega,q,\gamma,p,t_1} \int_{\mathbb{R}^n} \Big(\sum_{k\in\mathbb{Z}} |f_k(x)|^2 \Big)^{p/2} (\mathrm{M}_{t_1} M_{\lambda,t_1}^{\tilde\sigma} u + \mathrm{M}_{t_1} M_{\lambda-1,t_1}^{\tilde\sigma} \mathrm{M}_{t_1} u)(x)\,dx. \end{aligned} \tag{3-41}$$

Since $\mathrm{M}_{t_1} M^{\tilde{\sigma}}_{\lambda,t_1} u + \mathrm{M}_{t_1} M^{\tilde{\sigma}}_{\lambda-1,t_1} \mathrm{M}_{t_1} u \in A_1$, by the weighted Littlewood–Paley theory, (3-41) yields that

$$\begin{aligned}
(3\text{-}42)\quad \|G_{\lambda,j} f\|_{L^p(u)} &= \Big\| \Big(\sum_{k\in\mathbb{Z}} |\omega_{k,\lambda} * S^3_{j+k,\lambda} f|^2 \Big)^{1/2} \Big\|_{L^p(u)} \\
&\le C_{h,\Omega,q,\gamma,p,t_1} \Big\| \Big(\sum_{k\in\mathbb{Z}} |S^3_{j+k,\lambda} f|^2 \Big)^{1/2} \Big\|_{L^p(\mathrm{M}_{t_1} M^{\tilde{\sigma}}_{\lambda,t_1} u + \mathrm{M}_{t_1} M^{\tilde{\sigma}}_{\lambda-1,t_1} \mathrm{M}_{t_1} u)} \\
&\le C_{h,\Omega,q,\gamma,p,\lambda,t_1} \|f\|_{L^p(\mathrm{M}_{t_1} M^{\tilde{\sigma}}_{\lambda,t_1} u + \mathrm{M}_{t_1} M^{\tilde{\sigma}}_{\lambda-1,t_1} \mathrm{M}_{t_1} u)}
\end{aligned}$$

for all $1 < p < 2$. Substituting u^{1/t_1} for u in (3-42), one finds

$$(3\text{-}43)\qquad \|G_{\lambda,j} f\|_{L^p(u^{1/t_1})} \le C_{h,\Omega,q,\gamma,p,\lambda,t_1} \|f\|_{L^p((\mathrm{M} M^{\tilde{\sigma}}_{\lambda} u + \mathrm{M} M^{\tilde{\sigma}}_{\lambda-1} \mathrm{M} u)^{1/t_1})}$$

for all $1 < p < 2$. On the other hand, by (3-8) and the arguments similar to those used in deriving (3-21),

$$\begin{aligned}
(3\text{-}44)\quad & \int_{\mathbb{R}^n} |\omega_{k,\lambda} * S_{j+k,\lambda} w(x)|^2 u(x)\, dx \\
&\qquad \le C_{h,\Omega,q,\gamma,s} 2^{-(1-1/s)/(2q'\gamma')|j|} \int_{\mathbb{R}^n} |w(x)|^2 \mathrm{M}_s M^{\tilde{\omega}}_{\lambda,s} u(x)\, dx
\end{aligned}$$

for any function w and any $s > 1$. By (3-44) with $w = S^2_{j+k,\lambda} f$ and (3-17), we obtain that

$$\begin{aligned}
(3\text{-}45)\quad \|G_{\lambda,j} f\|^2_{L^2(u)} &= \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |\omega_{k,\lambda} * S^3_{j+k,\lambda} f(x)|^2 u(x)\, dx \\
&\le \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} |\omega_{k,\lambda} * S^3_{j+k,\lambda} f(x)|^2 u(x)\, dx \\
&\le C_{h,\Omega,q,\gamma,s} 2^{-(1-1/s)/(2q'\gamma')|j|} \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |S^2_{j+k,\lambda} f(x)|^2 \mathrm{M}_s M^{\tilde{\omega}}_{\lambda,s} u(x)\, dx \\
&\le C_{h,\Omega,q,\gamma,s} 2^{-(1-1/s)/(2q'\gamma')|j|} \int_{\mathbb{R}^n} |f(x)|^2 \mathrm{M}_s M^{\tilde{\omega}}_{\lambda,s} u(x)\, dx.
\end{aligned}$$

Take $s = t_1$. Substituting u^{1/t_1} for u in (3-45), we get

$$(3\text{-}46)\qquad \|G_{\lambda,j} f\|_{L^2(u^{1/t_1})} \le C_{h,\Omega,q,\gamma,\lambda,t_1} 2^{-(1-1/t_1)/(4q'\gamma')|j|} \|f\|_{L^2((\mathrm{M} M^{\tilde{\omega}}_{\lambda} u)^{1/t_1})}.$$

It follows from (3-9) that

$$(3\text{-}47)\qquad \mathrm{M} M^{\tilde{\omega}}_{\lambda} u \le \mathrm{M} M^{\tilde{\sigma}}_{\lambda} |u| + \mathrm{M}^2 M^{\tilde{\sigma}}_{\lambda-1} |u| \le \mathrm{M} M^{\tilde{\sigma}}_{\lambda} \mathrm{M} u + \mathrm{M}^2 M^{\tilde{\sigma}}_{\lambda-1} \mathrm{M} u.$$

Formula (3-47) together with (3-46) implies that

$$\|G_{\lambda,j}f\|_{L^2(u^{1/t_1})} \leq C_{h,\Omega,q,\gamma,\lambda,t_1}2^{-(1-1/t_1)/(4q'\gamma')|j|}\|f\|_{L^2((\mathrm{M}M_\lambda^{\tilde{\sigma}}\mathrm{M}u+\mathrm{M}^2M_{\lambda-1}^{\tilde{\sigma}}\mathrm{M}u)^{1/t_1})}. \tag{3-48}$$

By an interpolation between (3-43) and (3-48), we obtain

$$\|G_{\lambda,j}f\|_{L^p(u^{1/t_1})} \leq C_{h,\Omega,q,\gamma,p,\lambda,t_1}2^{-\beta(p,q,\gamma,t_1)|j|}\|f\|_{L^p((\mathrm{M}M_\lambda^{\tilde{\sigma}}\mathrm{M}u+\mathrm{M}^2M_{\lambda-1}^{\tilde{\sigma}}\mathrm{M}u)^{1/t_1})} \tag{3-49}$$

for all $1 < p < 2$, where $\beta(p, q, \gamma, t_1) > 0$. We get from (3-49) and (3-34) that

$$\|G_\lambda^\omega f\|_{L^p(u^{1/t_1})} \leq C_{h,\Omega,q,\gamma,p,\lambda,t_1}\|f\|_{L^p((\mathrm{M}M_\lambda^{\tilde{\sigma}}\mathrm{M}u+\mathrm{M}^2M_{\lambda-1}^{\tilde{\sigma}}\mathrm{M}u)^{1/t_1})} \tag{3-50}$$

for all $1 < p < 2$. Combining (3-50) with (3-32), (3-33), we now have

$$\begin{aligned}\int_{\mathbb{R}^n}(M_\lambda^\sigma f(x))^p u^{1/t_1}(x)\,dx \leq C_{h,\Omega,q,\gamma,p,\lambda,t_1}\bigg(&\int_{\mathbb{R}^n}(M_{\lambda-1}^\sigma|f|(x))^p(\mathrm{M}u)^{1/t_1}(x)\,dx \\ &+\int_{\mathbb{R}^n}|f(x)|^p(\mathrm{M}M_\lambda^{\tilde{\sigma}}\mathrm{M}u+\mathrm{M}^2M_{\lambda-1}^{\tilde{\sigma}}\mathrm{M}u)^{1/t_1}(x)\,dx\bigg)\end{aligned} \tag{3-51}$$

for all $1 < p < 2$. We want to show that

$$\int_{\mathbb{R}^n}(M_\lambda^\sigma f(x))^p u^{1/t_1}(x)\,dx \leq C_{h,\Omega,q,\gamma,p,\lambda,t_1}\int_{\mathbb{R}^n}|f(x)|^p(\mathrm{M}^\lambda u+\mathrm{M}^2\widetilde{\mathrm{M}^\lambda}u+H_\lambda u)^{1/t_1}(x)\,dx \tag{3-52}$$

for all $1 \leq \lambda \leq N$ and $1 < p < 2$. When $\lambda = 1$, (3-1) and (3-51) imply

$$\begin{aligned}\int_{\mathbb{R}^n}(M_1^\sigma f(x))^p u^{1/t_1}(x)\,dx &\leq C_{h,\Omega,q,\gamma,p,t_1}\bigg(\int_{\mathbb{R}^n}(M_0^\sigma|f|(x))^p(\mathrm{M}u)^{1/t_1}(x)\,dx \\ &\qquad+\int_{\mathbb{R}^n}|f(x)|^p(\mathrm{M}M_1^{\tilde{\sigma}}\mathrm{M}u+\mathrm{M}^2M_0^{\tilde{\sigma}}\mathrm{M}u)^{1/t_1}(x)\,dx\bigg) \\ &\leq C_{h,\Omega,q,\gamma,p,t_1}\int_{\mathbb{R}^n}|f(x)|^p(\mathrm{M}u+\mathrm{M}^2\widetilde{\mathrm{M}u}+\mathrm{M}M_1^{\tilde{\sigma}}\mathrm{M}u)^{1/t_1}(x)\,dx \\ &\leq C_{h,\Omega,q,\gamma,p,t_1}\int_{\mathbb{R}^n}|f(x)|^p(\mathrm{M}u+\mathrm{M}^2\widetilde{\mathrm{M}u}+H_1u)^{1/t_1}(x)\,dx\end{aligned}$$

for any $1 < p < 2$. This yields (3-52) for $\lambda = 1$. Assume that (3-52) holds for $\lambda = \iota - 1$ with $\iota \in \{2, \dots, N\}$. We obtain, from (3-51) and our assumption,

$$\begin{aligned}
&\int_{\mathbb{R}^n} (M_\iota^\sigma f(x))^p u^{1/t_1}(x)\,dx \\
&\quad \le C_{h,\Omega,q,\gamma,p,\iota,t_1}\biggl(\int_{\mathbb{R}^n} (M_{\iota-1}^\sigma |f|(x))^p (\mathrm{M}u)^{1/t_1}(x)\,dx \\
&\qquad\qquad + \int_{\mathbb{R}^n} |f(x)|^p (\mathrm{M}M_\iota^{\tilde\sigma}\mathrm{M}u + \mathrm{M}^2 M_{\iota-1}^{\tilde\sigma}\mathrm{M}u)^{1/t_1}(x)\,dx \biggr) \\
&\quad \le C_{h,\Omega,q,\gamma,p,\iota,t_1}\biggl(\int_{\mathbb{R}^n} |f(x)|^p (\mathrm{M}^\iota\mathrm{M}u + \mathrm{M}^2\widetilde{\mathrm{M}^{\iota-1}\mathrm{M}}u + H_{\iota-1}\mathrm{M}u)^{1/t_1}(x)\,dx \\
&\qquad\qquad + \int_{\mathbb{R}^n} |f(x)|^p (\mathrm{M}M_\iota^{\tilde\sigma}\mathrm{M}u + \mathrm{M}^2 M_{\iota-1}^{\tilde\sigma}\mathrm{M}u)^{1/t_1}(x)\,dx \biggr) \\
&\quad \le C_{h,\Omega,q,\gamma,p,\iota,t_1} \int_{\mathbb{R}^n} |f(x)|^p (\mathrm{M}^\iota u + \mathrm{M}^2\widetilde{\mathrm{M}^\iota}u + H_\iota\mathrm{M}u)^{1/t_1}(x)\,dx
\end{aligned}$$

for all $1 < p < 2$. This yields (3-52) for $\lambda = \iota$. Thus, (3-52) is proved. Inequality (3-52) together with (3-9) and (3-33) yields that

$$\begin{aligned}
(3\text{-}53)\quad &\int_{\mathbb{R}^n} (\sup_{k\in\mathbb{Z}} |\omega_{k,\lambda} * f(x)|)^p u^{1/t_1}(x)\,dx \\
&\quad \le \int_{\mathbb{R}^n} (M_\lambda^\omega |f|(x))^p u^{1/t_1}(x)\,dx \\
&\quad \le \int_{\mathbb{R}^n} (M_\lambda^\sigma |f|(x))^p u^{1/t_1}(x)\,dx + \int_{\mathbb{R}^n} (\mathrm{M}M_{\lambda-1}^\sigma |f|(x))^p u^{1/t_1}(x)\,dx \\
&\quad \le C_{h,\Omega,q,\gamma,p,\lambda,t_1} \int_{\mathbb{R}^n} |f(x)|^p (\mathrm{M}^{\lambda+1}u + \mathrm{M}^2\widetilde{\mathrm{M}^\lambda}u + H_\lambda\mathrm{M}u)^{1/t_1}(x)\,dx
\end{aligned}$$

for all $1 < p < 2$. Since $\mathrm{M}M_\lambda^{\tilde\sigma}u + \mathrm{M}M_{\lambda-1}^{\tilde\sigma}\mathrm{M}u \le H_\lambda u$, an interpolation between (3-38) and (3-53) yields

$$\begin{aligned}
(3\text{-}54)\quad &\int_{\mathbb{R}^n} \Bigl(\sum_{k\in\mathbb{Z}} |\omega_{k,\lambda} * f_k(x)|^2 \Bigr)^{p/2} u^{1/t_2}(x)\,dx \\
&\quad \le C_{h,\Omega,q,\gamma,p,\lambda,t_2} \int_{\mathbb{R}^n} \Bigl(\sum_{k\in\mathbb{Z}} |f_k(x)|^2 \Bigr)^{p/2} (\mathrm{M}^\lambda u + \mathrm{M}^2\widetilde{\mathrm{M}^\lambda}u + H_\lambda u)^{1/t_2}(x)\,dx
\end{aligned}$$

for all $1 < p < 2$, where $1/t_2 = 1/t_1 + \frac{1}{2}p(1 - 1/t_1)$. Using (3-54) and arguments similar to those used in deriving (3-52), we obtain

$$\begin{aligned}
(3\text{-}55)\quad &\int_{\mathbb{R}^n} (M_\lambda^\sigma f(x))^p u^{1/t_2}(x)\,dx \\
&\qquad \le C_{h,\Omega,q,\gamma,p,\lambda,t_2} \int_{\mathbb{R}^n} |f(x)|^p (\mathrm{M}^\lambda u + \mathrm{M}^2\widetilde{\mathrm{M}^\lambda}u + H_\lambda u)^{1/t_2}(x)\,dx
\end{aligned}$$

for all $1 \le \lambda \le N$ and $1 < p < 2$. As the same reason as above, we can obtain a strictly decreasing sequence $\{t_k\}_{k\in\mathbb{N}}$ by the recursion formula

$$t_1 = \frac{2}{p}, \quad \frac{1}{t_{k+1}} = \frac{1}{t_k} + \frac{p}{2}\Big(1 - \frac{1}{t_k}\Big), \quad k = 2, 3, \dots$$

such that

$$\int_{\mathbb{R}^n} (M_\lambda^\sigma f(x))^p u^{1/t_k}(x)\,dx \le C_{h,\Omega,q,\gamma,p,\lambda,t_k} \int_{\mathbb{R}^n} |f(x)|^p (\mathrm{M}^\lambda u + \mathrm{M}^2\widetilde{\mathrm{M}^\lambda u} + H_\lambda u)^{1/t_k}(x)\,dx \tag{3-56}$$

for all $1 \le \lambda \le N$, $1 < p < 2$, and all $k \in \mathbb{N}$. Using (3-12), (3-56), and the well-known Fefferman–Stein inequality for M (see (3-102) below), we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} (M_\lambda^\mu f(x))^p u^{1/t_k}(x)\,dx \\
&\le \int_{\mathbb{R}^n} (\mathrm{M}^{N-\lambda} M_\lambda^\sigma |f|(x))^p u^{1/t_k}(x)\,dx \\
&\qquad\qquad + \int_{\mathbb{R}^n} (\mathrm{M}^{N-\lambda+1} M_{\lambda-1}^\sigma |f|(x))^p u^{1/t_k}(x)\,dx \\
&\le C_p\bigg(\int_{\mathbb{R}^n} (M_\lambda^\sigma |f|(x))^p (\mathrm{M}^{N-\lambda} u)^{1/t_k}(x)\,dx \\
&\qquad\qquad + \int_{\mathbb{R}^n} (M_{\lambda-1}^\sigma |f|(x))^p (\mathrm{M}^{N-\lambda+1} u)^{1/t_k}(x)\,dx\bigg) \\
&\le C_{h,\Omega,q,\gamma,p,\lambda,t_k}\bigg(\int_{\mathbb{R}^n} |f(x)|^p \Big(\mathrm{M}^\lambda(\mathrm{M}^{N-\lambda} u) \\
&\qquad\qquad + \mathrm{M}^2\mathrm{M}^\lambda\widetilde{(\mathrm{M}^{N-\lambda} u)} + H_\lambda(\mathrm{M}^{N-\lambda} u)\Big)^{1/t_k}(x)\,dx \\
&\quad + \int_{\mathbb{R}^n} |f(x)|^p \Big(\mathrm{M}^{\lambda-1}(\mathrm{M}^{N-\lambda+1} u) \\
&\qquad\qquad + \mathrm{M}^2\mathrm{M}^{\lambda-1}\widetilde{(\mathrm{M}^{N-\lambda+1} u)} + H_{\lambda-1}(\mathrm{M}^{N-\lambda+1} u)\Big)^{1/t_k}(x)\,dx\bigg) \\
&\le C_{h,\Omega,q,\gamma,p,\lambda,t_k} \int_{\mathbb{R}^n} |f(x)|^p (\mathrm{M}^N u + \mathrm{M}^2\widetilde{\mathrm{M}^N u} + H_N u)^{1/t_k}(x)\,dx.
\end{aligned} \tag{3-57}$$

By (3-57) and the lemma in [Zhang 2008, p.1574] we can get (3-31). □

Proof of Theorem 1.2. For $1 \le \lambda \le N$, let $\Theta_{\lambda,s}$ be given as in Theorem 1.2. We shall prove Theorem 1.2 by combining the method used in the proof of [Zhang 2008, Theorem 1.2] with ideas from [Duoandikoetxea and Rubio de Francia 1986; Fan et al. 1999]. For any $\epsilon > 0$, there exists an integer k such that $2^{k-1} \le \epsilon < 2^k$. We now write

$$T_{P_N}^* f(x) \le M_N^\sigma f(x) + \sup_{k\in\mathbb{Z}} \bigg|\sum_{j=k}^{\infty} \sigma_{j,N} * f(x)\bigg|. \tag{3-58}$$

We shall prove Theorem 1.2 by considering the following two steps:

Step 1: The proof of (i) of Theorem 1.2. By (3-58), to prove (1-9), it suffices to show that

$$\|M_N^\sigma f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,N,s}\|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s u)} \tag{3-59}$$

and

$$\left\|\sup_{k\in\mathbb{Z}}\left|\sum_{j=k}^{\infty}\sigma_{j,N}*f\right|\right\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,N,s}\|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s u+\Theta_{N,s}\mathrm{M}_s^2 u)} \tag{3-60}$$

for all $2\le p<\infty$, $s>1$, and any nonnegative measurable function u on $\mathbb{R}^n$.

Let us first prove (3-59). Fix $u\in A_1$ and $1\le\lambda\le N$. By arguments similar to those used in deriving (3-25),

$$\left\|\left(\sum_{k\in\mathbb{Z}}|\omega_{k,\lambda}*g_k|^2\right)^{1/2}\right\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,s}\left\|\left(\sum_{k\in\mathbb{Z}}|g_k|^2\right)^{1/2}\right\|_{L^p(M^{\tilde{\omega}}_{\lambda,s}u)} \tag{3-61}$$

for all $2<p<\infty$ and any $s>1$. It follows from (3-61) that

$$\begin{aligned}\|G_{\lambda,j}f\|_{L^p(u)} &= \left\|\left(\sum_{k\in\mathbb{Z}}|\omega_{k,\lambda}*S^3_{j+k}f|^2\right)^{1/2}\right\|_{L^p(u)}\\ &\le C_{h,\Omega,q,\gamma,p,s}\left\|\left(\sum_{k\in\mathbb{Z}}|S^3_{j+k}f|^2\right)^{1/2}\right\|_{L^p(\mathrm{M}_s M^{\tilde{\omega}}_{\lambda,s}u)}\\ &\le C_{h,\Omega,q,\gamma,p,\lambda,s}\|f\|_{L^p(\mathrm{M}_s M^{\tilde{\omega}}_{\lambda,s}u)}\end{aligned} \tag{3-62}$$

for all $2<p<\infty$ and any $s>1$. In the last inequality of (3-62), we used the weighted Littlewood–Paley theory and the fact that $\mathrm{M}_s M^{\tilde{\omega}}_{\lambda,s}u\in A_1$. On the other hand, it follows from (3-45) that

$$\|G_{\lambda,j}f\|_{L^2(u)} \le C_{h,\Omega,q,\gamma,\lambda,s}2^{-(1-1/s)/(4q'\gamma')|j|}\|f\|_{L^2(\mathrm{M}_s M^{\tilde{\omega}}_{\lambda,s}u)}. \tag{3-63}$$

By (3-62), (3-63), and an interpolation (see [Bergh and Löfström 1976, Corollary 5.5.4]), we have

$$\|G_{\lambda,j}f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s}2^{-\vartheta(p,q,\gamma,s)|j|}\|f\|_{L^p(\mathrm{M}_s M^{\tilde{\omega}}_{\lambda,s}u)} \tag{3-64}$$

for all $2\le p<\infty$ and $s>1$, where $\vartheta(p,q,\gamma,s)>0$ depends on p, q, γ and s. Combining (3-64) with (3-34) yields that

$$\|G^\omega_\lambda f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s}\|f\|_{L^p(\mathrm{M}_s M^{\tilde{\omega}}_s u)} \tag{3-65}$$

for all $2\le p<\infty$ and $s>1$. We get from (3-9) that

$$\mathrm{M}_s M^{\tilde{\omega}}_{\lambda,s}u \le C(\mathrm{M}_s M^{\tilde{\sigma}}_{\lambda,s}|u| + \mathrm{M}_s^2 M^{\tilde{\sigma}}_{\lambda-1,s}|u|). \tag{3-66}$$

Inequality (3-66) together with (3-65) yields

$$\|G_\lambda^\omega f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s}\|f\|_{L^p(\mathrm{M}_s M_{\lambda,s}^{\tilde{\sigma}}|u|+\mathrm{M}_s^2 M_{\lambda-1,s}^{\tilde{\sigma}}|u|)} \tag{3-67}$$

for all $2 \le p < \infty$ and $s > 1$. On the other hand, from (3-10), (3-67), and the well-known Fefferman–Stein inequality for M (see (3-102) below) we have

$$\begin{aligned}\|M_\lambda^\sigma f\|_{L^p(u)} &\le \|\mathrm{M}M_{\lambda-1}^\sigma|f|\|_{L^p(u)} + \|G_\lambda^\omega f\|_{L^p(u)}\\ &\le C_p\|M_{\lambda-1}^\sigma|f|\|_{L^p(\mathrm{M}u)} + C_{h,\Omega,q,\gamma,p,\lambda,s}\|f\|_{L^p(\mathrm{M}_s M_{\lambda,s}^{\tilde{\sigma}}|u|+\mathrm{M}_s^2 M_{\lambda-1,s}^{\tilde{\sigma}}|u|)}\end{aligned} \tag{3-68}$$

for all $2 \le p < \infty$ and any $s > 1$. Formula (3-68) together with (3-1), (3-2), and an induction argument implies that

$$\|M_\lambda^\sigma f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s}\|f\|_{L^p(\mathrm{M}^\lambda u+I_{\lambda,s}u+J_{\lambda,s}u)} \quad \forall 1 \le \lambda \le N. \tag{3-69}$$

Since $u \le \mathrm{M}_s u$ and $\mathrm{M}_s u \le A_1$, (3-69) leads to

$$\|M_\lambda^\sigma f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s}\|f\|_{L^p(\mathrm{M}_s^{\lambda+1}u+I_{\lambda,s}\mathrm{M}_s u+J_{\lambda,s}\mathrm{M}_s u)} \tag{3-70}$$

for all $2 \le p < \infty$, $s > 1$, and any nonnegative function u on $\mathbb{R}^n$. Inequality (3-70) yields that (3-59) holds for all $2 \le p < \infty$.

We now prove (3-60). It follows from (3-11) that

$$\sup_{k\in\mathbb{Z}}\Big|\sum_{j=k}^{\infty}\sigma_{j,N} * f(x)\Big| \le \sum_{\lambda=1}^{N}\sup_{k\in\mathbb{Z}}\Big|\sum_{j=k}^{\infty}\mu_{j,\lambda} * f(x)\Big| =: \sum_{\lambda=1}^{N} T_\lambda^* f(x). \tag{3-71}$$

Fix $1 \le \lambda \le N$, let $\psi_{k,\lambda}$ be given as in (3-7). We write

$$\sum_{j=k}^{\infty}\mu_{j,\lambda}*f(x) = \psi_{k,\lambda}*T_\lambda f(x) - \psi_{k,\lambda}*\sum_{j=-\infty}^{k-1}\mu_{j,\lambda}*f(x) + (\delta-\psi_{k,\lambda})*\sum_{j=k}^{\infty}\mu_{j,\lambda}*f(x).$$

Here, δ is the Dirac-delta and T_λ is given as in (3-14). It follows that

$$\begin{aligned} T_\lambda^* f(x) &\le \sup_{k\in\mathbb{Z}}|\psi_{k,\lambda} * T_\lambda f(x)| + \sup_{k\in\mathbb{Z}}\Big|\psi_{k,\lambda} * \sum_{j=-\infty}^{k-1}\mu_{j,\lambda} * f(x)\Big|\\ &\qquad + \sup_{k\in\mathbb{Z}}\Big|(\delta-\psi_{k,\lambda}) * \sum_{j=k}^{\infty}\mu_{j,\lambda} * f(x)\Big|\\ &=: A_{1,\lambda}f(x) + A_{2,\lambda}f(x) + A_{3,\lambda}f(x).\end{aligned} \tag{3-72}$$

For $A_{1,\lambda} f$, by the well-known Fefferman–Stein inequality for M (see (3-102) below) and (3-15), we obtain

$$
\begin{aligned}
\|A_{1,\lambda} f\|_{L^p(u)} &\le \|\mathrm{M}(T_\lambda f)\|_{L^p(u)} \le C_p \|T_\lambda f\|_{L^p(\mathrm{M}u)} \\
&\le C_{h,\Omega,q,\gamma,p,\lambda,s} \|f\|_{L^p(\Lambda_{N,s}\mathrm{M}u)} \\
&\le C_{h,\Omega,q,\gamma,p,\lambda,s} \|f\|_{L^p(\Theta_{N,s}\mathrm{M}u)}
\end{aligned}
\tag{3-73}
$$

for all $2 \le p < \infty$, $s > 1$, and any nonnegative measurable function u on $\mathbb{R}^n$.

For $A_{2,\lambda} f$, it is clear that

$$
\begin{aligned}
A_{2,\lambda} f(x) &= \sup_{k\in\mathbb{Z}} \left| \sum_{j=1}^{\infty} \psi_{k,\lambda} * \mu_{k-j,\lambda} * f(x) \right| \\
&\le \sum_{j=1}^{\infty} \sup_{k\in\mathbb{Z}} |\psi_{k,\lambda} * \mu_{k-j,\lambda} * f(x)| =: \sum_{j=1}^{\infty} I_j f(x).
\end{aligned}
$$

Consequently,

$$
\|A_{2,\lambda} f\|_{L^p(u)} \le \sum_{j=1}^{\infty} \|I_j f\|_{L^p(u)} \tag{3-74}
$$

for all $1 < p < \infty$ and any nonnegative measurable function u on $\mathbb{R}^n$. We shall show that

$$
\|I_j f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s} \|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s^2 u)} \tag{3-75}
$$

for all $2 \le p < \infty$, any $s > 1$, and any nonnegative measurable function u on $\mathbb{R}^n$. We get by the well-known Fefferman–Stein inequality for M (see (3-102) below), (3-12), and (3-70), that

$$
\begin{aligned}
&\|I_j f\|_{L^p(u)} \\
&\quad \le \|\mathrm{M} M_\lambda^\mu |f|\|_{L^p(u)} \le C_p \|M_\lambda^\mu |f|\|_{L^p(\mathrm{M}u)} \\
&\quad \le C_p (\|M_\lambda^\sigma |f|\|_{L^p(\mathrm{M}^{N-\lambda+1}u)} + \|M_{\lambda-1}^\sigma |f|\|_{L^p(\mathrm{M}^{N-\lambda+2}u)}) \\
&\quad \le C_{h,\Omega,q,\gamma,p,\lambda,s} \|f\|_{L^p(\mathrm{M}_s^{N+2}u + I_{\lambda,s}\mathrm{M}_s^{N-\lambda+2}u + I_{\lambda,s}\mathrm{M}_s^{N-\lambda+3}u + J_{\lambda,s}\mathrm{M}_s^{N-\lambda+2}u + J_{\lambda-1,s}\mathrm{M}_s^{N-\lambda+3}u)} \\
&\quad \le C_{h,\Omega,q,\gamma,p,\lambda,s} \|f\|_{L^p(\mathrm{M}_s^{N+2}u + I_{N,s}\mathrm{M}_s^2 u + J_{N,s}\mathrm{M}_s^2 u)}
\end{aligned}
$$

for all $2 \le p < \infty$, any $s > 1$, and any nonnegative measurable function u on $\mathbb{R}^n$. This proves (3-75).

On the other hand, by (3-13) and Plancherel's theorem, we get

$$\begin{aligned}\|I_j f\|_{L^2(\mathbb{R}^n)}^2 &\le \Big\|\Big(\sum_{k\in\mathbb{Z}} |\psi_{k,\lambda} * \mu_{k-j,\lambda} * f|^2\Big)^{1/2}\Big\|_{L^2(\mathbb{R}^n)}^2 \\ &\le \sum_{k\in\mathbb{Z}} \int_{\{|a_\lambda \xi|\le 2^{-k\lambda}\}} |\hat{\mu}_{k-j,\lambda}(\xi)|^2 |\hat{f}(\xi)|^2\, d\xi \\ &\le C \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |\hat{\mu}_{k-j,\lambda}(\xi)|^2 \chi_{\{|a_\lambda\xi|\le 2^{-k\lambda}\}} |\hat{f}(\xi)|^2\, d\xi \\ &\le C_{h,\Omega,q,\gamma} \sup_{\xi\in\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |a_\lambda 2^{\lambda(k-j)}\xi|^{1/(2\lambda q'\gamma')} \chi_{\{|a_\lambda\xi|\le 2^{-k\lambda}\}} \|f\|_{L^2(\mathbb{R}^n)}^2 \\ &\le C_{h,\Omega,q,\gamma}^{-j/(2q'\gamma')} \sup_{\xi\in\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |2^{k\lambda} a_\lambda \xi|^{1/(2\lambda q'\gamma')} \chi_{\{|a_\lambda\xi|\le 2^{-k\lambda}\}} \|f\|_{L^2(\mathbb{R}^n)}^2 \\ &\le C_{h,\Omega,q,\gamma} 2^{-j/(2q'\gamma')} \|f\|_{L^2(\mathbb{R}^n)}^2,\end{aligned}$$

where the last inequality is obtained by the properties of the lacunary sequence. It follows that

$$\|I_j f\|_{L^2(\mathbb{R}^n)} \le C_{h,\Omega,q,\gamma} 2^{-j/(4q'\gamma')} \|f\|_{L^2(\mathbb{R}^n)}. \tag{3-76}$$

On the other hand, by (3-75) with $p=2$ and u replacing u^s, we get

$$\|I_j f\|_{L^2(u^s)} \le C_{h,\Omega,q,\gamma,\lambda,s} \|f\|_{L^2(\Theta_{N,s} \mathrm{M}_s^2 u^s)}. \tag{3-77}$$

An interpolation between (3-76) and (3-77) yields

$$\begin{aligned}\|I_j f\|_{L^2(u)} &\le C_{h,\Omega,q,\gamma,\lambda,s} 2^{-(1-1/s)/(4q'\gamma')j} \|f\|_{L^2((\Theta_{N,s}\mathrm{M}_s^2 u^s)^{1/s})} \\ &\le C_{h,\Omega,q,\gamma,\lambda,s} 2^{-(1-1/s)/(4q'\gamma')j} \|f\|_{L^2(\Theta_{N,s^2}\mathrm{M}_{s^2}^2 u)}.\end{aligned} \tag{3-78}$$

Take s^2 replacing s. Formula (3-78) leads to

$$\|I_j f\|_{L^2(u)} \le C_{h,\Omega,q,\gamma,\lambda,s} 2^{-(1-1/\sqrt{s})/(4q'\gamma')} \|f\|_{L^2(\Theta_{N,s}\mathrm{M}_s^2 u)}. \tag{3-79}$$

Interpolating between (3-79) and (3-75) (see [Bergh and Löfström 1976, Corollary 5.5.4]) yields

$$\|I_j f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s} 2^{-\varsigma(p,q,\gamma,s)j} \|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s^2 u)}, \tag{3-80}$$

for all $2 \le p < \infty$, where $\varsigma(p,q,\gamma,s) > 0$. Thus, we get from (3-80) and (3-74),

$$\|A_{2,\lambda} f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s} \|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s^2 u)} \tag{3-81}$$

for all $2 \le p < \infty$, $s > 1$, and any nonnegative measurable function u on $\mathbb{R}^n$.

Finally we estimate $A_{3,\lambda}f$. Obviously,

$$\begin{aligned}A_{3,\lambda}f(x)&=\sup_{k\in\mathbb{Z}}\Big|\sum_{j=0}^{\infty}(\delta-\psi_{k,\lambda})*\mu_{k+j,\lambda}*f(x)\Big|\\&\le\sum_{j=0}^{\infty}\sup_{k\in\mathbb{Z}}|(\delta-\psi_{k,\lambda})*\mu_{k+j,\lambda}*f(x)|\\&=:\sum_{j=0}^{\infty}J_jf(x).\end{aligned}$$

It follows that

$$\|A_{3,\lambda}f\|_{L^p(u)}\le\sum_{j=0}^{\infty}\|J_jf\|_{L^p(u)}\tag{3-82}$$

for all $1<p<\infty$ and any nonnegative measurable function u on $\mathbb{R}^n$. By the argument similar to those used to derive (3-75),

$$\|J_jf\|_{L^p(u)}\le C_{h,\Omega,q,\gamma,p,\lambda,s}\|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s^2u)}\tag{3-83}$$

for all $2\le p<\infty$, any $s>1$, and any nonnegative measurable function u on $\mathbb{R}^n$. On the other hand, using (3-13) and the Plancherel theorem, we can obtain

$$\begin{aligned}\|J_jf\|_{L^2(\mathbb{R}^n)}^2&\le\bigg\|\Big(\sum_{k\in\mathbb{Z}}|(\delta-\psi_{k,\lambda})*\mu_{j+k,\lambda}*f|^2\Big)^{1/2}\bigg\|_{L^2(\mathbb{R}^n)}^2\\&\le\sum_{k\in\mathbb{Z}}\int_{\{2^{k\lambda}a_\lambda\xi|\ge1\}}|\hat{\mu}_{j+k,\lambda}(\xi)|^2|\hat{f}(\xi)|^2\,d\xi\\&\le\sum_{k\in\mathbb{Z}}\sum_{i=-\infty}^{k}\int_{\{2^{-\lambda i}\le|a_\lambda\xi|<2^{-\lambda(i-1)}\}}|\hat{\mu}_{j+k,\lambda}(\xi)|^2|\hat{f}(\xi)|^2\,d\xi\\&\le C_{h,\Omega,q,\gamma}\sum_{k\in\mathbb{Z}}\sum_{i=-\infty}^{k}2^{-(j+k-i)/(2q'\gamma')}\int_{\{2^{-\lambda i}\le|a_\lambda\xi|<2^{-\lambda(i-1)}\}}|\hat{f}(\xi)|^2\,d\xi\\&\le C_{h,\Omega,q,\gamma}2^{-j/(2q'\gamma')}\sum_{i=0}^{\infty}2^{-i/(2q'\gamma')}\|f\|_{L^2(\mathbb{R}^n)}^2\\&\le C_{h,\Omega,q,\gamma}2^{-j/(2q'\gamma')}\|f\|_{L^2(\mathbb{R}^n)}^2.\end{aligned}$$

It follows that

$$\|J_jf\|_{L^2(\mathbb{R}^n)}\le C_{h,\Omega,q,\gamma}2^{-j/(4q'\gamma')}\|f\|_{L^2(\mathbb{R}^n)}.\tag{3-84}$$

By (3-83), (3-84), and arguments similar to those used in deriving (3-80),

$$\|J_j f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s} 2^{-\tau(p,q,\gamma,s)j} \|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s^2 u)} \tag{3-85}$$

for all $2 \le p < \infty$ and $s > 1$, where $\tau(p,q,\gamma,s) > 0$. Equation (3-85) together with (3-82) yields

$$\|A_{3,\lambda} f\|_{L^p(u)} \le C_{h,\Omega,q,\lambda,\gamma} \|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s^2 u)} \tag{3-86}$$

for all $2 \le p < \infty$ and $s > 1$. Then (3-60) follows from (3-71)–(3-73), (3-81), and (3-86). □

Step 2: The proof of (ii) of Theorem 1.2. Let $1 < p < 2$ and $\{t_k\}_{k\in\mathbb{N}}$ be given as in Theorem 1.2. Fix a nonnegative measurable function u on $\mathbb{R}^n$. By (3-58), to prove (1-10), it suffices to show that

$$\|M_N^\sigma f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,N,s,t_k} \|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s u)} \quad \forall s > t_k \tag{3-87}$$

and

$$\Big\| \sup_{k\in\mathbb{Z}} \Big| \sum_{j=k}^{\infty} \sigma_{j,N} * f \Big| \Big\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,N,s,t_k} \|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s u + \Theta_{N,s}\mathrm{M}_s^2 u)} \quad \forall s > t_k \tag{3-88}$$

for all $k \in \mathbb{N}$.

We first prove (3-87). Fix $k \in \mathbb{N}$. Substitute u^{t_k} for u in (3-56), one has

$$\int_{\mathbb{R}^n} (M_\lambda^\sigma f(x))^p u(x)\,dx \le C_{h,\Omega,q,\gamma,p,\lambda,t_k} \int_{\mathbb{R}^n} |f(x)|^p (\mathrm{M}^\lambda u^{t_k} + \mathrm{M}^2 \widetilde{\mathrm{M}^\lambda u^{t_k}} + H_\lambda u^{t_k})^{1/t_k}(x)\,dx \tag{3-89}$$

for all $1 \le \lambda \le N$. Notice that

$$(\mathrm{M}^\lambda u^{t_k} + \mathrm{M}^2 \widetilde{\mathrm{M}^\lambda u^{t_k}} + H_\lambda u^{t_k})^{1/t_k}(x) \le C_{s,t_k} (\mathrm{M}^\lambda u^s + \mathrm{M}^2 \widetilde{\mathrm{M}^\lambda u^s} + H_\lambda u^s)^{1/s}(x)$$

for any $s > t_k$ by Hölder's inequality. Then (3-89) yields that

$$\int_{\mathbb{R}^n} (M_\lambda^\sigma f(x))^p u(x)\,dx \le C_{h,\Omega,q,\gamma,p,\lambda,t_k} \int_{\mathbb{R}^n} |f(x)|^p (\mathrm{M}_s^\lambda u + \mathrm{M}_s^2 \widetilde{\mathrm{M}_s^\lambda} u + H_{\lambda,s} u)(x)\,dx \quad \forall s > t_k \tag{3-90}$$

holds for all $1 \le \lambda \le N$ and any fixed positive integer k, which gives (3-87).

Below we prove (3-88). For $A_{1,\lambda} f$, by the well-known Fefferman–Stein inequality for M (see (3-102) below) and (3-30), we obtain

$$
\begin{aligned}
\|A_{1,\lambda} f\|_{L^p(u)} &\le C\|\mathrm{M}(T_\lambda f)\|_{L^p(u)} \le C_p\|T_\lambda f\|_{L^p(\mathrm{M}u)} \\
&\le C_{h,\Omega,q,\gamma,p,\lambda,s,t_k}\|f\|_{L^p(\Lambda_{N,s}\mathrm{M}u)} \\
&\le C_{h,\Omega,q,\gamma,p,\lambda,s,t_k}\|f\|_{L^p(\Theta_{N,s}\mathrm{M}u)} \quad \forall s > t_k
\end{aligned} \tag{3-91}
$$

for any fixed positive integer k.

For $A_{2,\lambda} f$, it follows from the well-known Fefferman–Stein inequality for M (see (3-102) below), (3-12), and (3-91) that

$$
\begin{aligned}
&\|I_j f\|_{L^p(u)} \\
&\quad\le C\|\mathrm{M}M_\lambda^\mu f\|_{L^p(u)} \le C_p\|M_\lambda^\mu f\|_{L^p(\mathrm{M}u)} \\
&\quad\le C_p(\|M_\lambda^\sigma|f|\|_{L^p(\mathrm{M}^{N-\lambda+1}u)} + \|M_{\lambda-1}^\sigma|f|\|_{L^p(\mathrm{M}^{N-\lambda+2}u)}) \\
&\quad\le C_{h,\Omega,q,\gamma,p,\lambda,s,t_k}\|f\|_{L^p(\mathrm{M}_s^N Mu+\mathrm{M}_s^2\widetilde{\mathrm{M}_s^N\mathrm{M}}u+H_{\lambda,s}\mathrm{M}^{N-\lambda+1}u+H_{\lambda-1,s}\mathrm{M}^{N-\lambda+2}u)} \\
&\quad\le C_{h,\Omega,q,\gamma,p,\lambda,s,t_k}\|f\|_{L^p(\mathrm{M}_s^N\mathrm{M}u+\mathrm{M}_s^2\widetilde{\mathrm{M}_s^N\mathrm{M}}u+H_{N,s}\mathrm{M}u)} \\
&\quad\le C_{h,\Omega,q,\gamma,p,\lambda,s,t_k}\|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s^2u)} \quad \forall s > t_k
\end{aligned} \tag{3-92}
$$

for any fixed positive integer k. Interpolating between (3-79) and (3-92) (see [Bergh and Löfström 1976, Corollary 5.5.4]) yields

$$
\|I_j f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s,t_k} 2^{-\delta(p,q,\gamma,s)j}\|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s^2u)} \quad \forall s > t_k, \tag{3-93}
$$

where $\delta(p,q,\gamma,s) > 0$. Thus, we get from (3-93) and (3-74) that

$$
\|A_{2,\lambda} f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s,t_k}\|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s^2u)} \quad \forall s > t_k \tag{3-94}
$$

for any fixed positive integer k.

For $A_{3,\lambda} f$, by the argument similar to those used to derive (3-92),

$$
\|J_j f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s,t_k}\|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s^2u)} \quad \forall s > t_k \tag{3-95}
$$

for any fixed positive integer k. By (3-95), (3-84), and arguments similar to those used in deriving (3-85),

$$
\|J_j f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s,t_k} 2^{-o(p,q,\gamma,s)j}\|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s^2u)} \quad \forall s > t_k \tag{3-96}
$$

for any fixed positive integer k, where $o(p,q,\gamma,s) > 0$. Inequality (3-96) together with (3-82) yields

$$
\|A_{3,\lambda} f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,\lambda,s,t_k}\|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s^2u)} \quad \forall s > t_k \tag{3-97}
$$

for any fixed positive integer k. Then (3-88) follows from (3-71), (3-72), (3-92), (3-94), and (3-97).

We now turn to proving Corollaries 1.3–1.5.

Proof of Corollary 1.3. By (3-6), one finds that

$$\|\Lambda_{N,s} f\|_{L^r(\mathbb{R}^n)} \le C_{h,\Omega,q,\gamma} \|f\|_{L^r(\mathbb{R}^n)} \tag{3-98}$$

for any $1 < s < \infty$ and $r > s$. We let $\{t_k\}$ be the sequence as in (ii) of Theorem 1.1 when $1 < p < 2$, and, for the sake of convenience, we set $\{t_k\} = \{1 + 1/k\}$ when $2 \le p < \infty$. It is clear that $\{t_k\}_{k\in\mathbb{N}}$ is strictly decreasing and $\lim_{k\to\infty} t_k = 1$. It follows from (1-7) and (1-8) that for $1 < p < \infty$, it holds that

$$\|T_{P_N} f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,s,N,t_k} \|f\|_{L^p(\Lambda_{N,s}u)} \quad \forall s > t_k \tag{3-99}$$

for any fixed positive integer k and any nonnegative measurable function u on $\mathbb{R}^n$. By (3-98), (3-99), and Proposition 2.1, we have (1-11) and (1-12) for the case of $1 < \tilde{p} < p < \infty$. On the other hand, by duality, we can obtain (1-11) and (1-12) for the case of $1 < p < \tilde{p} < \infty$. Taking $u = 1$, we obtain $\Lambda_{N,s}u \le C$. This together with (3-99) yields that T_{P_N} is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. It leads to (1-11) for the case $p = \tilde{p}$ by (1-1). The inequality (1-12) for the case $p = \tilde{p}$ follows from (1-1) and (1-15). This completes the proof of Corollary 1.3. $\square$

Proof of Corollary 1.4. By (3-6), one finds that

$$\|\Theta_{N,s}\mathrm{M}_s u + \Theta_{N,s}\mathrm{M}_s^2 u\|_{L^r(\mathbb{R}^n)} \le C_{h,\Omega,q,\gamma} \|u\|_{L^r(\mathbb{R}^n)} \tag{3-100}$$

for any $1 < s < \infty$ and $r > s$. We let $\{t_k\}$ be the sequence as in (ii) of Theorem 1.1 when $1 < p < 2$, and, for the sake of convenience, we set $\{t_k\} = \{1 + 1/k\}$ when $2 \le p < \infty$. Then Theorem 1.2 yields

$$\|T^*_{P_N} f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p,s,N,t_k} \|f\|_{L^p(\Theta_{N,s}\mathrm{M}_s u+\Theta_{N,s}\mathrm{M}_s^2 u)} \quad \forall s > t_k \tag{3-101}$$

for any fixed positive integer k and any nonnegative measurable function u on $\mathbb{R}^n$. By (3-100), (3-101), and Proposition 2.1, we obtain (1-13) and (1-14) for the case $1 < \tilde{p} < p < \infty$. It was known that $T^*_{P_N}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. This together with (1-1) yields (1-13) for the case of $p = \tilde{p}$. The inequality (1-14) for the case of $p = \tilde{p}$ follows from (1-1) and (1-16). This finishes the proof of Corollary 1.4. $\square$

Proof of Corollary 1.5. By (3-98), (3-99), and Proposition 2.1, we obtain (1-15) for the case of $1 < \tilde{p} < p < \infty$. On the other hand, a duality argument yields (1-15) for the case of $1 < p < \tilde{p} < \infty$. Inequality (1-15) for the case $p = \tilde{p}$ follows from (3-98), (3-99), and the L^p bounds for T_{P_N}. Similarly, we can obtain (1-16) for the case of $1 < \tilde{p} \le p < \infty$ by (3-100), (3-101), and the L^p boundedness of $T^*_{P_N}$. This completes the proof of Corollary 1.5. $\square$

We want to make a few remarks before ending the paper. Since Proposition 2.1 plays a crucial rule to show the boundedness of an operator T on the mixed radial-angular space $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)$, we expect to establish a suitable weighted L^p inequality for T. To this end, for the operator $T_{h,\Omega}$, we need to treat some technical difficulties for different assumptions on Ω. This is a key step, but is definitely not trivial. For instance, we have no idea how to establish a suitable weighted L^p inequality for $T_{h,\Omega}$, although the $L^p(\mathbb{R}^n)$ boundedness of $T_{h,\Omega}$ is well known, if Ω is a function in the function class $L\log L(\mathrm{S}^{n-1})$. For the singular integral T_Ω, another roughness assumption on Ω is that Ω lies in the Grafakos–Stefanov class $\mathscr{F}_\alpha(\mathrm{S}^{n-1})$, where

$$\begin{aligned}&\mathscr{F}_\alpha(\mathrm{S}^{n-1})\\&:=\left\{\Omega\in L^1(\mathrm{S}^{n-1}):\sup_{\xi\in\mathrm{S}^{n-1}}\int_{\mathrm{S}^{n-1}}|\Omega(y')|\left(\log\frac{1}{|\xi\cdot y'|}\right)^\alpha d\sigma(y')<\infty\right\}\quad\text{for}\quad\alpha>0,\end{aligned}$$

and this class was originally introduced by Grafakos and Stefanov [1998] in the study of L^p boundedness of T_Ω. With the help of the established weighted L^p inequality for T_Ω (see [Zhang 2008, Lemma 2]) applying [Zhang 2008, Theorems 1 and 2], and Proposition 2.1, we can show that both T_Ω and its maximal operator T^*_Ω are bounded on $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)$ for any $1<p<\infty$ and $1<\tilde{p}<\infty$ provided $\Omega\in\mathscr{F}_\alpha(\mathrm{S}^{n-1})$ for all $\alpha>1$.

Not only for rough singular integrals, Proposition 2.1 also works for all linear or sublinear operators. The Hardy–Littlewood maximal function M is bounded on $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)$, based on Proposition 2.1 and the well-known Fefferman–Stein [1971] weighted norm inequality

$$\|\mathrm{M}f\|_{L^p(u)}\le C_p\|f\|_{L^p(\mathrm{M}u)}. \tag{3-102}$$

Also, any Calderón–Zygmund operator T is bounded on $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)$ for any $1<p<\infty$ and $1<\tilde{p}<\infty$ because of Proposition 2.1 and the well-known inequality

$$\|\mathrm{T}f\|_{L^p(u)}\le C_p\|f\|_{L^p(\mathrm{M}_s u)},\qquad s>1.$$

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FENG LIU
COLLEGE OF MATHEMATICS AND SYSTEMS SCIENCE
SHANDONG UNIVERSITY OF SCIENCE AND TECHNOLOGY
QINGDAO
CHINA
liufeng860314@163.com

DASHAN FAN
DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF WISCONSIN-MILWAUKEE
MILWAUKEE, WI
UNITED STATES
fan@uwm.edu

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