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$\infty$ -TILTING THEORY

LEONID POSITSELSKI AND JAN ŠTOVÍČEK

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**We define the notion of an infinitely generated tilting object of infinite homological dimension in an abelian category. A one-to-one correspondence between  $\infty$ -tilting objects in complete, cocomplete abelian categories with an injective cogenerator and  $\infty$ -cotilting objects in complete, cocomplete abelian categories with a projective generator is constructed. We also introduce  $\infty$ -tilting pairs, consisting of an  $\infty$ -tilting object and its  $\infty$ -tilting class, and obtain a bijective correspondence between  $\infty$ -tilting and  $\infty$ -cotilting pairs. Finally, we discuss the related derived equivalences and t-structures.**

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**Introduction**

The phrase “tilting theory” is often used to refer to a well-developed general machinery for producing equivalences between triangulated categories (see [Angeleri Hügel et al. 2007] for an introduction, history and applications). Such equivalences are often represented by a distinguished object, a so-called tilting object, and it is crucial to most of the theory that such a tilting object is homologically small. If  $A$  is an abelian category with exact coproducts (e.g., a category of modules over a ring or sheaves on a topological space) and  $T$  is a tilting object, the smallness typically translates at least to the assumptions that  $T$  is finitely generated and of finite projective dimension.

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In this paper we introduce and systematically develop  $\infty$ -tilting theory, where all homological smallness assumptions are dropped. This brings under one roof various concepts and results from the literature:

- (1) Wakamatsu tilting modules [Mantese and Reiten 2004; Wakamatsu 1988; 1990] over finite-dimensional algebras.
- (2) Semidualizing bimodules and the Foxby equivalence [Christensen 2000; Holm and White 2007; Positselski 2017a].
- (3) The comodule–contramodule [Positselski 2010, Section 0.2 and Chapter 5] and the semimodule–semicontramodule [Positselski 2010, Sections 0.3.7 and 6.3] correspondences.

These results come with rather different motivations: from criteria for stable equivalences of finite-dimensional self-injective algebras in (1), through Gorenstein homological algebra in (2), to the representation theory of infinite-dimensional Lie algebras (e.g., the Virasoro or Kac–Moody algebras) in (3).

A part of the work has been done in our previous paper [Positselski and Šťovíček 2019], where we explained how the finite generation assumption can be naturally dropped with help of additive monads and, in several cases of interest, with topological rings.

In this paper we focus on dropping the assumption of finite homological dimension. It turns out that we still obtain triangulated equivalences and (co)tilting  $t$ -structures, but in general not for the conventional derived categories of two abelian categories, but rather for a so-called *pseudo-coderived* category of one of them and a *pseudo-contraderived* category of the other.

Here, a pseudo-coderived category of an abelian category  $A$  is a certain triangulated category  $D$  to which  $A$  fully embeds as the heart of a  $t$ -structure and such that  $\text{Ext}_A^i(X, Y)$  is canonically isomorphic to  $\text{Hom}_D(X, Y[i])$  for all  $X, Y \in A$  and  $i \geq 0$ . The term “pseudo-coderived” comes from the fact that, under reasonable assumptions satisfied in particular in the situations (1)–(3) above, the pseudo-coderived category is an intermediate Verdier quotient between the conventional derived category  $D(A)$  and the coderived category  $D^{\text{co}}(A)$  (which is none other than the homotopy category  $\text{Hot}(A_{\text{inj}})$  of complexes of injective objects if  $A$  is a locally Noetherian Grothendieck category). A pseudo-contraderived category has formally dual properties. Pseudo-co/contraderived categories are in fact not determined uniquely by their abelian hearts, but depend on a certain parameter, so that we often do not get just a single triangulated equivalence, but rather a family of compatible triangulated equivalences. We refer to [Positselski 2017a] for an in-depth discussion of this new class of triangulated categories.

To put our results into context, we briefly recall the history of tilting theory, which evolved through a series of successive generalizations in several directions.

The definition of what is now known as a finitely generated tilting module of projective dimension 1 over a finite-dimensional associative algebra first appeared in [Happel and Ringel 1982] (see also [Bongartz 1981]), which built upon [Brenner and Butler 1980]. The main result was the so-called *tilting theorem*, or the *Brenner–Butler theorem*, establishing equivalences between certain additive subcategories of the categories of finitely generated modules over an algebra  $R$  and over the endomorphism algebra  $S$  of a tilting  $R$ -module. Happel [1987] proved that a tilting module induces a triangulated equivalence between the derived categories of finitely generated  $R$ -modules and  $S$ -modules.

Finitely presented tilting modules of projective dimension 1 over arbitrary rings were discussed by Colby and Fuller [1990], while finitely presented tilting modules of arbitrary finite projective dimension  $n$  were studied by Miyashita [1986] and Cline, Parshall and Scott [Cline et al. 1986]. The tilting theorem (for categories of infinitely generated modules) was proved in [Miyashita 1986], and the related derived equivalence was constructed in [Cline et al. 1986]. Infinitely generated tilting modules of projective dimension 1 (now also known as *big 1-tilting modules*) were defined by Colpi and Trlifaj [1995]. The tilting theorem for self-small tilting objects of projective dimension 1 in Grothendieck abelian categories was obtained by Colpi [1999]. Cotilting modules of injective dimension 1 were introduced by Colby and Fuller [1990] and Colpi, D’Este and Tonolo [Colpi et al. 1997a] (see also [Colpi et al. 1997b]). Finally, infinitely generated tilting modules of projective dimension  $n$  and cotilting modules of injective dimension  $n$  (*big  $n$ -tilting* and  *$n$ -cotilting modules*) were defined by Angeleri Hügel and Coelho [2001] and characterized by Bazzoni [2004].

The main results of the infinitely generated tilting theory claim that all  $n$ -tilting modules are of finite type [Bazzoni and Šťovíček 2007] and all  $n$ -cotilting modules are pure-injective [Šťovíček 2006]. The tilting theorem for big 1-tilting modules was obtained in some form by Gregorio and Tonolo [2001]. Another approach, based on a previous work by Facchini, was developed by Bazzoni [2010], who also proved that the derived category of  $R$ -modules is equivalent to a full subcategory and a quotient category of the derived category of  $S$ -modules when  $S$  is the endomorphism ring of a big 1-tilting  $R$ -module. This was extended to big  $n$ -tilting modules by Bazzoni, Mantese, and Tonolo [Bazzoni et al. 2011]. A correspondence between  $n$ -cotilting modules and small  $n$ -tilting objects in Grothendieck abelian categories together with the related derived equivalence were constructed in [Šťovíček 2014a]. Big  $n$ -tilting objects in abelian categories were defined and the related derived equivalence was obtained by Nicolás, Saorín and Zvonareva [Nicolás et al. 2019] and Fiorot, Mattiello and Saorín [Fiorot et al. 2017] (see also Psaroudakis and Vitória [2018]). Finally, a correspondence between big  $n$ -tilting and  $n$ -cotilting objects in abelian categories was constructed in [Positselski and Šťovíček 2019].

The main innovation in [Positselski and Šťovíček 2019], which allows one to obtain very naturally derived equivalences from big  $n$ -tilting objects and which is based on the ideas previously developed in [Šťovíček 2014a; Psaroudakis and Vitória 2018; Nicolás et al. 2019; Positselski and Rosický 2017], is that to a big tilting object  $T$  in an abelian category  $A$  one can assign a richer structure than its ring of endomorphisms  $\text{Hom}_A(T, T)$ . For any set  $X$ , consider the set of all morphisms  $T \rightarrow T^{(X)}$  in  $A$ , where  $T^{(X)}$  denotes the coproduct of  $X$  copies of  $T$ . Then the endofunctor  $X \mapsto \text{Hom}_A(T, T^{(X)})$  is a monad on the category of sets. The tilting heart  $B$  corresponding to the tilting object  $T \in A$  is the abelian category of all algebras (which we also call modules) over this monad. In many naturally occurring situations, one can equip  $\text{Hom}_A(T, T)$  with a complete and separated topology so that  $\text{Hom}_A(T, T^{(X)})$  identifies with families of elements of  $\text{Hom}_A(T, T)$  indexed by  $X$  which converge to zero.

A notion of a finitely generated tilting module of infinite projective dimension (now known as *Wakamatsu tilting modules*) was introduced in the representation theory of finite-dimensional algebras by Wakamatsu in [Wakamatsu 1988; 1990] and it was studied further by Mantese and Reiten in [2004].

In the present paper we work out a common generalization of two lines of thought described above, namely of big  $n$ -tilting/cotilting modules and finite-dimensional Wakamatsu tilting modules. We develop a theory of big tilting and cotilting objects of possibly infinite homological dimension in abelian categories. Our goal is also to put on a rigorous footing the discussion of “ $\infty$ -tilting objects” in [Positselski and Šťovíček 2019, Sections 10.1, 10.2 and 10.3]. The structure of the paper is as follows.

To a complete, cocomplete abelian category  $A$  with an injective cogenerator  $J$  and an  $\infty$ -tilting object  $T$  we associate in Section 2 a complete, cocomplete abelian category  $B$  with a projective generator  $P$  and an  $\infty$ -cotilting object  $W$ . We do so in such a way that, up to equivalence, this induces a bijective correspondence between the triples  $(A, T, J)$  and  $(B, P, W)$ .

In order to obtain the announced version of derived equivalences, we need to associate to each  $\infty$ -tilting object a certain coresolving subcategory  $E \subset A$  which plays the role of the tilting class in [Positselski and Šťovíček 2019]. This is discussed in Section 3. Such a class  $E$  is in general not unique, but the possible choices form a complete lattice with respect to the inclusion. Having chosen  $E$ , we already obtain a uniquely determined full subcategory  $F \subset B$  which plays the role of a cotilting class, and equivalences  $E \simeq F$  and  $D(E) \simeq D(F)$ . Each of  $D(E)$  and  $D(F)$  comes naturally equipped with two t-structures, and the two abelian categories  $A$  and  $B$  are the hearts of these two t-structures (see Section 5).

If, moreover, the  $\infty$ -tilting class  $E$  is closed under coproducts in  $A$  and the  $\infty$ -cotilting class  $F$  is closed under products in  $B$ , then we show in Section 4 that

$D(E)$  is a pseudo-coderived category and  $D(F)$  is a pseudo-contraderived category in the sense of [Positselski 2017a]. Although the above closure properties of  $E$  and  $F$  are not automatic in our setup, they are satisfied for our motivating classes of examples mentioned above and examined in detail in Section 6.

### 1. The tilted and cotilted abelian categories

Given an additive category  $C$  with set-indexed coproducts and an object  $M \in C$ , we denote by  $\text{Add}(M) \subset C$  the full subcategory formed by the direct summands of coproducts of copies of  $M$  in  $C$ . Similarly, given an additive category  $C$  with set-indexed products and an object  $L \in C$ , we denote by  $\text{Prod}(L) \subset C$  the full subcategory formed by the direct summands of products of copies of  $L$  in  $C$ . Given a set  $X$ , the coproduct of  $X$  copies of  $M$  is denoted by  $M^{(X)} \in \text{Add}(M)$  and the product of  $X$  copies of  $L$  is denoted by  $L^X \in \text{Prod}(L)$ .

We say that an additive category is *idempotent-complete* (or, in other terminology, Karoubian or pseudo-abelian) if it contains the images of all idempotent endomorphisms of its objects.

**Theorem 1.1.** (a) *Let  $C$  be an idempotent-complete additive category with coproducts and  $M \in C$  be an object. Then there exists a unique abelian category  $B$  with enough projective objects such that the full subcategory of projective objects  $B_{\text{proj}} \subset B$  is equivalent to the full subcategory  $\text{Add}(M) \subset C$ . The abelian category  $B$  has products, coproducts, and a natural projective generator  $P \in B_{\text{proj}}$  corresponding to the object  $M \in \text{Add}(M)$ .*

(b) *Let  $C$  be an idempotent-complete additive category with products and  $L \in C$  be an object. Then there exists a unique abelian category  $A$  with enough injective objects such that the full subcategory of injective objects  $A_{\text{inj}} \subset A$  is equivalent to the full subcategory  $\text{Prod}(L) \subset C$ . The abelian category  $A$  has products, coproducts, and a natural injective cogenerator  $J \in A_{\text{inj}}$  corresponding to the object  $L \in \text{Prod}(L)$ .*

*Proof.* (a) The category  $B$  is unique, because an abelian category with enough projective objects is determined by its full subcategory of projective objects [Šťovíček 2014a, proof of Theorem 6.2; Positselski 2015, proof of Theorem 3.6].

To prove existence, one can construct  $B$  as the category of finitely presented (coherent) contravariant functors on  $\text{Add}(M)$ . This category can be also described as the quotient category of the category  $\text{Add}(M)^2$  of morphisms in  $\text{Add}(M)$  by the ideal of all morphisms in  $\text{Add}(M)^2$  which factorize through objects of the full subcategory in  $\text{Add}(M)^2$  consisting of all the split epimorphisms in  $\text{Add}(M)$  [Beligiannis 2000]. The category  $B$  is abelian, because the additive category  $\text{Add}(M)$  is right coherent (has weak kernels) [Freyd 1966, Corollary 1.5; Krause 1998, Lemma 2.2, Proposition 2.3; Beligiannis 2000, Proposition 4.5(1); Krause 2002, Lemma 1(1)] (see also [Fiorot 2016, Appendix B] for a further discussion

and references). Indeed, if  $f : M' \rightarrow M''$  is a morphism in  $\text{Add}(M)$  and  $X$  is the set of all morphisms  $M \rightarrow M'$  whose composition with  $f$  vanishes, then the natural morphism  $M^{(X)} \rightarrow M'$  is a weak kernel of  $f$  in  $\text{Add}(M)$  (cf. [Krause 2002, Lemma 2(1)]).

Even more explicitly,  $\mathbb{B}$  is the category of modules over the monad

$$\mathbb{T} : X \mapsto \text{Hom}_{\mathbb{C}}(M, M^{(X)})$$

on the category of sets (we call modules here what are often called monadic  $\mathbb{T}$ -algebras, since they generalize ordinary modules over a ring; see the discussions in the introduction to [Positselski and Rosický 2017], [Positselski 2017b, Lemma 1.1 and Example 1.2(2)], and [Positselski and Šťovíček 2019, Sections 6.1 and 6.3]).

Coproducts in the category of coherent functors exist [Krause 2002, Lemma 1(2)]; more generally, whenever the category of projective objects  $\mathbb{B}_{\text{proj}}$  in an abelian category  $\mathbb{B}$  with enough projective objects has coproducts, the coproducts in  $\mathbb{B}$  can be constructed in terms of the coproducts in  $\mathbb{B}_{\text{proj}}$  (and the embedding functor  $\mathbb{B}_{\text{proj}} \rightarrow \mathbb{B}$  preserves coproducts). Products exist in the category of algebras over every monad  $\mathbb{T} : \text{Sets} \rightarrow \text{Sets}$  and are preserved by the forgetful functor from the category of  $\mathbb{T}$ -algebras to  $\text{Sets}$  (coproducts also exist in the category of  $\mathbb{T}$ -algebras, but are not preserved by the forgetful functor). The natural projective generator  $P \in \mathbb{B}_{\text{proj}}$  is the free  $\mathbb{T}$ -algebra/module with one generator.

(b) This is dual to the proof of (a). Explicitly,  $\mathbb{A}$  is the opposite category to the category of coherent covariant functors on  $\text{Prod}(L)$ , or the opposite category to the category of modules over the monad  $\mathbb{T} : X \mapsto \text{Hom}_{\mathbb{C}}(L^X, L)$  on the category of sets.  $\square$

We will use the notation  $\mathbb{B} = \sigma_M(\mathbb{C})$  and  $\mathbb{A} = \pi_L(\mathbb{C})$ . Assuming that  $M \in \mathbb{C}$  is a “tilting object” in one sense or another (cf. Section 2), one can call  $\mathbb{B}$  the *abelian category tilted from  $\mathbb{C}$  at  $M$* . Similarly, assuming that  $L \in \mathbb{C}$  is a “cotilting object” in some sense, one can call  $\mathbb{A}$  the *abelian category cotilted from  $\mathbb{C}$  at  $L$* .

Now let us assume that  $\mathbb{C}$  is an abelian category. Then, in the context of Theorem 1.1(a), the additive embedding functor  $\Phi_{\text{proj}} : \mathbb{B}_{\text{proj}} \simeq \text{Add}(M) \rightarrow \mathbb{C}$  can be uniquely extended to a right exact functor  $\Phi : \mathbb{B} \rightarrow \mathbb{C}$ . To compute the object  $\Phi(B) \in \mathbb{C}$  for a given object  $B \in \mathbb{B}$ , one can present  $B$  as the cokernel of a morphism of projective objects  $f : P'' \rightarrow P'$  in  $\mathbb{B}$  and put  $\Phi(B) = \text{coker } \Phi_{\text{proj}}(f)$ .

The additive embedding functor  $\text{Add}(M) \simeq \mathbb{B}_{\text{proj}} \rightarrow \mathbb{B}$  can be extended to a (left exact) functor  $\Psi : \mathbb{C} \rightarrow \mathbb{B}$  right adjoint to  $\Phi$ . Representing the objects of  $\mathbb{B}$  as modules over the monad  $\mathbb{T} : X \mapsto \text{Hom}_{\mathbb{C}}(M, M^{(X)})$  on the category of sets, one can compute the functor  $\Psi$  as the functor  $N \mapsto \text{Hom}_{\mathbb{C}}(M, N)$ , with the  $\mathbb{T}$ -module structure on the set  $\text{Hom}_{\mathbb{C}}(M, N)$  constructed as explained in [Positselski and Šťovíček 2019, Section 6.3] (in this case  $\text{Hom}_{\mathbb{C}}(M, N)$  of course carries the structure of an abelian group, even a right  $\mathbb{T}(\ast)$ -module, where  $\ast$  stands for a one-element set).

Indeed, let us show that the functor  $\Phi$  is left adjoint to  $\Psi$ . First of all, the natural projective generator  $P \in \mathbb{B}$  (corresponding to the object  $M \in \text{Add}(M)$ ) corepresents the forgetful functor from the category  $\mathbb{B} \simeq \mathbb{T}\text{-mod}$  to the category of sets or abelian groups, that is, for any object  $B \in \mathbb{T}\text{-mod}$  one has  $\text{Hom}_{\mathbb{B}}(P, B) \simeq B$ . In particular, for any object  $N \in \mathbb{C}$  we have a natural isomorphism of the Hom groups

$$\text{Hom}_{\mathbb{B}}(P, \Psi(N)) = \text{Hom}_{\mathbb{B}}(P, \text{Hom}_{\mathbb{C}}(M, N)) \simeq \text{Hom}_{\mathbb{C}}(M, N) = \text{Hom}_{\mathbb{C}}(\Phi(P), N).$$

Hence for any set  $X$  there are natural isomorphisms

$$\text{Hom}_{\mathbb{B}}(P^{(X)}, \Psi(N)) \simeq \text{Hom}_{\mathbb{C}}(M, N)^X \simeq \text{Hom}_{\mathbb{C}}(M^{(X)}, N) \simeq \text{Hom}_{\mathbb{C}}(\Phi(P^{(X)}), N).$$

Passing to the direct summands, we get a natural isomorphism of the Hom groups

$$\text{Hom}_{\mathbb{B}}(P', \Psi(N)) \simeq \text{Hom}_{\mathbb{C}}(\Phi(P'), N)$$

for all objects  $P' \in \mathbb{B}_{\text{proj}}$  and  $N \in \mathbb{C}$ . This isomorphism is clearly functorial in an object  $N \in \mathbb{C}$ ; and the construction of the action of the monad  $\mathbb{T}$  on the set  $\Psi(N) = \text{Hom}_{\mathbb{C}}(M, N)$  in [Positselski and Šťovíček 2019, proof of Proposition 6.2 and Remark 6.4] is designed so as to make these isomorphisms compatible with all the morphisms  $P'' \rightarrow P'$  in the category  $\mathbb{B}_{\text{proj}} \simeq \text{Add}(M)$ . Finally, both the contravariant functors  $\text{Hom}_{\mathbb{B}}(-, \Psi(N))$  and  $\text{Hom}_{\mathbb{C}}(\Phi(-), N)$  take the cokernels of morphisms in  $\mathbb{B}$  to the kernels of morphisms of abelian groups, so our isomorphism of the Hom groups extends from  $P' \in \mathbb{B}_{\text{proj}}$  to all objects  $B \in \mathbb{B}$ .

Similarly, in the context of Theorem 1.1(b), the additive embedding functor  $\Psi_{\text{inj}} : A_{\text{inj}} \simeq \text{Prod}(L) \rightarrow \mathbb{C}$  can be uniquely extended to a left exact functor  $\Psi : A \rightarrow \mathbb{C}$ . The additive embedding functor  $\text{Prod}(L) \simeq A_{\text{inj}} \rightarrow A$  can be extended to a (right exact) functor  $\Phi : \mathbb{C} \rightarrow A$  left adjoint to  $\Psi$ .

For more explicit descriptions of abelian categories  $\mathbb{B}$  arising in connection with objects  $M$  in more specific classes of additive categories  $\mathbb{C}$  in Theorem 1.1(a), we refer to [Positselski and Šťovíček 2019, Theorems 7.1, 9.9, and 9.11, and Proposition 9.1].

The following question will be addressed in Section 2: given an abelian category  $A$  with coproducts and an object  $M \in A$ , under which assumptions is there an object  $L$  in the abelian category  $\mathbb{B} = \sigma_M(A)$  such that  $\pi_L(\mathbb{B}) = A$ ? Similarly, given an abelian category  $\mathbb{B}$  with products and an object  $L \in \mathbb{B}$ , under which assumptions is there an object  $M$  in the abelian category  $A = \pi_L(\mathbb{B})$  such that  $\sigma_M(A) = \mathbb{B}$ ?

## 2. $\infty$ -tilting-cotilting correspondence

Let  $A$  be an abelian category with coproducts. We will say that an object  $T \in A$  is *weakly tilting* if one has

$$\text{Ext}_A^i(T, T^{(X)}) = 0 \quad \text{for all sets } X \text{ and all integers } i > 0.$$

Given two objects  $T' \in \text{Add}(T) \subset A$  and  $A \in A$ , a morphism  $t : T' \rightarrow A$  is said to be an *Add}(T)-precover* if every morphism  $t'' : T'' \rightarrow A$  with  $T'' \in \text{Add}(T)$  factorizes through the morphism  $t$ . Equivalently, this means that the map of abelian groups  $\text{Hom}_A(T, t) : \text{Hom}_A(T, T') \rightarrow \text{Hom}_A(T, A)$  is surjective. For every object  $A \in A$ , the natural morphism  $T^{\text{Hom}_A(T, A)} \rightarrow A$  is an *Add}(T)-precover*.

Let  $T \in A$  be a weakly tilting object. By the definition, the full subcategory  $E_{\max}(T) \subset A$  consists of all the objects  $E \in A$  satisfying the following two conditions:

- (i<sub>max</sub>)  $\text{Ext}_A^i(T, E) = 0$  for all  $i > 0$ .
- (ii<sub>max</sub>) There exists an exact sequence

$$\cdots \rightarrow T_2 \rightarrow T_1 \rightarrow T_0 \rightarrow E \rightarrow 0$$

in  $A$  such that  $T_j \in \text{Add}(T)$  for all  $j \geq 0$  and the sequence remains exact after applying the functor  $\text{Hom}_A(T, -)$ .

Notice that the condition of exactness of the sequence of abelian groups obtained by applying  $\text{Hom}_A(T, -)$  in (ii<sub>max</sub>) can be equivalently restated as the condition that the images  $Z_j$  of the morphisms  $T_{j+1} \rightarrow T_j$  satisfy  $\text{Ext}_A^1(T, Z_j) = 0$  for all  $j \geq 0$ . In this case, assuming (i<sub>max</sub>), one also has  $\text{Ext}_A^i(T, Z_j) = 0$  for all  $j \geq 0$  and  $i > 0$ . As (ii<sub>max</sub>) is obviously satisfied for  $Z_j$ , it follows that  $Z_j \in E_{\max}(T)$  for all  $j \geq 0$ .

Conversely, given a short exact sequence  $0 \rightarrow Z_0 \rightarrow T_0 \rightarrow E \rightarrow 0$  with  $E$  satisfying (i<sub>max</sub>),  $Z_0$  satisfying (ii<sub>max</sub>),  $T_0 \in \text{Add}(T)$ , and  $\text{Hom}_A(T, T_0) \rightarrow \text{Hom}_A(T, E)$  a surjective map, one clearly has  $E \in E_{\max}(T)$ .

The following lemma is a generalization of [Wakamatsu 1990, Proposition 2.6].

**Lemma 2.1.** *For any weakly tilting object  $T \in A$ , the full subcategory  $E_{\max}(T)$  in the abelian category  $A$  is closed under*

- (a) *extensions,*
- (b) *the cokernels of monomorphisms,*
- (c) *the kernels of those epimorphisms which remain epimorphisms after applying the functor  $\text{Hom}_A(T, -)$ , and*
- (d) *direct summands.*

*Proof.* To prove parts (a)–(c), consider a short exact sequence  $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$  in the abelian category  $A$ .

(a) Clearly, the object  $E$  satisfies the condition (i<sub>max</sub>) whenever the objects  $E'$  and  $E''$  do. Suppose that  $T'_0 \rightarrow E'$  and  $T''_0 \rightarrow E''$  are epimorphisms onto the objects  $E'$  and  $E''$  from objects  $T'_0, T''_0 \in \text{Add}(T)$  that remain epimorphisms after applying the functor  $\text{Hom}_A(T, -)$ . Since  $\text{Ext}_A^1(T''_0, E') = 0$ , the morphism  $T''_0 \rightarrow E''$  can be lifted to a morphism  $T''_0 \rightarrow E$ . Hence we obtain a morphism from the split short exact sequence  $0 \rightarrow T'_0 \rightarrow T'_0 \oplus T''_0 \rightarrow T''_0 \rightarrow 0$  to the short exact sequence

$0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ . Being an epimorphism at the leftmost and rightmost terms, this morphism of short exact sequences is also an epimorphism at the middle term. The short sequence of kernels  $0 \rightarrow Z'_0 \rightarrow Z_0 \rightarrow Z''_0 \rightarrow 0$  is then also exact, and the vanishing of  $\text{Ext}_A^i(T, Z'_0)$  and  $\text{Ext}_A^i(T, Z''_0)$  implies the same of  $\text{Ext}_A^i(T, Z_0)$ . We can thus proceed with the construction of a resolution as in  $(ii_{\max})$  inductively.

(b) Clearly, the object  $E''$  satisfies the condition  $(i_{\max})$  whenever the objects  $E'$  and  $E$  do. Moreover, the epimorphism  $E \rightarrow E''$  remains an epimorphism after applying  $\text{Hom}_A(T, -)$ , since  $\text{Ext}_A^1(T, E') = 0$ . Let  $T_0 \rightarrow E$  be an epimorphism onto  $E$  from an object  $T_0 \in \text{Add}(T)$  that remains an epimorphism after applying  $\text{Hom}_A(T, -)$ . Then the composition  $T_0 \rightarrow E \rightarrow E''$  has the same property. Let  $Z_0$  and  $Z''_0$  be the kernels of the epimorphisms  $T_0 \rightarrow E$  and  $T_0 \rightarrow E''$ . Then there is a short exact sequence  $0 \rightarrow Z_0 \rightarrow Z''_0 \rightarrow E' \rightarrow 0$ . Assuming that  $Z_0 \in E_{\max}(T)$ , one can apply part (a) in order to conclude that  $Z''_0 \in E_{\max}(T)$ ; hence  $E'' \in E_{\max}(T)$ .

(c) Let us first show that the kernel of every  $\text{Add}(T)$ -precover  $t' : T' \rightarrow E$  belongs to  $E_{\max}(T)$  whenever  $E \in E_{\max}(T)$ . By the definition, there exists an  $\text{Add}(T)$ -precover  $t_0 : T_0 \rightarrow E$  with the kernel  $Z_0$  belonging to  $E_{\max}(T)$ . Consider the following pullback diagram:

$$\begin{array}{ccccc}
 & & Z' & \xlongequal{\quad} & Z' \\
 & & \downarrow & & \downarrow \\
 Z_0 & \longrightarrow & S & \twoheadrightarrow & T' \\
 \parallel & & \downarrow & & \downarrow t' \\
 Z_0 & \longrightarrow & T_0 & \xrightarrow{t_0} & E
 \end{array}$$

As  $Z_0$  and  $T'$  are in  $E_{\max}(T)$ , we have  $S \in E_{\max}(T)$  by part (a). Furthermore, since  $t'$  stays an epimorphism after applying  $\text{Hom}_A(T, -)$  and  $T', E$  satisfy  $(i_{\max})$ , it follows that  $Z'$  satisfies  $(i_{\max})$  and the middle column splits. Hence there exists a short exact sequence  $0 \rightarrow T_0 \rightarrow S \rightarrow Z' \rightarrow 0$  and  $Z' \in E_{\max}(T)$  by part (b).

Now we can return to our short exact sequence  $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ . Clearly, if the objects  $E$  and  $E''$  satisfy  $(i_{\max})$  and the map  $\text{Hom}_A(T, E) \rightarrow \text{Hom}_A(T, E'')$  is surjective, then the object  $E'$  also satisfies  $(i_{\max})$ . Furthermore, if  $T_0 \rightarrow E$  is an  $\text{Add}(T)$ -precover with the kernel  $Z_0$  and if  $Z''_0$  is the kernel of the composition  $T_0 \rightarrow E \rightarrow E''$ , then  $Z_0, Z''_0 \in E_{\max}(T)$  by the previous paragraph. It remains to apply part (b) to the short exact sequence  $0 \rightarrow Z_0 \rightarrow Z''_0 \rightarrow E' \rightarrow 0$  in order to conclude that  $E' \in E_{\max}(T)$ .

(d) Let  $E'$  and  $E''$  be two objects in  $A$  for which  $E = E' \oplus E'' \in E_{\max}(T)$ . Then it is obvious that  $E'$  and  $E''$  satisfy  $(i_{\max})$ . Starting from the exact sequence  $(ii_{\max})$  for the object  $E$ , we will simultaneously construct similar exact sequences for the two

objects  $E'$  and  $E''$ . Applying the construction of part (b) to the short exact sequence  $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ , we get an epic  $\text{Add}(T)$ -precover  $T_0 \rightarrow E''$  with the kernel  $Z_0''$  included in a short exact sequence  $0 \rightarrow Z_0 \rightarrow Z_0'' \rightarrow E' \rightarrow 0$ . Applying the same construction to the short exact sequence  $0 \rightarrow E'' \rightarrow E \rightarrow E' \rightarrow 0$ , we have an epic  $\text{Add}(T)$ -precover  $T_0 \rightarrow E'$  with the kernel  $Z_0'$  included in a short exact sequence  $0 \rightarrow Z_0 \rightarrow Z_0' \rightarrow E'' \rightarrow 0$ .

Continuing with an epic  $\text{Add}(T)$ -precover  $T_1 \rightarrow Z_0$  and applying the construction of part (a), we obtain an epic  $\text{Add}(T)$ -precover  $T_1 \oplus T_0 \rightarrow Z_0''$  with the kernel  $Z_1''$  included in a short exact sequence  $0 \rightarrow Z_1 \rightarrow Z_1'' \rightarrow Z_0' \rightarrow 0$ . Proceeding in this way, we obtain an epic  $\text{Add}(T)$ -precover  $T_2 \oplus T_1 \oplus T_0 \rightarrow Z_1''$  with the kernel  $Z_2''$  included in a short exact sequence  $0 \rightarrow Z_2 \rightarrow Z_2'' \rightarrow Z_1' \rightarrow 0$ , an epic  $\text{Add}(T)$ -precover  $T_3 \oplus T_2 \oplus T_1 \oplus T_0 \rightarrow Z_2''$ , etc. Hence we obtain a long exact sequence satisfying the requirements of  $(ii_{\max})$  for  $E''$  of the form

$$\cdots \rightarrow T_2 \oplus T_1 \oplus T_0 \rightarrow T_0 \oplus T_1 \rightarrow T_0 \rightarrow E'' \rightarrow 0,$$

and there is a similar sequence of the same form for  $E'$ .  $\square$

It follows from [Lemma 2.1\(c\)](#) that, given an object  $E \in E_{\max}(T)$ , one can construct an exact sequence  $(ii_{\max})$  for it by choosing an arbitrary  $\text{Add}(T)$ -precover  $T_0 \rightarrow E$ , taking its kernel  $Z_0$ , choosing an arbitrary  $\text{Add}(T)$ -precover  $T_1 \rightarrow Z_0$ , etc. Whichever  $\text{Add}(T)$ -precovers one chooses, all the subsequent  $\text{Add}(T)$ -precovers will be epimorphisms, so one will not encounter any problems in this process.

In view of [Lemma 2.1\(a\)](#), for any weakly tilting object  $T \in A$ , the full subcategory  $E_{\max}(T) \subset A$  inherits a Quillen exact category structure from the abelian category  $A$ . There are enough projective objects in the exact category  $E_{\max}(T)$ , and the full subcategory of projective objects in  $E_{\max}(T)$  coincides with  $\text{Add}(T) \subset E_{\max}(T) \subset A$ .

Given a full subcategory  $E$  of an idempotent complete exact category  $A$ , we will call  $E$  a *coresolving* subcategory provided that

- (a)  $E$  is closed under extensions, cokernels of admissible monomorphisms, and direct summands in  $A$ , and
- (b)  $E$  is cogenerating in  $A$ , i.e., each  $A \in A$  admits an admissible monomorphism  $A \rightarrow E$  in  $A$  with  $E \in E$ .

Coresolving subcategories provide a suitable framework to speak of coresolution dimensions of objects [[Šťovíček 2014a](#), §2; [Auslander and Bridger 1969](#), Chapter 3].

Let now  $A$  be an abelian category with set-indexed products and an injective cogenerator  $J \in A$ . Then set-indexed coproducts exist and are exact in  $A$  [[Positselski and Šťovíček 2019](#), Section 2]. The full subcategory of injective objects in  $A$  can be described as  $A_{\text{inj}} = \text{Prod}(J)$ .

We will say that an object  $T \in A$  is  $\infty$ -tilting (or *big Wakamatsu tilting*) if  $T$  is weakly tilting and  $A_{\text{inj}} \subset E_{\max}(T)$ . In this case, the full subcategory  $E_{\max}(T) \subset A$  is

coresolving, there are enough injective objects in the exact category  $E_{\max}(T)$ , and these are precisely the injective objects of the ambient abelian category  $A$ .

Now let us present the dual definitions. Let  $B$  be an abelian category with products. We will say that an object  $W \in B$  is *weakly cotilting* if one has

$$\text{Ext}_B^i(W^X, W) = 0 \quad \text{for all sets } X \text{ and all integers } i > 0.$$

Let  $W \in B$  be a weakly cotilting object. By the definition, the full subcategory  $F_{\max}(W) \subset B$  consists of all the objects  $F \in B$  satisfying the following two conditions:

(i $_{\max}^*$ )  $\text{Ext}_B^i(F, W) = 0$  for all  $i > 0$ .

(ii $_{\max}^*$ ) There exists an exact sequence

$$0 \rightarrow F \rightarrow W^0 \rightarrow W^1 \rightarrow W^2 \rightarrow \dots$$

in  $B$  such that  $W^j \in \text{Prod}(W)$  for all  $j \geq 0$  and the sequence remains exact after applying the contravariant functor  $\text{Hom}_B(-, W)$ .

**Lemma 2.2.** *For any weakly cotilting object  $W \in B$ , the full subcategory  $F_{\max}(T)$  in the abelian category  $B$  is closed under*

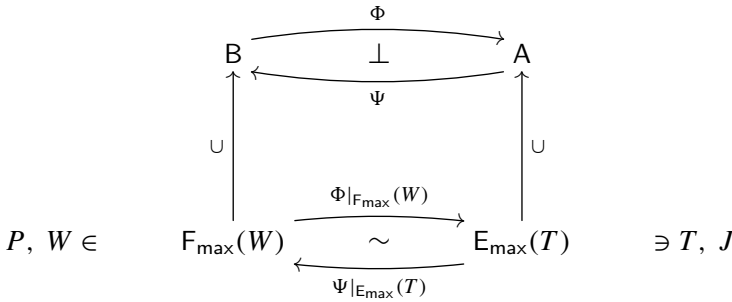
- (a) *extensions,*
- (b) *the kernels of epimorphisms,*
- (c) *the cokernels of those monomorphisms which are transformed into surjective maps by the contravariant functor  $\text{Hom}_B(-, W)$ , and*
- (d) *direct summands.*

*Proof.* Dual to [Lemma 2.1](#). □

The definition of a  $\text{Prod}(W)$ -preenvelope in  $B$  is dual to the above definition of an  $\text{Add}(T)$ -precover in  $A$ . The morphism  $F \rightarrow W^0$  in an exact sequence (ii $_{\max}^*$ ) is a  $\text{Prod}(W)$ -preenvelope. Denoting the cokernel of this morphism by  $Z^0$ , the morphism  $Z^0 \rightarrow W^1$  is also a  $\text{Prod}(W)$ -preenvelope, etc.

Conversely, it follows from [Lemma 2.2\(c\)](#) that, given any object  $F \in F_{\max}(W)$ , one can construct an exact sequence (ii $_{\max}^*$ ) for it by choosing some arbitrary  $\text{Prod}(W)$ -preenvelope  $F \rightarrow W^0$ , taking its cokernel  $Z^0$ , choosing an arbitrary  $\text{Prod}(W)$ -preenvelope  $Z^0 \rightarrow W^1$ , etc. Whichever  $\text{Prod}(W)$ -preenvelopes one chooses in this process, all the subsequent  $\text{Prod}(W)$ -preenvelopes will be monomorphisms, so one will not encounter any problems.

In view of [Lemma 2.2\(a\)](#), for any weakly cotilting object  $W \in B$ , the full subcategory  $F_{\max}(W) \subset B$  inherits an exact category structure from the abelian category  $B$ . There are enough injective objects in the exact category  $F_{\max}(W)$ , and the full subcategory of injective objects in  $F_{\max}(W)$  coincides with  $\text{Prod}(W)$ .



**Figure 1.** Illustration of the  $\infty$ -tilting-cotilting correspondence (see Theorems 2.3 and 2.4 and Corollary 2.5).

Let  $B$  be an abelian category with set-indexed coproducts and a projective generator  $P \in B$ . Then set-indexed products exist and are exact in  $B$ . The full subcategory of projective objects in  $B$  can be described as  $B_{\text{proj}} = \text{Add}(P)$ .

We will say that an object  $W \in B$  is  $\infty$ -cotilting (or *big Wakamatsu cotilting*) if  $W$  is weakly cotilting and  $B_{\text{proj}} \subset F_{\text{max}}(W)$ .

When the object  $W$  is  $\infty$ -cotilting, the full subcategory  $F_{\text{max}}(W) \subset B$  is *resolving* (i.e., generating and closed under extensions, kernels of epimorphisms and direct summands). In this case, there are enough projective objects in the exact category  $F_{\text{max}}(W)$ , and these are precisely the projective objects of the ambient abelian category  $B$ .

**Theorem 2.3.** *Let  $A$  be a complete, cocomplete abelian category with an injective cogenerator  $J$  and an  $\infty$ -tilting object  $T \in A$ . Put  $B = \sigma_T(A)$ , and let  $\Phi : B \rightarrow A$  be the right exact functor identifying the full subcategory of projective objects  $B_{\text{proj}} \subset B$  with the full subcategory  $\text{Add}(T) \subset A$ . Let  $\Psi : A \rightarrow B$  be the left exact functor right adjoint to  $\Phi$ ; so  $P = \Psi(T)$  is a projective generator of  $B$ . Set  $W = \Psi(J) \in B$ .*

*Then  $W$  is an  $\infty$ -cotilting object in  $B$ , and the restrictions of the functors  $\Psi$  and  $\Phi$  induce a pair of inverse equivalences of exact categories between  $E_{\text{max}}(T)$  and  $F_{\text{max}}(W)$  (see Figure 1), which identify the  $\infty$ -tilting object  $T \in A$  with the projective generator  $P \in B$  and the  $\infty$ -cotilting object  $W \in B$  with the injective cogenerator  $J \in A$ .*

*Proof.* The functor  $\Psi|_{E_{\text{max}}(T)} : E_{\text{max}}(T) \rightarrow B$  is exact, because the functor  $\Psi$  can be computed as  $\text{Hom}_A(T, -)$ , and the condition  $(i_{\text{max}})$  is imposed.

To check that the functor  $\Psi|_{E_{\text{max}}(T)}$  is fully faithful, one can choose for any two objects  $E'$  and  $E'' \in E_{\text{max}}(T)$  two initial fragments  $T'_1 \rightarrow T'_0 \rightarrow E' \rightarrow 0$  and  $T''_1 \rightarrow T''_0 \rightarrow E'' \rightarrow 0$  of exact sequences  $(ii_{\text{max}})$ . The two sequences being exact in the exact category  $E_{\text{max}}(T)$  and the objects of  $\text{Add}(T)$  being projective in  $E_{\text{max}}(T)$ , one can compute the group  $\text{Hom}_A(E', E'')$  as the group of all morphisms  $T'_0 \rightarrow T''_0$  forming a commutative square with some morphism  $T'_1 \rightarrow T''_1$ , modulo

those morphisms that come from some morphism  $T'_0 \rightarrow T''_1$ . The functor  $\Psi$  takes the exact sequences  $T_1^{(k)} \rightarrow T_0^{(k)} \rightarrow E^{(k)} \rightarrow 0$ ,  $k = 1, 2$ , to exact sequences  $\Psi(T_1^{(k)}) \rightarrow \Psi(T_0^{(k)}) \rightarrow \Psi(E^{(k)}) \rightarrow 0$  with the objects  $\Psi(T_j^{(k)})$  belonging to  $B_{\text{proj}}$ , so the groups  $\text{Hom}_B(\Psi(E'), \Psi(E''))$  can be computed similarly in terms of morphisms between the objects  $\Psi(T_j^{(k)})$ . It remains to recall that the functor  $\Psi|_{\text{Add}(T)}$  is fully faithful (see [Section 1](#)).

Furthermore, since the functor  $\Psi|_{E_{\max}(T)}$  is exact and fully faithful, and takes the projective objects of  $E_{\max}(T)$  to projective objects in  $B$ , and since there are enough projectives in  $E_{\max}(T)$ , it follows that the functor  $\Psi|_{E_{\max}(T)}$  induces isomorphisms of the Ext groups

$$\text{Ext}_{E_{\max}(T)}^i(E', E'') \simeq \text{Ext}_B^i(\Psi(E'), \Psi(E''))$$

for all objects  $E'$  and  $E'' \in E_{\max}(T)$  and all  $i \geq 0$ . Similarly, as there are enough injectives in  $E_{\max}(T)$  and the injectives of  $E_{\max}(T)$  are injective in  $A$ , one has

$$\text{Ext}_{E_{\max}(T)}^i(E', E'') \simeq \text{Ext}_A^i(E', E''), \quad E', E'' \in E_{\max}(T), \quad i \geq 0.$$

The functor  $\Psi$ , being right adjoint, preserves products; so the equations  $\text{Prod}(J) = A_{\text{inj}}$  and  $W = \Psi(J)$  imply  $\text{Prod}(W) = \Psi(A_{\text{inj}})$ . In particular,  $W^X = \Psi(J^X)$  for any set  $X$ . As  $A_{\text{inj}} \subset E_{\max}(T)$  and  $\text{Ext}_A^i(J^X, J) = 0$  for  $i > 0$ , it follows that  $\text{Ext}_B^i(W^X, W) = 0$ . So the object  $W \in B$  is weakly cotilting.

Moreover, for the same reasons one has  $\text{Ext}_B^i(\Psi(E), W) = 0$  for all  $E \in E_{\max}(T)$  and  $i > 0$ . In other words, the objects  $\Psi(E) \in B$  satisfy the condition  $(i_{\max}^*)$ . Let us show that they also satisfy  $(ii_{\max}^*)$ , that is  $\Psi(E_{\max}(T)) \subset F_{\max}(W)$ . Let  $0 \rightarrow E \rightarrow J^0 \rightarrow J^1 \rightarrow J^2 \rightarrow \dots$  be an injective coresolution of  $E$  in  $A$ . In view of [Lemma 2.1\(b\)](#), this coresolution is an acyclic complex in the exact category  $E_{\max}(T)$ . The object  $J \in A$  being injective, this coresolution is taken to an acyclic complex of abelian groups by the contravariant functor  $\text{Hom}_A(-, J)$ . Hence, applying the fully faithful exact functor  $\Psi|_{E_{\max}(T)}$ , we obtain a coresolution  $(ii_{\max}^*)$  for the object  $\Psi(E)$ . Thus  $B_{\text{proj}} = \Psi(\text{Add}(T)) \subset \Psi(E_{\max}(T)) \subset F_{\max}(W)$ , and we have shown that the object  $W$  is  $\infty$ -cotilting in  $B$ .

There are enough injective objects in the category  $A$ , and the left exact functor  $\Psi$  establishes an equivalence  $A_{\text{inj}} \simeq \text{Prod}(W)$ . Hence we have  $A = \pi_W(B)$ . The assertions dual to what we have already proved now tell that the functor  $\Phi$  is exact and fully faithful in restriction to  $F_{\max}(W)$  and that  $\Phi(F_{\max}(W)) \subset E_{\max}(T)$ . Being an adjoint pair of exact and fully faithful functors,  $\Psi|_{E_{\max}(T)}$  and  $\Phi|_{F_{\max}(W)}$  are equivalences of the exact categories  $E_{\max}(T)$  and  $F_{\max}(W)$ .  $\square$

**Theorem 2.4.** *Let  $B$  be a complete, cocomplete abelian category with a projective generator  $P$  and an  $\infty$ -cotilting object  $W \in B$ . Put  $A = \pi_W(B)$ , and let  $\Psi : A \rightarrow B$  be the left exact functor identifying the full subcategory of injective objects  $A_{\text{inj}} \subset A$*

with the full subcategory  $\text{Prod}(W) \subset B$ . Let  $\Phi : B \rightarrow A$  be the right exact functor left adjoint to  $\Psi$ ; so  $J = \Phi(W)$  is a injective cogenerator of  $A$ . Set  $T = \Phi(P) \in A$ .

Then  $T$  is an  $\infty$ -tilting object in  $A$ , and the restrictions of the functors  $\Phi$  and  $\Psi$  induce a pair of inverse equivalences of exact categories between  $F_{\max}(W)$  and  $E_{\max}(T)$  (see Figure 1), which identify the  $\infty$ -cotilting object  $W \in B$  with the injective cogenerator  $J \in A$  and the  $\infty$ -tilting object  $T \in A$  with the projective generator  $P \in B$ .

*Proof.* Dual to Theorem 2.3. □

**Corollary 2.5.** *The constructions of Theorems 2.3 and 2.4 establish a one-to-one correspondence between equivalence classes of*

- (1) complete, cocomplete abelian categories  $A$  with an injective cogenerator  $J$  and an  $\infty$ -tilting object  $T$ , and
- (2) complete, cocomplete abelian categories  $B$  with a projective generator  $P$  and an  $\infty$ -cotilting object  $W$ . □

### 3. $\infty$ -tilting and $\infty$ -cotilting pairs

As above, let  $A$  be an abelian category with set-indexed products and an injective cogenerator  $J \in A$ . Let  $T \in A$  be an object and  $E \subset A$  be a full subcategory. We will say that  $(T, E)$  is an  $\infty$ -tilting pair in  $A$  if the following conditions hold:

- (i)  $A_{\text{inj}} \subset E$ .
- (ii)  $\text{Add}(T) \subset E$ .
- (iii)  $\text{Ext}_A^1(T, E) = 0$  for all  $E \in E$ .
- (iv)  $E$  is closed under the cokernels of monomorphisms and extensions in  $A$ .
- (v) Every  $\text{Add}(T)$ -precover  $T' \rightarrow E$  of an object  $E \in E$  is an epimorphism in  $A$  with the kernel belonging to  $E$ .

Due to the condition (iv), the full subcategory  $E \subset A$  inherits an exact category structure from the abelian category  $A$ . According to the condition (i), there are enough injective objects in the exact category  $E$ , and these are precisely the injective objects of the ambient abelian category  $A$ , that is,  $E_{\text{inj}} = A_{\text{inj}}$ .

It follows from the condition (iii) together with the condition (i) and the first part of the condition (iv) that

$$\text{Ext}_A^i(T, E) = 0 \quad \text{for all } E \in E \text{ and all integers } i > 0.$$

Hence, in view of the condition (ii), the object  $T \in A$  has to be weakly tilting.

From the conditions (ii) and (iii) we see that the objects of  $\text{Add}(T)$  are projective in the exact category  $E$ . It follows from the condition (v) that there are enough projective objects belonging to  $\text{Add}(T)$  in  $E$ . Hence there are enough projective

objects in  $E$  and the class of all projective objects in  $E$  coincides with  $\text{Add}(T)$ , that is  $E_{\text{proj}} = \text{Add}(T)$ .

Now it is clear that all the objects  $E \in E$  satisfy the conditions  $(i_{\max})$  and  $(ii_{\max})$ ; so we have  $E \subset E_{\max}(T) \subset A$ . From the condition (i) we conclude that  $A_{\text{inj}} \subset E_{\max}(T)$ . Thus the object  $T \in A$  has to be  $\infty$ -tilting. Conversely, according to [Lemma 2.1](#), for any  $\infty$ -tilting object  $T \in A$ , the pair  $(T, E_{\max}(T))$  is an  $\infty$ -tilting pair in  $A$ . To summarize, we have shown the following.

**Lemma 3.1.** *Let  $A$  be a complete, cocomplete abelian category with an injective cogenerator. Then an object  $T \in A$  is a part of an  $\infty$ -tilting pair  $(T, E)$  in  $A$  if and only if it is an  $\infty$ -tilting object. The full subcategory  $E = E_{\max}(T)$  is the maximal of all full subcategories  $E \subset A$  forming an  $\infty$ -tilting pair with  $T \in A$ .  $\square$*

In general, we do not assume that  $E$  is closed under direct summands. However, we can add that assumption whenever convenient (e.g., in [Sections 4](#) or [5](#)):

**Lemma 3.2.** *If  $(T, E)$  is an  $\infty$ -tilting pair in  $A$  and  $E'$  is the closure of  $E$  under direct summands, then  $(T, E')$  is also an  $\infty$ -tilting pair and  $E'$  is a coresolving subcategory in  $A$ .*

*Proof.* The conditions (i)–(iii) are obviously true for  $E'$ . To prove (iv), suppose that we have an exact sequence

$$0 \rightarrow E'_1 \rightarrow E_1 \rightarrow E''_1 \rightarrow 0$$

with  $E'_1, E''_1 \in E'$ , i.e., there exist  $E'_2, E''_2 \in A$  such that  $E' = E'_1 \oplus E'_2$  and  $E'' = E''_1 \oplus E''_2$  belong to  $E$ . Then  $E_1 \oplus E'_2 \oplus E''_2$  is an extension of  $E'$  by  $E''$  in  $A$ , and hence  $E_1 \in E'$ . Similarly, if  $f_1 : E'_1 \rightarrow E_1$  is a monomorphism in  $A$  with  $E'_1, E_1 \in E'$ , then there is a split monomorphism  $f_2 : E'_2 \rightarrow E_2$  such that  $f_1 \oplus f_2$  is a monomorphism in  $A$  between objects of  $E$ . Finally, to prove (v), it suffices to note that if  $E = E_1 \oplus E_2 \in E$  and if  $t_1 : T_1 \rightarrow E_1$  and  $t_2 : T_2 \rightarrow E_2$  are  $\text{Add}(T)$ -precovers, then also  $t_1 \oplus t_2 : T_1 \oplus T_2 \rightarrow E$  is an  $\text{Add}(T)$ -precover.  $\square$

Now we present the dual definitions. Let  $B$  be an abelian category with set-indexed coproducts and a projective generator  $P \in B$ . Let  $W \in B$  be an object and  $F \subset B$  be a full subcategory. We will say that  $(W, F)$  is an  $\infty$ -cotilting pair in  $B$  if the following conditions hold:

- (i\*)  $B_{\text{proj}} \subset F$ .
- (ii\*)  $\text{Prod}(W) \subset F$ .
- (iii\*)  $\text{Ext}_B^1(F, W) = 0$  for all  $F \in F$ .
- (iv\*)  $F$  is closed under the kernels of epimorphisms and extensions in  $B$ .
- (v\*) Every  $\text{Prod}(W)$ -preenvelope  $F \rightarrow W'$  of an object  $F \in F$  is a monomorphism in  $B$  with the cokernel belonging to  $F$ .

As above, it follows from the conditions (i\*)–(v\*) that

$$\text{Ext}_{\mathbb{B}}^i(F, W) = 0 \quad \text{for all } F \in \mathbb{F} \text{ and all integers } i > 0,$$

the object  $W \in \mathbb{B}$  is weakly cotilting, and the full subcategory  $\mathbb{F} \subset \mathbb{B}$  inherits an exact category structure from the abelian category  $\mathbb{B}$ . The exact category  $\mathbb{F}$  has both enough projective and enough injective objects; the full subcategories of projective and injective objects in  $\mathbb{F}$  are described as  $\mathbb{F}_{\text{proj}} = \mathbb{B}_{\text{proj}}$  and  $\mathbb{F}_{\text{inj}} = \text{Prod}(W)$ . Moreover, as before one also has:

**Lemma 3.3.** *Let  $\mathbb{B}$  be a complete, cocomplete abelian category with a projective generator. Then an object  $W \in \mathbb{B}$  is a part of an  $\infty$ -cotilting pair  $(W, \mathbb{F})$  in  $\mathbb{B}$  if and only if it is an  $\infty$ -cotilting object. The full subcategory  $\mathbb{F} = \mathbb{F}_{\text{max}}(W)$  is the maximal of all full subcategories  $\mathbb{F} \subset \mathbb{B}$  forming an  $\infty$ -cotilting pair with  $W \in \mathbb{B}$ .*

*Moreover, if  $(W, \mathbb{F})$  is an  $\infty$ -cotilting pair and  $\mathbb{F}'$  is the closure of  $\mathbb{F}$  under direct summands, then  $(W, \mathbb{F}')$  is also an  $\infty$ -cotilting pair and  $\mathbb{F}'$  is a resolving subcategory of  $\mathbb{B}$ .*

*Proof.* This is dual to Lemmas 3.1 and 3.2. □

The  $\infty$ -tilting-cotilting correspondence from the last section now extends to one between  $\infty$ -tilting and  $\infty$ -cotilting pairs.

**Proposition 3.4.** *In the context of Corollary 2.5 (see also Figure 1), the assignments  $\mathbb{F} = \Psi(\mathbb{E})$  and  $\mathbb{E} = \Phi(\mathbb{F})$  establish a bijective correspondence between*

- (1) *the full subcategories  $\mathbb{E} \subset \mathbb{E}_{\text{max}}(T)$  forming an  $\infty$ -tilting pair with  $T \in \mathbb{A}$  and*
- (2) *the full subcategories  $\mathbb{F} \subset \mathbb{F}_{\text{max}}(W)$  forming an  $\infty$ -cotilting pair with  $W \in \mathbb{B}$ .*

*Proof.* Let  $(T, \mathbb{E})$  be an  $\infty$ -tilting pair in the category  $\mathbb{A}$ . Put  $\mathbb{F} = \Psi(\mathbb{E})$ . We have to show that  $(W, \mathbb{F})$  is an  $\infty$ -cotilting pair in the category  $\mathbb{B}$ .

Indeed, the condition (i\*) follows from (ii) and the condition (ii\*) follows from (i), as  $\mathbb{B}_{\text{proj}} = \Psi(\text{Add}(T))$  and  $\text{Prod}(W) = \Psi(\mathbb{A}_{\text{inj}})$ . The condition (iii\*) holds, since  $\mathbb{F} = \Psi(\mathbb{E}) \subset \Psi(\mathbb{E}_{\text{max}}(T)) = \mathbb{F}_{\text{max}}(W)$  and all  $F \in \mathbb{F}_{\text{max}}(W)$  satisfy (iii\*).

The full subcategory  $\mathbb{F}$  is closed under extensions in  $\mathbb{F}_{\text{max}}(W)$ , because  $\Psi : \mathbb{E}_{\text{max}}(T) \rightarrow \mathbb{F}_{\text{max}}(W)$  is an equivalence of exact categories and the full subcategory  $\mathbb{E}$  is closed under extensions in  $\mathbb{E}_{\text{max}}(W)$ . Since the full subcategory  $\mathbb{F}_{\text{max}}(W)$  is closed under extensions in  $\mathbb{B}$  by Lemma 2.2(a), it follows that  $\mathbb{F}$  is closed under extensions in  $\mathbb{B}$ .

Let  $f : F' \rightarrow F''$  be an epimorphism in  $\mathbb{B}$  between two objects  $F', F'' \in \mathbb{F}$ . Then there exists a morphism  $e : E' \rightarrow E''$  in  $\mathbb{E}$  such that  $F^{(s)} \simeq \Psi(E^{(s)})$ ,  $s = 1, 2$ , and  $f = \Psi(e)$ . The map of abelian groups  $\text{Hom}_{\mathbb{E}}(T, e)$  is surjective, since the map  $\text{Hom}_{\mathbb{F}}(P, f)$  is and  $P = \Psi(T)$ . Let  $T_0 \rightarrow E'$  be an  $\text{Add}(T)$ -precover; then the composition  $T_0 \rightarrow E' \rightarrow E''$  is also an  $\text{Add}(T)$ -precover. Denote the kernels of the morphisms  $T_0 \rightarrow E'$  and  $T_0 \rightarrow E''$  by  $Z'_0$  and  $Z''_0$ , respectively. Then  $Z'_0, Z''_0 \in \mathbb{E}$  by

the condition (v) and the natural morphism  $Z'_0 \rightarrow Z''_0$  is a monomorphism. Hence  $\ker(e) = \text{coker}(Z'_0 \rightarrow Z''_0) \in E$  by the condition (iv) and  $\ker(f) = \Psi(\ker e) \in F$ . This proves that the full subcategory  $F \subset B$  is closed under the kernels of epimorphisms in  $B$  and finishes the proof of the condition (iv\*).

To prove the condition (v\*), let  $f : F \rightarrow W^0$  be a  $\text{Prod}(W)$ -preenvelope of an object  $F \in F$ . Then  $f = \Psi(e)$ , where  $e : E \rightarrow J^0$  is a morphism in  $E$  and  $J^0 \in A_{\text{inj}}$ . The map  $\text{Hom}_E(e, J)$  is surjective, since the map  $\text{Hom}_F(f, W)$  is and  $W = \Psi(J)$ . Since  $J$  is an injective cogenerator of  $A$ , it follows that  $e$  is a monomorphism in  $A$ . By the condition (iv), the cokernel of  $e$  belongs to  $E$ , so  $e$  is a monomorphism in  $E$ . Since the functor  $\Psi|_E$  is exact, it follows that  $f$  is a monomorphism in  $B$  with the cokernel belonging to  $F$ .

To summarize these arguments, the conditions (iv) and (v) essentially say that the full subcategory  $E$  is closed under the cokernels of monomorphisms, extensions, and kernels of epimorphisms in  $E_{\max}(T)$ , while the conditions (iv\*) and (v\*) mean that the full subcategory  $F$  is closed under the kernels of epimorphisms, extensions, and cokernels of monomorphisms in  $F_{\max}(W)$ .  $\square$

**Corollary 3.5.** *The constructions of Theorems 2.3 and 2.4 and Proposition 3.4 establish a one-to-one correspondence between equivalence classes of*

- (1) *quadruples  $(A, E, T, J)$ , where  $A$  is a complete, cocomplete abelian category with an injective cogenerator  $J$  and  $(T, E)$  is an  $\infty$ -tilting pair in  $A$ , and*
- (2) *quadruples  $(B, F, P, W)$ , where  $B$  is a complete, cocomplete abelian category with a projective generator  $P$  and  $(W, F)$  is an  $\infty$ -cotilting pair in  $B$ .*

*In this correspondence, the exact categories  $E$  and  $F$  are naturally equivalent,  $E \simeq F$ , and the equivalence identifies  $T$  with  $P$  and  $W$  with  $J$ .*  $\square$

In general, there can be many classes  $E$  which form an  $\infty$ -tilting pair with a given  $\infty$ -tilting object  $T \in A$ . Thanks to the following lemma, we know that they form a complete lattice.

**Lemma 3.6.** *Let  $A$  be a complete, cocomplete abelian category with an injective cogenerator and let  $T \in A$  be an  $\infty$ -tilting object. If  $E_i \subset A$ ,  $i \in I$ , is a collection of full subcategories such that  $(T, E_i)$  is an  $\infty$ -tilting pair for each  $i \in I$ , then  $(T, E)$  is an  $\infty$ -tilting pair with  $E = \bigcap_{i \in I} E_i$ .*

*Dually, if  $B$  is a complete, cocomplete abelian category with a projective generator,  $W \in B$  is an  $\infty$ -cotilting object and  $(W, F_j)$  are  $\infty$ -cotilting pairs,  $j \in J$ , then  $(W, F)$  is an  $\infty$ -cotilting pair with  $F = \bigcap_{j \in J} F_j$ .*

*Proof.* It is straightforward to check that each of the conditions (i)–(v) and (i\*)–(v\*) is preserved by intersections of classes.  $\square$

**Example 3.7.** In particular, whenever  $T$  is an  $\infty$ -tilting object in  $A$ , there exists a unique minimal full subcategory  $E_{\min}(T) \subset A$  for which  $(T, E_{\min}(T))$  is an  $\infty$ -tilting

pair in  $A$ . In fact, the full subcategory  $E_{\min}(T)$  consists of all the objects in  $A$  that can be obtained from the objects of  $A_{\text{inj}} \subset E_{\min}(A)$  and  $\text{Add}(T) \subset E_{\min}(A)$  by applying iteratively the operations of the passage to the cokernel of a monomorphism, an extension, or the kernel of an  $\text{Add}(T)$ -precover. For every  $\infty$ -tilting pair  $(T, E)$  in  $A$ , one then has  $E_{\min}(T) \subset E$ .

Similarly, whenever  $W$  is an  $\infty$ -cotilting object in  $B$ , there exists a unique minimal full subcategory  $F_{\min}(W) \subset B$  such that  $(W, F_{\min}(W))$  is an  $\infty$ -cotilting pair in  $B$ . For every  $\infty$ -cotilting pair  $(W, F)$  in  $B$ , one has  $F_{\min}(W) \subset F$ .

In the situation of [Corollary 2.5](#) (and [Figure 1](#)), the full subcategories  $E_{\min}(T) \subset A$  and  $F_{\min}(W) \subset B$  are transformed into each other by the functors  $\Psi$  and  $\Phi$ , that is

$$F_{\min}(W) = \Psi(E_{\min}(T)) \quad \text{and} \quad E_{\min}(T) = \Phi(F_{\min}(W)).$$

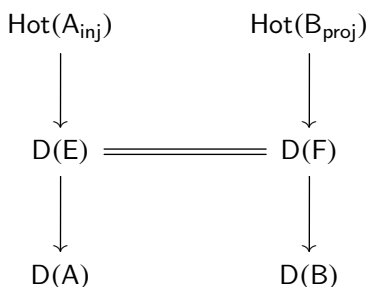
**Remark 3.8.** There is a certain similarity between our results in Sections 1–3 of this paper and those in [\[Enomoto 2017, Sections 2–3\]](#). Let us explain the connection and the differences between our approaches. The paper [\[Enomoto 2017\]](#) is a far-reaching development of the traditional point of view in Wakamatsu tilting theory, in which finitely generated modules over Artinian algebras are the main objects of study. Enomoto [\[2017\]](#) works with skeletally small exact categories, and essentially never considers infinite products or coproducts. The definition of a projective generator in [\[Enomoto 2017, paragraph before Corollary 2.14\]](#) presumes an exact category with enough projective objects in which every projective object is a direct summand of a *finite* direct sum of copies of the (single) generator.

Nevertheless, the generality level in [\[Enomoto 2017, Sections 2–3\]](#) exceeds that of our exposition. In particular, our [Lemma 2.2](#) is but a particular case of [\[Enomoto 2017, Proposition 3.2\]](#) (while our [Lemma 2.1](#) is dual). Enomoto achieves this generality by working with arbitrary (skeletally small) additive categories in place of our classes  $\text{Add}(T)$  and  $\text{Prod}(W)$ . An exact category playing the role of our  $F$  is generally denoted by  $\mathcal{E}$  in [\[Enomoto 2017\]](#), an additive category playing the role of our  $\text{Add}(T) = F_{\text{proj}} = B_{\text{proj}}$  is denoted by  $\mathcal{C}$ , an additive category in the role of our  $\text{Prod}(W) = F_{\text{inj}}$  is denoted by  $\mathcal{W}$ , and the exact category in the role of our  $F_{\max}(W)$  is denoted by  $X_{\mathcal{W}}$ . (The reader should be warned that Enomoto calls “Wakamatsu tilting” what we would call “Wakamatsu cotilting” or “ $\infty$ -cotilting”.)

Finally, in the role of our abelian category  $B$ , Enomoto has an exact category which he denotes by  $\text{mod } \mathcal{C}$ . This difference occurs because our observation that the category  $\text{Add}(T)$  always has weak kernels (as pointed out in the proof of [Theorem 1.1](#)) has no counterpart in [\[Enomoto 2017\]](#).

#### 4. $\infty$ -tilting-cotilting derived equivalences

Unlike in [\[Positselski and Šťovíček 2019, Sections 3–5\]](#), in our present situation the coresolution dimensions of objects of the category  $A$  with respect to its coresolving



**Figure 2.** Equivalences induced by  $\infty$ -tilting/ $\infty$ -cotilting pairs in general.

subcategory  $E$  or  $E_{\max}(T)$  can well be infinite, and so can the resolution dimensions of objects of the category  $B$  with respect to its resolving subcategory  $F$  or  $F_{\max}(W)$ . Hence the equivalence of exact categories  $E \simeq F$  does not generally lead to any equivalence between the derived categories  $D(A)$  and  $D(B)$ . All one can say is that there is the commutative diagram formed by triangulated functors and a triangulated equivalence in [Figure 2](#). Here  $\text{Hot}(A_{\text{inj}})$  and  $\text{Hot}(B_{\text{proj}})$  are the homotopy categories of (unbounded complexes in) the additive categories  $A_{\text{inj}}$  and  $B_{\text{proj}}$ , while  $D(E)$  and  $D(F)$  are the (unbounded) derived categories of the exact categories  $E$  and  $F$ , and  $D(A)$  and  $D(B)$  are the similar derived categories of the abelian categories  $A$  and  $B$ .

If  $A$  is a Grothendieck category, the canonical functor  $\text{Hot}(A_{\text{inj}}) \rightarrow D(A)$  in the left-hand side column of [Figure 2](#) is a Verdier quotient functor. This follows, e.g., from [\[Alonso Tarrío et al. 2000, Theorem 5.4\]](#). If the full subcategories  $E \subset A$  and  $F \subset B$  have additional closure properties, we will obtain a similar diagram below where all the functors are Verdier quotients.

One issue here is that, unlike for tilting modules of finite projective dimension, the class  $E$  in the definition of a tilting pair need not be closed under coproducts in  $A$  (cf. [\[Positselski and Šťovíček 2019, Lemma 5.3\]](#)). Dually, the class  $F$  need not be closed under products. There are some elementary relations between the closure properties of  $E$  and  $F$ , however.

**Lemma 4.1.** *In the context of [Corollary 3.5](#), if the full subcategory  $E \subset A$  is closed under products, then the full subcategory  $F \subset B$  is closed under products. If the full subcategory  $F \subset B$  is closed under coproducts, then the full subcategory  $E \subset A$  is closed under coproducts.*

*Proof.* The first assertion holds, since  $F = \Psi(E)$  and the functor  $\Psi : A \rightarrow B$  preserves products (see [Figure 1](#)). The second assertion holds, since  $E = \Phi(F)$  and the functor  $\Phi : B \rightarrow A$  preserves coproducts. □

It would be interesting to know whether the converse assertions to those of [Lemma 4.1](#) are true.

Suppose now that  $E$  is a part of an  $\infty$ -tilting pair in a complete, cocomplete

abelian category  $A$  with an injective cogenerator. Then  $A$  has exact coproducts ([Mitchell 1965, Exercise III.2]) and, if  $E$  is closed under coproducts,  $E$  has exact coproducts too. In such a situation the following definition from [Positselski 2010, Sections 2.1 and 4.1] or [Positselski 2012, Section A.1] applies and gives a more adequate replacement of  $\text{Hot}(A_{\text{inj}}) = \text{Hot}(E_{\text{inj}})$  in Figure 2.

If  $E$  is an exact category with arbitrary coproducts which are exact, we call a complex *coacyclic* if it belongs to the smallest localizing subcategory of  $\text{Hot}(E)$  which contains the total complexes of short exact sequences of complexes over  $E$ . The *coderived category* of  $E$ , which we denote by  $D^{\text{co}}(E)$ , is defined as the Verdier quotient category of  $\text{Hot}(E)$  by the subcategory of coacyclic complexes.

Note that it follows from the above definition that each coacyclic complex is exact and, thus, we have a Verdier quotient functor  $D^{\text{co}}(E) \rightarrow D(E)$ . On the other hand, if  $E$  in addition has enough injectives, the natural functor  $\text{Hot}(E_{\text{inj}}) \rightarrow D^{\text{co}}(E)$  is fully faithful by [Positselski 2012, Lemma A.1.3]. To summarize, we have triangulated functors

$$\text{Hot}(E_{\text{inj}}) \xrightarrow{\text{fully faithful}} D^{\text{co}}(E) \twoheadrightarrow D(E),$$

where the first functor is fully faithful and the second one is a Verdier quotient. In fact, the fully faithful functor was proved to be an equivalence in some cases [Positselski 2017c, Theorem 2.4].

If  $F$  is an exact category with arbitrary products which are exact, the class of *contraacyclic* complexes in  $\text{Hot}(F)$  and the *contraderived category*  $D^{\text{ctr}}(F)$  of  $F$  are defined dually, and we have triangulated functors

$$\text{Hot}(F_{\text{proj}}) \xrightarrow{\text{fully faithful}} D^{\text{ctr}}(F) \twoheadrightarrow D(F).$$

As above, the fully faithful functor in the leftmost arrow is known to be an equivalence in some cases [Positselski 2017c, Theorem 4.4(b)].

Now we can state the main result of the section (see also Figure 3 below).

**Proposition 4.2.** (a) *Let  $A$  be an exact category where set-indexed coproducts exist and are exact, and let  $E \subset A$  be a coresolving subcategory closed under coproducts. Then the functor between the coderived categories  $D^{\text{co}}(E) \rightarrow D^{\text{co}}(A)$  induced by the embedding of exact categories  $E \rightarrow A$  is a triangulated equivalence. The triangulated functor between the conventional derived categories  $D(E) \rightarrow D(A)$  induced by the same exact embedding is a Verdier quotient functor.*

(b) *Let  $B$  be an exact category where set-indexed products exist and are exact, and let  $F \subset B$  be a resolving subcategory closed under products. Then the functor between the contraderived categories  $D^{\text{ctr}}(F) \rightarrow D^{\text{ctr}}(B)$  induced by the embedding of exact categories  $F \rightarrow B$  is a triangulated equivalence. The triangulated functor between the conventional derived categories  $D(F) \rightarrow D(B)$  induced by the same exact embedding is a Verdier quotient functor.*

$$\begin{array}{ccc}
 D^{\text{co}}(A) & & D^{\text{ctr}}(B) \\
 \downarrow & & \downarrow \\
 \Downarrow & & \Downarrow \\
 D(E) & \xlongequal{\quad\quad\quad} & D(F) \\
 \downarrow & & \downarrow \\
 \Downarrow & & \Downarrow \\
 D(A) & & D(B)
 \end{array}$$

**Figure 3.** Equivalences induced by  $\infty$ -tilting/ $\infty$ -cotilting pairs when  $E$  is closed under coproducts and  $F$  under products.

*Proof.* The first assertion of part (b) is [Positselski 2012, Proposition A.3.1(b)], and the first assertion of part (a) is the dual result.

To prove the second assertion of part (a), notice that we have a commutative diagram of triangulated functors  $D^{\text{co}}(E) = D^{\text{co}}(A) \rightarrow D(E) \rightarrow D(A)$ , where both the functors  $D^{\text{co}}(A) \rightarrow D(E)$  and  $D^{\text{co}}(A) \rightarrow D(A)$  are Verdier quotient functors. It follows that the functor  $D(E) \rightarrow D(A)$  is also a Verdier quotient.  $\square$

In particular, Proposition 4.2 tells that, when in the situation of Corollary 3.5 the full subcategory  $E \subset A$  is closed under coproducts and the full subcategory  $F \subset B$  is closed under products, we have a commutative diagram formed by Verdier quotient functors and a triangulated equivalence as in Figure 3.

**Remark 4.3.** Let  $A$  be an exact category with exact coproducts, and let  $E' \subset E'' \subset A$  be two coresolving subcategories closed under coproducts. Then one has  $D^{\text{co}}(E') \simeq D^{\text{co}}(E'') \simeq D^{\text{co}}(A)$ , while the natural functors between the conventional derived

$$\begin{array}{ccc}
 D^{\text{co}}(A) & & D^{\text{ctr}}(B) \\
 \downarrow & & \downarrow \\
 \Downarrow & & \Downarrow \\
 D(E') & \xlongequal{\quad\quad\quad} & D(F') \\
 \downarrow & & \downarrow \\
 \Downarrow & & \Downarrow \\
 D(E'') & \xlongequal{\quad\quad\quad} & D(F'') \\
 \downarrow & & \downarrow \\
 \Downarrow & & \Downarrow \\
 D(A) & & D(B)
 \end{array}$$

**Figure 4.** Compatible equivalences for different choices of  $\infty$ -tilting/ $\infty$ -cotilting pairs.

categories  $D(E') \rightarrow D(E'') \rightarrow D(A)$  are Verdier quotient functors. Thus, when a coproduct-closed coresolving subcategory is being enlarged, its derived category gets deflated. In other words, the larger the subcategory  $E \subset A$ , the smaller its derived category  $D(E)$ .

Similarly, let  $B$  be an exact category with exact products, and let  $F' \subset F'' \subset B$  be two resolving subcategories closed under products. Then one has

$$D^{\text{ctr}}(F') \simeq D^{\text{ctr}}(F'') \simeq D^{\text{ctr}}(B),$$

while the natural functors between the conventional derived categories

$$D(F') \rightarrow D(F'') \rightarrow D(B)$$

are Verdier quotient functors.

In particular, when in the situation of [Proposition 3.4](#) there are two  $\infty$ -tilting pairs  $(T, E')$  and  $(T, E'')$  with  $E' \subset E'' \subset A$ , and the corresponding two  $\infty$ -cotilting pairs are  $(W, F')$  and  $(W, F'')$ , so  $F' \subset F'' \subset B$ , we obtain the commutative diagram of Verdier quotient functors and triangulated equivalences as in [Figure 4](#). We refer to [\[Positselski 2017a, Section 1\]](#) for a further discussion.

## 5. $\infty$ -tilting and $\infty$ -cotilting t-structures

The aim of the section is to lift the canonical t-structures from  $D(A)$  and  $D(B)$  to  $D(E)$  and  $D(F)$ , respectively, in [Figures 2](#) or [3](#) in the previous section. By doing this, we obtain a picture very similar to the classical tilting theory, where both  $A$  and  $B$  can be viewed as full subcategories of  $D(E)$  such that  $E = A \cap B$  (since  $E \simeq F$ , we of course obtain the same picture in  $D(F)$ ).

We start with a lemma showing that t-structures can be lifted with respect to certain triangulated functors with partial adjoints.

**Lemma 5.1.** *Let  $D$  and  $'D$  be triangulated categories and let  $(D^{\leq 0}, D^{\geq 0})$  be a t-structure on  $D$ . Let  $F : 'D \rightarrow D$  be a triangulated functor such that a right adjoint functor to  $F$  is defined on  $D^{\geq 0} \subset D$ , that is, for every object  $X \in D^{\geq 0}$  there exists an object  $G(X) \in 'D$  such that the functors  $\text{Hom}_D(F(-), X)$  and  $\text{Hom}_D(-, G(X))$  are isomorphic on  $'D$ . Assume that the adjunction morphism  $\varepsilon_X : FG(X) \rightarrow X$  is an isomorphism in  $D$  for all objects  $X \in D^{\geq 0}$ .*

*Set  $'D^{\leq 0} = F^{-1}(D^{\leq 0}) \subset 'D$  to be the full preimage of  $D^{\leq 0}$  under  $F$  and  $'D^{\geq 0} = G(D^{\geq 0}) \subset 'D$  to be the essential image of  $D^{\geq 0}$  under  $G$ . Then the pair of full subcategories  $({}'D^{\leq 0}, {}'D^{\geq 0})$  is a t-structure on  $'D$ . The functors  $F$  and  $G$  restrict to mutually inverse equivalences between the abelian hearts  $A = D^{\leq 0} \cap D^{\geq 0} \subset D$  and  $'A = {}'D^{\leq 0} \cap {}'D^{\geq 0} \subset 'D$  of the two t-structures.*

*Proof.* One can easily check that the functor  $G$  commutes with the shift functors  $[-1]$  on  $'D$  and  $D$  (since the functor  $F$  does). Let us show that  $\text{Hom}_D(X, 'Y) = 0$

for all  $'X \in 'D^{\leq 0}$  and  $'Y \in 'D^{\geq 1}$ . Indeed, we have  $F('X) \in D^{\leq 0}$  and  $'Y = G(Y)$  for some  $Y \in D^{\geq 1}$ . Hence

$$\mathrm{Hom}_{\mathcal{D}}('X, 'Y) = \mathrm{Hom}_{\mathcal{D}}('X, G(Y)) = \mathrm{Hom}_{\mathcal{D}}(F('X), Y) = 0.$$

Now let  $'X \in 'D$  be an arbitrary object. Set  $X = F('X) \in D$ , and consider a distinguished triangle

$$(1) \quad \tau_{\leq 0}X \rightarrow X \rightarrow \tau_{\geq 1}X \rightarrow (\tau_{\leq 0}X)[1]$$

in  $D$  with  $\tau_{\leq 0}X \in D^{\leq 0}$  and  $\tau_{\geq 1}X \in D^{\geq 1}$ . Put  $\tau_{\geq 1}'X = G(\tau_{\geq 1}X) \in 'D^{\geq 1}$ . Then the morphism  $F('X) = X \rightarrow \tau_{\geq 1}X$  in  $D$  corresponds to a certain morphism

$$'X \rightarrow G(\tau_{\geq 1}X) = \tau_{\geq 1}'X$$

in  $'D$ . Denote by  $\tau_{\leq 0}'X$  a cocone of the latter morphism, so that we have a distinguished triangle

$$(2) \quad \tau_{\leq 0}'X \rightarrow 'X \rightarrow \tau_{\geq 1}'X \rightarrow (\tau_{\leq 0}'X)[1]$$

in  $'D$ . Applying the functor  $F$  to the morphism  $'X \rightarrow \tau_{\geq 1}'X$  produces the morphism  $X = F('X) \rightarrow F(\tau_{\geq 1}'X) = FG(\tau_{\geq 1}X) = \tau_{\geq 1}(X)$ . Thus the triangulated functor  $F$  takes the distinguished triangle (2) to the distinguished triangle (1), and it follows that the object  $F(\tau_{\leq 0}'X)$  is isomorphic to  $\tau_{\leq 0}X$ . In other words, we have  $\tau_{\leq 0}'X \in 'D^{\leq 0}$  and  $\tau_{\geq 1}'X \in 'D^{\geq 1} := 'D^{\geq 0}[-1]$  in (2). It follows that  $( 'D^{\leq 0}, 'D^{\geq 0} )$  is a t-structure.

Furthermore, the functors  $F$  and  $G$  restrict to an equivalence between the coaisles  $D^{\geq 0} \subset D$  and  $'D^{\geq 0} \subset 'D$ . Indeed, if  $'X \in 'D^{\geq 0}$ , then  $'X = G(X)$  for some  $X \in D^{\geq 0}$  and  $F('X) = FG(X) = X \in D^{\geq 0}$ . Thus the functor  $F$  restricts to  $F : 'D^{\geq 0} \rightarrow D^{\geq 0}$ , the functor  $G : D^{\geq 0} \rightarrow 'D^{\geq 0}$  is its (honest) right adjoint, and the composition

$$D^{\geq 0} \xrightarrow{G} 'D^{\geq 0} \xrightarrow{F} D^{\geq 0}$$

is the identity functor by assumption. Hence the functor  $G$  is fully faithful; and its essential image coincides with  $'D^{\geq 0}$  by the definition. Finally, for any  $'X \in 'D^{\geq 0}$  we have  $'X \in 'A$  if and only if  $F('X) \in A$ , because we have  $'X \in 'D^{\leq 0}$  if and only if  $F('X) \in D^{\leq 0}$ .  $\square$

**Remark 5.2.** In the special case where the functor  $F : 'D \rightarrow D$  from the former lemma is a part of a recollement

$$\begin{array}{ccc} \mathcal{D} & \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{F} \\ \xrightarrow{\quad} \end{array} & 'D & \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} & ''D, \end{array}$$

the t-structure  $( 'D^{\leq 0}, 'D^{\geq 0} )$  coincides with the result of gluing  $( D^{\leq 0}, D^{\geq 0} )$  with the trivial t-structure  $( ''D, 0 )$  on  $''D$  in the sense of [Beilinson et al. 1982, Théorème 1.4.10].

We recall that for any t-structure  $(D^{\leq 0}, D^{\geq 0})$  on a triangulated category  $D$  with the abelian heart  $A = D^{\leq 0} \cap D^{\geq 0} \subset D$  there are natural maps

$$(3) \theta_{A,D}^i = \theta_{A,D}^i(X, Y) : \text{Ext}_A^i(X, Y) \rightarrow \text{Hom}_D(X, Y[i]) \quad \text{for all } X, Y \in A, i \geq 0.$$

A t-structure  $(D^{\leq 0}, D^{\geq 0})$  is said to be *of the derived type* if the maps  $\theta_{A,D}^i(X, Y)$  are isomorphisms for all  $X, Y \in A$  and  $i \geq 0$  (see [Beilinson et al. 1982, Remarque 3.1.17; Positselski 2011, Corollary A.17; Positselski and Šťovíček 2019, Section 4] for further details).

**Lemma 5.3.** *In the context of Lemma 5.1, the t-structure  $({}'D^{\leq 0}, {}'D^{\geq 0})$  on the triangulated category  $'D$  is of the derived type if and only if the t-structure  $(D^{\leq 0}, D^{\geq 0})$  on the triangulated category  $D$  is.*

*Proof.* According to Lemma 5.1, the functor  $F : {}'A \rightarrow A$  is an equivalence of categories. So, according to the proof of Lemma 5.1, is the functor  $F : {}'D^{\geq 0} \rightarrow D^{\geq 0}$ . It remains to observe that the domain of the map (3) is an Ext group computed in the abelian heart of the t-structure, while the codomain is a Hom group in the coaisle:  $\text{Hom}_D(X, Y[i]) = \text{Hom}_D(X[-i], Y)$ , and both the objects  $X[-i]$  and  $Y$  belong to  $D^{\geq 0}$ .  $\square$

Lemma 5.4 describes the situation in which we want to apply Lemma 5.1.

**Lemma 5.4.** *Let  $A$  be an abelian category and  $E \subset A$  be a coresolving subcategory, viewed as an exact category with the exact category structure inherited from  $A$ . Then the functor between the derived categories of bounded below complexes  $D^+(E) \rightarrow D^+(A)$  induced by the exact embedding functor  $E \rightarrow A$  is a triangulated equivalence. The inverse functor to this equivalence  $D(A) \supset D^+(A) \rightarrow D^+(E) \subset D(E)$  is a partially defined right adjoint functor (in a sense analogous to the statement of Lemma 5.1) to the functor between the unbounded derived categories  $D(E) \rightarrow D(A)$  induced by the exact embedding  $E \rightarrow A$ .*

*Proof.* For any bounded below complex  $A^\bullet$  in  $A$  there exists a bounded below complex  $E^\bullet$  in  $E$  together with a quasi-isomorphism  $A^\bullet \rightarrow E^\bullet$  of complexes in  $A$  [Hartshorne 1966, Lemma I.4.6(1)]. Thus, the functor  $D^+(E) \rightarrow D^+(A)$  is essentially surjective.

Since  $E$  is closed under the cokernels of monomorphisms, any bounded below complex in  $E$  that is acyclic in  $A$  is also acyclic in  $E$ . From this we will deduce that for any complex  $E^\bullet$  in  $E$  and any bounded below complex  $F^\bullet$  in  $E$  the natural map

$$(4) \quad \text{Hom}_{D(E)}(E^\bullet, F^\bullet) \rightarrow \text{Hom}_{D(A)}(E^\bullet, F^\bullet)$$

is an isomorphism, which implies both that the functor  $D^+(E) \rightarrow D^+(A)$  is fully faithful (hence, a triangulated equivalence) and that the inverse functor to it is partially right adjoint to the canonical functor  $D(E) \rightarrow D(A)$ .

Indeed, an arbitrary morphism  $E^\bullet \rightarrow F^\bullet$  in the derived category  $D(A)$  can be represented by a fraction of morphisms of complexes  $E^\bullet \rightarrow X^\bullet \leftarrow F^\bullet$ , where  $X^\bullet$  is a complex in  $A$  and  $F^\bullet \rightarrow X^\bullet$  is a quasi-isomorphism of complexes in  $A$ . Now the complex  $X^\bullet$  is acyclic in low cohomological degrees, so for  $n \ll 0$  the natural morphism from  $X^\bullet$  to its canonical truncation  $X^\bullet \rightarrow \tau_{\geq n} X^\bullet$  is a quasi-isomorphism of complexes in  $A$ . The complex  $\tau_{\geq n} X^\bullet$  is bounded below, so there exists a bounded below complex  $G^\bullet$  in  $E$  together with a quasi-isomorphism  $\tau_{\geq n} X^\bullet \rightarrow G^\bullet$  of complexes in  $A$ . Then the composition  $F^\bullet \rightarrow X^\bullet \rightarrow \tau_{\geq n} X^\bullet \rightarrow G^\bullet$  is a quasi-isomorphism of complexes in the exact category  $E$ . This allows us to represent our morphism  $E^\bullet \rightarrow F^\bullet$  in  $D(A)$  by a fraction  $E^\bullet \rightarrow G^\bullet \leftarrow F^\bullet$  of morphisms of complexes in  $E$ . This proves surjectivity of the map (4).

The injectivity is similar. If a fraction  $E^\bullet \rightarrow X^\bullet \leftarrow F^\bullet$  vanishes in the group  $\text{Hom}_{D(A)}(E^\bullet, F^\bullet)$ , then there exists a quasi-isomorphism  $X^\bullet \rightarrow G^\bullet$  of complexes in  $A$  such that  $E^\bullet \rightarrow X^\bullet \rightarrow G^\bullet$  is null-homotopic. As above, we can choose  $X^\bullet \rightarrow G^\bullet$  so that  $G^\bullet$  is a bounded below complex in  $E$ , and it follows that the fraction vanishes in  $\text{Hom}_{D(E)}(E^\bullet, F^\bullet)$  as well.  $\square$

Given an abelian category  $A$  with a coresolving subcategory  $E \subset A$ , for any complex  $E^\bullet$  in  $E$  we denote by  $H_A^n(E^\bullet) \in A$  the cohomology objects of the complex  $E^\bullet$  viewed as a complex in  $A$ . Consider the following two full subcategories in the unbounded derived category  $D(E)$ :

- $D_A^{\leq 0}(E) \subset D(E)$  is the full subcategory of all complexes  $E^\bullet$  in  $E$  such that  $H_A^n(E^\bullet) = 0$  for all  $n > 0$ .
- $D^{\geq 0}(E) \subset D(E)$  is the full subcategory of all objects in  $D(E)$  that can be represented by complexes  $E^\bullet$  in  $E$  with  $E^n = 0$  for all  $n < 0$ .

As in the usual notation, for any  $n \in \mathbb{Z}$  we set  $D_A^{\leq n}(E) = D_A^{\leq 0}(E)[-n] \subset D(E)$  and  $D^{\geq n}(E) = D^{\geq 0}(E)[-n] \subset D(E)$ .

**Proposition 5.5.** *Let  $A$  be an abelian category and  $E \subset A$  be a coresolving subcategory. Then the pair of full subcategories  $(D_A^{\leq 0}(E), D^{\geq 0}(E))$  is a t-structure on the unbounded derived category  $D(E)$  of the exact category  $E$ . Moreover, this is a t-structure of the derived type, and the triangulated functor  $D(E) \rightarrow D(A)$  induced by the exact embedding  $E \rightarrow A$  identifies its heart  $D_A^{\leq 0}(E) \cap D^{\geq 0}(E)$  with the abelian category  $A$ .*

*Proof.* We apply Lemma 5.1 to the situation described in Lemma 5.4, where  $'D = D(E)$ ,  $D = D(A)$ , and  $F : D(E) \rightarrow D(A)$  is the canonical functor. Moreover, we set  $(D^{\leq 0}, D^{\geq 0})$  to be the canonical t-structure on  $D(A)$ , which is certainly of the derived type. Then  $G = F|_{D^{\geq 0}(E)}^{-1} : D^{\geq 0} \rightarrow D^{\geq 0}(E) \subset D(E)$  is a partially defined right adjoint to  $F$  in the sense of Lemma 5.1 and  $(D_A^{\leq 0}(E), D^{\geq 0}(E))$  is precisely

the lifted t-structure from the conclusion of the lemma. It is of the derived type by [Lemma 5.3](#).

For clarity, we summarize the construction of the t-structure truncations  $\tau_{\leq 0}^E E^\bullet$  and  $\tau_{\geq 1}^E E^\bullet$  for a given complex  $E^\bullet$  over  $E$ . One first considers its canonical truncation  $\tau_{\geq 1}^A E^\bullet$  as a complex in  $A$ , in the standard t-structure on  $D(A)$ . So  $\tau_{\geq 1}^A E^\bullet$  is a complex in  $A$  with the terms concentrated in the cohomological degrees  $\geq 1$ ; hence there exists a complex  $F^\bullet$  in  $E$  with the terms concentrated in the cohomological degrees  $\geq 1$  endowed with a quasi-isomorphism  $\tau_{\geq 1}^A E^\bullet \rightarrow F^\bullet$  of complexes in  $A$ . One sets  $\tau_{\geq 1}^E E^\bullet = F^\bullet$ , and  $\tau_{\leq 0}^E E^\bullet$  is a cocone of the morphism of complexes  $E^\bullet \rightarrow F^\bullet$  in  $D(E)$ .  $\square$

**Remark 5.6.** It is useful to look into (non)degeneracy properties of the t-structure  $(D_A^{\leq 0}(E), D^{\geq 0}(E))$  on  $D(E)$ . The intersection  $\bigcap_{n \geq 0} D^{\geq n}(E) \subset D(E)$  consists of some bounded below complexes in  $E$  with vanishing cohomology in  $A$ . All such complexes are acyclic in  $E$ , so this intersection is a zero category. On the other hand, the intersection  $\bigcap_{n \leq 0} D_A^{\leq n}(E) \subset D(E)$  consists of all the complexes in  $E$  with vanishing cohomology in  $A$ . This is precisely the kernel of the triangulated functor  $D(E) \rightarrow D(A)$ , and it can very well be nontrivial. Indeed, let  $k$  be a field,  $A = k[x]/(x^2)\text{-mod}$  and  $E = A_{i_{\text{inj}}}$  (see also [Example 6.3](#) below). Since the complex

$$\dots \rightarrow k[x]/(x^2) \xrightarrow{x} k[x]/(x^2) \xrightarrow{x} k[x]/(x^2) \rightarrow \dots$$

is acyclic but not contractible, it is nonzero in  $D(E) = \text{Hot}(A_{i_{\text{inj}}})$ , but it becomes zero in  $D(A)$ .

Let us formulate the dual assertions. Given an abelian category  $B$  with a resolving subcategory  $F \subset B$ , for any complex  $F^\bullet$  in  $F$  we denote by  $H_B^n(F^\bullet) \in B$  the cohomology objects of the complex  $F^\bullet$  viewed as a complex in  $B$ . Consider the following two subcategories in the unbounded derived category  $D(F)$ :

- $D^{\leq 0}(F) \subset D(F)$  is the full subcategory of all objects in  $D(F)$  that can be represented by complexes  $F^\bullet$  in  $F$  with  $F^n = 0$  for all  $n > 0$ .
- $D_B^{\geq 0}(F) \subset D(F)$  is the full subcategory of all complexes  $F^\bullet$  in  $F$  such that  $H_B^n(F^\bullet) = 0$  for all  $n < 0$ .

**Proposition 5.7.** *Let  $B$  be an abelian category and  $F \subset B$  be a resolving subcategory. Then the pair of full subcategories  $(D^{\leq 0}(F), D_B^{\geq 0}(F))$  is a t-structure on the unbounded derived category  $D(F)$  of the exact category  $F$ . Moreover, this is a t-structure of the derived type, and the triangulated functor  $D(F) \rightarrow D(B)$  induced by the exact embedding  $F \rightarrow B$  identifies its heart  $D^{\leq 0}(F) \cap D_B^{\geq 0}(F)$  with the abelian category  $B$ .*

*Proof.* Dual to [Proposition 5.5](#).  $\square$

Now we are well-equipped for the discussion of  $\infty$ -tilting and  $\infty$ -cotilting  $t$ -structures. Let  $A$  be a complete, cocomplete abelian category with an injective cogenerator  $J$  and an  $\infty$ -tilting pair  $(T, E)$ , and let  $B$  be the corresponding complete, cocomplete abelian category with a projective generator  $P$  and an  $\infty$ -cotilting pair  $(W, F)$ , as in [Corollary 3.5](#). Suppose further for convenience that  $E$ , and hence also  $F$ , are idempotent complete. Then the exact category  $E \simeq F$  is simultaneously a coresolving subcategory in  $A$  and a resolving subcategory in  $B$ .

Thus we have two  $t$ -structures

$$(D_A^{\leq 0}(E), D_A^{\geq 0}(E)) \quad \text{and} \quad (D_B^{\leq 0}(F), D_B^{\geq 0}(F))$$

on the unbounded derived category  $D(E) = D = D(F)$ . The hearts of these  $t$ -structures are the abelian categories  $A$  and  $B$ , respectively.

From the point of view of the category  $A$ , the  $t$ -structure  $(D_A^{\leq 0}(E), D_A^{\geq 0}(E))$  on the triangulated category  $D$  can be called the *standard  $t$ -structure*, and the  $t$ -structure  $(D^{\leq 0}(F), D_B^{\geq 0}(F))$  is the  *$\infty$ -tilting  $t$ -structure*. Looking from the point of view of the category  $B$ , the  $t$ -structure  $(D^{\leq 0}(F), D_B^{\geq 0}(F))$  on the triangulated category  $D$  is the *standard  $t$ -structure*, and the  $t$ -structure  $(D_A^{\leq 0}(E), D^{\geq 0}(E))$  is the  *$\infty$ -cotilting  $t$ -structure*. The abelian category  $B$  is the  *$\infty$ -tilting heart*, and the abelian category  $A$  is the  *$\infty$ -cotilting heart*.

## 6. Examples

**Example 6.1.** Let  $A$  be a complete, cocomplete abelian category with an injective cogenerator  $J$  and an  $\infty$ -tilting object  $T \in A$ , and let  $B$  be the corresponding complete, cocomplete abelian category with a projective generator  $P$  and an  $\infty$ -cotilting object  $W \in B$ , as in [Corollary 2.5](#). In this context, if *both* the projective dimension of the  $\infty$ -tilting object  $T \in A$  and the injective dimension of the  $\infty$ -cotilting object  $W \in B$  are finite, then they are equal to each other,  $\text{pd}_A T = n = \text{id}_B W$ . Furthermore, this holds if and only if the object  $T \in A$  is  $n$ -tilting if and only if the object  $W \in B$  is  $n$ -cotilting (both in the sense of [\[Positselski and Šťovíček 2019, Sections 2 and 4\]](#)).

Indeed, suppose that  $\text{pd}_A T < \infty$  and  $\text{id}_B W < \infty$  and denote by  $n$  the maximum of the two values. Then the left exact functor  $\Psi : A \rightarrow B$  has finite homological dimension, since it can be computed as the functor  $\text{Hom}_A(T, -)$ ; and the right exact functor  $\Phi : B \rightarrow A$  has finite homological dimension, since it can be computed as the functor  $\text{Hom}_B(-, W)^{\text{op}}$ . Denote by  $E_T \subset A$  the full subcategory of all objects  $E \in A$  such that  $\text{Ext}_A^i(T, E) = 0$  for all  $i > 0$ , and by  $F_W \subset B$  the full subcategory of all objects  $F \in B$  such that  $\text{Ext}_B^i(F, W) = 0$  for all  $i > 0$ . (By the definition, we have  $E_{\max}(T) \subset E_T$  and  $F_{\max}(W) \subset F_W$ .)

Then the functor  $\Psi$  is exact on the exact category  $E_T$  and the functor  $\Phi$  is exact on the exact category  $F_W$ . The full subcategory  $E_T$  is coresolving in  $A$ , and the full subcategory  $F_W$  is resolving in  $B$ , with both the (co)resolution dimensions bounded

by the finite constant  $n$ . The latter fact is due to the observation that, thanks to a simple dimension-shifting argument, any  $n$ -th cosyzygy object in  $A$  belongs to  $E_T$  and any  $n$ -th syzygy object in  $B$  belongs to  $F_W$ .

Let us show that the functors  $\Phi$  and  $\Psi$  restrict to mutually inverse equivalences between the exact categories  $E_T$  and  $F_W$ . Given an object  $E \in E_T$ , choose an exact sequence  $0 \rightarrow E \rightarrow J^0 \rightarrow \dots \rightarrow J^{d-1} \rightarrow E' \rightarrow 0$  in  $A$  with  $J^i \in A_{\text{inj}}$  with  $d \geq \max(n, 2)$ . Then the sequence

$$0 \rightarrow \Psi(E) \rightarrow \Psi(J^0) \rightarrow \dots \rightarrow \Psi(J^{d-1}) \rightarrow \Psi(E') \rightarrow 0$$

is exact in  $B$ , and the objects  $\Psi(J^i)$  belong to the full subcategory  $\text{Prod}(W) \subset F_W \subset B$ . Hence  $\Psi(E) \in F_W$  by dimension shifting.

Furthermore, we have  $E' \in E_T$ , hence  $\Psi(E') \in F_W$ . It follows that the sequence  $0 \rightarrow \Phi\Psi(E) \rightarrow \Phi\Psi(J^0) \rightarrow \dots \rightarrow \Phi\Psi(J^{d-1}) \rightarrow \Phi\Psi(E') \rightarrow 0$  is exact in  $A$ . Since the adjunction morphisms  $\Phi\Psi(J^i) \rightarrow J^i$  are isomorphisms for  $i$  equal to 0 and 1, so is the adjunction morphism  $\Phi\Psi(E) \rightarrow E$ . Similarly one shows that  $\Phi(F) \in E_T$  for all  $F \in F_W$ , and the adjunction morphism  $F \rightarrow \Psi\Phi(F)$  is an isomorphism.

According to [Bondal and van den Bergh 2003, Lemmas 5.4.1, 5.4.2; Fiorot et al. 2017, Proposition 1.5; Positselski and Šťovíček 2019, Theorem 5.5], the triangulated functors  $D(E_T) \rightarrow D(A)$  and  $D(F_W) \rightarrow D(B)$  induced by the exact embedding functors  $E_T \rightarrow A$  and  $F_W \rightarrow B$  are equivalences of triangulated categories. Thus we obtain a triangulated equivalence

$$D(A) \simeq D(E_T) = D(F_W) \simeq D(B).$$

Applying, e.g., [Positselski and Šťovíček 2019, Proposition 2.5 and Corollary 4.4(b)], one can conclude that the conditions (i)–(iii) and (i\*)–(iii\*) of [Positselski and Šťovíček 2019, Sections 2 and 4] hold for  $T$  and  $W$ , respectively. That is,  $T$  is  $n$ -tilting,  $W$  is  $n$ -cotilting and, moreover,  $\text{pd}_A T = n = \text{id}_B W$  by [Positselski and Šťovíček 2019, Corollary 4.12].

Following [Positselski and Šťovíček 2019, Lemma 5.1], the two conditions (i<sub>max</sub>) and (ii<sub>max</sub>) defining the full subcategory  $E_{\text{max}}(T) \subset A$  are equivalent in this case. Similarly, the two conditions (i\*<sub>max</sub>) and (ii\*<sub>max</sub>) defining the full subcategory  $F_{\text{max}}(W) \subset B$  are equivalent. So either one of the two conditions is sufficient to define these classes in the  $n$ -(co)tilting case, and we actually have  $E_{\text{max}}(T) = E_T$  and  $F_{\text{max}}(W) = F_W$ . It is only in the  $\infty$ -(co)tilting situation that we need to impose both the conditions. The full subcategory  $E = E_{\text{max}}(T)$  is the  $n$ -tilting class of an  $n$ -tilting object  $T$ , and the full subcategory  $F = F_{\text{max}}(W)$  is the  $n$ -cotilting class of an  $n$ -cotilting object  $W$ , as discussed in [Positselski and Šťovíček 2019, Sections 3–4]. According to [Positselski and Šťovíček 2019, Lemma 5.3 and Remark 5.4], both the full subcategories  $E$  and  $F$  are closed under both the infinite products and coproducts in  $A$  and  $B$ .

Finally, note that if  $T$  is  $n$ -tilting, then  $W$  is  $n$ -cotilting and vice versa by [Positselski and Šťovíček 2019, Corollary 4.12]. Thus, both the projective dimension of  $T$  and the injective dimension of  $W$  need to be finite for either of the two objects to be  $n$ -(co)tilting.

**Example 6.2.** Let  $A$  be a complete, cocomplete abelian category with an injective cogenerator  $J$  and an  $\infty$ -tilting pair  $(T, E)$ , and let  $B$  be the corresponding complete, cocomplete abelian category with a projective generator  $P$  and an  $\infty$ -cotilting pair  $(W, F)$ , as in Corollary 3.5. Suppose that the full subcategory  $E \subset A$  is closed under coproducts and the full subcategory  $F \subset B$  is closed under products. Then, by Proposition 4.2, the triangulated functors  $D(E) \rightarrow D(A)$  and  $D(F) \rightarrow D(B)$  induced by the exact embeddings  $E \rightarrow A$  and  $F \rightarrow B$  are Verdier quotient functors.

Assume that only *one* of the objects  $T$  and  $W$  has finite homological dimension, or more specifically, that  $\text{pd}_A T < \infty$ . Then the left exact functor  $\Psi : A \rightarrow B$  has finite homological dimension and the full subcategory

$$E_T = \{E \in A \mid \text{Ext}_A^i(T, E) = 0 \text{ for all } i > 0\}$$

of  $A$  has finite coresolution dimension, as in the previous example. In particular, the complex  $\Psi(E^\bullet)$  is acyclic in  $B$  for any complex  $E^\bullet$  in the category  $E$  that is acyclic in  $A$ . So the composition of triangulated functors  $D(E) \simeq D(F) \rightarrow D(B)$  factorizes through the Verdier quotient functor  $D(E) \rightarrow D(A)$ , or in other words, the triangulated equivalence  $D(E) \simeq D(F)$  descends to a triangulated functor  $D(A) \rightarrow D(B)$  in Figure 3. This is also a Verdier quotient functor (since such is the functor  $D(F) \rightarrow D(B)$ ).

Similarly, assume that  $\text{id}_B W < \infty$ . Then the right exact functor  $\Phi : B \rightarrow A$  has finite homological dimension. In particular, the complex  $\Phi(F^\bullet)$  is acyclic in  $A$  for any complex  $F^\bullet$  in the category  $F$  that is acyclic in  $B$ . Hence the triangulated equivalence  $D(F) \simeq D(E)$  descends to a triangulated Verdier quotient functor  $D(B) \rightarrow D(A)$ .

In the representation theory of finite-dimensional algebras, it is an open problem whether a finite-dimensional  $\infty$ -tilting module of finite projective dimension is already  $n$ -tilting for some  $n$ . It goes under the name of the Wakamatsu tilting conjecture, and it is a member of a family of long standing so-called homological conjectures for finite-dimensional algebras [Mantese and Reiten 2004, Section 4; Beligiannis and Reiten 2007, §IV.3].

**Example 6.3.** Let  $A$  be a locally Noetherian Grothendieck abelian category (see [Positselski and Šťovíček 2019, Section 10.2]). Choose an injective object  $J \in A$  such that  $A_{\text{inj}} = \text{Add}(J)$ ; then it follows that  $J$  is an injective cogenerator of  $A$ , and one also has  $A_{\text{inj}} = \text{Prod}(J)$ . Set  $T = J$  and  $E = A_{\text{inj}} \subset A$ . Then  $(T, E)$  is an  $\infty$ -tilting pair in  $A$ .

In the corresponding abelian category  $B$  with a natural projective generator  $P$  [Positselski 2015, Theorem 3.6], one has  $B_{\text{proj}} = \text{Add}(P) = \text{Prod}(P)$  (see also Lemma 4.1). The related  $\infty$ -cotilting pair in  $B$  is  $(W, F)$ , where  $W = P$  and  $F = B_{\text{proj}}$ . So both the full subcategories  $E \subset A$  and  $F \subset B$  are closed under both the products and coproducts. As always in the context of Corollary 3.5, one has an equivalence of additive/exact categories  $E \simeq F$ .

The derived category  $D(E)$  is simply the homotopy category  $\text{Hot}(A_{\text{inj}})$ ; it is equivalent to the coderived category  $D^{\text{co}}(A)$  (see the argument for [Positselski 2017c, Theorem 2.4]). The derived category  $D(F)$  is simply the homotopy category  $\text{Hot}(B_{\text{proj}})$ ; it is equivalent to the contraderived category  $D^{\text{ctr}}(B)$  (cf. [Positselski 2017c, Theorem 4.4(b)] and [Positselski 2012, Corollary A.6.2]). Hence the derived equivalence

$$D^{\text{co}}(A) \simeq \text{Hot}(A_{\text{inj}}) = \text{Hot}(B_{\text{proj}}) \simeq D^{\text{ctr}}(B).$$

These are the *minimal*  $\infty$ -tilting and  $\infty$ -cotilting pair for the  $\infty$ -tilting object  $T \in A$  and the  $\infty$ -cotilting object  $W \in B$ , in the sense of Example 3.7: one has  $E_{\text{min}}(T) = E = A_{\text{inj}}$  and  $F_{\text{min}}(W) = F = B_{\text{proj}}$ .

**Example 6.4.** In the context of the previous example, it is also instructive to consider the *maximal*  $\infty$ -tilting pair  $(T, E_{\text{max}}(T))$  for the  $\infty$ -tilting object  $T = J$  in the category  $A$  and the maximal  $\infty$ -cotilting pair  $(W, F_{\text{max}}(W))$  for the  $\infty$ -cotilting object  $W = P$  in the category  $B$ .

The full subcategory  $E_{\text{max}}(T) \subset A$  consists of all the objects  $E \in A$  for which there exists an unbounded acyclic complex of injective objects

$$\dots \rightarrow J^{-2} \rightarrow J^{-1} \rightarrow J^0 \rightarrow J^1 \rightarrow J^2 \rightarrow \dots$$

such that the complex  $\text{Hom}_A(J, J^\bullet)$  is acyclic and  $E$  is the image of the morphism  $J^{-1} \rightarrow J^0$ . This is known as the full subcategory of *Gorenstein injective objects* in the abelian category  $A$ .

Similarly, the full subcategory  $F_{\text{max}}(W) \subset B$  consists of all the objects  $F \in B$  for which there exists an unbounded acyclic complex of projective objects

$$\dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow P_{-1} \rightarrow P_{-2} \rightarrow \dots$$

such that the complex  $\text{Hom}_B(P_\bullet, P)$  is acyclic and  $F$  is the image of the morphism  $P_0 \rightarrow P_{-1}$ . This is known as the full subcategory of *Gorenstein projective objects* in the abelian category  $B$  (cf. [Enomoto 2017, Definition 3.7]).

Hence we can conclude from Theorems 2.3 and 2.4 that the exact categories of Gorenstein injective objects in  $A$  and Gorenstein projective objects in  $B$  are naturally equivalent.

If  $A$  has a generating set of objects of finite projective dimension, then, by [Gillespie 2017, Theorem 5.7] or [Šťovíček 2014b, Lemma 7.2], the class of acyclic

complexes of injectives is closed under products (although products may not be exact in  $A$ ). In particular,  $E_{\max}(T) \subset A$  is closed under products, and so is  $F_{\max}(W) \subset B$  by [Lemma 4.1](#). Dually, if  $B$  has a cogenerating set of objects of finite injective dimension, then both  $E_{\max}(T) \subset A$  and  $F_{\max}(W) \subset B$  are closed under coproducts.

In particular, if  $A$  is the category of quasi-coherent sheaves on a quasi-compact semiseparated scheme  $X$ , then any quasi-coherent sheaf on  $X$  is the quotient of one of the so-called very flat quasi-coherent sheaves [[Positselski 2012](#), Lemma 4.1.1] (see [[Murfet 2006](#), Section 2.4] or [[Efimov and Positselski 2015](#), Lemma A.1] for the more widely known, but weaker assertion with flat sheaves in place of the very flat ones). If  $X$  is covered by  $n$  affine open subschemes, then the projective dimension of any very flat quasi-coherent sheaf, as an object of  $A$ , does not exceed  $n$ , as one can show using a Čech resolution for the affine covering, together with the fact that the projective dimension of a very flat module does not exceed 1 (cf. [[Positselski and Slávik 2017](#), properties (VF5) and (VF6)]). Thus the class of acyclic complexes of injectives is closed under products in  $A$ . If  $X$  is also Noetherian, then  $A$  is a locally Noetherian category, and the discussion in the previous paragraph applies.

**Example 6.5.** In the context of [Examples 6.3](#) and [6.4](#), one can say that a locally Noetherian Grothendieck abelian category  $A$  is  $n$ -Gorenstein if the  $\infty$ -tilting object  $T = J$  is  $n$ -tilting. This means that  $\text{pd}_A T = \text{id}_B W \leq n$  (cf. [[Positselski and Šťovíček 2019](#), Theorem 10.3]).

In this case, we have the minimal  $\infty$ -tilting and  $\infty$ -cotilting pair  $(T, E_{\min}(T))$  and  $(W, F_{\min}(W))$  with  $E_{\min}(T) = A_{\text{inj}}$  and  $F_{\min}(W) = B_{\text{proj}}$ , as in [Example 6.3](#). We also have the maximal  $\infty$ -tilting and  $\infty$ -cotilting pair  $(T, E_{\max}(T))$  and  $(W, F_{\max}(W))$  with  $E_{\max}(T) = E_T$  being the  $n$ -tilting class of the  $n$ -tilting object  $T \in A$  (consisting of all the Gorenstein injectives in  $A$ ) and  $F_{\max}(W) = F_W$  being the  $n$ -cotilting class of the  $n$ -cotilting object  $W \in B$  (consisting of all the Gorenstein projectives in  $B$ ), as in [Examples 6.1](#) and [6.4](#).

The two related derived equivalences (as in [Section 4](#)) form a commutative diagram with the natural Verdier quotient functors as follows:

$$\begin{array}{ccccccc}
 D^{\text{co}}(A) & \xlongequal{\quad} & \text{Hot}(A_{\text{inj}}) & \xlongequal{\quad} & \text{Hot}(B_{\text{proj}}) & \xlongequal{\quad} & D^{\text{ctr}}(B) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 D(A) & \xlongequal{\quad} & D(E_T) & \xlongequal{\quad} & D(F_W) & \xlongequal{\quad} & D(B)
 \end{array}$$

**Example 6.6.** Let  $A$  and  $B$  be associative rings, and let  $C$  be an  $A$ - $B$ -bimodule. One says that  $C$  is a *semidualizing bimodule* (in the terminology of [[Holm and White 2007](#)]) or a *pseudo-dualizing bimodule* (in the terminology of [[Positselski 2017a](#)], which we adopt here) for the rings  $A$  and  $B$  if the following conditions are satisfied:

- The left  $A$ -module  $C$  has a projective resolution by finitely generated projective left  $A$ -modules, and the right  $B$ -module  $C$  has a projective resolution by finitely generated projective right  $B$ -modules.
- The homothety maps  $A \rightarrow \text{Ext}_{B^{\text{op}}}^*(C, C)$  and  $B^{\text{op}} \rightarrow \text{Ext}_A^*(C, C)$  are isomorphisms of graded rings (where  $B^{\text{op}}$  denotes the opposite ring to  $B$ ).

This definition is (essentially) obtained by dropping the finite injective dimension condition in the definition of a dualizing module over a pair of associative rings.

Let  $C$  be a pseudo-dualizing  $A$ - $B$ -bimodule. Set  $A = A\text{-mod}$  and  $B = B\text{-mod}$  to be the abelian categories of left modules over the rings  $A$  and  $B$ . Then  $T = C$  is a (finitely generated)  $\infty$ -tilting object in  $A$ . The related maximal  $\infty$ -tilting class  $E_{\max}(T) \subset A$  is known as the *Bass class* [Holm and White 2007, Theorem 6.1], and it contains the injective left  $A$ -modules by [Holm and White 2007, Lemma 4.1].

The corresponding tilted abelian category is  $\sigma_T(A) = B$ , and its natural projective generator is  $P = B$ . Choosing  $J = \text{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$  as the injective cogenerator of  $A$ , the corresponding  $\infty$ -cotilting object in  $B$  is  $W = \text{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z})$ . The related maximal  $\infty$ -cotilting class  $F_{\max}(W) \subset B$  is known as the *Auslander class* [Holm and White 2007, Theorem 2]. The objects of the full subcategory  $\text{Add}(T) \subset A$  are called  $C$ -projectives in [Holm and White 2007], and the objects of the full subcategory  $\text{Prod}(W) \subset B$  are called  $C$ -injectives. The equivalence of exact categories

$$E_{\max}(T) \simeq F_{\max}(W)$$

is a part of what is known as the *Foxby equivalence* [Holm and White 2007, Theorem 1 or Proposition 4.1].

Both the full subcategories  $E_{\max}(T) \subset A$  and  $F_{\max}(W) \subset B$  are closed under both the infinite products and coproducts [Holm and White 2007, Proposition 4.2], so the results of our Section 4 apply and provide a commutative diagram of a triangulated equivalence and Verdier quotient functors:

$$\begin{array}{ccc}
 D^{\text{co}}(A\text{-mod}) & & D^{\text{ctr}}(B\text{-mod}) \\
 \downarrow & & \downarrow \\
 D(E_{\max}(T)) & \xlongequal{\quad} & D(F_{\max}(W)) \\
 \downarrow & & \downarrow \\
 D(A\text{-mod}) & & D(B\text{-mod})
 \end{array}$$

The paper [Positselski 2017a] is devoted to generalizing this theory to the case of a pseudo-dualizing *complex* of bimodules. In particular, (a coproduct and product-closed version of) the *minimal*  $\infty$ -tilting and  $\infty$ -cotilting classes for  $T$  and  $W$  are discussed in [Positselski 2017a, Section 5].

**Example 6.7.** Let  $\mathcal{C}$  be a coassociative, counital coring over an associative ring  $A$  (see [Positselski and Šťovíček 2019, Section 10.3]). Assume that  $\mathcal{C}$  is a projective left and a flat right  $A$ -module. Let  $A = \mathcal{C}\text{-comod}$  be the category of left  $\mathcal{C}$ -comodules; it is a Grothendieck abelian category. Set  $T \in A$  to be the cofree left  $\mathcal{C}$ -comodule  $T = \mathcal{C}$ . We claim that  $T$  is an  $\infty$ -tilting object in  $A$ .

Indeed, it was explained in [Positselski and Šťovíček 2019, Section 10.3] that  $T$  is weakly tilting, so it remains to show that the injective objects of  $A$  satisfy the condition (ii<sub>max</sub>). A left  $\mathcal{C}$ -comodule is injective if and only if it is a direct summand of a  $\mathcal{C}$ -comodule  $\mathcal{C} \otimes_A I$  coinduced from an injective left  $A$ -module  $I$  [Positselski 2010, Sections 1.1.2 and 5.1.5]. Now applying the coinduction functor  $\mathcal{C} \otimes_A -$  to a projective resolution of the  $A$ -module  $I$  produces an  $\text{Add}(T)$ -resolution of the  $\mathcal{C}$ -comodule  $\mathcal{C} \otimes_A I$  as in (ii<sub>max</sub>). This resolution remains exact after applying the functor  $\text{Hom}_A(T, -)$ , because  $\text{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{C} \otimes_A V) \simeq \text{Hom}_A(\mathcal{C}, V)$  and  $\mathcal{C}$  is a projective left  $A$ -module.

The abelian category  $B = \sigma_T(A)$  is the category of left  $\mathcal{C}$ -contramodules,  $B = \mathcal{C}\text{-contra}$  [Positselski and Šťovíček 2019, Section 10.3]. The natural projective generator is  $P = \text{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{C}) = \text{Hom}_A(\mathcal{C}, A)$ . Given an injective cogenerator  $I$  of the category of left  $A$ -modules, one can choose  $J = \mathcal{C} \otimes_A I$  as the injective cogenerator of  $A = \mathcal{C}\text{-comod}$ ; then the related cotilting object in  $B = \mathcal{C}\text{-contra}$  is  $W = \text{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{C} \otimes_A I) = \text{Hom}_A(\mathcal{C}, I)$ .

When the left homological dimension of the ring  $A$  is finite, we can describe the minimal class  $E_{\min}(T)$  which forms an  $\infty$ -tilting pair with  $T$  (Example 3.7) more explicitly as the full subcategory  $E \subset A$  of all  $\mathcal{C}/A$ -injective left  $\mathcal{C}$ -comodules [Positselski 2010, Sections 5.1.4 and 5.3], [Positselski 2015, Section 3.4]. Here, a left  $\mathcal{C}$ -comodule  $\mathcal{M}$  is called  $\mathcal{C}/A$ -injective if  $\text{Ext}_{\mathcal{C}}^i(\mathcal{L}, \mathcal{M}) = 0$  for all  $i > 0$  and all left  $\mathcal{C}$ -comodules  $\mathcal{L}$  with projective underlying left  $A$ -modules. The class  $E$  is coresolving and contains all the coinduced  $\mathcal{C}$ -comodules  $\mathcal{C} \otimes_A M$ ,  $M \in A\text{-mod}$ . In particular,  $\text{Add}(T) \subset E$ , objects of  $\text{Add}(T)$  are by definition projective in  $E$ , and by [Positselski 2010, Lemma 5.2(a) and the proof of Lemma 5.3.2(a)], there are enough such projectives. On the other hand,  $E$  has enough injectives,  $E_{\text{inj}} = A_{\text{inj}}$ , and the proof of [Positselski 2010, Theorem 5.3] reveals that any object of  $E$  has finite injective dimension bounded by the left homological dimension of  $A$ . Now we can use the following observation.

**Lemma 6.8.** *Let  $(T, E)$  be an  $\infty$ -tilting pair in a complete, cocomplete abelian category  $A$  with an injective cogenerator. If  $E$  has finite homological dimension as an exact category, then  $E = E_{\min}(T)$ .*

*Proof.* If  $n$  is the homological dimension of  $E$ , then any object  $E \in E$  admits a long exact sequence

$$0 \rightarrow E \rightarrow J^0 \rightarrow \dots \rightarrow J^n \rightarrow 0$$

in  $E$  with  $J^0, \dots, J^n \in E_{\text{inj}} = A_{\text{inj}}$ . Since this sequence remains exact after applying  $\text{Hom}_A(T, -)$ , it follows that  $E \in E_{\text{min}}(T)$  by the conditions (i) and (v) from Section 3.  $\square$

Dually, the full subcategory  $F \subset B$  in the related  $\infty$ -cotilting pair  $(W, F)$  consists of all the  $\mathcal{C}/A$ -projective left  $\mathcal{C}$ -contramodules. This is analogously the minimal  $\infty$ -cotilting pair for the  $\infty$ -cotilting object  $W \in B$ . Since the class of  $\mathcal{C}/A$ -injective comodules is closed under products and the class of  $\mathcal{C}/A$ -projective contramodules is closed under coproducts, both the full subcategories  $E \subset A$  and  $F \subset B$  are closed under both the infinite products and coproducts by Lemma 4.1. The related derived equivalence [Positselski 2010, Section 5.4] is

$$D^{\text{co}}(\mathcal{C}\text{-comod}) \simeq D(E) = D(F) \simeq D^{\text{ctr}}(\mathcal{C}\text{-contra}).$$

For comparison, when  $\mathcal{C}$  is a left Gorenstein coring in the sense of [Positselski and Šťovíček 2019, Section 10.3], i.e.,  $T \in A$  is an  $n$ -tilting object, considering the corresponding tilting and cotilting classes  $E_{\text{max}}(T) \subset A$  and  $F_{\text{max}}(W) \subset B$  produces a triangulated equivalence between the conventional derived categories,  $D(\mathcal{C}\text{-comod}) \simeq D(\mathcal{C}\text{-contra})$ .

**Example 6.9.** The case of a coassociative coalgebra  $\mathcal{C}$  over a field  $k$  is a common particular case of Examples 6.3, 6.4 and 6.7. It is also a particular case of Example 6.10.

In this case, one has  $A = \mathcal{C}\text{-comod}$  and  $B = \mathcal{C}\text{-contra}$ . The  $\infty$ -tilting object  $T = \mathcal{C} = J$  is the natural injective cogenerator of the locally Noetherian Grothendieck abelian category  $A$ , and the  $\infty$ -cotilting object  $W = \mathcal{C}^* = \text{Hom}_k(\mathcal{C}, k) = P$  is the natural projective generator of the abelian category  $B$ .

When  $\mathcal{C}$  is a Gorenstein coalgebra, we are in the situation of Example 6.5 (see [Positselski and Šťovíček 2019, Section 10.1]).

**Example 6.10.** Let  $\mathcal{S}$  be a semiassociative, semiunital semialgebra over a coassociative, counital coalgebra  $\mathcal{C}$  over a field  $k$  (see [Positselski and Šťovíček 2019, Section 10.3]). Assume that  $\mathcal{S}$  is an injective left and right  $\mathcal{C}$ -comodule. Let  $A = \mathcal{S}\text{-simod}$  be the category of left  $\mathcal{S}$ -semimodules; it is a Grothendieck abelian category. Set  $T \in A$  to be the semifree left  $\mathcal{S}$ -semimodule  $T = \mathcal{S}$ , and take  $E \subset A$  to be the full subcategory of all left  $\mathcal{S}$ -semimodules whose underlying left  $\mathcal{C}$ -comodules are injective,  $E = \mathcal{S}\text{-simod}_{\mathcal{C}\text{-inj}}$ . Then  $(T, E)$  is an  $\infty$ -tilting pair in  $A$ .

The related abelian category  $B = \sigma_T(A)$  is the category of left  $\mathcal{S}$ -semicontramodules,  $B = \mathcal{S}\text{-sicontr}$  [Positselski and Šťovíček 2019, Section 10.3]. The natural projective generator is  $P = \text{Hom}_{\mathcal{S}}(\mathcal{S}, \mathcal{S}) \in \mathcal{S}\text{-sicontr}$ . The full subcategory  $F = \Psi(E) \subset B$  consists of all left  $\mathcal{S}$ -semicontramodules whose underlying left  $\mathcal{C}$ -contramodules are projective,  $F = \mathcal{S}\text{-sicontr}_{\mathcal{C}\text{-proj}}$ . The  $\infty$ -cotilting object  $W \in B$  corresponding to the natural choice of an injective cogenerator  $J \in A$  is  $W = \mathcal{S}^* =$

$\text{Hom}_k(\mathcal{S}, k) \in \mathcal{S}\text{-slictr}$ . Both the full subcategories  $E \subset A$  and  $F \subset B$  are closed under both the products and coproducts. A detailed discussion of the equivalence of exact categories  $\mathcal{S}\text{-simod}_{e\text{-inj}} \simeq \mathcal{S}\text{-slictr}_{e\text{-proj}}$  can be found in [Positselski 2015, Section 3.5].

The derived category  $D(E)$  of the exact category  $E$  is called the *semiderived category of left  $\mathcal{S}$ -semimodules* and denoted by  $D(\mathcal{S}\text{-simod}_{e\text{-inj}}) = D^{\text{si}}(\mathcal{S}\text{-simod})$  [Positselski 2010, Section 0.3.3]. Generally speaking, it is properly intermediate between the coderived category  $D^{\text{co}}(\mathcal{S}\text{-simod})$  and the derived category  $D(\mathcal{S}\text{-simod})$ . Similarly, the derived category  $D(F)$  of the exact category  $F$  is called the *semiderived category of left  $\mathcal{S}$ -semicontramodules* and denoted by  $D(\mathcal{S}\text{-slictr}_{e\text{-proj}}) = D^{\text{si}}(\mathcal{S}\text{-slictr})$  [Positselski 2010, Section 0.3.6]. Generally speaking, it is properly intermediate between the contraderived category  $D^{\text{ctr}}(\mathcal{S}\text{-slictr})$  and the derived category  $D(\mathcal{S}\text{-slictr})$ .

The triangulated equivalence  $D^{\text{si}}(\mathcal{S}\text{-simod}) \simeq D^{\text{si}}(\mathcal{S}\text{-slictr})$  is called the *derived semimodule-semicontramodule correspondence* [Positselski 2010, Sections 0.3.7 and 6.3]. For an application to representation theory of infinite-dimensional Lie algebras (such as the Virasoro or Kac–Moody algebras), see [Positselski 2010, Corollary D.3.1].

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LEONID POSITSELSKI  
 INSTITUTE OF MATHEMATICS  
 CZECH ACADEMY OF SCIENCES  
 PRAGUE  
 CZECH REPUBLIC

and

LABORATORY OF ALGEBRAIC GEOMETRY  
 NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS  
 MOSCOW  
 RUSSIA

and

SECTOR OF ALGEBRA AND NUMBER THEORY  
 INSTITUTE FOR INFORMATION TRANSMISSION PROBLEMS  
 MOSCOW  
 RUSSIA

and

DEPARTMENT OF MATHEMATICS  
 FACULTY OF NATURAL SCIENCES  
 UNIVERSITY OF HAIFA  
 MOUNT CARMEL

ISRAEL

[posic@mccme.ru](mailto:posic@mccme.ru)

JAN ŠTŮVÍČEK  
 DEPARTMENT OF ALGEBRA  
 FACULTY OF MATHEMATICS AND PHYSICS  
 CHARLES UNIVERSITY IN PRAGUE  
 PRAHA  
 CZECH REPUBLIC  
[stovicek@karlin.mff.cuni.cz](mailto:stovicek@karlin.mff.cuni.cz)

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Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[blasius@math.ucla.edu](mailto:blasius@math.ucla.edu)

Paul Balmer  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[balmer@math.ucla.edu](mailto:balmer@math.ucla.edu)

Vyjayanthi Chari  
Department of Mathematics  
University of California  
Riverside, CA 92521-0135  
[chari@math.ucr.edu](mailto:chari@math.ucr.edu)

Matthias Aschenbrenner  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[matthias@math.ucla.edu](mailto:matthias@math.ucla.edu)

Daryl Cooper  
Department of Mathematics  
University of California  
Santa Barbara, CA 93106-3080  
[cooper@math.ucsb.edu](mailto:cooper@math.ucsb.edu)

Wee Teck Gan  
Mathematics Department  
National University of Singapore  
Singapore 119076  
[matgwt@nus.edu.sg](mailto:matgwt@nus.edu.sg)

Kefeng Liu  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[liu@math.ucla.edu](mailto:liu@math.ucla.edu)

Jiang-Hua Lu  
Department of Mathematics  
The University of Hong Kong  
Pokfulam Rd., Hong Kong  
[jhlu@maths.hku.hk](mailto:jhlu@maths.hku.hk)

Sorin Popa  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[popa@math.ucla.edu](mailto:popa@math.ucla.edu)

Jie Qing  
Department of Mathematics  
University of California  
Santa Cruz, CA 95064  
[qing@cats.ucsc.edu](mailto:qing@cats.ucsc.edu)

Paul Yang  
Department of Mathematics  
Princeton University  
Princeton NJ 08544-1000  
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