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The Fang–Lu formula is an identity relating the Weil–Petersson metric, the generalized Hodge metrics and the BCOV torsion on the moduli space of polarized Calabi–Yau manifolds. In this note, we extend this formula to the compactification of the moduli space of polarized Calabi–Yau manifolds assuming the logarithm of BCOV torsion is locally L^1-integrable. On the other hand, we use this extended formula to study global numerical properties for polarized families of Calabi–Yau manifolds.

1. Introduction

Reidemeister torsion (R-torsion) is an invariant that can distinguish between closed manifolds which are homotopy equivalent but not homeomorphic. Analytic torsion (or Ray–Singer torsion) is an invariant of Riemannian manifolds defined by Ray and Singer [1971; 1973] as an analytic analogue of Reidemeister torsion. These two torsions naturally coincide, which is known as the Cheeger–Müller theorem [Müller 1978; Cheeger 1979].

Bershadsky, Cecotti, Ooguri and Vafa [Bershadsky et al. 1993; 1994] conjectured an equivalence between the physical quantity F_1 of a Calabi–Yau threefold and a linear combination of the holomorphic analytic torsion, which is now called BCOV torsion. Motivated by their conjecture, Fang, Lu and Yoshikawa [Fang et al. 2008] considered a modification of BCOV torsion, called the BCOV invariant, and conducted a detailed study of the asymptotic behavior of that invariant for Calabi–Yau threefold. See also [Yoshikawa 2015; 2017].

By using the curvature formula for Quillen metrics, Bershadsky, Cecotti, Ooguri and Vafa [Bismut et al. 1988a; 1988b; 1988c] obtained a variational formula for the BCOV torsion of Ricci-flat Calabi–Yau manifolds. (In our terminology, a compact connected Kähler manifold X is called a *Calabi–Yau* manifold if $H^q(X, \mathcal{O}_X) = 0$ for $0 < q < \dim X$ and $K_X \cong \mathcal{O}_X$.) Fang and Lu [2005] expressed the variation of the BCOV torsion T of Ricci-flat Calabi–Yau manifolds as a linear combination

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of the Weil–Petersson metric ω_{WP} and the generalized Hodge metrics ω_{H^i} (see Section 2 for precise definitions):

$$\sum_{i=1}^{n}(-1)^i\omega_{\mathrm{H}^i} - \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log T = \frac{\chi}{12}\omega_{\mathrm{WP}}, \tag{1-1}$$

where χ is the Euler characteristic number. As has been pointed out in [Fang and Lu 2005], it is an interesting problem to extend this formula to the compactified moduli space. The main aim of this note is to establish this extension. In fact, let $\mathcal{M}$ be a connected component of the moduli space of polarized Calabi–Yau manifolds. Since $\mathcal{M}$ is quasiprojective, the smooth part $\mathcal{M}_{\mathrm{reg}}$ of $\mathcal{M}$ admits a compactification $\overline{\mathcal{M}}$, where $\overline{\mathcal{M}}$ is a smooth projective manifold such that $D = \overline{\mathcal{M}} - \mathcal{M}_{\mathrm{reg}}$ is a simple normal crossing divisor. Suppose that $D = \overline{\mathcal{M}} - \mathcal{M}_{\mathrm{reg}} = \sum_v D_v$, where the sum is locally finite and each D_v is a irreducible hypersurface of $\overline{\mathcal{M}}$. We will show that the following holds:

Theorem 1.1. *Assume* $\log T$ *is locally* L^1*-integrable on* $\overline{\mathcal{M}}$*. For each component* D_v *of the boundary divisor* D*, we set*

$$a_v := \lim_{p\to D_v}\frac{\log T}{\log|f_v|^2}, \tag{1-2}$$

where $p \in \mathcal{M}_{\mathrm{reg}}$ *and* f_v *is a local defining function of the hypersurface* D_v*. Then the following equation of currents on* $\overline{\mathcal{M}}$ *holds*:

$$dd^c\log T + \sum_{i=1}^{n}(-1)^{i-1}T_{\omega_{\mathrm{H}^i}} + \frac{\chi}{12}T_{\omega_{\mathrm{WP}}} = \sum_v a_v[D_v], \tag{1-3}$$

where the currents $T_{\omega_{\mathrm{WP}}}$ *and* $T_{\omega_{\mathrm{H}^i}}$ *are the trivial extensions of* ω_{WP} *and* ω_{H^i} *from* $\mathcal{M}_{\mathrm{reg}}$ *to* $\overline{\mathcal{M}}$*, and for each* v*,* $[D_v] = \int_{D_v}$ *is the current associated to the hypersurface* D_v*.*

For polarized families of Calabi–Yau manifolds we have a similar result. In fact, let $\mathcal{X}$ be a smooth projective variety of dimension $n+m$, let S be a smooth projective variety of dimension m and let $f : \mathcal{X} \to S$ be a surjective, flat holomorphic map with generic fiber Calabi–Yau n-fold. Let $f^0 : \mathcal{X}^0 \to S^0$ be the smooth part of f and assume that the discriminant locus $E := S \setminus S^0$ of f is a simple normal crossing divisor.

Theorem 1.2. *Assume* $\log T$ *is locally* L^1*-integrable on* S*. Let* $E = \sum_v E_v$ *be the irreducible decomposition of* E *and set*

$$a_v := \lim_{p\to E_v}\frac{\log T}{\log|f_v|^2}, \tag{1-4}$$

where $p \in S^0$ *and* f_v *is a local defining function of the hypersurface* E_v*. Then the*

following equation of currents on S holds:

$$dd^c \log T + \sum_{i=1}^{n} (-1)^{i-1} T_{\omega_{\mathrm{H}^i}} + \frac{\chi}{12} T_{\omega_{\mathrm{WP}}} = \sum_v a_v [E_v], \tag{1-5}$$

where the currents $T_{\omega_{\mathrm{WP}}}$ and $T_{\omega_{\mathrm{H}^i}}$ are the trivial extensions of ω_{WP} and ω_{H^i} from S^0 to S, $[E_v] = \int_{E_v}$ is the current associated to the hypersurface E_v and χ is the topological Euler number of a general fiber of f.

At first glance, Theorem 1.2 is a direct consequence of Theorem 1.1. However, this is not the case. First of all, unlike the space $\mathcal{M}$, the compactification $\overline{\mathcal{M}}$ itself is not a moduli space, so we do not have an induced map $S \to \overline{\mathcal{M}}$. Secondly, the pull back of currents may not be well-defined in general [Demailly 2012, pp. 18]. This is why we state them as two separate theorems. Nevertheless, Theorem 1.2 can be proved in the same way as Theorem 1.1. Hence we will only give the proof of Theorem 1.1 and leave the proof of Theorem 1.2 to the readers.

The current equation (1-5) in the case where $n = 3$, $m = 1$ was proved in [Fang et al. 2008, Theorem 10.1]. The proof of these two theorems is a modification of the arguments in [Fang et al. 2008, Section 7]. There are two crucial points here. The first is the assumption that $\log T$ is locally L^1-integrable.[1] This assumption is natural in the sense that it has been shown to be valid when $n = 3$ [Fang et al. 2008, Theorem 9.1]. Under this assumption, we will show that $\log T$ has at most logarithmic growth (see (3-18)) when approaching the boundary divisor D (in Theorem 1.1) and the discriminant locus E (in Theorem 1.2) so that the asymptotic values a_v in these two theorems make sense. The second is that each term in the Fang–Lu formula (1-1) is bounded by the Poincaré metric near the boundary of the moduli space [Fang and Lu 2005, Theorem A.1].

Next, we use an observation in [Liu and Xia 2019] to get the following:

Corollary 1.3. *We assume the same conditions as in Theorem 1.2. Suppose that the local monodromies of the polarized family $f^0 : \mathcal{X}^0 \to S^0$ are all unipotent. Then*

$$\begin{aligned} \sum_{i=1}^{n} (-1)^{i-1} \sum_{0 \le p \le i} p(\deg P\mathcal{H}_e^{p,i-p} + \deg P\mathcal{H}_e^{p-1,i-p-1} + \cdots) + \frac{\chi}{12} \deg P\mathcal{H}_e^{n,0} \\ = \sum_v a_v \deg E_v, \end{aligned} \tag{1-6}$$

where $P\mathcal{H}_e^{p,q} \to S$ denotes the canonical Deligne extensions of the Hodge bundles $P\mathcal{H}^{p,q} = PR^q f_*^0 \Omega^p_{\mathcal{X}^0/S^0} \to S^0$. In particular, for polarized families of Calabi–Yau

[1] The recent work of Eriksson, Freixas and Mourougane has provided further evidence for this assumption. See [Eriksson et al. 2018a, Theorem B].

3-*folds*, *we have*

$$(\chi+36)\deg P\mathcal{H}_e^{3,0}+12\deg P\mathcal{H}_e^{2,1}=12\sum_v a_v\deg E_v, \tag{1-7}$$

and for polarized families of Calabi–Yau 4-*folds*, *we have*

$$(\chi-48)\deg P\mathcal{H}_e^{4,0}-24\deg P\mathcal{H}_e^{3,1}+12\deg P\mathcal{H}_e^{2,1}=12\sum_v a_v\deg E_v. \tag{1-8}$$

The identity (1-7) for $m=1$ was proved in [Liu and Xia 2019, Theorem 4.4]. These identities are closely related to the Grothendieck–Riemann–Roch theorem and it is our belief that the asymptotic value a_v can be read out directly from the data of the corresponding singular fiber, see [Liu and Xia 2019; Eriksson et al. 2018b; Green et al. 2009]. Moreover, combining (1-8) with the Arakelov inequality in [Liu and Xia 2019, Theorem 5.1], we get:

Corollary 1.4. *We assume the same conditions as in Theorem 1.2 and we let* $n=4, m=1$. *If* $f:\mathcal{X}\to S$ *is not isotrivial and the local monodromies of the polarized family* $f^0:\mathcal{X}^0\to S^0$ *are all unipotent*, *then*

$$\begin{aligned}1+\frac{\sum_v a_v-\deg P\mathcal{H}_e^{2,1}}{2\deg P\mathcal{H}_e^{4,0}}&\le\frac{\chi}{24}\\&\le\frac{2\pi(2g-2+s)(h^{3,1}+4)+\sum_v a_v-\deg P\mathcal{H}_e^{2,1}}{2\deg P\mathcal{H}_e^{4,0}}.\end{aligned} \tag{1-9}$$

This article is structured as follows. In Section 2 we recall basic definitions and fix our notations. In Section 3 we give the proofs of our main results. In Section 4 we apply our main results to polarized families of Calabi–Yau manifolds.

2. Preliminaries

2A. ***BCOV torsion.*** Let (M,g) be a compact Kähler manifold of dimension n with Kähler form ω. Let $\square_{p,q}=(\bar\partial+\bar\partial^*)^2$ be the $\bar\partial$-Laplacian acting on C^∞ (p,q)-forms on M or equivalently $(0,q)$-forms on M with values in Ω_M^p, where Ω_M^1 is the holomorphic cotangent bundle of M and $\Omega_M^p:=\Lambda^p\Omega_M^1$. Let $\{\lambda_j\}$ be the eigenvalues of $\square_{p,q}$; then it is well known that $0=\lambda_0\le\lambda_1\le\lambda_2\le\cdots\le\lambda_j\longrightarrow+\infty$. The *spectral zeta function* of $\square_{p,q}$ is defined as

$$\zeta_{p,q}(s):=\sum_{\lambda_j>0}\lambda_j^{-s}, \tag{2-1}$$

where the multiplicities of the eigenvalues are taken into account.[2] Then $\zeta_{p,q}(s)$ converges on the half-plane $\{s \in \mathbb{C};\ \Re s > \dim M\}$, extends to a meromorphic function on $\mathbb{C}$, and is holomorphic at $s = 0$. From [Ray and Singer 1973], the (holomorphic) *analytic torsion* of (M, Ω^p_M) is the real number defined as

$$\tau(M, \Omega^p_M) := \exp\Big\{-\sum_{q\geq 0}(-1)^q q\, \zeta'_{p,q}(0)\Big\}. \tag{2-2}$$

Note that $\tau(M, \Omega^p_M)$ depends not only on the complex structure of M but also on the metric g.

Bershadsky, Cecotti, Ooguri and Vafa [Bershadsky et al. 1994] introduced the following combination of analytic torsion:

Definition 2.1. The *BCOV torsion* of (M, g) is the real number defined as

$$T_{\mathrm{BCOV}}(M, g) := \prod_{p\geq 0} \tau(M, \Omega^p_M)^{p(-1)^p} = \exp\Big\{-\sum_{p,q\geq 0}(-1)^{p+q} pq\, \zeta'_{p,q}(0)\Big\}. \tag{2-3}$$

Recall that a compact connected Kähler manifold X is said to be *Calabi–Yau* if $H^q(X, \mathcal{O}_X) = 0$ for $0 < q < \dim X$ and $K_X \cong \mathcal{O}_X$, where K_X is the canonical line bundle of X. In general, $T_{\mathrm{BCOV}}(M, g)$ is only a spectrum invariant. But when M is a Calabi–Yau threefold, it is possible to construct a holomorphic invariant of M from $T_{\mathrm{BCOV}}(M, g)$ by multiplying by a correction factor, see [Fang et al. 2008]. In this note we will only consider BCOV torsion with respect to the unique Ricci-flat metric [Yau 1978] on a polarized Calabi–Yau manifold.

2B. ***The moduli space of the polarized Calabi–Yau manifold.*** Let $(X, [\omega])$ be a polarized Calabi–Yau manifold of dimension $n \geq 3$, that is, X is a Calabi–Yau manifold and $[\omega] = c_1(L) \in H^2(X, \mathbb{Z})$ is the first Chern class of an ample line bundle L on X. Let $\mathcal{M}$ be the (coarse) moduli space of the polarized Calabi–Yau manifold $(X, [\omega])$. Locally, $\mathcal{M}$ is identified as a finite discrete quotient of the local versal deformation space Def (Kuranishi space) of X. By the Bogomolov–Tian–Todorov theorem [Tian 1987; Todorov 1989], the base space Def of the Kuranishi family

$$\pi : (\mathfrak{X}, X) \to (\mathrm{Def}, 0) \tag{2-4}$$

is smooth (a priori it is only a complex analytic space). Indeed, Def is an open subset of the linear space $H^1(X, \Theta_X)$, where Θ_X is the holomorphic tangent bundle of X, so we may assume Def is contractible. Since $H^0(X, \Theta_X) \cong H^0(X, \Omega^{n-1}_X) = 0$, the Kuranishi family π is universal. Define

$$H^1(X, \Theta_X)_\omega := \{\theta \in H^1(X, \Theta_X) \mid \theta \lrcorner \omega = 0 \in H^2(X, \mathcal{O}_X)\}, \tag{2-5}$$

[2] By this we mean that if an eigenvalue λ_j has multiplicity k, then it will appear k times in the series $\sum_{\lambda_j>0} \lambda_j^{-s}$.

and $\mathrm{Def}_\omega := \mathrm{Def} \cap H^1(X, \Theta_X)_\omega$. Then each point $\theta \in \mathrm{Def}_\omega$ stands for a fiber of the Kuranishi family $\pi : (\mathfrak{X}, X) \to (\mathrm{Def}, 0)$ with a polarization

$$[\omega_\theta] = [\omega] \in H^2(X_\theta, \mathbb{Z}) = H^2(X, \mathbb{Z})$$

[Tian 1987]. In fact, since $H^2(X, \mathcal{O}_X) = 0$, we have $\mathrm{Def}_\omega = \mathrm{Def}$. We thus have the associated period mapping $\mathrm{Def} \to D$, where D is the period domain, i.e., the classifying space of polarized (weight k) Hodge structure of $(X, [\omega])$. $\mathrm{Def} \to D$ is a holomorphic mapping, and is an immersion (local Torelli) for $k = n$, see [Griffiths 1968, Corollary 3.6]. We remark that $\mathcal{M}$ is a complex orbifold and is locally covered by Def, so when we work with metrics, and curvatures of $\mathcal{M}$, we can treat these notions on Def instead.

We recall some natural metrics on $\mathcal{M}$, see [Fang and Lu 2005] for more detail.

2C. ***Weil–Petersson metric.*** Let $t \in \mathrm{Def}$. The Kodaira–Spencer map is now an isomorphism

$$\rho : T_t \,\mathrm{Def} \xrightarrow{\cong} H^1(X_t, \Theta_t), \tag{2-6}$$

where Θ_t is the holomorphic tangent bundle of X_t.

Let $(t_1, \dots, t_m)$ be a local holomorphic coordinate system of $\mathcal{M}$; we define a Hermitian inner product on $T_t\mathcal{M}$ by

$$\left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial \bar{t}_j}\right)_{\mathrm{WP}} = \int_{X_t} A^\alpha_{i\bar\beta} \cdot \overline{A^\gamma_{j\bar\delta}} g^{\delta\bar\beta} g_{\alpha\bar\gamma}\, dV_{X_t}, \tag{2-7}$$

where $A_i = A^\alpha_{i\bar\beta}(\partial/\partial t_\alpha) \otimes d\bar{t}^\beta$ $(i = 1, \dots, m)$ are the harmonic representations of $\rho(\partial/\partial t_i)$. This inner product on each T_tU for $t \in \mathcal{M}$ gives a Hermitian metric on the moduli space $\mathcal{M}$, which is called the Weil–Petersson metric. Equipped with the Weil–Petersson metric, $\mathcal{M}$ is a Kähler orbifold.

Let Ω be a (nonzero) holomorphic $(n, 0)$-form on X_t. Define $\Omega \lrcorner \rho(\partial/\partial t_i)$ to be the contraction of Ω and $\rho(\partial/\partial t_i)$. The Weil–Petersson metric can be rewritten (see [Tian 1987] as

$$\left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial \bar{t}_j}\right)_{\mathrm{WP}} = -\frac{\int_{X_t} \Omega \lrcorner \rho\left(\frac{\partial}{\partial t_i}\right) \wedge \overline{\Omega \lrcorner \rho\left(\frac{\partial}{\partial t_j}\right)}}{\int_{X_t} \Omega \wedge \overline{\Omega}}. \tag{2-8}$$

2D. ***Generalized Hodge metrics and Hodge bundles.*** Recall that, for all $0 \le k \le n$ and $p+q = k$ there are natural holomorphic vector bundles $P\mathcal{H}^{p,q} := PR^q\pi_*\Omega^p_{\mathfrak{X}/\mathrm{Def}}$, called Hodge bundles, on Def (hence on $\mathcal{M}$), whose fiber is

$$(PR^q\pi_*\Omega^p_{\mathfrak{X}/\mathrm{Def}})_t = PH^q(X_t, \Omega^p_{X_t}), \tag{2-9}$$

where $PH^q(X_t, \Omega^p_{X_t})$ is the primitive cohomology of $(X_t, [\omega_t])$. By abuse of notation, we will always use the same symbol $P\mathcal{H}^{p,q}$ to denote the Hodge bundle on Def and on $\mathcal{M}$.

By differentiating harmonic representatives, we have a holomorphic bundle map

$$\frac{\partial}{\partial t_i} : P\mathcal{F}^p \to P\mathcal{H}/P\mathcal{F}^p, \tag{2-10}$$

where $P\mathcal{F}^p = P\mathcal{H}^{p,k-p} \oplus P\mathcal{H}^{p+1,k-p-1} \oplus \cdots \oplus P\mathcal{H}^{k,0}$ and $P\mathcal{H} := PR^k\pi_*(\mathbb{C})$. In this way, we get a natural holomorphic bundle map

$$T(\mathrm{Def}) \to \bigoplus_{1\le p\le k} \mathrm{Hom}\,(P\mathcal{F}^p, P\mathcal{H}/P\mathcal{F}^p). \tag{2-11}$$

We remark that this bundle map is just the differential of the period mapping $\mathrm{Def} \to D$. There are natural metrics on the Hodge bundles $P\mathcal{F}^p$, and hence on each $\mathrm{Hom}\,(P\mathcal{F}^p, P\mathcal{H}/P\mathcal{F}^p)$, induced by the Riemann–Hodge bilinear relations. Let h_{PH^k} be the pull back of the metric on

$$\bigoplus_{1\le p\le k} \mathrm{Hom}\,(P\mathcal{F}^p, P\mathcal{H}/P\mathcal{F}^p).$$

Then, by (2-11), we have that h_{PH^k} is semipositive. We use ω_{PH^k} to denote the associated (1, 1)-form of the Hermitian symmetric bilinear form h_{PH^k} for all $k \le n$. We let

$$\omega_{\mathrm{H}^k} := \omega_{PH^k} + \omega_{PH^{k-2}} + \cdots . \tag{2-12}$$

We call both ω_{H^k} and ω_{PH^k} generalized Hodge metrics. We note that when $k = n$, ω_{PH^n} is a positive (1, 1)-form by local Torelli. It's just the pull back of the usual Hodge metric on D and we call $\omega_{\mathrm{H}} := \omega_{PH^n}$ the Hodge metric.

2E. ***Weil–Petersson form and generalized Hodge forms.*** Let $f : \mathcal{X} \to S$ be a smooth polarized family of Calabi–Yau manifolds with $(X_0 := f^{-1}(0), [\omega_0]) \cong (X, [\omega])$, where $0 \in S$ and $[\omega_0]$ is the polarization of X_0 induced from that of $\mathcal{X}$. Since $\mathcal{M}$ is the moduli space of the polarized Calabi–Yau manifold $(X, [\omega])$, we have a natural commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{\phi} & D/\Gamma \\ \downarrow{\scriptstyle\psi} & & \downarrow \\ \mathcal{M} & \xrightarrow{\varphi} & D/G_{\mathbb{Z}} \end{array}$$

where Γ is the monodromy group of the family f, and $G_{\mathbb{Z}} = \mathrm{Aut}(H_{\mathbb{Z}}, Q)$, see [Griffiths 1984] for more information. To keep our notation simple, we will just use ω_{WP} and ω_{H^i} to denote the pull back forms $\psi^*\omega_{\mathrm{WP}}$ and $\psi^*\omega_{\mathrm{H}^i}$, and call them

the Weil–Petersson form and generalized Hodge forms on S, respectively. Similarly, we will use $P\mathcal{H}^{p,q}$ to denote the pull back bundle $\psi^* P\mathcal{H}^{p,q}$, which is isomorphic to $PR^q f_*\Omega^p_{\mathcal{X}/S}$, and call them Hodge bundles on S.

3. Proof of Theorem 1.1 and Theorem 1.2

After some preparations in Section 3A and 3B, we will prove Theorem 1.1 and Theorem 1.2 in Section 3C and 3D respectively.

3A. *Extension of dd^c-closed functions.* In this subsection we show that dd^c-closed functions can be locally extended through a hypersurface when some suitable growth conditions are satisfied.[3] The proof is a slight modification to that of the one variable case [Fang et al. 2008, Section 7.2]. We use Δ to denote the unit open disc in the complex plane, Δ^* to denote the punctured unit disc and $\mathcal{O}$ to denote the sheaf of holomorphic functions. Recall that $d^c := (\sqrt{-1}/4\pi)(\bar\partial - \partial)$ and $dd^c = (\sqrt{-1}/2\pi)\partial\bar\partial$. It is easy to see that the real or imaginary part of a holomorphic function is dd^c-closed and in the one dimensional case dd^c-closed functions are just harmonic functions.

Lemma 3.1. *Let $H(z)$ be a real-valued dd^c-closed function on $U \times \Delta^*$, where U is an open subset in $\mathbb{C}^k$. Then we have that*:

(1) *There exist $c \in \mathbb{R}$, $F(z) \in \mathcal{O}(U \times \Delta^*)$ and a real-valued dd^c-closed function g on U with*

$$H(z) = c \log|t|^2 + 2\,\mathrm{Re}\, F(z) + g(u),$$

where $z = (u, t) = (u_1, \dots, u_k, t) \in U \times \Delta^*$.

(2) *If for any fixed $u \in U$, there exist $\gamma = \gamma(u) \in \mathbb{R}$ such that $|t|^\gamma e^{H(z)}$ is a locally L^1-integrable function on Δ, then for the function F in* (1), *we have $F(z) \in \mathcal{O}(U \times \Delta)$.*

(3) *If for any fixed $u \in U$, $H(u, t) = O(\log(-\log|t|))$ as $t \to 0$, then for the functions F, g in* (1), *we have $H(z) = 2\,\mathrm{Re}\, F(z) + g(u)$ and $H(z)$ extends to a dd^c-closed function on $U \times \Delta$.*

Proof. (1) Since $H(z)$ is dd^c-closed, we know that for all i, j, the derivatives $\partial H/\partial u_i$ and $f(z) := \partial H/\partial t$ are holomorphic functions on $U \times \Delta^*$. Let $f(z) = f(u, t) = \sum_{n\in\mathbb{Z}} a_n(u)\, t^n$ be the Laurent expansion of f with respect to t and define $F(z) \in \mathcal{O}(U \times \Delta^*)$ by $F(z) := \sum_{n\neq -1} a_n(u)\, t^{n+1}/(n+1)$. Noting that $f = a_{-1}(u)/t + \partial F/\partial t$ and using the fact that $H(z)$ is real-valued, we have

$$d_t H(z) = \frac{\partial H}{\partial t} dt + \frac{\partial H}{\partial \bar t} d\bar t = f\,dt + \bar f\, d\bar t = a_{-1}\frac{dt}{t} + \bar a_{-1}\frac{d\bar t}{\bar t} + d_t F + d_t \bar F, \tag{3-1}$$

[3] For more background material of Lemma 3.1, we refer the readers to [Axler 1986; Kodaira 2007].

where d_t means differential with respect to the t variable. Integrating both sides over the circle $|t| = 1/2$, for all $u \in U$ we get $a_{-1}(u) \in \mathbb{R}$ by the Stokes' theorem, and we conclude that $a_{-1}(u)$ is a real constant because we know that it is holomorphic. So $d_t H(z) = a_{-1} d \log|t|^2 + 2d_t\{\mathrm{Re}\, F(z)\}$ and $H(z) = a_{-1} \log|t|^2 + 2\mathrm{Re}\, F(z) + g(u)$ for some real-valued dd^c-closed function g on U. This proves (1).

(2) First we fix $u \in U$. Then $e^{H(z)} = |t|^{2c}|e^{F(z)}|^2 e^{g(u)}$. By assumption, there exists $\gamma \in \mathbb{R}$ such that

$$e^{g(u)} \int_{|t|<1/2} |t|^{\gamma+2c}\, |e^{F(z)}|^2 \sqrt{-1}\, dt \wedge d\bar{t} < +\infty, \tag{3-2}$$

where F, g and c are as given in (1). Since $e^{F(z)}$ is holomorphic in $t \in \Delta^*$, we deduce from (3-2) that $e^{F(z)}$ is a meromorphic function in $t \in \Delta$. Set $F_u(t) := F(z)$. There exist $N \in \mathbb{Z}$ and a nowhere vanishing holomorphic function $\epsilon_u(t) \in \mathcal{O}(\Delta)$ with $e^{F(z)} = t^N \epsilon_u(t)$, where $\epsilon_u(t)$ is holomorphic in u. Then $F_u'(t) = Nt^{-1} + \epsilon_u'(t)\epsilon_u(t)^{-1}$. Since $e^{F(z)}$, and thus $F_u(t)$ is a meromorphic function in $t \in \Delta$, the residue of $F_u'(t)$ must vanish, i.e., $N = 0$. Therefore $F_u'(t) = \epsilon_u'(t)\epsilon_u(t)^{-1}$. Since $\epsilon_u(t) \in \mathcal{O}(\Delta)$ and is nowhere vanishing, from the Laurent series expansions of $F_u(t)$ and $F_u'(t)$ on Δ^* we deduce that $t = 0$ is a removable singularity of the function $F_u(t)$. Hence, for any $t \in \Delta$ we may set

$$F_u(t) := \int_0^t \frac{\epsilon_u'(s)}{\epsilon_u(s)}\, ds + F_u(0), \quad \text{where } F_u(0) := \lim_{t\to 0} F_u(t).$$

It follows that $F(z) \in \mathcal{O}(U \times \Delta)$.

(3) From the assumption, we easily see that for any fixed $u \in U$, there exists $\gamma \in \mathbb{R}$ such that $|t|^\gamma e^{H(u,t)} \in L^1_{\mathrm{loc}}(\Delta)$. Indeed, it suffices to choose any positive $\gamma \in \mathbb{R}$. By (1) and (2), we have $F(z) \in \mathcal{O}(U \times \Delta)$ and so $H(z) - c \log|t|^2 = 2\,\mathrm{Re}\, F(z) + g(u)$ is a dd^c-closed function on $U \times \Delta$, where F, g and c are as given in (1). Since $H(u,t) = O(\log(-\log|t|))$ as $t \to 0$ and $\lim_{t\to 0}(\log|t|^2/\log(-\log|t|)) = \infty$, we get $c = 0$. This completes the proof. □

Lemma 3.2. *Let $T(z) = T(u,t)$ be a positive function on $U \times \Delta^*$ such that for any fixed $u \in U$, $T(u,t)$ is locally L^1-integrable on Δ. Let $P(z)$ be a real-valued function on $U \times \Delta^*$ such that for any fixed $u \in U$, $P(z) = P(u,t) \le C(-\log|t|)$, where $C = C(u) \in \mathbb{R}$ is a constant, and if $\log T(z) + P(z)$ is dd^c-closed on $U \times \Delta^*$, then there exists $a \in \mathbb{R}$ such that for any fixed $u \in U$,*

$$\log T(u,t) = a \log|t|^2 + O(|P(u,t)|), \qquad t \to 0. \tag{3-3}$$

Proof. Set $H(z) = \log T(z) + P(z)$. By Lemma 3.1 (1), there exist $a \in \mathbb{R}$ and $F(z) \in \mathcal{O}(U \times \Delta^*)$ and a real-valued dd^c-closed function g on U with $H(z) = a \log|t|^2 + 2\,\mathrm{Re}\, F(z) + g(u)$. Since for any fixed $u \in U$, $P(u,t) \le C(-\log|t|)$, we

have $\log T(z) = H(z) - P(z) \geq H(z) - C(-\log|t|)$. By our assumption on $T(z)$, we have that for any fixed $u \in U$,

$$\int_{|t|<1/2} |t|^C e^{H(u,t)} \sqrt{-1}\, dt \wedge d\bar{t} \leq \int_{|t|<1/2} T(u,t) \sqrt{-1}\, dt \wedge d\bar{t} < +\infty. \tag{3-4}$$

By Lemma 3.1 (2), we deduce that $F(z) \in \mathcal{O}(U \times \Delta)$, and hence $\log T(u,t) = H(z) - P(z) = a \log|t|^2 + O(|P(u,t)|),\ t \to 0$. □

Lemma 3.3. *Let $T(z) = T(u,t)$ be a positive function on $U \times \Delta^*$ such that for any fixed $u \in U$, $T(u,t)$ is locally L^1-integrable on Δ. Let $P(z)$ be a real-valued function on $U \times \Delta^*$ such that for any fixed $u \in U$, $P(u,t) = O(\log(-\log|t|))$ as $t \to 0$, and if $\log T(z) + P(z) + f(u)$ is dd^c-closed on $U \times \Delta^*$ where $f(u)$ is a smooth function independent of t, then*:

(1) *There exists $a \in \mathbb{R}$ such that for any fixed $u \in U$,*

$$\log T(u,t) = a \log|t|^2 + O(\log(-\log|t|)), \qquad t \to 0. \tag{3-5}$$

(2) $\log T(z) + P(z) + f(u) - a \log|t|^2$ *extends to a dd^c-closed function on $U \times \Delta$.*

Proof. First we fix $u \in U$. By assumption, $P(z) + f(u) = O(\log(-\log|t|))$ as $t \to 0$, and so $P(z) + f(u) \leq C\,(-\log|t|)$ for some constant C. Applying Lemma 3.2, we get

$$\log T(u,t) = a \log|t|^2 + O(|P(u,t) + f(u)|) = a \log|t|^2 + O(\log(-\log|t|)), \tag{3-6}$$

as $t \to 0$. This proves (1). For (2), we set $H(z) = \log T(z) + P(z) + f(u) - a \log|t|^2$. Then $H(z) = O(\log(-\log|t|))$ as $t \to 0$. It follows from Lemma 3.1 (3) that $H(z)$ extends to a dd^c-closed function on $U \times \Delta$. □

3B. *The Poincaré metric and trivial current extensions of* (1, 1)*-forms.* The Poincaré metric on $\Delta^k \times \Delta^{*l}$ is defined by

$$\omega_{\mathrm{P}} = \sqrt{-1} \sum_{j=1}^{k} dz_j \wedge d\bar{z}_j + \sqrt{-1} \sum_{j=k+1}^{k+l} \frac{dz_j \wedge d\bar{z}_j}{|z_j|^2 \cdot \log^2|z_j|^2}. \tag{3-7}$$

Let M be a complex manifold of dimension n and u a $(1,1)$-current on M. Then u is *positive* if $u(\eta \wedge \bar{\eta}) \geq 0$ for all $\eta \in A_c^{n-1,0}(M)$, where $A_c^{n-1,0}(M)$ is the vector space of smooth $(n-1,0)$-forms on M with compact support. A real-valued function $f \in L^1_{\mathrm{loc}}(M)$ is *plurisubharmonic* if f is upper semicontinuous and $dd^c f$ is positive as currents on M.

A C^∞ real $(1,1)$-form α on $\Delta^k \times \Delta^{*l}$ has *Poincaré growth* if there exists $C > 0$ with

$$-C\,\omega_{\mathrm{P}} \leq \alpha \leq C\,\omega_{\mathrm{P}}. \tag{3-8}$$

In this case, the coefficient of α lies in $L^1_{\mathrm{loc}}(\Delta^k \times \Delta^{*l})$. If α has Poincaré growth, the $(1,1)$-current T_α on Δ^{k+l} defined by

$$T_\alpha(\psi) := \int_{\Delta^{k+l}} \alpha \wedge \psi, \qquad \psi \in A_c^{k+l-1,k+l-1}(\Delta^{k+l}), \tag{3-9}$$

is called the *trivial extension* of α from $\Delta^k \times \Delta^{*l}$ to Δ^{k+l}. By definition, if α is semipositive, is d-closed and has Poincaré growth on $\Delta^k \times \Delta^{*l}$, then its trivial extension T_α is a closed, positive $(1,1)$-current on Δ^{k+l}. Note also that

$$dd^c\left\{2\pi\left(\sum_{j=1}^{k}|z_j|^2 - \sum_{j=1}^{l}\log(-\log|z_{k+j}|^2)\right)\right\} = T_{\omega_{\mathrm{P}}} \tag{3-10}$$

as currents on Δ^{k+l}.

We will follow the convention that $\widehat{z_{k+i}}$ means z_{k+i} has been omitted.

Proposition 3.4. *Let ω_i, $i = 1, \dots, N$, be closed, semipositive $(1,1)$-forms on $\Delta^k \times \Delta^{*l}$ with $\omega_i \le C\omega_{\mathrm{P}}$ for some positive constant C. Let T be a positive smooth function on $\Delta^k \times \Delta^{*l}$ such that $\log T$ is a locally L^1-integrable function on Δ^{k+l} and $dd^c \log T + \sum_{i=1}^N A_i\omega_i = 0$ on $\Delta^k \times \Delta^{*l}$. Then we have that*:

(1) *ω_i extend trivially to closed positive $(1,1)$-currents T_{ω_i} on Δ^{k+l}.*

(2) *There exist constants $a_i \in \mathbb{R}$ such that for any fixed $(z_1, \dots, \widehat{z_{k+i}}, \dots, z_{k+l}) \in \Delta^k \times \Delta^{*l-1}$, and for $i = 1, \dots, l$,*

$$\log T = a_i \log|z_{k+i}|^2 + O(\log(-\log|z_{k+i}|)), \qquad z_{k+i} \to 0. \tag{3-11}$$

(3) *The following equation of currents on Δ^{k+l} holds*:

$$dd^c \log T + \sum_{i=1}^{N} A_i T_{\omega_i} = \sum_{i=1}^{l} a_i dd^c \log|z_{k+i}|^2. \tag{3-12}$$

Proof. The proof of (1) is obvious since all the ω_i are semipositive, d-closed and have Poincaré growth on $\Delta^k \times \Delta^{*l}$. For (2), by the dd^c-Poincaré lemma [Griffiths and Harris 1978, pp. 387], there exists a plurisubharmonic function $\varphi_i \in \mathrm{Psh}(\Delta^{k+l})$ with $dd^c\varphi_i = T_{\omega_i}$ for each $i = 1, \dots, N$. From $\omega_i \le C\omega_{\mathrm{P}}$ we infer that

$$dd^c\left\{2\pi C\left(\sum_{j=1}^{k}|z_j|^2 - \sum_{j=1}^{l}\log(-\log|z_{k+j}|^2)\right) - \varphi_i\right\} = CT_{\omega_{\mathrm{P}}} - T_{\omega_i} \ge 0, \tag{3-13}$$

where $T_{\omega_{\mathrm{P}}}$ is the trivial current extension of the Poincaré metric ω_{P} from $\Delta^k \times \Delta^{*l}$ to Δ^{k+l}. Therefore $2\pi C\big(\sum_{j=1}^k|z_j|^2 - \sum_{j=1}^l \log(-\log|z_{k+j}|^2)\big) - \varphi_i$ is a plurisubharmonic function on Δ^{k+l}; combining this with the fact $\varphi_i \in \mathrm{Psh}(\Delta^{k+l})$, it follows

that there exist constants C_1, C_2, such that

$$2\pi C\left(\sum_{j=1}^{k}|z_j|^2-\sum_{j=1}^{l}\log(-\log|z_{k+j}|^2)\right)-C_1\le\varphi_i\le C_2 \quad \text{on } \Delta^{k+l}\left(\tfrac{1}{2}\right), \tag{3-14}$$

where $\Delta^{k+l}\left(\frac{1}{2}\right):=\{z=(z_1,\dots,z_{k+l})\in\Delta^{k+l}:|z_j|<\frac{1}{2},\ j=1,\dots,k+l\}$, and if we set $P=\sum_{i=1}^{N}A_i\varphi_i$, we easily see that for any fixed $(z_1,\dots,\widehat{z_{k+i}},\dots,z_{k+l})\in\Delta^{k+l-1}\left(\frac{1}{2}\right)$, $i\ge 1$,

$$-C_3\log(-\log|z_{k+i}|)\le P\le C_3\log(-\log|z_{k+i}|) \tag{3-15}$$

for small $|z_{k+i}|$ and some constant $C_3=C_3(z_1,\dots,\widehat{z_{k+i}},\dots,z_{k+l})$. Now since $\log T+P$ is dd^c-closed and $P_{z_1,\dots,\widehat{z_{k+i}},\dots,z_{k+l}}(z_{k+i})=O(\log(-\log|z_{k+i}|))$, $z_{k+i}\to 0$ for each $i=1,\dots,l$, we get that (2) follows from Lemma 3.3 (1). By applying Lemma 3.3 (2) with $f=0$, it follows that $\log T+P-a_1\log|z_{k+1}|^2$ extends to a dd^c-closed function on $\Delta^{k+1}\times\Delta^{*l-1}$. Next by applying Lemma 3.3 (2) with $f=a_1\log|z_{k+1}|^2$, it follows that $\log T+P-a_1\log|z_{k+1}|^2-a_2\log|z_{k+2}|^2$ extends to a dd^c-closed function on $\Delta^{k+2}\times\Delta^{*l-2}$. Continuing this way, (3) follows easily. □

3C. ***Current equation on the compactification of the moduli space of polarized Calabi–Yau manifolds.*** In this subsection we prove a current equation which extends the Fang–Lu formula to the compactification of the moduli space of polarized Calabi–Yau manifolds. Let $\mathcal{M}$ be a connected component of the moduli space of polarized Calabi–Yau manifolds. By [Viehweg 1995], $\mathcal{M}$ is quasiprojective. The smooth part $\mathcal{M}_{\mathrm{reg}}$ of $\mathcal{M}$ admits a compactification $\overline{\mathcal{M}}$, where $\overline{\mathcal{M}}$ is a smooth projective manifold such that $D=\overline{\mathcal{M}}-\mathcal{M}_{\mathrm{reg}}$ is a simple normal crossing divisor. On $\mathcal{M}_{\mathrm{reg}}$, there exists a Kähler metric ω that is locally equivalent to the Poincaré metric ω_p [Zucker 1979]. More precisely, we can choose a local coordinate chart $(W;z_1,\dots,z_m)$ of $\overline{\mathcal{M}}$ such that

(1) D is given by $z_{k+1}\cdots z_m=0$;

(2) $W\cong\Delta^m$;

(3) $\mathcal{M}_{\mathrm{reg}}\cap W\cong\Delta^k\times\Delta^{*m-k}$;

(4) $\omega\sim\omega_{\mathrm{P}}=\sqrt{-1}\sum_{j=1}^{k}dz_j\wedge d\bar{z}_j+\sqrt{-1}\sum_{j=k+1}^{m}\dfrac{dz_j\wedge d\bar{z}_j}{|z_j|^2\log^2|z_j|^2}$.

It is known from [Fang and Lu 2005, Theorem A.1] that

$$\omega_{\mathrm{H}^i}\le C\omega_{\mathrm{P}},\quad i=0,\dots,n \quad\text{and}\quad \omega_{\mathrm{WP}}\le C\omega_{\mathrm{P}}, \tag{3-16}$$

for some positive constant C depending on the lower bound of $\mathrm{Ric}(\omega)$, the dimension m of $\mathcal{M}$ and the dimension n of Calabi–Yau manifolds. Next, recall that on $\mathcal{M}$,

we have the Fang–Lu formula [2005, Theorem 1.1]:

$$\sum_{i=1}^{n}(-1)^i\omega_{\mathrm{H}^i} - \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log T = \frac{\chi}{12}\omega_{\mathrm{WP}}, \tag{3-17}$$

where ω_{H^i} are the generalized Hodge metrics defined in Section 2D, T is the BCOV torsion with respect to the unique Ricci-flat metric in the integral cohomology class that defines the polarization and χ is the Euler number. Note that T is a smooth positive function on $\mathcal{M}$.

Because of (3-16), we may let $T_{\omega_{\mathrm{H}^i}}$, $T_{\Omega_{\mathrm{WP}}}$ be the trivial current extensions of ω_{H^i}, ω_{WP} from $\mathcal{M}_{\mathrm{reg}}$ to $\overline{\mathcal{M}}$. $\overline{\mathcal{M}}$ can be covered by coordinate charts $\{(W \cong \Delta^m; z_1, \dots, z_m)\}$ such that either $W \cap D = \varnothing$ or $W \cap \mathcal{M}_{\mathrm{reg}} \cong \Delta^k \times \Delta^{*m-k}$ and D is given by $z_{k+1}\cdots z_m = 0$. In the second case, by (3-16), (3-17) and Proposition 3.4 (2), we know that if $\log T$ is locally L^1-integrable on $\overline{\mathcal{M}}$ then there exist constants $a_i \in \mathbb{R}$ such that for any fixed $(z_1, \dots, \widehat{z_i}, \dots, z_{k+l}) \in \Delta^k \times \Delta^{*l}$,

$$\log T = a_i \log|z_i|^2 + O(\log(-\log|z_i|)), \quad z_i \to 0, \text{ for all } i \geq k+1. \tag{3-18}$$

It's easy to see that

$$a_i = \lim_{z_i \to 0} \frac{\log T}{\log|z_i|^2}. \tag{3-19}$$

Suppose now that $D = \overline{\mathcal{M}} - \mathcal{M}_{\mathrm{reg}} = \sum_v D_v$, where the sum is locally finite and each D_v is an irreducible hypersurface of $\overline{\mathcal{M}}$. We may associate a real number a_v with each component D_v of the divisor D by setting

$$a_v := \lim_{p \to D_v} \frac{\log T}{\log|f_v|^2}, \tag{3-20}$$

where $p \in \mathcal{M}_{\mathrm{reg}}$ and f_v is a local defining function of the hypersurface D_v. This is well defined because of (3-18) and (3-19).

Theorem 3.5 (=Theorem 1.1). *Assume* $\log T$ *is locally* L^1*-integrable on* $\overline{\mathcal{M}}$*. Then the following equation of currents on* $\overline{\mathcal{M}}$ *holds*:

$$dd^c \log T + \sum_{i=1}^{n}(-1)^{i-1}T_{\omega_{\mathrm{H}^i}} + \frac{\chi}{12}T_{\omega_{\mathrm{WP}}} = \sum_v a_v[D_v], \tag{3-21}$$

where for each v, $[D_v] = \int_{D_v}$ *is the current associated to the hypersurface* D_v.

Proof. The statement is local. On a coordinate chart W with $W \cap D = \varnothing$, the equality (3-21) reduces to (3-17) because $\operatorname{supp}\big(\sum_v a_v[D_v]\big) \subseteq D$. So it's enough to prove it on coordinate charts W with $W \cap D \neq \varnothing$. First, it follows immediately

from (3-16), (3-17) and Proposition 3.4 (3) that

$$dd^c \log T + \sum_{i=1}^{n} (-1)^{i-1} T_{\omega_{\mathrm{H}^i}} + \frac{\chi}{12} T_{\omega_{\mathrm{WP}}} = \sum_{i=k+1}^{m} a_i dd^c \log |z_i|^2 \tag{3-22}$$

holds on coordinate chart $(W; z_1, \dots, z_m)$ of $\overline{\mathcal{M}}$ such that D is given by $W \cong \Delta^m$, $z_{k+1} \cdots z_m = 0$ and $\mathcal{M}_{\mathrm{reg}} \cap W \cong \Delta^k \times \Delta^{*m-k}$. Now, (3-21) follows from (3-20), (3-22) and the Poincaré–Lelong formula. □

Since $\overline{\mathcal{M}}$ is projective, we may set ω_{FS} to be the Kähler form of the Fubini–Study metric restricted to $\overline{\mathcal{M}}$.

Corollary 3.6. *Under the same conditions as in Theorem 3.5 we have*

$$\int_{\overline{\mathcal{M}}} \left(\sum_{i=1}^{n} (-1)^{i-1} T_{\omega_{\mathrm{H}^i}} + \frac{\chi}{12} T_{\omega_{\mathrm{WP}}} \right) \wedge \omega_{\mathrm{FS}}^{m-1} = \sum_{v} a_v \deg D_v, \tag{3-23}$$

where $\deg D_v := \int_{\overline{\mathcal{M}}} c_1(D_v) \wedge \omega_{\mathrm{FS}}^{m-1}$ *and* $c_1(D_v)$ *is the first Chern class of the line bundle associated to* D_v.

Proof. We note that $dd^c \log T \wedge \omega_{\mathrm{FS}}^{m-1}$ is d-exact on $\overline{\mathcal{M}}$. By integrating the identity (3-21) over $\overline{\mathcal{M}}$, (3-23) follows immediately. □

3D. *Current equation for polarized family of Calabi–Yau manifolds.* In this subsection, we prove a similar current equation for a polarized family of Calabi–Yau manifolds.

Let $\mathcal{X}$ be a smooth projective variety of dimension $n+m$, let S be a smooth projective variety of dimension m and let $f : \mathcal{X} \to S$ be a surjective, flat holomorphic map with generic fiber a Calabi–Yau n-fold. Let $f^0 : \mathcal{X}^0 \to S^0$ be the smooth part of f, that is, each fiber $f^{-1}(s)$ is smooth for $s \in S^0$ and singular for $s \notin S^0$, $\mathcal{X}^0 = f^{-1}(S^0)$ and f^0 is the restriction of f to $\mathcal{X}^0$. With the polarization induced from the embedding of $\mathcal{X}$ into projective space, $f^0 : \mathcal{X}^0 \to S^0$ becomes a smooth polarized family of Calabi–Yau manifolds. We further assume that the discriminant locus $E := S \setminus S^0$ of f is a simple normal crossing divisor.

We make the following two observations. First, by pulling back the Fang–Lu formula via the induced holomorphic map $\psi : S^0 \to \mathcal{M}$, we know that (3-17) still holds on S^0:

$$\sum_{i=1}^{n} (-1)^i \omega_{\mathrm{H}^i} - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log T = \frac{\chi}{12} \omega_{\mathrm{WP}}, \tag{3-24}$$

where, by our conventions (see Section 2E), we use the same symbols to denote the forms after pulling back. In particular, T is the BCOV torsion of the fibers of $f^0 : \mathcal{X}^0 \to S^0$. Second, from the proof of [Fang and Lu 2005, Theorem A.1], we

easily see that (3-16) still holds near E. So we can proceed as in Theorem 3.5 and get the following:

Theorem 3.7 (=Theorem 1.2). *Assume* $\log T$ *is locally* L^1*-integrable on* S. *Let* $E = \sum_v E_v$ *be the irreducible decomposition of* E *and set*

$$a_v := \lim_{p \to E_v} \frac{\log T}{\log |f_v|^2}, \tag{3-25}$$

where $p \in S^0$ *and* f_v *is a local defining function of the hypersurface* E_v. *Then the following equation of currents on* S *holds*:

$$dd^c \log T + \sum_{i=1}^{n} (-1)^{i-1} T_{\omega_{\mathrm{H}^i}} + \frac{\chi}{12} T_{\omega_{\mathrm{WP}}} = \sum_v a_v [E_v], \tag{3-26}$$

where the currents $T_{\omega_{\mathrm{WP}}}$ *and* $T_{\omega_{\mathrm{H}^i}}$ *are the trivial extensions of* ω_{WP} *and* ω_{H^i} *from* S^0 *to* S, $[E_v] = \int_{E_v}$ *is the current associated to the hypersurface* E_v *and* χ *is the topological Euler number of a general fiber of* f.

Proof. The proof is almost the same as Theorem 3.5 and is omitted. □

Remark 3.8. For $n = 3$, the BCOV torsion T here differs from the BCOV invariant τ in [Fang et al. 2008] by a positive constant depending only on the polarization [Yoshikawa 2017, pp. 284]; it follows that the asymptotic value a_v is the same for T and τ.

4. Applications to polarized families of Calabi–Yau manifolds

Now we want to integrate (3-26) over S to get global numerical results, but before that we make the following useful observations as in [Liu and Xia 2019].

Firstly, if we assume the local monodromies of the polarized family $f^0 : \mathcal{X}^0 \to S^0$ (with the polarization induced from the projective embedding of $\mathcal{X}$) are all unipotent, there is then the canonical Deligne extensions $P\mathcal{H}_e^{p,q}$ of the Hodge bundles $P\mathcal{H}^{p,q}$ from S^0 to S. Recall that the Hodge bundles are defined by $P\mathcal{H}^{p,q} := PR^q\pi_*\Omega^p_{\mathcal{X}^0/S^0}$ and $(PR^q\pi_*\Omega^p_{\mathcal{X}^0/S^0})_t = PH^q(X_t, \Omega^p_{X_t})$ for any $t \in S$, where $PH^q(X_t, \Omega^p_{X_t})$ is the primitive cohomology of $(X_t, [\omega_t])$. The first Chern forms of $P\mathcal{H}^{p,q}$ are related to the Weil–Petersson form and generalized Hodge forms ([Tian 1987, Theorem 2; Fang and Lu 2005, Proposition 2.8]):

$$\begin{aligned} \omega_{\mathrm{WP}} &= c_1(P\mathcal{H}^{n,0}), \\ \omega_{\mathrm{H}^i} &= \sum_{0 \le p \le i} p(c_1(P\mathcal{H}^{p,i-p}) + c_1(P\mathcal{H}^{p-1,i-p-1}) + \cdots). \end{aligned} \tag{4-1}$$

Secondly, it follows from the work of Cattani, Kaplan and Schmid [Cattani et al. 1986, Corollary 5.23] that:

(i) The forms $c_1(P\mathcal{H}^{p,q})$ are integrable and they define closed, (1, 1)-currents $\overline{c_1(P\mathcal{H}^{p,q})}$ (trivial extensions) on the completion S.

(ii) $\overline{c_1(P\mathcal{H}^{p,q})} = c_1(P\mathcal{H}_e^{p,q})$ in $H^2_{\mathrm{DR}}(S)$.

Hence we have:

Lemma 4.1. *As cohomology classes in $H^2_{\mathrm{DR}}(S)$, we have*

$$T_{\omega_{\mathrm{WP}}} = \overline{c_1(P\mathcal{H}^{n,0})} = c_1(P\mathcal{H}_e^{n,0}), \tag{4-2}$$

and

$$T_{\omega_{\mathrm{H}^i}} = \sum_{0\le p\le i} p(\overline{c_1(P\mathcal{H}^{p,i-p})} + \overline{c_1(P\mathcal{H}^{p-1,i-p-1})} + \cdots) \tag{4-3}$$

$$= \sum_{0\le p\le i} p(c_1(P\mathcal{H}_e^{p,i-p}) + c_1(P\mathcal{H}_e^{p-1,i-p-1}) + \cdots). \tag{4-4}$$

Proof. This follows from (4-1) and (ii) above. □

Now we can get the following:

Corollary 4.2. *We assume the same conditions as in Theorem 3.7. Suppose that the local monodromies of the polarized family $f^0 : \mathcal{X}^0 \to S^0$ are all unipotent. Then*

$$\sum_{i=1}^{n}(-1)^{i-1}\sum_{0\le p\le i} p(\deg P\mathcal{H}_e^{p,i-p} + \deg P\mathcal{H}_e^{p-1,i-p-1} + \cdots) + \frac{\chi}{12}\deg P\mathcal{H}_e^{n,0} = \sum_v a_v \deg E_v. \tag{4-5}$$

In particular, for polarized family of Calabi–Yau 3-folds, we have

$$(\chi + 36)\deg P\mathcal{H}_e^{3,0} + 12\deg P\mathcal{H}_e^{2,1} = 12\sum_v a_v \deg E_v, \tag{4-6}$$

and for polarized family of Calabi–Yau 4-folds, we have

$$(\chi - 48)\deg P\mathcal{H}_e^{4,0} - 24\deg P\mathcal{H}_e^{3,1} + 12\deg P\mathcal{H}_e^{2,1} = 12\sum_v a_v \deg E_v. \tag{4-7}$$

Proof. By Theorem 3.7 and Lemma 4.1, we have the following equality of cohomology classes in $H^2_{\mathrm{DR}}(S)$:

$$\begin{aligned} &dd^c \log T \\ &\quad + \sum_{i=1}^{n}(-1)^{i-1}\sum_{0\le p\le i} p(c_1(P\mathcal{H}_e^{p,i-p}) + c_1(P\mathcal{H}_e^{p-1,i-p-1}) + \cdots) + \frac{\chi}{12}c_1(P\mathcal{H}_e^{n,0}) \\ &= \sum_v a_v[E_v]. \end{aligned} \tag{4-8}$$

Since S is projective, let ω_{FS} be the restriction of the Kähler form of the Fubini–Study metric to S, by integration on S, and noting that $dd^c \log T \wedge \omega_{\mathrm{FS}}^{m-1}$ is d-exact, (4-5) follows. □

Now assume $m = 1$, that is, S is a compact Riemann surface. In this case, $E = \{s_v\}$ is a set of points and we have $\deg s_v = 1$, for all v. We see that for any n, $12\sum_v a_v$ is an integer which is not clear from the definition. In fact, for Calabi–Yau 3-folds, each individual a_v is known to be a rational number [Yoshikawa 2015, Theorem 0.1]. On the other hand, by applying the Grothendieck–Riemann–Roch formula to the family $f : \mathcal{X} \to S$ (see [Green et al. 2009] for $n = 3$), the sum $12\sum_v a_v$ can be given a geometrical meaning. For example, if the family f has only semistable singular fibers and is relatively minimal, then $12\sum_v a_v$ is equal to an expression involving the Euler characteristic number and intersection data of the singular fibers. In [Liu and Xia 2019], we conjectured that each individual a_v should be determined only by the data of the corresponding singular fiber. This statement is true for ordinary double point singularities. Indeed, Yoshikawa [2015, Theorem 5.2] proved that $a_v = \frac{1}{6}\#\mathrm{Sing}\, X_{s_v}$, where $\#\mathrm{Sing}\, X_{s_v}$ is the number of ordinary double points on the singular fiber X_{s_v}. In this direction, Eriksson, Freixas and Mourougane [Eriksson et al. 2018b] obtained interesting results under the assumption that the total space $\mathcal{X}$ has trivial canonical bundle which is equivalent to f being relatively minimal because they work locally.

We now use the identity (4-7) to prove an Euler number bound for a polarized family of Calabi–Yau 4-folds over a compact Riemann surface:

Corollary 4.3. *We assume the same conditions as in Theorem 3.7 and we let $n = 4, m = 1$. If $f : \mathcal{X} \to S$ is not isotrivial and the local monodromies of the polarized family $f^0 : \mathcal{X}^0 \to S^0$ are all unipotent, then*

$$\begin{aligned} 1+\frac{\sum_v a_v - \deg P\mathcal{H}_e^{2,1}}{2\deg P\mathcal{H}_e^{4,0}} &\le \frac{\chi}{24} \\ &\le \frac{2\pi(2g-2+s)(h^{3,1}+4)+\sum_v a_v - \deg P\mathcal{H}_e^{2,1}}{2\deg P\mathcal{H}_e^{4,0}}. \end{aligned} \tag{4-9}$$

Proof. We have the following inequality [Liu and Xia 2019, Theorem 5.1]:

$$0 < \deg P\mathcal{H}_e^{4,0} \le 2\deg P\mathcal{H}_e^{4,0} + \deg P\mathcal{H}_e^{3,1} \le \pi(2g-2+s)(h^{3,1}+4). \tag{4-10}$$

Now (4-9) follows from (4-7) and (4-10). □

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