Pacific Journal of Mathematics

CLASSIFICATION OF GRADIENT EXPANDING AND STEADY RICCI SOLITONS

FEI YANG, SHOUWEN FANG AND LIANGDI ZHANG

Volume 301 No. 1 July 2019

PACIFIC JOURNAL OF MATHEMATICS
Vol. 301, No. 1, 2019
dx.doi.org/10.2140/pjm.2019.301.371

CLASSIFICATION OF GRADIENT EXPANDING AND STEADY RICCI SOLITONS

FEI YANG, SHOUWEN FANG AND LIANGDI ZHANG

In this paper, we prove some classification theorems for gradient expanding and steady Ricci solitons. We show that a complete noncompact radially Ricci flat (i.e., $\mathrm{Ric}(\nabla f, \nabla f) = 0$) gradient expanding Ricci soliton with nonnegative Ricci curvature is a finite quotient of $\mathbb{R}^n$. Moreover, we prove that a complete noncompact gradient expanding Ricci soliton with $\mathrm{Ric} \geq 0$ and $\mathrm{div}^4 \mathrm{Rm} = 0$ is a finite quotient of $\mathbb{R}^n$. For a nontrivial complete noncompact radially Ricci flat (i.e., $\mathrm{Ric}(\nabla f, \nabla f) = 0$) gradient steady Ricci soliton with $\int |\nabla R|^2 e^{\alpha f} < +\infty$ for some $\alpha \in \mathbb{R}$, we show that it is Einstein with vanishing Ricci curvature or a quotient of $\mathbb{R}^n$ or of the product $\mathbb{R}^k \times N^{n-k}$ with $1 \leq k \leq n-1$, where N is Einstein with vanishing Ricci curvature.

1. Introduction

A complete Riemannian manifold (M^n, g) is called a gradient Ricci soliton if there exists a smooth function f on M^n such that the Ricci tensor Ric of the metric g satisfies the equation

$$\mathrm{Ric} + \mathrm{Hess}\, f = \lambda g \tag{1-1}$$

for some constant λ. For $\lambda > 0$ the Ricci soliton is shrinking, for $\lambda = 0$ it is steady and for $\lambda < 0$ expanding.

An Einstein manifold with constant potential function is called a *trivial* gradient Ricci soliton. When $f = \frac{\lambda}{2}|x|^2$ on $\mathbb{R}^n$, $\mathrm{Hess}\, f = \lambda g$ and therefore yields a gradient soliton where the background metric is flat. This example is called a *Gaussian* soliton.

Taking a product $N \times \mathbb{R}^k$ with N being Einstein with Einstein constant λ and $f = \frac{\lambda}{2}|x|^2$ on $\mathbb{R}^k$ yields a mixed gradient soliton. A gradient soliton is *rigid* if it is of the type $N \times_\Gamma \mathbb{R}^k$, where Γ acts freely on N and by orthogonal transformations on $\mathbb{R}^k$ (no translational components).

This work is partially supported by Natural Science Foundation of China (No. 11601495) and Science Foundation for the Excellent Young Scholars of Central Universities (No. CUGL170213).
MSC2010: 53C21, 53C25.
Keywords: gradient expanding Ricci soliton, gradient steady Ricci soliton.

Throughout the paper, for gradient expanding Ricci solitons we normalize the constant $\lambda = -\frac{1}{2}$ so that (1-1) becomes

$$\text{Ric} + \text{Hess}\, f = -\tfrac{1}{2} g. \tag{1-2}$$

In this paper, we shall focus our attention on gradient expanding and steady Ricci solitons (M^n, g, f). It turns out that a compact gradient steady or expanding Ricci soliton is necessarily an Einstein metric (see [Hamilton 1995; Ivey 1994]).

Some properties of gradient expanding Ricci solitons have been proved in recent years. G. Catino, P. Mastrolia and D. D. Monticelli [Catino et al. 2017] showed that a gradient expanding Ricci soliton with nonnegative Ricci curvature and fourth order divergence-free Weyl tensor has harmonic Weyl curvature. H. D. Cao et al. [2014] estimated the potential function f of a complete noncompact gradient expanding soliton with nonnegative Ricci curvature, that is $-f$ is of quadratic growth. They also showed that the condition of $\text{Ric} \geq 0$ can be relaxed to $\text{Rc} \geq -\left(\frac{1}{2} - \varepsilon\right) g$ for any small $\varepsilon > 0$.

A 3-dimensional complete gradient expanding Ricci soliton with constant scalar curvature is classified and indeed it is a finite quotient of $\mathbb{R}^3$, $\mathbb{H}^2 \times \mathbb{R}$, and $\mathbb{H}^3$ (see [Petersen and Wylie 2010]). For a 3-dimensional complete gradient expanding Ricci soliton with nonnegative Ricci curvature and the scalar curvature $R \in L^1(M^3)$, Catino, Mastrolia and Monticelli [Catino et al. 2016] showed that it is isometric to a quotient of the Gaussian soliton $\mathbb{R}^3$. Moreover, Cao et al. [2014] proved that a 3-dimensional complete expanding gradient Ricci solitons with divergence-free Bach tensor and nonnegative Ricci curvature is rotationally symmetric. For the n-dimensional case, they also proved that a complete Bach-flat gradient expanding Ricci soliton with nonnegative Ricci curvature is rotationally symmetric.

Some properties of gradient steady Ricci solitons are as follows: P. Petersen and W. Wylie [2009] proved that a gradient steady soliton whose scalar curvature achieves its minimum is Ricci flat. Moreover, if f is not constant then it is a product of a Ricci flat manifold with $\mathbb{R}$. O. Munteanu and N. Sesum [2013] showed that any gradient steady Ricci soliton has at least linear volume growth and at most growth rate of $e^{\sqrt{r}}$. Moreover, they proved that a gradient steady Ricci soliton has at most one nonparabolic end. P. Wu [2013] proved that the infimum of the potential function of a gradient steady Ricci soliton must decay linearly.

Cao et al. [2014] proved that a 3-dimensional gradient steady Ricci soliton with divergence-free Bach tensor is either flat or isometric to the Bryant soliton up to a scaling factor. Catino, Mastrolia and Monticelli [Catino et al. 2016] proved that a 3-dimensional complete gradient steady Ricci soliton with

$$\liminf_{r \to +\infty} \frac{1}{r} \int_{B_r(O)} R = 0$$

is isometric to a quotient of $\mathbb{R}^3$ or $\mathbb{R} \times \Sigma^2$, where Σ^2 is the cigar soliton. Under the condition of κ-noncollapsed, S. Brendle [2013] proved that a 3-dimensional complete nonflat gradient steady Ricci soliton is isometric to the Bryant soliton up to scaling.

In higher dimensions, Cao and Q. Chen [2012] proved that an n-dimensional complete noncompact locally conformally flat gradient steady Ricci soliton is either flat or isometric to the Bryant soliton. Moreover, Cao et al. [2014] showed that a Bach-flat gradient steady Ricci soliton with positive Ricci curvature such that the scalar curvature R attains its maximum at some interior point is isometric to the Bryant soliton up to a scaling factor. Brendle [2014] proved that a steady gradient Ricci soliton of dimension n $(n \geq 4)$ is rotationally symmetric if it has positive sectional curvature and is asymptotically cylindrical. In particular, it is isometric to the Bryant soliton up to scaling.

The aim of this paper is to obtain some classification theorems of gradient expanding and steady Ricci solitons. In order to state our results precisely, we introduce the following definitions for the Riemannian curvature:

$$(\operatorname{div} \mathrm{Rm})_{ijk} := \nabla_l R_{ijkl}, \qquad (\operatorname{div}^2 \mathrm{Rm})_{ik} := \nabla_j \nabla_l R_{ijkl},$$
$$(\operatorname{div}^3 \mathrm{Rm})_i := \nabla_k \nabla_j \nabla_l R_{ijkl}, \qquad \operatorname{div}^4 \mathrm{Rm} := \nabla_i \nabla_k \nabla_j \nabla_l R_{ijkl}.$$

The main results of this paper are the following theorems for gradient expanding and steady Ricci solitons.

For a complete noncompact gradient expanding Ricci soliton with nonnegative Ricci curvature, we have the following classification theorem.

Theorem 1.1. *Let (M^n, g, f) be a complete noncompact gradient expanding Ricci soliton with nonnegative Ricci curvature. Then, under any of the additional conditions*

(i) *(M^n, g, f) is radially Ricci flat, or*

(ii) $\operatorname{div}^4 \mathrm{Rm} = 0$, *or*

(iii) $\operatorname{tr} \operatorname{div}^2 \mathrm{Rm} = 0$,

(M^n, g, f) is a finite quotient of the Gaussian expanding soliton $\mathbb{R}^n$.

For a nontrivial complete noncompact gradient steady Ricci soliton, we will prove the following classification theorem.

Theorem 1.2. *Let (M^n, g, f) be a nontrivial complete noncompact gradient steady Ricci soliton. Then, under any of the additional conditions*

(i) *(M^n, g, f) is radially Ricci flat and $\int |\nabla R| e^{\alpha f} < +\infty$ for some $\alpha \in \mathbb{R}$, or*

(ii) $\operatorname{div}^4 \mathrm{Rm} = 0$ *and $\int |\mathrm{Rm}| e^{\alpha f} < +\infty$ for some $\alpha \neq 0$, or*

(iii) $\operatorname{tr} \operatorname{div}^2 \mathrm{Rm} = 0$ *and $\int |\nabla R| e^{\alpha f} < +\infty$ for some $\alpha \neq 0$,*

(M^n, g, f) is Einstein with vanishing Ricci curvature or a quotient of $\mathbb{R}^n$ or of the product $\mathbb{R}^k \times N^{n-k}$ with $1 \le k \le n-1$, where N is Einstein with vanishing Ricci curvature.

Remark 1.3. As will be clear from the proof, the scalar assumptions on the vanishing of $\operatorname{div}^4 \mathrm{Rm}$ in Theorem 1.1 and Theorem 1.2 can be trivially relaxed to a (suitable) inequality. The condition of $\operatorname{div}^4 \mathrm{Rm} = 0$ in Theorem 1.1 can be relaxed to $\int \operatorname{div}^4 \mathrm{Rm}\, e^f \le 0$. Moreover, the condition of $\operatorname{div}^4 \mathrm{Rm} = 0$ in Theorem 1.2 can be relaxed to $\int \operatorname{div}^4 \mathrm{Rm}\, e^{\alpha f} \le 0$ for some $\alpha \neq 0$.

The rest of this paper is organized as follows. In Section 2, we recall some background material which will be needed in the proof of the main theorems. In Section 3, we prove an integral identity for complete noncompact gradient expanding Ricci solitons with nonnegative Ricci curvature. In Section 4, we finish the proof of Theorem 1.1. In Section 5, we deal with Theorem 1.2. In the Appendix, we show that a complete noncompact gradient expanding or steady Ricci soliton with $\operatorname{div}^3 \mathrm{Rm}(\nabla f) = 0$ is rigid.

2. Preliminaries

We recall the following formulas for gradient Ricci solitons.

Proposition 2.1 [Yang and Zhang 2017]. *Let (M^n, g, f) be a gradient Ricci soliton. We have the following identities*:

$$(\operatorname{div}^2 \mathrm{Rm})_{ik} = 2\lambda R_{ik} + \nabla_l R_{ik} \nabla_l f - \tfrac{1}{2}\nabla_i \nabla_k R - R_{ik}^2 - R_{ijkl} R_{jl}, \tag{2-1}$$

$$(\operatorname{div}^3 \mathrm{Rm})_i = -R_{ijkl} \nabla_k R_{jl}, \tag{2-2}$$

$$(\operatorname{div}^3 \mathrm{Rm})(\nabla f) = -\tfrac{1}{2}|\operatorname{div} \mathrm{Rm}|^2. \tag{2-3}$$

Next we list the results that will be needed in the proof of the main theorems.

Lemma 2.2 [Cao et al. 2014]. *Let (M^n, g_{ij}, f) $(n \ge 3)$ be a complete noncompact gradient expanding soliton with nonnegative Ricci curvature* $\mathrm{Rc} \ge 0$. *Then there exist some constants $c_1 > 0$ and $c_2 > 0$ such that the potential function f satisfies the estimates*

$$\tfrac{1}{4}\big(r(x) - c_1\big)^2 - c_2 \le -f(x) \le \tfrac{1}{4}\big(r(x) + 2\sqrt{-f(O)}\,\big)^2, \tag{2-4}$$

where $r(x)$ is the distance function from any fixed base point in M^n. In particular, f is a strictly concave exhaustion function achieving its maximum at some interior point O, which we take as the base point, and the underlying manifold M^n is diffeomorphic to $\mathbb{R}^n$.

Lemma 2.3 [Petersen and Wylie 2009]. *The following conditions for a gradient expanding soliton* $\mathrm{Ric} + \mathrm{Hess}\, f = \lambda g$ *all imply that the metric is radially flat and has constant scalar curvature.*

(1) *The scalar curvature is constant and* $\sec(E, \nabla f) \le 0$.

(2) *The scalar curvature is constant and* $\lambda g \le \mathrm{Ric} \le 0$.

(3) *The curvature tensor is harmonic.*

(4) $\mathrm{Ric} \le 0$ *and* $\sec(E, \nabla f) = 0$.

Lemma 2.4 [Petersen and Wylie 2009]. *A gradient soliton is rigid if and only if it has constant scalar curvature and is radially flat*, *that is*, $\sec(E, \nabla f) = 0$.

3. An integral identity for gradient expanding Ricci solitons

We prove a useful integral identity (see Lemma 3.2 below), which will be needed in the proof of Theorem 1.1. The first step is to obtain the following proposition.

Proposition 3.1. *Let* (M^n, g, f) *be a complete noncompact gradient expanding Ricci soliton with* $\mathrm{Ric} \ge 0$; *then*

$$\left|\int \nabla_{\nabla f} R e^f\right| < +\infty. \tag{3-1}$$

Proof. Since $\mathrm{Ric} \ge 0$, $|\mathrm{Ric}| \le R$ and the Bishop comparison theorem implies that the volume of a geodesic ball is at most Euclidean growth. By Lemma 2.2, $-f$ is of quadratic growth. Note that $R + |\nabla f|^2 + f = \text{Const.}$, $|\mathrm{Ric}||\nabla f|^2$ of at most polynomial growth. Therefore, we have

$$\left|\int \nabla_{\nabla f} R e^f\right| = 2\left|\int \mathrm{Ric}(\nabla f, \nabla f) e^f\right| \le 2\int |\mathrm{Ric}||\nabla f|^2 e^f < +\infty. \qquad \square$$

Lemma 3.2. *Let* (M^n, g, f) *be a complete noncompact gradient expanding Ricci soliton with* $\mathrm{Ric} \ge 0$; *then we have*

$$\int \nabla_{\nabla f} R e^f = -\int \Delta R e^f. \tag{3-2}$$

Proof. Let $\phi(t) = 1$ on $(0, s)$, $\phi(t) = \frac{2s-t}{s}$ on $(s, 2s)$ and $\phi \equiv 0$ on $[2s, +\infty)$ for any fixed $s > 0$. Since $\mathrm{Ric} \ge 0$, Lemma 2.2 implies that $-f$ is of quadratic growth. Therefore, $\phi(-f)$ has compact support for any fixed $s > 0$. Define the compact set $D(s) := \{x \in M^n \mid -f(x) \le s\}$.

By direct computation, we have

$$\begin{aligned} \int \nabla_{\nabla f} R\phi(-f) e^f &= \int \langle \nabla R, \nabla e^f\rangle \phi(-f) \\ &= -\int \Delta R\phi(-f) e^f + \int \langle \nabla R, \nabla f\rangle \phi'(-f) e^f \\ &= -\int \Delta R\phi(-f) e^f - \frac{1}{s}\int_{D(2s)\setminus D(s)} \nabla_{\nabla f} R e^f. \end{aligned} \tag{3-3}$$

It follows from Proposition 3.1 that

$$\lim_{s\to+\infty}\frac{1}{s}\int_{D(2s)\setminus D(s)}\nabla_{\nabla f}Re^{f}=0.$$

Therefore, (3-2) follows by taking $s\to+\infty$ in (3-3). □

4. Proof of the main result for gradient expanding Ricci solitons

In this section, we prove Theorem 1.1.

Theorem 4.1. *Let (M^n, g, f) be a complete noncompact gradient expanding Ricci soliton with* $\mathrm{Ric}\geq 0$ *and* $\mathrm{Ric}(\nabla f,\nabla f)=0$. *Then (M^n, g, f) is a finite quotient of the Gaussian expanding soliton $\mathbb{R}^n$.*

Proof. Since $\nabla_{\nabla f}R=2\,\mathrm{Ric}(\nabla f,\nabla f)=0$, Lemma 3.2 implies that

$$\int \Delta Re^{f}=0. \tag{4-1}$$

Noting that $\Delta_f R=-R-2|\mathrm{Ric}|^2$ and $\Delta_f R=\Delta R-\nabla_{\nabla f}R=\Delta R$, we have

$$\Delta R=-R-2|\mathrm{Ric}|^2. \tag{4-2}$$

Applying (4-2) to (4-1), we obtain

$$\int |\mathrm{Ric}|^2e^{f}=-\frac{1}{2}\int Re^{f}. \tag{4-3}$$

Since $\mathrm{Ric}\geq 0$, $R\geq 0$. From (4-3), we know that $|\mathrm{Ric}|=0$ on M^n, i.e., (M^n, g, f) has vanishing Ricci curvature.

Hence, condition (2) in Lemma 2.3 holds. It follows from Lemma 2.3 that (M^n, g, f) is radially flat and has constant scalar curvature. By Lemma 2.4, we have (M^n, g, f) is rigid. Since $\mathrm{Ric}=0$ on M^n, (M^n, g, f) is a finite quotient of the Gaussian expanding soliton $\mathbb{R}^n$. □

Theorem 4.2. *Let (M^n, g, f) be a complete noncompact gradient expanding Ricci soliton. If* $\mathrm{div}^4\,\mathrm{Rm}=0$ *and* $\mathrm{Ric}\geq 0$, *then (M^n, g, f) is a finite quotient of the Gaussian expanding soliton $\mathbb{R}^n$.*

Proof. Let $\phi(t)=1$ on $[0,s]$, $\phi(t)=\frac{2s-t}{s}$ on $(s,2s)$ and $\phi\equiv 0$ on $[2s,+\infty)$ for any fixed $s>0$. Since $\mathrm{Ric}\geq 0$, Lemma 2.2 implies that $-f$ is of quadratic growth. Therefore, $\phi(-f)$ has compact support for any fixed $s>0$. Define the compact set $D(s):=\{x\in M^n \mid -f(x)\leq s\}$.

Integrating by parts, we have

$$\begin{aligned}\text{(4-4)}\quad \int \operatorname{div}^4 \mathrm{Rm}\,\phi(-f)e^f &= -\int \operatorname{div}^3 \mathrm{Rm}(\nabla f)\phi(-f)e^f + \int \operatorname{div}^3 Rm(\nabla f)\phi'(-f)e^f \\ &= \frac{1}{2}\int |\operatorname{div}\mathrm{Rm}|^2\phi(-f)e^f + \frac{1}{2s}\int_{D(2s)\setminus D(s)} |\operatorname{div}\mathrm{Rm}|^2 e^f,\end{aligned}$$

where we used (2-3) in the second equality.

Since $\operatorname{div}^4 \mathrm{Rm} = 0,\ \phi(-f) \geq 0$ on M^n, it follows from (4-4) that

$$\int |\operatorname{div}\mathrm{Rm}|^2\phi(-f)e^f = 0. \tag{4-5}$$

Note that $\phi(-f) = 1$ on the compact set $D(s) := \{x \in M^n \mid -f(x) \leq s\}$. From (4-5), we know that

$$\int_{D(s)} |\operatorname{div}\mathrm{Rm}|^2 e^f = 0. \tag{4-6}$$

Taking $s \to +\infty$ in (4-6), we have

$$\int |\operatorname{div}\mathrm{Rm}|^2 e^f = 0,$$

that is, $|\operatorname{div}\mathrm{Rm}| = 0$ on M^n.

Note that

$$\begin{aligned}\text{(4-7)}\qquad 0 = \nabla_l R_{ijkl} &= \nabla_j R_{ik} - \nabla_i R_{jk} \\ &= -\nabla_j\nabla_i\nabla_k f + \nabla_i\nabla_j\nabla_k f \\ &= R_{ijkl}\nabla_l f.\end{aligned}$$

It follows that M^n is radially flat.

Tracing $\operatorname{div}\mathrm{Rm}$, we have

$$0 = g^{ik}\nabla_l R_{ijkl} = \nabla_l R_{jl} = \tfrac{1}{2}\nabla_j R, \tag{4-8}$$

that is, M^n has a constant scalar curvature.

By Lemma 2.4, we have (M^n, g, f) is rigid. Since $\mathrm{Ric} \geq 0$, we conclude that (M^n, g, f) is a finite quotient of the Gaussian expanding soliton $\mathbb{R}^n$. □

Theorem 4.3. *Let (M^n, g, f) be a complete noncompact gradient expanding Ricci soliton with* $\mathrm{Ric} \geq 0$ *and* $\operatorname{tr}\operatorname{div}^2 \mathrm{Rm} = 0$ *then (M^n, g, f) is a finite quotient of the Gaussian expanding soliton $\mathbb{R}^n$.*

Proof. Tracing (2-1), we have

$$0 = g^{ik}(\operatorname{div}^2 \mathrm{Rm})_{ik} = -R + \nabla_{\nabla f} R - \tfrac{1}{2}\Delta R - 2|\mathrm{Ric}|^2 = -R + \tfrac{1}{2}\Delta R - \Delta_f R - 2|\mathrm{Ric}|^2 = \tfrac{1}{2}\Delta R, \tag{4-9}$$

where we used the fact that $\Delta_f R = -R - 2|\mathrm{Ric}|^2$.

It follows that

$$\Delta R = 0. \tag{4-10}$$

Applying (4-10) to Lemma 3.2, we obtain that $\int \Delta R e^f = \int \nabla_{\nabla f} R e^f = 0$. Noting that $\Delta R - \nabla_{\nabla f} R = \Delta_f R = -R - 2|\mathrm{Ric}|^2$, we have

$$\int |\mathrm{Ric}|^2 e^f = -\frac{1}{2}\int R e^f \le 0. \tag{4-11}$$

It follows that $|\mathrm{Ric}| = 0$ on M^n, i.e., (M^n, g, f) has vanishing Ricci curvature.

Hence, condition (2) in Lemma 2.3 holds. It follows from Lemma 2.3 that (M^n, g, f) is radially flat and has constant scalar curvature. By Lemma 2.4, we have that (M^n, g, f) is rigid. Since $\mathrm{Ric} = 0$ on M^n, (M^n, g, f) is a finite quotient of the Gaussian expanding soliton $\mathbb{R}^n$. □

Theorem 1.1 follows directly from Theorems 4.1–4.3.

5. Proof of the main result for gradient steady Ricci solitons

In this section, we prove Theorem 1.2.

Theorem 5.1. *Let (M^n, g, f) be a nontrivial complete noncompact radially Ricci flat* (*i.e.*, $\mathrm{Ric}(\nabla f, \nabla f) = 0$) *gradient steady Ricci soliton with $\int |\nabla R| e^{\alpha f} < +\infty$ for some $\alpha \in \mathbb{R}$. Then (M^n, g, f) is Einstein with vanishing Ricci curvature or a quotient of $\mathbb{R}^n$ or of the product $\mathbb{R}^k \times N^{n-k}$ with $1 \le k \le n-1$, where N is Einstein with vanishing Ricci curvature.*

Proof. Let B_r be a geodesic ball with radius r and let ν be the unit outward normal vector field to ∂B_r. Integrating by parts, we obtain

$$\begin{aligned}(\alpha+1)\int_{B_r} \nabla_{\nabla f} R e^{\alpha f} &= \int_{B_r} \langle \nabla R, \nabla e^{\alpha f}\rangle + \int_{B_r} \nabla_{\nabla f} R e^{\alpha f} \\ &= \int_{\partial B_r} \nabla_\nu R e^{\alpha f} - \int_{B_r} \Delta R e^{\alpha f} + \int_{B_r} \nabla_{\nabla f} R e^{\alpha f} \\ &= \int_{\partial B_r} \nabla_\nu R e^{\alpha f} - \int_{B_r} \Delta_f R e^{\alpha f} \\ &= \int_{\partial B_r} \nabla_\nu R e^{\alpha f} + 2\int_{B_r} |\mathrm{Ric}|^2 e^{\alpha f},\end{aligned} \tag{5-1}$$

where we used the fact that $\Delta_f R = -2|\mathrm{Ric}|^2$.

Note that $\nabla_{\nabla f} R = 2\,\mathrm{Ric}(\nabla f, \nabla f) = 0$. It follows from (5-1) that

$$\int_{B_r} |\mathrm{Ric}|^2 e^{\alpha f} = -\int_{\partial B_r} \nabla_\nu R e^{\alpha f} \le \int_{\partial B_r} |\nabla R| e^{\alpha f}. \tag{5-2}$$

Since $\int |\nabla R| e^{\alpha f} < +\infty$, we have

$$\lim_{r\to+\infty} \int_{\partial B_r} |\nabla R| e^{\alpha f} = 0. \tag{5-3}$$

Taking $r \to +\infty$ in (5-2) and using (5-3), we obtain

$$\int |\mathrm{Ric}|^2 e^{\alpha f} = 0, \tag{5-4}$$

that is, $|\mathrm{Ric}| = 0$ on M^n. It follows that (M^n, g, f) has vanishing scalar curvature.

Moreover, we have

$$\begin{aligned} R_{ijkl}\nabla_l f &= \nabla_i\nabla_j\nabla_k f - \nabla_j\nabla_i\nabla_k f \\ &= -\nabla_i R_{jk} + \nabla_j R_{ik} = 0, \end{aligned} \tag{5-5}$$

where we used (1-1) in the second equality and $\mathrm{Ric} = 0$ on M^n. It follows that $\sec(E, \nabla f) = R_{ijkl} E_i E_k \nabla_j f \nabla_l f = 0$, i.e., (M^n, g, f) is radially flat.

Since M has vanishing scalar curvature and is radially flat, Lemma 2.4 implies (M^n, g, f) is rigid. To conclude, (M^n, g, f) is Einstein with vanishing Ricci curvature or a quotient of $\mathbb{R}^n$ or of the product $\mathbb{R}^k \times N^{n-k}$ with $1 \le k \le n-1$, where N is Einstein with vanishing Ricci curvature. □

Theorem 5.2. *Let (M^n, g, f) be a nontrivial complete noncompact gradient steady Ricci soliton with $\int |\mathrm{Rm}| e^{\alpha f} < +\infty$ for some $\alpha \ne 0$. If in addition* $\mathrm{div}^4\,\mathrm{Rm} = 0$, *then it is Einstein with vanishing Ricci curvature or a quotient of $\mathbb{R}^n$ or of the product $\mathbb{R}^k \times N^{n-k}$ with $1 \le k \le n-1$, where N is Einstein with vanishing Ricci curvature.*

Proof. Let B_r be a geodesic ball with radius r and let ν be the outward unit normal vector field to ∂B_r. Integrating by parts, we obtain

$$\begin{aligned} \int_{B_r} \mathrm{div}^4\,\mathrm{Rm}\, e^{\alpha f} &\equiv \int_{B_r} \nabla_i\nabla_k\nabla_j\nabla_l R_{ijkl} e^{\alpha f} \\ &= \int_{\partial B_r} \nabla_k\nabla_j\nabla_l R_{ijkl}\nu_i e^{\alpha f} - \alpha \int_{B_r} \nabla_k\nabla_j\nabla_l R_{ijkl}\nabla_i f e^{\alpha f} \\ &= -\int_{\partial B_r} R_{ijkl}\nabla_k R_{jl}\nu_i e^{\alpha f} + \frac{\alpha}{2}\int_{B_r} |\mathrm{div}\,\mathrm{Rm}|^2 e^{\alpha f}, \end{aligned}$$

where we used (2-2) and (2-3) in the last equality.

Since $\operatorname{div}^4 \mathrm{Rm} = 0$, we have

$$\frac{\alpha}{2}\int_{B_r} |\operatorname{div}\mathrm{Rm}|^2 e^{\alpha f} = \int_{\partial B_r} R_{ijkl}\nabla_k R_{jl}\nu_i e^{\alpha f}. \tag{5-6}$$

Next, we prove

$$\lim_{r\to+\infty}\int_{\partial B_r} R_{ijkl}\nabla_k R_{jl}\nu_i e^{\alpha f} = 0. \tag{5-7}$$

By direct computation, we have

$$\begin{aligned}\nabla_p R_{lkjp} &= \nabla_k R_{lj} - \nabla_l R_{kj}\\ &= -\nabla_k\nabla_l\nabla_j f + \nabla_l\nabla_k\nabla_j f\\ &= R_{lkjp}\nabla_p f,\end{aligned} \tag{5-8}$$

where we used the second Bianchi identity in the first equality and (1-1) in the second.

Noting that $R \ge 0$ (cf. B. L. Chen [2009]) and $R + |\nabla f|^2 = \text{Const.}$, we have that $|\nabla f|$ is bounded. By direct computation, we obtain

$$\begin{aligned}\left|\int R_{ijkl}\nabla_k R_{jl}\nu_i e^{\alpha f}\right| &= \frac{1}{2}\left|\int R_{ijkl}(\nabla_k R_{jl} - \nabla_l R_{jk})\nu_i e^{\alpha f}\right|\\ &= \frac{1}{2}\left|\int R_{ijkl}\nabla_p R_{lkjp}\nu_i e^{\alpha f}\right|\\ &= \frac{1}{2}\left|\int R_{ijkl}R_{lkjp}\nabla_p f\,\nu_i e^{\alpha f}\right|\\ &\le \frac{1}{2}\int |\mathrm{Rm}|^2|\nabla f| e^{\alpha f}\\ &\le c\int |\mathrm{Rm}|^2 e^{\alpha f} < +\infty,\end{aligned}$$

where we used (5-8) in the third equality and the assumption of

$$\int |\mathrm{Rm}|^2 e^{\alpha f} < +\infty$$

in the last. Then (5-7) follows.

Taking $r \to +\infty$ in (5-6), we have

$$\int |\operatorname{div}\mathrm{Rm}|^2 e^{\alpha f} = 0,$$

that is, $|\operatorname{div}\mathrm{Rm}| = 0$ on M.

By direct computation, we have

$$R_{ijkl}\nabla_l f = \nabla_i\nabla_j\nabla_k f - \nabla_j\nabla_i\nabla_k f \\ = -\nabla_i R_{jk} + \nabla_j R_{ik} \\ = \nabla_l R_{ijkl} \\ = 0, \tag{5-9}$$

where we used (1-1) in the second equality and $\mathrm{Ric} = 0$ on M^n. It follows that $\sec(E, \nabla f) = R_{ijkl}E_i E_k \nabla_j f \nabla_l f = 0$, i.e., (M^n, g, f) is radially flat.

Moreover, we have

$$\nabla_l R = 2\nabla_j R_{jl} = 2g^{ik}\nabla_l R_{ijkl} = 0, \tag{5-10}$$

that is, R is a constant on M^n.

Since M^n is radially flat and has a constant scalar curvature, Lemma 2.4 implies that (M^n, g) is rigid. To conclude, (M^n, g, f) is Einstein with vanishing Ricci curvature or a quotient of $\mathbb{R}^n$ or of the product $\mathbb{R}^k \times N^{n-k}$ with $1 \le k \le n-1$, where N is Einstein with vanishing Ricci curvature. □

Theorem 5.3. *Let (M^n, g, f) be a nontrivial complete noncompact gradient steady Ricci soliton with $\int |\nabla R| e^{\alpha f} < +\infty$ for some $\alpha \neq 0$. If in addition* $\mathrm{tr}\,\mathrm{div}^2\,\mathrm{Rm} = 0$, *then (M^n, g, f) is a quotient of $\mathbb{R}^n$ or of the product $\mathbb{R}^k \times N^{n-k}$ with $1 \le k \le n-1$, where N is Einstein with vanishing Ricci curvature.*

Proof. From the proof of Theorem 5.1, we only need to show that M^n has vanishing Ricci curvature.

Let B_r be a geodesic ball with radius r and let ν be the unit outward normal vector field to ∂B_r. Integrating by parts, we obtain

$$\alpha \int_{B_r} \nabla_{\nabla f} R e^{\alpha f} = \int_{B_r} \langle \nabla R, \nabla e^{\alpha f} \rangle \\ = \int_{\partial B_r} \nabla_\nu R e^{\alpha f} - \int_{B_r} \Delta R e^{\alpha f}. \tag{5-11}$$

Tracing (2-1), we have

$$g^{ik}(\mathrm{div}^2\,\mathrm{Rm})_{ik} = \nabla_{\nabla f} R - \tfrac{1}{2}\Delta R - 2|\mathrm{Ric}|^2 = \tfrac{1}{2}\Delta R - \Delta_f R - 2|\mathrm{Ric}|^2 = \tfrac{1}{2}\Delta R, \tag{5-12}$$

where we used $\Delta_f R = -2|\mathrm{Ric}|^2$.

Since $\mathrm{tr}\,\mathrm{div}^2\,\mathrm{Rm} = 0$, it follows from (5-12) that

$$\Delta R = 0. \tag{5-13}$$

On the other hand,

$$g^{ik}(\operatorname{div}^2\mathrm{Rm})_{ik} = \nabla_{\nabla f}R - \tfrac{1}{2}\Delta R - 2|\mathrm{Ric}|^2 = \tfrac{1}{2}\nabla_{\nabla f}R - \tfrac{1}{2}\Delta_f R - 2|\mathrm{Ric}|^2 = \tfrac{1}{2}\nabla_{\nabla f}R - |\mathrm{Ric}|^2, \tag{5-14}$$

where we used the fact that $\Delta_f R = -2|\mathrm{Ric}|^2$.

It follows from $\operatorname{tr}\operatorname{div}^2\mathrm{Rm} = 0$ and (5-14) that

$$\nabla_{\nabla f}R = 2|\mathrm{Ric}|^2. \tag{5-15}$$

Applying (5-13) and (5-15) to (5-11), and noting that $\alpha \neq 0$, we obtain

$$\int_{B_r}|\mathrm{Ric}|^2 e^{\alpha f} = \frac{1}{2\alpha}\int_{\partial B_r}\nabla_\nu R e^{\alpha f} \le \frac{1}{2|\alpha|}\int_{\partial B_r}|\nabla R|e^{\alpha f}. \tag{5-16}$$

Since $\int|\nabla R|e^{\alpha f} < +\infty$, we have

$$\lim_{r\to+\infty}\int_{\partial B_r}\nabla_\nu R e^{\alpha f} = 0.$$

By taking $r \to +\infty$ in (5-16), we obtain

$$\int|\mathrm{Ric}|^2 e^{\alpha f} = 0, \tag{5-17}$$

that is, $|\mathrm{Ric}| = 0$ on M^n, i.e., (M^n, g, f) has vanishing Ricci curvature.

This completes the proof of Theorem 5.3. □

Theorem 1.2 follows directly from Theorems 5.1–5.3.

Appendix

We prove a rigid result for complete noncompact gradient steady and expanding Ricci solitons in this section. It was implicitly proved by Yang and Zhang [2017]. For readers' convenience, we include a proof here.

Theorem A.1. *Let (M^n, g, f) be a complete noncompact gradient steady or expanding Ricci soliton with* $\operatorname{div}^3\mathrm{Rm}(\nabla f) = 0$; *then (M^n, g, f) is rigid.*

Proof. Since $\operatorname{div}^3\mathrm{Rm}(\nabla f) = 0$, it follows from (2-3) that

$$|\operatorname{div}\mathrm{Rm}|^2 = -2\operatorname{div}^3\mathrm{Rm}(\nabla f) = 0.$$

Therefore, M is radially flat. Moreover, we have

$$\nabla_i R = 2\nabla_l R_{il} = -2g^{jk}\nabla_l R_{ijkl} = 0,$$

that is, R is a constant on M.

Since M^n is radially flat and has constant scalar curvature, Lemma 2.4 implies that (M^n, g, f) is rigid. □

Acknowledgement

The authors would like to thank the referee for many useful comments on improving the previous versions of the paper.

References

[Brendle 2013] S. Brendle, "Rotational symmetry of self-similar solutions to the Ricci flow", *Invent. Math.* **194**:3 (2013), 731–764. MR Zbl

[Brendle 2014] S. Brendle, "Rotational symmetry of Ricci solitons in higher dimensions", *J. Differential Geom.* **97**:2 (2014), 191–214. MR Zbl

[Cao and Chen 2012] H.-D. Cao and Q. Chen, "On locally conformally flat gradient steady Ricci solitons", *Trans. Amer. Math. Soc.* **364**:5 (2012), 2377–2391. MR Zbl

[Cao et al. 2014] H.-D. Cao, G. Catino, Q. Chen, C. Mantegazza, and L. Mazzieri, "Bach-flat gradient steady Ricci solitons", *Calc. Var. Partial Differential Equations* **49**:1-2 (2014), 125–138. MR Zbl

[Catino et al. 2016] G. Catino, P. Mastrolia, and D. D. Monticelli, "Classification of expanding and steady Ricci solitons with integral curvature decay", *Geom. Topol.* **20**:5 (2016), 2665–2685. MR Zbl

[Catino et al. 2017] G. Catino, P. Mastrolia, and D. D. Monticelli, "Gradient Ricci solitons with vanishing conditions on Weyl", *J. Math. Pures Appl.* (9) **108**:1 (2017), 1–13. MR Zbl

[Chen 2009] B.-L. Chen, "Strong uniqueness of the Ricci flow", *J. Differential Geom.* **82**:2 (2009), 363–382. MR Zbl

[Hamilton 1995] R. S. Hamilton, "The formation of singularities in the Ricci flow", pp. 7–136 in *Surveys in differential geometry, II* (Cambridge, MA, 1993), edited by S.-T. Yau, Int. Press, Cambridge, MA, 1995. MR Zbl

[Ivey 1994] T. Ivey, "New examples of complete Ricci solitons", *Proc. Amer. Math. Soc.* **122**:1 (1994), 241–245. MR Zbl

[Munteanu and Sesum 2013] O. Munteanu and N. Sesum, "On gradient Ricci solitons", *J. Geom. Anal.* **23**:2 (2013), 539–561. MR Zbl

[Petersen and Wylie 2009] P. Petersen and W. Wylie, "Rigidity of gradient Ricci solitons", *Pacific J. Math.* **241**:2 (2009), 329–345. MR Zbl

[Petersen and Wylie 2010] P. Petersen and W. Wylie, "On the classification of gradient Ricci solitons", *Geom. Topol.* **14**:4 (2010), 2277–2300. MR Zbl

[Wu 2013] P. Wu, "On the potential function of gradient steady Ricci solitons", *J. Geom. Anal.* **23**:1 (2013), 221–228. MR Zbl

[Yang and Zhang 2017] F. Yang and L. Zhang, "Rigidity of gradient shrinking Ricci solitons", preprint, 2017. arXiv

Received July 21, 2017. Revised August 13, 2018.

Fei Yang
School of Mathematics and Physics
China University of Geosciences
Wuhan
China
yangfei810712@163.com

Shouwen Fang
School of Mathematical Science
Yangzhou University
Yangzhou
China
shwfang@163.com

Liangdi Zhang
Center of Mathematical Sciences
Zhejiang University
Hangzhou
China
ldzhang91@163.com

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2019 is US $490/year for the electronic version, and $665/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 301 No. 1 July 2019

0030-8730(201907)301:1;1-I