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We study the long-time dynamics, in the sense of pullback attractors, of solutions for semilinear nonautonomous thermoelastic plate systems in a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 2$. Using the theory of uniform sectorial operators, in the sense of P. Sobolevskiĭ (1961), we will prove existence, uniform boundedness, regularity and upper semicontinuity of pullback attractors for the evolution system

$$\begin{cases} u_{tt} + \Delta^2 u + a\Delta\theta = f(u), & t > \tau,\ x \in \Omega, \\ \theta_t - \kappa(t)\Delta\theta - a\Delta u_t = 0, & t > \tau,\ x \in \Omega, \end{cases}$$

subject to boundary conditions

$$u = \Delta u = \theta = 0, \quad t > \tau, \quad x \in \partial\Omega,$$

with respect to the functional parameter κ.

1. Introduction

In this paper we study a model that describes the small vibrations of a homogeneous, elastic and thermal isotropic Euler–Bernoulli plate. In fact we consider the initial-boundary value problem

$$\begin{cases} u_{tt} + \Delta^2 u + a\Delta\theta = f(u), & t > \tau,\ x \in \Omega, \\ \theta_t - \kappa(t)\Delta\theta - a\Delta u_t = 0, & t > \tau,\ x \in \Omega, \end{cases} \tag{1-1}$$

subject to boundary conditions

$$\begin{cases} u = \Delta u = 0,\ t > \tau,\ x \in \partial\Omega, \\ \theta = 0,\ t > \tau,\ x \in \partial\Omega, \end{cases} \tag{1-2}$$

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and initial conditions

$$u(\tau,x)=u_0(x),\ u_t(\tau,x)=v_0(x)\ \text{ and }\ \theta(\tau,x)=\theta_0(x),\quad x\in\Omega,\ \tau\in\mathbb{R}, \tag{1-3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $N\geqslant 2$, where the boundary $\partial\Omega$ is assumed to be regular enough and $a>0$.

Next we exhibit conditions under which the nonautonomous problem (1-1)–(1-3) is locally and globally well posed in some appropriate space that we will specify later.

We assume that κ is continuously differentiable in $\mathbb{R}$ and satisfies

$$0<\kappa_0\leqslant\kappa(t),\kappa'(t)\leqslant\kappa_1\quad \text{for all } t\in\mathbb{R}, \tag{1-4}$$

for some positive constants κ_0 and κ_1.

Suppose that $f:\mathbb{R}\to\mathbb{R}$ is locally Lipschitz satisfying

$$\limsup_{|s|\to\infty}\frac{f(s)}{s}<\lambda_1 \tag{1-5}$$

uniformly in $t\in\mathbb{R}$, where $\lambda_1>0$ is the first eigenvalue of negative Laplacian operator with homogeneous Dirichlet boundary condition. Furthermore, the function f satisfies the subcritical growth condition; that is,

$$|f'(s)|\leqslant C(1+|s|^{\rho-1})\quad \text{for all } s\in\mathbb{R}, \tag{1-6}$$

where $1\leqslant\rho<\frac{N}{N-4}$, with $N\geqslant 5$, and $C>0$ independent of $t\in\mathbb{R}$. In this case, the embedding $H^2(\Omega)\cap H_0^1(\Omega)\hookrightarrow L^{2N/(N-4)}(\Omega)$ is compact and this will be used in analysis of the energy functionals. We will justify these restrictions later in the paper. If $N=2,3,4$, we suppose the growth condition (1-6) with $\rho\geqslant 1$.

Using the theory of uniform sectorial operators, in the sense of [Sobolevskiĭ 1961], the authors proved in [Bezerra et al. 2018] the local and global well-posedness of the nonautonomous problem (1-1)–(1-3) (under conditions (1-5) and (1-6)), the existence of pullback attractors and uniform bounds for these pullback attractors when $\kappa(t)\equiv\kappa$.

The main goal of this paper is to prove the regularity of the pullback attractors and their upper semicontinuity with respect to the functional parameter κ. For completeness, under the additional condition (1-4) we prove the local and global posedness for (1-1)–(1-3) as well the existence and uniform boundedness of pullback attractors for this problem.

We emphasize that no additional damping in first evolution equation in (1-1) is required in the present work.

To formulate the nonautonomous problem (1-1)–(1-3) in the nonlinear evolution process setting, we introduce some notation. Here, we denote $X=L^2(\Omega)$ and

$\Lambda : D(\Lambda) \subset X \to X$ to be the biharmonic operator defined by

$$D(\Lambda) = \{u \in H^4(\Omega);\ u = \Delta u = 0 \text{ on } \partial\Omega\}$$

and

$$\Lambda u = (-\Delta)^2 u \quad \text{for all } u \in D(\Lambda). \tag{1-7}$$

Then Λ is a positive self-adjoint operator in X with compact resolvent and therefore $-\Lambda$ generates a compact analytic semigroup on X (that is, Λ is a sectorial operator, in the sense of [Henry 1981]). Denote by X^α, $\alpha > 0$, the fractional power spaces associated with the operator Λ; that is, $X^\alpha = D(\Lambda^\alpha)$ endowed with the graph norm. With this notation, we have $X^{-\alpha} = (X^\alpha)'$ for all $\alpha > 0$, (see [Amann 1995]). Of special interest is the case $\alpha = \frac{1}{2}$, since $-\Lambda^{\frac{1}{2}}$ is the Laplacian operator with homogeneous Dirichlet boundary conditions.

If we denote $v = u_t$, then we can rewrite the nonautonomous problem (1-1)–(1-3) in the abstract form

$$\begin{cases} w_t = A_{(\kappa)}(t)w + F(w),\ t > \tau, \\ w(\tau) = w_0,\ \tau \in \mathbb{R}, \end{cases} \tag{1-8}$$

where $w = w(t)$ for all $t \in \mathbb{R}$, and $w_0 = w(\tau)$ are given by

$$w = \begin{bmatrix} u \\ v \\ \theta \end{bmatrix}, \quad \text{and} \quad w_0 = \begin{bmatrix} u_0 \\ v_0 \\ \theta_0 \end{bmatrix}, \tag{1-9}$$

and, for each $t \in \mathbb{R}$, the unbounded linear operator $A_{(\kappa)}(t) : D(A_{(\kappa)}(t)) \subset Y \to Y$ is defined by

$$A_{(\kappa)}(t) \begin{bmatrix} u \\ v \\ \theta \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ -\Lambda & 0 & -a\Lambda^{\frac{1}{2}} \\ 0 & a\Lambda^{\frac{1}{2}} & \kappa(t)\Lambda^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} u \\ v \\ \theta \end{bmatrix} = \begin{bmatrix} v \\ -\Lambda u - a\Lambda^{\frac{1}{2}}\theta \\ a\Lambda^{\frac{1}{2}} v + \kappa(t)\Lambda^{\frac{1}{2}}\theta \end{bmatrix}, \tag{1-10}$$

where

$$Y = (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega) \times L^2(\Omega)$$

is the phase space of the problem (1-1)–(1-3) and the domain of the operator $A_{(\kappa)}(t)$ is defined by the space

$$D(A_{(\kappa)}(t)) = X^1 \times X^{\frac{1}{2}} \times X^{\frac{1}{2}}, \tag{1-11}$$

with $X^1 = \{u \in H^4(\Omega);\ u = \Delta u = 0 \text{ on } \partial\Omega\}$ and $X^{\frac{1}{2}} = H^2(\Omega) \cap H_0^1(\Omega)$.

The nonlinearity F is given by

$$F(w) = \begin{bmatrix} 0 \\ f^e(u) \\ 0 \end{bmatrix}, \tag{1-12}$$

where $f^e(u)$ is the Nemytskii operator associated with $f(u)$; that is,

$$f^e(u)(x) := f(u(x)) \quad \text{for all } x \in \Omega.$$

This paper is organized as follows: in Section 2 we recall concepts and results about singularly nonautonomous problems. Section 3 is devoted to studying the existence of local and global solutions in some appropriate space, as well as the existence of pullback attractors for (1-1)–(1-3). In Section 4 we present results on regularity of the pullback attractors, following Carvalho, Langa, Robinson [Carvalho et al. 2013]. Finally, in Section 5 we prove that the family of pullback attractors behave upper semicontinuously with respect to the functional parameter κ.

2. Abstract linear problem

Throughout the paper, $L(\mathcal{Z})$ will denote the space of linear and bounded operators defined in a Banach space $\mathcal{Z}$. Let $\mathcal{B}(t)$, $t \in \mathbb{R}$, be a family of unbounded closed linear operators defined on a fixed dense subspace D of $\mathcal{Z}$.

2A. ***Nonautonomous abstract linear problem.*** Consider the singularly nonautonomous abstract linear parabolic problem of the form

$$\begin{cases} \dfrac{du}{dt} = -\mathcal{B}(t)u, \ t > \tau, \\ u(\tau) = u_0 \in D. \end{cases}$$

We assume that:

(a) The family of operators $\mathcal{B}(t) : D \subset \mathcal{Z} \to \mathcal{Z}$ is *uniformly sectorial*, that is, $\mathcal{B}(t)$ is closed densely defined (the domain D is fixed) and there is a constant $C > 0$ (independent of $t \in \mathbb{R}$) such that

$$\|(\mathcal{B}(t) + \lambda I)^{-1}\|_{L(\mathcal{Z})} \leqslant \frac{C}{|\lambda| + 1} \quad \text{for all } \lambda \in \mathbb{C} \text{ with } \operatorname{Re}\lambda \geqslant 0.$$

(b) The map $\mathbb{R} \ni t \mapsto \mathcal{B}(t)$ is *uniformly Hölder continuous*, that is, there are constants $C > 0$ and $\varepsilon_0 > 0$ such that, for any $t, \tau, s \in \mathbb{R}$,

$$\|[\mathcal{B}(t) - \mathcal{B}(\tau)]\mathcal{B}^{-1}(s)\|_{L(\mathcal{Z})} \leqslant C(t-\tau)^{\varepsilon_0}, \quad \varepsilon_0 \in (0, 1].$$

Denote by $\mathcal{B}_0$ the operator $\mathcal{B}(t_0)$ for some $t_0 \in \mathbb{R}$ fixed. If $\mathcal{Z}^\alpha$ denotes the domain of $\mathcal{B}_0^\alpha$, $\alpha > 0$, with the graph norm and $\mathcal{Z}^0 := \mathcal{Z}$, denote by $\{\mathcal{Z}^\alpha; \alpha \geqslant 0\}$ the fractional power scale associated with $\mathcal{B}_0$.

From (a), $-\mathcal{B}(t)$ is the generator of an analytic semigroup $\{e^{-\tau\mathcal{B}(t)} \in L(\mathcal{Z}) : \tau \geqslant 0\}$. Using this and the fact that $0 \in \rho(\mathcal{B}(t))$, it follows that

$$\|e^{-\tau\mathcal{B}(t)}\|_{L(\mathcal{Z})} \leqslant C, \quad \tau \geqslant 0, \ t \in \mathbb{R},$$

and

$$\|\mathcal{B}(t)e^{-\tau\mathcal{B}(t)}\|_{L(\mathcal{Z})} \leqslant C\tau^{-1}, \quad \tau > 0,\ t \in \mathbb{R}.$$

It follows from (b) that $\|\mathcal{B}(t)\mathcal{B}^{-1}(\tau)\|_{L(\mathcal{Z})} \leqslant C$, for all $(t,\tau) \in I$, for some $I \subset \mathbb{R}^2$ bounded. Also, the semigroup $e^{-\tau\mathcal{B}(t)}$ generated by $-\mathcal{B}(t)$ satisfies the estimate

$$\|e^{-\tau\mathcal{B}(t)}\|_{L(\mathcal{Z}^\beta,\mathcal{Z}^\alpha)} \leqslant M\tau^{\beta-\alpha}, \tag{2-1}$$

where $0 \leqslant \beta \leqslant \alpha < 1+\varepsilon_0$.

Next we recall the definition of a linear evolution process associated with a family of operators $\{\mathcal{B}(t) : t \in \mathbb{R}\}$.

Definition 2.1. A family $\{L(t,\tau) : t \geqslant \tau \in \mathbb{R}\} \subset L(\mathcal{Z})$ satisfying

(1) $L(\tau,\tau) = I$,

(2) $L(t,\sigma)L(\sigma,\tau) = L(t,\tau)$ for any $t \geqslant \sigma \geqslant \tau$,

(3) $\mathcal{P} \times \mathcal{Z} \ni ((t,\tau),u_0) \mapsto L(t,\tau)v_0 \in \mathcal{Z}$ is continuous, where $\mathcal{P} = \{(t,\tau) \in \mathbb{R}^2 : t \geqslant \tau\}$

is called a *linear evolution process* (*process* for short) or *family of evolution operators*.

If the operator $\mathcal{B}(t)$ is uniformly sectorial and uniformly Hölder continuous, then there exists a linear evolution process $\{L(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ associated with $\mathcal{B}(t)$, which is given by

$$L(t,\tau) = e^{-(t-\tau)\mathcal{B}(\tau)} + \int_\tau^t L(t,s)[\mathcal{B}(\tau) - \mathcal{B}(s)]e^{-(s-\tau)\mathcal{B}(\tau)}\,ds.$$

The evolution process $\{L(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ satisfies the condition

$$\|L(t,\tau)\|_{\mathcal{L}(\mathcal{Z}^\beta,\mathcal{Z}^\alpha)} \leqslant C(\alpha,\beta)(t-\tau)^{\beta-\alpha}, \tag{2-2}$$

where $0 \leqslant \beta \leqslant \alpha < 1+\varepsilon_0$. For more details see [Carvalho and Nascimento 2009] and [Sobolevskiĭ 1961].

2B. *Abstract results on pullback attractors.* In this subsection we will present basic definitions and results of the theory of pullback attractors for nonlinear evolution processes. For more details, we refer to [Caraballo et al. 2010], [Carvalho et al. 2013] and [Chepyzhov and Vishik 2002].

We consider the singularly nonautonomous abstract parabolic problem

$$\begin{cases} \dfrac{du}{dt} = -\mathcal{B}(t)u + g(u), \ t > \tau, \\ u(\tau) = u_0 \in D, \end{cases} \tag{2-3}$$

where the operator $\mathcal{B}(t)$ is uniformly sectorial and uniformly Hölder continuous and the nonlinearity g satisfies conditions which will be specified later. The nonlinear evolution process $\{S(t,\tau): t \geqslant \tau \in \mathbb{R}\}$ associated with $\mathcal{B}(t)$ is given by

$$S(t,\tau)u_0 = L(t,\tau)u_0 + \int_\tau^t L(t,s)g(S(s,\tau))\,ds \quad \text{for all } t \geqslant \tau.$$

Definition 2.2. Let $g : \mathbb{R} \times X^\alpha \to X^\beta,\ \alpha \in [\beta, \beta+1)$, be a continuous function. We say that a continuous function $u : [\tau, \tau+t_0] \to X^\alpha$ is a (*local*) *solution* of (2-3) starting in $u_0 \in X^\alpha$ if $u \in C([\tau, \tau+t_0], X^\alpha) \cap C^1((\tau, \tau+t_0], X^\alpha),\ u(\tau) = u_0$, $u(t) \in D(\mathcal{B}(t))$ for all $t \in (\tau, \tau+t_0]$ and (2-3) is satisfied for all $t \in (\tau, \tau+t_0)$.

We can now state the following result, from [Caraballo et al. 2011]. We also refer to [Carvalho and Nascimento 2009] for a more general version that includes the critical growth case.

Theorem 2.3. *Suppose that the family of operators $\mathcal{B}(t)$ is uniformly sectorial and uniformly Hölder continuous in X^β. If $g : X^\alpha \to X^\beta,\ \alpha \in [\beta, \beta+1)$, is a Lipschitz continuous map in bounded subsets of X^α, then, given $r > 0$, there is a time $t_0 > 0$ such that for all $u_0 \in B_{X^\alpha}(0; r)$ (open ball of radius r centered at the origin of X^α) there exists a unique solution of the problem* (2-3) *starting in u_0 and defined in $[\tau, \tau+t_0]$. Moreover, such solutions are continuous with respect the initial data in $B_{X^\alpha}(0; r)$.*

Next we present several definitions from the theory of pullback attractors, which can be found in [Caraballo et al. 2010; 2013; Chepyzhov and Vishik 2002].

We begin by recalling the definition of Hausdorff semidistance between two subsets A and B of a metric space (X, d):

$$\operatorname{dist}_H(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b).$$

Definition 2.4. Let $\{S(t,\tau): t \geqslant \tau \in \mathbb{R}\}$ be an evolution process in a metric space X. Given A and B subsets of X, we say that A *pullback attracts* B at time t if

$$\lim_{\tau \to -\infty} \operatorname{dist}_H(S(t,\tau)B, A) = 0,$$

where $S(t,\tau)B := \{S(t,\tau)x \in X : x \in B\}$.

Definition 2.5. The *pullback orbit* of a subset $B \subset X$ relative to the evolution process $\{S(t,\tau): t \geqslant \tau \in \mathbb{R}\}$ in the time $t \in \mathbb{R}$ is defined by $\gamma_p(B, t) := \bigcup_{\tau \leqslant t} S(t,\tau)B$.

Definition 2.6. An evolution process $\{S(t,\tau): t \geqslant \tau \in \mathbb{R}\}$ in X is *pullback strongly bounded* if, for each $t \in \mathbb{R}$ and each bounded subset B of X, $\bigcup_{\tau \leqslant t} \gamma_p(B, \tau)$ is bounded.

Definition 2.7. An evolution process $\{S(t,\tau): t \geqslant \tau \in \mathbb{R}\}$ in X is *pullback asymptotically compact* if, for each $t \in \mathbb{R}$, each sequence $\{\tau_n\}$ in $(-\infty, t]$ with $\tau_n \to -\infty$

as $n \to \infty$ and each bounded sequence $\{x_n\}$ in X such that $\{S(t, \tau_n)x_n\} \subset X$ is bounded, the sequence $\{S(t, \tau_n)x_n\}$ is relatively compact in X.

Definition 2.8. We say that a family of bounded subsets $\{B(t) : t \in \mathbb{R}\}$ of X is *pullback absorbing* for the evolution process $\{S(t, \tau) : t \geqslant \tau \in \mathbb{R}\}$ if, for each $t \in \mathbb{R}$ and for any bounded subset B of X, there exists $\tau_0(t, B) \leqslant t$ such that

$$S(t, \tau)B \subset B(t) \quad \text{for all } \tau \leqslant \tau_0(t, B).$$

Definition 2.9. A family of subsets $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ of X is called a *pullback attractor* for the evolution process $\{S(t, \tau) : t \geqslant \tau \in \mathbb{R}\}$ if it is invariant (that is, $S(t, \tau)\mathcal{A}(\tau) = \mathcal{A}(t)$, for any $t \geqslant \tau$), $\mathcal{A}(t)$ is compact for all $t \in \mathbb{R}$, and pullback attracts bounded subsets of X at time t, for each $t \in \mathbb{R}$.

In applications, to prove a process has a pullback attractor, we use Theorem 2.11, proved in [Caraballo et al. 2010], which gives a sufficient condition for existence of a compact pullback attractor. For this, we will need the concept of pullback strongly bounded dissipativeness.

Definition 2.10. An evolution process $\{S(t, \tau) : t \geqslant \tau \in \mathbb{R}\}$ in X is *pullback strongly bounded dissipative* if, for each $t \in \mathbb{R}$, there is a bounded subset $B(t)$ of X which pullback absorbs bounded subsets of X at time s for each $s \leqslant t$; that is, given a bounded subset B of X and $s \leqslant t$, there exists $\tau_0(s, B)$ such that $S(s, \tau)B \subset B(t)$ for all $\tau \leqslant \tau_0(s, B)$.

Now we can present the result which guarantees the existence of pullback attractors for nonautonomous problems; see [Caraballo et al. 2010].

Theorem 2.11. *If an evolution process $\{S(t, \tau) : t \geqslant \tau \in \mathbb{R}\}$ in the metric space X is pullback strongly bounded dissipative and pullback asymptotically compact, then $\{S(t, \tau) : t \geqslant \tau \in \mathbb{R}\}$ has a pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ with the property that $\bigcup_{\tau \leqslant t} \mathcal{A}(\tau)$ is bounded for each $t \in \mathbb{R}$.*

The next result gives sufficient conditions for pullback asymptotic compactness, and its proof can be found in [Caraballo et al. 2010].

Theorem 2.12. *Let $\{S(t, s) : t \geqslant s \in \mathbb{R}\}$ be a pullback strongly bounded evolution process such that $S(t, s) = L(t, s) + U(t, s)$, where there exists a nonincreasing function $k : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$, with $k(\sigma, r) \to 0$ when $\sigma \to \infty$, and for all $s \leqslant t$ and $x \in X$ with $\|x\| \leqslant r$,*

$$\|L(t, s)x\| \leqslant k(t - s, r),$$

and $U(t, s)$ is compact. Then, the family of evolution process $\{S(t, s) : t \geqslant s \in \mathbb{R}\}$ is pullback asymptotically compact.

3. Existence results

In this section we study the existence of global solutions for (1-8). For this, we consider the linear problem associated with (1-1)–(1-3),

$$\begin{cases} w_t = A_{(\kappa)}(t)w, \ t > \tau, \\ w(\tau) = w_0, \ \tau \in \mathbb{R}, \end{cases}$$

where w and w_0 are defined in (1-9) and the linear unbounded operator $A_{(\kappa)}$ is defined by (1-10) and (1-11).

We use the term singularly nonautonomous to express the fact that the unbounded operator $A_{(\kappa)}(t)$ is time-dependent and generates a semigroup that satisfies an estimate as in (2-1).

It is not difficult to see that $0 \in \rho(A_{(\kappa)}(t))$ for any $t \in \mathbb{R}$. Moreover, the operator $A^{-1}_{(\kappa)}(t) : D(A^{-1}_{(\kappa)}(t)) \subset Y \to Y$ is defined by

$$D(A^{-1}_{(\kappa)}(t)) = L^2(\Omega) \times H^{-2}(\Omega) \times H^{-2}(\Omega),$$

where $H^{-2}(\Omega)$ denotes the dual $X^{-\frac{1}{2}}$ of $X^{\frac{1}{2}}$ and

$$\begin{aligned} A^{-1}_{(\kappa)}(t) \begin{bmatrix} u \\ v \\ \theta \end{bmatrix} &= \begin{bmatrix} \frac{a^2}{\kappa(t)}\Lambda^{-\frac{1}{2}} & -\Lambda^{-1} & -\frac{a}{\kappa(t)}\Lambda^{-1} \\ I & 0 & 0 \\ -\frac{a}{\kappa(t)}I & 0 & \frac{1}{\kappa(t)}\Lambda^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} u \\ v \\ \theta \end{bmatrix} \\ &= \begin{bmatrix} \frac{a^2}{\kappa(t)}\Lambda^{-\frac{1}{2}}u - \Lambda^{-1}v - \frac{a}{\kappa(t)}\Lambda^{-1}\theta \\ u \\ -\frac{1}{\kappa(t)}au + \frac{1}{\kappa(t)}\Lambda^{-\frac{1}{2}}\theta \end{bmatrix}. \end{aligned}$$

Proposition 3.1. *Denote by Y_{-1} the extrapolation space of $Y = X^{\frac{1}{2}} \times X \times X$ generated by operator $A^{-1}_{(\kappa)}(t)$. The following equality holds*:

$$Y_{-1} = X \times X^{-\frac{1}{2}} \times X^{-\frac{1}{2}}.$$

Proof. This proof follows the same ideas of the proof in [Bezerra et al. 2018, Proposition 3.1]. □

Remark. Following the same ideas from [Baroun et al. 2009] and [Lasiecka and Triggiani 1998], we conclude that for all t, there exists a positive constant M (independent of t), such that

$$\|(\lambda I + A_{(\kappa)}(t))^{-1}\|_{L(Y)} \leqslant \frac{M}{1+|\lambda|} \quad \text{for all } \lambda \in \mathbb{C} \text{ with } \operatorname{Re}\lambda \geqslant 0.$$

From this we can conclude that $A_{(\kappa)}(t)$ is uniformly sectorial (in Y).

Note that the operator $A_{(\kappa)}(t)$ can be extended to its closed Y_{-1}-realization (see [Amann 1995, p. 262]), which we will still denote by the same symbol so that $A_{(\kappa)}(t)$ considered in Y_{-1} is then the sectorial positive operator (see [Carvalho and Cholewa 2002]). Our next concern will be to obtain embedding of the spaces from the fractional powers scale $Y_{\alpha-1},\ \alpha \geqslant 0$, generated by $(A_{(\kappa)}(t), Y_{-1})$.

Theorem 3.2. *The operators $A_{(\kappa)}(t)$ are uniformly sectorial and the map $\mathbb{R} \ni t \mapsto A_{(\kappa)}(t) \in \mathcal{L}(Y, Y_{-1})$ is uniformly Hölder continuous. Then, there exists a process*

$$\{L(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$$

(*or simply $L(t,\tau)$*) *associated with the operator $A_{(\kappa)}(t)$, that is given by*

$$L(t,\tau)=e^{-(t-\tau)A_{(\kappa)}(\tau)}+\int_\tau^t L(t,s)[A_{(\kappa)}(\tau)-A_{(\kappa)}(s)]e^{-(s-\tau)A_{(\kappa)}(\tau)}\,ds \quad \textit{for all } t\geqslant\tau.$$

The linear evolution operator $\{L(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ satisfies the condition (2-2).

Proof. Following the same ideas from [Carvalho and Cholewa 2002] and [Lasiecka and Triggiani 1998], we can conclude that the operator $A_{(\kappa)}(t)$ is a sectorial positive operator in Y_{-1}. It is not difficult to see that it is also closed and densely defined. Note that for $[u\ \ v\ \ \theta]^T \in X^{\frac{1}{2}} \times X \times X$, and $t, s \in \mathbb{R}$, we can estimate the norm $\|[(A_{(\kappa)}(t) - A_{(\kappa)}(s))[u\ \ v\ \ \theta]^T\|_{X\times X^{-\frac{1}{2}}\times X^{-\frac{1}{2}}}$ using (1-4). In fact,

$$\left\|(A_{(\kappa)}(t)-A_{(\kappa)}(s))\begin{bmatrix}u\\v\\\theta\end{bmatrix}\right\|_{Y_{-1}}=|\kappa(t)-\kappa(s)|\|(-\Delta)\theta\|_{X^{-\frac{1}{2}}}\leqslant c|t-s|^\beta\left\|\begin{bmatrix}u\\v\\\theta\end{bmatrix}\right\|_{X^{\frac{1}{2}}\times X\times X}$$

for any $t, s \in \mathbb{R}$; hence the application $\mathbb{R} \ni t \mapsto A_{(\kappa)}(t) \in \mathcal{L}(Y)$ is uniformly Hölder continuous, and this argument shows that

$$\|A_{(\kappa)}(t) - A_{(\kappa)}(s)\|_{\mathcal{L}(Y,Y_{-1})} \leqslant c|t-s|^\beta.$$

Therefore, there exists a linear evolution process $\{L(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ associated with the operator $A_{(a)}(t)$, that is given by

$$L(t,\tau)=e^{-(t-\tau)A_{(\kappa)}(\tau)}+\int_\tau^t L(t,s)[A_{(\kappa)}(\tau)-A_{(\kappa)}(s)]e^{-(s-\tau)A_{(\kappa)}(\tau)}\,ds \quad \text{for all } t\geqslant\tau.$$

Furthermore, the process $\{L(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ satisfies the condition (2-2). □

The following result is a direct consequence of (1-6), see [Carbone et al. 2011, Lemma 2.4].

Lemma 3.3. *Let $f \in C^1(\mathbb{R})$ be a function such that the condition* (1-6) *holds. Then*

$$|f(s_1) - f(s_2)| \leqslant 2^{\rho-1}c|s_1 - s_2|(1 + |s_1|^{\rho-1} + |s_2|^{\rho-1}) \quad \textit{for all } s_1, s_2 \in \mathbb{R}.$$

Lemma 3.4 [Carbone et al. 2011]. *Assume that* $1 < \rho < \frac{N+4}{N-4}$ *and let* $f \in C^1(\mathbb{R})$ *be a function such that*

$$|f'(s)| \leqslant C(1+|s|^{\rho-1}) \quad \textit{for all } s \in \mathbb{R}.$$

Then there exists $s \in (0,1)$ *such that the Nemytskii operator* $f^e : X^{\frac{1}{2}} \to X^{-\frac{s}{2}}$ *is Lipschitz continuous in bounded subsets of* $X^{\frac{1}{2}}$ *uniformly in* $t \in \mathbb{R}$.

Remark. Since $L^{2N/(N-4)}(\Omega) \hookrightarrow L^2(\Omega)$, it follows from the proof of [Carbone et al. 2011, Lemma 2.5] that $f^e : X^{\frac{1}{2}} \to L^2(\Omega)$ is Lipschitz continuous in bounded subsets; that is,

$$\|f^e(u) - f^e(v)\|_{L^2(\Omega)} \leqslant \tilde{c}\, \|f^e(u) - f^e(v)\|_{L^{\frac{2N}{(N-4)\rho}}(\Omega)} \leqslant \tilde{\tilde{c}}\|u-v\|_{X^{1/2}},$$

with $\tilde{\tilde{c}} = \tilde{\tilde{c}}(\|u\|_{X^{1/2}}, \|v\|_{X^{1/2}})$. The scheme below describes this situation:

$$X^{\frac{1}{2}} \hookrightarrow H^2(\Omega) \hookrightarrow L^{\frac{2N}{N-4}}(\Omega) \xmapsto{f(u)\approx u^\rho} L^{\frac{2N}{(N-4)\rho}}(\Omega) \hookrightarrow L^2(\Omega),$$

where in the last inclusion we use that $\rho < \frac{N}{N-4}$.

Proposition 3.5. *The operator* $A_{(\kappa)}(t)$ *given in* (1-10) *is maximal accretive.*

Proof. This proof is analogous to the proof [Bezerra et al. 2018, Proposition 4.3], and so we omit it. □

Remark. Below we have a partial description of the fractional power spaces scale for $A_{(\kappa)}(t)$. For convenience we denote Y by Y_0, then

$$Y_0 \hookrightarrow Y_{\alpha-1} \hookrightarrow Y_{-1} \quad \text{for all } 0 < \alpha < 1,$$

where

$$Y_{\alpha-1} = [Y_{-1}, Y_0]_\alpha = X^{\frac{\alpha}{2}} \times X^{\frac{\alpha-1}{2}} \times X^{\frac{\alpha-1}{2}},$$

where $[\,\cdot\,,\cdot\,]_\alpha$ denotes the complex interpolation functor (see [Triebel 1978]). The first equality follows from Proposition 3.5 (since $0 \in \rho(A_{(\kappa)}(t))$) (see [Amann 1995, Example 4.7.3(b)]) and the second equality follows from [Carvalho and Cholewa 2002, Proposition 2].

Corollary 3.6. *If* f *is as in Lemma 3.4, then the function* $F : Y \to Y_{\alpha-1}$ ($\alpha \in (0,1)$), *given by* (1-12), *is Lipschitz continuous in bounded subsets of* Y.

Now, Theorem 2.3 guarantees local well posedness for the problem (1-8) in the energy space Y.

Corollary 3.7. *If* f *and* F *are as in Corollary 3.6, then given* $r > 0$, *there is a time* $\tau = \tau(r) > 0$, *such that for all* $w_0 \in B_Y(0; r)$ *there exists a unique solution* $w : [t_0, t_0+\tau] \to Y$ *of the problem* (1-8) *starting in* w_0. *Moreover, such solutions are continuous with respect the initial data in* $B_Y(0; r)$.

Since τ can be chosen uniformly in bounded subsets of Y, the solutions which do not blow up in Y must exist globally. Alternatively, we obtain a uniform in time estimate of $\|(u(t), \partial_t u(t), \theta(t))\|_Y$; such an estimate is needed to justify global solvability of the problem (1-8) in Y.

The total energy of the system $\mathcal{E}(t)$ associated with the solution $(u(t), \partial_t u(t), \theta(t))$ of (1-1)–(1-3) in Y is defined by

$$\mathcal{E}(t) = \tfrac{1}{2}\|u(t)\|^2_{X^{1/2}} + \tfrac{1}{2}\|u_t(t)\|^2_X + \tfrac{1}{2}\|\theta(t)\|^2_X - \int_\Omega \int_0^u f(s)\,ds\,dx. \tag{3-1}$$

It is not difficult to see that the function $t \mapsto \mathcal{E}(t)$ is monotone decreasing along solutions. In fact, using (1-1), we can show that there exists a positive constant c such that

$$\mathcal{E}'(t) \leqslant 0.$$

We obtain (from (1-5)) that for each $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that if

$$\int_\Omega \int_0^{u(\,\cdot\,,t)} f(s)\,ds\,dx \leqslant \varepsilon\|u(\,\cdot\,,t)\|^2_X + C_\varepsilon, \tag{3-2}$$

the property $\mathcal{E}(t) \leqslant \mathcal{E}(\tau)$ offers an a priori estimate of the solution $(u(t), \partial_t u(t), \theta(t))$ in Y. In fact,

$$\frac{1}{2}\left\|\begin{bmatrix} u \\ v \\ \theta \end{bmatrix}\right\|^2_Y \leqslant c\mathcal{E}(\tau) + c\varepsilon_0\|u(\,\cdot\,,t)\|^2_X + C_{\varepsilon_0} \leqslant c\mathcal{E}(\tau) + c\varepsilon_0\left\|\begin{bmatrix} u \\ v \\ \theta \end{bmatrix}\right\|^2_Y + C_{\varepsilon_0},$$

and, if we choose $0 < \varepsilon_0 < \frac{1}{2c}$, we get boundedness as desired; that is,

$$\limsup_{t\to+\infty}\left\|\begin{bmatrix} u \\ v \\ \theta \end{bmatrix}\right\|_Y < +\infty.$$

With this, we ensure that there exists a global solution $w(t)$ for Cauchy problem (1-8) in Y and it defines an evolution process $\{S(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$, that is,

$$S(t,\tau)w_0 = w(t) \quad \text{for all } t \geqslant \tau \in \mathbb{R}.$$

According to [Carvalho and Nascimento 2009],

$$S(t,\tau)w_0 = L(t,\tau)w_0 + \int_\tau^t L(t,s)F(s, S(s,\tau)w_0)\,ds \quad \text{for all } t \geqslant \tau \in \mathbb{R}, \tag{3-3}$$

where $\{L(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ is the linear evolution process associated with the homogeneous problem (1-8).

In order to prove the existence of pullback attractors for (1-1)–(1-3) we use the modified energy method.

Theorem 3.8. *Let $\mathcal{L}$ be the energy functional associated to* (1-1)–(1-3) *given by*

$$\mathcal{L}(t) = M\mathcal{E}(t) + \delta_1 \int_\Omega u u_t \, dx - \delta_2 \int_\Omega u_t \Delta^{-1}\theta \, dx,$$

where $\mathcal{E}$ is defined in (3-1), *and* $0 < \delta_1 < \delta_2 < 1$ *and* $M > 0$ *are appropriate constants.*

(a) *There exist constants* $M_1, M_2 > 0$ *such that*

$$\mathcal{L}'(t) \leqslant -M_1\mathcal{E}(t) + M_2 \tag{3-4}$$

for any $t \geqslant 0$.

(b) *For* $M > 0$ *sufficiently large, there exist constants* $\beta_1, \beta_2, \beta_3 > 0$ *and* $\beta_4 > 0$ *such that*

$$\beta_3\mathcal{E}(t) - \beta_4 \leqslant \mathcal{L}(t) \leqslant \beta_1\mathcal{E}(t) + \beta_2 \tag{3-5}$$

for any $t \geqslant 0$.

Proof. See [Bezerra et al. 2018, Theorems 5.1 and 5.2]. □

Remark. For every $t \in \mathbb{R}$, from (3-2) we have

$$\begin{aligned}\mathcal{E}(t) &= \tfrac{1}{2}\|u(t)\|^2_{X^{1/2}} + \tfrac{1}{2}\|u_t(t)\|^2_X + \tfrac{1}{2}\|\theta(t)\|^2_X - \int_\Omega \int_0^u f(t,s)\, ds\, dx \\ &\geqslant \big(\tfrac{1}{2} - \tfrac{1}{2}\varepsilon C_0\big)\|u(t)\|^2_{X^{1/2}} + \tfrac{1}{2}\|u_t(t)\|^2_X + \tfrac{1}{2}\|\theta(t)\|^2_X - C_\varepsilon,\end{aligned}$$

where ε is such that $\varepsilon < 1/C_0$; that is

$$\|\Delta u(t)\|^2_X + \|u_t(t)\|^2_X + \|\theta(t)\|^2_X \leqslant C_1\mathcal{E}(t) + C'_\varepsilon,$$

where $C_1^{-1} = \min\big\{\big(\tfrac{1}{2} - \tfrac{1}{2}\varepsilon C_0\big), \tfrac{1}{2}\big\}$.

Corollary 3.9. *Under the same conditions as in Theorem 3.8, if $B \subset Y$ is a bounded set, and $(u, v, \theta) : [\tau, \tau + T] \to Y$, $T > 0$, is the solution of* (1-1)–(1-3) *starting in $(u_0, v_0, \theta_0) \in B$, there exist positive constants $\bar{\omega}$, $\gamma_1 = \gamma_1(B)$ and γ_2, such that*

$$\|\Delta u(t)\|^2_X + \|u_t(t)\|^2_X + \|\theta(t)\|^2_X \leqslant \gamma_1 e^{-\bar{\omega}(t-\tau)} + \gamma_2 \tag{3-6}$$

for any $t \in [\tau, \tau + T]$.

Proof. From (3-4) and (3-5), we obtain

$$\mathcal{L}'(t) \leqslant -\sigma_1\mathcal{L}(t) + \sigma_2,$$

where $\sigma_1 = M_1/\beta_1$ and $\sigma_2 = M_1\beta_2/\beta_1 + M_2$, and thus

$$\mathcal{L}(t) \leqslant \mathcal{L}(\tau)e^{-\sigma_1(t-\tau)} + \sigma_2 e^{-\sigma_1 t}\int_\tau^t e^{\sigma_1 s}\, ds \leqslant \mathcal{L}(\tau)e^{-\sigma_1(t-\tau)} + \frac{\sigma_2}{\sigma_1}.$$

Again, by (3-5) together with the remark on page 406, we conclude that

$$\|\Delta u(t)\|_X^2 + \|u_t(t)\|_X^2 + \|\theta(t)\|_X^2 \leqslant \gamma_1 e^{-\sigma_1(t-\tau)} + \gamma_2,$$

where $\gamma_1 = \gamma_1(\mathcal{L}(\tau)) > 0$ and $\gamma_2 > 0$. □

Theorem 3.10. *Under the same conditions as in Theorem 3.8, the problem* (1-1)–(1-3) *has a pullback attractor* $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ *in* Y *and*

$$\bigcup_{t\in\mathbb{R}} \mathcal{A}(t) \subset Y.$$

Proof. From estimate (3-6), it is easy to check that the evolution process $\{S(t, \tau) : t \geqslant \tau \in \mathbb{R}\}$ associated with (1-1)–(1-3) is pullback strongly bounded dissipative in Y.

Hence, applying the same ideas of the proofs of [Bezerra et al. 2018, Theorems 5.1 and 5.2], we conclude that the family of evolution process $\{S(t, \tau) : t \geqslant \tau \in \mathbb{R}\}$ is pullback asymptotically compact (see Theorem 2.12). In fact, from (3-3) we write

$$S(t, \tau)w_0 = L(t, \tau)w_0 + U(t, \tau)w_0,$$

where

$$U(t, \tau)w_0 := \int_\tau^t L(t, s)F(S(t, s)w_0)\, ds \tag{3-7}$$

for any initial condition $w_0 \in Y$.

With the same arguments used in [Bezerra et al. 2018, Theorem 5.1] with $f \equiv 0$ in (1-1) and with the functionals

$$\mathcal{E}(t) = \tfrac{1}{2}\|u(t)\|_{X^{1/2}}^2 + \tfrac{1}{2}\|u_t(t)\|_X^2 + \tfrac{1}{2}\|\theta(t)\|_X^2$$

and

$$\mathcal{L}(t) = M\mathcal{E}(t) + \delta_1\langle u, u_t\rangle_X - \delta_2\langle u_t, (\Delta^{-1}\theta)\rangle_X,$$

we get from (3-4) that there exists $c_1 > 0$ such that

$$\mathcal{L}'(t) \leqslant -c_1\mathcal{E}(t).$$

From arguments used in the proof of [Bezerra et al. 2018, Theorem 5.2] with $f \equiv 0$ in (1-1), by (3-5) we get $c_2, c_3 > 0$ such that

$$c_2\mathcal{E}(t) \leqslant \mathcal{L}(t) \leqslant c_3\mathcal{E}(t) \tag{3-8}$$

and hence

$$\mathcal{L}'(t) \leqslant -c_0\mathcal{L}(t)$$

for some $c_0 > 0$. From this, we obtain

$$\mathcal{L}(t) \leqslant \mathcal{L}(\tau)e^{-c_0(t-\tau)},$$

and thanks to (3-8) we get

$$\mathcal{E}(t) \leqslant \frac{c_3}{c_2}\mathcal{E}(\tau)e^{-c_0(t-\tau)},$$

for some $c_0 > 0$. This ensures that there exist constants $K, \alpha > 0$ such that

$$\|L(t,\tau)\|_{\mathcal{L}(Y)} \leqslant Ke^{-\alpha(t-\tau)} \quad \text{for all } t \geqslant \tau. \tag{3-9}$$

The family of evolution processes $\{U(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ is compact from Y into Y. In fact, the compactness of $U(t,\tau)$ follows easily from

$$X^{1/2} \xrightarrow{f^e} X^{-s/2} \hookrightarrow X^{-1/2},$$

being the last inclusion compact (since $s < 1$; see Lemma 3.4). Thanks to the assumptions on the nonlinearity of f, it follows that f^e is compact from $X^{\frac{1}{2}}$ into $X^{-\frac{1}{2}}$. Taking into account that F is given by (1-12), compactness of f^e implies that F is also compact from Y into Y_{-1}, and since $L(t,\tau)$ is a bounded linear operator from Y_{-1} to Y, the operator $U(t,\tau)$ is compact from Y into Y (see [Hale 1988, Theorem 4.6.1]).

Now, applying Theorem 2.11, we get that the problem (1-1)–(1-3) has a pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ in Y and that $\bigcup_{t\in\mathbb{R}} \mathcal{A}(t) \subset Y$ is bounded. □

4. Regularity of the pullback attractors

In this section we investigate the regularity of the pullback attractors; in fact, we prove that $\bigcup_{t\in\mathbb{R}} \mathcal{A}(t)$ is a bounded subset of Y^1.

Theorem 4.1. *The pullback attractor* $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ *for the problem* (1-1)–(1-3), *obtained in Theorem 3.8, lies in a more regular space than Y; in fact,*

$$\bigcup_{t\in\mathbb{R}} \mathcal{A}(t)$$

is a bounded subset of Y^1.

Proof. The main idea is to use the argument of progressive increases of regularity, following Babin and Vishik [1992] (see also [Carvalho et al. 2013, Chapter 15]).

Let $\xi : \mathbb{R} \to Y$ be a global bounded solution of (1-1). Then, the set $\{\xi(t);\ t \in \mathbb{R}\}$ is a bounded subset of Y. First, observe that we already know that

$$\bigcup_{t\in\mathbb{R}} \mathcal{A}(t) \text{ is bounded in } Y.$$

Hence, if $\xi(\cdot) = (u(\cdot), u_t(\cdot), \theta(\cdot)) : \mathbb{R} \to Y$ is such that $\xi(t) \in \mathcal{A}(t)$ for all $t \in \mathbb{R}$, then

$$\xi(t) = L(t,s)\xi(s) + \int_s^t L(t,\theta)F(\xi(\theta))\,d\theta,$$

and, using the decay of $L(t,s)$ in (3-9) and letting $s \to -\infty$ it follows that

$$\xi(t) = \int_{-\infty}^{t} L(t,\theta) F(\xi(\theta))\, d\theta. \tag{4-1}$$

Now fix $s \in \mathbb{R}$, set $(\mu_0, \mu_1, \vartheta_0) = \xi(s)$, and consider

$$\begin{bmatrix} \mu(t) \\ \mu_t(t) \\ \vartheta(t) \end{bmatrix} = U(t,s) \begin{bmatrix} \mu_0 \\ \mu_1 \\ \vartheta_0 \end{bmatrix} = \int_s^t L(t,\theta) F\left(S(\theta,s) \begin{bmatrix} \mu_0 \\ \mu_1 \\ \vartheta_0 \end{bmatrix} \right) d\theta,$$

where $U(\,\cdot\,,\cdot\,)$ is defined in (3-7). Note that $(\mu(\cdot), \vartheta(\cdot))$ solves the system

$$\begin{cases} \mu_{tt} + \Delta^2 \mu + a\Delta\vartheta = f(\mu(t,s;\mu_0)), & t > s,\ x \in \Omega, \\ \vartheta_t - \kappa(t)\Delta\vartheta - a\Delta\mu_t = 0, & t > s,\ x \in \Omega, \end{cases} \tag{4-2}$$

with

$$\mu(s,x) = \mu_t(s,x) = 0 \quad \text{and} \quad \vartheta(s,x) = 0,\ x \in \Omega. \tag{4-3}$$

This happens inside the pullback attractor $\mathcal{A}(\cdot)$.

To estimate the solution of (4-2)–(4-3) for $(\mu_0, \mu_1, \vartheta_0)$ in a bounded subset B of Y, we again consider the energy functional

$$\begin{aligned} \mathcal{L}_\delta(t) = \tfrac{1}{2} M \|\mu(t)\|^2_{X^{1/2}} + \tfrac{1}{2} M \|\mu_t(t)\|^2_X + \tfrac{1}{2} M \|\vartheta(t)\|^2_X& \\ + \langle \mu(t), \mu_t(t)\rangle_X - \delta_2 \langle \mu_t(t), \Delta^{-1}\vartheta(t)\rangle_X&, \end{aligned}$$

to obtain (we omitted t on the right side in order to simplify the notation)

$$\begin{aligned} \mathcal{L}'_\delta(t) = M\langle \mu, \mu_t\rangle_X + M\langle \mu_t, \mu_{tt}\rangle_X + M\langle \vartheta, \vartheta_t\rangle_X + \|\mu_t\|^2_X& \\ + \langle \mu, \mu_{tt}\rangle_X - \delta_2\langle \mu_{tt}, \Delta^{-1}\vartheta\rangle_X - \delta_2 \langle \mu_t, \Delta^{-1}\vartheta_t\rangle_X&, \end{aligned}$$

and by (4-2) we get

$$\begin{aligned} \mathcal{L}'_\delta(t) = M\langle \mu, \mu_t\rangle_X - M\langle \mu_t, \Delta^2\mu\rangle_X + M\langle \mu_t, f(\mu)\rangle_X - \kappa(t) M \|\vartheta\|^2_{H^1_0(\Omega)}& \\ + (1 - a\delta_2)\|\mu_t\|^2_X - \|\mu\|^2_{X^{1/2}} + \langle \mu, f(\mu)\rangle_X + (\delta_2 - a)\langle \Delta\mu, \vartheta\rangle_X& \\ + a\delta_2 \|\vartheta\|^2_X - \delta_2\langle f(\mu), \Delta^{-1}\vartheta\rangle_X - \delta_2\kappa(t)\langle \mu_t, \vartheta\rangle_X&. \end{aligned}$$

From Poincaré and Young inequalities

$$\begin{aligned} \mathcal{L}'_\delta(t) \leqslant \big(\tfrac{1}{2}\nu_0 M + \tfrac{1}{2}\nu_1 C_a\big)\|\mu\|^2_{X^{1/2}} + \Big(\frac{M}{2\nu_0} + C_a + \tfrac{1}{2}\nu_2\delta_2\kappa_1 + \tfrac{1}{2}M\Big)\|\mu_t\|^2_X& \\ + \Big(+\frac{C_a}{2\nu_1} + a\delta_2\Big)\|\vartheta\|^2_X - \kappa_0\lambda_1 M \|\vartheta\|^2_{H^1_0(\Omega)} + \tfrac{1}{2}\delta_2 \int_\Omega |\Delta^{-1}\vartheta|^2\, dx& \\ + \big(\tfrac{1}{2}M + \tfrac{1}{2}\delta_2\big)\int_\Omega |f(\mu)|^2\, dx + \int_\Omega f(\mu)\mu\, dx&, \end{aligned}$$

where $C_a = 1 - a\delta_2$ and $C_a = \delta_2 - a$.

To deal with the integral terms, just notice that from dissipativeness condition (1-5), for each $\nu > 0$ there exists $C_\nu > 0$ such that

$$\int_\Omega f(\mu)\mu\, dx \leqslant \nu \|\mu\|_X^2 + C_\nu \leqslant m_0 \nu \|\mu\|_{X^{1/2}}^2 + C_\nu,$$

where $m_0 > 0$ is the embedding constant for $\|\cdot\|_X \leqslant m_0 \|\cdot\|_{X^{1/2}}$.

From (1-6), there exists $C > 0$ such that

$$\int_\Omega |f(\mu)|^2\, dx \leqslant C\|\mu\|_X^2 + C\|\mu\|_{L^{2\rho}(\Omega)}^{2\rho}.$$

Since the condition $1 \leqslant \rho < \frac{N}{N-4}$ implies $X^{\frac{1}{2}} \hookrightarrow L^{2\rho}(\Omega)$, we get

$$\int_\Omega |f(\mu)|^2\, dx \leqslant C\|\mu\|_X^2 + \bar{C} \leqslant \bar{C}_1 \|\mu\|_{X^{1/2}}^2 + \bar{C}_2,$$

whenever $\|\mu\|_{X^{1/2}} \leqslant r$ (see [Carbone et al. 2011] and [Carvalho et al. 2009]).

From this it follows that

$$\bigcup_{s\leqslant\tau\leqslant t} U(\tau, s)B \text{ is a bounded subset of } Y. \tag{4-4}$$

Hence $(\varpi, \zeta) = (\mu_t, \vartheta_t)$ solves the system

$$\begin{cases} \varpi_{tt} + \Delta^2\varpi + a\Delta\zeta = f'(\mu(t, s; \mu_0))\varpi(t, s; \mu_0), & t > s,\ x \in \Omega, \\ \zeta_t - \kappa(t)\Delta\zeta - \kappa'(t)\Delta\vartheta - a\Delta\varpi_t = 0, & t > s,\ x \in \Omega, \end{cases} \tag{4-5}$$

with $\varpi(s) = 0$, $\varpi_t(s) = f(\mu_0)$, and $\zeta(s) = 0$.

Finally, now we would like to estimate $(\varpi, \varpi_t, \zeta)$ in Y, but solutions are not regular enough to allow this directly. Instead we work "towards" Y by progressive increases of regularity. For $\alpha > 0$, we define the fractional power spaces $X^\alpha = D(\Lambda^\alpha)$ with the graph norm, and let $X^{-\alpha} = (X^\alpha)'$; see (1-7).

For

$$(\varpi, \varpi_t, \zeta) \in X^{\frac{1-\alpha}{2}} \times X^{-\frac{\alpha}{2}} \times X^{-\frac{\alpha}{2}},$$

we define

$$\begin{aligned} \mathcal{L}_\alpha(t) = \tfrac{1}{2}M\big(\|\varpi(t)\|^2_{X^{\frac{1-\alpha}{2}}} + \|\varpi_t(t)\|^2_{X^{-\frac{\alpha}{2}}} + \|\zeta(t)\|^2_{X^{-\frac{\alpha}{2}}}\big) \\ + \delta_1\langle\varpi, \varpi_t\rangle_{X^{-\frac{\alpha}{2}}} - \delta_2\langle\varpi_t, \Delta^{-1}\zeta\rangle_{X^{-\frac{\alpha}{2}}}. \end{aligned} \tag{4-6}$$

$$\begin{aligned} \mathcal{L}'_\alpha(t) = M\langle\varpi_t, \varpi\rangle_{X^{\frac{1-\alpha}{2}}} + M\langle\varpi_{tt}, \varpi_t\rangle_{X^{-\frac{\alpha}{2}}} + M\langle\zeta_t, \zeta\rangle_{X^{-\frac{\alpha}{2}}} + \delta_1\langle\varpi_t, \varpi_t\rangle_{X^{-\frac{\alpha}{2}}} \\ + \delta_1\langle\varpi, \varpi_{tt}\rangle_{X^{-\frac{\alpha}{2}}} - \delta_2\langle\varpi_{tt}, \Delta^{-1}\zeta\rangle_{X^{-\frac{\alpha}{2}}} \\ - \delta_2\langle\varpi_t, \Delta^{-1}\zeta_t\rangle_{X^{-\frac{\alpha}{2}}}. \end{aligned}$$

Note that from (1-2)–(1-7) (that is, $\Lambda^{\frac{1}{2}} = -\Delta$), (4-5) and (4-6) we have

$$\begin{aligned}\mathcal{L}'_\alpha(t) = {} & M\langle \varpi_t, \varpi\rangle_{X^{\frac{1-\alpha}{2}}} - M\langle \varpi_t, \varpi\rangle_{X^{\frac{1-\alpha}{2}}} - Ma\langle \varpi_t, \Delta\zeta\rangle_{X^{-\frac{\alpha}{2}}} \\ & + M\langle \varpi_t, f'(\mu)\varpi\rangle_{X^{-\frac{\alpha}{2}}} + M\kappa'(t)\langle \zeta, \Delta\vartheta\rangle_{X^{-\frac{\alpha}{2}}} + M\kappa(t)\langle \zeta, \Delta\zeta\rangle_{X^{-\frac{\alpha}{2}}} \\ & + Ma\langle \zeta, \Delta\varpi_t\rangle_{X^{-\frac{\alpha}{2}}} + \delta_1\|\varpi_t\|^2_{X^{-\frac{\alpha}{2}}} - \delta_1\|\varpi\|^2_{X^{\frac{1-\alpha}{2}}} - a\delta_1\langle \varpi, \Delta\zeta\rangle_{X^{-\frac{\alpha}{2}}} \\ & + \delta_1\langle \varpi, f'(\mu)\varpi\rangle_{X^{-\frac{\alpha}{2}}} + \delta_2\langle \zeta, \Lambda^{\frac{1}{2}}\varpi\rangle_{X^{-\frac{\alpha}{2}}} + a\delta_2\|\zeta\|^2_{X^{-\frac{\alpha}{2}}} \\ & - \delta_2\langle \Lambda^{-\frac{1}{2}}\zeta, f'(\mu)\varpi\rangle_{X^{-\frac{\alpha}{2}}} - \delta_2\kappa(t)\langle \varpi_t, \zeta\rangle_{X^{-\frac{\alpha}{2}}} \\ & - \delta_2\kappa'(t)\langle \varpi_t, \vartheta\rangle_{X^{-\frac{\alpha}{2}}} - a\delta_2\|\varpi_t\|^2_{X^{-\frac{\alpha}{2}}};\end{aligned}$$

in other words,

$$\begin{aligned}\mathcal{L}'_\alpha(t) = {} & M\langle \varpi_t, f'(\mu)\varpi\rangle_{X^{-\frac{\alpha}{2}}} + M\kappa'(t)\langle \zeta, \Delta\vartheta\rangle_{X^{-\frac{\alpha}{2}}} + M\kappa(t)\langle \zeta, \Delta\zeta\rangle_{X^{-\frac{\alpha}{2}}} \\ & + (\delta_1 - a\delta_2)\|\varpi_t\|^2_{X^{-\frac{\alpha}{2}}} - \delta_1\|\varpi\|^2_{X^{\frac{1-\alpha}{2}}} - a\delta_1\langle \varpi, \Delta\zeta\rangle_{X^{-\frac{\alpha}{2}}} \\ & + \delta_1\langle \varpi, f'(\mu)\varpi\rangle_{X^{-\frac{\alpha}{2}}} + \delta_2\langle \zeta, \Lambda^{\frac{1}{2}}\varpi\rangle_{X^{-\frac{\alpha}{2}}} \\ & + a\delta_2\|\zeta\|^2_{X^{-\frac{\alpha}{2}}} - \delta_2\langle \Lambda^{-\frac{1}{2}}\zeta, f'(\mu)\varpi\rangle_{X^{-\frac{\alpha}{2}}} \\ & - \delta_2\kappa(t)\langle \varpi_t, \zeta\rangle_{X^{-\frac{\alpha}{2}}} - \delta_2\kappa'(t)\langle \varpi_t, \vartheta\rangle_{X^{-\frac{\alpha}{2}}}.\end{aligned} \tag{4-7}$$

Next, we collect estimates of the terms that appear on the right side of (4-7). First, we deal with the three terms in which the nonlinearity of f' appears explicitly.

Let

$$\alpha_1 := \frac{(\rho - 1)(N - 4)}{4}.$$

Note that since $\rho < \frac{N}{N-4}$, we obtain $\alpha_1 < 1$. We observe that

$$\langle \varpi_t, f'(\mu)\varpi\rangle_{X^{-\frac{\alpha}{2}}} \leqslant \|\varpi_t\|_{X^{-\frac{\alpha}{2}}} \|f'(\mu)\varpi\|_{X^{-\frac{\alpha}{2}}},$$

and using the embedding $X^{\frac{\alpha}{2}} = H^{2\alpha}(\Omega) \hookrightarrow L^p(\Omega)$ (or equivalently $L^{\frac{p}{p-1}}(\Omega) \hookrightarrow X^{-\frac{\alpha}{2}}$) for any $1 < p \leqslant \frac{2N}{N-4\alpha}$ $(0 < \alpha \leqslant \alpha_1)$ and (1-6), we have for some $c_4 > 0$,

$$\begin{aligned}\|f'(\mu)\varpi\|_{X^{-\frac{\alpha}{2}}} & \leqslant c_4\|f'(\mu)\varpi\|_{L^{\frac{2N}{N+4\alpha}}(\Omega)} \\ & \leqslant c_4 C\|\varpi(1 + |\mu|^{\rho-1})\|_{L^{\frac{2N}{N+4\alpha}}(\Omega)} \\ & \leqslant c_4 C\|\varpi\|_X\|1 + |\mu|^{\rho-1}\|_{L^{\frac{N}{2\alpha}}(\Omega)}\end{aligned} \tag{4-8}$$

and so

$$\|f'(\mu)\varpi\|^2_{X^{-\frac{\alpha}{2}}} \leqslant c_4^2 C^2\|\varpi\|^2_X\|1 + |\mu|^{\rho-1}\|^2_{L^{\frac{N}{2\alpha}}(\Omega)}$$

and from (4-4) μ remains in a bounded subset of $X^{\frac{1}{2}}$ and $X^{\frac{1}{2}} \hookrightarrow L^{\frac{(\rho-1)N}{2\alpha}}(\Omega)$ for any $1 < \rho < \frac{N-4+4\alpha}{N-4}$. This implies

$$\begin{aligned}\int_\Omega (1+|\mu|^{\rho-1})^{\frac{N}{2\alpha}}\,dx &\leqslant |\Omega| + \|\mu\|_{L^{\frac{(\rho-1)N}{2\alpha}}(\Omega)}^{\frac{(\rho-1)N-2\alpha}{N(\rho-1)}} \\ &\leqslant |\Omega| + c_5\|\mu\|_{X^{1/2}}^{\frac{(\rho-1)N-2\alpha}{N(\rho-1)}} \\ &\leqslant c_5,\end{aligned}$$

for some $c_5 > 0$. From this, there exists a positive constant $C_{f,1} > 0$ such that

$$\|f'(\mu)\varpi\|^2_{X^{-\frac{\alpha}{2}}} \leqslant C_{f,1}. \tag{4-9}$$

With this we have

$$\begin{aligned}M\langle\varpi_t, f'(\mu)\varpi\rangle_{X^{-\frac{\alpha}{2}}} &\leqslant \frac{\varepsilon_0}{2}\|\varpi_t\|^2_{X^{-\frac{\alpha}{2}}} + \frac{M^2}{2\varepsilon_0}\|f'(\mu)\varpi\|^2_{X^{-\frac{\alpha}{2}}} \\ &\leqslant \frac{\varepsilon_0}{2}\|\varpi_t\|^2_{X^{-\frac{\alpha}{2}}} + \frac{C_{f,1}M^2}{2\varepsilon_0}\end{aligned} \tag{4-10}$$

for some $\varepsilon_0 > 0$.

Again, from (4-9) we obtain

$$\begin{aligned}\delta_1\langle\varpi, f'(\mu)\varpi\rangle_{X^{-\frac{\alpha}{2}}} &\leqslant \tfrac{1}{2}\big(\|\varpi\|^2_{X^{-\frac{\alpha}{2}}} + \delta_1^2\|f'(\mu)\varpi\|^2_{X^{-\frac{\alpha}{2}}}\big) \\ &\leqslant \frac{\varepsilon_1}{2}\|\varpi\|^2_{X^{\frac{1-\alpha}{2}}} + \frac{C_{f,1}\delta_1^2}{2\varepsilon_1}\end{aligned} \tag{4-11}$$

for all $\varepsilon_1 > 0$.

We have the embedding $X^\alpha = H^{4\alpha}(\Omega) \hookrightarrow L^p(\Omega)$ (or equivalently $L^{\frac{p}{p-1}}(\Omega) \hookrightarrow X^{-\alpha}$) for any $1 < p \leqslant \frac{2N}{N-4(1+\alpha)}$. From this and using (1-6), it follows that

$$\begin{aligned}\|f'(\mu)\varpi\|_{X^{-\frac{1+\alpha}{2}}} &\leqslant c_6\|f'(\mu)\varpi\|_{L^{\frac{2N}{N+4(1+\alpha)}}(\Omega)} \\ &\leqslant Cc_6\|\varpi(1+|\mu|^{\rho-1})\|_{L^{\frac{2N}{N+4(1+\alpha)}}(\Omega)} \\ &\leqslant Cc_6\|\varpi\|_X\|1+|\mu|^{\rho-1}\|_{L^{\frac{N}{2(1+\alpha)}}(\Omega)},\end{aligned} \tag{4-12}$$

and so

$$\|f'(\mu)\varpi\|^2_{X^{-\frac{1+\alpha}{2}}} \leqslant C^2c_6^2\|\varpi\|^2_X\|1+|\mu|^{\rho-1}\|^2_{L^{\frac{N}{2(1+\alpha)}}(\Omega)},$$

where

$$\int_\Omega (1+|\mu|^{\rho-1})^{\frac{N}{2(1+\alpha)}}\,dx \leqslant |\Omega| + \|\mu\|^{\frac{2(1+\alpha)}{N(\rho-1)}}_{L^{\frac{(\rho-1)N}{2(1+\alpha)}}(\Omega)} \leqslant |\Omega| + c_7\|\mu\|^{\frac{2(1+\alpha)}{N(\rho-1)}}_{X^{1/2}}. \tag{4-13}$$

In the last estimate we used the embedding

$$X^{\frac{1}{2}} \hookrightarrow L^{\frac{(\rho-1)N}{2(1+\alpha)}}(\Omega), \quad 1 < \rho < \frac{N+4\alpha}{N-4}.$$

From this, there exists a positive constant c_8 such that

$$\|1+|\mu|^{\rho-1}\|^2_{L^{\frac{N}{2(1+\alpha)}}(\Omega)} \leqslant c_8.$$

From (4-12) and (4-13), we have

$$-\delta_2\langle\Lambda^{-\frac{1}{2}}\zeta, f'(\mu)\varpi\rangle_{X^{-\frac{\alpha}{2}}} = -\delta_2\langle\Lambda^{-\frac{1}{2}-\frac{\alpha}{2}}\zeta, \Lambda^{-\frac{\alpha}{2}} f'(\mu)\varpi\rangle_X \leqslant \delta_2\|\zeta\|_{X^{-\frac{\alpha}{2}}}\|f'(\mu)\varpi\|_{X^{-\frac{1+\alpha}{2}}} \leqslant \tfrac{1}{2}\|\zeta\|^2_{X^{-\frac{\alpha}{2}}} + \tfrac{1}{2}\delta_2^2 C_{f,2} \tag{4-14}$$

for some $C_{f,2} > 0$.

Finally, we consider the last term:

$$-\delta_2\kappa(t)\langle\varpi_t, \zeta\rangle_{X^{-\frac{\alpha}{2}}} \leqslant \delta_2\kappa_1\|\varpi_t\|_{X^{-\frac{\alpha}{2}}}\|\zeta\|_{X^{-\frac{\alpha}{2}}} \leqslant \frac{\delta_2\kappa_1}{2}\big(\|\varpi_t\|^2_{X^{-\frac{\alpha}{2}}} + \|\zeta\|^2_{X^{-\frac{\alpha}{2}}}\big).$$

Since ϑ remains in a bounded subset of X (see (4-4)), for $\frac{1}{2} \leqslant \alpha < 1$ we have the embedding $L^2 = X^0 \hookrightarrow X^{\frac{1}{4}-\frac{\alpha}{2}}$ $\big(\frac{1}{4}-\frac{\alpha}{2} \leqslant 0\big)$ and

$$\begin{aligned} M\kappa'(t)\langle\zeta, \Delta\vartheta\rangle_{X^{-\frac{\alpha}{2}}} &= -M\kappa'(t)\langle\zeta, \Lambda^{\frac{1}{2}}\vartheta\rangle_{X^{-\frac{\alpha}{2}}} \\ &= -M\kappa'(t)\langle\Lambda^{\frac{1}{4}-\frac{\alpha}{2}}\zeta, \Lambda^{\frac{1}{4}-\frac{\alpha}{2}}\vartheta\rangle_X \\ &\leqslant M\kappa_1\|\Lambda^{\frac{1}{4}-\frac{\alpha}{2}}\zeta\|_X\|\Lambda^{\frac{1}{4}-\frac{\alpha}{2}}\vartheta\|_X \\ &= M\kappa_1\|\zeta\|_{X^{\frac{1}{4}-\frac{\alpha}{2}}}\|\vartheta\|_{X^{\frac{1}{4}-\frac{\alpha}{2}}} \\ &\leqslant \tfrac{1}{2}M\kappa_1 c(\|\zeta\|^2_X + \|\vartheta\|^2_X) \leqslant C_3 \end{aligned}$$

for some $c > 0$ and $C_3 > 0$.

It is not difficult to see that

$$\langle\zeta, \Delta\zeta\rangle_{X^{-\frac{\alpha}{2}}} = -\|\zeta\|^2_{X^{\frac{1-2\alpha}{4}}}.$$

From this we conclude that

$$M\kappa(t)\langle\zeta, \Delta\zeta\rangle_{X^{-\frac{\alpha}{2}}} \leqslant -M\kappa_0\|\zeta\|^2_{X^{\frac{1-2\alpha}{4}}} \leqslant -Mc_2\kappa_0\|\zeta\|^2_{X^{-\frac{\alpha}{2}}}.$$

Using Cauchy and Young inequalities we obtain

$$\begin{aligned} -a\delta_1\langle\varpi, \Delta\zeta\rangle_{X^{-\frac{\alpha}{2}}} &= a\delta_1\langle\varpi, \Lambda^{\frac{1}{2}}\zeta\rangle_{X^{-\frac{\alpha}{2}}} = a\delta_1\langle\Lambda^{\frac{1}{2}}\varpi, \zeta\rangle_{X^{-\frac{\alpha}{2}}} \\ &\leqslant a\delta_1\|\varpi\|_{X^{\frac{1-\alpha}{2}}}\|\zeta\|_{X^{-\frac{\alpha}{2}}} \leqslant \tfrac{1}{2}a\delta_1\varepsilon_2\|\varpi\|^2_{X^{\frac{1-\alpha}{2}}} + \frac{a\delta_1}{2\varepsilon_2}\|\zeta\|^2_{X^{-\frac{\alpha}{2}}} \end{aligned}$$

for all $\varepsilon_2 > 0$, and

$$\delta_2\langle\zeta, \Lambda^{\frac{1}{2}}\varpi\rangle_{X^{-\frac{\alpha}{2}}} \leqslant \delta_2\|\varpi\|_{X^{\frac{1-\alpha}{2}}}\|\zeta\|_{X^{-\frac{\alpha}{2}}} \leqslant \tfrac{1}{2}\delta_2\varepsilon_3\|\varpi\|^2_{X^{\frac{1-\alpha}{2}}} + \frac{\delta_2}{2\varepsilon_3}\|\zeta\|^2_{X^{-\frac{\alpha}{2}}}$$

for all $\varepsilon_3 > 0$.

Finally, from $X \hookrightarrow X^{-\frac{\alpha}{2}}$ we obtain

$$
\begin{aligned}
-\delta_2\kappa'(t)\langle\varpi_t,\vartheta\rangle_{X^{-\frac{\alpha}{2}}} &\leqslant \delta_2\kappa_0\|\varpi_t\|_{X^{-\frac{\alpha}{2}}}\|\vartheta\|_{X^{-\frac{\alpha}{2}}}\\
&\leqslant \tfrac{1}{2}\delta_2\kappa_0\big(\|\varpi_t\|^2_{X^{-\frac{\alpha}{2}}}+\|\vartheta\|^2_{X^{-\frac{\alpha}{2}}}\big)\\
&\leqslant \tfrac{1}{2}\delta_2\kappa_0\big(\|\varpi_t\|^2_{X^{-\frac{\alpha}{2}}}+c\|\vartheta\|^2_{X}\big)\\
&\leqslant \tfrac{1}{2}\delta_2\kappa_0\|\varpi_t\|^2_{X^{-\frac{\alpha}{2}}}+c
\end{aligned}\tag{4-15}
$$

for some $c > 0$.

Now combining (4-7) with (4-10), (4-11) and (4-14)–(4-15) we conclude that

$$
\begin{aligned}
\mathcal{L}'_\alpha(t) \leqslant{}& -\tfrac{1}{2}\delta_1\|\varpi\|^2_{X^{\frac{1-\alpha}{2}}} - \big(\tfrac{1}{2}\delta_1-\tfrac{1}{2}\varepsilon_1-\tfrac{1}{2}a\delta_1\varepsilon_2-\tfrac{1}{2}\delta_2\varepsilon_3\big)\|\varpi\|^2_{X^{\frac{1-\alpha}{2}}}\\
&-\big(a\delta_2-\tfrac{1}{2}\varepsilon_0-\delta_1-\tfrac{1}{2}\delta_2\kappa_1-\tfrac{1}{2}\delta_2\kappa_0\big)\|\varpi_t\|^2_{X^{-\frac{\alpha}{2}}}\\
&-\Big(Mc_2\kappa_0-\tfrac{1}{2}-\frac{a\delta_1}{2\varepsilon_2}-a\delta_2-\frac{\delta_2}{2\varepsilon_3}-\tfrac{1}{2}\delta_2\kappa_1\Big)\|\zeta\|^2_{X^{-\frac{\alpha}{2}}}\\
&+\frac{C_{f,1}M^2}{2\varepsilon_0}+\frac{C_{f,1}\delta_1^2}{2\varepsilon_1}+\tfrac{1}{2}\delta_2^2C_{f,2}+C_3+c.
\end{aligned}
$$

In other words,

$$
\begin{aligned}
\mathcal{L}'_\alpha(t) \leqslant{}& -\tfrac{1}{2}\delta_1\|\varpi\|^2_{X^{\frac{1-\alpha}{2}}} - \big(\tfrac{1}{2}\delta_1-\tfrac{1}{2}\varepsilon_1-\tfrac{1}{2}a\delta_1\varepsilon_2-\tfrac{1}{2}\delta_2\varepsilon_3\big)\|\varpi\|^2_{X^{\frac{1-\alpha}{2}}}\\
&-\big(a\delta_2-\tfrac{1}{2}\varepsilon_0-\delta_1-\delta_2\kappa_1\big)\|\varpi_t\|^2_{X^{-\frac{\alpha}{2}}}\\
&-\Big(Mc_2\kappa_0-\tfrac{1}{2}-\frac{a\delta_1}{2\varepsilon_2}-a\delta_2-\frac{\delta_2}{2\varepsilon_3}-\tfrac{1}{2}\delta_2\kappa_1\Big)\|\zeta\|^2_{X^{-\frac{\alpha}{2}}}\\
&+\frac{C_{f,1}M^2}{2\varepsilon_0}+\frac{C_{f,1}\delta_1^2}{2\varepsilon_1}+\tfrac{1}{2}\delta_2^2C_{f,2}+C_3+c.
\end{aligned}
$$

Now, it is enough to choose $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$, respectively, such that

$$
\varepsilon_1=\frac{\delta_1}{3},\quad \varepsilon_2=\frac{1}{3a},\quad \text{and}\quad \varepsilon_3=\frac{\delta_1}{3\delta_2},
$$

and so

$$
\begin{aligned}
\mathcal{L}'_\alpha(t) \leqslant{}& -\tfrac{1}{2}\delta_1\|\varpi\|^2_{X^{\frac{1-\alpha}{2}}} - \big(a\delta_2-\tfrac{1}{2}\varepsilon_0-\delta_1-\delta_2\kappa_1\big)\|\varpi_t\|^2_{X^{-\frac{\alpha}{2}}}\\
&-\Big(Mc_2\kappa_0-\tfrac{1}{2}-\frac{a\delta_1}{2\varepsilon_2}-a\delta_2-\frac{\delta_2}{2\varepsilon_3}-\tfrac{1}{2}\delta_2\kappa_1\Big)\|\zeta\|^2_{X^{-\frac{\alpha}{2}}}\\
&+\frac{C_{f,1}M^2}{2\varepsilon_0}+\frac{C_{f,1}\delta_1^2}{2\varepsilon_1}+\tfrac{1}{2}\delta_2^2C_{f,2}+C_3+c.
\end{aligned}
$$

Since $\delta_1 < \delta_2$, if we assume that $a < 1+\kappa_1$, then choosing $\varepsilon_0 > 0$ such that

$$
\varepsilon_0 < 2(a-\kappa_1)\delta_2-2\delta_1 < 2(\delta_2-\delta_1),
$$

we conclude that

$$
a\delta_2-\tfrac{1}{2}\varepsilon_0-\delta_1-\delta_2\kappa_1 > 0.
$$

Now, it is enough to choose $M > 0$ sufficiently large such that

$$Mc_2\kappa_0 - \tfrac{1}{2} - \frac{a\delta_1}{2\varepsilon_2} - a\delta_2 - \frac{\delta_2}{2\varepsilon_3} - \tfrac{1}{2}\delta_2\kappa_1 > 0,$$

and so there exist $\ell_1 > 0$ and $\ell_2 > 0$ such that

$$\mathcal{L}'_\alpha(t) \leqslant -\ell_1(\|\varpi\|^2_{X^{\frac{1-\alpha}{2}}} + \|\varpi_t\|^2_{X^{-\frac{\alpha}{2}}} + \|\zeta\|^2_{X^{-\frac{\alpha}{2}}}) + \ell_2.$$

From this, (4-1), and the fact $\mathcal{A}(t) = \{\xi(t); \xi(t)$ is a global bounded solution$\}$ we obtain

$$\bigcup_{t\in\mathbb{R}} \mathcal{A}(t) \text{ is bounded in } X^{\frac{2-\alpha_1}{2}} \times X^{\frac{1-\alpha_1}{2}} \times X^{\frac{1-\alpha_1}{2}}. \tag{4-16}$$

Using (4-16) and restarting from (4-8) and (4-12) with $\alpha_2 = (1+\rho)\alpha_1 - \rho$ it follows that

$$\bigcup_{t\in\mathbb{R}} \mathcal{A}(t) \text{ is bounded in } X^{\frac{2-\alpha_2}{2}} \times X^{\frac{1-\alpha_2}{2}} \times X^{\frac{1-\alpha_2}{2}}.$$

Iterating this procedure a finite number of times, we can now show that

$$\bigcup_{t\in\mathbb{R}} \mathcal{A}(t) \text{ is bounded in } X^1 \times X^{\frac{1}{2}} \times X^{\frac{1}{2}},$$

which implies

$$\sup_{\xi\in\mathcal{A}} \sup_{t\in\mathbb{R}} \{\|\xi(t)\|_Y, \|\xi(t)\|_{Y^1}, \|\xi_t(t)\|_Y\} < \infty,$$

where $\mathcal{A}$ is the set of global bounded solutions for (1-8). □

5. Upper semicontinuity of the pullback attractors

From the results obtained in the previous section, we can prove a result on upper semicontinuity of the pullback attractors with respect to the functional parameter κ. Let $\{\kappa_\varepsilon : \varepsilon \in [0, 1]\}$ be the family of real valued functions of one real variable satisfying (1-4), and denote by $S_{(\kappa_\varepsilon)}(\cdot\,,\cdot)$ and $\{\mathcal{A}_{(\kappa_\varepsilon)}(t) : t \in \mathbb{R}\}$, respectively, the evolution process and its pullback attractor associated with problem (1-1)–(1-3).

Moreover, we will assume that

$$\|\kappa_\varepsilon - \kappa_0\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{as } \varepsilon \to 0^+.$$

Now we are able to present the main result of this section.

Theorem 5.1. *For each $a > 0$ and $\varepsilon \in [0, 1]$, let $w^{(\varepsilon)}(\cdot) = S_{(\kappa_\varepsilon)}(\cdot\,, \tau)w_0$ be the solution of* (1-8) *in Y. Then, for each $T > 0$, $w^{(\varepsilon)}$ converges to $w^{(0)}$ in $C([0, T]; Y)$ as $\varepsilon \to 0^+$. Moreover, the family of pullback attractors $\{\mathcal{A}_{(\kappa_\varepsilon)}(t) : t \in \mathbb{R}\}$ is upper semicontinuous in $\varepsilon = 0$.*

Proof. For each $w_0 \in Y$, consider $w^{(\varepsilon)} = S_{(\kappa_\varepsilon)}(t,\tau)w_0$ and $w^{(0)} = S_{(\kappa_0)}(t,\tau)w_0$. Let $w = w^{(\varepsilon)} - w^{(0)}$, with $w^{(\varepsilon)} = (u^{(\varepsilon)}, {u^{(\varepsilon)}}_t, \theta^{(\varepsilon)})$ and $w^{(0)} = (u^{(0)}, {u^{(0)}}_t, \theta^{(0)})$ ($u = u^{(\varepsilon)} - u^{(0)}$ and $\theta = \theta^{(\varepsilon)} - \theta^{(0)}$). Then, for all $t > \tau$ and $x \in \Omega$,

$$\begin{cases} u_{tt} + \Delta^2 u + a\Delta\theta = f(u^{(\varepsilon)}) - f(u^{(0)}), \\ \theta_t - \kappa_\varepsilon(t)\Delta\theta^{(\varepsilon)} + \kappa_0(t)\Delta\theta^{(0)} - a\Delta u_t = 0. \end{cases}$$

Multiplying the first equation by u_t and multiplying the second equation by θ, we get

$$\tfrac{1}{2}\frac{d}{dt}\int_\Omega |u_t|^2\,dx + \tfrac{1}{2}\frac{d}{dt}\int_\Omega |\Delta u|^2\,dx + a\int_\Omega \Delta\theta u_t\,dx = \int_\Omega [f(u^{(\varepsilon)}) - f(u^{(0)})]u_t\,dx,$$

$$\tfrac{1}{2}\frac{d}{dt}\int_\Omega |\theta|^2\,dx + \kappa_\varepsilon(t)\int_\Omega |\nabla\theta|^2\,dx - (\kappa_\varepsilon - \kappa_0)(t)\int_\Omega \Delta\theta^{(0)}\theta\,dx - a\int_\Omega \Delta u_t\theta\,dx = 0.$$

Since

$$\int_\Omega \Delta\theta u_t\,dx = \int_\Omega \Delta u_t\theta\,dx,$$

it follows that

$$\begin{aligned} \text{(5-1)}\quad \frac{d}{dt}&\left(\tfrac{1}{2}\int_\Omega |u_t|^2\,dx + \tfrac{1}{2}\int_\Omega |\Delta u|^2\,dx + \tfrac{1}{2}\int_\Omega |\theta|^2\,dx\right) \\ &= -\kappa_\varepsilon(t)\int_\Omega |\nabla\theta|^2\,dx \\ &\qquad + (\kappa_\varepsilon - \kappa_0)(t)\int_\Omega \Delta\theta^{(0)}\theta\,dx + \int_\Omega [f(u^{(\varepsilon)}) - f(u^{(0)})]u_t\,dx \\ &\leqslant (\kappa_\varepsilon - \kappa_0)(t)\int_\Omega \Delta\theta^{(0)}\theta\,dx + \int_\Omega [f(u^{(\varepsilon)}) - f(u^{(0)})]u_t\,dx. \end{aligned}$$

Using Young's inequality

$$\int_\Omega \nabla\theta^{(0)}\nabla\theta\,dx \leqslant \tfrac{1}{2}\int_\Omega |\nabla\theta^{(0)}|^2\,dx + \tfrac{1}{2}\int_\Omega |\nabla\theta|^2\,dx,$$

by (5-1) we conclude that

$$\begin{aligned} \text{(5-2)}\quad \frac{d}{dt}&\big(\|u_t\|_X^2 + \|u\|_{X^{1/2}}^2 + \|\theta\|_X^2\big) \\ &\leqslant \|\theta^{(0)}\|_{H_0^1(\Omega)}^2\|\kappa_\varepsilon - \kappa_0\|_{L^\infty(\mathbb{R})} + \|\theta\|_{H_0^1(\Omega)}^2\|\kappa_\varepsilon \\ &\qquad - \kappa_0\|_{L^\infty(\mathbb{R})} + 2\int_\Omega [f(u^{(\varepsilon)}) - f(u^{(0)})]u_t\,dx, \end{aligned}$$

and from Section 4 we have $w = w^{(\varepsilon)} - w^{(0)}$, with $w^{(\varepsilon)} = (u^{(\varepsilon)}, u_t^{(\varepsilon)}, \theta^{(\varepsilon)})$ and $w^{(0)} = (u^{(0)}, {u^{(0)}}_t, \theta^{(0)})$ ($u = u^{(\varepsilon)} - u^{(0)}$ and $\theta = \theta^{(\varepsilon)} - \theta^{(0)}$) bounded in $X^1 \times X^{\frac{1}{2}} \times X^{\frac{1}{2}}$.

Hence there exists $C > 0$ independent of $\varepsilon > 0$ such that

$$\|\theta\|_{H_0^1(\Omega)} \leqslant C \quad (\text{and } \|\theta^{(0)}\|_{H_0^1(\Omega)} \leqslant C) \tag{5-3}$$

for any $\varepsilon \in [0, 1)$.

Combining (5-2) and (5-3) we conclude that

$$\frac{d}{dt}(\|u_t\|_X^2+\|u\|_{X^{1/2}}^2+\|\theta\|_X^2)\leqslant C\|\kappa_\varepsilon-\kappa_0\|_{L^\infty(\mathbb{R})}+2\int_\Omega [f(u^{(\varepsilon)})-f(u^{(0)})]u_t\,dx, \tag{5-4}$$

where $C > 0$ is independent of ε.

From the mean value theorem, assumption (1-6) and $\frac{(\rho-1)}{2\rho} + \frac{1}{2\rho} + \frac{1}{2} = 1$, we obtain

$$\begin{aligned}\int_\Omega [f(u^{(\varepsilon)}) - f(u^{(0)})]u_t\,dx & \\ &\leqslant \|f'(\xi u^{(\varepsilon)} + (1-\xi)u^{(0)})\|_{L^{\frac{2\rho}{\rho-1}}(\Omega)} \|u^{(\varepsilon)} - u^{(0)}\|_{L^{2\rho}(\Omega)} \|u_t\|_{L^2(\Omega)} \\ &\leqslant C_{0,f}\|u^{(\varepsilon)} - u^{(0)}\|_{L^{2\rho}(\Omega)}\|u_t\|_X,\end{aligned}$$

and so

$$\int_\Omega [f(u^{(\varepsilon)}) - f(u^{(0)})]u_t\,dx \leqslant C_{0,f}\|u\|_{L^{2\rho}(\Omega)}\|u_t\|_X \leqslant C_{0,f}\|u\|_{X^{1/2}}\|u_t\|_X$$

for some $\xi \in [0, 1]$ and such that $C_{0,f} > 0$ is a constant depending on the initial data.

Hence, from Young's inequality,

$$\int_\Omega [f(u^{(\varepsilon)}) - f(u^{(0)})]u_t\,dx \leqslant C'(\|u\|_{X^{1/2}}^2 + \|u_t\|_X^2 + \|\theta\|_X^2) \tag{5-5}$$

for some $C' > 0$ independent of ε.

Therefore, by (5-4) and (5-5)

$$\frac{d}{dt}(\|u\|_{X^{1/2}} + \|u_t\|_X + \|\theta\|_X) \leqslant C\|\kappa_\varepsilon - \kappa_0\|_{L^\infty(\mathbb{R})} + C''(\|u\|_{X^{1/2}}^2 + \|u_t\|_X^2 + \|\theta\|_X^2),$$

and consequently

$$\|u_t\|_X^2 + \|u\|_{X^{1/2}}^2 + \|\theta\|_X^2 \leqslant C\|\kappa_\varepsilon - \kappa_0\|_{L^\infty(\mathbb{R})}(t-\tau)e^{C''(t-\tau)}, \quad t > \tau, \tag{5-6}$$

that is, $w^{(\varepsilon)}(= S_{(\kappa_\varepsilon)}(t, \tau)w_0)$ goes to $w^{(0)}$ $(= S_{(\kappa_0)}(t, \tau)w_0)$ as $\varepsilon \to 0^+$ in compact subsets of $\mathbb{R}$ uniformly for w_0 in bounded subsets of Y.

For $\delta > 0$ given, let $\tau \in \mathbb{R}$ be such that $\operatorname{dist}(S_{(\kappa_0)}(t, \tau)B, \mathcal{A}_{(\kappa_0)}(t)) < \frac{\delta}{2}$ for all $t \in \mathbb{R}$, $B \supset \bigcup_{s\leqslant t} \mathcal{A}_{(\kappa_\varepsilon)}(s)$, is a bounded set in Y whose existence is guaranteed by Theorem 2.11.

Using (5-6), there exists $\varepsilon_0 > 0$ such that

$$\sup_{u_\varepsilon \in \mathcal{A}_{(\kappa_\varepsilon)}(\tau)} \|S_{(\kappa_\varepsilon)}(t, \tau)u_\varepsilon - S_{(\kappa_0)}(t, \tau)u_\varepsilon\|_Y < \frac{\delta}{2}$$

for all $\varepsilon < \varepsilon_0$. Finally,

$$\begin{aligned}
&\operatorname{dist}_H(\mathcal{A}_{(\kappa_\varepsilon)}(t), \mathcal{A}_{(\kappa_0)}(t)) \\
&\qquad \leqslant \operatorname{dist}_H(S_{(\kappa_\varepsilon)}(t,\tau)\mathcal{A}_{(\kappa_\varepsilon)}(\tau), S_{(\kappa_0)}(t,\tau)\mathcal{A}_{(\kappa_\varepsilon)}(\tau)) \\
&\qquad\qquad\qquad + \operatorname{dist}_H(S_{(\kappa_0)}(t,\tau)\mathcal{A}_{(\kappa_\varepsilon)}(\tau), S_{(\kappa_0)}(t,\tau)\mathcal{A}_{(\kappa_0)}(\tau)) \\
&\qquad \leqslant \sup_{u_\varepsilon \in \mathcal{A}_{(\kappa_\varepsilon)}(\tau)} \operatorname{dist}_H(S_{(\kappa_\varepsilon)}(t,\tau)u_\varepsilon, S_{(\kappa_0)}(t,\tau)u_\varepsilon) \\
&\qquad\qquad\qquad + \operatorname{dist}_H(S_{(\kappa_0)}(t,\tau)\mathcal{A}_{(\kappa_\varepsilon)}(\tau), \mathcal{A}_{(\kappa_0)}(t)) \\
&\qquad < \frac{\delta}{2} + \frac{\delta}{2} = \delta,
\end{aligned}$$

which proves the upper semicontinuity of the family of attractors. □

Remark. Observe that, if we assume that a is continuously differentiable in $\mathbb{R}$, and there exist positive constants a_0 and a_1 such that

$$0 < a_0 \leqslant a(t), a'(t) \leqslant a_1 \quad \text{for all } t \in \mathbb{R},$$

then all the calculations in this paper remain valid for $a(t)$ instead of a.

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FLANK D. M. BEZERRA
DEPARTAMENTO DE MATEMÁTICA
UNIVERSIDADE FEDERAL DA PARAÍBA
JOÃO PESSOA
BRAZIL
flank@mat.ufpb.br

VERA L. CARBONE
carbone@dm.ufscar.br

MARCELO J. D. NASCIMENTO
marcelo@dm.ufscar.br

KARINA SCHIABEL
schiabel@dm.ufscar.br

(*all*)
DEPARTAMENTO DE MATEMÁTICA
UNIVERSIDADE FEDERAL DE SÃO CARLOS
SÃO CARLOS
BRAZIL

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