SELF-DUAL EINSTEIN ACH METRICS AND CR GJMS
OPERATORS IN DIMENSION THREE

Taiji Marugame

SELF-DUAL EINSTEIN ACH METRICS AND CR GJMS OPERATORS IN DIMENSION THREE

Taiji Marugame

By refining Matsumoto's construction of Einstein ACH metrics, we construct a one-parameter family of ACH metrics which solve the Einstein equation to infinite order and have a given three-dimensional CR structure at infinity. When the parameter is 0, the metric is self-dual to infinite order. As an application, we give another proof of the fact that three-dimensional CR manifolds admit CR invariant powers of the sublaplacian (CR GJMS operators) of all orders, which has been proved by Gover and Graham. We also prove the convergence of the formal solutions when the CR structure is real analytic.

1. Introduction

The GJMS operator $\mathcal{P}_{2k}$ on a conformal manifold of dimension N is an invariant linear differential operator acting on conformal densities of weight $k - N/2$ whose principal part is the power Δ^k of the laplacian [Graham et al. 1992]. It plays an important role in geometric analysis on conformal manifolds, and is also related to a fundamental curvature quantity, called the Q-curvature, whose integral gives a global conformal invariant [Fefferman and Graham 2002; Fefferman and Hirachi 2003; Graham and Zworski 2003]. The GJMS operator is constructed via the (Fefferman–Graham) ambient metric [2012] or equivalently via the Poincaré metric whose boundary at infinity is the given conformal manifold [Fefferman and Graham 2002; Graham and Zworski 2003]. The ambient metric is a formal solution to the Ricci flat equation, which corresponds to the Einstein equation for the Poincaré metric. When the dimension N is odd, the equation can be solved to infinite order and $\mathcal{P}_{2k}$ is defined for all $k \geq 1$. On the other hand, when N is even, an obstruction to the existence of a formal solution appears, and $\mathcal{P}_{2k}$ can only be defined for $1 \leq k \leq N/2$ due to the ambiguity of the ambient metric at higher orders. Moreover, it is known that this result of the existence of $\mathcal{P}_{2k}$ is sharp [Gover and Hirachi 2004].

The CR counterpart of these operators are CR invariant powers of the sublaplacian

$$P_{2k} : \mathcal{E}\left(\frac{k-n-1}{2}, \frac{k-n-1}{2}\right) \longrightarrow \mathcal{E}\left(\frac{-k-n-1}{2}, \frac{-k-n-1}{2}\right)$$

on a $(2n+1)$-dimensional CR manifold M, which are called the CR GJMS operators or the Gover–Graham operators [Gover and Graham 2005; Hislop et al. 2008]. Here, $\mathcal{E}(w, w')$ is a complex line bundle over M called the CR density of weight (w, w'); see Section 2A for the definition. One can associate a conformal structure to a circle bundle over M, called the Fefferman conformal structure [1976], and apply the GJMS construction to produce P_{2k} for $1 \leq k \leq n+1$. Gover and Graham [2005] gave more operators by using techniques of CR tractor calculus; they proved that for each (w, w') such that $k = w + w' + n + 1 \in \mathbb{N}_+$ and $(w, w') \notin \mathbb{N} \times \mathbb{N}$, there exists a CR invariant linear differential operator $P_{w,w'} : \mathcal{E}(w, w') \to \mathcal{E}(w - k, w' - k)$ whose principal part is Δ_b^k. In cases where $w = w'$, these operators provide CR invariant modifications of Δ_b^k for all k with $k \equiv n \mod 2$. When $n = 1$, even more operators can be constructed: CR structure is a Cartan geometry modeled on the CR sphere $S^{2n+1} = \mathrm{SU}(n+1, 1)/P$, where P is the isotropy subgroup of a point in S^{2n+1}, and three-dimensional CR structure has a special feature from this viewpoint in that P is a Borel subgroup. Then the BGG machinery developed in [Čap et al. 2001] gives operators $P_{w,w'}$ for $(w, w') \in \mathbb{N} \times \mathbb{N}$ when $n = 1$. Thus one has:

Theorem 1.1 [Gover and Graham 2005, Theorem 1.3]. *Suppose M is a three-dimensional strictly pseudoconvex CR manifold. For each (w, w') such that $k = w + w' + 2 \in \mathbb{N}_+$, there exists a CR invariant linear differential operator $P_{w,w'} : \mathcal{E}(w, w') \to \mathcal{E}(w - k, w' - k)$ on M, whose principal part is Δ_b^k.*

In this paper, we provide a unified proof of Theorem 1.1 for cases in which $w = w'$. To this end, we construct an ACH (asymptotically complex hyperbolic) metric on a manifold with boundary M whose Taylor expansion along M is completely determined by local data of M. Our ACH metric is a refinement of the ACH Einstein metric which Matsumoto [2013; 2014] constructed for partially integrable CR manifolds. To state the results, let us recall some basic notions related to ACH metrics. Let M be a $(2n+1)$-dimensional strictly pseudoconvex partially integrable CR manifold. Namely, M has a contact distribution $H \subset TM$ together with an almost complex structure $J \in \mathrm{End}(H)$, and the eigenspace $T^{1,0}M \subset \mathbb{C}H$ with the eigenvalue i satisfies the partial integrability: $[\Gamma(T^{1,0}M), \Gamma(T^{1,0}M)] \subset \Gamma(\mathbb{C}H)$. A Θ-structure on a manifold $\overline{X}$ with boundary M is a conformal class $[\Theta]$ of sections $\Theta \in \Gamma(M, T^*\overline{X})$ such that $\Theta|_{TM}$ is a contact form on M. A diffeomorphism which preserves a Θ-structure is called a Θ-diffeomorphism. On the product $M \times [0, \infty)_\rho$, we define the standard Θ-structure by extending each contact form θ on M to Θ so that $\Theta(\partial/\partial\rho) = 0$. Fix a contact form θ on M and let $\{T, Z_\alpha\}$ be an admissible frame. We take the local frame $\{\mathbf{Z}_\infty = \rho\partial_\rho, \mathbf{Z}_0 = \rho^2 T, \mathbf{Z}_\alpha = \rho Z_\alpha, \mathbf{Z}_{\overline{\alpha}} = \rho Z_{\overline{\alpha}}\}$ and

its dual coframe $\{\boldsymbol{\theta}^\infty, \boldsymbol{\theta}^0, \boldsymbol{\theta}^\alpha, \boldsymbol{\theta}^{\overline{\alpha}}\}$ on $M \times (0, \infty)_\rho$. Then for any ACH metric g on X, there exists a Θ-diffeomorphism $\Phi : M \times [0, \infty)_\rho \to \overline{X}$ which is defined near M and restricts to the identity on M, such that $\Phi^*g = g_{IJ}\boldsymbol{\theta}^I\boldsymbol{\theta}^J$ satisfies

$$g_{\infty\infty} = 4, \qquad g_{\infty 0} = g_{\infty\alpha} = 0, \quad g_{00} = 1 + O(\rho),$$
$$g_{0\alpha} = O(\rho), \quad g_{\alpha\beta} = O(\rho), \qquad g_{\alpha\overline{\beta}} = h_{\alpha\overline{\beta}} + O(\rho),$$

where $h_{\alpha\overline{\beta}}$ is the Levi form on M. The CR manifold M is called the *CR structure at infinity* of g. Matsumoto [2013; 2014] proved that for any partially integrable CR manifold M, there exists an ACH metric g on $M \times [0, \infty)_\rho$ which satisfies

$$E_{IJ} := \mathrm{Ric}_{IJ} + \frac{n+2}{2}g_{IJ} = O(\rho^{2n+2}),$$
$$\mathrm{Scal} = -(n+1)(n+2) + O(\rho^{2n+3}),$$

where Ric is the Ricci tensor and Scal is the scalar curvature. Up to the pull-back by a Θ-diffeomorphism which fixes M, such a metric is unique modulo tensors which have $O(\rho^{2n+2})$ coefficients and $O(\rho^{2n+3})$ trace in the frame $\{Z_I\}$. The order $O(\rho^{2n+2})$ in the above equation is optimal in general since $(\rho^{-2n-2}E_{\alpha\beta})|_M$ is independent of the choice of a solution g and defines a CR invariant tensor $\mathcal{O}_{\alpha\beta} \in \mathcal{E}_{\alpha\beta}(-n, -n)$, called the *CR obstruction tensor*. Matsumoto [2016] generalized the CR GJMS operators P_{2k} to the partially integrable case by using Dirichlet-to-Neumann type operators for the eigenvalue equations of the laplacian of g, but the order is again restricted to $1 \le k \le n+1$ due to the presence of the obstruction.

If we confine ourselves to the case where M is an integrable CR manifold, there is a possibility to refine the construction of ACH metrics. In fact, the CR obstruction tensor vanishes for integrable CR manifolds, in particular for three-dimensional CR manifolds since the CR structure is always integrable in this dimension. However, we need an additional normalization condition on the metric to ensure the uniqueness since the Einstein equation does not determine the $O(\rho^{2n+2})$-term of the metric. A possible normalization is the Kähler condition; Fefferman [1976] constructed an approximate solution to the complex Monge–Ampère equation on a strictly pseudoconvex domain Ω with boundary M and defined a Kähler metric which satisfies $E_{IJ} = O(\rho^{2n+4})$ as an ACH metric on the "square root" of Ω. However, this construction also has an obstruction $\mathcal{O} \in \mathcal{E}(-n-2, -n-2)$, called the *CR obstruction density*, and the metric is only determined modulo $O(\rho^{2n+4})$.

In this paper, we show that the self-dual equation $W^- = 0$ works as a better normalization when M is three-dimensional. The anti-self-dual part W^- of the Weyl curvature is connected to the Ricci tensor by the Bianchi identity

(1-1) $$\nabla^I W^-_{IJKL} = C^-_{JKL},$$

where C^-_{IJK} is the anti self-dual part of the Cotton tensor C_{IJK}, which is defined by

$C_{IJK} := \nabla_K P_{IJ} - \nabla_J P_{IK}$ with the Schouten tensor $P_{IJ} = \frac{1}{2} \operatorname{Ric}_{IJ} - \frac{1}{12} \operatorname{Scal} g_{IJ}$. It follows from (1-1) that the equation $E_{IJ} = O(\rho^4)$ implies $W^-_{IJKL} = O(\rho^4)$, and it turns out that the further normalization $W^-_{IJKL} = O(\rho^5)$ determines g_{IJ} modulo $O(\rho^5)$. We can then solve $E_{IJ} = O(\rho^6)$, which implies $W^-_{IJKL} = O(\rho^6)$. In the next step, besides the Einstein equation, we have freedom to prescribe the value of

$$\eta := \left(\rho^{-6} W^-_{\infty 0 \infty 0} \right) \big|_M .$$

If the Taylor coefficients of g_{IJ} along M have universal expressions in terms of pseudohermitian structure, η defines a CR invariant of weight $(-3, -3)$ (see Lemma 4.1). Thus, we should prescribe η to be a CR invariant in order to obtain a CR invariant normalization condition. It is known that a CR invariant in $\mathcal{E}(-3, -3)$ on a three-dimensional CR manifold is unique up to a constant multiple [Graham 1987], so there is no choice but to set $\eta = \lambda \mathcal{O}$ with a constant $\lambda \in \mathbb{R}$. After this step, the Einstein equation determines g_{IJ} to infinite order, and in the case $\lambda = 0$, the self-duality follows automatically from (1-1). Thus our main theorem reads as follows:

Theorem 1.2. *Let M be a three-dimensional strictly pseudoconvex CR manifold, and let $\lambda \in \mathbb{R}$. Then there exists an ACH metric g^λ_{IJ} on $M \times [0, \infty)_\rho$ which has M as the CR structure at infinity and satisfies*

$$\operatorname{Ric}_{IJ} + \tfrac{3}{2} g^\lambda_{IJ} = O(\rho^\infty), \quad W^-_{IJKL} = O(\rho^6), \quad \eta = \lambda \mathcal{O},$$

where η is the density defined by (4-1). The metric g^λ_{IJ} is unique modulo $O(\rho^\infty)$ up to the pull-back by a Θ-diffeomorphism which fixes M. Moreover, g^0_{IJ} satisfies $W^-_{IJKL} = O(\rho^\infty)$.

The Taylor coefficients of g^λ_{IJ} along the boundary have universal expressions in terms of the pseudohermitian structure for a fixed contact form.

By applying the construction of the CR GJMS operators via the ACH metric [Matsumoto 2016], we obtain the following theorem, which is a special case of Theorem 1.1:

Theorem 1.3. *Let M be a three-dimensional strictly pseudoconvex CR manifold, and let $\lambda \in \mathbb{R}$. Then, there exists a CR invariant linear differential operator*

$$P^\lambda_{2k} : \mathcal{E}(k/2 - 1, \ k/2 - 1) \to \mathcal{E}(-k/2 - 1, \ -k/2 - 1)$$

which is a polynomial in λ of degree $\leq k/3$, and has the principal part Δ^k_b.

Let us mention a similar construction in conformal geometry. Fefferman and Graham [2012] constructed a formal solution to the self-dual Einstein equation for the Poincaré metric with a given three-dimensional conformal manifold $\mathcal{M}$ as its conformal infinity. Thus our result is a CR analogue of their construction. When $\mathcal{M}$ is real analytic, LeBrun [1982] showed by twistor methods that there exists a real

analytic self-dual Einstein metric on $\mathcal{M} \times (0, \epsilon)$ with the conformal infinity $\mathcal{M}$. The metric of Fefferman and Graham gives the Taylor expansion of LeBrun's metric. In the CR case, Biquard [2007] showed the existence of a self-dual Einstein ACH metric with a given real analytic CR three-manifold as its infinity by using twistor methods. Thus our formal solution g_{IJ}^0 gives the Taylor expansion of Biquard's metric. In this paper, we prove the convergence of g_{IJ}^λ by applying the results of Baouendi and Goulaouic [1976] on singular nonlinear Cauchy problems.

Theorem 1.4. *Suppose M is a real analytic strictly pseudoconvex CR manifold of dimension three. Then the formal solution g_{IJ}^λ in Theorem 1.2 converges to a real analytic ACH metric near M.*

This paper is organized as follows. In Section 2, we review pseudohermitian geometry on a CR manifold and basic notions on ACH metrics. By following Matsumoto [2013], we describe the Levi-Civita connection of an ACH metric in terms of the extended Tanaka–Webster connection. In Section 3, we clarify the relationship between the Einstein equation and the self-dual equation, and compute the variation of curvature quantities under a perturbation of the metric. Section 4 is devoted to the proof of Theorem 1.2; we construct a one-parameter family of formal solutions to the Einstein equation and examine the dependence on the parameter. Then, in Section 5 we use these metrics to construct the CR GJMS operators and prove Theorem 1.3. Finally, in Section 6 we show the convergence of the formal solutions in the case where M is a real analytic CR manifold.

2. CR structure and ACH metric

2A. *Pseudohermitian geometry.* Let M be a $(2n+1)$-dimensional C^∞ manifold. A pair (H, J) is called a *CR structure* on M if H is a rank $2n$ subbundle of TM and J is an almost complex structure on H which satisfies the (formal) integrability condition

$$[\Gamma(T^{1,0}M), \Gamma(T^{1,0}M)] \subset \Gamma(T^{1,0}M),$$

where $T^{1,0}M \subset \mathbb{C}H$ is the eigenspace of J with the eigenvalue i. We note that the integrability condition automatically holds when M is three-dimensional. For any real 1-form θ such that $\mathrm{Ker}\,\theta = H$, we define the Levi form h_θ by

$$h_\theta(Z, \overline{W}) = -\,\mathrm{id}\,\theta(Z, \overline{W})$$

for $Z, W \in T^{1,0}M$. We say the CR structure is *strictly pseudoconvex* if h_θ is positive definite for some θ. Since $h_{f\theta} = f h_\theta$ for any function f, such θ is determined up to a multiple by a positive function. When M is strictly pseudoconvex, H defines a contact structure, so we call θ a contact form. The *Reeb vector field* is the real

vector field T uniquely determined by the conditions

$$\theta(T) = 1, \quad T \lrcorner d\theta = 0.$$

Let $\{Z_\alpha\}$ be a local frame for $T^{1,0}M$. If we put $Z_{\overline{\alpha}} := \overline{Z}_\alpha$, then $\{T, Z_\alpha, Z_{\overline{\alpha}}\}$ gives a local frame for $\mathbb{C}TM$, which we call an *admissible frame*. The dual coframe $\{\theta, \theta^\alpha, \theta^{\overline{\alpha}}\}$ is called an *admissible coframe* and satisfies

$$d\theta = i h_{\alpha\overline{\beta}} \theta^\alpha \wedge \theta^{\overline{\beta}},$$

where $h_{\alpha\overline{\beta}} = h_\theta(Z_\alpha, Z_{\overline{\beta}})$.

The *CR canonical bundle* is defined by $K_M := \wedge^{n+1}(T^{0,1}M)^\perp \subset \wedge^{n+1}\mathbb{C}T^*M$, where $T^{0,1}M := \overline{T^{1,0}M}$. When K_M^{-1} admits an $(n+2)$-nd root $\mathcal{E}(1,0)$, the *CR density bundle* is defined by

$$(2\text{-}1) \qquad\qquad \mathcal{E}(w, w') = \mathcal{E}(1,0)^{\otimes w} \otimes \overline{\mathcal{E}(1,0)}^{\otimes w'}$$

for each $(w, w') \in \mathbb{C}^2$ with $w - w' \in \mathbb{Z}$. In this paper, we restrict ourselves to the cases $w = w'$. In these cases, the definition (2-1) is independent of the choice of $\mathcal{E}(1,0)$ so we can define $\mathcal{E}(w,w)$ without assuming the global existence of $\mathcal{E}(1,0)$. We also denote the space of sections of these bundles by the same symbols, and call them *CR densities*.

For any contact form θ, there exists a local nonvanishing section ζ of K_M, unique up to a multiple of a $U(1)$-valued function, which satisfies

$$\theta \wedge (d\theta)^n = i^{n^2} n! \theta \wedge (T \lrcorner \zeta) \wedge (T \lrcorner \overline{\zeta}).$$

Then, the weighted contact form $\boldsymbol{\theta} := \theta \otimes |\zeta|^{-2/(n+2)} \in \Gamma(T^*M \otimes \mathcal{E}(1,1))$ is defined globally and independent of the choice of θ. Thus, there is a one-to-one correspondence between the set of contact forms and the set of positive sections $\tau \in \mathcal{E}(1,1)$, called *CR scales*. We define the CR invariant weighted Levi form $\boldsymbol{h}_{\alpha\overline{\beta}} := \tau h_{\alpha\overline{\beta}}$ by putting a weight to h_θ with the CR scale τ corresponding to θ. We raise and lower the indices of tensors on $\mathbb{C}H$ by $\boldsymbol{h}_{\alpha\overline{\beta}}$ and its inverse $\boldsymbol{h}^{\alpha\overline{\beta}}$, which has weight $(-1, -1)$.

For a fixed contact form θ, we can define a canonical linear connection ∇ on TM, called the *Tanaka–Webster connection*. It preserves $T^{1,0}M$ and satisfies $\nabla T = 0$, $\nabla h_\theta = 0$. In an admissible frame $\{T, Z_\alpha, Z_{\overline{\alpha}}\}$, the connection 1-forms $\omega_\beta{}^\alpha$ satisfy the structure equation

$$d\theta^\alpha = \theta^\beta \wedge \omega_\beta{}^\alpha + A^\alpha{}_{\overline{\beta}} \boldsymbol{\theta} \wedge \theta^{\overline{\beta}}.$$

The tensor $A_{\alpha\beta} := \overline{A_{\overline{\alpha}\overline{\beta}}}$ satisfies $A_{\alpha\beta} = A_{\beta\alpha}$ and is called the *Tanaka–Webster torsion tensor*. We use the index 0 for the direction of T, and we denote the components of covariant derivatives of a tensor by indices preceded by a comma,

e.g., $A_{\alpha\gamma,\overline{\beta}} = \nabla_{\overline{\beta}} A_{\alpha\gamma}$. We omit the comma for covariant derivatives of a function. The curvature form $\Omega_\alpha{}^\beta = d\omega_\alpha{}^\beta - \omega_\alpha{}^\gamma \wedge \omega_\gamma{}^\beta$ is given by

$$(2\text{-}2) \quad \Omega_\alpha{}^\beta = R_\alpha{}^\beta{}_{\gamma\overline{\mu}}\theta^\gamma \wedge \theta^{\overline{\mu}} + A_{\alpha\gamma,}{}^\beta\theta^\gamma \wedge \boldsymbol{\theta} - A^\beta{}_{\overline{\gamma},\alpha}\theta^{\overline{\gamma}} \wedge \boldsymbol{\theta}$$
$$-i A_{\alpha\gamma}\theta^\gamma \wedge \theta^\beta + i \boldsymbol{h}_{\alpha\overline{\gamma}} A^\beta{}_{\overline{\mu}}\theta^{\overline{\gamma}} \wedge \theta^{\overline{\mu}}.$$

The tensor $R_\alpha{}^\beta{}_{\gamma\overline{\mu}}$ is called the *Tanaka–Webster curvature tensor*. Taking traces with the weighted Levi form, we define the Tanaka–Webster Ricci tensor $\mathrm{Ric}_{\alpha\overline{\beta}} := R_\gamma{}^\gamma{}_{\alpha\overline{\beta}}$ and the Tanaka–Webster scalar curvature $\mathrm{Scal} := \mathrm{Ric}_\alpha{}^\alpha$. The *sublaplacian* is the differential operator $\Delta_b : \mathcal{E}(w, w') \to \mathcal{E}(w-1, w'-1)$ defined by

$$\Delta_b f = -\boldsymbol{h}^{\alpha\overline{\beta}}(\nabla_\alpha\nabla_{\overline{\beta}} + \nabla_{\overline{\beta}}\nabla_\alpha)f.$$

If we rescale the contact form as $\widehat{\theta} = e^\Upsilon \theta$, the Tanaka–Webster connection and its curvature quantities satisfy transformation formulas involving the derivatives of the scaling factor Υ; see, e.g., [Lee 1988]. We note that in dimension three the rank of $T^{1,0}M$ is 1 and the curvature form (2-2) is reduced to

$$\Omega_1{}^1 = \mathrm{Scal}\,\boldsymbol{h}_{1\overline{1}}\theta^1 \wedge \theta^{\overline{1}} + A_{11,}{}^1\theta^1 \wedge \boldsymbol{\theta} - A^1{}_{\overline{1},1}\theta^{\overline{1}} \wedge \boldsymbol{\theta}.$$

Also, in this dimension, M is locally CR diffeomorphic to the standard sphere S^3 if and only if the *Cartan tensor*

$$Q_{11} := \frac{1}{6}\,\mathrm{Scal}_{11} + \frac{i}{2}\,\mathrm{Scal}\,A_{11} - A_{11,0} - \frac{2i}{3}A_{11,\overline{1}}{}^{\overline{1}}$$

vanishes identically. The Cartan tensor is a CR invariant tensor of weight $(-1, -1)$. We also have a CR invariant density defined by

$$(2\text{-}3) \qquad \mathcal{O} := (\nabla^1\nabla^1 - iA^{11})\,Q_{11} \in \mathcal{E}(-3, -3),$$

called the *obstruction density*. It follows from the Bianchi identity for the Cartan tensor that $\mathcal{O}$ is a real density [Cheng and Lee 1990]. There is also a CR invariant density, called the obstruction density, on higher-dimensional CR manifolds and it appears as the logarithmic coefficient in the asymptotic expansion of the solution to the complex Monge–Ampère equation on a strictly pseudoconvex domain [Lee and Melrose 1982]. In dimension three, a CR invariant of weight $(-3, -3)$ is unique up to a constant multiple [Graham 1987], so it is necessarily a multiple of $\mathcal{O}$.

2B. *ACH metrics.*

The ACH metric was introduced by Epstein, Melrose and Mendoza [Epstein et al. 1991] as a generalization of the complex hyperbolic metric on the ball. In this paper, we define it by using the characterization via the normal form.

Let X be the interior of a $(2n+2)$-dimensional C^∞ manifold whose boundary M is equipped with a strictly pseudoconvex CR structure (H, J). A conformal class $[\Theta]$

in $\Gamma(M, T^*\overline{X})$ is called a Θ-*structure* if $\Theta|_{TM}$ gives a contact form on M for each $\Theta \in [\Theta]$. We call $(\overline{X}, [\Theta])$ a Θ-*manifold*. Let $(\overline{X}', [\Theta'])$ be another Θ-manifold with the same boundary M. Then, a diffeomorphism Φ from a neighborhood of M in $\overline{X}$ to a neighborhood of M in $\overline{X}'$ is called a Θ-*diffeomorphism* if it fixes M and satisfies $[\Phi^*\Theta'] = [\Theta]$. We take a boundary-defining function $\rho \in C^\infty(\overline{X})$ which is positive on X. A vector field V on $\overline{X}$ is called a Θ-*vector field* if it satisfies

$$V|_M = 0, \quad \widetilde{\Theta}(V) = O(\rho^2),$$

where $\widetilde{\Theta}$ is an arbitrary extension of a $\Theta \in [\Theta]$. Note that the definition is independent of the choice of Θ and $\widetilde{\Theta}$. We extend $\{d\rho, \widetilde{\Theta}\}$ to a local coframe $\{d\rho, \widetilde{\Theta}, \alpha^1, \dots, \alpha^{2n}\}$ for $T^*\overline{X}$ near M. Let $\{N, T, Y_1, \dots, Y_{2n}\}$ be the dual frame. Then, any Θ-vector field V can be written as

$$V = V^\infty(\rho N) + V^0(\rho^2 T) + V^i(\rho Y_i), \quad V^\infty, V^0, V^i \in C^\infty(\overline{X}).$$

If we take another local coframe $\{d\rho', \widetilde{\Theta}', \alpha'^i\}$ and its dual $\{N', T', Y_i'\}$, then the transition function between $\{\rho N, \rho^2 T, \rho Y_i\}$ and $\{\rho' N', \rho'^2 T', \rho' Y_i'\}$ is smooth and nondegenerate up to M, so there exists a vector bundle $^\Theta T\overline{X}$ over $\overline{X}$ for which $\{\rho N, \rho^2 T, \rho Y_i\}$ gives a local frame. A Θ-vector field is identified with a section of this bundle and we call $^\Theta T\overline{X}$ the Θ-*tangent bundle*. A fiber metric on $^\Theta T\overline{X}$ is called a Θ-*metric*. Since the restriction $^\Theta T\overline{X}|_X$ is canonically isomorphic to TX, a Θ-metric defines a Riemannian metric on X. A local frame $\{Z_I\}$ of $^\Theta T\overline{X}$ is called a Θ-*frame*. We also consider the dual $^\Theta T^*\overline{X}$ of the Θ-tangent bundle and various tensor bundles, whose sections are called Θ-*tensors*. A Θ-tensor is said to be $O(\rho^m)$ if each component in a Θ-frame is $O(\rho^m)$. The Θ-vector fields are closed under the Lie bracket, and those which vanish at a fixed point $p \in M$ form an ideal. Thus the fiber $^\Theta T_p\overline{X}$ becomes a Lie algebra, which we call the *tangent algebra*.

The product $M \times [0, \infty)_\rho$ has a canonical Θ-structure, called the *standard Θ-structure*, which is defined by extending each contact form θ on M to $\Theta \in \Gamma(M, T^*\overline{X})$ with $\Theta(\partial/\partial\rho) = 0$. Let θ be a contact form and $\{T, Z_\alpha, Z_{\overline{\alpha}}\}$ an admissible frame for $\mathbb{C}TM$. We extend $\{T, Z_\alpha, Z_{\overline{\alpha}}\}$ to $M \times [0, \infty)_\rho$ in the trivial way, and define a (complexified) Θ-frame $\{Z_I\}$ by

$$Z_\infty = \rho\partial_\rho, \quad Z_0 = \rho^2 T, \quad Z_\alpha = \rho Z_\alpha, \quad Z_{\overline{\alpha}} = \rho Z_{\overline{\alpha}},$$

where $\partial_\rho = \partial/\partial\rho$. A Θ-metric g on $M \times [0, \infty)_\rho$ is called a *normal form ACH metric* if the components $g_{IJ} = g(Z_I, Z_J)$ satisfy

$$(2\text{-}4) \qquad \begin{aligned} & g_{\infty\infty} = 4, \qquad g_{\infty 0} = g_{\infty\alpha} = 0, \qquad g_{00} = 1 + O(\rho), \\ & g_{0\alpha} = O(\rho), \quad g_{\alpha\beta} = O(\rho), \qquad g_{\alpha\overline{\beta}} = h_{\alpha\overline{\beta}} + O(\rho), \end{aligned}$$

where $h_{\alpha\overline{\beta}} = h_\theta(Z_\alpha, Z_{\overline{\beta}})$. On a general Θ-manifold $(\overline{X}, [\Theta])$, the ACH metric is defined as follows:

Definition 2.1. A Θ-metric g on $\overline{X}$ is called an ACH metric if for any contact form θ on M, there exist a neighborhood $U \subset \overline{X}$ of M and a Θ-diffeomorphism $\Phi_\theta : M \times [0, \infty)_\rho \to U$ such that $\Phi_\theta^* g$ is a normal form ACH metric.

We remark that there is an alternative definition of the ACH metric which involves only the boundary value of g; see [Matsumoto 2013, Definition 4.6].

The germ of Φ_θ along M is unique, and we call $\rho \circ \Phi_\theta^{-1}$ the *model defining function* for θ. We identify a neighborhood of M in $\overline{X}$ with $M \times [0, \epsilon)_\rho$ through Φ_θ and regard $\{Z_I\}$ as a Θ-frame on $\overline{X}$. The following proposition will be used in the proof of Lemma 4.1.

Proposition 2.2. *The boundary values $Z_\infty|_M, Z_0|_M$ are independent of θ and determined only by the ACH metric g.*

Proof. By strict pseudoconvexity of (H, J), the derived Lie algebras of the tangent algebra $^\Theta T_p \overline{X}$ at a point $p \in M$ are given by

$$\mathcal{D}^1 := [^\Theta T_p \overline{X}, {}^\Theta T_p \overline{X}] = \text{span}\{(Z_0)_p, (Z_\alpha)_p, (Z_{\overline{\alpha}})_p\},$$
$$\mathcal{D}^2 := [\mathcal{D}^1, \mathcal{D}^1] = \text{span}\{(Z_0)_p\}.$$

Thus, $(Z_\infty)_p$ and $(Z_0)_p$ are oriented bases of $(\mathcal{D}^1)^\perp$ and $\mathcal{D}^2$ respectively. Since they are normalized by $|(Z_\infty)_p|_g^2 = 4$ and $|(Z_0)_p|_g^2 = 1$, they are independent of θ. $\square$

Let $\theta, \widehat{\theta} = e^\Upsilon \theta$ be contact forms on M and ρ and $\widehat{\rho}$ be the corresponding model-defining functions. Then there exists a positive function f on $\overline{X}$ such that $\widehat{\rho} = f\rho$. Since the Reeb vector fields are related as $\widehat{T} = e^{-\Upsilon}(T - ih^{\alpha\overline{\gamma}}\Upsilon_{\overline{\gamma}}Z_\alpha + ih^{\gamma\overline{\alpha}}\Upsilon_\gamma Z_{\overline{\alpha}})$, we have

$$\widehat{Z}_0 = \widehat{\rho}^2\widehat{T} = e^{-\Upsilon}f^2 Z_0 + O(\rho)$$

as a Θ-vector field, where we regard Υ as a function on a neighborhood of M. It follows from $\widehat{Z}_0|_M = Z_0|_M$ that $f|_M = e^{\Upsilon/2}$. Thus we have

$$(2\text{-}5) \qquad\qquad \widehat{\rho} = e^{\Upsilon/2}\rho + O(\rho^2).$$

In particular, a contact form is recovered from the 1-jet of the corresponding model-defining function along the boundary.

2C. *The Levi-Civita connection.* Let g be an ACH metric on a Θ-manifold $(\overline{X}, [\Theta])$ with boundary M. Here and after, we assume that M is three-dimensional. We lower and raise the indices of Θ-tensors by g_{IJ} and its inverse g^{IJ}. In order to describe the Levi-Civita connection of g, we introduce an extension of the Tanaka–Webster connection by following Matsumoto. We refer the reader to [Matsumoto 2013, §6.2] or [Matsumoto 2014, §4] for a more detailed exposition.

Let θ be a contact form on M. We identify a neighborhood of M in $\overline{X}$ with $M \times [0, \epsilon)_\rho$ by the Θ-diffeomorphism determined by θ. We take an admissible frame $\{T, Z_1, Z_{\overline{1}}\}$ and define the *extended Tanaka–Webster connection* $\overline{\nabla}$ on $T\overline{X}$ by

$$\overline{\nabla}\partial_\rho = 0, \qquad \overline{\nabla}_{\partial_\rho}T = \overline{\nabla}_{\partial_\rho}Z_1 = 0,$$

$$\overline{\nabla}_T Z_1 = \nabla_T^{\mathrm{TW}} Z_1, \qquad \overline{\nabla}_{Z_1} Z_1 = \nabla_{Z_1}^{\mathrm{TW}} Z_1, \qquad \overline{\nabla}_{Z_{\overline{1}}} Z_1 = \nabla_{Z_{\overline{1}}}^{\mathrm{TW}} Z_1,$$

where ∇^{TW} denotes the Tanaka–Webster connection associated with θ. Then, $\overline{\nabla}$ is a *Θ-connection* in the sense that if V, W are Θ-vector fields, so is the covariant derivative $\overline{\nabla}_V W$. We take the Θ-frame $\{Z_I\} = \{\rho\partial_\rho, \rho^2 T, \rho Z_1, \rho Z_{\overline{1}}\}$ and define the Christoffel symbols $\overline{\Gamma}_{IJ}{}^K$ by $\overline{\nabla}_{Z_I} Z_J = \overline{\Gamma}_{IJ}{}^K Z_K$. A simple calculation shows that

(2-6)
$$\overline{\Gamma}_{\infty\infty}{}^\infty = 1, \qquad \overline{\Gamma}_{\infty 0}{}^0 = 2, \qquad \overline{\Gamma}_{\infty 1}{}^1 = 1,$$
$$\overline{\Gamma}_{01}{}^1 = \rho^2 \Gamma_{01}{}^1, \qquad \overline{\Gamma}_{11}{}^1 = \rho\Gamma_{11}{}^1, \qquad \overline{\Gamma}_{\overline{1}1}{}^1 = \rho\Gamma_{\overline{1}1}{}^1,$$

where $\Gamma_{ij}{}^k$ are the Christoffel symbols of ∇^{TW} with respect to $\{T, Z_1, Z_{\overline{1}}\}$; the components which cannot be obtained by taking complex conjugates of (2-6) are 0. It follows from (2-6) that the components of the covariant derivative of a Θ-tensor $S_{I_1 \cdots I_p}{}^{J_1 \cdots J_q}$ are computed as

(2-7)
$$\overline{\nabla}_\infty S_{I_1 \cdots I_p}{}^{J_1 \cdots J_q} = \left(\rho\partial_\rho - \#(I_1 \cdots I_p) + \#(J_1 \cdots J_q)\right) S_{I_1 \cdots I_p}{}^{J_1 \cdots J_q},$$
$$\overline{\nabla}_0 S_{I_1 \cdots I_p}{}^{J_1 \cdots J_q} = \rho^2 \nabla_0^{\mathrm{TW}} S_{I_1 \cdots I_p}{}^{J_1 \cdots J_q},$$
$$\overline{\nabla}_1 S_{I_1 \cdots I_p}{}^{J_1 \cdots J_q} = \rho\nabla_1^{\mathrm{TW}} S_{I_1 \cdots I_p}{}^{J_1 \cdots J_q},$$

where $\#(I_1 \cdots I_p) := p + (\text{the number of 0s})$ and we regard S as a tensor on $\mathbb{C}H$ when we apply ∇^{TW} to it [Matsumoto 2013, Lemma 6.2; 2014, (4.9)].

The torsion tensor $\overline{T}_{IJ}{}^K$ and the curvature tensor $\overline{R}_I{}^J{}_{KL}$ of $\overline{\nabla}$ are defined by

$$(\overline{\nabla}_V W - \overline{\nabla}_W V - [V, W])^K = \overline{T}_{IJ}{}^K V^I W^J,$$
$$(\overline{\nabla}_V \overline{\nabla}_W Y - \overline{\nabla}_W \overline{\nabla}_V Y - \overline{\nabla}_{[V,W]} Y)^J = \overline{R}_I{}^J{}_{KL} Y^I V^K W^L,$$

respectively. In the Θ-frame $\{Z_I\}$, the components are given by

(2-8)
$$\overline{T}_{1\overline{1}}{}^0 = ih_{1\overline{1}}, \qquad \overline{T}_{01}{}^{\overline{1}} = \rho^2 A_1{}^{\overline{1}},$$

and

(2-9)
$$\overline{R}_1{}^1{}_{1\overline{1}} = \rho^2 \mathrm{Scal}^{\mathrm{TW}} h_{1\overline{1}}, \qquad \overline{R}_1{}^1{}_{01} = -\rho^3 A_{11}{}^{,1}, \qquad \overline{R}_1{}^1{}_{0\overline{1}} = \rho^3 A^1{}_{\overline{1},1},$$

where $\mathrm{Scal}^{\mathrm{TW}}$ denotes the Tanaka–Webster scalar curvature, and we have removed the CR weights in the Tanaka–Webster tensors by the CR scale corresponding to θ. The components which cannot be obtained from (2-8) and (2-9) by the symmetries of $\overline{T}$ and $\overline{R}$, or by taking the complex conjugates, are all 0. The nonzero components

of the Ricci tensor $\bar{R}_{IJ} = \bar{R}_I{}^K{}_{KJ}$ are given by

$$\bar{R}_{1\bar{1}} = \rho^2 \operatorname{Scal}^{\mathrm{TW}} h_{1\bar{1}}, \quad \bar{R}_{10} = \rho^2 A_{11,}{}^1.$$

Let ∇ be the Levi-Civita connection of g, which is also a Θ-connection ([Matsumoto 2013, Proposition 4.4]). We define the *difference Θ-tensor* $D_{IJ}{}^K$ by

$$\nabla_I V^K = \overline{\nabla}_I V^K + D_{IJ}{}^K V^J.$$

Since ∇ is torsion-free, we have

(2-10) $$D_{IJ}{}^K = D_{JI}{}^K + \bar{T}_{JI}{}^K.$$

Using this relation and the fact $\nabla g = 0$, we obtain

(2-11) $$2D_{IJK} = \overline{\nabla}_I g_{JK} + \overline{\nabla}_J g_{KI} - \overline{\nabla}_K g_{IJ} - \bar{T}_{IJK} + \bar{T}_{JKI} - \bar{T}_{KIJ}.$$

We will compute $D_{IJ}{}^K$ by these formulas. Since the components g_{IJ} satisfy (2-4), g is described by ρ-dependent tensors φ_{ij} on M defined by

$$g_{00} = 1 + \varphi_{00}, \quad g_{01} = \varphi_{01}, \quad g_{11} = \varphi_{11}, \quad g_{1\bar{1}} = h_{1\bar{1}} + \varphi_{1\bar{1}}.$$

In the construction of a formal solution to the self-dual Einstein equation, we need to examine the effect of a perturbation

(2-12) $$\varphi_{ij} \mapsto \varphi_{ij} + \psi_{ij}, \quad \psi_{ij} = O(\rho^m)$$

on the curvature quantities of g. It is useful in the computation to ignore irrelevant terms on which the perturbation causes only changes in higher orders. Such terms are of the form

(2-13) $$O(\rho) \cdot (\rho \partial_\rho)^l \mathcal{D}\varphi_{ij},$$

where $\mathcal{D}$ is a ρ-dependent differential operator on M. These are called *negligible terms*. In fact, a negligible term changes by $O(\rho^{m+1})$ under the perturbation (2-12). Thus, it suffices to compute $D_{IJ}{}^K$ modulo negligible terms. For simplicity, we assume that the admissible frame $\{Z_1\}$ is unitary with respect to the Levi form; namely $h_{1\bar{1}} = 1$. Noting that $\varphi_{ij} = O(\rho)$, we have

(2-14) $$g^{\infty\infty} = \tfrac{1}{4}, \quad g^{\infty 0} = g^{\infty 1} = 0, \quad g^{00} \equiv 1 - \varphi_{00}, \quad g^{1\bar{1}} \equiv -\varphi_{1\bar{1}}, \quad g^{11} \equiv -\varphi_{\bar{1}\bar{1}}$$

modulo negligible terms. By computing with (2-7), (2-8), (2-10), (2-11), (2-14) we obtain the following result:

Lemma 2.3 [Matsumoto 2013, Lemma 6.4; 2014, Table 1]. *Let $\{T, Z_1, Z_{\bar{1}}\}$ be a unitary admissible frame and $\{\mathbf{Z}_I\} = \{\rho\partial_\rho, \rho^2 T, \rho Z_1, \rho Z_{\bar{1}}\}$ the associated Θ-frame.*

Then, modulo negligible terms, the components $D_{IJ}{}^K$ are given by

$$D_{\infty\infty}{}^\infty \equiv -1, \qquad\qquad D_{\infty 0}{}^\infty \equiv D_{\infty 1}{}^\infty \equiv 0,$$

$$D_{00}{}^\infty \equiv \tfrac{1}{2} - \tfrac{1}{8}(\rho\partial_\rho - 4)\varphi_{00}, \qquad D_{01}{}^\infty \equiv -\tfrac{1}{8}(\rho\partial_\rho - 3)\varphi_{01},$$

$$D_{1\bar{1}}{}^\infty \equiv \tfrac{1}{4} - \tfrac{1}{8}(\rho\partial_\rho - 2)\varphi_{1\bar{1}}, \qquad D_{11}{}^\infty \equiv -\tfrac{1}{8}(\rho\partial_\rho - 2)\varphi_{11},$$

$$D_{\infty\infty}{}^1 \equiv D_{00}{}^1 \equiv D_{\overline{11}}{}^1 \equiv 0, \quad D_{\infty\bar{1}}{}^1 \equiv \tfrac{1}{2}\rho\partial_\rho\varphi_{\overline{11}}, \qquad D_{0\bar{1}}{}^1 \equiv \tfrac{i}{2}\varphi_{\overline{11}},$$

$$D_{\infty 0}{}^1 \equiv \tfrac{1}{2}(\rho\partial_\rho + 1)\varphi_{0\bar{1}}, \qquad D_{01}{}^1 \equiv \tfrac{i}{2}(1 + \varphi_{00} - \varphi_{1\bar{1}}), \quad D_{1\bar{1}}{}^1 \equiv \tfrac{i}{2}\varphi_{0\bar{1}},$$

$$D_{\infty 1}{}^1 \equiv -1 + \tfrac{1}{2}\rho\partial_\rho\varphi_{1\bar{1}}, \qquad D_{\bar{1}0}{}^1 \equiv \tfrac{i}{2}\varphi_{\overline{11}} + \rho^2 A_{\overline{11}}, \qquad D_{11}{}^1 \equiv i\varphi_{01},$$

$$D_{\infty\infty}{}^0 \equiv D_{00}{}^0 \equiv 0, \qquad D_{1\bar{1}}{}^0 \equiv -\tfrac{i}{2}, \qquad\qquad D_{11}{}^0 \equiv -\rho^2 A_{11},$$

$$D_{\infty 0}{}^0 \equiv -2 + \tfrac{1}{2}\rho\partial_\rho\varphi_{00}, \qquad D_{\infty 1}{}^0 \equiv \tfrac{1}{2}(\rho\partial_\rho - 1)\varphi_{01}, \qquad D_{01}{}^0 \equiv -\tfrac{i}{2}\varphi_{01}.$$

The components which are not displayed are obtained by taking the complex conjugates or using the relation (2-10)*.*

Remark 2.4. We have modified a typographical error in [Matsumoto 2013, Table 6.2; 2014, Table 1]; the value of $D_{01}{}^1$ above differs by $-\tfrac{i}{2}\varphi_{1\bar{1}}$ from that in [Matsumoto 2013; 2014]. (Note that $D_{IJ}{}^K$ is denoted by $D^K{}_{IJ}$ in [Matsumoto 2013] and by $D_J{}^K{}_I$ in [Matsumoto 2014].) The correct value is used in the other computations in [Matsumoto 2013; 2014].

3. The self-dual Einstein equation

Let g be an ACH metric on a four-dimensional Θ-manifold $(\overline{X}, [\Theta])$ which has a strictly pseudoconvex CR manifold M as its boundary. We fix a contact form θ on M and identify a neighborhood of M as $M \times [0, \epsilon)_\rho$, where ρ is the model-defining function for θ. We take a unitary admissible frame $\{T, Z_1, Z_{\bar{1}}\}$ on M and work in the associated Θ-frame $\{\boldsymbol{Z}_I\} = \{\rho\partial_\rho, \rho^2 T, \rho Z_1, \rho Z_{\bar{1}}\}$.

3A. *The Einstein equation.* We will recall from [Matsumoto 2013; 2014] the computation of the Einstein tensor modulo negligible terms which is needed in the construction of the Einstein ACH metric. We set

$$E_{IJ} := \mathrm{Ric}_{IJ} + \tfrac{3}{2}g_{IJ}.$$

In terms of the extended Tanaka–Webster connection and the difference Θ-tensor, the curvature tensor of g is expressed as

$$(3\text{-}1) \quad R_I{}^J{}_{KL} = \overline{R}_I{}^J{}_{KL} + \overline{\nabla}_K D_{LI}{}^J$$
$$- \overline{\nabla}_L D_{KI}{}^J + D_{KM}{}^J D_{LI}{}^M - D_{LM}{}^J D_{KI}{}^M + \overline{T}_{KL}{}^M D_{MI}{}^J.$$

Hence, the Ricci tensor is given by

$$(3\text{-}2) \quad \mathrm{Ric}_{IJ} = R_J{}^K{}_{KI}$$
$$= \bar{R}_{JI} + \overline{\nabla}_K D_{IJ}{}^K - \overline{\nabla}_I D_{KJ}{}^K$$
$$+ D_{KM}{}^K D_{IJ}{}^M - D_{IM}{}^K D_{KJ}{}^M + \bar{T}_{KI}{}^M D_{MJ}{}^K$$
$$= \bar{R}_{JI} + \overline{\nabla}_K D_{IJ}{}^K - \overline{\nabla}_I D_{KJ}{}^K + D_{KM}{}^K D_{IJ}{}^M - D_{MI}{}^K D_{KJ}{}^M.$$

In the last equality, we have used (2-10). With this formula and Lemma 2.3, we can compute E_{IJ} modulo negligible terms:

Lemma 3.1 [Matsumoto 2013, Lemma 6.5; 2014, Lemma 4.2]. *Let $\{T, Z_1, Z_{\bar{1}}\}$ be a unitary admissible frame and $\{Z_I\} = \{\rho \partial_\rho, \rho^2 T, \rho Z_1, \rho Z_{\bar{1}}\}$ the associated Θ-frame. Then, modulo negligible terms, the components of the Einstein tensor E_{IJ} are given by*

$$E_{\infty\infty} \equiv -\tfrac{1}{2}\rho\partial_\rho(\rho\partial_\rho - 4)\varphi_{00} - \rho\partial_\rho(\rho\partial_\rho - 2)\varphi_{1\bar{1}},$$
$$E_{\infty 0} \equiv 0,$$
$$E_{\infty 1} \equiv -\tfrac{i}{2}(\rho\partial_\rho + 1)\varphi_{01},$$
$$E_{00} \equiv -2\rho^4|A|^2 - \tfrac{1}{8}\big((\rho\partial_\rho)^2 - 6\rho\partial_\rho - 4\big)\varphi_{00} + \tfrac{1}{2}(\rho\partial_\rho - 2)\varphi_{1\bar{1}},$$
$$E_{01} \equiv \rho^3 A_{11},{}^1 - \tfrac{1}{8}(\rho\partial_\rho + 1)(\rho\partial_\rho - 5)\varphi_{01},$$
$$E_{1\bar{1}} \equiv \rho^2 \mathrm{Scal}^{\mathrm{TW}} - \tfrac{1}{8}\big((\rho\partial_\rho)^2 - 6\rho\partial_\rho - 8\big)\varphi_{1\bar{1}} + \tfrac{1}{8}(\rho\partial_\rho - 4)\varphi_{00},$$
$$E_{11} \equiv i\rho^2 A_{11} - \rho^4 A_{11,0} - \tfrac{1}{8}\rho\partial_\rho(\rho\partial_\rho - 4)\varphi_{11}.$$

The components which are not displayed are obtained by the symmetry or by taking the complex conjugates.

Remark 3.2. We have corrected the value of E_{00} in [Matsumoto 2013, Lemma 6.5; 2014, Lemma 4.2], where the term $-2\rho^4|A|^2$ is missed, though this modification has no significant effect on the construction of the Einstein ACH metric.

3B. *The self-dual equation.* Let $\{\theta^I\}$ be the dual Θ-coframe of $\{Z_I\}$. We take the orientation of $\overline{X}$ such that $\theta \wedge d\theta \wedge d\rho > 0$, and define a skew symmetric Θ-tensor ε_{IJKL} by

$$\mathrm{vol}_g = \frac{1}{4!}\varepsilon_{IJKL}\boldsymbol{\theta}^I \wedge \boldsymbol{\theta}^J \wedge \boldsymbol{\theta}^K \wedge \boldsymbol{\theta}^L,$$

where vol_g is the volume form of g. Since $\det(g_{IJ}) \equiv -4(1 + \varphi_{00} + 2\varphi_{1\bar{1}})$ modulo negligible terms, we have

$$\mathrm{vol}_g = |\det(g_{IJ})|^{1/2}\, i\boldsymbol{\theta}^0 \wedge \boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}^{\bar{1}} \wedge \boldsymbol{\theta}^\infty$$
$$\equiv (2i + i\varphi_{00} + 2i\varphi_{1\bar{1}})\,\boldsymbol{\theta}^0 \wedge \boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}^{\bar{1}} \wedge \boldsymbol{\theta}^\infty,$$

and hence

$$(3\text{-}3) \qquad\qquad \varepsilon_{01\bar{1}\infty} \equiv 2i + i\varphi_{00} + 2i\varphi_{1\bar{1}}.$$

Let $P_{IJ} = \frac{1}{2} \operatorname{Ric}_{IJ} - \frac{1}{12} \operatorname{Scal} g_{IJ}$ be the Schouten tensor, and let

$$W_{IJKL} = R_{IJKL} + g_{IK} P_{JL} - g_{JK} P_{IL} + g_{JL} P_{IK} - g_{IL} P_{JK}$$

be the Weyl curvature. Since $\overline{X}$ is four-dimensional, we can define the anti-self-dual part of the Weyl curvature, which is given by

$$W_{IJKL}^- = \frac{1}{2}\left(W_{IJKL} - \frac{1}{2}\varepsilon_{KL}{}^{PQ} W_{IJPQ}\right).$$

Note that W_{IJKL}^- has the same symmetry as the Weyl curvature and satisfies

$$\frac{1}{2}\varepsilon_{KL}{}^{PQ} W_{IJPQ}^- = -W_{IJKL}^-.$$

Thus, by (2-14) and (3-3), we have

$$(3\text{-}4) \qquad W_{\infty 0 \infty 0}^- \equiv -W_{\infty 1 \infty \bar{1}}^- - W_{\infty \bar{1} \infty 1}^- = -2 W_{\infty 1 \infty \bar{1}}^-,$$

$$(3\text{-}5) \qquad W_{IJ01}^- = -\varepsilon_{01\bar{1}\infty} W_{IJ}^-{}^{\bar{1}\infty} \equiv -\frac{i}{2} W_{IJ1\infty}^-,$$

$$(3\text{-}6) \qquad W_{IJ1\bar{1}}^- = -\varepsilon_{1\bar{1}0\infty} W_{IJ}^-{}^{0\infty} \equiv -\frac{i}{2} W_{IJ0\infty}^-$$

modulo $O(\rho) \cdot W_{IJKL}^-$. Since $W_{IJKL}^- = W_{KLIJ}^-$, we also have

$$W_{01KL}^- \equiv -\frac{i}{2} W_{1\infty KL}^-,$$

$$W_{1\bar{1}KL}^- \equiv -\frac{i}{2} W_{0\infty KL}^-$$

modulo $O(\rho) \cdot W_{IJKL}^-$. As a consequence, we have the following lemma:

Lemma 3.3. *Let m be a positive integer. If $W_{\infty 1 \infty 1}^-, W_{\infty 0 \infty 1}^-, W_{\infty 0 \infty 0}^- = O(\rho^m)$, then $W_{IJKL}^- = O(\rho^m)$.*

Thus, in order to solve the self-dual equation $W_{IJKL}^- = O(\rho^\infty)$, we only have to deal with the three components indicated above.

Next, we consider the Bianchi identity which relates the self-dual equation to the Einstein equation. Let $C_{IJK} := \nabla_K P_{IJ} - \nabla_J P_{IK}$ be the Cotton tensor of g and define the anti-self-dual part C_{IJK}^- by

$$C_{IJK}^- = \frac{1}{2}\left(C_{IJK} - \frac{1}{2}\varepsilon_{JK}{}^{PQ} C_{IPQ}\right).$$

Then, since $\nabla_I \varepsilon_{JKLM} = 0$, the Bianchi identity $\nabla^I W_{IJKL} = C_{JKL}$ yields

$$\nabla^I W_{IJKL}^- = C_{JKL}^-.$$

If g satisfies $E_{IJ} = O(\rho^m)$ for some $m \geq 1$, then we have $P_{IJ} = -\frac{1}{4} g_{IJ} + O(\rho^m)$ and hence $C_{IJK}^- = O(\rho^m)$ since the covariant differentiation does not decrease the vanishing order of a Θ-tensor. Therefore, it holds that

$$E_{IJ} = O(\rho^m) \Rightarrow \nabla^I W_{IJKL}^- = O(\rho^m).$$

To derive the consequence of the latter equation, we will compute

$$(3\text{-}7) \quad \nabla^I W^-_{IJKL} = \overline{\nabla}^I W^-_{IJKL} - W^-_{MJKL} D^I{}_I{}^M - W^-_{IMKL} D^I{}_J{}^M$$
$$- W^-_{IJML} D^I{}_K{}^M - W^-_{IJKM} D^I{}_L{}^M$$

modulo $O(\rho) \cdot \mathcal{D} W^-_{IJKL}$, where $\mathcal{D}$ is a ρ-dependent differential operator on M. By computations similar to (3-4), (3-5), (3-6), we have

$$(3\text{-}8) \qquad\qquad \nabla^I W^-_{I0\infty0} \equiv -2 \nabla^I W^-_{I1\infty\overline{1}},$$

$$(3\text{-}9) \qquad\qquad \nabla^I W^-_{IJ01} \equiv -\frac{i}{2} \nabla^I W^-_{IJ1\infty},$$

$$(3\text{-}10) \qquad\qquad \nabla^I W^-_{IJ1\overline{1}} \equiv -\frac{i}{2} \nabla^I W^-_{IJ0\infty}$$

modulo $O(\rho) \cdot \mathcal{D} W^-_{IJKL}$. By (3-9) and (3-10), it suffices to consider the cases where $K = \infty$. Then, taking complex conjugates we may assume that $L = 0$ or 1, and the case $(J, K, L) = (\overline{1}, \infty, 0)$ is reduced to the case $(J, K, L) = (1, \infty, 0)$. Moreover, by (3-8) the case $(J, K, L) = (\overline{1}, \infty, 1)$ is reduced to the case $(J, K, L) = (0, \infty, 0)$. Thus, it suffices to compute (3-7) for

$$(J, K, L) = (1, \infty, 1), (0, \infty, 0), (0, \infty, 1), (1, \infty, 0), (\infty, \infty, 1), (\infty, \infty, 0).$$

By (2-7), we have

$$\overline{\nabla}^I W^-_{IJKL} = \overline{\nabla}^\infty W^-_{\infty JKL} + \overline{\nabla}^1 W^-_{1JKL} + \overline{\nabla}^{\overline{1}} W^-_{\overline{1}JKL} + \overline{\nabla}^0 W^-_{0JKL}$$
$$\equiv \tfrac{1}{4} \left(\rho \partial_\rho - \#(\infty J K L) \right) W^-_{\infty JKL}.$$

The other terms in the right-hand side of (3-7) can be computed by Lemma 2.3. The final results are:

$$\nabla^I W^-_{I1\infty1} \equiv \frac{1}{4} (\rho \partial_\rho - 4) W^-_{\infty 1 \infty 1}, \qquad \nabla^I W^-_{I0\infty0} \equiv \frac{1}{4} (\rho \partial_\rho - 6) W^-_{\infty 0 \infty 0},$$

$$(3\text{-}11) \quad \nabla^I W^-_{I0\infty1} \equiv \frac{1}{4} (\rho \partial_\rho - 6) W^-_{\infty 0 \infty 1}, \qquad \nabla^I W^-_{I1\infty0} \equiv \frac{1}{4} (\rho \partial_\rho - 5) W^-_{\infty 1 \infty 0},$$

$$\nabla^I W^-_{I\infty\infty1} \equiv \frac{i}{2} W^-_{\infty 0 \infty 1}, \qquad\qquad \nabla^I W^-_{I\infty\infty0} \equiv 0.$$

Consequently, by an inductive argument, we have the following implication:

$$E_{IJ} = O(\rho^4) \Rightarrow W^-_{IJKL} = O(\rho^4).$$

Moreover, if $E_{IJ} = O(\rho^5)$ then $W^-_{\infty 0 \infty 0}$ and $W^-_{\infty 0 \infty 1} = O(\rho^5)$, but we cannot conclude that $W^-_{\infty 1 \infty 1} = O(\rho^5)$. Thus, we may use the equation $W^-_{\infty 1 \infty 1} = O(\rho^5)$ as a normalization on the metric which is independent of the Einstein equation. We will also use a normalization on the ρ^6-term in $W^-_{\infty 0 \infty 0}$ whose vanishing is not imposed by the Einstein equation. To make sure that such normalizations

in fact work, we must calculate the variations of $W^-_{\infty 1 \infty 1}$ and $W^-_{\infty 0 \infty 0}$ under the perturbation (2-12).

First, we calculate the relevant components of the curvature tensor modulo negligible terms. Since the curvature tensor is given by (2-9) and $R_{IJ\infty K} = -4R_K{}^\infty{}_{IJ}$, we obtain the following result by a straightforward computation using (2-7) and Lemma 2.3:

$$
\begin{aligned}
&R_{\infty 0 \infty 0} \equiv 4 + \tfrac{1}{2}\big((\rho\partial_\rho)^2 - 4\rho\partial_\rho + 8\big)\varphi_{00}, && R_{01\infty 0} \equiv \tfrac{i}{4}(\rho\partial_\rho + 1)\varphi_{01}, \\
&R_{\infty 1 \infty 1} \equiv \tfrac{1}{2}\big((\rho\partial_\rho)^2 - 2\rho\partial_\rho + 2\big)\varphi_{11}, && R_{01\infty 1} \equiv \rho^2 A_{11} - \tfrac{i}{4}\rho\partial_\rho\varphi_{11}, \\
&R_{0\bar1\infty 1} \equiv -\tfrac{i}{2} + \tfrac{i}{4}(\rho\partial_\rho - 2)\varphi_{00} - \tfrac{i}{4}\rho\partial_\rho\varphi_{1\bar1}, && R_{1\bar1\infty 1} \equiv \tfrac{3i}{4}(\rho\partial_\rho - 1)\varphi_{01}, \\
&R_{1\bar1\infty 0} \equiv -i + \tfrac{i}{2}(\rho\partial_\rho - 2)\varphi_{00} - \tfrac{i}{2}\rho\partial_\rho\varphi_{1\bar1}.
\end{aligned}
$$

(3-12)

These equations enable us to compute the variations of the curvature components under the perturbation (2-12), which we denote by putting "δ" to each component. For example, by the first equation in (3-12), we have

$$
\delta R_{\infty 0 \infty 0} = \tfrac{1}{2}(m^2 - 4m + 8)\psi_{00} + O(\rho^{m+1}).
$$

Next, we calculate the variation of the Schouten tensor

$$
P_{IJ} = \tfrac{1}{2}E_{IJ} - \tfrac{1}{12}(E_K{}^K + 3)g_{IJ}.
$$

Since $E_{IJ} = O(\rho)$ by Lemma 3.1, we have

$$
\delta P_{IJ} = \tfrac{1}{2}\delta E_{IJ} - \tfrac{1}{12}g^{KL}(\delta E_{KL})g_{IJ} - \tfrac{1}{4}\delta g_{IJ} + O(\rho^{m+1}),
$$

which yields

$$
\begin{aligned}
&\delta P_{\infty\infty} = -\tfrac{1}{6}(m^2 - 3m - 1)\psi_{00} - \tfrac{1}{6}(2m^2 - m + 2)\psi_{1\bar1} + O(\rho^{m+1}), \\
&\delta P_{\infty 0} = O(\rho^{m+1}), \\
&\delta P_{\infty 1} = -\tfrac{i}{4}(m+1)\psi_{01} + O(\rho^{m+1}), \\
&\delta P_{00} = -\tfrac{1}{24}(m^2 - 6m - 1)\psi_{00} + \tfrac{1}{24}(m^2 + m - 14)\psi_{1\bar1} + O(\rho^{m+1}), \\
&\delta P_{01} = -\tfrac{1}{16}(m^2 - 4m - 1)\psi_{01} + O(\rho^{m+1}), \\
&\delta P_{1\bar1} = \tfrac{1}{48}(m^2 - 3m - 10)\psi_{00} - \tfrac{1}{48}(m^2 - 8m - 8)\psi_{1\bar1} + O(\rho^{m+1}), \\
&\delta P_{11} = -\tfrac{1}{16}(m^2 - 4m + 4)\psi_{11} + O(\rho^{m+1}).
\end{aligned}
$$

(3-13)

From (2-14), (3-12), and (3-13), we have

$$W_{\infty 1}{}^{0\bar{1}} = O(\rho),$$

$$W_{\infty 0}{}^{\bar{1}1} = -i + O(\rho),$$

$$\delta W_{\infty 1}{}^{0\bar{1}} = -\frac{i}{4}(m-2)\psi_{11} + O(\rho^{m+1}),$$

(3-14)

$$\delta W_{\infty 0}{}^{\bar{1}1} = \frac{i}{2}(m-2)\psi_{00} - \frac{i}{2}(m-4)\psi_{1\bar{1}} + O(\rho^{m+1}),$$

$$\delta W_{\infty 1 \infty 1} = \frac{1}{4}(m^2 - 4)\psi_{11} + O(\rho^{m+1}),$$

$$\delta W_{\infty 0 \infty 0} = \frac{1}{6}(m^2 - 3m + 20)\psi_{00} - \frac{1}{6}(m^2 - 2m + 16)\psi_{1\bar{1}} + O(\rho^{m+1}).$$

Finally, by (3-3) and (3-14), we obtain

$$(3\text{-}15) \quad \delta W_{\infty 1 \infty 1}^{-} = \tfrac{1}{2}(\delta W_{\infty 1 \infty 1} - \delta\varepsilon_{\infty 10\bar{1}} \cdot W_{\infty 1}{}^{0\bar{1}} - \varepsilon_{\infty 10\bar{1}} \cdot \delta W_{\infty 1}{}^{0\bar{1}}) + O(\rho^{m+1})$$

$$= \tfrac{1}{8}(m^2 - 2m)\psi_{11} + O(\rho^{m+1}),$$

$$(3\text{-}16) \quad \delta W_{\infty 0 \infty 0}^{-} = \tfrac{1}{2}(\delta W_{\infty 0 \infty 0} - \delta\varepsilon_{\infty 0\bar{1}1} \cdot W_{\infty 0}{}^{\bar{1}1} - \varepsilon_{\infty 0\bar{1}1} \cdot \delta W_{\infty 0}{}^{\bar{1}1}) + O(\rho^{m+1})$$

$$= \tfrac{1}{12}(m^2 + 3m + 2)\psi_{00} - \tfrac{1}{12}(m^2 + 4m + 4)\psi_{1\bar{1}} + O(\rho^{m+1}).$$

3C. *Bianchi identities.* Since the Einstein equation is an overdetermined system, we need some relations which are satisfied by the components of the Einstein tensor in order to construct a formal solution to the Einstein equation. Some of them are given by the Bianchi identity $g^{IJ}\nabla_K E_{IJ} = 2g^{IJ}\nabla_I E_{JK}$:

Lemma 3.4 [Matsumoto 2013, Lemma 6.6; 2014, Lemma 6.1]. *Suppose g satisfies* $E_{IJ} = O(\rho^m)$ *for an integer* $m \geq 1$. *Then, we have*

$$(3\text{-}17) \qquad (m-8)E_{\infty\infty} - 4(m-4)E_{00} - 8(m-2)E_{1\bar{1}} = O(\rho^{m+1}),$$

$$(3\text{-}18) \qquad\qquad\qquad (m-6)E_{\infty 0} = O(\rho^{m+1}),$$

$$(3\text{-}19) \qquad\qquad\qquad (m-5)E_{\infty 1} - 4i E_{01} = O(\rho^{m+1}).$$

We will also use some equations obtained from the Bianchi identity $\nabla^I W_{IJKL}^{-} = C_{JKL}^{-}$ in the construction of g. Since the Cotton tensor is given by

$$C_{IJK} = \tfrac{1}{2}(\nabla_K E_{IJ} - \nabla_J E_{IK}) - \tfrac{1}{12}\big((\nabla_K E_L{}^L)g_{IJ} - (\nabla_J E_L{}^L)g_{IK}\big),$$

we can compute the components C_{IJK}^{-} in terms of E_{IJ} by using (2-7), (2-14), (3-3), and Lemma 2.3. As a result, we have the following lemma:

Lemma 3.5. *Suppose g satisfies $E_{IJ} = O(\rho^m)$ for an integer $m \geq 1$. Then,*

$$(3\text{-}20) \qquad C^-_{1\infty 1} = -\tfrac{1}{4}(m-2)E_{11} + O(\rho^{m+1}),$$

$$(3\text{-}21) \qquad C^-_{0\infty 0} = -\tfrac{5}{24}mE_{00} + \tfrac{1}{96}(m-12)E_{\infty\infty} + \tfrac{1}{12}(m+6)E_{1\bar{1}} + O(\rho^{m+1}),$$

$$(3\text{-}22) \qquad C^-_{\infty\infty 0} = -\tfrac{1}{4}(m-2)E_{\infty 0} + O(\rho^{m+1}).$$

4. Construction of the metric

4A. *The formal solution to the self-dual Einstein equation.* Let M be a three-dimensional strictly pseudoconvex CR manifold. We fix a contact form θ and construct a one-parameter family of ACH metrics g^λ on $\overline{X} = M \times [0, \infty)_\rho$ which are in normal form with respect to θ and satisfy the Einstein equation to infinite order. The parameter $\lambda \in \mathbb{R}$ is involved in the normalization on the ρ^6-term in g^λ, and if $\lambda = 0$ the metric is self-dual to infinite order. As in the previous section, we take the Θ-frame $\{Z_I\} = \{\rho\partial_\rho, \rho^2 T, \rho Z_1, \rho Z_{\bar{1}}\}$ associated with a unitary admissible frame $\{T, Z_1, Z_{\bar{1}}\}$ on M. We suppress the superscript λ in the following.

First we show a lemma which assures that our normalization condition is independent of the choice of θ.

Lemma 4.1. *Suppose that an ACH metric g on $\overline{X}$ satisfies $W^-_{IJKL} = O(\rho^6)$, and let ρ_θ be the model-defining function associated with a contact form θ. Then,*

$$(4\text{-}1) \qquad\qquad \eta_\theta := \left(\rho_\theta^{-6} W^-_{\infty 0\infty 0}\right)\big|_M$$

satisfies $\eta_{\widehat{\theta}} = e^{-3\Upsilon}\eta_\theta$ for the rescaling $\widehat{\theta} = e^\Upsilon \theta$.

Proof. By Proposition 2.2, $Z_\infty|_M$ and $Z_0|_M$ are determined by g and independent of θ. Thus, we have

$$\widehat{W}^-_{\infty 0\infty 0} = W^-_{\infty 0\infty 0} + O(\rho^7).$$

Since $\rho_{\widehat{\theta}} = e^{\Upsilon/2}\rho_\theta + O(\rho^2)$ by (2-5), we obtain $\eta_{\widehat{\theta}} = e^{-3\Upsilon}\eta_\theta$. $\square$

This lemma implies that if η_θ has a universal expression in terms of the Tanaka–Webster connection, then it defines a CR invariant $\eta \in \mathcal{E}(-3, -3)$. Since such a CR invariant is necessarily a multiple of the obstruction density [Graham 1987], we are led to the CR invariant normalization $\eta = \lambda \mathcal{O}$.

Now we construct the metric and prove Theorem 1.2. We start with an arbitrary normal form ACH metric $g^{(1)}_{IJ}$, which automatically satisfies $E_{IJ} = O(\rho)$ by Lemma 3.1. Supposing that we have a normal form ACH metric $g^{(m)}_{IJ}$ such that $E_{IJ} = O(\rho^m)$, we consider a perturbed metric

$$g^{(m+1)}_{IJ} = g^{(m)}_{IJ} + \psi_{IJ}, \quad \psi_{\infty J} = 0, \ \psi_{IJ} = O(\rho^m)$$

and try to solve $E_{IJ} = O(\rho^{m+1})$. We also take W^-_{IJKL} into consideration in each inductive step by using the following equations modulo $O(\rho) \cdot \mathcal{D}W^-_{IJKL}$ from (3-11):

$$(4\text{-}2) \qquad \nabla^I W^-_{I1\infty 1} \equiv \frac{1}{4}(\rho\partial_\rho - 4)W^-_{\infty 1\infty 1},$$

$$(4\text{-}3) \qquad \nabla^I W^-_{I0\infty 0} \equiv \frac{1}{4}(\rho\partial_\rho - 6)W^-_{\infty 0\infty 0},$$

$$(4\text{-}4) \qquad \nabla^I W^-_{I\infty\infty 1} \equiv \frac{i}{2} W^-_{\infty 0\infty 1},$$

$$(4\text{-}5) \qquad \nabla^I W^-_{I\infty\infty 0} \equiv 0.$$

By Lemma 3.1, the variation of E_{IJ} is given by

$$(4\text{-}6) \qquad \delta E_{\infty\infty} = -\frac{1}{2}m(m-4)\psi_{00} - m(m-2)\psi_{1\bar{1}} + O(\rho^{m+1}),$$

$$(4\text{-}7) \qquad \delta E_{\infty 0} = O(\rho^{m+1}),$$

$$(4\text{-}8) \qquad \delta E_{\infty 1} = -\frac{i}{2}(m+1)\psi_{01} + O(\rho^{m+1}),$$

$$(4\text{-}9) \qquad \delta E_{00} = -\frac{1}{8}(m^2 - 6m - 4)\psi_{00} + \frac{1}{2}(m-2)\psi_{1\bar{1}} + O(\rho^{m+1}),$$

$$(4\text{-}10) \qquad \delta E_{01} = -\frac{1}{8}(m+1)(m-5)\psi_{01} + O(\rho^{m+1}),$$

$$(4\text{-}11) \qquad \delta E_{1\bar{1}} = -\frac{1}{8}(m^2 - 6m - 8)\psi_{1\bar{1}} + \frac{1}{8}(m-4)\psi_{00} + O(\rho^{m+1}),$$

$$(4\text{-}12) \qquad \delta E_{11} = -\frac{1}{8}m(m-4)\psi_{11} + O(\rho^{m+1}).$$

The determinant of the coefficients of (4-9) and (4-11) as a system of linear equations for ψ_{00} and $\psi_{1\bar{1}}$ is

$$\det\begin{pmatrix} -\frac{1}{8}(m^2 - 6m - 4) & \frac{1}{2}(m-2) \\ \frac{1}{8}(m-4) & -\frac{1}{8}(m^2 - 6m - 8) \end{pmatrix} = \frac{1}{64}m(m+2)(m-6)(m-8).$$

First we consider the case of $m \leq 5$, where the determinant is nonzero. We determine ψ_{00} and $\psi_{1\bar{1}}$ (modulo $O(\rho^{m+1})$) by (4-9) and (4-11) so that E_{00}, $E_{1\bar{1}} = O(\rho^{m+1})$ holds. Then, by the Bianchi identities (3-17) and (3-18), we have $E_{\infty\infty}$ and $E_{\infty 0} = O(\rho^{m+1})$. We determine ψ_{01} by (4-8) to obtain $E_{\infty 1} = O(\rho^{m+1})$. Then, (3-19) gives $E_{01} = O(\rho^{m+1})$. When $m \leq 3$, (4-12) determines ψ_{11} so that $E_{11} = O(\rho^{m+1})$, thus we have $E_{IJ} = O(\rho^{m+1})$. Moreover, by (4-2)–(4-4) and Lemma 3.3, we also have $W^-_{IJKL} = O(\rho^{m+1})$. When $m = 4$, we cannot use (4-12) to obtain $E_{11} = O(\rho^5)$. However, since $W^-_{IJKL} = O(\rho^4)$, it follows from (4-2) that

$$C^-_{1\infty 1} = \nabla^I W^-_{I1\infty 1} = \tfrac{1}{4}(4-4)W^-_{\infty 1\infty 1} + O(\rho^5) = O(\rho^5),$$

so we have $E_{11} = O(\rho^5)$ by (3-20). (This also follows from the fact that the CR obstruction tensor $\mathcal{O}_{11} = (\rho^{-4}E_{11})|_M$ vanishes in three dimensions; see [Matsumoto 2013; 2014].) Thus, we have $E_{IJ} = O(\rho^5)$ and by (4-2)–(4-4), it holds that

$$W^-_{\infty 1 \infty 1} = O(\rho^4), \quad W^-_{\infty 0 \infty 0}, W^-_{\infty 0 \infty 1} = O(\rho^5).$$

We can choose ψ_{11} so that $W^-_{\infty 1 \infty 1} = O(\rho^5)$ holds since

$$\delta W^-_{\infty 1 \infty 1} = \psi_{11} + O(\rho^5)$$

by (3-15). Thus we obtain unique $g^{(5)}_{IJ}$ modulo $O(\rho^5)$ with $E_{IJ}, W^-_{IJKL} = O(\rho^5)$. When $m = 5$, we can construct $g^{(6)}_{IJ}$ with $E_{IJ} = O(\rho^6)$ in the same way as for $m \leq 3$ and we also have $W^-_{IJKL} = O(\rho^6)$ by (4-2)–(4-4).

Next we consider the case of $m = 6$, where the equations (4-6), (4-9), (4-11) are not pairwise independent. We determine ψ_{01} by (4-8) so that $E_{\infty 1} = O(\rho^7)$. Then we also have $E_{01} = O(\rho^7)$ by (3-19). We determine ψ_{11} by (4-12) and obtain $E_{11} = O(\rho^7)$. By (3-16), we have

$$(4\text{-}13) \qquad \delta W^-_{\infty 0 \infty 0} = \tfrac{14}{3}\psi_{00} - \tfrac{16}{3}\psi_{1\bar{1}} + O(\rho^7).$$

We use this equation and (4-6) to determine $\psi_{00}, \psi_{1\bar{1}}$ so that

$$E_{\infty\infty} = O(\rho^7), \quad \eta = \lambda\mathcal{O}$$

holds. Thus we have determined $g^{(7)}_{IJ}$ and we must check that it also satisfies $E_{00}, E_{1\bar{1}}, E_{\infty 0} = O(\rho^7)$. Since $W^-_{IJKL} = O(\rho^6)$, by (4-3) we have

$$C^-_{0\infty 0} = \tfrac{1}{4}(6-6)W^-_{\infty 0 \infty 0} + O(\rho^7) = O(\rho^7).$$

Then it follows from (3-21) that

$$-\tfrac{5}{4}E_{00} + E_{1\bar{1}} = O(\rho^7).$$

Also, (3-17) gives

$$E_{00} + 4E_{1\bar{1}} = O(\rho^7).$$

Therefore, we have $E_{00}, E_{1\bar{1}} = O(\rho^7)$. Moreover, by $W^-_{IJKL} = O(\rho^6)$ and (4-5), it holds that $C^-_{\infty\infty 0} = O(\rho^7)$, which implies $E_{\infty 0} = O(\rho^7)$ by (3-22). Thus, $g^{(7)}_{IJ}$ satisfies $E_{IJ} = O(\rho^7)$, $W^-_{IJKL} = O(\rho^6)$, and $\eta = \lambda\mathcal{O}$. We note that it satisfies $W^-_{IJKL} = O(\rho^7)$ when $\lambda = 0$.

When $m = 7$, we can determine $g^{(8)}_{IJ}$ so that it satisfies $E_{IJ} = O(\rho^8)$ in the same way as in $m \leq 3$. If $\lambda = 0$, it also satisfies $W^-_{IJKL} = O(\rho^8)$ by (4-2)–(4-4).

Let us consider the case of $m = 8$. In this case, (4-9) and (4-11) are not independent. We use (4-6) and (4-9) to determine ψ_{00} and $\psi_{1\bar{1}}$ so that $E_{\infty\infty}, E_{00} = O(\rho^9)$. Then (3-17) gives $E_{1\bar{1}} = O(\rho^9)$. We determine ψ_{01} and ψ_{11} by (4-8) and (4-12) respectively and obtain $E_{\infty 1}, E_{11} = O(\rho^9)$. By (3-18), (3-19), we have

$E_{\infty 0}$, $E_{01} = O(\rho^9)$. Thus we have constructed $g_{IJ}^{(9)}$ with $E_{IJ} = O(\rho^9)$, which satisfies $W_{IJKL}^- = O(\rho^9)$ when $\lambda = 0$ by (4-2)–(4-4).

Finally, let $m \geq 9$. In this case, the equation $E_{IJ} = O(\rho^{m+1})$ determines $g_{IJ}^{(m+1)}$ in the same way as in $m \leq 3$, and it satisfies $W_{IJKL}^- = O(\rho^{m+1})$ by (4-2)–(4-4) when $\lambda = 0$.

Consequently, we can construct all $g_{IJ}^{(m+1)}$ inductively, and by Borel's lemma we obtain a solution g_{IJ}^λ to

$$E_{IJ} = O(\rho^\infty), \quad W_{IJKL}^- = O(\rho^6), \quad \eta = \lambda\mathcal{O},$$

which is unique modulo $O(\rho^\infty)$. By the construction, g_{IJ}^0 satisfies $W_{IJKL}^- = O(\rho^\infty)$. Thus we complete the proof of Theorem 1.2.

4B. *Dependence on λ.* We can read off the dependence of g_{IJ}^λ on the parameter λ from the construction.

Proposition 4.2. *The metric g_{IJ}^λ admits the following asymptotic expansion:*

$$g_{IJ}^\lambda \sim g_{IJ}^0 + \sum_{k=1}^\infty \lambda^k \rho^{6k} \phi_{IJ}^{(k)}(\rho), \quad \phi_{I\infty}^{(k)} = \phi_{01}^{(k)} = \phi_{11}^{(k)} = 0.$$

Here, $\phi_{IJ}^{(k)}(\rho)$ is a formal power series in ρ.

Proof. We write the Taylor expansion of $g_{IJ}^\lambda - g_{IJ}^0$ as

$$g_{IJ}^\lambda - g_{IJ}^0 \sim \sum_{k=0}^\infty \rho^k \Phi_{IJ}^{\lambda,k}, \quad \Phi_{I\infty}^{\lambda,k} = 0.$$

Then, it suffices to show that $\Phi_{01}^{\lambda,k} = \Phi_{11}^{\lambda,k} = 0$ and each $\Phi_{IJ}^{\lambda,k}$ is a polynomial in λ of degree $\leq k/6$. First, we note that $g_{IJ}^\lambda - g_{IJ}^0 = O(\rho^6)$, so $\Phi_{IJ}^{\lambda,k} = 0$ for $k \leq 5$. Since both g_{IJ}^λ and g_{IJ}^0 satisfy $E_{\infty 1}$, $E_{11} = O(\rho^\infty)$, we also have that $\Phi_{01}^{\lambda,k} = \Phi_{11}^{\lambda,k} = 0$ for $k \geq 6$, by (4-8), (4-12). From (4-6), (4-13), we see $\Phi_{00}^{\lambda,6}$ and $\Phi_{1\bar{1}}^{\lambda,6}$ are determined by

$$-6\Phi_{00}^{\lambda,6} - 24\Phi_{1\bar{1}}^{\lambda,6} = 0,$$

$$\tfrac{14}{3}\Phi_{00}^{\lambda,6} - \tfrac{16}{3}\Phi_{1\bar{1}}^{\lambda,6} = \lambda\mathcal{O}.$$

Thus we have deg $\Phi_{00}^{\lambda,6} = $ deg $\Phi_{1\bar{1}}^{\lambda,6} = 1$. Now we prove deg $\Phi_{00}^{\lambda,k}$, deg $\Phi_{1\bar{1}}^{\lambda,k} \leq k/6$ by the induction on k. When $k \geq 7$, $\Phi_{00}^{\lambda,k}$ and $\Phi_{1\bar{1}}^{\lambda,k}$ are determined by the condition $\partial_\rho^k E_{00}|_{\rho=0} = \partial_\rho^k E_{1\bar{1}}|_{\rho=0} = 0$ for $k \neq 8$ and $\partial_\rho^k E_{\infty\infty}|_{\rho=0} = \partial_\rho^k E_{00}|_{\rho=0} = 0$ for $k = 8$. These conditions can be regarded as a system of linear equations for $\Phi_{00}^{\lambda,k}$ and $\Phi_{1\bar{1}}^{\lambda,k}$, and in view of (2-7), (2-11), (3-2), the terms involving the other components are linear combinations of

$$\mathcal{D}_1\Phi_{I_1 J_1}^{\lambda,l_1} \cdots \mathcal{D}_p\Phi_{I_p J_p}^{\lambda,l_p}, \quad (l_1 + \cdots + l_p \leq k, \ l_j < k),$$

where $\mathcal{D}_j$ is a differential operator on M. Then, by the induction hypothesis,

$$\deg \Phi_{IJ}^{\lambda,k} \leq \frac{l_1 + \cdots + l_p}{6} \leq \frac{k}{6}.$$

Thus, we complete the proof. □

4C. *Evenness.* Let g be a normal form ACH metric on $M \times [0, \infty)_\rho$. Then it can be written in the form

(4-14)
$$g = \frac{h_\rho + 4d\rho^2}{\rho^2},$$

where h_ρ is a family of Riemannian metrics on M. We say g is *even* when h_ρ has even Laurent expansion at $\rho = 0$. In other words, g is even if and only if the components g_{00}, g_{11}, $g_{1\bar{1}}$ are even in ρ, and g_{01} is odd in ρ. An ACH metric is said to be even if its normal form is even for any choice of θ.

Proposition 4.3. *The ACH metric g^λ is even.*

Proof. Fix a contact form θ and suppose g^λ is in the normal form as (4-14). By using the Laurent expansion of h_ρ, we can regard the right-hand side of (4-14) as an ACH metric g_-^λ defined on $M \times (-\infty, 0]_\rho$. Then, g_-^λ also satisfies

$$E_{IJ} = O(\rho^\infty), \quad W_{IJKL}^- = O(\rho^6), \quad \eta = \lambda\mathcal{O},$$

with respect to the orientation satisfying

$$i\boldsymbol{\theta}^0 \wedge \boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}^{\bar{1}} \wedge \boldsymbol{\theta}^\infty = i\rho^{-5}\theta \wedge \theta^1 \wedge \theta^{\bar{1}} \wedge d\rho > 0.$$

We consider the ACH metric $\iota^* g_-^\lambda$ on $M \times [0, \infty)_\rho$, where $\iota(x, \rho) := (x, -\rho)$. Since ι preserves the orientation, $\iota^* g_-^\lambda$ satisfies

$$E_{IJ} = O(\rho^\infty), \quad W_{IJKL}^- = O(\rho^6).$$

Noting that $\iota_* \mathbf{Z}_\infty = \mathbf{Z}_\infty$ and $\iota_* \mathbf{Z}_0 = \mathbf{Z}_0$, we have

$$\begin{aligned}
\rho^{-6} W^-[\iota^* g_-^\lambda]_{\infty 0 \infty 0} &= (\iota^*\rho)^{-6}(\iota^* W^-[g_-^\lambda])(\mathbf{Z}_\infty, \mathbf{Z}_0, \mathbf{Z}_\infty, \mathbf{Z}_0) \\
&= \iota^*\big(\rho^{-6} W^-[g_-^\lambda](\iota_* \mathbf{Z}_\infty, \iota_* \mathbf{Z}_0, \iota_* \mathbf{Z}_\infty, \iota_* \mathbf{Z}_0)\big) \\
&= \iota^*(\rho^{-6} W^-[g_-^\lambda]_{\infty 0 \infty 0}).
\end{aligned}$$

Thus, $\iota^* g_-^\lambda$ also satisfies $\eta = \lambda\mathcal{O}$. Therefore, by the uniqueness we obtain

$$\iota^* g_-^\lambda = g^\lambda + O(\rho^\infty),$$

which implies that g^λ is even. □

5. CR GJMS operators

Matsumoto [2016] generalized the CR GJMS operators to partially integrable CR manifolds via Dirichlet-to-Neumann type operators associated with eigenvalue equations for the laplacian of the ACH metric. In dimension three, it is stated as follows:

Theorem 5.1 [Matsumoto 2016, Theorem 3.3]. *Let M be a three-dimensional strictly pseudoconvex CR manifold and g an ACH metric on a Θ-manifold $\overline{X}$ with the boundary M. Let θ be a contact form on M and let ρ be the model-defining function associated with θ. Then, for any $k \in \mathbb{N}_+$ and $f \in C^\infty(M)$, there exist $F, G \in C^\infty(\overline{X})$ with $F|_M = f$ such that the function $u := \rho^{-k+2}F + (\rho^{k+2}\log\rho)G$ satisfies*

$$\left(\Delta + \frac{k^2}{4} - 1\right)u = O(\rho^\infty),$$

where $\Delta = -g^{IJ}\nabla_I\nabla_J$ is the laplacian of g. The function G is unique modulo $O(\rho^\infty)$ and $P_{2k}f := (-1)^{k+1}k!(k-1)!/2 \cdot G|_M$ defines a formally self-adjoint linear differential operator $\mathcal{E}(k/2-1, k/2-1) \to \mathcal{E}(-k/2-1, -k/2-1)$ which is independent of the choice of θ and has the principal part Δ_b^k.

We apply this theorem to our metric g^λ. Since g^λ is determined to infinite order and the Taylor expansion has a universal expression in terms of the pseudohermitian structure, the operator P_{2k}^λ has a universal expression in terms of the Tanaka–Webster connection. Thus, we obtain the CR GJMS operators P_{2k}^λ for all $k \geq 1$.

In order to prove that P_{2k}^λ is a polynomial in λ of degree $\leq k/3$, we will review the details of its construction. A linear differential operator on $\overline{X}$ is called a Θ-*differential operator* if it is the sum of linear differential operators of the form $aY_1 \cdots Y_N$, where $a \in C^\infty(\overline{X})$ and $Y_j \in \Gamma(^\Theta T\overline{X})$. Note that a Θ-differential operator preserves the subspace $\rho^m C^\infty(\overline{X}) \subset C^\infty(\overline{X})$ for each $m \geq 1$. We fix a contact form θ and denote the associated Tanaka–Webster connection by ∇^{TW}. Suppose that g^λ is of the normal form

$$g^\lambda = k_\rho + 4\frac{d\rho^2}{\rho^2}$$

for θ, where k_ρ is a family of Riemannian metrics on M. Then, the laplacian Δ of g^λ is written as

$$(5\text{-}1) \qquad \Delta = -\tfrac{1}{4}(\rho\partial_\rho)^2 + \rho\partial_\rho + \rho^2\Delta_b - \rho^4 T^2 + \rho\Psi$$

with the Θ-differential operator Ψ defined by

$$\Psi f = -\tfrac{1}{8}(\partial_\rho \log\det k_\rho)\rho\partial_\rho f - \rho^{-1}\big((k_\rho^{-1})^{ij} - (k_0^{-1})^{ij}\big)\nabla_i^{\mathrm{TW}}\nabla_j^{\mathrm{TW}} f$$
$$+ \tfrac{1}{2}(k_\rho^{-1})^{ij}(k_\rho^{-1})^{kl}\rho^{-1}\big(\nabla_i^{\mathrm{TW}}(k_\rho)_{jk} + \nabla_j^{\mathrm{TW}}(k_\rho)_{ik} - \nabla_k^{\mathrm{TW}}(k_\rho)_{ij}\big)\nabla_l^{\mathrm{TW}} f.$$

Here, the components are with respect to a Θ-frame $\{\mathbf{Z}_I\}$, and we note that $(k_\rho^{-1})^{ij} - (k_0^{-1})^{ij}$ and $\nabla_i^{\mathrm{TW}}(k_\rho)_{jk} + \nabla_j^{\mathrm{TW}}(k_\rho)_{ik} - \nabla_k^{\mathrm{TW}}(k_\rho)_{ij}$ are $O(\rho)$ by (2-6). In particular, Ψ involves $\partial_\rho g_{IJ}^\lambda$ but not higher order derivatives.

Given a function $f \in C^\infty(M)$, we try to solve the equation

$$\left(\Delta + \frac{k^2}{4} - 1\right)(\rho^{-k+2}F) = 0$$

for $F \in C^\infty(\overline{X})$ with $F|_M = f$. Let $F \sim \sum_{j=0}^\infty f^{(j)}\rho^j$, $(f^{(j)} \in C^\infty(M))$ be the Taylor expansion of F along M. By (5-1), we have

$$(5\text{-}2) \quad \left(\Delta + \frac{k^2}{4} - 1\right)(\rho^{-k+2+j} f^{(j)}) = \rho^{-k+2+j}\left(-\frac{1}{4}j(j-2k)f^{(j)} + \rho \mathcal{D}_j f^{(j)}\right),$$

where $\mathcal{D}_j$ is a ρ-dependent linear differential operator on M. Starting with $f^{(0)} = f$, we inductively define $f^{(j)}$ so that F satisfies

$$\left(\Delta + \frac{k^2}{4} - 1\right)(\rho^{-k+2}F) = O(\rho^{-k+3+j}).$$

Let $\mathcal{D}_j \sim \sum_{l=0}^\infty \mathcal{D}_j^{(l)}\rho^l$ be the Taylor expansion of $\mathcal{D}_j$. Then, by (5-2), $f^{(j)}$ is determined for $j \leq 2k-1$ as

$$f^{(j)} = \frac{4}{j(j-2k)}\sum_{l=0}^{j-1}\mathcal{D}_l^{(j-1-l)} f^{(l)}.$$

We cannot define $f^{(2k)}$ due to the vanishing of the coefficient of $f^{(2k)}$ in (5-2), and we need to introduce the logarithmic term $(\rho^{k+2}\log\rho)G$ in which the coefficient $G|_M$ is a multiple of

$$\left(\rho^{-k-2}\left(\Delta + \frac{k^2}{4} - 1\right)(\rho^{-k+2}F)\right)\Big|_M .$$

Therefore, up to a constant multiple, $P_{2k}^\lambda f$ is given by

$$\sum_{j=0}^{2k-1}\mathcal{D}_j^{(2k-1-j)} f^{(j)}.$$

Since Ψ involves only g_{IJ}^λ and their first order derivatives in ρ, $\mathcal{D}_j^{(l)}$ involves $\partial_\rho^m g_{IJ}^\lambda$ for $m \leq l+1$. Consequently, P_{2k}^λ is written in terms of $\partial_\rho^m g_{IJ}^\lambda$ ($m \leq 2k$), and by Proposition 4.2 it is a polynomial in λ of degree $\leq k/3$. Thus we complete the proof of Theorem 1.3.

6. Convergence of the formal solutions

We will prove Theorem 1.4, which asserts that the formal solution g^λ converges to a real analytic ACH metric near M when M is a real analytic CR manifold. In

the case of $\lambda = 0$, this recovers the result of Biquard [2007]. The key tool is the result of Baouendi and Goulaouic [1976] on the unique existence of the solution to a singular nonlinear Cauchy problem. Let us state their theorem in a form which fits to our setting.

We regard local coordinates (x, ρ) of $M \times [0, \infty)_\rho$ as complex variables and consider an equation for a $\mathbb{C}^N$-valued holomorphic function $v(x, \rho)$ of the form

$$(6\text{-}1) \quad (\rho \partial_\rho)^m v + A_{m-1}(\rho \partial_\rho)^{m-1} v + \cdots + A_0 v$$
$$= F(x, \rho, \{(\rho \partial_\rho)^l \partial_x^\alpha (\rho v)\}_{l+|\alpha| \leq m, \, l < m}),$$

where A_j is an $N \times N$ matrix and $F(x, \rho, \{y_{l,\alpha}\}_{l+|\alpha| \leq m, \, l < m})$ is a holomorphic function near 0. For each $k \in \mathbb{N}$, we set

$$\mathcal{P}(k) := k^m I + k^{m-1} A_{m-1} + \cdots + A_0,$$

where I is the identity matrix of size N. Then, by [Baouendi and Goulaouic 1976, Theorem 3.1] we have the following theorem:

Theorem 6.1. *If* $\det \mathcal{P}(k) \neq 0$ *for all* $k \in \mathbb{N}$, (6-1) *has a unique holomorphic solution* $v(x, \rho)$ *near* $(0, 0)$.

In the original statement of [Baouendi and Goulaouic 1976, Theorem 3.1], the right-hand side of the (6-1) is replaced by $G(\rho, \{(\rho \partial_\rho)^l \partial_x^\alpha (\rho v)\}_{l+|\alpha| \leq m, \, l < m})$ with G a C^∞-map

$$G : \mathbb{C} \times B^{N'} \to B,$$

where B is the Banach space of $\mathbb{C}^N$-valued bounded holomorphic functions of x on a fixed polydisc, and N' is the number of multiindices (l, α) such that $l + |\alpha| \leq m$, $l < m$. Also, the solution v is given as a C^∞-function of ρ valued in B. In our (6-1), G is given by $G(\rho, \{y_{l,\alpha}\}) := F(x, \rho, \{y_{l,\alpha}(x)\})$. Since this is analytic in ρ, it follows from [Baouendi and Goulaouic 1976, Remark 2.2] and the proof of [Baouendi and Goulaouic 1976, Theorem 3.1] that the solution $v(x, \rho)$ is C^∞ and $v(x, \rho^m)$ is holomorphic, which implies that $v(x, \rho)$ itself is holomorphic. Thus we obtain Theorem 6.1 as a special case of their theorem.

Now we apply this theorem to our case. We assume that M is a real analytic CR manifold. Let g_{IJ}^λ be the components of the formal solution g^λ in a Θ-frame $\{Z_I\}$, and let

$$g_{IJ}^{(k)} := \frac{1}{k!} \partial_\rho^k g_{IJ}^\lambda \big|_M$$

be the Taylor coefficients, which are analytic functions on M. We consider an ACH metric of the form

$$\widetilde{g}_{IJ}^\lambda = \sum_{k=0}^{8} \rho^k g_{IJ}^{(k)} + \rho^9 \widetilde{\varphi}_{IJ},$$

which automatically satisfies $E_{IJ} = O(\rho^9)$. Then, we consider the equation

$$(6\text{-}2) \qquad -8\rho^{-9}(E_{00}, E_{1\bar{1}}, E_{01}, E_{11}) = 0$$

for $v = (\widetilde{\varphi}_{00}, \widetilde{\varphi}_{1\bar{1}}, \widetilde{\varphi}_{01}, \widetilde{\varphi}_{11})$. We shall show that this equation is written in the form (6-1) for $m = 2$ and satisfies the assumption of Theorem 6.1; then we can conclude that g_{IJ}^λ converges since it gives the Taylor expansion of the solution v.

We see that in Lemma 3.1 the negligible term which we ignored in the computation of E_{IJ} is an analytic function in

$$x, \ \rho, \ \rho(\rho\partial_\rho)^l \partial_x^\alpha(\rho^9\widetilde{\varphi}) \quad \text{for } l + |\alpha| \le 2, \ l < 2.$$

Thus, it can be written in the form

$$f_{IJ}^{(1)}(x, \rho) + \rho^9 f_{IJ}^{(2)}(x, \rho, \{(\rho\partial_\rho)^l \partial_x^\alpha(\rho\widetilde{\varphi})\}_{l+|\alpha|\le 2, \ l<2})$$

with analytic functions $f_{IJ}^{(1)}, f_{IJ}^{(2)}$. Then, by Lemma 3.1, we have

$$-8E_{00} = I_1(\rho\partial_\rho)\varphi_{00} + I_2(\rho\partial_\rho)\varphi_{1\bar{1}}$$
$$+ 16\rho^4|A|^2 + f_{00}^{(1)}(x, \rho) + \rho^9 f_{00}^{(2)}(x, \rho, \{(\rho\partial_\rho)^l \partial_x^\alpha(\rho\widetilde{\varphi})\}_{l+|\alpha|\le 2, \ l<2}),$$

where

$$\varphi_{IJ} = \sum_{k=1}^{8} \rho^k g_{IJ}^{(k)} + \rho^9 \widetilde{\varphi}_{IJ}$$

and

$$I_1(t) = t^2 - 6t - 4, \quad I_2(t) = -4(t-2).$$

Since $E_{00} = O(\rho^9)$, we have

$$I_1(\rho\partial_\rho)\left(\sum_{k=1}^{8} \rho^k g_{00}^{(k)}\right) + I_2(\rho\partial_\rho)\left(\sum_{k=1}^{8} \rho^k g_{1\bar{1}}^{(k)}\right) + 16\rho^4|A|^2 + f_{00}^{(1)}(x, \rho) = \rho^9 f_{00}^{(0)}(x, \rho)$$

with some analytic function $f_{00}^{(0)}$. Therefore, the equation $-8\rho^{-9}E_{00} = 0$ is written

$$I_1(\rho\partial_\rho + 9)\widetilde{\varphi}_{00} + I_2(\rho\partial_\rho + 9)\widetilde{\varphi}_{1\bar{1}} + F_{00}(x, \rho, \{(\rho\partial_\rho)^l \partial_x^\alpha(\rho\widetilde{\varphi})\}_{l+|\alpha|\le 2, \ l<2}) = 0$$

with an analytic function F_{00}.

Similarly, the equations $-8\rho^{-9}E_{IJ} = 0$ for $(I, J) = (1, \bar{1}), (0, 1), (1, 1)$ are respectively written as

$$I_3(\rho\partial_\rho + 9)\widetilde{\varphi}_{00} + I_4(\rho\partial_\rho + 9)\widetilde{\varphi}_{1\bar{1}} + F_{1\bar{1}}(x, \rho, \{(\rho\partial_\rho)^l \partial_x^\alpha(\rho\widetilde{\varphi})\}_{l+|\alpha|\le 2, \ l<2}) = 0,$$
$$I_5(\rho\partial_\rho + 9)\widetilde{\varphi}_{01} + F_{01}(x, \rho, \{(\rho\partial_\rho)^l \partial_x^\alpha(\rho\widetilde{\varphi})\}_{l+|\alpha|\le 2, \ l<2}) = 0,$$
$$I_6(\rho\partial_\rho + 9)\widetilde{\varphi}_{11} + F_{11}(x, \rho, \{(\rho\partial_\rho)^l \partial_x^\alpha(\rho\widetilde{\varphi})\}_{l+|\alpha|\le 2, \ l<2}) = 0,$$

where $F_{1\bar{1}}$, F_{01}, F_{11} are analytic functions and

$$I_3(t) = -t + 4, \quad I_4(t) = t^2 - 6t - 8, \quad I_5(t) = (t+1)(t-5), \quad I_6(t) = t(t-4).$$

Hence (6-2) is of the form (6-1), and we have

$$
\det \mathcal{P}(k) = \det \begin{pmatrix} I_1(k+9) & I_2(k+9) & & \\ I_3(k+9) & I_4(k+9) & & \\ & & I_5(k+9) & \\ & & & I_6(k+9) \end{pmatrix}
$$
$$
= (k+1)(k+3)(k+4)(k+5)(k+9)^2(k+10)(k+11)
$$
$$
\neq 0
$$

for any $k \in \mathbb{N}$. Thus, by Theorem 6.1, (6-2) has a unique holomorphic solution and we complete the proof of Theorem 1.4.

Acknowledgements

References

[Baouendi and Goulaouic 1976] M. S. Baouendi and C. Goulaouic, "Singular nonlinear Cauchy problems", *J. Differential Equations* **22**:2 (1976), 268–291. MR Zbl

[Biquard 2007] O. Biquard, "Sur les variétés CR de dimension 3 et les twisteurs", *Ann. Inst. Fourier (Grenoble)* **57**:4 (2007), 1161–1180. MR Zbl

[Čap et al. 2001] A. Čap, J. Slovák, and V. Souček, "Bernstein–Gelfand–Gelfand sequences", *Ann. of Math.* (2) **154**:1 (2001), 97–113. MR Zbl

[Cheng and Lee 1990] J. H. Cheng and J. M. Lee, "The Burns–Epstein invariant and deformation of CR structures", *Duke Math. J.* **60**:1 (1990), 221–254. MR Zbl

[Epstein et al. 1991] C. L. Epstein, R. B. Melrose, and G. A. Mendoza, "Resolvent of the Laplacian on strictly pseudoconvex domains", *Acta Math.* **167**:1-2 (1991), 1–106. MR Zbl

[Fefferman 1976] C. L. Fefferman, "Monge–Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains", *Ann. of Math.* (2) **103**:2 (1976), 395–416. Correction in **104**:3 (1976), 393–394. MR Zbl

[Fefferman and Graham 2002] C. Fefferman and C. R. Graham, "Q-curvature and Poincaré metrics", *Math. Res. Lett.* **9**:2-3 (2002), 139–151. MR Zbl

[Fefferman and Graham 2012] C. Fefferman and C. R. Graham, *The ambient metric*, Ann. of Math. Studies **178**, Princeton Univ. Press, 2012. MR Zbl

[Fefferman and Hirachi 2003] C. Fefferman and K. Hirachi, "Ambient metric construction of Q-curvature in conformal and CR geometries", *Math. Res. Lett.* **10**:5-6 (2003), 819–831. MR Zbl

[Gover and Graham 2005] A. R. Gover and C. R. Graham, "CR invariant powers of the sub-Laplacian", *J. Reine Angew. Math.* **583** (2005), 1–27. MR Zbl

[Gover and Hirachi 2004] A. R. Gover and K. Hirachi, "Conformally invariant powers of the Laplacian: a complete nonexistence theorem", *J. Amer. Math. Soc.* **17**:2 (2004), 389–405. MR Zbl

[Graham 1987] C. R. Graham, "Scalar boundary invariants and the Bergman kernel", pp. 108–135 in *Complex analysis, II* (College Park, MD, 1985/86), edited by C. A. Berenstein, Lect. Notes in Math. **1276**, Springer, 1987. MR Zbl

[Graham and Zworski 2003] C. R. Graham and M. Zworski, "Scattering matrix in conformal geometry", *Invent. Math.* **152**:1 (2003), 89–118. MR Zbl

[Graham et al. 1992] C. R. Graham, R. Jenne, L. J. Mason, and G. A. J. Sparling, "Conformally invariant powers of the Laplacian, I: Existence", *J. London Math. Soc.* (2) **46**:3 (1992), 557–565. MR Zbl

[Hislop et al. 2008] P. D. Hislop, P. A. Perry, and S.-H. Tang, "CR-invariants and the scattering operator for complex manifolds with boundary", *Anal. PDE* **1**:2 (2008), 197–227. MR Zbl

[LeBrun 1982] C. R. LeBrun, "$\mathcal{H}$-space with a cosmological constant", *Proc. Roy. Soc. London Ser. A* **380**:1778 (1982), 171–185. MR Zbl

[Lee 1988] J. M. Lee, "Pseudo-Einstein structures on CR manifolds", *Amer. J. Math.* **110**:1 (1988), 157–178. MR Zbl

[Lee and Melrose 1982] J. M. Lee and R. Melrose, "Boundary behaviour of the complex Monge–Ampère equation", *Acta Math.* **148** (1982), 159–192. MR Zbl

[Matsumoto 2013] Y. Matsumoto, *Asymptotically complex hyperbolic Einstein metrics and CR geometry*, Ph.D. thesis, University of Tokyo, 2013, Available at https://tinyurl.com/matsthes.

[Matsumoto 2014] Y. Matsumoto, "Asymptotics of ACH-Einstein metrics", *J. Geom. Anal.* **24**:4 (2014), 2135–2185. MR Zbl

[Matsumoto 2016] Y. Matsumoto, "GJMS operators, Q-curvature, and obstruction tensor of partially integrable CR manifolds", *Differential Geom. Appl.* **45** (2016), 78–114. MR Zbl

TAIJI MARUGAME
INSTITUTE OF MATHEMATICS, ACADEMIA SINICA
TAIPEI
TAIWAN
marugame@gate.sinica.edu.tw

PACIFIC JOURNAL OF MATHEMATICS