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WATARU TAKEDA

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We consider the upper bound of the Piltz divisor problem over number fields. The Piltz divisor problem is known as a generalization of the Dirichlet divisor problem. We deal with this problem over number fields and improve the error term of this function for many cases. Our proof uses the estimate of exponential sums. We also show uniform results for the ideal counting function and relatively r-prime lattice points as one of its applications.

1. Introduction

The behavior of arithmetic functions has long been studied, and it is one of the most important areas of research in analytic number theory. But many arithmetic functions $f(n)$ fluctuate as n increases, and it becomes difficult to deal with them. Thus, many authors study partial sums $\sum_{n\le x} f(n)$ to obtain some information about arithmetic functions $f(n)$. In this paper we consider the Piltz divisor function $I_K^m(x)$ over a number field. Let K be a number field with extension degree $[K:\mathbb{Q}]=n$, and let $\mathcal{O}_K$ be its ring of integers. Let D_K be the absolute value of the discriminant of K. Then the Piltz divisor function $I_K^m(x)$ counts the number of m-tuples of ideals $(\mathfrak{a}_1,\mathfrak{a}_2,\dots,\mathfrak{a}_m)$ such that the product of their ideal norms $\mathfrak{N}\mathfrak{a}_1\cdots\mathfrak{N}\mathfrak{a}_m\le x$. It is known that

$$I_K^m(x)\sim \operatorname*{Res}_{s=1}\biggl(\zeta_K(s)^m\frac{x^s}{s}\biggr). \tag{1-1}$$

We denote by $\Delta_K^m(x)$ the error term of $I_K^m(x)$, that is, $I_K^m(x)-\mathrm{Res}_{s=1}\bigl(\zeta_K(s)^m\frac{x^s}{s}\bigr)$.

In the case $m=1$, this function is the ordinary ideal counting function over K. For simplicity we substitute $I_K(x)$ and $\Delta_K(x)$ for $I_K^1(x)$ and $\Delta_K^1(x)$, respectively. There are many results about $I_K(x)$ from the 1900s. In the case $K=\mathbb{Q}$, integer ideals of $\mathbb{Z}$ and positive integers are in one-to-one correspondence, so $I_{\mathbb{Q}}(x)=[x]$, where $[\cdot]$ is the Gauss symbol. For the general case, the best estimate of $\Delta_K(x)$ hitherto is the following theorem.

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Theorem 1-2. *The following estimates hold for all $\varepsilon > 0$:*

$n = [K : \mathbb{Q}]$	$\Delta_K(x)$	
2	$O\big(x^{\frac{131}{416}}(\log x)^{\frac{18627}{8320}}\big)$	[Huxley 2002]
3	$O\big(x^{\frac{43}{96}+\varepsilon}\big)$	[Müller 1988]
4	$O\big(x^{\frac{41}{72}+\varepsilon}\big)$	[Bordellès 2015]
$5 \le n \le 10$	$O\big(x^{1-\frac{4}{2n+1}+\varepsilon}\big)$	[Bordellès 2015]
$11 \le n$	$O\big(x^{1-\frac{3}{n+6}+\varepsilon}\big)$	[Lao 2010]

There are also many results about $I_{\mathbb{Q}}^m$ from the 1800s. In 1849 Dirichlet showed that

$$I_{\mathbb{Q}}^2(x) = x\log x + (2\gamma - 1)x + O\big(x^{\frac{1}{2}}\big),$$

where γ is the Euler constant, defined by

$$\gamma = \lim_{n\to\infty}\bigg(\sum_{k=1}^{n}\frac{1}{k} - \log n\bigg).$$

The O-term has been improved by many researchers many times; the best estimate hitherto is $x^{\frac{517}{1648}+\varepsilon}$ [Bourgain and Watt 2017].

As we have mentioned above, there exist many results about other divisor problems, but it seems that there are not many results about the Piltz divisor problem over number fields.

Theorem 1-3 [Nowak 1993]. *When $n = [K : \mathbb{Q}] \ge 2$, then we get*

$$\Delta_K^m(x) = \begin{cases} O_K\big(x^{1-\frac{2}{mn}+\frac{8}{mn(5mn+2)}}(\log x)^{m-1-\frac{10(m-2)}{5n+2}}\big) & \textit{for } 3 \le mn \le 6, \\ O_K\big(x^{1-\frac{2}{mn}+\frac{3}{2m^2n^2}}(\log x)^{m-1-\frac{2(m-2)}{mn}}\big) & \textit{for } mn \ge 7. \end{cases}$$

For the estimate of the lower bound, Girstmair, Kühleitner, Müller, and Nowak obtain the following Ω-results:

Theorem 1-4 [Girstmair et al. 2005]. *For any fixed number field K with $n = [K : \mathbb{Q}] \ge 2$,*

$$\Delta_K^m(x) = \Omega\big(x^{\frac{1}{2}-\frac{1}{2mn}}(\log x)^{\frac{1}{2}-\frac{1}{2mn}}(\log\log x)^{\kappa}(\log\log\log x)^{-\lambda}\big), \tag{1-5}$$

where κ and λ are constants depending on K. To be more precise, let K^{gal} be the Galois closure of $K/\mathbb{Q}$, $G = Gal(K^{\mathrm{gal}}/\mathbb{Q})$ its Galois group, and $H = Gal(K^{\mathrm{gal}}/K)$ the subgroup of G corresponding to K. Then

$$\kappa = \frac{mn+1}{2mn}\bigg(\sum_{\nu=1}^{n}\delta_\nu \nu^{\frac{2mn}{mn+1}} - 1\bigg) \quad \textit{and} \quad \lambda = \frac{mn+1}{4mn}R + \frac{mn-1}{2mn},$$

where

$$\delta_\nu = \frac{\bigl|\{\tau \in G \mid |\{\sigma \in G \mid \tau \in \sigma H \sigma^{-1}\}| = \nu |H|\}\bigr|}{|G|}$$

and R *is the number of* $1 \le \nu \le n$ *with* $\delta_\nu > 0$.

We know the following conditional result. If we assume the Lindelöf hypothesis for the Dedekind zeta function, it holds that for all $\varepsilon > 0$, for all K, and for all m,

(1-6) $$\Delta_K^m(x) = O_\varepsilon\bigl(x^{\frac{1}{2}+\varepsilon} D_K^\varepsilon\bigr).$$

In this paper we estimate the error term of $\Delta_K^m(x)$ by using exponential sums. In [Nowak 1993; Girstmair et al. 2005], they use other approaches, so we expect new development for the Piltz divisor problem over number fields. As a results, we improve the estimate of upper bound of $\Delta_K^m(x)$ for many K and many m.

In Section 2, we show some auxiliary theorems to consider the upper bound of the error term $\Delta_K^m(x)$. First we give a review of the convexity bound for the Dedekind zeta function and generalized Atkinson lemma [1941]. Next we show Proposition 2-6, which reduces an ideal counting problem to a problem of exponential sums. This proposition plays a crucial role in our computing $\Delta_K^m(x)$.

In Section 3, we prove the following theorem about the error term $\Delta_K^m(x)$ by using estimates of exponential sums.

Theorem 1-7. *For every* $\varepsilon > 0$ *the following estimate holds. When* $mn \ge 4$, *then*

$$\Delta_K^m(x) = O_{n,m,\varepsilon}\bigl(x^{\frac{2mn-3}{2mn+1}+\varepsilon} D_K^{\frac{2m}{2mn+1}+\varepsilon}\bigr).$$

This theorem gives improvement of the upper bound of $\Delta_K^m(x)$ for $mn \ge 4$.

In Section 4, we give some applications. First we give a uniform estimate for the ideal counting function over number fields. Second we show a good uniform upper bound of the distribution of relatively r-prime lattice points over number fields as a corollary of the first application.

In Section 5, we consider a conjecture about estimates for the Piltz divisor functions over number fields. It is proposed that for all number fields K and for all m the best upper bound of the error term is better than that on the assumption of the Lindelöf hypothesis (1-6).

2. Auxiliary theorem

In this section, we show some important lemmas for our argument. Let $s = \sigma + it$ and $n = [K : \mathbb{Q}]$. We use the convexity bound of the Dedekind zeta function to obtain an upper bound of the error term of the Piltz divisor function $\Delta_K^m(x)$.

It is well known that the Dedekind zeta function satisfies the functional equation

$$\zeta_K(1-s) = D_K^{s-\frac{1}{2}} 2^{n(1-s)} \pi^{-ns} \Gamma(s)^n \left(\cos\frac{\pi s}{2}\right)^{r_1+r_2} \left(\sin\frac{\pi s}{2}\right)^{r_2} \zeta_K(s), \tag{2-1}$$

where r_1 is the number of real embeddings of K and r_2 is the number of pairs of complex embeddings.

The Phragmen–Lindelöf principle and (2-1) give the well known convexity bound of the Dedekind zeta function [Rademacher 1959]: for any $\varepsilon > 0$ and $n = [K:\mathbb{Q}]$,

$$\zeta_K(\sigma+it) = \begin{cases} O_{n,\varepsilon}\left(|t|^{\frac{n}{2}-n\sigma+\varepsilon} D_K^{\frac{1}{2}-\sigma+\varepsilon}\right) & \text{if } \sigma \le 0, \\ O_{n,\varepsilon}\left(|t|^{\frac{n(1-\sigma)}{2}+\varepsilon} D_K^{\frac{1-\sigma}{2}+\varepsilon}\right) & \text{if } 0 \le \sigma \le 1, \\ O_{n,\varepsilon}(|t|^{\varepsilon} D_K^{\varepsilon}) & \text{if } 1 \le \sigma \end{cases} \tag{2-2}$$

as $|t|^n D_K \to \infty$, where K runs through number fields with $[K:\mathbb{Q}] = n$. In the previous papers, we also use this convexity bound (2-2) to estimate the distribution of ideals. In the following sections, we show some estimates for $\Delta_K^m(x)$ in the similar way to our previous papers.

Lemma 2-3 states the growth of the product of the Gamma function and trigonometric functions in the functional equation (2-1) of the Dedekind zeta function.

Lemma 2-3. *Let $\tau \in \{\cos, \sin\}$ and n be a positive integer. Then*

$$\frac{\Gamma(s)^n}{1-s}\left(\cos\frac{\pi s}{2}\right)^{r_1+r_2}\left(\sin\frac{\pi s}{2}\right)^{r_2} = C n^{-ns}\Gamma\left(ns - \frac{n+1}{2}\right)\tau\left(\frac{n\pi s}{2}\right) + O_n\left(|t|^{-2+n\sigma-\frac{n}{2}}\right),$$

where C is a constant and $s = \sigma + it$.

Proof. This lemma is shown from the Stirling formula and estimate for trigonometric functions. □

Next we introduce the generalized Atkinson lemma. This lemma is quite useful for calculating integrals of the Dedekind zeta function.

Lemma 2-4 [Atkinson 1941]. *Let $y > 0$, $1 < A \le B$, and $\tau \in \{\cos, \sin\}$, and define*

$$I = \frac{1}{2\pi i}\int_{A-iB}^{A+iB} \Gamma(s)\tau\left(\frac{\pi s}{2}\right) y^{-s}\, ds.$$

If $y \le B$, then

$$I = \tau(y) + O\left(y^{-\frac{1}{2}}\min\left(\left(\log\frac{B}{y}\right)^{-1}, B^{\frac{1}{2}}\right) + y^{-A}B^{A-\frac{1}{2}} + y^{-\frac{1}{2}}\right).$$

If $y > B$, then

$$I = O\left(y^{-A}\left(B^{A-\frac{1}{2}}\min\left(\left(\log\frac{y}{B}\right)^{-1}, B^{\frac{1}{2}}\right) + A^{A-\frac{1}{2}}\right)\right).$$

Finally we introduce the following lemma to reduce the ideal counting problem to an exponential sum problem.

Lemma 2-5 [Bordellès 2015]. *Let $1 \le L \le R$ be a real number and f be an arithmetical function satisfying $f(m) = O(m^\varepsilon)$, and let $\boldsymbol{e}(x) = \exp(2\pi i x)$ and $F = f * \mu$, where $*$ is the Dirichlet product symbol. For $a \in \mathbb{R} - \{1\}$ and $b, x \in \mathbb{R}$ and for every $\varepsilon > 0$ the following estimate holds*:

$$\sum_{m \le R} \frac{f(m)}{m^a} \tau(2\pi x m^b) = O_{n,\varepsilon}\Bigg(L^{1-a} + R^\varepsilon \max_{L<S\le R} S^{-a} \times \max_{S<S_1\le 2S} \max_{\substack{M,N\le S_1 \\ MN \asymp S}} \max_{\substack{M\le M_1 \le 2M \\ N\le N_1\le 2N}} \Bigg| \sum_{M<m\le M_1} F(m) \sum_{N<n\le N_1} \boldsymbol{e}(x(mn)^b) \Bigg| \Bigg).$$

The next proposition plays a crucial role in our computing $I_K^m(x)$. We consider the distribution of ideals of $\mathbb{O}_K$, where K runs through extensions with $[K:\mathbb{Q}] = n$ and some conditions. The detail of the conditions will be determined later, but they state the relation of the principal term and the error term.

Proposition 2-6. *Let $F_K = I_K^m * \mu$. For every $\varepsilon > 0$ the following estimate holds*:

$$\Delta_K^m(x) = O_{n,m,\varepsilon}\Bigg(L^{1-\alpha} + x^{\frac{mn-1}{2mn}} D_K^{\frac{1}{2n}} R^\varepsilon \max_{L\le S\le R} S^{-\frac{mn+1}{2mn}} \times \max_{S<S_1\le 2S} \max_{\substack{M,N\le S_1 \\ MN \asymp S}} \max_{\substack{M\le M_1 \le 2M \\ N\le N_1\le 2N}} \Bigg| \sum_{M<l\le M_1} F_K(m) \sum_{N<k\le N_1} \boldsymbol{e}\Bigg(mn\Big(\frac{xlk}{D_K}\Big)^{\frac{1}{mn}}\Bigg) \Bigg| + x^{\frac{mn-2}{2mn}+\varepsilon} D_K^{\frac{1}{n}+\varepsilon} R^{\frac{mn-2}{2mn}+\varepsilon} + x^{\frac{mn-1}{mn}+\varepsilon} D_K^{\frac{1}{n}+\varepsilon} R^{-\frac{1}{mn}+\varepsilon} \Bigg),$$

where K runs through number fields with $[K:\mathbb{Q}] = n$ and some conditions.

Proof. Let $d_K^m(l)$ be the number of m-tuples of ideals $(\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_m)$ such that the product of their ideal norms $\mathfrak{N}\mathfrak{a}_1 \cdots \mathfrak{N}\mathfrak{a}_m = l$. Then one can easily check that

$$\zeta_K(s)^m = \sum_{l=1}^{\infty} \frac{d_K^m(l)}{l^s} \quad \text{for } \Re s > 1 \tag{2-7}$$

and

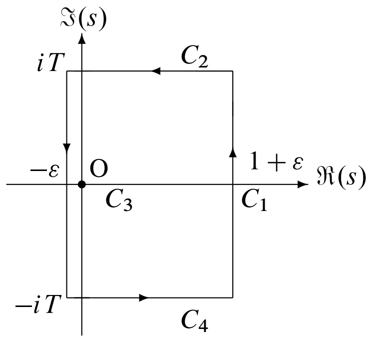
$$I_K^m(x) = \sum_{l \le x} d_K^m(l).$$

Thus, Perron's formula plays a crucial role in this proof.

We consider the integral

$$\frac{1}{2\pi i} \int_C \zeta_K(s)^m \frac{x^s}{s}\, ds,$$

where C is the contour $C_1 \cup C_2 \cup C_3 \cup C_4$ shown below:



In a way similar to the well known proof of Perron's formula, we estimate

$$\frac{1}{2\pi i}\int_{C_1}\zeta_K(s)^m\frac{x^s}{s}\,ds = I_K^m(x)+O_\varepsilon\left(\frac{x^{1+\varepsilon}}{T}\right). \tag{2-8}$$

We can select the large T, so that the O-term in the right-hand side is sufficiently small. For estimating the left-hand side by using estimate (2-2), we divide it into the integrals over C_2, C_3, and C_4.

First we consider the integrals over C_2 and C_4 as

$$\left|\frac{1}{2\pi i}\int_{C_2\cup C_4}\zeta_K(s)^m\frac{x^s}{s}\,ds\right| \le \frac{1}{2\pi}\int_{-\varepsilon}^{1+\varepsilon}|\zeta_K(\sigma+iT)|^m\frac{x^\sigma}{T}\,d\sigma+\frac{1}{2\pi}\int_{-\varepsilon}^{1+\varepsilon}|\zeta_K(\sigma-iT)|^m\frac{x^\sigma}{T}\,d\sigma.$$

It holds by the convexity bound of the Dedekind zeta function (2-2) that their sum is estimated as

$$\left|\frac{1}{2\pi i}\int_{C_2\cup C_4}\zeta_K(s)^m\frac{x^s}{s}\,ds\right| = O_{n,m,\varepsilon}\left(\int_{-\varepsilon}^{1+\varepsilon}(T^{mn}D_K^m)^{\frac{1-\sigma}{2}+\varepsilon}\frac{x^\sigma}{T}\,d\sigma\right) = O_{n,m,\varepsilon}\left(\frac{x^{1+\varepsilon}D_K^\varepsilon}{T^{1-\varepsilon}}+T^{\frac{mn}{2}-1+\varepsilon}D_K^{\frac{m}{2}+\varepsilon}x^{-\varepsilon}\right). \tag{2-9}$$

By the Cauchy residue theorem, (2-8), and (2-9) we obtain

$$\Delta_K^m(x)=\int_{C_3}\zeta_K(s)^m\frac{x^s}{s}\,ds+O_{n,m,\varepsilon}\left(\frac{x^{1+\varepsilon}D_K^\varepsilon}{T^{1-\varepsilon}}+T^{\frac{mn}{2}-1+\varepsilon}D_K^{\frac{m}{2}+\varepsilon}x^{-\varepsilon}\right). \tag{2-10}$$

Thus, it suffices to consider the integral over C_3 as

$$\frac{1}{2\pi i}\int_{C_3}\zeta_K(s)^m\frac{x^s}{s}\,ds=\frac{1}{2\pi i}\int_{-\varepsilon-iT}^{-\varepsilon+iT}\zeta_K(s)^m\frac{x^s}{s}\,ds.$$

Changing the variable s to $1-s$, we have

$$\frac{1}{2\pi i}\int_{C_3}\zeta_K(s)^m\frac{x^s}{s}\,ds=\frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT}\zeta_K(1-s)^m\frac{x^{1-s}}{1-s}\,ds.$$

From functional equation (2-1), it holds that

$$\frac{1}{2\pi i}\int_{C_3}\zeta_K(s)^m\frac{x^s}{s}\,ds=\frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT}\Big(D_K^{s-\frac{1}{2}}2^{n(1-s)}\pi^{-ns}\Gamma(s)^n\Big(\cos\frac{\pi s}{2}\Big)^{r_1+r_2}\times\Big(\sin\frac{\pi s}{2}\Big)^{r_2}\zeta_K(s)\Big)^m\frac{x^{1-s}}{1-s}\,ds.$$

By Lemma 2-3 the integral over C_3 can be expressed as

$$\begin{aligned}&\frac{1}{2\pi i}\int_{C_3}\zeta_K(s)\frac{x^s}{s}\,ds\\&=\frac{Cx}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT}D_K^{-\frac{m}{2}}\Big(\frac{(2n)^{mn}\pi^{mn}x}{D_K^m}\Big)^{-s}\Gamma\Big(mns-\frac{mn+1}{2}\Big)\tau\Big(\frac{mn\pi s}{2}\Big)\zeta_K(s)\,ds\\&\qquad+O_{n,m,\varepsilon}\big(D_K^{\frac{m}{2}+\varepsilon}T^{\frac{mn}{2}-1+\varepsilon}x^{-\varepsilon}\big).\end{aligned}$$

Changing the variable $mns-\frac{mn+1}{2}$ to s, we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{C_3}\zeta_K(s)\frac{x^s}{s}\,ds=\frac{Cx^{\frac{mn-1}{2mn}}D_K^{\frac{1}{2n}}}{2\pi i}&\int_{\frac{mn-1}{2}+mn\varepsilon-mniT}^{\frac{mn-1}{2}+mn\varepsilon+mniT}\Big(2mn\pi\Big(\frac{x}{D_K^m}\Big)^{\frac{1}{mn}}\Big)^{-s}\\&\times\Gamma(s)\tau\Big(\frac{\pi s}{2}+\frac{(mn+1)\pi}{4}\Big)\zeta_K\Big(\frac{s}{mn}+\frac{mn+1}{2mn}\Big)\,ds\\&+O_{n,m,\varepsilon}\big(D_K^{\frac{m}{2}+\varepsilon}T^{\frac{mn}{2}-1+\varepsilon}x^{-\varepsilon}\big).\end{aligned}$$

From (2-7) the function $\zeta_K(s)^m$ can be expressed as a Dirichlet series. It is absolutely and uniformly convergent on compact subsets on $\Re(s)>1$. Therefore, we can interchange the order of summation and integral. Thus, we obtain

$$\begin{aligned}&\int\Big(2mn\pi\Big(\frac{x}{D_K^m}\Big)^{\frac{1}{mn}}\Big)^{-s}\Gamma(s)\tau\Big(\frac{\pi s}{2}+\frac{(mn+1)\pi}{4}\Big)\zeta_K\Big(\frac{s}{mn}+\frac{mn+1}{2mn}\Big)\,ds\\&\qquad=\sum_{l=1}^{\infty}\frac{d_K^m(l)}{l^{\frac{mn+1}{2mn}}}\int\Big(2mn\pi\Big(\frac{lx}{D_K^m}\Big)^{\frac{1}{mn}}\Big)^{-s}\Gamma(s)\tau\Big(\frac{\pi s}{2}+\frac{(mn+1)\pi}{4}\Big)\,ds,\end{aligned}$$

where the integration is on the vertical line from $\frac{mn-1}{2}+mn\varepsilon-mniT$ to $\frac{mn-1}{2}+mn\varepsilon+mniT$. Properties of trigonometric functions lead to

$$\tau\Big(\frac{\pi s}{2}+\frac{(mn+1)\pi}{4}\Big)=\pm\begin{cases}\tau\big(\frac{\pi s}{2}\big)&\text{if } mn \text{ is odd},\\ \frac{1}{\sqrt{2}}\big(\tau\big(\frac{\pi s}{2}\big)\pm\tau_1\big(\frac{\pi s}{2}\big)\big)&\text{if } mn \text{ is even},\end{cases}$$

where $\{\tau, \tau_1\} = \{\sin, \cos\}$. Hence, it holds that

$$\frac{1}{2\pi i}\int_{C_3} \zeta_K(s)^m \frac{x^s}{s}\,ds = \frac{C x^{\frac{mn-1}{2mn}} D_K^{\frac{1}{2n}}}{2\pi i} \sum_{l=1}^{\infty} \frac{d_K^m(l)}{l^{\frac{mn+1}{2mn}}} \times \int_{\frac{mn-1}{2}+mn\varepsilon-mniT}^{\frac{mn-1}{2}+mn\varepsilon+mniT} \left(2mn\pi\left(\frac{lx}{D_K^m}\right)^{\frac{1}{mn}}\right)^{-s} \Gamma(s)\tau\left(\frac{\pi s}{2}\right) ds + O_{n,m,\varepsilon}\big(D_K^{\frac{m}{2}+\varepsilon} T^{\frac{mn}{2}-1+\varepsilon} x^{-\varepsilon}\big).$$

Applying Lemma 2-4 to this integral with $y = 2mn\pi\big(\frac{lx}{D_K^m}\big)^{\frac{1}{mn}}$, $A = \frac{mn-1}{2} + mn\varepsilon$, $B = mnT$, and $T = 2\pi\big(\frac{xR}{D_K^m}\big)^{\frac{1}{mn}}$, this becomes

$$\begin{aligned}\frac{1}{2\pi i}\int_{C_3} \zeta_K(s)^m \frac{x^s}{s}\,ds &= \frac{C x^{\frac{mn-1}{2mn}} D_K^{\frac{1}{2n}}}{2\pi i} \sum_{l\le R} \frac{d_K^m(l)}{l^{\frac{mn+1}{2mn}}} \tau\left(2mn\pi\left(\frac{lx}{D_K^m}\right)^{\frac{1}{mn}}\right) \\ &+ O_{n,m,\varepsilon}\left(x^{\frac{mn-2}{2mn}} D_K^{\frac{1}{n}} \sum_{l\le R} \frac{d_K^m(l)}{l^{\frac{mn+2}{2mn}}} \min\left\{\left(\log\frac{R}{l}\right)^{-1}, \left(\frac{Rx}{D_K^m}\right)^{\frac{1}{2mn}}\right\}\right) \\ &+ O_{n,m,\varepsilon}\left(x^{\frac{mn-2}{2mn}} D_K^{\frac{1}{n}} \sum_{l\le R} \frac{d_K^m(l)}{l^{\frac{mn+2}{2mn}}} \left(\left(\frac{R}{l}\right)^{\frac{mn-2}{2mn}} + 1\right)\right) \\ &+ O_{n,m,\varepsilon}\left(x^{\frac{mn-2}{2mn}} D_K^{\frac{1}{n}} R^{\frac{mn-2}{2mn}+\varepsilon} \sum_{l> R} \frac{d_K^m(l)}{l^{1+\varepsilon}} \min\left\{\left(\log\frac{l}{R}\right)^{-1}, \left(\frac{Rx}{D_K^m}\right)^{\frac{1}{2mn}}\right\}\right) \\ &+ O_{n,m,\varepsilon}\big(x^{\frac{mn-2}{2mn}+\varepsilon} D_K^{\frac{1}{n}+\varepsilon} R^{\frac{mn-2}{2mn}+\varepsilon}\big).\end{aligned}$$

We evaluate three O-terms as follows. First we consider the first O-term. One can estimate $\big(\log\frac{R}{l}\big)^{-1} = O\big(\frac{R}{R-l}\big)$, so we obtain

$$\begin{aligned}&O_{n,m,\varepsilon}\left(x^{\frac{mn-2}{2mn}} D_K^{\frac{1}{n}} \sum_{l\le R} \frac{d_K^m(l)}{l^{\frac{mn+2}{2mn}}} \min\left\{\left(\log\frac{R}{l}\right)^{-1}, \left(\frac{Rx}{D_K^m}\right)^{\frac{1}{2mn}}\right\}\right) \\ &= O_{n,m,\varepsilon}\left(x^{\frac{mn-2}{2mn}} D_K^{\frac{1}{n}} \sum_{l\le [R]-1} \frac{d_K^m(l)}{l^{\frac{mn+2}{2mn}}} \left(\log\frac{R}{l}\right)^{-1} + x^{\frac{mn-2}{2mn}} D_K^{\frac{1}{n}} \sum_{[R]\le l\le R} \frac{d_K^m(l)}{l^{\frac{mn+2}{2mn}}} \left(\frac{Rx}{D_K^m}\right)^{\frac{1}{2mn}}\right) \\ &= O_{n,m,\varepsilon}\left(x^{\frac{mn-2}{2mn}} D_K^{\frac{1}{n}} \sum_{l\le [R]-1} \frac{d_K^m(l)}{l^{\frac{mn+2}{2mn}}} \frac{R}{R-l} + x^{\frac{mn-1}{2mn}} D_K^{\frac{1}{2n}} R^{\frac{1}{2mn}} \sum_{[R]\le l\le R} \frac{d_K^m(l)}{l^{\frac{mn+2}{2mn}}}\right) \\ &= O_{n,m,\varepsilon}\big(x^{\frac{mn-2}{2mn}} D_K^{\frac{1}{n}} R^{\frac{mn-2}{2mn}+\varepsilon} + x^{\frac{mn-1}{2mn}} D_K^{\frac{1}{2n}} R^{-\frac{mn+1}{2mn}}\big).\end{aligned}$$

Next we calculate the second O-term

$$O_{n,m,\varepsilon}\left(x^{\frac{mn-2}{2mn}} D_K^{\frac{1}{n}} \sum_{l\le R} \frac{d_K^m(l)}{l^{\frac{mn+2}{2mn}}}\left(\left(\frac{R}{l}\right)^{\frac{mn-2}{2mn}}+1\right)\right) = O_{n,m,\varepsilon}\left(x^{\frac{mn-2}{2mn}} D_K^{\frac{1}{n}} R^{\frac{mn-2}{2mn}} \sum_{l\le R} \frac{d_K^m(l)}{l}\right).$$

Since it is well known that $d_K^m(l)=O(l^{\varepsilon})$, we get

$$\begin{aligned} O_{n,m,\varepsilon}\left(x^{\frac{mn-2}{2mn}} D_K^{\frac{1}{n}} \sum_{l\le R} \frac{d_K^m(l)}{l^{\frac{mn+2}{2mn}}}\left(\left(\frac{R}{l}\right)^{\frac{mn-2}{2mn}}+1\right)\right) &= O_{n,m,\varepsilon}\left(x^{\frac{mn-2}{2mn}} D_K^{\frac{1}{n}} R^{\frac{mn-2}{2mn}} \int_1^R \frac{t^{\varepsilon}}{t}\,dt\right) \\ &= O_{n,m,\varepsilon}\left(x^{\frac{mn-2}{2mn}} D_K^{\frac{1}{n}} R^{\frac{mn-2}{2mn}+\varepsilon}\right). \end{aligned}$$

Finally we estimate the third O-term in a similar way to calculate the first O-term. One can estimate $\left(\log \frac{l}{R}\right)^{-1} = O\left(\frac{R}{l-R}\right)$, so we obtain

$$\begin{aligned} &O_{n,m,\varepsilon}\left(x^{\frac{mn-2}{2mn}} D_K^{\frac{1}{n}} R^{\frac{mn-2}{2mn}+\varepsilon} \sum_{l>R} \frac{d_K^m(l)}{l^{1+\varepsilon}} \min\left\{\left(\log\frac{l}{R}\right)^{-1}, \left(\frac{Rx}{D_K^m}\right)^{\frac{1}{2mn}}\right\}\right) \\ &= O_{n,m,\varepsilon}\left(x^{\frac{mn-2}{2mn}} D_K^{\frac{1}{n}} R^{\frac{mn-2}{2mn}+\varepsilon}\left(\sum_{R<l\le[R]+1} \frac{d_K^m(l)}{l^{1+\varepsilon}}\left(\frac{Rx}{D_K^m}\right)^{\frac{1}{2mn}} + \sum_{[R]+2\le l} \frac{d_K^m(l)}{l^{1+\varepsilon}}\left(\log\frac{l}{R}\right)^{-1}\right)\right) \\ &= O_{n,m,\varepsilon}\left(x^{\frac{mn-1}{2mn}} D_K^{\frac{1}{2n}} R^{\frac{mn-1}{2mn}+\varepsilon} \sum_{R<l\le[R]+1} \frac{d_K^m(l)}{l^{1+\varepsilon}} + x^{\frac{mn-2}{2mn}} D_K^{\frac{1}{n}} R^{\frac{mn-2}{2mn}+\varepsilon} \sum_{[R]+2\le l} \frac{d_K^m(l)}{l^{1+\varepsilon}} \frac{R}{l-R}\right) \\ &= O_{n,m,\varepsilon}\left(x^{\frac{mn-1}{2mn}} D_K^{\frac{1}{2n}} R^{-\frac{mn+1}{2mn}+\varepsilon} + x^{\frac{mn-2}{2mn}} D_K^{\frac{1}{n}} R^{\frac{mn-2}{2mn}+\varepsilon}\right). \end{aligned}$$

From above results, we obtain

$$\frac{1}{2\pi i}\int_{C_3} \zeta_K(s)^m \frac{x^s}{s}\,ds = \frac{C x^{\frac{mn-1}{2mn}} D_K^{\frac{1}{2n}}}{2\pi i} \sum_{l\le R} \frac{d_K^m(l)}{l^{\frac{mn+1}{2mn}}} \tau\left(2n\pi\left(\frac{lx}{D_K^m}\right)^{\frac{1}{mn}}\right) + O_{n,m,\varepsilon}\left(x^{\frac{mn-1}{2mn}} D_K^{\frac{1}{2n}} R^{-\frac{mn+1}{2mn}+\varepsilon} + x^{\frac{mn-2}{2mn}} D_K^{\frac{1}{n}} R^{\frac{mn-2}{2mn}+\varepsilon}\right). \tag{2-11}$$

From estimates (2-10) and (2-11), it is obtained that

$$\Delta_K^m(x) = \frac{Cx^{\frac{mn-1}{2mn}} D_K^{\frac{1}{2n}}}{2\pi i} \sum_{l \le R} \frac{d_K^m(l)}{l^{\frac{mn+1}{2mn}}} \tau\left(2mn\pi\left(\frac{lx}{D_K^m}\right)^{\frac{1}{mn}}\right) + O_{n,m,\varepsilon}\left(x^{\frac{mn-2}{2mn}+\varepsilon} D_K^{\frac{1}{n}+\varepsilon} R^{\frac{mn-2}{2mn}+\varepsilon} + x^{\frac{mn-1}{mn}+\varepsilon} D_K^{\frac{1}{n}+\varepsilon} R^{-\frac{1}{mn}+\varepsilon}\right).$$

Next we consider the above sum. Let $F_K = d_K^m * \mu$, where $*$ is the Dirichlet product symbol. From Lemma 2-5 this becomes

$$\Delta_K^m(x) = O_{n,m,\varepsilon}\Bigg(L^{1-\alpha} + x^{\frac{mn-1}{2mn}} D_K^{\frac{1}{2n}} R^{\varepsilon} \max_{L \le S \le R} S^{-\frac{mn+1}{2mn}} \times \max_{S < S_1 \le 2S} \max_{\substack{M,N \le S_1 \\ MN \asymp S}} \max_{\substack{M \le M_1 \le 2M \\ N \le N_1 \le 2N}} \left| \sum_{M < l \le M_1} F_K(l) \sum_{N < k \le N_1} \boldsymbol{e}\left(mn\left(\frac{xlk}{D_K^m}\right)^{\frac{1}{mn}}\right)\right| + x^{\frac{mn-2}{2mn}+\varepsilon} D_K^{\frac{1}{n}+\varepsilon} R^{\frac{mn-2}{2mn}+\varepsilon} + x^{\frac{mn-1}{mn}+\varepsilon} D_K^{\frac{1}{n}+\varepsilon} R^{-\frac{1}{mn}+\varepsilon}\Bigg). \qquad \square$$

Let $\mathcal{S}_K(x, S)$ be the sum in the O-term, that is,

$$S^{-\frac{mn+1}{2mn}} \max_{S < S_1 \le 2S} \max_{\substack{M,N \le S_1 \\ MN \asymp S}} \max_{\substack{M \le M_1 \le 2M \\ N \le N_1 \le 2N}} \left| \sum_{M < l \le M_1} F_K(l) \sum_{N < k \le N_1} \boldsymbol{e}\left(mn\left(\frac{xlk}{D_K^m}\right)^{\frac{1}{mn}}\right)\right|.$$

This proposition reduces the initial problem to an exponential sums problem. There are many results to estimate an exponential sums. In the next section, we estimate the Piltz divisor function by using some results for exponential sums established by many authors.

3. Estimate of counting function

In the last section, we showed that the error term of the Piltz divisor function $\Delta_K^m(x)$ can be expressed as an exponential sum. Let $X > 1$ be a real number, $1 \le M < M_1 \le 2M$ and $1 \le N < N_1 \le 2N$ be integers, and $(a_m), (b_n) \subset \mathbb{C}$ be sequences of complex numbers, and let $\alpha, \beta \in \mathbb{R}$. Then we define

$$\mathcal{S} = \sum_{M < m \le M_1} a_m \sum_{N < n \le N_1} b_n \boldsymbol{e}\left(X\left(\frac{m}{M}\right)^{\alpha}\left(\frac{n}{N}\right)^{\beta}\right). \tag{3-1}$$

Lemma 3-2 [Wu 1998]. *Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha\beta(\alpha-1)(\beta-1) \neq 0$, and $|a_m| \le 1$ and $|b_n| \le 1$ and $\mathcal{L} = \log(XMN + 2)$. Then*

$$\mathcal{L}^{-2}\mathcal{S} = O\left((XM^3N^4)^{\frac{1}{5}} + (X^4M^{10}N^{11})^{\frac{1}{16}} + (XM^7N^{10})^{\frac{1}{11}} + MN^{\frac{1}{2}} + (X^{-1}M^{14}N^{23})^{\frac{1}{22}} + X^{-\frac{1}{2}}MN\right).$$

Next Bordellès also shows this lemma by using estimates for triple exponential sums by Robert and Sargos.

Lemma 3-3 [Bordellès 2015]. *Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha\beta(\alpha-1)(\beta-1) \neq 0$, and $|a_m| \leq 1$ and $|b_n| \leq 1$. If $X = O(M)$, then*

$$(MN)^{-\varepsilon}\mathcal{S} = O\big((XM^5N^7)^{\frac{1}{8}} + N(X^{-2}M^{11})^{\frac{1}{12}} + (X^{-3}M^{21}N^{23})^{\frac{1}{24}} + M^{\frac{3}{4}}N + X^{-\frac{1}{4}}MN\big).$$

The following Srinivasan result is important for our estimating $\Delta_K^m(x)$.

Lemma 3-4 [Srinivasan 1962]. *Let N and P be positive integers and $u_n \geq 0$, $v_p > 0$, A_n, and B_p denote constants for $1 \leq n \leq N$ and $1 \leq p \leq P$. Then there exists q with properties*

$$Q_1 \leq q \leq Q_2$$

and

$$\sum_{n=1}^{N} A_n q^{u_n} + \sum_{p=1}^{P} B_p q^{-v_p} = O\Bigg(\sum_{n=1}^{N}\sum_{p=1}^{P} \sqrt[u_n+v_p]{A_n^{v_p} B_p^{u_n}} + \sum_{n=1}^{N} A_n Q_1^{u_n} + \sum_{p=1}^{P} B_p Q_2^{-v_p}\Bigg).$$

The constant involved in the O-symbol is less than $N + P$.

Srinivasan [1962] remarks that the inequality in Lemma 3-4 corresponds to the “best possible” choice of q in the range $Q_1 \leq q \leq Q_2$. We apply Lemma 3-4 to improve the error term $\Delta_K^m(x)$.

Theorem 3-5. *For every $\varepsilon > 0$ the following estimate holds. When $mn \geq 4$, then*

$$\Delta_K^m(x) = O_{n,m,\varepsilon}\big(x^{\frac{2mn-3}{2mn+1}+\varepsilon} D_K^{\frac{2m}{2mn+1}+\varepsilon}\big)$$

as x tends to infinity.

Proof. We note that

$$\Bigg|\sum_{M<l\leq M_1} F_K(l) \sum_{N<k\leq N_1} \boldsymbol{e}\Bigg(mn\bigg(\frac{xlk}{D_K^m}\bigg)^{\frac{1}{mn}}\Bigg)\Bigg| = \Bigg|\sum_{M<l\leq M_1} F_K(l) \sum_{N<k\leq N_1} \boldsymbol{e}\Bigg(mn\bigg(\frac{xMN}{D_K^m}\bigg)^{\frac{1}{mn}}\bigg(\frac{l}{M}\bigg)^{\frac{1}{mn}}\bigg(\frac{k}{N}\bigg)^{\frac{1}{mn}}\Bigg)\Bigg|.$$

We use the above lemmas with $X = mn\big(\frac{xMN}{D_K^m}\big)^{\frac{1}{mn}} > 0$. Let $0 \le \alpha \le \frac{1}{3}$; we consider four cases:

$$\begin{aligned} &\text{case 1,} && S^{\alpha} \ll N \ll S^{\frac{1}{2}},\\ &\text{case 2,} && S^{\frac{1}{2}} \ll N \ll S^{1-\alpha},\\ &\text{case 3,} && S^{1-\alpha} \ll N,\\ &\text{case 4,} && N \ll S^{\alpha}.\end{aligned}$$

When $S^{\alpha} \ll N \ll S^{\frac{1}{2}}$, we apply Lemma 3-2 and this gives

$$\begin{aligned} &S^{-\varepsilon} x^{\frac{mn-1}{2mn}} D_K^{\frac{1}{2n}} \mathcal{S}_K(x,S) \\ &= O_{n,m,\varepsilon}\big(x^{\frac{5mn-3}{10mn}} D_K^{\frac{3}{10n}} R^{\frac{2mn-3}{10mn}} + x^{\frac{2mn-1}{4mn}} D_K^{\frac{1}{4n}} R^{\frac{5mn-8}{32mn}} + x^{\frac{11mn-9}{22mn}} D_K^{\frac{9}{22n}} R^{\frac{6mn-9}{22mn}} \\ &\qquad + x^{\frac{mn-1}{2mn}} D_K^{\frac{1}{2n}} R^{\frac{mn-1}{2mn}-\frac{1}{2}\alpha} + x^{\frac{11mn-12}{22mn}} D_K^{\frac{6}{11n}} R^{\frac{15mn-24}{44mn}} + x^{\frac{mn-2}{2mn}} D_K^{\frac{1}{n}} R^{\frac{mn-2}{2mn}}\big). \end{aligned} \tag{3-6}$$

When $S^{\frac{1}{2}} \ll N \ll S^{1-\alpha}$ we use Lemma 3-2 again reversing the role of M and N. We obtain the same estimate for the case that $S^{\alpha} \ll N \ll S^{\frac{1}{2}}$. For case 3, we use Lemma 3-3:

$$\begin{aligned} &S^{-\varepsilon} x^{\frac{mn-1}{2mn}} D_K^{\frac{1}{2n}} \mathcal{S}_K(x,S) \\ &\quad = O_{n,m,\varepsilon}\big(x^{\frac{4mn-3}{8mn}} D_K^{\frac{3}{8n}} R^{\frac{mn-3}{8mn}+\frac{1}{4}\alpha} + x^{\frac{3mn-4}{6mn}} D_K^{\frac{2}{3n}} R^{\frac{5mn-8}{12mn}+\frac{1}{12}\alpha} \\ &\quad + x^{\frac{4mn-5}{8mn}} D_K^{\frac{5}{8n}} R^{\frac{3mn-5}{8mn}-\frac{1}{12}\alpha} + x^{\frac{mn-1}{2mn}} D_K^{\frac{1}{2n}} R^{\frac{mn-2}{4mn}+\frac{1}{4}\alpha} + x^{\frac{2mn-3}{4mn}} D_K^{\frac{3}{4n}} R^{\frac{2mn-3}{4mn}}\big). \end{aligned} \tag{3-7}$$

If $x^{\frac{1}{mn(1-\alpha)-1}} D_K^{-\frac{m}{mn(1-\alpha)-1}} \ll S$, the condition of Lemma 3-3, $X = O(N)$, is satisfied. Therefore, it suffices to choose $L = x^{\frac{1}{mn(1-\alpha)-1}} D_K^{-\frac{m}{mn(1-\alpha)-1}}$. For case 4, we use Lemma 3-3 again reversing the role of M and N. We obtain the same estimate for the case that $N \ll S^{\alpha}$. Combining (3-6) and (3-7) with Proposition 2-6, we obtain

$$\begin{aligned} \Delta_K^m(x) = O_{n,m,\varepsilon}\big(&x^{\frac{5mn-3}{10mn}} D_K^{\frac{3}{10n}} R^{\frac{2mn-3}{10mn}+\varepsilon} + x^{\frac{2mn-1}{4mn}} D_K^{\frac{1}{4n}} R^{\frac{5mn-8}{32mn}+\varepsilon} \\ &+ x^{\frac{11mn-9}{22mn}} D_K^{\frac{9}{22n}} R^{\frac{6mn-9}{22mn}+\varepsilon} + x^{\frac{mn-1}{2mn}} D_K^{\frac{1}{2n}} R^{\frac{mn-1}{2mn}-\frac{1}{2}\alpha+\varepsilon} \\ &+ x^{\frac{11mn-12}{22mn}} D_K^{\frac{6}{11n}} R^{\frac{15mn-24}{44mn}+\varepsilon} + x^{\frac{mn-2}{2mn}} D_K^{\frac{1}{n}} R^{\frac{mn-2}{2mn}+\varepsilon} \\ &+ x^{\frac{4mn-3}{8mn}} D_K^{\frac{3}{8n}} R^{\frac{mn-3}{8mn}+\frac{1}{4}\alpha+\varepsilon} + x^{\frac{3mn-4}{6mn}} D_K^{\frac{2}{3n}} R^{\frac{5mn-8}{12mn}+\frac{1}{12}\alpha+\varepsilon} \\ &+ x^{\frac{4mn-5}{8mn}} D_K^{\frac{5}{8n}} R^{\frac{3mn-5}{8mn}+\frac{1}{12}\alpha+\varepsilon} + x^{\frac{2mn-3}{4mn}} D_K^{\frac{3}{4n}} R^{\frac{2mn-3}{4mn}+\varepsilon} \\ &+ x^{\frac{mn-1}{mn}+\varepsilon} D_K^{\frac{1}{n}+\varepsilon} R^{-\frac{1}{mn}+\varepsilon} + x^{\frac{1-\alpha}{mn(1-\alpha)-1}} D_K^{-\frac{m(1-\alpha)}{mn(1-\alpha)-1}}\big). \end{aligned} \tag{3-8}$$

By Lemma 3-4 with $x^{\frac{1}{mn(1-\alpha)-1}} D_K^{-\frac{m}{mn(1-\alpha)-1}} \le R \le xD$ there exists R such that the error term of estimate (3-8) is much less than

$$
\begin{aligned}
&x^{\frac{2mn}{2mn+7}+\varepsilon} D_K^{\frac{2m}{2mn+7}+\varepsilon} + x^{\frac{5mn+3}{5mn+24}+\varepsilon} D_K^{\frac{5m}{5mn+24}+\varepsilon} + x^{\frac{6mn-4}{6mn+13}+\varepsilon} D_K^{\frac{6m}{6mn+13}+\varepsilon}\\
&\quad + x^{\frac{(1-\alpha)mn+\alpha-1}{(1-\alpha)mn+1}+\varepsilon} D_K^{\frac{(1-\alpha)m}{(1-\alpha)mn+1}+\varepsilon} + x^{\frac{15mn-17}{15mn+20}+\varepsilon} D_K^{\frac{3m}{3mn+4}+\varepsilon} + x^{\frac{mn-2}{mn}+\varepsilon} D_K^{\frac{1}{n}+\varepsilon}\\
&\quad + x^{\frac{(2\alpha+1)mn-2\alpha}{(2\alpha+1)mn+5}+\varepsilon} D_K^{\frac{(2\alpha+1)m}{(2\alpha+1)mn+5}+\varepsilon} + x^{\frac{(\alpha+5)mn-\alpha-7}{(\alpha+5)mn+4}+\varepsilon} D_K^{\frac{(\alpha+5)m}{(\alpha+5)mn+4}+\varepsilon}\\
&\quad + x^{\frac{(2\alpha+9)mn-2\alpha-12}{(2\alpha+9)mn+9}+\varepsilon} D_K^{\frac{(2\alpha+9)m}{(2\alpha+9)mn+9}+\varepsilon} + x^{\frac{2mn-3}{2mn+1}+\varepsilon} D_K^{\frac{2m}{2mn+1}+\varepsilon}\\
&\quad + x^{\frac{5mn(1-\alpha)-6+3\alpha}{10mn(1-\alpha)-10}+\varepsilon} D_K^{\frac{m-3m\alpha}{10mn(1-\alpha)-10}+\varepsilon} + x^{\frac{16mn(1-\alpha)-19+8\alpha}{32mn(1-\alpha)-32}+\varepsilon} D_K^{\frac{3m-8m\alpha}{32mn(1-\alpha)-32}+\varepsilon}\\
&\quad + x^{\frac{11mn(1-\alpha)-14+9\alpha}{22mn(1-\alpha)-22}+\varepsilon} D_K^{\frac{3m-9m\alpha}{22mn(1-\alpha)-22}+\varepsilon} + x^{\frac{1}{2}+\varepsilon}\\
&\quad + x^{\frac{22mn(1-\alpha)-31+24\alpha}{44mn(1-\alpha)-44}+\varepsilon} D_K^{\frac{9m-24m\alpha}{44mn(1-\alpha)-44}+\varepsilon} + x^{\frac{mn(1-\alpha)-2+2\alpha}{2mn(1-\alpha)-2}+\varepsilon} D_K^{\frac{m-2m\alpha}{2mn(1-\alpha)-2}+\varepsilon}\\
&\quad + x^{\frac{4mn(1-\alpha)-6+5\alpha}{8mn(1-\alpha)-8}+\varepsilon} D_K^{\frac{2m-5m\alpha}{8mn(1-\alpha)-8}+\varepsilon} + x^{\frac{6mn(1-\alpha)-9+9\alpha}{12mn(1-\alpha)-12}+\varepsilon} D_K^{\frac{3m-9m\alpha}{12mn(1-\alpha)-12}+\varepsilon}\\
&\quad + x^{\frac{12mn(1-\alpha)-18+17\alpha}{24mn(1-\alpha)-24}+\varepsilon} D_K^{\frac{6m-17m\alpha}{24mn(1-\alpha)-24}+\varepsilon} + x^{\frac{2mn(1-\alpha)-3+3\alpha}{4mn(1-\alpha)-4}+\varepsilon} D_K^{\frac{m-3m\alpha}{4mn(1-\alpha)-4}+\varepsilon}\\
&\quad + x^{\frac{1-\alpha}{mn(1-\alpha)-1}} D_K^{-\frac{m(1-\alpha)}{mn(1-\alpha)-1}}.
\end{aligned}
$$

When $mn \ge 4$ and $\alpha = \frac{mn+3}{7mn-5}$, then we have

$$\Delta_K^m(x) = O_{n,m,\varepsilon}\left(x^{\frac{2mn-3}{2mn+1}+\varepsilon} D_K^{\frac{2m}{2mn+1}+\varepsilon}\right). \qquad \square$$

For $mn \ge 4$ this theorem gives new results for the Piltz divisor problem over number fields. In particular, if we fix K with $[K:\mathbb{Q}] = 4$, then we improve the estimate for $\Delta_K(x)$ as follows:

Corollary 3-9. *For any number field K with $[K:\mathbb{Q}] = 4$,*

$$\Delta_K(x) = O_{K,\varepsilon}\left(x^{\frac{5}{9}+\varepsilon}\right).$$

This result is better than Bordellès' result.

4. Application

In this section we introduce some applications of our theorems. First we obtain a uniform estimate for the ideal counting function $I_K(x)$. From the proof of Theorem 3-5, we obtain the following theorem.

Theorem 4-1. *For all $\varepsilon > 0$ for any fixed $0 \le \beta \le \frac{8}{2n+5} - \varepsilon$ and $C > 0$ the following holds. If K runs through number fields with $[K:\mathbb{Q}] \le n$ and $D_K \le Cx^\beta$, then*

$$\Delta_K(x) = O_{C,n,\varepsilon}\big(x^{\frac{2n-3+2\beta}{2n+1}+\varepsilon}\big).$$

The condition $D_K \le Cx^\beta$ is caused by the relation between the principal term and the error term. It is well known that $I_K(x)$ is very important to estimate the distribution of relatively r-prime lattice points. We regard an ℓ-tuple of ideals $(\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_\ell)$ of $\mathbb{O}_K$ as a lattice point in K^ℓ. We say that a lattice point $(\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_\ell)$ is *relatively r-prime* for a positive integer r if there exists no prime ideal $\mathfrak{p}$ such that $\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_\ell \subset \mathfrak{p}^r$. Let $V_\ell^r(x, K)$ denote the number of relatively r-prime lattice points $(\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_\ell)$ such that their ideal norm $\mathfrak{N}\mathfrak{a}_i \le x$.

B. D. Sittinger [2010] shows that

$$V_\ell^r(x, K) \sim \frac{\rho_K^\ell}{\zeta_K(r\ell)} x^\ell,$$

where ρ_K is the residue of ζ_K at $s = 1$. It is well known that

$$\rho_K = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K \sqrt{D_K}}, \tag{4-2}$$

where h_K is the class number of K, R_K is the regulator of K, and w_K is the number of roots of unity in $\mathbb{O}_K^*$.

After that we show some results for the error term:

$$E_\ell^r(x, K) = V_\ell^r(x, K) - \frac{\rho_K^\ell}{\zeta_K(r\ell)} x^\ell.$$

In [Takeda 2017; Takeda and Koyama 2018] we consider the relation between the relatively r-prime problem and other mathematical problems. If we assume the Lindelöf hypothesis for $\zeta_K(s)$, then it holds that for all $\varepsilon > 0$

$$E_\ell^r(x, K) = \begin{cases} O_\varepsilon\big(x^{\frac{1}{r}(\frac{3}{2}+\varepsilon)}\big) & \text{if } r\ell = 2, \\ O_\varepsilon\big(x^{\ell-\frac{1}{2}+\varepsilon}\big) & \text{otherwise.} \end{cases} \tag{4-3}$$

From easy calculation, we obtain the following corollary.

Corollary 4-4. *For all $\varepsilon > 0$ and for any fixed $0 \le \beta \le \frac{8}{2n+5} - \varepsilon$ and $C > 0$ the following holds. If K runs through number fields with $[K:\mathbb{Q}] \le n$ and $D_K \le Cx^\beta$, then*

$$E_\ell^r(x, K) = \begin{cases} O_{C,n,\varepsilon}\big(x^{\frac{4n-2}{r(2n+1)}+\frac{4}{2n+1}\beta+\varepsilon}\big) & \text{if } r\ell = 2, \\ O_{C,n,\varepsilon}\big(x^{\ell-\frac{4}{2n+1}+\frac{2n+5-(2n+1)\ell}{2(2n+1)}\beta+\varepsilon}\big) & \text{otherwise.} \end{cases}$$

For the proof of this corollary, please see the proof of Theorem 4.1 of [Takeda and Koyama 2018].

5. Conjecture

Theorem 4-1 states good uniform upper bounds. It is proposed that for all number fields K the best uniform upper bound of the error term is better than that on the assumption of the Lindelöf hypothesis (1-6).

Conjecture 5-1. *If K runs through number fields with $D_K < x$, then*

$$\Delta_K^m(x) = o\big(x^{\frac{1}{2}}\big).$$

From estimate (1-5), this conjecture may give the best estimate for uniform upper bound of $\Delta_K^m(x)$. As we remarked above (Theorem 1-2) this conjecture is very difficult even when K is fixed and $m = 1$.

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WATARU TAKEDA
DEPARTMENT OF MATHEMATICS
NAGOYA UNIVERSITY
CHIKUSA-KU
NAGOYA
JAPAN
d18002r@math.nagoya-u.ac.jp

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