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OPTIMAL DECAY ESTIMATE OF STRONG SOLUTIONS FOR THE 3D INCOMPRESSIBLE OLDROYD-B MODEL WITHOUT DAMPING

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OPTIMAL DECAY ESTIMATE OF STRONG SOLUTIONS FOR THE 3D INCOMPRESSIBLE OLDROYD-B MODEL WITHOUT DAMPING

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We obtain the decay estimate of the global solution for the 3D incompressible Oldroyd-B model with only dissipation. The decay rate is optimal in the sense that this rate coincides with that of the linear system, which improves upon work by Zhu (*J. Funct. Anal.* 274:7 (2018) 2039–2060.)

1. Introduction

In this paper, we consider the Cauchy problem for the three-dimensional (3D) incompressible Oldroyd-B model given by

$$\begin{cases} \partial_t u + u\cdot\nabla u - \nu\Delta u + \nabla p = \mu_1 \operatorname{div}\tau, \ (t,x)\in\mathbb{R}^+\times\mathbb{R}^3 \\ \partial_t \tau + u\cdot\nabla\tau + a\tau + Q(\tau,\nabla u) = \mu_2 D(u), \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0(x), \quad \tau|_{t=0} = \tau_0(x), \end{cases} \tag{1-1}$$

where $u=(u_1,u_2,u_3)$ stands for the 3D velocity field, p the pressure and τ the non-Newtonian part of the stress tensor which can be seen as a symmetric matrix here. The values ν, a, μ_1 and μ_2 are nonnegative parameters, where we call μ_1 and μ_2 the coupling parameters. $D(u)$ and $W(u)$ are the deformation tensor and vorticity tensor, which can be given by

$$D(u) \triangleq \frac{\nabla u + (\nabla u)^\top}{2}, \quad W(u) \triangleq \frac{\nabla u - (\nabla u)^\top}{2},$$

and

$$Q(\tau,\nabla u) = \tau W(u) - W(u)\tau - \alpha(D(u)\tau + \tau D(u)), \ \alpha\in[-1,1].$$

The initial data are (u_0,τ_0) satisfying $\operatorname{div} u_0 = 0$ and $(\tau_0)_{ij} = (\tau_0)_{ji}$.

We refer to [Bird et al. 1977; Chemin and Masmoudi 2001; Oldroyd 1958] for the details about the derivation of (1-1). Guillopé and Saut [1990a; 1990b] obtained

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local existence and uniqueness, when the initial data is in Sobolev space, which was extended to the Lebesgue space in [Fernández Cara et al. 1994]. Lions and Masmoudi [2000] obtained global existence of the weak solution for the case $\alpha = 0$. However, to the best of our knowledge, whether the case $\alpha \neq 0$ can yield a global weak solution is an open question. Considering the initial data in critical Besov space, Chemin and Masmoudi [2001] showed local well-posedness of the solution and global well-posedness with small initial data, where they required the small coupling parameter for the global result. We refer to [Chen and Miao 2008; Zi et al. 2014] for the results in generalized space. By using some techniques developed in the study of compressible Navier–Stokes, Zi et al. [2014] removed the condition concerning the small coupling parameter. Recently, Fang and Zi [2016] proved global well-posedness with a new class of large initial data admitting large initial vertical velocity. By using a decomposition technique, we [Wan 2019] improved the initial condition in [Fang and Zi 2016] to the more generalized initial condition. In particular, for the 2D case, [Wan 2019] proved global well-posedness with large initial velocity, which improved upon the corresponding work in [Fang and Zi 2016].

We point out that all the above works are based on $a > 0$. If $a = 0$, it seems difficult to proved the global well-posedness even under small initial data. By using a new method, i.e., constructing the time-weighted energies, Zhu [2018] first proved global well-posedness in 3D under small initial data, which can be showed as follows:

Theorem 1.1 [Zhu 2018]. *Let $\nu = \mu_1 = \mu_2 = 1$ and $a = 0$. Let $N \geq 7$ be a big enough constant. Assume we have the initial data $(|D|^{-1}u_0, |D|^{-1}\tau_0) \in H^N(\mathbb{R}^3) \times H^N(\mathbb{R}^3)$. There exists a small enough constant ϵ such that if*

$$\||D|^{-1}u_0\|_{H^N} + \||D|^{-1}\tau_0\|_{H^N} < \epsilon, \tag{1-2}$$

the system (1-1) has a unique global solution (u, τ) satisfying

$$\|u(t)\|_{H^{N-1}} + \|\tau(t)\|_{H^{N-1}} \lesssim \epsilon \quad \text{for all } t > 0.$$

Remark 1.2. Zhu [2018] only assumed $N = 3$; here the assumption of $N \geq 7$ in Theorem 1.1 is used to get the faster decay rate of the solution in the following context. For instance, we need high regularity of the solution when showing the decay estimate of $\|\nabla u(t)\|_{L^\infty}$; see Section 5 for the details.

Let us remark that the approach in [Zhu 2018] seems difficult to use for the 2D case, since $\|u(t)\|_{L^\infty(\mathbb{R}^2)}$ may yield a weakened decay, which is not integrable in time and may bring the growth of the solutions. Recently, based on a time-space approach, a new system concerning $u - 2\mathbb{P}\,\mathrm{div}\tau$ and the estimate in Besov space, [Wan 2017b] proved the global small solution for the 2D case.

It is known that long-time behavior of the global solution is an interesting and important issue in the studies of many fluids. Notice that Zhu [2018] obtained the decay estimate in the 3D case by constructing some time-weighted estimates such as

$$\|u(t)\|_{L^2} \lesssim (1+t)^{-\frac{1}{2}}, \qquad \|\nabla u(t)\|_{L^2} \lesssim (1+t)^{-1}. \tag{1-3}$$

So a natural question is whether the decay estimate (1-3) is optimal. Motivated by this, in this paper, we will focus on the long-time behavior of the global solution for (1-1) with $a=0$. Let us first introduce some notation:

$$X(t) \triangleq \sup_{0\le\tau\le t} \big\{\langle\tau\rangle^{\frac{3}{4}}\|u(\tau)\|_{L^2} + \langle\tau\rangle^{\frac{3}{2}}\|\widehat{u}(\tau)\|_{L^1}\big\},$$

$$Y(t) \triangleq \sup_{0\le\tau\le t} \big\{\langle\tau\rangle^{\frac{5}{4}}(\|\nabla u(\tau)\|_{L^2} + \|\mathcal{A}(\tau)\|_{L^2}) + \langle\tau\rangle^{2}\||\xi|\widehat{u}(\tau)\|_{L^1}\big\},$$

where $\mathcal{A} = \mathbb{P}\,\mathrm{div}\tau$. Now, we state our main result.

Theorem 1.3. *Let (u,τ) be the solution of (1-1) obtained in Theorem 1.1 with the initial data satisfying (1-2) and $\|(u_0,\tau_0)\|_{L^1} < \epsilon$. Then we have*

$$X(t) + Y(t) \lesssim \epsilon.$$

In particular, for all $t>0$, the following decay estimates hold:

$$\|u(t)\|_{L^2} \lesssim \epsilon\langle t\rangle^{-\frac{3}{4}}, \qquad \|\nabla u(t)\|_{L^2} + \|\mathcal{A}(t)\|_{L^2} \lesssim \epsilon\langle t\rangle^{-\frac{5}{4}}$$

and

$$\|u(t)\|_{L^\infty} \lesssim \epsilon\langle t\rangle^{-\frac{3}{2}}, \qquad \|\nabla u(t)\|_{L^\infty} \lesssim \epsilon\langle t\rangle^{-2}.$$

Remark 1.4. One can also get $\|\mathcal{A}(t)\|_{L^\infty} \lesssim \epsilon\langle t\rangle^{-2}$ by following the same idea. Since this decay estimate does not play an essential role in the estimate of $X(t)$ and $Y(t)$, we omit it.

Remark 1.5. One can easily find the obtained decay rate is faster than (1-3). In addition, this faster decay rate yields many better time-weighted estimates. For instance, by using our decay estimate we get that

$$\int_0^t \langle t'\rangle^{a}\|\nabla u(t')\|_{L^2}\,dt' \lesssim \epsilon \quad \text{for all } a \in \big(0, \tfrac{1}{4}\big),$$

while the associated result in [Zhu 2018, page 7] can only yield

$$\int_0^t \langle t'\rangle^{a}\|\nabla u(t')\|_{L^2}\,dt' \lesssim \epsilon \quad \text{for all } a < 0.$$

Remark 1.6. By using interpolation inequality, we can obtain the explicit decay rate in L^p $(2 \le p \le \infty)$ space, which is optimal since it is consistent with the linear part of system (1-1).

The paper is structured as follows. In Section 2, we provide some definitions of spaces and several lemmas. Section 3 is devoted to giving the integral representation. Section 4 bounds the estimate of $\|u(t)\|_{H^1}$. Section 5 provides the estimate of $\|\hat{u}(t)\|_{L^1}$, $\||\xi|\hat{u}(t)\|_{L^1}$ and $\|\mathcal{A}(t)\|_{L^2}$. In the last section, we prove Theorem 1.3. In the Appendix, we provide the proof of Lemma 4.1 and (5-3).

Let us complete this section by describing the notation we shall use in this paper.

Notation. For two operators A and B, we denote by $[A, B] = AB - BA$ the commutator between A and B, $\langle t\rangle$ means $1 + |t|$, and $A \lesssim B$ means that there exists a constant C such that $A \le CB$.

2. Preliminaries

In this section, we give some necessary definitions, propositions and lemmas in d dimensions.

The fractional Laplacian operator $|D|^\alpha = (-\Delta)^{\frac{\alpha}{2}}$ is defined through the Fourier transform, namely,

$$\widehat{|D|^\alpha f}(\xi) \triangleq |\xi|^\alpha \hat{f}(\xi),$$

where the Fourier transform is given by

$$\hat{f}(\xi) \triangleq \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\,dx.$$

Let $\mathfrak{C} = \left\{\xi \in \mathbb{R}^d,\ \frac{3}{4} \le |\xi| \le \frac{8}{3}\right\}$. Choose a smooth radial function φ supported on $\mathfrak{C}$ such that

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}.$$

We denote $\varphi_j = \varphi(2^{-j}\xi)$, $h = \mathfrak{F}^{-1}\varphi$, where $\mathfrak{F}^{-1}$ stands for the inverse Fourier transform. Then the dyadic blocks Δ_j and S_j can be defined as

$$\Delta_j f = \varphi(2^{-j}D)f = 2^{jd}\int_{\mathbb{R}^d} h(2^j y) f(x-y)\,dy, \quad S_j f = \sum_{k\le j-1} \Delta_k f.$$

We can easily verify that with our choice of φ

$$\Delta_j\Delta_k f = 0 \text{ if } |j-k| \ge 2 \quad \text{and} \quad \Delta_j(S_{k-1} f \Delta_k f) = 0 \text{ if } |j-k| \ge 5.$$

Let us recall the definition of the Besov space.

Definition 2.1. Let $s \in \mathbb{R}$ and $(p, q) \in [1, \infty]^2$. The homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^d)$ is defined by

$$\dot{B}^s_{p,q}(\mathbb{R}^d) = \{f \in \mathfrak{S}'(\mathbb{R}^d);\ \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} = \begin{cases} \Big(\sum_{j \in \mathbb{Z}} 2^{sqj} \|\Delta_j f\|^q_{L^p(\mathbb{R}^d)}\Big)^{\frac{1}{q}} & \text{for } 1 \le q < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{sj} \|\Delta_j f\|_{L^p(\mathbb{R}^d)} & \text{for } q = \infty, \end{cases}$$

and $\mathfrak{S}'(\mathbb{R}^d)$ denotes the dual space of

$$\mathfrak{S}(\mathbb{R}^d) = \{f \in \mathcal{S}(\mathbb{R}^d); \partial^\alpha \hat{f}(0) = 0, \text{ for all } \alpha \in \mathbb{N}^d \text{ multi-index}\}$$

and can be identified by the quotient space of $\mathcal{S}'/\mathcal{P}$ with the polynomial space $\mathcal{P}$.

Thanks to the definition of Δ_j, we have

$$\|[\Delta_j, f]g\|_{L^p} \lesssim 2^{-j} \|\nabla f\|_{L^{p_1}} \|g\|_{L^{p_2}}, \tag{2-1}$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Equation (2-1) will be used in the Appendix.

The following proposition provides Bernstein type inequalities. For more details about Besov space such as some useful embedding relations, see, e.g., [Bahouri et al. 2011; Stein 1970].

Proposition 2.2. *Let $1 \le p \le q \le \infty$. Then for any $\beta, \gamma \in (\mathbb{N} \cup \{0\})^d$, there exists a constant C independent of f, j such that*:

(1) *If f satisfies*

$$\operatorname{supp} \hat{f} \subset \{\xi \in \mathbb{R}^d : |\xi| \le \mathcal{K} 2^j\},$$

then

$$\|\partial^\gamma f\|_{L^q(\mathbb{R}^d)} \le C 2^{j|\gamma| + jd\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^d)}.$$

(2) *If f satisfies*

$$\operatorname{supp} \hat{f} \subset \{\xi \in \mathbb{R}^d : \mathcal{K}_1 2^j \le |\xi| \le \mathcal{K}_2 2^j\}$$

then

$$\|f\|_{L^p(\mathbb{R}^d)} \le C 2^{-j|\gamma|} \sup_{|\beta| = |\gamma|} \|\partial^\beta f\|_{L^p(\mathbb{R}^d)}.$$

For the special case $p = q = 2$, we have

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} \approx \|f\|_{\dot{B}^s_{2,2}(\mathbb{R}^d)}.$$

The $\dot{H}^s(\mathbb{R}^d)$ and $H^s(\mathbb{R}^d)$ norm of f can be also defined as follows:

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} \triangleq \||D|^s f\|_{L^2(\mathbb{R}^d)}, \quad s \in \mathbb{R},$$
$$\|f\|_{H^s(\mathbb{R}^d)} \triangleq \|f\|_{L^2(\mathbb{R}^d)} + \||D|^s f\|_{L^2(\mathbb{R}^d)}, \quad s > 0.$$

Let us introduce the homogeneous Bony's decomposition:

$$uv = T_u v + T_v u + R(u, v),$$

where

$$T_u v = \sum_{j\in\mathbb{Z}} S_{j-1} u \Delta_j v, \quad T_v u = \sum_{j\in\mathbb{Z}} \Delta_j u S_{j-1} v, \quad R(u, v) = \sum_{j\in\mathbb{Z}} \Delta_j u \tilde{\Delta}_j v;$$

here $\tilde{\Delta}_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1}$.

Lemma 2.3. [Kato and Ponce 1988] (i) *Let $s > 0$ and $1 \le p, r \le \infty$, then*

$$\|fg\|_{\dot{B}^s_{p,r}(\mathbb{R}^d)} \le C\{\|f\|_{L^{p_1}(\mathbb{R}^d)}\|g\|_{\dot{B}^s_{p_2,r}(\mathbb{R}^d)} + \|g\|_{L^{r_1}(\mathbb{R}^d)}\|f\|_{\dot{B}^s_{r_2,r}(\mathbb{R}^d)}\}, \tag{2-2}$$

where $1 \le p_1, r_1 \le \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2}$.

[Kenig et al. 1991] (ii) *Let $s > 0$, and $1 < p < \infty$, then*

$$\begin{aligned}\|[|D|^s, f]g\|_{L^p(\mathbb{R}^d)} \\ \le C\{\|\nabla f\|_{L^{p_1}(\mathbb{R}^d)}\||D|^{s-1}g\|_{L^{p_2}(\mathbb{R}^d)} + \||D|^s f\|_{L^{p_3}(\mathbb{R}^d)}\|g\|_{L^{p_4}(\mathbb{R}^d)}\},\end{aligned} \tag{2-3}$$

where $1 < p_2, p_3 < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$.

The inequalities below are used frequently in the proof.

Lemma 2.4 [Wan 2017a]. *If $0 < s_1 \le s_2$, then*

$$\int_0^t \langle t-\tau\rangle^{-s_1}\langle\tau\rangle^{-s_2}\,d\tau \le \begin{cases} C\langle t\rangle^{-s_1} & \text{if } s_2 > 1, \\ C\langle t\rangle^{-s_1}\ln(1+t) & \text{if } s_2 = 1, \\ C\langle t\rangle^{1-s_1-s_2} & \text{if } s_2 < 1. \end{cases}$$

Remark 2.5. For the case $s_1 \ge s_2$, since one can get a similar result by using the change of variable, we omit the details.

Let us introduce a generalized estimate of the solution to the heat equation.

Lemma 2.6. *If $f \in L^1(\mathbb{R}^d)$, we have*

$$\|e^{t\Delta} f\|_{L^2(\mathbb{R}^d)} \lesssim t^{-\frac{d}{4}}\|f\|_{\dot{B}^0_{1,\infty}(\mathbb{R}^d)}.$$

Proof. Thanks to the interpolation inequality

$$\|g\|_{L^2} \lesssim \|g\|^{\frac{1}{2}}_{\dot{B}^{-1}_{2,\infty}}\|g\|^{\frac{1}{2}}_{\dot{B}^1_{2,\infty}},$$

we have

$$\|e^{t\Delta}f\|_{L^2} \lesssim \|e^{t\Delta}f\|^{\frac{1}{2}}_{\dot{B}^{-1}_{2,\infty}} \|e^{t\Delta}f\|^{\frac{1}{2}}_{\dot{B}^{1}_{2,\infty}}.$$

So the aim reduces to the estimate of $\|e^{t\Delta}f\|_{\dot{B}^a_{2,\infty}}$, $a=-1,1$. By Bernstein's inequality, for all $a \geq -\frac{3}{2}$, we have

$$\begin{aligned}
\text{(2-4)}\quad \|e^{t\Delta}f\|_{\dot{B}^a_{2,\infty}} &\lesssim \sup_{j\in\mathbb{Z}} 2^{aj}\|\Delta_j e^{t\Delta}f\|_{L^2} \lesssim \sup_{j\in\mathbb{Z}} 2^{aj} e^{-2^{2j-1}t}\|\Delta_j f\|_{L^2} \\
&\leq \sup_{j\in\mathbb{Z}} 2^{\left(a+\frac{d}{2}\right)j} e^{-2^{2j-1}t}\|\Delta_j f\|_{L^1} \lesssim t^{-\left(\frac{a}{2}+\frac{d}{4}\right)}\|f\|_{\dot{B}^0_{1,\infty}}.
\end{aligned}$$

Setting $a=-1$ and $a=1$ in (2-4), respectively, we can conclude the proof. □

3. Spectral analysis

In this section, we give the integral representation of the solution to (1-1). In fact, we repeat the procedure in Section 3 of [Wan 2017b]. We use v_i to stand for the i-th component of the vector v in the following context. Denote

$$\mathcal{A} \triangleq \mathbb{P}\,\mathrm{div}\tau.$$

Let us investigate the spectrum properties of the following system:

$$\text{(3-1)}\qquad \begin{cases} \partial_t u_i - \Delta u_i = \mathcal{A}_i + G_i, \\ \partial_t \mathcal{A}_i = \frac{1}{2}\Delta u_i + F_i + H_i, \\ G = -\mathbb{P}\,(u\cdot\nabla u), \quad F = -\mathbb{P}\,\mathrm{div}(u\cdot\nabla\tau), \\ H = -\mathbb{P}\,\mathrm{div}(Q(\tau,\nabla u)), \quad i=1,2,3, \end{cases}$$

where $\mathbb{P}$ is the Leray operator. Denote

$$A \triangleq \begin{pmatrix} -|\xi|^2 & 1 \\ -\frac{|\xi|^2}{2} & 0 \end{pmatrix},$$

then the eigenvalues of the matrix A can be given as

$$\lambda_\pm = \begin{cases} \frac{1}{2}(-|\xi|^2 \pm i|\xi|\sqrt{2-|\xi|^2}) & \text{when } |\xi|<\sqrt{2}, \\ \frac{1}{2}(-|\xi|^2 \pm |\xi|\sqrt{|\xi|^2-2}) & \text{when } |\xi|\geq\sqrt{2}, \end{cases}$$

where $i=\sqrt{-1}$. After Fourier transform, (3-1) reduces to

$$\text{(3-2)}\qquad \partial_t \begin{pmatrix} \widehat{u_i} \\ \widehat{\mathcal{A}_i} \end{pmatrix}(\xi) = A\begin{pmatrix} \widehat{u_i} \\ \widehat{\mathcal{A}_i} \end{pmatrix}(\xi) + \begin{pmatrix} \widehat{G_i} \\ \widehat{F_i}+\widehat{H_i} \end{pmatrix}(\xi),$$

By using the standard method of diagonalization via the eigenvalues and eigenvectors, we can get from (3-2) that

$$
\begin{aligned}
u(t,x) &= M_{11}(\partial,t)u_0(x) + M_{12}(\partial,t)\mathcal{A}_0(x) \\
&\qquad + \int_0^t M_{11}(\partial,t-s)G\,ds + \int_0^t M_{12}(\partial,t-s)(F+H)\,ds \\
\mathcal{A}(t,x) &= M_{21}(\partial,t)u_0(x) + M_{22}(\partial,t)\mathcal{A}_0(x) \\
&\qquad + \int_0^t M_{21}(\partial,t-s)G\,ds + \int_0^t M_{22}(\partial,t-s)(F+H)\,ds,
\end{aligned} \tag{3-3}
$$

where

$$\widehat{M_{ij}f}(\xi,t) \triangleq \widehat{M_{ij}}(\xi,t)\hat{f}(\xi), \quad (i,j)\in\{1,2\}^2,$$

and

$$\begin{pmatrix} \widehat{M_{11}}(\xi,t) & \widehat{M_{12}}(\xi,t) \\ \widehat{M_{21}}(\xi,t) & \widehat{M_{22}}(\xi,t) \end{pmatrix} \triangleq \begin{pmatrix} \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} & \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \\ \frac{|\xi|^2}{2}\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_- - \lambda_+} & \frac{\lambda_- e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t}}{\lambda_- - \lambda_+} \end{pmatrix}.$$

To bound $M_{ij}(\partial,t)$, we split the whole space $\mathbb{R}^3$ into the following four regions:

$$
\begin{aligned}
D_1 &\triangleq \{\xi\in\mathbb{R}^3 : |\xi|<1\}, \\
D_2 &\triangleq \{\xi\in\mathbb{R}^3 : 1\le|\xi|<\sqrt{2}\}, \\
D_3 &\triangleq \{\xi\in\mathbb{R}^3 : \sqrt{2}\le|\xi|<2\}, \\
D_4 &\triangleq \{\xi\in\mathbb{R}^3 : |\xi|\ge 2\}.
\end{aligned}
$$

Let us keep the fact that $|\xi|\approx 1$ when $\xi\in D_2\cup D_3$ in mind. Next, a proposition devoted to the estimates of $\widehat{M_{ij}}(\xi,t)$ is given as follows.

Proposition 3.1 [Wan 2017b]. *For all $(i,j)\in\{1,2\}^2$, $\widehat{M_{ij}}(\xi,t)$ satisfies the following estimates*:

(1) *When $\xi\in D_1$,*

$$
\begin{aligned}
&|\widehat{M_{11}}(\xi,t)| \lesssim e^{-\frac{|\xi|^2}{2}t}, && |\widehat{M_{12}}(\xi,t)| \lesssim |\xi|^{-1}e^{-\frac{|\xi|^2}{2}t}, \\
&|\widehat{M_{21}}(\xi,t)| \lesssim |\xi|e^{-\frac{|\xi|^2}{2}t}, && |\widehat{M_{22}}(\xi,t)| \lesssim e^{-\frac{|\xi|^2}{2}t}.
\end{aligned} \tag{3-4}
$$

(2) *When $\xi\in D_2$,*

$$
\begin{aligned}
&|\widehat{M_{11}}(\xi,t)| \lesssim e^{-\frac{|\xi|^2}{4}t}, && |\widehat{M_{12}}(\xi,t)| \lesssim |\xi|^{-1}e^{-\frac{|\xi|^2}{4}t}, \\
&|\widehat{M_{21}}(\xi,t)| \lesssim |\xi|e^{-\frac{|\xi|^2}{4}t}, && |\widehat{M_{22}}(\xi,t)| \lesssim e^{-\frac{|\xi|^2}{4}t}.
\end{aligned}
$$

(3) *When* $\xi \in D_3$,

$$|\widehat{M_{11}}(\xi,t)| \lesssim e^{-\frac{|\xi|^2}{16}t}, \qquad |\widehat{M_{12}}(\xi,t)| \lesssim |\xi|^{-1}e^{-\frac{|\xi|^2}{16}t},$$
$$|\widehat{M_{21}}(\xi,t)| \lesssim |\xi|e^{-\frac{|\xi|^2}{16}t}, \quad |\widehat{M_{22}}(\xi,t)| \lesssim e^{-\frac{|\xi|^2}{16}t}.$$

(4) *When* $\xi \in D_4$,

$$\begin{aligned} &|\widehat{M_{11}}(\xi,t)| \lesssim e^{-\frac{t}{2}}, \quad |\widehat{M_{12}}(\xi,t)| \lesssim |\xi|^{-2}e^{-\frac{t}{2}},\\ &|\widehat{M_{21}}(\xi,t)| \lesssim e^{-\frac{t}{2}}, \quad |\widehat{M_{22}}(\xi,t)| \lesssim e^{-\frac{t}{2}}. \end{aligned} \tag{3-5}$$

4. The estimate of $\|u(t)\|_{H^1}$

We will give the estimate of $\|u(t)\|_{L^2}$ and $\|\nabla u(t)\|_{L^2}$ in order.

The estimate of $\|u(t)\|_{L^2}$. Thanks to $\|\widehat{f}\|_{L^2} = \|f\|_{L^2}$, we have

$$\|u(t)\|_{L^2} = \|\hat{u}(t)\|_{L^2} = \sum_{i=1}^{4} \|\hat{u}(t)\|_{L^2(D_i)},$$

which is sufficient for us to estimate the four terms on the right-hand side.

4A. ***The estimate of*** $\|\hat{u}(t)\|_{L^2(\mathbb{R}^3\setminus D_4)}$. Using (3-3), we can get

$$\begin{aligned} \|\hat{u}(t)\|_{L^2(D_1)} \le \|\widehat{M_{11}}(t)\widehat{u_0}\|_{L^2(D_1)} &+ \|\widehat{M_{12}}(t)\widehat{\mathcal{A}_0}\|_{L^2(D_1)}\\ &+\int_0^t \|\widehat{M_{11}}(t-s)\widehat{G}\|_{L^2(D_1)}\,ds + \int_0^t \|\widehat{M_{12}}(t-s)\widehat{F}\|_{L^2(D_1)}\,ds\\ &+\int_0^t \|\widehat{M_{12}}(t-s)\widehat{H}\|_{L^2(D_1)}\,ds \triangleq I_1 + I_2 + \cdots + I_5. \end{aligned}$$

For I_1, using (3-4) and the estimate of the solution for the heat equation given by

$$\|e^{t\Delta} f\|_{L^2} \lesssim t^{-\frac{3}{4}}\|f\|_{L^1}, \tag{4-1}$$

we get

$$I_1 \lesssim \|e^{-\frac{|\xi|^2}{2}t}\widehat{u_0}\|_{L^2(D_1)} \lesssim \|e^{-\frac{|\xi|^2}{2}t}\widehat{u_0}\|_{L^2} = \|e^{\frac{1}{2}\Delta t}u_0\|_{L^2} \lesssim t^{-\frac{3}{4}}\|u_0\|_{L^1}.$$

Together with $I_1 \lesssim \|u_0\|_{L^2}$, this yields

$$I_1 \lesssim \langle t\rangle^{-\frac{3}{4}}\|u_0\|_{L^1\cap L^2}.$$

A similar method leads to

$$I_2 \lesssim \langle t\rangle^{-\frac{3}{4}}\|\tau_0\|_{L^1\cap L^2}.$$

Before we proceed, note that

$$\|\widehat{M_{11}}(t-s)\widehat{G}\|_{L^2(D_1)} \le \|\widehat{M_{11}}(t-s)\,\widehat{u\cdot\nabla u}\,\|_{L^2(D_1)}.$$

For I_3, using (3-4) and (4-1), we have

$$I_3 \lesssim \int_0^t (t-s)^{-\frac{3}{4}}\|u\cdot\nabla u\|_{L^1}\,ds,$$

which, with the estimate $I_3 \lesssim \int_0^t \|u\cdot\nabla u\|_{L^2}\,ds$, implies

$$\begin{aligned} I_3 &\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{4}}(\|u\cdot\nabla u\|_{L^1}+\|u\cdot\nabla u\|_{L^2})\,ds \\ &\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{4}}\|u\|_{L^2}\|\nabla u\|_{L^2\cap L^\infty}\,ds \\ &\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{4}}(\langle s\rangle^{-\frac{11}{4}}+\langle s\rangle^{-2})\,ds\;X(t)Y(t) \\ &\lesssim \langle t\rangle^{-\frac{3}{4}}X(t)Y(t). \end{aligned}$$

For I_4, using

$$\mathbb{P}\,\mathrm{div}(u\cdot\nabla\tau) = \sum_{i=1,2,3}\partial_i\mathbb{P}\,\mathrm{div}(u_i\tau)$$

and (3-4), we get

$$I_4 \lesssim \int_0^t \|e^{-\frac{|\xi|^2}{2}(t-s)}|\xi||\widehat{u\otimes\tau}|\|_{L^2(D_1)}\,ds.$$

Using the spherical coordinate system

$$\xi_1 = r\sin\varphi\cos\theta,\quad \xi_2 = r\sin\varphi\sin\theta,\quad \xi_3 = r\cos\varphi,$$

we can obtain

$$\begin{aligned} \|e^{-\frac{|\xi|^2}{2}t}|\xi|\hat{f}\|_{L^2(D_1)} &= \left(\int_{|\xi|\le 1} e^{-|\xi|^2 t}|\xi|^2|\hat{f}|^2\,d\xi\right)^{\frac{1}{2}} \\ &= \left(\int_0^{2\pi} d\theta\int_0^{\pi} d\varphi\int_0^1 e^{-r^2t}r^4\sin\varphi|\hat{f}(r,\theta,\varphi)|\,dr\right)^{\frac{1}{2}} \\ &\lesssim \left(\int_0^1 e^{-r^2t}r^4dr\right)^{\frac{1}{2}}\|\hat{f}\|_{L^\infty} \\ &\lesssim t^{-\frac{5}{4}}\|f\|_{L^1}\left(\int_0^{\sqrt{t}} e^{-r^2}r^4\,dr\right)^{\frac{1}{2}} \\ &\lesssim t^{-\frac{5}{4}}\|f\|_{L^1}. \end{aligned}$$

This yields

$$I_4 \lesssim \int_0^t (t-s)^{-\frac{5}{4}} \|u \otimes \tau\|_{L^1}\, ds \lesssim \int_0^t (t-s)^{-\frac{5}{4}} \|u\|_{L^2} \|\tau\|_{L^2}\, ds.$$

Direct computations lead to

$$I_4 \lesssim \int_0^t \|\widehat{u \otimes \tau}\|_{L^2(D_1)}\, ds \lesssim \int_0^t \|u \otimes \tau\|_{L^2}\, ds.$$

So

$$\begin{aligned} I_4 &\lesssim \int_0^t \langle t-s\rangle^{-\frac{5}{4}} \|u\|_{L^2 \cap L^\infty} \|\tau\|_{L^2}\, ds \\ &\lesssim \int_0^t \langle t-s\rangle^{-\frac{5}{4}} (\langle s\rangle^{-\frac{3}{4}} + \langle s\rangle^{-\frac{3}{2}})\, ds (X(t)+Y(t)) \|\tau\|_{L^\infty_t(L^2)} \\ &\lesssim \langle t\rangle^{-\frac{3}{4}} (X(t)+Y(t)) \|\tau\|_{L^\infty_t(L^2)}. \end{aligned}$$

For I_5, using (3-4) and (4-1), we have

$$\begin{aligned} I_5 &\lesssim \int_0^t \|e^{-\frac{|\xi|^2}{2}(t-s)} \widehat{Q(\tau,\nabla u)}\|_{L^2(D_1)}\, ds \lesssim \int_0^t \|e^{-\frac{|\xi|^2}{2}(t-s)} \widehat{Q(\tau,\nabla u)}\|_{L^2}\, ds \\ &\lesssim \int_0^t \|e^{\frac{1}{2}\Delta(t-s)} Q(\tau,\nabla u)\|_{L^2}\, ds \lesssim \int_0^t (t-s)^{-\frac{3}{4}} \|Q(\tau,\nabla u)\|_{L^1}\, ds \\ &\lesssim \int_0^t (t-s)^{-\frac{3}{4}} \|\tau\|_{L^2} \|\nabla u\|_{L^2}\, ds. \end{aligned}$$

We also have

$$I_5 \lesssim \int_0^t \|Q(\tau,\nabla u)\|_{L^2}\, ds \lesssim \int_0^t \|\tau\|_{L^2} \|\nabla u\|_{L^\infty}\, ds,$$

so

$$\begin{aligned} I_5 &\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{4}} \|\tau\|_{L^2} \|\nabla u\|_{L^2 \cap L^\infty}\, ds \\ &\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{4}} (\langle s\rangle^{-\frac{5}{4}} + \langle s\rangle^{-2})\, ds (X(t)+Y(t)) \|\tau\|_{L^\infty_t(L^2)} \\ &\lesssim \langle t\rangle^{-\frac{3}{4}} (X(t)+Y(t)) \|\tau\|_{L^\infty_t(L^2)}. \end{aligned}$$

Collecting the above estimates of I_i yields

$$\begin{aligned} \|\hat{u}\|_{L^2(D_1)} \lesssim \langle t\rangle^{-\frac{3}{4}} \|(u_0,\tau_0)\|_{L^1 \cap L^2} + \langle t\rangle^{-\frac{3}{4}} X(t) Y(t) \\ + \langle t\rangle^{-\frac{3}{4}} (X(t)+Y(t)) \|\tau\|_{L^\infty_t(L^2)}. \end{aligned} \tag{4-2}$$

Since the estimates of $\widehat{M_{ij}}$ in $D_2 \cup D_3$ are similar to that in D_1, repeating the above procedure yields

$$\sum_{i=2,3} \|\hat{u}\|_{L^2(D_i)} \lesssim \langle t\rangle^{-\frac{3}{4}} \|(u_0,\tau_0)\|_{L^1\cap L^2} + \langle t\rangle^{-\frac{3}{4}} X(t)Y(t) + \langle t\rangle^{-\frac{3}{4}} (X(t)+Y(t))\|\tau\|_{L^\infty_t(L^2)}. \tag{4-3}$$

4B. *The estimate of $\|\hat{u}(t)\|_{L^2(D_4)}$*. For the estimate in D_4, we will use (3-5) in the following context.

$$\begin{aligned}
\|\hat{u}(t)\|_{L^2(D_4)} \le{}& \|\widehat{M_{11}}(t)\widehat{u_0}\|_{L^2(D_4)} + \|\widehat{M_{12}}(t)\widehat{\mathcal{A}_0}\|_{L^2(D_4)} \\
&+ \int_0^t \|\widehat{M_{11}}(t-s)\widehat{G}\|_{L^2(D_4)}\,ds + \int_0^t \|\widehat{M_{12}}(t-s)\widehat{F}\|_{L^2(D_4)}\,ds \\
&+ \int_0^t \|\widehat{M_{12}}(t-s)\widehat{H}\|_{L^2(D_4)}\,ds \\
={}& I_1' + I_2' + \cdots + I_5'.
\end{aligned}$$

We have

$$I_1' + I_2' \lesssim e^{-\frac{t}{2}} \|(u_0,\tau_0)\|_{L^2}.$$

For I_3', we have

$$\begin{aligned}
I_3' &\lesssim \int_0^t e^{-\frac{t-s}{2}} \|u\cdot\nabla u\|_{L^2}\,ds \lesssim \int_0^t e^{-\frac{t-s}{2}} \|u\|_{L^2}\|\nabla u\|_{L^\infty}\,ds \\
&\lesssim \int_0^t e^{-\frac{t-s}{2}} \langle s\rangle^{-\frac{11}{4}}\,ds\; X(t)Y(t) \\
&\lesssim \langle t\rangle^{-\frac{11}{4}} X(t)Y(t).
\end{aligned}$$

For I_4', we have

$$\begin{aligned}
I_4' &\lesssim \int_0^t e^{-\frac{t-s}{2}} \|u\otimes\tau\|_{L^2}\,ds \lesssim \int_0^t e^{-\frac{t-s}{2}} \|u\|_{L^\infty}\|\tau\|_{L^2}\,ds \\
&\lesssim \int_0^t e^{-\frac{t-s}{2}} \langle s\rangle^{-\frac{3}{2}}\,ds\; X(t)\|\tau\|_{L^\infty_t L^2} \\
&\lesssim \langle t\rangle^{-\frac{3}{2}} X(t)\|\tau\|_{L^\infty_t L^2}.
\end{aligned}$$

For I_5', we get

$$\begin{aligned}
I_5' &\lesssim \int_0^t e^{-\frac{t-s}{2}} \|Q(\tau,\nabla u)\|_{L^2}\,ds \lesssim \int_0^t e^{-\frac{t-s}{2}} \|\nabla u\|_{L^\infty}\|\tau\|_{L^2}\,ds \\
&\lesssim \int_0^t e^{-\frac{t-s}{2}} \langle s\rangle^{-2}\,ds\; Y(t)\|\tau\|_{L^\infty_t L^2} \\
&\lesssim \langle t\rangle^{-2} Y(t)\|\tau\|_{L^\infty_t L^2}.
\end{aligned}$$

Thus we can infer

$$\|\widehat{u}(t)\|_{L^2(D_4)} \lesssim e^{-\frac{t}{2}}\|(u_0,\tau_0)\|_{L^2} + \langle t\rangle^{-\frac{3}{2}}\big(X(t)Y(t)+(X(t)+Y(t))\|\tau\|_{L^\infty_t L^2}\big). \tag{4-4}$$

Combining (4-2), (4-3) and (4-4) implies that

$$\|u(t)\|_{L^2} \lesssim \langle t\rangle^{-\frac{3}{4}}\|(u_0,\tau_0)\|_{L^1\cap L^2} + \langle t\rangle^{-\frac{3}{4}}X(t)Y(t) + \langle t\rangle^{-\frac{3}{4}}(X(t)+Y(t))\|\tau\|_{L^\infty_t(L^2)}. \tag{4-5}$$

The estimate of $\|\nabla u(t)\|_{L^2}$. Before we begin this estimate, let us introduce a lemma dealing with a commutator estimate, which plays an important role in the following proof.

Lemma 4.1. *Let* $i = 1, 2, 3$, *then*

$$\|[\mathbb{P}\operatorname{div}, u_i]\tau\|_{\dot{B}^0_{1,\infty}} \lesssim (\|\nabla u_i\|_{L^2}\|\tau\|_{L^2} + \|u_i\|_{L^2}\|\mathcal{A}\|_{L^2}).$$

The proof of this lemma will be postponed until the Appendix. As in the previous procedure, we have

$$\|\nabla u(t)\|_{L^2} = \|\widehat{\nabla u}(t)\|_{L^2} \lesssim \sum_{i=1}^{4}\||\xi|\widehat{u}\|_{L^2(D_i)}.$$

4C. ***The estimate of*** $\||\xi|\hat{u}(t)\|_{L^2(\mathbb{R}^3\setminus D_4)}$**.** Using (3-3), we have

$$\begin{aligned}\||\xi|\widehat{u}(t)\|_{L^2(D_1)} &\le \|\widehat{M_{11}}(t)\widehat{u_0}\|_{L^2(D_1)} + \|\widehat{M_{12}}(t)\widehat{\mathcal{A}_0}\|_{L^2(D_1)} \\ &\quad + \int_0^t \|\widehat{M_{11}}(t-s)\widehat{G}\|_{L^2(D_1)}\,ds + \int_0^t \|\widehat{M_{12}}(t-s)\widehat{F}\|_{L^2(D_1)}\,ds \\ &\quad + \int_0^t \|\widehat{M_{12}}(t-s)\widehat{H}\|_{L^2(D_1)}\,ds \\ &= J_1+J_2+\cdots+J_5.\end{aligned}$$

Applying (3-4) and (4-1), we get

$$J_1 \lesssim \||\xi|e^{-\frac{|\xi|^2}{2}t}\widehat{u_0}\|_{L^2(D_1)} \lesssim t^{-\frac{1}{2}}\|e^{-\frac{|\xi|^2}{4}t}\widehat{u_0}\|_{L^2} \lesssim t^{-\frac{1}{2}}\|e^{\frac{1}{4}t\Delta}u_0\|_{L^2} \lesssim t^{-\frac{5}{4}}\|u_0\|_{L^1}.$$

With another estimate

$$J_1 \lesssim \||\xi|e^{-\frac{|\xi|^2}{2}t}\widehat{u_0}\|_{L^2(D_1)} \lesssim \|\widehat{u_0}\|_{L^2} \lesssim \|u_0\|_{L^2},$$

we infer

$$J_1 \lesssim \langle t\rangle^{-\frac{5}{4}}\|u_0\|_{L^1\cap L^2}.$$

A similar process leads to

$$J_2 \lesssim \langle t\rangle^{-\frac{5}{4}} \|\tau_0\|_{L^1\cap L^2}.$$

Getting the estimate of J_1 in the same way, we can infer that

$$\begin{aligned} J_3 &\lesssim \int_0^t \langle t-s\rangle^{-\frac{5}{4}} (\|u\cdot\nabla u\|_{L^1} + \|u\cdot\nabla u\|_{L^2})\,ds \\ &\lesssim \int_0^t \langle t-s\rangle^{-\frac{5}{4}} \|u\|_{L^2}(\|\nabla u\|_{L^2} + \|\nabla u\|_{L^\infty})\,ds \\ &\lesssim \int_0^t \langle t-s\rangle^{-\frac{5}{4}} (\langle s\rangle^{-2} + \langle s\rangle^{-\frac{11}{4}})\,ds X(t)Y(t) \lesssim \langle t\rangle^{-\frac{5}{4}} X(t)Y(t) \end{aligned}$$

and

$$\begin{aligned} J_5 &\lesssim \int_0^t \langle t-s\rangle^{-\frac{5}{4}} (\|Q(\tau,\nabla u)\|_{L^1} + \|Q(\tau,\nabla u)\|_{L^2})\,ds \\ &\lesssim \int_0^t \langle t-s\rangle^{-\frac{5}{4}} \|\nabla u\|_{L^2}(\|\tau\|_{L^2} + \|\tau\|_{L^\infty})\,ds \\ &\lesssim \int_0^t \langle t-s\rangle^{-\frac{5}{4}} \langle s\rangle^{-\frac{5}{4}}\,ds Y(t)\|\tau\|_{L_t^\infty(L^2\cap L^\infty)} \lesssim \langle t\rangle^{-\frac{5}{4}} Y(t)\|\tau\|_{L_t^\infty(L^2\cap L^\infty)}. \end{aligned}$$

We shall use a different approach to get the estimates of J_4. Otherwise, a bad term $\|[\mathbb{P}\,\mathrm{div}, u_i]\tau\|_{L^1}$ will appear when we use the standard estimate of the solution for the heat equation (4-1). It is known that this bad term cannot be bounded by a standard commutator estimate. Our idea uses Lemmas 2.6 and 4.1. Now, we begin the estimate of J_4. Thanks to (3-4) again, and using

$$\mathbb{P}\,\mathrm{div}(u\cdot\nabla\tau) = \sum_{i=1,2,3} \partial_i \mathbb{P}\,\mathrm{div}(u_i\tau) = \sum_{i=1,2,3} (\partial_i[\mathbb{P}\,\mathrm{div}, u_i]\tau + \partial_i(u_i\mathcal{A})), \tag{4-6}$$

we have

$$\begin{aligned} J_4 &\lesssim \int_0^t \|e^{-\frac{|\xi|^2}{2}(t-s)} \widehat{\mathbb{P}\,\mathrm{div}(u\cdot\nabla\tau)}\|_{L^2(D_1)}\,ds \\ &\lesssim \int_0^t \|e^{-\frac{|\xi|^2}{2}(t-s)} |\xi|\,\widehat{u\otimes\mathcal{A}}\|_{L^2(D_1)}\,ds \\ &\qquad + \underbrace{\sum_{i=1,2,3} \int_0^t \|e^{-\frac{|\xi|^2}{2}(t-s)} |\xi_i|\,\widehat{[\mathbb{P}\,\mathrm{div}, u_i]\tau}\|_{L^2(D_1)}\,ds}_{\Upsilon_1}. \end{aligned}$$

By the previous approach, we can get

$$\int_0^t \|e^{-\frac{|\xi|^2}{2}(t-s)} |\xi|\widehat{u\otimes\mathcal{A}}\|_{L^2(D_1)}\,ds \lesssim \int_0^t (t-s)^{-\frac{5}{4}} \|u\otimes\mathcal{A}\|_{L^1}\,ds,$$

and we also have

$$\int_0^t \|e^{-\frac{|\xi|^2}{2}(t-s)}|\xi|\widehat{u\otimes\mathcal{A}}\|_{L^2(D_1)}\,ds \lesssim \int_0^t \|u\otimes\mathcal{A}\|_{L^2}\,ds.$$

So we can get

$$\begin{aligned}
\int_0^t \|e^{-\frac{|\xi|^2}{2}(t-s)}|\xi|\widehat{u\otimes\mathcal{A}}\|_{L^2(D_1)}\,ds
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{5}{4}}(\|u\otimes\mathcal{A}\|_{L^1}+\|u\otimes\mathcal{A}\|_{L^2})\,ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{5}{4}}\|\mathcal{A}\|_{L^2}(\|u\|_{L^2}+\|u\|_{L^\infty})\,ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{5}{4}}(\langle s\rangle^{-2}+\langle s\rangle^{-\frac{11}{4}})\,ds Y(t)(X(t)+Y(t))\\
&\lesssim \langle t\rangle^{-\frac{5}{4}}Y(t)(X(t)+Y(t)).
\end{aligned}$$

As for Υ_1, by Lemmas 2.6 and 4.1, we have

$$\begin{aligned}
\Upsilon_1 &\lesssim \int_0^t (t-s)^{-\frac{1}{2}}\|e^{-\frac{|\xi|^2}{4}(t-s)}\widehat{[\mathbb{P}\operatorname{div},u_i]\tau}\|_{L^2(D_1)}\,ds\\
&\lesssim \int_0^t (t-s)^{-\frac{1}{2}}\|e^{-\frac{1}{4}(t-s)\Delta}[\mathbb{P}\operatorname{div},u_i]\tau\|_{L^2}\,ds\\
&\lesssim \int_0^t (t-s)^{-\frac{5}{4}}\|[\mathbb{P}\operatorname{div},u_i]\tau\|_{\dot B^0_{1,\infty}}\,ds\\
&\lesssim \int_0^t (t-s)^{-\frac{5}{4}}(\|\nabla u\|_{L^2}\|\tau\|_{L^2}+\|u\|_{L^2}\|\mathcal{A}\|_{L^2})\,ds.
\end{aligned}$$

By using (2-3), we can also bound Υ_1 as follows:

$$\Upsilon_1 \lesssim \int_0^t \|[\mathbb{P}\operatorname{div},u_i]\tau\|_{L^2}\,ds \lesssim \int_0^t (\|\nabla u\|_{L^\infty}\|\tau\|_{L^2}+\|\tau\|_{L^\infty}\|\nabla u\|_{L^2})\,ds.$$

Finally, we infer

$$\begin{aligned}
\Upsilon_1 &\lesssim \int_0^t \langle t-s\rangle^{-\frac{5}{4}}(\|\nabla u\|_{L^2}\|\tau\|_{L^2}+\|u\|_{L^2}\|\mathcal{A}\|_{L^2}\\
&\qquad\qquad +\|\nabla u\|_{L^\infty}\|\tau\|_{L^2}+\|\tau\|_{L^\infty}\|\nabla u\|_{L^2})\,ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{5}{4}}\langle s\rangle^{-\frac{5}{4}}\,ds\; Y(t)(X(t)+\|\tau\|_{L^\infty_t(L^2\cap L^\infty)})\\
&\lesssim \langle t\rangle^{-\frac{5}{4}}Y(t)(X(t)+\|\tau\|_{L^\infty_t(L^2\cap L^\infty)}).
\end{aligned}$$

Thus

$$J_4 \lesssim \langle t\rangle^{-\frac{5}{4}}Y(t)(X(t)+Y(t)+\|\tau\|_{L^\infty_t(L^2\cap L^\infty)}).$$

Collecting the five estimates above, we get

$$\||\xi|\widehat{u}(t)\|_{L^2(D_1)} \lesssim \langle t\rangle^{-\frac{5}{4}}\|(u_0,\tau_0)\|_{L^1\cap L^2} + \langle t\rangle^{-\frac{5}{4}} Y(t) \times \{X(t)+Y(t)+\|\tau\|_{L^\infty_t(L^2\cap L^\infty)}\}. \tag{4-7}$$

Similarly,

$$\sum_{i=2,3} \||\xi|\widehat{u}(t)\|_{L^2(D_i)} \lesssim \langle t\rangle^{-\frac{5}{4}}\|(u_0,\tau_0)\|_{L^1\cap L^2} + \langle t\rangle^{-\frac{5}{4}} Y(t) \times \{X(t)+Y(t)+\|\tau\|_{L^\infty_t(H^2)}\}. \tag{4-8}$$

4D. ***The estimate of*** $\||\xi|\hat{u}(t)\|_{L^2(D_4)}$**.** As in the previous statement, we use (3-5) for the estimate in D_4:

$$\begin{aligned}\||\xi|\widehat{u}(t)\|_{L^2(D_4)} &\le \||\xi|\widehat{M_{11}}(t)\widehat{u_0}\|_{L^2(D_4)} + \||\xi|\widehat{M_{12}}(t)\widehat{\mathcal{A}_0}\|_{L^2(D_4)} \\ &\quad + \int_0^t \||\xi|\widehat{M_{11}}(t-s)\widehat{G}\|_{L^2(D_4)}\,ds + \int_0^t \||\xi|\widehat{M_{12}}(t-s)\widehat{F}\|_{L^2(D_4)}\,ds \\ &\quad + \int_0^t \||\xi|\widehat{M_{12}}(t-s)\widehat{H}\|_{L^2(D_4)}\,ds \\ &= J_1'+J_2'+\cdots+J_5'.\end{aligned}$$

For J_1' and J_2', we have

$$J_1'+J_2' \lesssim e^{-\frac{t}{2}}(\|\nabla u_0\|_{L^2}+\|\nabla\tau_0\|_{L^2}).$$

For J_3', using product estimate (2-2), we get

$$\begin{aligned}J_3' &\lesssim \int_0^t e^{-\frac{t-s}{2}}\|\nabla(u\cdot\nabla u)\|_{L^2}\,ds \lesssim \int_0^t e^{-\frac{t-s}{2}}\|u\otimes u\|_{\dot{H}^2}\,ds \\ &\lesssim \int_0^t e^{-\frac{t-s}{2}}\|u\|_{L^\infty}\|u\|_{\dot{H}^2}\,ds \lesssim \int_0^t e^{-\frac{t-s}{2}}\langle s\rangle^{-\frac{3}{2}}\,ds\; X(t)\|u\|_{L^\infty_t\dot{H}^2} \\ &\lesssim \langle t\rangle^{-\frac{3}{2}} X(t)\|u\|_{L^\infty_t\dot{H}^2}.\end{aligned}$$

For J_4', we have

$$\begin{aligned}J_4' &\lesssim \int_0^t e^{-\frac{t-s}{2}}\||\xi|^{-2}\widehat{\mathbb{P}\,\mathrm{div}(u\cdot\nabla\tau)}\|_{L^2(D_4)}\,ds \\ &\lesssim \int_0^t e^{-\frac{t-s}{2}}\||\xi|^{-2}\widehat{\mathbb{P}\,\mathrm{div}(u\cdot\nabla\tau)}\|_{L^2}\,ds \lesssim \int_0^t e^{-\frac{t-s}{2}}\|u\otimes\tau\|_{L^2}\,ds \\ &\lesssim \int_0^t e^{-\frac{t-s}{2}}\|u\|_{L^\infty}\|\tau\|_{L^2}\,ds \lesssim \int_0^t e^{-\frac{t-s}{2}}\langle s\rangle^{-\frac{3}{2}}\,ds X(t)\|\tau\|_{L^\infty_t L^2} \\ &\lesssim \langle t\rangle^{-\frac{3}{2}} X(t)\|\tau\|_{L^\infty_t L^2}.\end{aligned}$$

The last term J_5' can be bounded as follows:

$$\begin{aligned}J_5' &\lesssim \int_0^t e^{-\frac{t-s}{2}} \||\xi|^{-2}\widehat{\mathbb{P}\,\mathrm{div} Q(\tau,\nabla u)}\|_{L^2(D_4)}\,ds \lesssim \int_0^t e^{-\frac{t-s}{2}} \|Q(\tau,\nabla u)\|_{L^2}\,ds\\ &\lesssim \int_0^t e^{-\frac{t-s}{2}} \|\tau\|_{L^2}\|\nabla u\|_{L^\infty}\,ds \lesssim \int_0^t e^{-\frac{t-s}{2}} \langle s\rangle^{-2}\,ds Y(t)\|\tau\|_{L_t^\infty L^2}\\ &\lesssim \langle t\rangle^{-2} Y(t)\|\tau\|_{L_t^\infty L^2}.\end{aligned}$$

Combining the five estimates above can yield

$$\text{(4-9)}\quad \||\xi|\widehat{u}(t)\|_{L^2(D_4)} \lesssim e^{-\frac{t}{2}}\|(u_0,\tau_0)\|_{H^1} + \langle t\rangle^{-\frac{3}{2}} X(t)\|(u,\tau)\|_{L_t^\infty H^2} + \langle t\rangle^{-2} Y(t)\|\tau\|_{L_t^\infty L^2}.$$

Thanks to (4-7), (4-8) and (4-9), we can get

$$\text{(4-10)}\quad \||\xi|\widehat{u}(t)\|_{L^2} \lesssim \langle t\rangle^{-\frac{5}{4}}\|(u_0,\tau_0)\|_{L^1\cap H^2} + \langle t\rangle^{-\frac{5}{4}}(X(t)+Y(t)) \times\{X(t)+Y(t)+\|(u,\tau)\|_{L_t^\infty H^2}\}.$$

5. The estimate of $\|\hat{u}(t)\|_{L^1}$, $\||\xi|\hat{u}(t)\|_{L^1}$ and $\|\mathcal{A}(t)\|_{L^2}$

We will bound $\|\widehat{u}\|_{L^1}$ and $\||\xi|\widehat{u}\|_{L^1}$ in order.

The estimate of $\|\widehat{u}(t)\|_{L^1}$. As for the previous process, we have

$$\|\widehat{u}(t)\|_{L^1} \le \sum_{i=1}^4 \|\widehat{u}(t)\|_{L^1(D_i)}.$$

5A. ***The estimate of*** $\|\hat{u}\|_{L^1(\mathbb{R}^3\setminus D_4)}$. Thanks to (3-3), we have

$$\begin{aligned}\|\widehat{u}(t)\|_{L^1(D_1)} &\le \|\widehat{M_{11}}(t)\widehat{u_0}\|_{L^1(D_1)} + \|\widehat{M_{12}}(t)\widehat{\mathcal{A}_0}\|_{L^1(D_1)}\\ &\quad + \int_0^t \|\widehat{M_{11}}(t-s)\widehat{G}\|_{L^1(D_1)}\,ds + \int_0^t \|\widehat{M_{12}}(t-s)\widehat{F}\|_{L^1(D_1)}\,ds\\ &\quad + \int_0^t \|\widehat{M_{12}}(t-s)\widehat{H}\|_{L^1(D_1)}\,ds\\ &= K_1+K_2+\cdots+K_5.\end{aligned}$$

We will frequently use the estimate

$$\text{(5-1)}\quad \|e^{-b|\xi|^2 t}\|_{L^2(D_1)} = \left(\int_{|\xi|\le 1} e^{-2b|\xi|^2 t}\,d\xi\right)^{\frac{1}{2}} = t^{-\frac{3}{4}}\left(\int_{|\xi|\le\sqrt{t}} e^{-2bv^2}\,dv\right)^{\frac{1}{2}} \lesssim t^{-\frac{3}{4}} \quad \text{for all } b>0.$$

For K_1, using (3-4), (5-1) and (4-1), we have

$$K_1 \lesssim \|e^{-\frac{|\xi|^2}{2}t}\widehat{u_0}\|_{L^1(D_1)} \lesssim \|e^{-\frac{|\xi|^2}{4}t}\|_{L^2(D_1)}\|e^{-\frac{|\xi|^2}{4}t}\widehat{u_0}\|_{L^2(D_1)}$$
$$\lesssim t^{-\frac{3}{4}}\|e^{\frac{t}{4}\Delta}u_0\|_{L^2} \lesssim t^{-\frac{3}{2}}\|u_0\|_{L^1},$$

and we also have

$$K_1 \lesssim \|\widehat{u_0}\|_{L^1(D_1)} \lesssim \|u_0\|_{L^2},$$

so

$$K_1 \lesssim \langle t\rangle^{-\frac{3}{2}}\|u_0\|_{L^1\cap L^2}.$$

Similarly,

$$K_2 \lesssim \langle t\rangle^{-\frac{3}{2}}\|\tau_0\|_{L^1\cap L^2}.$$

For K_3, using (3-4), (5-1) and (4-1) again, we can obtain

$$\begin{aligned}
K_3 &\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{2}}(\|u\cdot\nabla u\|_{L^1} + \|u\cdot\nabla u\|_{L^2})\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{2}}\|u\|_{L^2}(\|\nabla u\|_{L^2} + \|\nabla u\|_{L^\infty})\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{2}}(\langle s\rangle^{-\frac{11}{4}} + \langle s\rangle^{-2})\,ds\,X(t)Y(t) \lesssim \langle t\rangle^{-\frac{3}{2}}X(t)Y(t).
\end{aligned} \tag{5-2}$$

Thanks to (4-6), this implies that

$$K_4 \le \mathcal{K}_{41} + \mathcal{K}_{42},$$

where

$$\mathcal{K}_{41} = \sum_{i=1,2,3}\int_0^t \||\xi_i|\widehat{M_{12}}(t-s)\widehat{u_i\mathcal{A}}\|_{L^1(D_1)}\,ds$$

and

$$\mathcal{K}_{42} = \sum_{i=1,2,3}\int_0^t \||\xi_i|\widehat{M_{12}}(t-s)\widehat{[\mathbb{P}\,\mathrm{div}, u_i]\tau}\|_{L^1(D_1)}\,ds.$$

In a similar way, we deduce

$$\begin{aligned}
\mathcal{K}_{41} &\lesssim \int_0^t \|e^{-\frac{|\xi|^2}{2}(t-s)}\widehat{u_i\mathcal{A}}\|_{L^1(D_1)}\,ds \lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{2}}(\|u\otimes\mathcal{A}\|_{L^1}+\|u\otimes\mathcal{A}\|_{L^2}) \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{2}}(\|u\|_{L^2}+\|u\|_{L^\infty})\|\mathcal{A}\|_{L^2}\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{2}}\langle s\rangle^{-2}\,ds\,X(t)Y(t) \\
&\lesssim \langle t\rangle^{-\frac{3}{2}}X(t)Y(t).
\end{aligned}$$

In the following context, we shall use a new approach to bound $\mathcal{K}_{42}$ and the last term K_5, due to the fact that τ does not have the decay property like $\mathcal{A}$.

Remark 5.1. If we follow the procedure yielding the estimate of K_3 and $\mathcal{K}_{41}$, we will face an integral of the following type:

$$\int_0^t \langle t-s\rangle^{-\frac{3}{2}} \|\nabla u\|_{L^2} \|\tau\|_{L^2}\, ds,$$

which yields a weaker decay rate $\langle t\rangle^{-\frac{5}{4}}$ than $\langle t\rangle^{-\frac{3}{2}}$, since we only have

$$\|\nabla u\|_{L^2} \lesssim \langle t\rangle^{-\frac{5}{4}} Y(t).$$

Now, we begin the estimate of $\mathcal{K}_{42}$. The new idea is splitting the integral interval $[0,t]$ into $[0,\frac{t}{2})$ and $[\frac{t}{2},t]$. That is

$$\mathcal{K}_{42} = K_{41} + K_{42},$$

where

$$K_{41} = \sum_{i=1,2,3} \int_0^{\frac{t}{2}} \| |\xi| \widehat{M_{12}}(t-s) \widehat{[\mathbb{P}\,\mathrm{div}, u_i]\tau} \|_{L^1(D_1)}\, ds$$

and

$$K_{42} = \sum_{i=1,2,3} \int_{\frac{t}{2}}^{t} \| |\xi| \widehat{M_{12}}(t-s) \widehat{[\mathbb{P}\,\mathrm{div}, u_i]\tau} \|_{L^1(D_1)}\, ds.$$

For K_{41}, using (3-4), (5-1) and Lemma 2.6, we have

$$\begin{aligned}
K_{41} &\lesssim \int_0^{\frac{t}{2}} \| e^{-\frac{|\xi|^2}{2}(t-s)} \widehat{[\mathbb{P}\,\mathrm{div}, u_i]\tau} \|_{L^1(D_1)}\, ds \\
&\lesssim \int_0^{\frac{t}{2}} \| e^{-\frac{|\xi|^2}{4}(t-s)} \|_{L^2(D_1)} \| e^{-\frac{|\xi|^2}{4}(t-s)} \widehat{[\mathbb{P}\,\mathrm{div}, u_i]\tau} \|_{L^2(D_1)}\, ds \\
&\lesssim \int_0^{\frac{t}{2}} (t-s)^{-\frac{3}{4}} \| e^{\frac{t-s}{4}\Delta} [\mathbb{P}\,\mathrm{div}, u_i]\tau \|_{L^2}\, ds \\
&\lesssim \int_0^{\frac{t}{2}} (t-s)^{-\frac{3}{2}} \| [\mathbb{P}\,\mathrm{div}, u_i]\tau \|_{\dot{B}^0_{1,\infty}}\, ds,
\end{aligned}$$

and we can also get

$$K_{41} \lesssim \int_0^t \| [\mathbb{P}\,\mathrm{div}, u_i]\tau \|_{L^2}\, ds.$$

Thus, by Lemma 4.1 and the commutator estimate, we have

$$\begin{aligned} K_{41} &\lesssim \int_0^{\frac{t}{2}} \langle t-s\rangle^{-\frac{3}{2}} (\|[\mathbb{P}\,\mathrm{div}, u_i]\tau\|_{\dot{B}^0_{1,\infty}} + \|[\mathbb{P}\,\mathrm{div}, u_i]\tau\|_{L^2})\, ds \\ &\lesssim \int_0^{\frac{t}{2}} \langle t-s\rangle^{-\frac{3}{2}} (\|\nabla u\|_{L^2\cap L^\infty}\|\tau\|_{L^2\cap L^\infty} + \|u\|_{L^2}\|\mathcal{A}\|_{L^2})\, ds \\ &\lesssim \langle t\rangle^{-\frac{3}{2}} \int_0^{\frac{t}{2}} \langle s\rangle^{-\frac{5}{4}}\, ds\; Y(t)(\|\tau\|_{L^\infty_t H^2} + X(t)) \\ &\lesssim \langle t\rangle^{-\frac{3}{2}} Y(t)(\|\tau\|_{L^\infty_t H^2} + X(t)). \end{aligned}$$

For K_{42}, applying (5-1), we can get

$$\begin{aligned} K_{42} &\lesssim \int_{\frac{t}{2}}^t (t-s)^{-\frac{3}{4}} \|e^{-\frac{|\xi|^2}{4}(t-s)} \widehat{[\mathbb{P}\,\mathrm{div}, u_i]\tau}\|_{L^2(D_1)}\, ds \\ &\lesssim \int_{\frac{t}{2}}^t (t-s)^{-\frac{3}{4}} \|[\mathbb{P}\,\mathrm{div}, u_i]\tau\|_{L^2}\, ds. \end{aligned}$$

This, together with

$$K_{42} \lesssim \int_{\frac{t}{2}}^t \|e^{-\frac{|\xi|^2}{2}(t-s)} \widehat{[\mathbb{P}\,\mathrm{div}, u_i]\tau}\|_{L^2(D_1)}\, ds \lesssim \int_{\frac{t}{2}}^t \|[\mathbb{P}\,\mathrm{div}, u_i]\tau\|_{L^2}\, ds$$

leads to

$$\begin{aligned} K_{42} &\lesssim \int_{\frac{t}{2}}^t \langle t-s\rangle^{-\frac{3}{4}} \|[\mathbb{P}\,\mathrm{div}, u_i]\tau\|_{L^2}\, ds \\ &\lesssim \int_{\frac{t}{2}}^t \langle t-s\rangle^{-\frac{3}{4}} \||\xi|\widehat{u}\|_{L^1}\|\tau\|_{L^2}\, ds \lesssim \int_{\frac{t}{2}}^t \langle t-s\rangle^{-\frac{3}{4}} \langle s\rangle^{-2}\, ds Y(t)\|\tau\|_{L^\infty_t L^2} \\ &\lesssim \langle t\rangle^{-2} Y(t)\|\tau\|_{L^\infty_t L^2} \int_{\frac{t}{2}}^t \langle t-s\rangle^{-\frac{3}{4}}\, ds \\ &\lesssim \langle t\rangle^{-\frac{7}{4}} Y(t)\|\tau\|_{L^\infty_t L^2}, \end{aligned}$$

where we have used

$$\|[\mathbb{P}\,\mathrm{div}, u_i]\tau\|_{L^2} \lesssim \||\xi|\widehat{u}\|_{L^1}\|\tau\|_{L^2}, \tag{5-3}$$

which will be proved in the Appendix, and

$$\int_{\frac{t}{2}}^t \langle t-s\rangle^{-\frac{3}{4}}\, ds \lesssim \int_0^{\frac{t}{2}} \langle s\rangle^{-\frac{3}{4}}\, ds \lesssim \langle t\rangle^{-\frac{1}{4}}.$$

So we have

$$K_4 \lesssim \langle t\rangle^{-\frac{3}{2}} Y(t)(\|\tau\|_{L^\infty_t H^2} + X(t)).$$

In the following, we will use the same strategy to bound K_5. We have

$$K_5 = K_{51} + K_{52},$$

where

$$K_{51} = \int_0^{\frac{t}{2}} \|\widehat{M_{12}}(t-s)\widehat{H}\|_{L^1(D_1)}\, ds$$

and

$$K_{52} = \int_{\frac{t}{2}}^{t} \|\widehat{M_{12}}(t-s)\widehat{H}\|_{L^1(D_1)}\, ds.$$

By (3-4), (5-1) and (4-1), we get

$$\begin{aligned} K_{51} &\lesssim \int_0^{\frac{t}{2}} \|e^{-\frac{|\xi|^2}{4}(t-s)}\|_{L^2(D_1)} \|e^{-\frac{|\xi|^2}{4}(t-s)} \widehat{Q(\tau,\nabla u)}\|_{L^2(D_1)}\, ds \\ &\lesssim \int_0^{\frac{t}{2}} (t-s)^{-\frac{3}{4}} \|e^{-\frac{|\xi|^2}{4}(t-s)} \widehat{Q(\tau,\nabla u)}\|_{L^2(D_1)}\, ds \\ &\lesssim \int_0^{\frac{t}{2}} (t-s)^{-\frac{3}{4}} \|e^{\frac{1}{4}(t-s)\Delta} Q(\tau,\nabla u)\|_{L^2}\, ds \\ &\lesssim \int_0^{\frac{t}{2}} (t-s)^{-\frac{3}{2}} \|Q(\tau,\nabla u)\|_{L^1}\, ds. \end{aligned}$$

We also have

$$\begin{aligned} K_{51} &\lesssim \int_0^{\frac{t}{2}} \|e^{-\frac{|\xi|^2}{2}(t-s)} \widehat{Q(\tau,\nabla u)}\|_{L^1(D_1)}\, ds \lesssim \int_0^{\frac{t}{2}} \|e^{-\frac{|\xi|^2}{2}(t-s)} \widehat{Q(\tau,\nabla u)}\|_{L^2}\, ds \\ &\lesssim \int_0^{\frac{t}{2}} \|Q(\tau,\nabla u)\|_{L^2}\, ds; \end{aligned}$$

thus

$$\begin{aligned} K_{51} &\lesssim \int_0^{\frac{t}{2}} \langle t-s\rangle^{-\frac{3}{2}} \|\tau\|_{L^2}(\|\nabla u\|_{L^2} + \|\nabla u\|_{L^\infty})\, ds \\ &\lesssim \int_0^{\frac{t}{2}} \langle t-s\rangle^{-\frac{3}{2}} (\langle s\rangle^{-\frac{5}{4}} + \langle s\rangle^{-2})\, ds\, Y(t) \|\tau\|_{L^\infty_t L^2} \lesssim \langle t\rangle^{-\frac{3}{2}} Y(t) \|\tau\|_{L^\infty_t L^2}. \end{aligned}$$

For K_{52}, by (3-4) and (5-1) again, we get

$$K_{52} \lesssim \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{3}{4}} \|Q(\tau,\nabla u)\|_{L^2}\, ds,$$

and, with the estimate

$$K_{52} \lesssim \int_{\frac{t}{2}}^{t} \|Q(\tau,\nabla u)\|_{L^2}\, ds,$$

we can infer

$$K_{52}\lesssim\int_{\frac{t}{2}}^{t}\langle t-s\rangle^{-\frac{3}{4}}\|\nabla u\|_{L^\infty}\|\tau\|_{L^2}\,ds\lesssim\int_{\frac{t}{2}}^{t}\langle t-s\rangle^{-\frac{3}{4}}\||\xi|\hat{u}\|_{L^1}\|\tau\|_{L^2}\,ds$$
$$\lesssim\int_{\frac{t}{2}}^{t}\langle t-s\rangle^{-\frac{3}{4}}\langle s\rangle^{-2}ds\,Y(t)\|\tau\|_{L^\infty_tL^2}\lesssim\langle t\rangle^{-2}Y(t)\|\tau\|_{L^\infty_tL^2}\int_{\frac{t}{2}}^{t}\langle t-s\rangle^{-\frac{3}{4}}\,ds$$
$$\lesssim\langle t\rangle^{-\frac{7}{4}}Y(t)\|\tau\|_{L^\infty_tL^2}.$$

Thus we can deduce that

$$K_5\lesssim\langle t\rangle^{-\frac{3}{2}}Y(t)\|\tau\|_{L^\infty_tL^2}.$$

Combining this with the estimates above implies

$$\|\hat{u}(t)\|_{L^1(D_1)}\lesssim\langle t\rangle^{-\frac{3}{2}}\|(u_0,\tau_0)\|_{L^1\cap L^2}+\langle t\rangle^{-\frac{3}{2}}Y(t)(\|\tau\|_{L^\infty_tH^2}+X(t)).\tag{5-4}$$

Similarly,

$$\sum_{i=2,3}\|\hat{u}(t)\|_{L^1(D_i)}\lesssim\langle t\rangle^{-\frac{3}{2}}\|(u_0,\tau_0)\|_{L^1\cap L^2}+\langle t\rangle^{-\frac{3}{2}}Y(t)(\|\tau\|_{L^\infty_tH^2}+X(t)).\tag{5-5}$$

5B. ***The estimate of $\|\hat{u}\|_{L^1(D_4)}$.*** Using (3-3), we have

$$\|\hat{u}(t)\|_{L^1(D_4)}\le\|\widehat{M_{11}}(t)\widehat{u_0}\|_{L^1(D_4)}+\|\widehat{M_{12}}(t)\widehat{\mathcal{A}_0}\|_{L^1(D_4)}$$
$$+\int_0^t\|\widehat{M_{11}}(t-s)\widehat{G}\|_{L^1(D_4)}\,ds+\int_0^t\|\widehat{M_{12}}(t-s)\widehat{F}\|_{L^1(D_4)}\,ds$$
$$+\int_0^t\|\widehat{M_{12}}(t-s)\widehat{H}\|_{L^1(D_4)}\,ds$$
$$=K_1'+K_2'+\cdots+K_5'.$$

For K_1', using (3-4), we get

$$K_1'\lesssim e^{-\frac{t}{2}}\|\widehat{u_0}\|_{L^1}\lesssim e^{-\frac{t}{2}}\|u_0\|_{H^2}.$$

In the same way,

$$K_2'\lesssim e^{-\frac{t}{2}}\|\widehat{\tau_0}\|_{L^1}\lesssim e^{-\frac{t}{2}}\|\tau_0\|_{H^2}.$$

For the three nonlinear terms' estimates, applying (3-4) and Young's inequality, we have

$$K_3'\lesssim\int_0^t e^{-\frac{t-s}{2}}\|\widehat{u\cdot\nabla u}\|_{L^1}\,ds\lesssim\int_0^t e^{-\frac{t-s}{2}}\|\hat{u}\|_{L^1}\||\xi|\hat{u}\|_{L^1}\,ds$$
$$\lesssim\int_0^t e^{-\frac{t-s}{2}}\langle s\rangle^{-\frac{7}{2}}\,ds\,X(t)Y(t)\lesssim\langle t\rangle^{-\frac{7}{2}}X(t)Y(t),$$

$$K_4' \lesssim \int_0^t e^{-\frac{t-s}{2}} \|\hat{u}\|_{L^1} \|\hat{\tau}\|_{L^1}\, ds \lesssim \int_0^t e^{-\frac{t-s}{2}} \langle s\rangle^{-\frac{3}{2}}\, ds X(t) \|\tau\|_{L^\infty_t H^2}$$
$$\lesssim \langle t\rangle^{-\frac{3}{2}} X(t) \|\tau\|_{L^\infty_t H^2}$$

$$K_5' \lesssim \int_0^t e^{-\frac{t-s}{2}} \||\xi|^{-1} \widehat{Q(\tau,\nabla u)}\|_{L^1(D_4)}\, ds \lesssim \int_0^t e^{-\frac{t-s}{2}} \|\hat{\tau}\|_{L^1} \||\xi|\hat{u}\|_{L^1}\, ds$$
$$\lesssim \int_0^t e^{-\frac{t-s}{2}} \langle s\rangle^{-2}\, ds Y(t) \|\tau\|_{L^\infty_t H^2} \lesssim \langle t\rangle^{-2} Y(t) \|\tau\|_{L^\infty_t H^2}.$$

Collecting the above estimates leads to

$$\|\hat{u}(t)\|_{L^1(D_4)} \lesssim e^{-\frac{t}{2}} \|(u_0,\tau_0)\|_{H^2} + \langle t\rangle^{-\frac{3}{2}} (X(t)+Y(t))(\|\tau\|_{L^\infty_t H^2} + X(t)). \tag{5-6}$$

Combining the estimates (5-4), (5-5) and (5-6), we can infer that

$$\|\hat{u}(t)\|_{L^1} \lesssim \langle t\rangle^{-\frac{3}{2}} \|(u_0,\tau_0)\|_{L^1\cap H^2} + \langle t\rangle^{-\frac{3}{2}} (X(t)+Y(t))(\|\tau\|_{L^\infty_t H^2} + X(t)). \tag{5-7}$$

The estimate of $\||\xi|\hat{u}(t)\|_{L^1}$. Thanks to

$$\||\xi|\hat{u}(t)\|_{L^1} \lesssim \sum_{i=1}^{4} \||\xi|\hat{u}(t)\|_{L^1(D_i)},$$

it is sufficient to bound the four terms on the right-hand side.

5C. *The estimate of* $\||\xi|\hat{u}(t)\|_{L^1(\mathbb{R}^3\setminus D_4)}$. Using (3-3), we get

$$\begin{aligned}\||\xi|\hat{u}(t)\|_{L^1(D_1)} &\le \||\xi|\widehat{M_{11}}(t)\widehat{u_0}\|_{L^1(D_1)} + \||\xi|\widehat{M_{12}}(t)\widehat{\mathcal{A}_0}\|_{L^1(D_1)} \\ &\quad + \int_0^t \||\xi|\widehat{M_{11}}(t-s)\widehat{G}\|_{L^1(D_1)}\, ds + \int_0^t \||\xi|\widehat{M_{12}}(t-s)\widehat{F}\|_{L^1(D_1)}\, ds \\ &\quad + \int_0^t \||\xi|\widehat{M_{12}}(t-s)\widehat{H}\|_{L^1(D_1)}\, ds \\ &= L_1 + L_2 + \cdots + L_5.\end{aligned}$$

For L_1, by (3-4), (5-1) and (4-1), we have

$$L_1 \lesssim \||\xi| e^{-\frac{|\xi|^2}{2}t} \widehat{u_0}\|_{L^1(D_1)} \lesssim t^{-\frac{1}{2}} \|e^{-\frac{|\xi|^2}{4}t} \widehat{u_0}\|_{L^1(D_1)}$$
$$\lesssim t^{-\frac{1}{2}} \|e^{-\frac{|\xi|^2}{8}t}\|_{L^2(D_1)} \|e^{-\frac{|\xi|^2}{8}t} \widehat{u_0}\|_{L^2(D_1)} \lesssim t^{-\frac{5}{4}} \|e^{\frac{t}{8}\Delta} u_0\|_{L^2} \lesssim t^{-2} \|u_0\|_{L^1},$$

which along with

$$L_1 \le \|\widehat{u_0}\|_{L^1(D_1)} \lesssim \|\widehat{u_0}\|_{L^2(D_1)} \lesssim \|u_0\|_{L^2}$$

yields

$$L_1 \lesssim \langle t\rangle^{-2}\|u_0\|_{L^1\cap L^2}.$$

Similarly, we can also get

$$L_2 \lesssim \langle t\rangle^{-2}\|\tau_0\|_{L^1\cap L^2},$$

and

$$\begin{aligned}
L_3 &\lesssim \int_0^t \langle t-s\rangle^{-2}(\|u\cdot\nabla u\|_{L^1}+\|u\cdot\nabla u\|_{L^2})\,ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-2}(\|u\|_{L^2}\|\nabla u\|_{L^2}+\|u\|_{L^2}\|\nabla u\|_{L^\infty})\,ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-2}(\langle s\rangle^{-2}+\langle s\rangle^{-\frac{11}{4}})\,ds\,X(t)Y(t)\lesssim \langle t\rangle^{-2}X(t)Y(t).
\end{aligned}$$

Next, we use the same approach as for dealing with the estimates of K_4 and K_5 to bound L_4 and L_5, respectively, using the same reasoning reason (see Remark 5.1 for the details). Indeed, we have

$$L_4 \le \mathcal{L}_{41}+\mathcal{L}_{42},$$

where

$$\mathcal{L}_{41}=\int_0^t \||\xi|^2\widehat{M_{12}}(t-s)\,\widehat{u\otimes\mathcal{A}}\,\|_{L^1(D_1)}\,ds$$

and

$$\mathcal{L}_{42}=\int_0^t \||\xi||\xi_i|\widehat{M_{12}}(t-s)\widehat{[\,\mathbb{P}\operatorname{div},u_i]\tau}\|_{L^1(D_1)}\,ds.$$

For $\mathcal{L}_{41}$, we have

$$\begin{aligned}
\mathcal{L}_{41} &\lesssim \int_0^t \langle t-s\rangle^{-2}(\|u\otimes\mathcal{A}\|_{L^1}+\|u\otimes\mathcal{A}\|_{L^2})\,ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-2}(\|u\|_{L^2}+\|u\|_{L^\infty})\|\mathcal{A}\|_{L^2}\,ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-2}(\langle s\rangle^{-2}+\langle s\rangle^{-\frac{11}{4}})\,ds\,X(t)Y(t)\lesssim \langle t\rangle^{-2}X(t)Y(t).
\end{aligned}$$

For $\mathcal{L}_{42}$, like the previous procedure for $\mathcal{K}_{42}$, we have

$$\mathcal{L}_{42}=L_{41}+L_{42},$$

where

$$\begin{aligned}
L_{41}&=\int_0^{\frac{t}{2}} \||\xi||\xi_i|\widehat{M_{12}}(t-s)\widehat{[\,\mathbb{P}\operatorname{div},u_i]\tau}\|_{L^1(D_1)}\,ds,\\
L_{42}&=\int_{\frac{t}{2}}^t \||\xi||\xi_i|\widehat{M_{12}}(t-s)\widehat{[\,\mathbb{P}\operatorname{div},u_i]\tau}\|_{L^1(D_1)}\,ds.
\end{aligned}$$

Using the same method as for the estimate of K_{41}, we infer

$$\begin{aligned}
L_{41} &\lesssim \int_0^{\frac{t}{2}} \langle t-s\rangle^{-2}(\|[\mathbb{P}\operatorname{div}, u_i]\tau\|_{\dot{B}^0_{1,\infty}} + \|[\mathbb{P}\operatorname{div}, u_i]\tau\|_{L^2})\,ds \\
&\lesssim \int_0^{\frac{t}{2}} \langle t-s\rangle^{-2}(\|\nabla u\|_{L^2\cap L^\infty}\|\tau\|_{L^2\cap L^\infty} + \|u\|_{L^2}\|\mathcal{A}\|_{L^2})\,ds \\
&\lesssim \langle t\rangle^{-2}\int_0^{\frac{t}{2}} \langle s\rangle^{-\frac{5}{4}}\,ds Y(t)(\|\tau\|_{L^\infty_t H^2} + X(t)) \\
&\lesssim \langle t\rangle^{-2} Y(t)(\|\tau\|_{L^\infty_t H^2} + X(t)).
\end{aligned}$$

Using the same method as for the estimate of K_{42}, we infer

$$\begin{aligned}
L_{42} &\lesssim \int_{\frac{t}{2}}^t \langle t-s\rangle^{-\frac{5}{4}}\|[\mathbb{P}\operatorname{div}, u_i]\tau\|_{L^2}\,ds \lesssim \int_{\frac{t}{2}}^t \langle t-s\rangle^{-\frac{5}{4}}\||\xi|\widehat{u}\|_{L^1}\|\tau\|_{L^2}\,ds \\
&\lesssim \int_{\frac{t}{2}}^t \langle t-s\rangle^{-\frac{5}{4}}\langle s\rangle^{-2}\,ds Y(t)\|\tau\|_{L^\infty_t L^2} \\
&\lesssim \langle t\rangle^{-2} Y(t)\|\tau\|_{L^\infty_t L^2}\int_{\frac{t}{2}}^t \langle t-s\rangle^{-\frac{5}{4}}\,ds \\
&\lesssim \langle t\rangle^{-2} Y(t)\|\tau\|_{L^\infty_t L^2}.
\end{aligned}$$

Following the process in the estimate of K_5, we have

$$L_5 = L_{51} + L_{52},$$

where

$$L_{51} = \int_0^{\frac{t}{2}} \||\xi|\widehat{M_{12}}(t-s)\widehat{H}\|_{L^1(D_1)}\,ds$$

and

$$L_{52} = \int_{\frac{t}{2}}^t \||\xi|\widehat{M_{12}}(t-s)\widehat{H}\|_{L^1(D_1)}\,ds.$$

Following the estimate of K_{51} and K_{52}, we can get

$$\begin{aligned}
L_{51} &\lesssim \int_0^{\frac{t}{2}} \langle t-s\rangle^{-2}(\|Q(\tau,\nabla u)\|_{L^1} + \|Q(\tau,\nabla u)\|_{L^2})\,ds \\
&\lesssim \int_0^{\frac{t}{2}} \langle t-s\rangle^{-2}\|\tau\|_{L^2}(\|\nabla u\|_{L^2} + \|\nabla u\|_{L^\infty})\,ds \\
&\lesssim \int_0^{\frac{t}{2}} \langle t-s\rangle^{-2}(\langle s\rangle^{-\frac{5}{4}} + \langle s\rangle^{-2})\,ds Y(t)\|\tau\|_{L^\infty_t L^2} \\
&\lesssim \langle t\rangle^{-2} Y(t)\|\tau\|_{L^\infty_t L^2},
\end{aligned}$$

and

$$\begin{aligned}L_{52} &\lesssim \int_{\frac{t}{2}}^{t} \langle t-s\rangle^{-\frac{5}{4}} \|Q(\tau,\nabla u)\|_{L^2}\,ds\\ &\lesssim \int_{\frac{t}{2}}^{t} \langle t-s\rangle^{-\frac{5}{4}} \|\tau\|_{L^2}\|\nabla u\|_{L^\infty}\,ds\\ &\lesssim \int_{\frac{t}{2}}^{t} \langle t-s\rangle^{-\frac{5}{4}} \langle s\rangle^{-2}\,ds Y(t)\|\tau\|_{L_t^\infty L^2}\\ &\lesssim \langle t\rangle^{-2} Y(t)\|\tau\|_{L_t^\infty L^2}.\end{aligned}$$

Thus we deduce

$$L_5 \lesssim \langle t\rangle^{-2} Y(t)\|\tau\|_{L_t^\infty L^2}.$$

Combining the estimates of L_i $(i=1,2,3,4,5)$, we have

$$\||\xi|\widehat{u}(t)\|_{L^1(D_1)} \lesssim \langle t\rangle^{-2}\|(u_0,\tau_0)\|_{L^1\cap L^2} + \langle t\rangle^{-2} Y(t)(\|\tau\|_{L_t^\infty H^2} + X(t)). \tag{5-8}$$

Repeating the above procedure, we can also obtain

$$\sum_{i=2,3} \||\xi|\widehat{u}(t)\|_{L^1(D_i)} \lesssim \langle t\rangle^{-2}\|(u_0,\tau_0)\|_{L^1\cap L^2} + \langle t\rangle^{-2} Y(t)(\|\tau\|_{L_t^\infty H^2} + X(t)). \tag{5-9}$$

5D. ***The estimate of*** $\||\xi|\hat{u}(t)\|_{L^1(D_4)}$**.** Using (3-3), we have

$$\begin{aligned}\||\xi|\widehat{u}(t)\|_{L^1(D_4)} &\le \||\xi|\widehat{M_{11}}(t)\widehat{u_0}\|_{L^1(D_4)} + \||\xi|\widehat{M_{12}}(t)\widehat{\mathcal{A}_0}\|_{L^1(D_4)}\\ &\quad + \int_0^t \||\xi|\widehat{M_{11}}(t-s)\widehat{G}\|_{L^1(D_4)}\,ds + \int_0^t \||\xi|\widehat{M_{12}}(t-s)\widehat{F}\|_{L^1(D_4)}\,ds\\ &\quad + \int_0^t \||\xi|\widehat{M_{12}}(t-s)\widehat{H}\|_{L^1(D_4)}\,ds\\ &= L_1' + L_2' + \cdots + L_5'.\end{aligned}$$

Applying (3-5), we have

$$L_1' \lesssim e^{-\frac{t}{2}}\||\xi|\widehat{u_0}\|_{L^1(D_4)} \lesssim e^{-\frac{t}{2}}\|u_0\|_{H^3}$$

and

$$L_2' \lesssim e^{-\frac{t}{2}}\||\xi|\widehat{\tau_0}\|_{L^1(D_4)} \lesssim e^{-\frac{t}{2}}\|\tau_0\|_{H^3}.$$

For L_3', by (3-5) and Young's inequality, we have

$$\begin{aligned} L_3' &\lesssim \int_0^t e^{-\frac{t-s}{2}} \| |\xi| \widehat{u\cdot\nabla u} \|_{L^1}\, ds \\ &\lesssim \int_0^t e^{-\frac{t-s}{2}} (\| |\xi| \widehat{u}\|_{L^1}^2 + \|\widehat{u}\|_{L^1} \| |\xi|^2 \widehat{u}\|_{L^1})\, ds \\ &\lesssim L_{31}' + L_{32}'. \end{aligned}$$

For L_{31}', we get

$$L_{31}' \lesssim \int_0^t e^{-\frac{t-s}{2}} \langle s\rangle^{-4}\, ds Y(t)^2 \lesssim \langle t\rangle^{-2} Y(t)^2.$$

We cannot directly bound the estimate L_{32}' as we can for L_{31}'. Let us first consider the estimate of $\| |\xi|^2 \widehat{u}\|_{L^1}$. We have

$$\begin{aligned} \| |\xi|^2 \widehat{u}\|_{L^1} &\leq \| |\xi|^2 \widehat{u}\|_{L^1(|\xi|\leq\langle s\rangle^{\frac{3}{2}})} + \| |\xi|^2 \widehat{u}\|_{L^1(|\xi|>\langle s\rangle^{\frac{3}{2}})} \\ &\leq \langle s\rangle^{\frac{3}{2}} \| |\xi| \widehat{u}\|_{L^1} + \langle s\rangle^{-\frac{1}{2}} \| |\xi|^{\frac{7}{3}} \widehat{u}\|_{L^1} \\ &\lesssim \langle s\rangle^{\frac{3}{2}} \| |\xi| \widehat{u}\|_{L^1} + \langle s\rangle^{-\frac{1}{2}} \|u\|_{H^4}. \end{aligned}$$

So we have

$$L_{32}' \lesssim \int_0^t e^{-\frac{t-s}{2}} \langle s\rangle^{-2}\, ds\, X(t)(Y(t)+\|u\|_{L_t^\infty H^4}) \lesssim \langle t\rangle^{-2} X(t)(Y(t)+\|u\|_{L_t^\infty H^4}).$$

Thus we have

$$L_3' \lesssim \langle t\rangle^{-2}(X(t)+Y(t))(Y(t)+\|u\|_{L_t^\infty H^4}).$$

For L_4', we have

$$\begin{aligned} L_4' &\lesssim \int_0^t e^{-\frac{t-s}{2}} (\| \widehat{[\mathbb{P}\,\mathrm{div}, u_i]\tau} \|_{L^1} + \| \widehat{u\otimes\mathcal{A}} \|_{L^1})\, ds \\ &\lesssim \int_0^t e^{-\frac{t-s}{2}} (\| |\xi| \widehat{u}\|_{L^1} \|\widehat{\tau}\|_{L^1} + \|\widehat{u}\|_{L^1} \|\widehat{\mathcal{A}}\|_{L^1})\, ds \\ &\lesssim L_{41}' + L_{42}', \end{aligned}$$

where we have used

$$\| \widehat{[\mathbb{P}\,\mathrm{div}, u_i]\tau} \|_{L^1} \lesssim \| |\xi| \widehat{u}\|_{L^1} \|\widehat{\tau}\|_{L^1},$$

which can be proved by a similar procedure as that which yielded the estimate of (5-3). For L_{41}', we have

$$L_{41}' \lesssim \int_0^t e^{-\frac{t-s}{2}} \langle s\rangle^{-2}\, ds\, Y(t)\|\tau\|_{L_t^\infty H^2} \lesssim \langle t\rangle^{-2} Y(t)\|\tau\|_{L_t^\infty H^2}.$$

For L'_{42}, we can get

$$\begin{aligned}
L'_{42} &\lesssim \int_0^t e^{-\frac{t-s}{2}} \|\widehat{u}\|_{L^1} \|\widehat{\mathcal{A}}\|_{L^1}\, ds \\
&\lesssim \int_0^t e^{-\frac{t-s}{2}} \|\widehat{u}\|_{L^1} (\|\widehat{\mathcal{A}}\|_{L^1(|\xi|\le\langle s\rangle^{\frac14})} + \|\widehat{\mathcal{A}}\|_{L^1(|\xi|>\langle s\rangle^{\frac14})})\, ds \\
&\lesssim \int_0^t e^{-\frac{t-s}{2}} \langle s\rangle^{\frac34} \|\widehat{u}\|_{L^1} \|\widehat{\mathcal{A}}\|_{L^2(|\xi|\le\langle s\rangle^{\frac14})}\, ds \\
&\qquad\qquad + \int_0^t e^{-\frac{t-s}{2}} \langle s\rangle^{-\frac12} \|\widehat{u}\|_{L^1} \||\xi|^2\widehat{\mathcal{A}}\|_{L^1(|\xi|>\langle s\rangle^{\frac14})}\, ds \\
&\lesssim \int_0^t e^{-\frac{t-s}{2}} \langle s\rangle^{-2}\, ds X(t)Y(t) \\
&\qquad\qquad + \int_0^t e^{-\frac{t-s}{2}} \langle s\rangle^{-2}\, ds X(t)\|\tau\|_{L^\infty_t H^5} \\
&\lesssim \langle t\rangle^{-2} X(t)(Y(t) + \|\tau\|_{L^\infty_t H^5}).
\end{aligned}$$

Thus we can obtain

$$L'_4 \lesssim \langle t\rangle^{-2}(X(t)+Y(t))(Y(t)+\|\tau\|_{L^\infty_t H^5}).$$

For L'_5, we have

$$\begin{aligned}
L'_5 &\lesssim \int_0^t e^{-\frac{t-s}{2}} \|\widehat{\tau}\|_{L^1} \||\xi|\widehat{u}\|_{L^1}\, ds \\
&\lesssim \int_0^t e^{-\frac{t-s}{2}} \langle s\rangle^{-2}\, ds Y(t)\|\tau\|_{L^\infty_t H^2} \\
&\lesssim \langle t\rangle^{-2} Y(t)\|\tau\|_{L^\infty_t H^2}.
\end{aligned}$$

Collecting the estimates of L'_i $(i = 1, 2, 3, 4, 5)$, we get

$$\begin{aligned}
&\||\xi|\widehat{u}(t)\|_{L^1(D_4)} \\
&\quad\lesssim e^{-\frac{t}{2}} \|(u_0,\tau_0)\|_{H^3} + \langle t\rangle^{-2}(X(t)+Y(t))(Y(t)+\|(u,\tau)\|_{L^\infty_t H^5}).
\end{aligned} \tag{5-10}$$

It follows from combining (5-8), (5-9) and (5-10) that

$$\begin{aligned}
&\||\xi|\widehat{u}\|_{L^1} \\
&\quad\lesssim \langle t\rangle^{-2}\|(u_0,\tau_0)\|_{L^1\cap H^3} + \langle t\rangle^{-2}(X(t)+Y(t))(Y(t)+\|(u,\tau)\|_{L^\infty_t H^5}).
\end{aligned} \tag{5-11}$$

The estimate of $\|\mathcal{A}(t)\|_{L^2}$. As for the previous procedure, we have

$$\|\mathcal{A}(t)\|_{L^2} = \|\widehat{\mathcal{A}}(t)\|_{L^2} \le \sum_{i=1}^{4} \|\widehat{\mathcal{A}}(t)\|_{L^2(D_i)}.$$

5E. ***The estimate of*** $\|\widehat{\mathcal{A}}(t)\|_{L^2(\mathbb{R}^2\setminus D_4)}$. Using (3-3), we have

$$
\begin{aligned}
\|\widehat{\mathcal{A}}(t)\|_{L^2(D_1)} \le \|\widehat{M_{21}}(t)\widehat{u_0}\|_{L^2(D_1)} &+ \|\widehat{M_{22}}(t)\widehat{\mathcal{A}_0}\|_{L^2(D_1)} \\
&+ \int_0^t \|\widehat{M_{21}}(t-s)\widehat{G}\|_{L^2(D_1)}\,ds + \int_0^t \|\widehat{M_{22}}(t-s)\widehat{F}\|_{L^2(D_1)}\,ds \\
&+ \int_0^t \|\widehat{M_{22}}(t-s)\widehat{H}\|_{L^2(D_1)}\,ds \\
&= N_1 + N_2 + \cdots + N_5.
\end{aligned}
$$

For N_1; using (3-4) and (4-1), we have

$$N_1 \lesssim t^{-\frac{1}{2}}\|e^{\frac{t}{4}\Delta}u_0\|_{L^2} \lesssim t^{-\frac{5}{4}}\|u_0\|_{L^1},$$

which, together with

$$N_1 \lesssim \||\xi|e^{-\frac{|\xi|^2}{2}t}\widehat{u_0}\|_{L^2(D_1)} \lesssim \|u_0\|_{L^2},$$

yields

$$N_1 \lesssim \langle t\rangle^{-\frac{5}{4}}\|u_0\|_{L^1\cap L^2}.$$

Similarly, we also have

$$N_2 \lesssim \langle t\rangle^{-\frac{5}{4}}\|\tau_0\|_{L^1\cap L^2}$$

and

$$
\begin{aligned}
N_3 &\lesssim \int_0^t \langle t-s\rangle^{-\frac{5}{4}}(\|u\cdot\nabla u\|_{L^1} + \|u\cdot\nabla u\|_{L^2})\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{5}{4}}\|u\|_{L^2}(\|\nabla u\|_{L^2} + \|\nabla u\|_{L^\infty})\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{5}{4}}(\langle s\rangle^{-2} + \langle s\rangle^{-\frac{11}{4}})\,ds X(t)Y(t) \\
&\lesssim \langle t\rangle^{-\frac{5}{4}}X(t)Y(t).
\end{aligned}
$$

For N_4, since

$$N_4 \lesssim \int_0^t \|e^{-\frac{|\xi|^2}{2}(t-s)}|\xi_i|\widehat{[\mathbb{P}\,\mathrm{div}, u_i]\tau}\|_{L^2}\,ds + \int_0^t \|e^{-\frac{|\xi|^2}{2}(t-s)}|\xi|\widehat{u\otimes\tau}\|_{L^2}\,ds,$$

we can get the estimate by repeating the estimate of J_4; in fact, we can infer that

$$N_4 \lesssim \langle t\rangle^{-\frac{5}{4}}Y(t)\big(X(t) + Y(t) + \|\tau\|_{L^\infty_t(L^2\cap L^\infty)}\big).$$

Like the estimate of J_5, we have

$$N_5 \lesssim \langle t\rangle^{-\frac{5}{4}}Y(t)\|\tau\|_{L^\infty_t(L^2\cap L^\infty)}.$$

Thus we have

$$\|\widehat{\mathcal{A}}\|_{L^2(D_1)} \lesssim \langle t\rangle^{-\frac{5}{4}}\|(u_0,\tau_0)\|_{L^1\cap L^2} + \langle t\rangle^{-\frac{5}{4}}Y(t)\|\tau\|_{L_t^\infty(L^2\cap L^\infty)}. \tag{5-12}$$

Similarly, we can also get

$$\sum_{i=2,3}\|\widehat{\mathcal{A}}\|_{L^2(D_i)} \lesssim \langle t\rangle^{-\frac{5}{4}}\|(u_0,\tau_0)\|_{L^1\cap L^2} + \langle t\rangle^{-\frac{5}{4}}Y(t)\|\tau\|_{L_t^\infty(L^2\cap L^\infty)}. \tag{5-13}$$

5F. ***The estimate of*** $\|\widehat{\mathcal{A}}(t)\|_{L^2(D_4)}$**.** We have

$$\begin{aligned}\|\widehat{\mathcal{A}}(t)\|_{L^2(D_4)} \le \|\widehat{M_{21}}(t)\widehat{u_0}\|_{L^2(D_4)} + \|\widehat{M_{22}}(t)\widehat{\mathcal{A}_0}\|_{L^2(D_4)}\\ + \int_0^t \|\widehat{M_{21}}(t-s)\widehat{G}\|_{L^2(D_4)}\,ds + \int_0^t \|\widehat{M_{22}}(t-s)\widehat{F}\|_{L^2(D_4)}\,ds\\ + \int_0^t \|\widehat{M_{22}}(t-s)\widehat{H}\|_{L^2(D_4)}\,ds\\ = N_1' + N_2' + \cdots + N_5'.\end{aligned}$$

Using (3-5), we have

$$\begin{aligned}N_1' + N_2' &\lesssim e^{-\frac{t}{2}}(\||\xi|\widehat{u_0}\|_{L^2(D_4)} + \||\xi|\widehat{\tau_0}\|_{L^2(D_4)})\\ &\lesssim e^{-\frac{t}{2}}\|(u_0,\tau_0)\|_{H^1}.\end{aligned}$$

Similarly,

$$\begin{aligned}N_3' &\lesssim \int_0^t e^{-\frac{t-s}{2}}\|u\cdot\nabla u\|_{L^2}\,ds\\ &\lesssim \int_0^t e^{-\frac{t-s}{2}}\|u\|_{L^2}\|\nabla u\|_{L^\infty}\,ds\\ &\lesssim \int_0^t e^{-\frac{t-s}{2}}\langle s\rangle^{-\frac{11}{4}}\,ds X(t)Y(t)\\ &\lesssim \langle t\rangle^{-\frac{11}{4}}X(t)Y(t),\end{aligned}$$

and

$$\begin{aligned}N_4' &\lesssim \int_0^t e^{-\frac{t-s}{2}}\|\mathbb{P}\operatorname{div}(u\cdot\nabla\tau)\|_{L^2}\,ds\\ &\lesssim \int_0^t e^{-\frac{t-s}{2}}(\|u\|_{L^\infty}\|\nabla\tau\|_{L^2} + \|\nabla u\|_{L^\infty}\|\tau\|_{L^2})\,ds\\ &\lesssim \int_0^t e^{-\frac{t-s}{2}}(\langle s\rangle^{-\frac{3}{2}} + \langle s\rangle^{-2})\,ds(X(t)+Y(t))\|\tau\|_{L_t^\infty H^1}\\ &\lesssim \langle t\rangle^{-\frac{3}{2}}(X(t)+Y(t))\|\tau\|_{L_t^\infty H^1}.\end{aligned}$$

For the last term N_5', we have

$$\begin{aligned}
N_5' &\lesssim \int_0^t e^{-\frac{t-s}{2}} (\|\nabla\tau\|_{L^2}\|\nabla u\|_{L^\infty} + \|\tau\|_{L^2}\|\Delta u\|_{L^2})\,ds\\
&\lesssim \int_0^t e^{-\frac{t-s}{2}} (\|\nabla\tau\|_{L^2}\|\nabla u\|_{L^\infty} + \|\tau\|_{L^2}\|\Delta u\|_{L^\infty})\,ds\\
&\lesssim \int_0^t e^{-\frac{t-s}{2}} (\|\nabla\tau\|_{L^2}\|\nabla u\|_{L^\infty} + \|\tau\|_{L^2}\||\xi|^2\widehat{u}\|_{L^1})\,ds\\
&\lesssim \int_0^t e^{-\frac{t-s}{2}} \Big\{\|\nabla\tau\|_{L^2}\|\nabla u\|_{L^\infty} + \|\tau\|_{L^2}(\||\xi|^2\widehat{u}\|_{L^1(|\xi|<\langle s\rangle^{\frac{3}{4}})}\\
&\qquad\qquad + \||\xi|^2\widehat{u}\|_{L^1(|\xi|\geq\langle s\rangle^{\frac{3}{4}})})\Big\}\,ds\\
&\lesssim \int_0^t e^{-\frac{t-s}{2}} \Big\{\|\nabla\tau\|_{L^2}\|\nabla u\|_{L^\infty} + \|\tau\|_{L^2}(\langle s\rangle^{\frac{3}{4}}\||\xi|\widehat{u}\|_{L^1}\\
&\qquad\qquad + \langle s\rangle^{-\frac{5}{4}}\||\xi|^{\frac{11}{3}}\widehat{u}\|_{L^1})\Big\}\,ds\\
&\lesssim \int_0^t e^{-\frac{t-s}{2}} (\langle s\rangle^{-2} + \langle s\rangle^{-\frac{5}{4}})\,ds(Y(t) + \|u\|_{L^\infty_t H^6})\|\tau\|_{L^\infty_t H^1}\\
&\lesssim \langle t\rangle^{-\frac{5}{4}}(Y(t) + \|u\|_{L^\infty_t H^6})\|\tau\|_{L^\infty_t H^1}.
\end{aligned}$$

Thus we have

$$\|\widehat{\mathcal{A}}(t)\|_{L^2(D_4)} \lesssim e^{-\frac{t}{2}}\|(u_0,\tau_0)\|_{H^1} + \langle t\rangle^{-\frac{5}{4}}(X(t)+Y(t) + \|u\|_{L^\infty_t H^6})\|\tau\|_{L^\infty_t H^1}. \tag{5-14}$$

Combining (5-12), (5-13) and (5-14) leads to

$$\|\widehat{\mathcal{A}}(t)\|_{L^2} \lesssim \langle t\rangle^{-\frac{5}{4}}\|(u_0,\tau_0)\|_{L^1\cap H^1} + \langle t\rangle^{-\frac{5}{4}}(X(t)+Y(t) + \|u\|_{L^\infty_t H^6})\|\tau\|_{L^\infty_t H^1}. \tag{5-15}$$

6. Proof of Theorem 1.3

In this section, we give the proof of our main result.

Proof of Theorem 1.3. Thanks to Sections 4 and 5, we can get

$$X(t) \lesssim \|(u_0,\tau_0)\|_{L^1\cap H^2} + (X(t)+Y(t))(\|\tau\|_{L^\infty_t H^2} + X(t)),$$

which can be obtained by combining (4-5), (5-7), and

$$Y(t) \lesssim \|(u_0,\tau_0)\|_{L^1\cap H^3} + \|u\|_{L^\infty_t H^6}\|\tau\|_{L^\infty_t H^1} + (X(t)+Y(t))(Y(t) + \|(u,\tau)\|_{L^\infty_t H^6}),$$

which can be deduced by using (4-10), (5-11) and (5-15). Since (u,τ) is the solution obtained by Theorem 1.1, we have

$$\|(u,\tau)\|_{L_t^\infty H^6} \lesssim \epsilon.$$

Using the small conditions of the initial data, we can get that there exists a positive constant C_1 such that

$$X(t)+Y(t) \le \ C_1\{\epsilon+\epsilon(X(t)+Y(t))+(X(t)+Y(t))^2\}.$$

Then one can get the desired result by using continuous arguments. This completes the proof of Theorem 1.3. □

Appendix

In this section, we show the proofs of Lemma 4.1 and (5-3) in order by using the Littlewood–Paley theory and Fourier analysis technique.

Proof of Lemma 4.1. Let $f=u_i$. Using

$$\Delta_j([\mathbb{P}\,\mathrm{div}, f]\tau) = [\Delta_j\mathbb{P}\,\mathrm{div}, f]\tau + f\Delta_j\mathcal{A} - \Delta_j(f\mathcal{A}),$$

we have

$$\|\Delta_j([\mathbb{P}\,\mathrm{div}, f]\tau)\|_{L^1} \le \|[\Delta_j\mathbb{P}\,\mathrm{div}, f]\tau\|_{L^1} + \|f\Delta_j\mathcal{A}-\Delta_j(f\mathcal{A})\|_{L^1} = F_1+F_2.$$

F_2 can be easily bounded as follows:

$$F_2 \lesssim \|f\|_{L^2}\|\mathcal{A}\|_{L^2}.$$

Then it remains to bound F_1. Using Bony's decomposition, we get

$$\begin{aligned}
&\|[\Delta_j\mathbb{P}\,\mathrm{div}, f]\tau\|_{L^1}\\
&\quad\le \sum_{|k-j|\le 4}\|[\Delta_j\mathbb{P}\,\mathrm{div}, S_{k-1}f]\Delta_k\tau\|_{L^1} + \sum_{|k-j|\le 4}\|\Delta_j\mathbb{P}\,\mathrm{div}(\Delta_k f S_{k-1}\tau)\|_{L^1}\\
&\qquad + \sum_{k\ge j-3}\|\Delta_k f S_{k+2}\Delta_j\mathcal{A}\|_{L^1} + \sum_{k\ge j-3}\|\Delta_j\mathbb{P}\,\mathrm{div}(\Delta_k f\tilde{\Delta}_k\tau)\|_{L^1}\\
&\quad= \gimel_1+\gimel_2+\gimel_3+\gimel_4.
\end{aligned}$$

Notice that

$$\sum_{|k-j|\le 4}\|[\Delta_j\mathbb{P}\,\mathrm{div}, S_{k-1}f]\Delta_k\tau\|_{L^1} = \sum_{|k-j|\le 4}\sum_{i=1,2,3}\|[\Delta_j\mathbb{P}\,\partial_{\mathrm{i}}, S_{k-1}f]\Delta_k\tau_{i\cdot}\|_{L^1}$$

and let $g=\tau_{i\cdot}$, and return to considering the estimate of $\|[\Delta_j\mathbb{P}\,\partial_{\mathrm{i}}, S_{k-1}f]\Delta_k g\|_{L^1}$. We can see there exists a vector function $h_i(x)$, the components of which are

Schwartz functions, such that

$$\Delta_j \mathbb{P}\, \partial_i g = 2^{4j} \int h_i(2^j(x-y)) g(y)\, dy.$$

Thus, letting $Q_i(x) = |x| h_i(x)$, we have

$$\begin{aligned}
&|[\Delta_j \mathbb{P}\, \partial_{\mathrm{i}}, S_{k-1} f]\Delta_k g| \\
&\quad = |2^{4j} \int h_i(2^j(x-y))\{(S_{k-1}f)(y) - (S_{k-1}f)(x)\}\Delta_k g(y)\, dy| \\
&\quad \le 2^{4j} \int_0^1 \int |h_i(2^j(x-y))||x-y||\nabla S_{k-1} f(sx+(1-s)y)||\Delta_k g(y)|\, dy\, ds \\
&\quad \le 2^{3j} \int_0^1 \int |Q_i(2^j(x-y))||\nabla S_{k-1} f(sx+(1-s)y)||\Delta_k g(y)|\, dy\, ds \\
&\quad = 2^{3j} \int_0^1 \int |Q_i(2^j z)||\nabla S_{k-1} f(x-z+sz)||\Delta_k g(x-z)|\, dz\, ds.
\end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned}
&\|[\Delta_j \mathbb{P}\, \partial_{\mathrm{i}}, S_{k-1} f]\Delta_k g\|_{L^1} \\
&\qquad \le 2^{3j} \int_0^1 \int |Q_i(2^j z)| \|\nabla S_{k-1} f(\cdot - z + sz)\|_{L^2(\cdot)} \|\Delta_k \tau(\cdot - z)\|_{L^2(\cdot)}\, dz\, ds \\
&\qquad \le 2^{3j} \int |Q_i(2^j z)|\, dz \|\nabla f\|_{L^2} \|\Delta_k g\|_{L^2} \lesssim \|\nabla f\|_{L^2} \|g\|_{L^2}.
\end{aligned}$$

Thus we can get

$$\gimel_1 \lesssim \|\nabla f\|_{L^2} \|\tau\|_{L^2}.$$

$\gimel_2$ can be easily bounded. Next, we bound $\gimel_3$ and $\gimel_4$. For $\gimel_3$, by Bernstein's inequality, we have

$$\begin{aligned}
\gimel_3 &\lesssim \sum_{k \ge j-3} \|\Delta_k f\|_{L^2} \|\Delta_j \mathcal{A}\|_{L^2} \lesssim 2^j \|\Delta_j \tau\|_{L^2} \sum_{k \ge j-3} 2^{-k} \|\nabla \Delta_k f\|_{L^2} \\
&\lesssim \|\nabla f\|_{L^2} \|\tau\|_{L^2}.
\end{aligned}$$

For $\gimel_4$, using Bernstein's inequality again, we have

$$\begin{aligned}
\gimel_4 &\lesssim 2^j \sum_{k \ge j-3} \|\Delta_k f\|_{L^2} \|\tilde{\Delta}_k \tau\|_{L^2} \lesssim 2^j \|\tau\|_{L^2} \sum_{k \ge j-3} 2^{-k} \|\Delta_k \nabla f\|_{L^2} \\
&\lesssim 2^j \|\nabla f\|_{L^2} \|\tau\|_{L^2} \sum_{k \ge j-3} 2^{-k} \lesssim \|\nabla f\|_{L^2} \|\tau\|_{L^2}.
\end{aligned}$$

Collecting the above four estimates leads to the desired results. □

Proof of (5-3). Let $f = u_i$, and using

$$[\mathbb{P}\,\mathrm{div}, f]\tau = \sum_{i=1,2,3} [\mathbb{P}\,\partial_{\mathrm{i}}, f]\tau_{i\cdot},$$

we return to the bound $\|[\mathbb{P}\,\partial_{\mathrm{i}}, f]g_i\|_{L^2}$, where $g_i = \tau_{i\cdot}$. Notice that if $\widehat{\mathbb{P}\,\partial_i} = \Phi(\xi, \xi_i)$, satisfying $|\nabla_\xi \Phi(\xi, \xi_i)| \lesssim 1$, we have

$$|\Phi(\xi, \xi_i) - \Phi(\xi - \eta, \xi_i - \eta_i)| \lesssim |\eta|.$$

Using the Fourier transform, we have

$$\begin{aligned}
|\widehat{[\mathbb{P}\,\partial_{\mathrm{i}}, f]g_i}| \\
&= \left| \Phi(\xi, \xi_i) \int \widehat{f}(\eta)\widehat{g_i}(\xi - \eta)\,d\eta - \int \widehat{f}(\eta)\Phi(\xi - \eta, \xi_i - \eta_i)\widehat{g_i}(\xi - \eta)\,d\eta \right| \\
&\le \left| \int \widehat{f}(\eta)|\Phi(\xi, \xi_i) - \Phi(\xi - \eta, \xi_i - \eta_i)|\widehat{g_i}(\xi - \eta)\,d\eta \right| \\
&\lesssim \left| \int |\eta|\widehat{f}(\eta)\widehat{g_i}(\xi - \eta)\,d\eta \right|,
\end{aligned}$$

which yields the desired result by using Young's inequality. □

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