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TRIANGULATED CATEGORIES WITH CLUSTER TILTING SUBCATEGORIES

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Dedicated to Professor Idun Reiten on the occasion of her 76th birthday

For a triangulated category $\mathscr{C}$ with a cluster tilting subcategory $\mathcal{T}$ which contains infinitely many indecomposable objects, the notion of weak $\mathcal{T}[1]$-cluster tilting subcategories of $\mathscr{C}$ is introduced. We use them to study the τ-tilting theory in the module category over $\mathcal{T}$. Inspired by the work of Iyama, Jørgensen and Yang (2014), we introduce the notion of τ-tilting subcategories of mod $\mathcal{T}$, and show that there exists a bijection between weak $\mathcal{T}[1]$-cluster tilting subcategories of $\mathscr{C}$ and support τ-tilting subcategories of mod $\mathcal{T}$. Moreover, we describe the subcategories of mod $\mathcal{T}$ which correspond to cluster tilting subcategories of $\mathscr{C}$. This generalizes and improves results by Adachi, Iyama and Reiten (2014), Beligiannis (2013), and Yang and Zhu (2019).

1. Introduction

The links between cluster tilting objects in a (2-Calabi–Yau) triangulated category and tilting modules over the cluster-tilted algebras have been studied for a relatively long time. They stemmed from the categorification of cluster algebras, see for

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examples: [Smith 2008; Fu and Liu 2009; Beaudet et al. 2014; Lasnier 2011; Beligiannis 2013]. Adachi, Iyama and Reiten [Adachi et al. 2014] established a bijection between cluster tilting objects in a 2-Calabi–Yau triangulated category and support τ-tilting modules over a cluster-tilted algebra (see also [Chang et al. 2015; Yang et al. 2017; Yang and Zhu 2019] for various versions of this bijection). They introduced the τ-tilting theory for finite-dimensional algebras. As a generalization of classical tilting theory, it completes tilting theory from the viewpoint of mutation. Nowadays the relationships between τ-tilting theory and the various aspects of the representation theory of finite-dimensional algebras have been studied.

In order to generalize the bijection in [Adachi et al. 2014] mentioned above to arbitrary triangulated categories with cluster tilting objects, two of us [Yang and Zhu 2019] introduced the notion of relative cluster tilting objects. An object M in a triangulated category $\mathscr{C}$ with a cluster tilting object T is called a $T[1]$-cluster tilting object provided that $|M| = |T|$ and $[T[1]](M, M[1]) = 0$, where $|X|$ denotes the number of the isomorphism classes of indecomposable direct summands of X and $[T[1]](X, X[1]))$ denotes the subgroup of $\mathrm{Hom}_{\mathscr{C}}(X, X[1])$ consisting of morphisms factoring through an object in $\mathsf{add}(T[1])$. It was proved that there is a bijection between the set of basic $T[1]$-cluster tilting objects and the set of basic support τ-tilting modules over the cluster tilted algebra $\mathrm{End}_{\mathscr{C}}(T)^{\mathrm{op}}$, see [Yang and Zhu 2019], which is the bijection in [Adachi et al. 2014] when $\mathscr{C}$ is 2-Calabi–Yau.

Although the (2-Calabi–Yau) triangulated categories with cluster tilting objects are the main sources for categorifying cluster algebras, the more general triangulated categories (not necessarily 2-Calabi–Yau) with cluster tilting subcategories (the number of nonisomorphic indecomposable objects in it is not finite) appear naturally, see for examples, [Jørgensen and Palu 2013; Ng 2010; Igusa and Todorov 2015a; 2015b; Holm and Jørgensen 2012; Liu and Paquette 2017; Chang et al. 2018; Gratz et al. 2019; Stovicek and van Roosmalen 2016; Jørgensen and Yakimov 2017]. It is natural to ask which classes of subcategories of $\mathscr{C}$ correspond bijectively to support τ-tilting subcategories of $\mathsf{mod}\ \mathcal{T}$ for 2-Calabi–Yau triangulated categories, higher Calabi–Yau triangulated categories or arbitrary triangulated categories, where $\mathcal{T}$ is a cluster tilting subcategory of $\mathscr{C}$. Iyama, Jørgensen and Yang [Iyama et al. 2014] gave a functor version of τ-tilting theory. They considered modules over a category and showed that for a triangulated category $\mathscr{C}$ with a silting subcategory $\mathcal{S}$, there exists a bijection between the set of silting subcategories of $\mathscr{C}$ which are in $\mathcal{S} * \mathcal{S}[1]$ and the set of support τ-tilting pairs of $\mathsf{mod}\,\mathcal{S}$.

Motivated by this question and the bijection given by Yang and Zhu [2019], we introduce the notions of $\mathcal{T}[1]$-cluster tilting subcategories (also called ghost cluster tilting subcategories) and weak $\mathcal{T}[1]$-cluster tilting subcategories of $\mathscr{C}$ (the precise definitions of these subcategories are given in Definition 3.1), which are generalizations of cluster tilting subcategories. When $\mathscr{C}$ has a cluster tilting object T, then weak $\mathsf{add}(T[1])$-cluster tilting subcategories coincide with $\mathsf{add}(T[1])$-cluster tilting subcategories, and are also the same as $T[1]$-cluster tilting objects introduced in [Yang and Zhu 2019].

The first part of our work is to develop a basic theory of ghost cluster tilting subcategories of $\mathscr{C}$. Some intrinsic properties and results on ghost cluster tilting subcategories will be presented. Some of our results can be summarized as follows.

$$\begin{tikzcd}
\boxed{\text{cluster tilting}} \arrow[d, shift left, "\text{Proposition 3.3}"] & \\
\boxed{\text{ghost cluster tilting}} \arrow[u, shift left, "\substack{\text{F-stable}\\ \text{Theorem 3.6}}"] \arrow[r, shift left, "\text{Corollary 3.11}"] \arrow[d, shift left, "\text{Proposition 3.7}"] & \boxed{\text{weak ghost cluster tilting}} \arrow[l, shift left, "\substack{\text{contravariantly finite}\\ \text{Definition 3.1}}"] \\
\boxed{\text{maximal ghost rigid}} \arrow[u, shift left, "\substack{\text{contravariantly finite}\\ \text{Theorem 3.10}}"] &
\end{tikzcd}$$

The second part of our paper is devoted to answering the question above. We have the following main result.

Theorem 1.1 (see Proposition 4.2 and Theorem 4.3). *Let $\mathscr{C}$ be a triangulated category with a cluster-tilting subcategory $\mathcal{T}$. The functor $\mathbb{H} : \mathscr{C} \to \mathsf{Mod}\ \mathcal{T}$ induces a bijection*

$$\Phi : \mathscr{X} \longmapsto (\mathbb{H}(\mathscr{X}), \mathcal{T} \cap \mathscr{X}[-1])$$

from the first of the following sets to the second:

(I) *$\mathcal{T}[1]$-rigid subcategories of $\mathscr{C}$.*

(II) *τ-rigid pairs of $\mathsf{mod}\ \mathcal{T}$.*

It restricts to a bijection from the first to the second of the following sets:

(I) *Weak $\mathcal{T}[1]$-tilting subcategories of $\mathscr{C}$.*

(II) *Support τ-tilting subcategories of $\mathsf{mod}\ \mathcal{T}$.*

Consequently, we also describe the subcategories of $\mathsf{mod}\ \mathcal{T}$ which correspond to cluster tilting subcategories of $\mathscr{C}$ (see Theorem 4.4). This generalizes and improves several results in the literature.

Inspired by Adachi, Iyama and Reiten [Adachi et al. 2014] and by Iyama, Jørgensen and Yang [Iyama et al. 2014], we introduce the notions of τ-tilting subcategories and tilting subcategories of $\mathsf{mod}\,\mathcal{T}$. In the third part of our paper, we give some close relationships between certain ghost cluster tilting subcategories of $\mathscr{C}$ and some important subcategories of $\mathsf{mod}\,\mathcal{T}$ (see Theorems 4.8 and 4.11).

The paper is organized as follows. In Section 2, we recall some elementary definitions and facts about cluster tilting subcategories and support τ-tilting subcategories. In Section 3, we will study the basic properties of ghost cluster tilting subcategories of $\mathscr{C}$. For a triangulated category with cluster tilting object, we will show that the definition of ghost cluster tilting objects in $\mathscr{C}$ is equivalent to the definition of relative cluster tilting objects in [Yang and Zhu 2019]. In Section 4, we explore the connections between ghost cluster tilting theory and τ-tilting theory.

We conclude this section with some conventions.

Throughout this article, k is an algebraically closed field. All modules we consider in this paper are left modules. Let $\mathscr{C}$ be an additive category. When we say that $\mathscr{D}$ is a subcategory of $\mathscr{C}$, we always assume that $\mathscr{D}$ is a full subcategory which is closed under isomorphisms, direct sums and direct summands. We denote by $[\mathscr{D}]$ the ideal of $\mathscr{C}$ consisting of morphisms which factor through objects in $\mathscr{D}$. Thus we get a new category $\mathscr{C}/[\mathscr{D}]$ whose objects are objects of $\mathscr{C}$ and whose morphisms are elements of $\mathscr{C}(X, Y)/[\mathscr{D}](X, Y)$ for $X, Y \in \mathscr{C}/[\mathscr{D}]$. For any object M, we denote by $\mathsf{add}\, M$ the full subcategory of $\mathscr{C}$ consisting of direct summands of direct sum of finitely many copies of M and simply denote $\mathscr{C}/[\mathsf{add}\, M]$ by $\mathscr{C}/[M]$. Let $\mathscr{X}$ and $\mathscr{Y}$ be subcategories of $\mathscr{C}$. We denote by $\mathscr{X} \vee \mathscr{Y}$ the smallest subcategory of $\mathscr{C}$ containing $\mathscr{X}$ and $\mathscr{Y}$. For two morphisms $f : M \to N$ and $g : N \to L$, the composition of f and g is denoted by $gf : M \to L$.

Let X be an object in $\mathscr{C}$. A morphism $f : D_0 \to X$ is called a *right $\mathscr{D}$-approximation* of X if $D_0 \in \mathscr{D}$ and $\mathrm{Hom}_{\mathscr{C}}(-, f)|_{\mathscr{D}}$ is surjective. If any object in $\mathscr{C}$ has a right $\mathscr{D}$-approximation, we call $\mathscr{D}$ *contravariantly finite* in $\mathscr{C}$. Dually, a *left $\mathscr{D}$-approximation* and a *covariantly finite subcategory* are defined. We say that $\mathscr{D}$ is *functorially finite* if it is both covariantly finite and contravariantly finite. For more details, we refer to [Auslander and Reiten 1991].

For any triangulated category $\mathscr{C}$, we assume that it is k-linear, Hom-finite, and satisfies the Krull–Remak–Schmidt property [Happel 1988]. For any object M in $\mathscr{C}$, we can write $M \simeq M_1 \oplus \cdots \oplus M_n$, where the endomorphism ring of M_i is local, for any $i = 1, 2, \ldots, n$. Then M is called basic if $M_i \not\simeq M_j$ for all $i \neq j$. In $\mathscr{C}$, we denote the shift functor by [1] and for objects X and Y, define $\mathrm{Ext}^i_{\mathscr{C}}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(X, Y[i])$. For two subcategories $\mathscr{X}, \mathscr{Y}$ of $\mathscr{C}$, we denote by $\mathrm{Ext}^1(\mathscr{X}, \mathscr{Y}) = 0$ when $\mathrm{Ext}^1(X, Y) = 0$ for any $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$. For a subcategory $\mathscr{X}$, we use $|\mathscr{X}|$ to denote the number of nonisomorphic indecomposable objects in $\mathscr{X}$. It is easy to see that $|\mathscr{X}| < \infty$ if and only if $\mathscr{X} = \mathrm{add}\, X$ for an object X. In this case, $|\mathscr{X}|$ is denoted simply by $|X|$.

2. Background and preliminary results

In this section, we give some background material and recall some results that will be used in this paper.

Cluster tilting subcategories and relative cluster tilting objects. Let $\mathscr{C}$ be a triangulated category. An important class of subcategories of $\mathscr{C}$ are the cluster tilting subcategories, which have many nice properties. We recall the definition of cluster tilting subcategories from [Buan et al. 2006; Keller and Reiten 2007; Koenig and Zhu 2008; Iyama and Yoshino 2008].

Definition 2.1. (1) A subcategory $\mathcal{T}$ of $\mathscr{C}$ is called *rigid* if $\mathrm{Hom}_{\mathscr{C}}(\mathcal{T}, \mathcal{T}[1]) = 0$.

(2) A subcategory $\mathcal{T}$ of $\mathscr{C}$ is *maximal rigid* if it is rigid and maximal with respect to this property, that is, $\mathcal{T} = \{M \in \mathscr{C} \mid \mathrm{Hom}_{\mathscr{C}}(\mathcal{T} \vee \mathsf{add}\, M, (\mathcal{T} \vee \mathsf{add}\, M)[1]) = 0\}$.

(3) A functorially finite subcategory $\mathcal{T}$ of $\mathscr{C}$ is called *cluster tilting* if

$$\mathcal{T} = \{M \in \mathscr{C} \mid \mathrm{Hom}_{\mathscr{C}}(\mathcal{T}, M[1]) = 0\} = \{M \in \mathscr{C} \mid \mathrm{Hom}_{\mathscr{C}}(M, \mathcal{T}[1]) = 0\}.$$

(4) An object T in $\mathscr{C}$ is *cluster tilting* if $\mathsf{add}\, T$ is a cluster tilting subcategory of $\mathscr{C}$.

Remark 2.2. It is easy to see that a subcategory $\mathcal{T}$ of $\mathscr{C}$ is cluster tilting if and only if it is contravariantly finite in $\mathscr{C}$ and $\mathcal{T} = \{M \in \mathscr{C} \mid \mathrm{Hom}_{\mathscr{C}}(\mathcal{T}, M[1]) = 0\}$, see for example [Koenig and Zhu 2008].

For two subcategories $\mathscr{X}$ and $\mathscr{Y}$ of $\mathscr{C}$, we denote by $\mathscr{X} * \mathscr{Y}$ the collection of objects in $\mathscr{C}$ consisting of all such $M \in \mathscr{C}$ with triangles

$$X \longrightarrow M \longrightarrow Y \longrightarrow X[1],$$

where $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$.

Recall from [Bondal and Kapranov 1989] that $\mathscr{C}$ has a Serre functor $\mathbb{S}$ provided $\mathbb{S} : \mathscr{C} \to \mathscr{C}$ is an equivalence and there exists a functorial isomorphism

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \simeq D\,\mathrm{Hom}_{\mathscr{C}}(B, \mathbb{S}A)$$

for any $A, B \in \mathscr{C}$, where D is the duality over k. Thus $\mathscr{C}$ has the Auslander–Reiten translation $\tau \simeq \mathbb{S}[-1]$, see [Reiten and Van den Bergh 2002]. Define an equivalence $F = \tau^{-1} \circ [1]$. An object M in $\mathscr{C}$ is called *F-stable* if $F(M) \simeq M$ and a subcategory $\mathcal{M}$ of $\mathscr{C}$ is called *F-stable* if $F(\mathcal{M}) = \mathcal{M}$. We say that $\mathscr{C}$ is 2-*Calabi–Yau* if $\mathbb{S} \simeq [2]$. Note that for a 2-Calabi–Yau category $\mathscr{C}$, $F = \mathrm{id}_{\mathscr{C}}$.

We have the following result [Keller and Reiten 2007; Koenig and Zhu 2008; Iyama and Yoshino 2008], which will be used frequently in this paper.

Proposition 2.3. *Let $\mathcal{T}$ be a cluster-tilting subcategory of $\mathscr{C}$ and C be an arbitrary object in $\mathscr{C}$. Then*:

(a) $\mathscr{C} = \mathcal{T} * \mathcal{T}[1]$.

(b) *$F\mathcal{T} = \mathcal{T}$ if $\mathscr{C}$ has a Serre functor.*

(c) *Let $C \to T_0$ be a left $\mathsf{add}\, T$-approximation of C. Let $C \to T_0 \to Y \to C[1]$ be a completed triangle. Then Y is in $\mathsf{add}\, T$.*

When $\mathscr{C}$ is a 2-Calabi–Yau triangulated category, cluster tilting objects have a very important property: if we remove some direct summand T_i from a cluster tilting object $T = T_1 \oplus T_2 \oplus \cdots \oplus T_n$ to get $T/T_i = \oplus_{j \neq i} T_j$ (which is called an almost complete cluster tilting object), then there is exactly one indecomposable object T_i^* such that $T_i^* \ncong T_i$ and $T/T_i \oplus T_i^*$ is a cluster-tilting object, which is called the mutation of T at T_i, see [Buan et al. 2006; Iyama and Yoshino 2008]. But the mutation of cluster tilting objects in triangulated categories which are not 2-Calabi–Yau is not always possible, see for example Section II1 in [Buan et al. 2009]. In order to generalize it in a more general triangulated category, Yang and Zhu [2019] introduced the notion of relative cluster tilting objects in triangulated categories as follows.

Definition 2.4 [Yang and Zhu 2019, Definition 3.1]. Let $\mathscr{C}$ be a triangulated category with a cluster tilting object.

- An object X in $\mathscr{C}$ is called *relative rigid* if there exists a cluster tilting object T such that $[T[1]](X, X[1]) = 0$. In this case, X is also called $T[1]$-*rigid*.
- An object X in $\mathscr{C}$ is called *relative cluster tilting* if there exists a cluster tilting object T such that X is $T[1]$-rigid and $|X| = |T|$. In this case, X is also called $T[1]$-*cluster tilting*.

Throughout this paper, we denote by $T[1]$-rigid $\mathscr{C}$ (respectively, $T[1]$-tilt $\mathscr{C}$) the set of isomorphism classes of basic $T[1]$-rigid (respectively, basic $T[1]$-cluster tilting) objects in $\mathscr{C}$.

Support τ-tilting modules and support τ-tilting subcategories. Let Λ be a finite-dimensional k-algebra and τ the Auslander–Reiten translation. We denote by $\mathsf{proj}\,\Lambda$ the subcategory of $\mathsf{mod}\,\Lambda$ consisting of projective Λ-modules. Support τ-tilting modules were introduced by Adachi, Iyama and Reiten [Adachi et al. 2014], they can be regarded as a generalization of tilting modules.

Definition 2.5. Let (X, P) be a pair with $X \in \mathsf{mod}\,\Lambda$ and $P \in \mathsf{proj}\,\Lambda$.

1. X is called τ-*rigid* if $\mathrm{Hom}_\Lambda(X, \tau X) = 0$.
2. X is called τ-*tilting* if X is τ-rigid and $|X| = |\Lambda|$.
3. (X, P) is called a τ-*rigid pair* if X is τ-rigid and $\mathrm{Hom}_\Lambda(P, X) = 0$.
4. (X, P) is a *support τ-tilting pair* if it is a τ-rigid pair and $|X| + |P| = |\Lambda|$. In this case, X is called a *support τ-tilting module*.

Throughout this paper, we denote by τ-rigid Λ the set of isomorphism classes of basic τ-rigid pairs of Λ, and by sτ-tiltΛ the set of isomorphism classes of basic support τ-tilting Λ-modules.

The following proposition gives a criterion for a τ-rigid Λ-module to be a support τ-tilting Λ-module.

Proposition 2.6 [Jasso 2015, Proposition 2.14]. *Let Λ be a finite-dimensional algebra and M a τ-rigid Λ-module. Then M is a support τ-tilting Λ-module if and only if there exists an exact sequence*

$$\Lambda \xrightarrow{f} M' \xrightarrow{g} M'' \to 0,$$

with $M', M'' \in \mathsf{add}\, M$ and f a left $(\mathsf{add}\, M)$-approximation of Λ.

Iyama, Jørgensen and Yang [Iyama et al. 2014, Definition 1.3] defined a functor version of $\tau-$tilting modules, and they extended the notion of support τ-tilting modules for finite dimensional algebras to that for essentially small additive categories. Let $\mathcal{T}$ be an additive category. We write $\mathsf{Mod}\ \mathcal{T}$ for the abelian category of contravariant additive functors from $\mathcal{T}$ to the category of abelian groups and $\mathsf{mod}\ \mathcal{T}$ for the full subcategory of finitely presented functors, see [Auslander 1974].

Definition 2.7 [Iyama et al. 2014, Definition 1.3]. Let $\mathcal{T}$ be an essentially small additive category.

(i) Let $\mathcal{M}$ be a subcategory of $\mathsf{mod}\ \mathcal{T}$. A class $\{\, P_1 \xrightarrow{\pi^M} P_0 \to M \to 0 \mid M \in \mathcal{M}\,\}$ of projective presentations in $\mathsf{mod}\ \mathcal{T}$ is said to have *Property (S)* if

$$\mathrm{Hom}_{\mathsf{mod}\ \mathcal{T}}(\pi^M, M') : \mathrm{Hom}_{\mathsf{mod}\ \mathcal{T}}(P_0, M') \to \mathrm{Hom}_{\mathsf{mod}\ \mathcal{T}}(P_1, M')$$

is surjective for any $M, M' \in \mathcal{M}$.

(ii) A subcategory $\mathcal{M}$ of $\mathsf{mod}\ \mathcal{T}$ is said to be *τ-rigid* if there is a class of projective presentations $\{P_1 \to P_0 \to M \to 0 \mid M \in \mathcal{M}\}$ which has Property (S).

(iii) A *τ-rigid pair* of $\mathsf{mod}\ \mathcal{T}$ is a pair $(\mathcal{M}, \mathcal{E})$, where $\mathcal{M}$ is a τ-rigid subcategory of $\mathsf{mod}\ \mathcal{T}$ and $\mathcal{E} \subseteq \mathcal{T}$ is a subcategory with $\mathcal{M}(\mathcal{E}) = 0$, that is, $M(E) = 0$ for each $M \in \mathcal{M}$ and $E \in \mathcal{E}$.

(iv) A τ-rigid pair $(\mathcal{M}, \mathcal{E})$ is *support τ-tilting* if $\mathcal{E} = \mathrm{Ker}\,(\mathcal{M})$ and for each $T \in \mathcal{T}$ there exists an exact sequence $\mathcal{T}(-, T) \xrightarrow{f} M^0 \to M^1 \to 0$ with $M^0, M^1 \in \mathcal{M}$ such that f is a left $\mathcal{M}$-approximation. In this case, $\mathcal{M}$ is called a *support τ-tilting subcategory* of $\mathsf{mod}\,\mathcal{T}$.

From triangulated categories to abelian categories. In this subsection, we assume that $\mathcal{T}$ is a cluster tilting subcategory of a triangulated category $\mathscr{C}$. A $\mathcal{T}$-module is a contravariant k-linear functor $F : \mathcal{T} \to \mathsf{Mod}\, k$. Then $\mathcal{T}$-modules form an abelian category $\mathsf{Mod}\,\mathcal{T}$. We denote by $\mathsf{mod}\,\mathcal{T}$ the subcategory of $\mathsf{Mod}\,\mathcal{T}$ consisting of finitely presented $\mathcal{T}$-modules. It is easy to see that $\mathsf{mod}\,\mathcal{T}$ is an abelian category. Moreover the restricted Yoneda functor

$$\mathbb{H} : \mathscr{C} \to \mathsf{Mod}\ \mathcal{T},\quad M \mapsto \mathrm{Hom}_{\mathscr{C}}(-, M)\mid_{\mathcal{T}}$$

is homological and induces an equivalence

$$\mathcal{T} \xrightarrow{\sim} \mathsf{proj}(\mathsf{mod}\ \mathcal{T}).$$

The following results are crucial in this paper.

Theorem 2.8. (i) $\mathbb{H}(\mathscr{C})$ *is a subcategory of* $\mathsf{mod}\,\mathcal{T}$.

(ii) [Auslander 1974] *For* $N \in \mathsf{Mod}\ \mathcal{T}$ *and* $T \in \mathcal{T}$, *there exists a natural isomorphism*

$$\mathrm{Hom}_{\mathsf{Mod}\ \mathcal{T}}(\mathbb{H}(T), N) \xrightarrow{\sim} N(T).$$

More explicitly, *if we have a map* $f : T \to T'$, *where* $T' \in \mathcal{T}$, *then we have the commutative diagram*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{mod}\ \mathcal{T}}(\mathbb{H}(T'), N) \arrow[r, "\circ\mathbb{H}(f)"] \arrow[d, "\simeq"'] & \mathrm{Hom}_{\mathsf{mod}\ \mathcal{T}}(\mathbb{H}(T), N) \arrow[d, "\simeq"] \\
N(T') \arrow[r, "N(f)"] & N(T).
\end{tikzcd}$$

(iii) [Keller and Reiten 2007; Koenig and Zhu 2008; Iyama and Yoshino 2008] *The functor* $\mathbb{H}$ *from* (i) *induces an equivalence*

$$\mathscr{C}/[\mathcal{T}[1]] \xrightarrow{\sim} \mathsf{mod}\ \mathcal{T},$$

and $\mathsf{mod}\ \mathcal{T}$ *is Gorenstein of dimension at most one.*

Proof of (i). Since $\mathcal{T}$ is cluster tilting, for any object $C \in \mathscr{C}$, there exists a triangle

$$T_0 \xrightarrow{f} T_1 \xrightarrow{g} C \xrightarrow{h} T_0[1],$$

where $T_0, T_1 \in \mathcal{T}$. Applying the functor $\mathbb{H}$ to the above triangle, we get an exact sequence

$$\mathbb{H}(T_0) \xrightarrow{f\circ} \mathbb{H}(T_1) \xrightarrow{g\circ} \mathbb{H}(C) \longrightarrow 0.$$

This shows that $\mathbb{H}(C) \in \mathsf{mod}\,\mathcal{T}$. □

If there exists an object $T \in \mathscr{C}$ such that $\mathcal{T} = \mathsf{add}\,T$, we obtain the following.

Corollary 2.9. *Let* T *be a cluster tilting object in* $\mathscr{C}$ *and* $\Lambda = \mathrm{End}_{\mathscr{C}}^{\mathrm{op}}(T)$. *Then the functor*

$$\mathrm{Hom}_{\mathscr{C}}(T, -) : \mathscr{C} \to \mathsf{mod}\ \Lambda \tag{2-1}$$

induces an equivalence

$$\overline{(-)} : \mathscr{C}/[T[1]] \xrightarrow{\sim} \mathsf{mod}\ \Lambda. \tag{2-2}$$

This equivalence gives a close relationship between the relative cluster tilting objects in $\mathscr{C}$ and support τ-tilting Λ-modules.

Theorem 2.10 [Yang and Zhu 2019, Theorem 3.6]. *Let $\mathscr{C}$ be a triangulated category with a Serre functor $\mathbb{S}$ and a cluster tilting object T, and let $\Lambda = \operatorname{End}^{\mathrm{op}}_{\mathscr{C}}(T)$. Then the functor* (2-1) *induces the bijections*

$$T[1]\text{-rigid}\,\mathscr{C} \overset{(a)}{\longleftrightarrow} \tau\text{-rigid}\,\Lambda, \qquad T[1]\text{-tilt}\,\mathscr{C} \overset{(b)}{\longleftrightarrow} \mathrm{s}\tau\text{-tilt}\,\Lambda.$$

3. Ghost cluster tilting subcategories

In this section, our aim is to define and study ghost cluster tilting subcategories in a triangulated category with cluster tilting subcategories, in particular, to compare them to the existing notions: cluster tilting subcategories [Keller and Reiten 2007; Koenig and Zhu 2008; Iyama and Yoshino 2008] and relative cluster tilting objects [Yang and Zhu 2019].

Ghost cluster tilting subcategories. We first give the definitions and then discuss connections between them.

Definition 3.1. Let $\mathscr{C}$ be a triangulated category with a cluster tilting subcategory.

(i) A subcategory $\mathscr{X}$ in $\mathscr{C}$ is called *ghost rigid* if there exists a cluster tilting subcategory $\mathcal{T}$ such that $[\mathcal{T}[1]](\mathscr{X}, \mathscr{X}[1]) = 0$. In this case, $\mathscr{X}$ is also called $\mathcal{T}[1]$-*rigid*.

(ii) A subcategory $\mathscr{X}$ in $\mathscr{C}$ is called *maximal ghost rigid* if there exists a cluster tilting subcategory $\mathcal{T}$ such that $\mathscr{X}$ is $\mathcal{T}[1]$-rigid and

$$[\mathcal{T}[1]](\mathscr{X} \vee \operatorname{\mathsf{add}} M, (\mathscr{X} \vee \operatorname{\mathsf{add}} M)[1]) = 0 \text{ implies } M \in \mathscr{X}.$$

In this case, $\mathscr{X}$ is also called *maximal* $\mathcal{T}[1]$-*rigid*.

(iii) A subcategory $\mathscr{X}$ in $\mathscr{C}$ is called *weak ghost cluster tilting* if there exists a cluster tilting subcategory $\mathcal{T}$ with $\mathcal{T} \subseteq \mathscr{X}[-1] * \mathscr{X}$ and

$$\mathscr{X} = \{M \in \mathscr{C} \mid [\mathcal{T}[1]](M, \mathscr{X}[1]) = 0 \quad \text{and} \quad [\mathcal{T}[1]](\mathscr{X}, M[1]) = 0\}.$$

In this case, $\mathscr{X}$ is also called *weak* $\mathcal{T}[1]$-*cluster tilting*.

(iv) A subcategory $\mathscr{X}$ in $\mathscr{C}$ is called *ghost cluster tilting* if $\mathscr{X}$ is contravariantly finite in $\mathscr{C}$ and there exists a cluster tilting subcategory $\mathcal{T}$ such that

$$\mathscr{X} = \{M \in \mathscr{C} \mid [\mathcal{T}[1]](M, \mathscr{X}[1]) = 0 \quad \text{and} \quad [\mathcal{T}[1]](\mathscr{X}, M[1]) = 0\}.$$

In this case, $\mathscr{X}$ is also called $\mathcal{T}[1]$-*cluster tilting*.

(v) An object X is called $\mathcal{T}[1]$-rigid, maximal $\mathcal{T}[1]$-rigid, weak $\mathcal{T}[1]$-cluster tilting, or $\mathcal{T}[1]$-cluster tilting if $\operatorname{\mathsf{add}} X$ is $\mathcal{T}[1]$-rigid, maximal $\mathcal{T}[1]$-rigid, weak $\mathcal{T}[1]$-cluster tilting, or $\mathcal{T}[1]$-cluster tilting respectively.

Remark 3.2. Since ghost cluster tilting subcategories are introduced in order to generalize the notion of relative cluster tilting objects, it is natural to compare the definition of relative cluster tilting objects of Definition 2.4 (originally given in [Yang and Zhu 2019]) to the definition of ghost cluster tilting subcategories of Definition 3.1(iv). When $\mathscr{C}$ has a cluster tilting object, we will show that ghost cluster tilting objects are exactly the relative cluster tilting objects in Theorem 3.16. Therefore when $|\mathcal{T}| = \infty$, we replace the condition "$|\mathscr{X}| = |\mathcal{T}|$" by the equation

$$\text{“}\mathscr{X} = \{M \in \mathscr{C} \mid [\mathcal{T}[1]](M, \mathscr{X}[1]) = 0 \quad \text{and} \quad [\mathcal{T}[1]](\mathscr{X}, M[1]) = 0\}.\text{”}$$

From here until the end of the section, we prove some properties of ghost cluster tilting subcategories. We first prove a cluster tilting subcategory is a ghost cluster tilting subcategory with respect to any cluster tilting subcategory.

Proposition 3.3. *Cluster tilting subcategories are ghost cluster tilting. More precisely, let $\mathscr{X}$ be a cluster tilting subcategory. Then $\mathscr{X}$ is $\mathcal{T}[1]$-cluster tilting subcategory for any cluster tilting subcategory $\mathcal{T}$.*

Proof. Let $\mathscr{X}$ be an arbitrary cluster tilting subcategory in $\mathscr{C}$. Clearly, $\mathscr{X}$ is contravariantly finite and

$$\mathscr{X} \subseteq \{M \in \mathscr{C} \mid [\mathcal{T}[1]](\mathscr{X}, M[1]) = 0 = [\mathcal{T}[1]](M, \mathscr{X}[1])\}.$$

For any object $M \in \{M \in \mathscr{C} \mid [\mathcal{T}[1]](\mathscr{X}, M[1]) = 0 = [\mathcal{T}[1]](M, \mathscr{X}[1])\}$, we need to prove that $M \in \mathscr{X}$. Since $\mathcal{T}$ is cluster tilting, there exists a triangle

$$T_1 \xrightarrow{f} T_0 \xrightarrow{g} M \xrightarrow{h} T_1[1],$$

where $T_0, T_1 \in \mathcal{T}$. Take a left $\mathscr{X}$-approximation of T_0 and complete it to a triangle

$$T_0 \xrightarrow{u} X_1 \xrightarrow{v} X_2 \xrightarrow{w} T_0[1],$$

where $X_1 \in \mathscr{X}$. Since $\mathscr{X}$ is cluster tilting, by Proposition 2.3(c), we have that $X_2 \in \mathscr{X}$. By the octahedral axiom, we have a commutative diagram

$$\begin{tikzcd}
T_1 \arrow[r, "f"] \arrow[d, equal] & T_0 \arrow[r, "g"] \arrow[d, "u"] & M \arrow[r, "h"] \arrow[d, "a"] & T_1[1] \arrow[d, equal] \\
T_1 \arrow[r, "x=uf"] & X_1 \arrow[r, "y"] \arrow[d, "v"] & N \arrow[r, "z"] \arrow[d, "b"] & T_1[1] \\
& X_2 \arrow[r, equal] \arrow[d, "w"] & X_2 \arrow[d, "c"] & \\
& T_0[1] \arrow[r, "{g[1]}"] & M[1] &
\end{tikzcd}$$

of triangles. We claim that x is a left $\mathscr{X}$-approximation of T_1. Indeed, for any morphism $\alpha : T_1 \to X'$, where $X' \in \mathscr{X}$, since $\alpha \circ h[-1] \in [\mathcal{T}](M[-1], \mathscr{X}) = 0$, there exists a morphism $\beta : T_0 \to X'$ such that $\alpha = \beta f$. Since u is a left $\mathscr{X}$-approximation of T_0 and $X' \in \mathscr{X}$, there exists a morphism $\gamma : X_1 \to X'$ such that $\beta = \gamma u$ and then

$\alpha = \gamma(uf) = \gamma x$. This shows that x is a left $\mathscr{X}$-approximation of T_1. Note that $\mathscr{X}$ is cluster tilting. Thus $N \in \mathscr{X}$. Since $c = g[1]w \in [\mathcal{T}[1]](\mathscr{X}, M[1]) = 0$, the triangle

$$M \xrightarrow{a} N \xrightarrow{b} X_2 \xrightarrow{c} M[1]$$

splits. It follows that M is a direct summand of N and therefore $M \in \mathscr{X}$. Thus

$$\mathscr{X} = \{M \in \mathscr{C} \mid [\mathcal{T}[1]](\mathscr{X}, M[1]) = 0 = [\mathcal{T}[1]](M, \mathscr{X}[1])\}$$

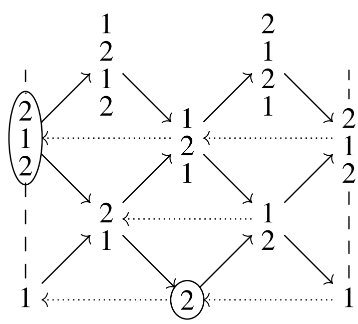
and hence $\mathscr{X}$ is $\mathcal{T}[1]$-cluster tilting. $\square$

The following example shows that ghost cluster tilting subcategories need not be cluster tilting.

Example 3.4. Let $A = kQ/I$ be a self-injective algebra given by the quiver

$$Q:\ 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

and $I = \langle \alpha\beta\alpha\beta, \beta\alpha\beta\alpha \rangle$. Let $\mathscr{C}$ be the stable module category $\underline{\mathrm{mod}}\, A$ of A. This is a triangulated category whose Auslander–Reiten quiver is the following (note that projective-injective modules should be deleted):



where the leftmost and rightmost columns are identified. It is easy to see that

$$\mathcal{T} := \mathsf{add}\left(2 \oplus \begin{smallmatrix}2\\1\\2\end{smallmatrix}\right)$$

is a cluster tilting subcategory of $\mathscr{C}$. Note that $\mathscr{X} := \mathsf{add}\big(2 \oplus \begin{smallmatrix}1\\2\end{smallmatrix}\big)$ is a $\mathcal{T}[1]$-cluster tilting subcategory of $\mathscr{C}$, but not a cluster tilting subcategory of $\mathscr{C}$, since $\mathrm{Hom}\big(\begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}[1]\big) = \mathrm{Hom}\big(\begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}\big) \neq 0$.

As we have seen in Example 3.4, ghost cluster tilting categories need not be cluster tilting categories, however the situation is much better when we assume that the triangulated category $\mathscr{C}$ has a Serre functor $\mathbb{S}$ as we will show in Theorem 3.6.

We need the following lemma in order to prove Theorem 3.6:

Lemma 3.5. *Let $\mathscr{C}$ be a triangulated category with a Serre functor $\mathbb{S}$ and a cluster tilting subcategory $\mathcal{T}$. For two objects M and N in $\mathscr{C}$,* $[\mathcal{T}[1]](M, N[1]) = 0$ *and*

$[\mathcal{T}[1]](N, \tau M) = 0$ *if and only if* $\mathrm{Hom}_{\mathscr{C}}(M, N[1]) = 0$. *In particular*, *if* $\mathscr{C}$ *is* 2-*Calabi–Yau*, *then* M *is* $\mathcal{T}[1]$-*rigid if and only if* M *is rigid.*

Proof. Our argument is similar to the proof of Proposition 3.4 in [Yang and Zhu 2019]. We give the proof for the convenience of the reader.

We show the "if" part. If $\mathrm{Hom}_{\mathscr{C}}(M, N[1]) = 0$, then $[\mathcal{T}[1]](M, N[1]) = 0$. By the Serre duality, we have

$$\mathrm{Hom}_{\mathscr{C}}(N, \tau M) \simeq \mathrm{Hom}_{\mathscr{C}}(N[1], \mathbb{S}M) \simeq D\,\mathrm{Hom}_{\mathscr{C}}(M, N[1]) = 0.$$

Thus we obtain $[\mathcal{T}[1]](N, \tau M) = 0$.

We now show the "only if" part. Since $\mathcal{T}$ is a cluster tilting subcategory, by Proposition 2.3(a), we have a triangle

$$T_0 \xrightarrow{g} N \xrightarrow{f} T_1[1] \xrightarrow{h} T_0[1]$$

with $T_0, T_1 \in \mathcal{T}$. Thus we have a commutative diagram of exact sequences:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(T_1[1], \tau M) \arrow[r, "\sim"] \arrow[d, "\cdot f"'] & D\,\mathrm{Hom}_{\mathscr{C}}(M, T_1[2]) \arrow[d, "D(f[1]\cdot)"] \\
\mathrm{Hom}_{\mathscr{C}}(N, \tau M) \arrow[r, "\sim"] \arrow[d] & D\,\mathrm{Hom}_{\mathscr{C}}(M, N[1]) \arrow[d, "D(g[1]\cdot)"] \\
\mathrm{Hom}_{\mathscr{C}}(T_0, \tau M) \arrow[r, "\sim"] \arrow[d] & D\,\mathrm{Hom}_{\mathscr{C}}(M, T_0[1]) \arrow[d, "D(h\cdot)"] \\
\mathrm{Hom}_{\mathscr{C}}(T_1, \tau M) \arrow[r, "\sim"] & D\,\mathrm{Hom}_{\mathscr{C}}(M, T_1[1])
\end{tikzcd}$$

Since $\mathrm{Im}(\cdot f) = \{af \mid a \in \mathrm{Hom}_{\mathscr{C}}(T_1[1], \tau M)\} \subseteq [\mathcal{T}[1]](N, \tau M) = 0$, we deduce

$$\mathrm{Ker} D(g[1]\cdot) = \mathrm{Im} D(f[1]\cdot) \simeq \mathrm{Im}(\cdot f) = 0. \tag{3-1}$$

$$\begin{tikzcd}
N \arrow[r, "f"] \arrow[d] & T_1[1] \arrow[r, "h"] \arrow[dl] & T_0[1] \arrow[r, "g{[1]}"] & N[1] \\
\tau M & & M \arrow[u, "b"'] \arrow[ul, dotted, "\exists c"] &
\end{tikzcd}$$

Take any $b \in \mathrm{Hom}_{\mathscr{C}}(M, T_0[1])$. Since $[\mathcal{T}[1]](M, N[1]) = 0$, we have $g[1]b = 0$. Thus there exists $c : M \to T_1[1]$ such that $b = hc$, which implies that

$$(h\cdot) : \mathrm{Hom}_{\mathscr{C}}(M, T_1[1]) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(M, T_0[1]), \qquad c \longmapsto hc = b$$

is surjective. Therefore, $D(h\cdot)$ is injective and

$$\operatorname{Im} D(g[1]\cdot) = \operatorname{Ker} D(h\cdot) = 0.$$

Combining this with (3-1), we deduce $D\operatorname{Hom}_{\mathscr{C}}(M, N[1]) = 0$ and $\operatorname{Hom}_{\mathscr{C}}(M, N[1])$ vanishes.

If $\mathscr{C}$ is 2-Calabi–Yau, then $\tau \simeq [1]$. The assertion is clear. □

The following result gives a characterization of cluster tilting subcategories in terms of ghost cluster tilting subcategories, which implies that in a 2-Calabi–Yau triangulated category, ghost cluster tilting subcategories coincide with cluster tilting subcategories.

Theorem 3.6. *Let $\mathscr{C}$ be a triangulated category with a Serre functor $\mathbb{S}$ and a cluster tilting subcategory. Then F-stable ghost cluster tilting subcategories of $\mathscr{C}$ are precisely cluster tilting subcategories, where $F = \tau^{-1}[1] = \mathbb{S}^{-1}[2]$.*

Proof. By Proposition 2.3, we have that cluster tilting subcategories are F-stable. By Proposition 3.3, we have that cluster tilting subcategories are ghost cluster tilting. Now we prove the other direction. Let $\mathscr{X}$ be a $\mathcal{T}[1]$-cluster tilting subcategory satisfying $F\mathscr{X} = \mathscr{X}$, where $\mathcal{T}$ is a cluster tilting subcategory. It follows that $\tau\mathscr{X} = \mathscr{X}[1]$.

(1) We show that $\mathscr{X}$ is a rigid subcategory of $\mathscr{C}$. For any two objects $M, N \in \mathscr{X}$, since $\mathscr{X}$ is $\mathcal{T}[1]$-tilting, we have

$$[\mathcal{T}[1]](M, N[1]) = 0. \tag{3-2}$$

Similarly, since $\tau\mathscr{X} = \mathscr{X}[1]$, we have $\tau M = M'[1]$, where $M' \in \mathscr{X}$. It follows that

$$[\mathcal{T}[1]](N, \tau M) = [\mathcal{T}[1]](N, M'[1]) = 0. \tag{3-3}$$

By Lemma 3.5, equalities (3-2) and (3-3) imply that $\operatorname{Hom}_{\mathscr{C}}(M, N[1]) = 0$.

(2) We show that $\mathscr{X} = \{M \in \mathscr{C} \mid \operatorname{Ext}^1_{\mathscr{C}}(\mathscr{X}, M) = 0\}$. The "$\subseteq$" part is clear. Assume that an object $M \in \mathscr{C}$ satisfies $\operatorname{Ext}^1_{\mathscr{C}}(\mathscr{X}, M) = 0$. Then

$$\begin{aligned}\operatorname{Hom}_{\mathscr{C}}(M, \mathscr{X}[1]) &\simeq D\operatorname{Hom}_{\mathscr{C}}(\mathscr{X}[1], \mathbb{S}M)\\ &\simeq D\operatorname{Hom}_{\mathscr{C}}(\tau\mathscr{X}, F\mathbb{S}M)\\ &\simeq D\operatorname{Hom}_{\mathscr{C}}(\mathscr{X}, M[1]) = 0.\end{aligned}$$

This implies that $[\mathcal{T}[1]](M, \mathscr{X}[1]) = 0$. Since $[\mathcal{T}[1]](\mathscr{X}, M[1]) = 0$ and $\mathscr{X}$ is $\mathcal{T}[1]$-cluster tilting, we obtain that $M \in \mathscr{X}$.

Note that $\mathscr{X}$ is contravariantly finite. It follows from Remark 2.2 that $\mathscr{X}$ is a cluster tilting subcategory of $\mathscr{C}$. □

Proposition 3.7. *Any ghost cluster tilting subcategory is a contravariantly finite maximal ghost rigid subcategory.*

Proof. Assume that $\mathscr{X}$ is a $\mathcal{T}[1]$-cluster tilting subcategory. If there exists an object $M \in \mathscr{C}$ such that

$$[\mathcal{T}[1]](\mathscr{X} \vee \mathsf{add}\, M, (\mathscr{X} \vee \mathsf{add}\, M)[1]) = 0,$$

then

$$[\mathcal{T}[1]](\mathscr{X}, M[1]) = 0 \quad \text{and} \quad [\mathcal{T}[1]](M, \mathscr{X}[1]) = 0.$$

Since $\mathscr{X}$ is $\mathcal{T}[1]$-cluster tilting, we obtain $M \in \mathscr{X}$. $\square$

The converse result to Proposition 3.7 will be given in Theorem 3.10. We need the following lemma to prove this theorem.

Lemma 3.8. (a) *Let $\mathcal{T}$ be a cluster tilting subcategory and $\mathscr{X}$ a maximal $\mathcal{T}[1]$-rigid subcategory in $\mathscr{C}$. Let $T_0 \in \mathcal{T}$, let $T_0 \xrightarrow{g} X_0$ be a left $\mathscr{X}$-approximation of T_0 and consider the associated triangle*:

$$M[-1] \xrightarrow{f} T_0 \xrightarrow{g} X_0 \xrightarrow{h} M.$$

Then $M \in \mathscr{X}$.

(b) *Let $\mathcal{T}$ be a cluster tilting subcategory and $\mathscr{X}$ a maximal $\mathcal{T}[1]$-rigid subcategory in $\mathscr{C}$. Let $T_0 \in \mathcal{T}$, let $\mathscr{X}_0[-1] \xrightarrow{f} T_0$ be a left $\mathscr{X}[-1]$-approximation of T_0 and consider the associated triangle*:

$$X_0[-1] \xrightarrow{f} T_0 \xrightarrow{g} M \xrightarrow{h} X_0.$$

Then $M \in \mathscr{X}$.

Proof. We only prove (a), the proof of (b) is similar. For any $x \in [\mathcal{T}](M[-1], \mathscr{X})$, there are two morphisms $x_1 : M[-1] \to T_1$ and $x_2 : T_1 \to X_1$ such that $x = x_2 x_1$, where $T_1 \in \mathcal{T}$ and $X_1 \in \mathscr{X}$.

$$\begin{tikzcd}
X_0[-1] \arrow[r, "{h[-1]}"] & M[-1] \arrow[r, "f"] \arrow[d, "x_1"'] & T_0 \arrow[r, "g"] \arrow[ddl, dotted, "a"] & X_0 \arrow[r, "h"] \arrow[ddll, dotted, "b"] & M \arrow[r, "{-f[1]}"] & T_0[1] \\
& T_1 \arrow[d, "x_2"'] & & & T_2 \arrow[u, "y_2"'] \arrow[ul, dotted, "c"'] & \\
& X_1 & & & X_2[-1] \arrow[u, "y_1"'] &
\end{tikzcd}$$

Since $\mathscr{X}$ is $\mathcal{T}[1]$-rigid, we have $xh[-1] = x_2(x_1 h[-1]) = 0$. Thus, there exists $a : T_0 \to X_1$ such that $x = af$. Because g is a left $\mathscr{X}$-approximation of T_0, we deduce that there exists $b : X_0 \to X_1$ such that $a = bg$. Therefore, $x = af = b(gf) = 0$ and

$$[\mathcal{T}[1]](M, \mathscr{X}[1]) = 0. \tag{3-4}$$

For $y \in [\mathcal{T}](\mathscr{X}[-1], M)$, there are two morphisms $y_1 : X_2[-1] \to T_2$ and $y_2 : T_2 \to M$ such that $y = y_2 y_1$, where $T_2 \in \mathcal{T}$ and $X_2 \in \mathscr{X}$. Since $f[1]y_2 = 0$, there exists $c : T_2 \to X_0$ such that $y_2 = hc$. Because $\mathscr{X}$ is $\mathcal{T}[1]$-rigid, we have $y = y_2 y_1 = h(cy_1) = 0$. Therefore,

$$[\mathcal{T}[1]](\mathscr{X}, M[1]) = 0. \tag{3-5}$$

For any $z \in [\mathcal{T}](M[-1], M)$, there are two morphisms $z_1 : M[-1] \to T_3$ and $z_2 : T_3 \to M$ such that $z = z_2 z_1$, where $T_3 \in \mathcal{T}$. Since $f[1]z_2 = 0$, there exists $d : T_3 \to X_0$ such that $z_2 = hd$. By equality (3-4), we have $z = z_2 z_1 = h(dz_1) = 0$. Thus,

$$[\mathcal{T}[1]](M, M[1]) = 0. \tag{3-6}$$

$$\begin{tikzcd}
T_0 \arrow[r, "g"] & X_0 \arrow[r, "h"] & M \arrow[r, "-f[1]"] & T_0[1] \\
 & & T_3 \arrow[u, "z_2"] \arrow[ul, dotted, "d"] & \\
 & & M[-1] \arrow[u, "z_1"] &
\end{tikzcd}$$

Using (3-4), (3-5) and (3-6), we get $[\mathcal{T}[1]](\mathscr{X} \vee \operatorname{add} M, (\mathscr{X} \vee \operatorname{add} M)[1]) = 0$. Note that $\mathscr{X}$ is maximal $\mathcal{T}[1]$-rigid. Hence $M \in \mathscr{X}$. □

This lemma immediately yields the following important conclusion:

Corollary 3.9. *Let $\mathcal{T}$ be a cluster tilting subcategory in a triangulated category $\mathscr{C}$ and $\mathscr{X}$ be a covariantly* (*or contravariantly*) *finite maximal $\mathcal{T}[1]$-rigid subcategory. Then*

$$\mathcal{T} \subseteq \mathscr{X}[-1] * \mathscr{X}.$$

Now we prove that the converse of Proposition 3.7 also holds, which generalizes a result of Zhou and Zhu [2011, Theorem 2.6].

Theorem 3.10. *Let $\mathscr{C}$ be a triangulated category with a cluster tilting subcategory $\mathcal{T}$. Then any contravariantly finite maximal $\mathcal{T}[1]$-rigid subcategory is a $\mathcal{T}[1]$-cluster tilting subcategory.*

Proof. Assume that $\mathscr{X}$ is a contravariantly finite maximal $\mathcal{T}[1]$-rigid subcategory in $\mathscr{C}$. Clearly,

$$\mathscr{X} \subseteq \{M \in \mathscr{C} \mid [\mathcal{T}[1]](\mathscr{X}, M[1]) = 0 = [\mathcal{T}[1]](M, \mathscr{X}[1])\}.$$

For any object $M \in \{M \in \mathscr{C} \mid [\mathcal{T}[1]](\mathscr{X}, M[1]) = 0 = [\mathcal{T}[1]](M, \mathscr{X}[1])\}$, since $\mathcal{T}$ is cluster tilting, there exists a triangle

$$T_1 \xrightarrow{f} T_0 \xrightarrow{g} M \xrightarrow{h} T_1[1],$$

where $T_0, T_1 \in \mathcal{T}$. By Corollary 3.9, there exists a triangle

$$T_0 \xrightarrow{u} X_1 \xrightarrow{v} X_2 \xrightarrow{w} T_0[1],$$

where $X_1, X_2 \in \mathscr{X}$. Since $\mathscr{X}$ is $\mathcal{T}[1]$-rigid, we have that u is a left $\mathscr{X}$-approximation of T_0. By the octahedral axiom, we have a commutative diagram

$$\begin{tikzcd}
T_1 \arrow[r, "f"] \arrow[d, equal] & T_0 \arrow[r, "g"] \arrow[d, "u"] & M \arrow[r, "h"] \arrow[d, "a"] & T_1[1] \arrow[d, equal] \\
T_1 \arrow[r, "x=uf"] & X_1 \arrow[r, "y"] \arrow[d, "v"] & N \arrow[r, "z"] \arrow[d, "b"] & T_1[1] \\
& X_2 \arrow[r, equal] \arrow[d, "w"] & X_2 \arrow[d, "c"] & \\
& T_0[1] \arrow[r, "g[1]"] & M[1] &
\end{tikzcd}$$

of triangles. Using similar arguments as in the proof of Proposition 3.3, we conclude that x is a left $\mathscr{X}$-approximation of T_1. By Lemma 3.8, we have $N \in \mathscr{X}$. Since

$$c = g[1]w \in [\mathcal{T}[1]](\mathscr{X}, M[1]) = 0.$$

This shows that the triangle

$$M \xrightarrow{a} N \xrightarrow{b} X_2 \xrightarrow{c} M[1].$$

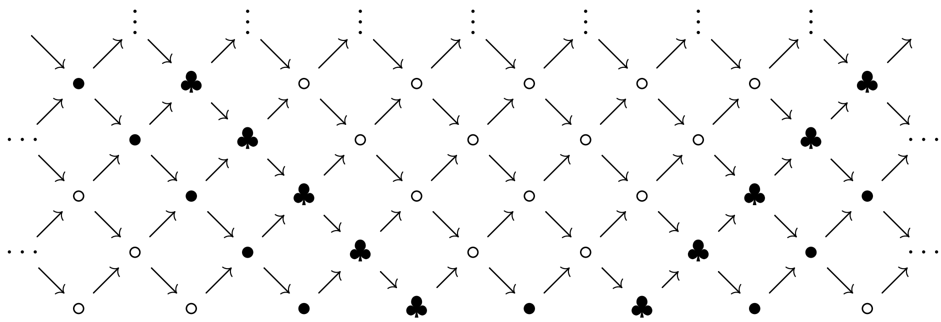
splits. It follows that M is a direct summand of N and thus $M \in \mathscr{X}$. Hence $\mathscr{X}$ is $\mathcal{T}[1]$-cluster tilting. □

Corollary 3.11. *Let $\mathscr{C}$ be a triangulated category with a cluster tilting subcategory $\mathcal{T}$. Then $\mathcal{T}[1]$-cluster tilting subcategories are weak $\mathcal{T}[1]$-cluster tilting subcategories.*

Proof. This follows from Theorem 3.10 and Corollary 3.9. □

The following example shows the converse is not true. More precisely, weak $\mathcal{T}[1]$-cluster tilting subcategories are not usually $\mathcal{T}[1]$-cluster tilting subcategories.

Example 3.12. The cluster category of type $\mathbb{A}_\infty$ was introduced in [Holm and Jørgensen 2012; Ng 2010]. This definition is completely analogous to the definition of the cluster category of type $\mathbb{A}_n$. Namely, it is the orbit category $D^f(\mathsf{mod}\,\Gamma)/\mathbb{S}[-2]$. Here Γ is a quiver of type $\mathbb{A}_\infty$ with zigzag orientation and $\mathbb{S}$ and $[1]$ are the Serre and shift functors of the finite derived category $D^f(\mathsf{mod}\,\Gamma)$. Let $\mathscr{C}$ be a cluster category of type $\mathbb{A}_\infty$. The Auslander–Reiten quiver of $\mathscr{C}$ is as follows:



Set $\mathscr{X}$ to be the subcategory whose indecomposable objects are marked by bullets here, and $\mathcal{T}$ to be the subcategory whose indecomposable objects are marked by clubsuits here. It is easy to see that $\mathcal{T}$ is a cluster tilting subcategory of $\mathscr{C}$ and $\mathcal{T} \subseteq \mathscr{X}[-1] * \mathscr{X}$. By [Holm and Jørgensen 2012, Theorem 4.3], we have that $\mathscr{X}$ is a weak cluster tilting subcategory of $\mathscr{C}$ since the corresponding set of arcs is a maximal set of noncrossing arcs. By [Holm and Jørgensen 2012, Theorem 4.4], we obtain that $\mathscr{X}$ is not contravariantly finite in $\mathscr{C}$ since the corresponding maximal set of noncrossing arcs has no right-fountain. That is to say, $\mathscr{X}$ is a weak ghost cluster tilting subcategory in the sense of Definition 3.1, but it is not ghost cluster tilting (=cluster tilting).

As an application of Theorem 3.10, we have the following:

Corollary 3.13 [Zhou and Zhu 2011, Theorem 2.6]. *Let $\mathscr{C}$ be a 2-Calabi–Yau triangulated category with a cluster tilting subcategory $\mathcal{T}$. Then every functorially finite maximal rigid subcategory is cluster-tilting.*

Proof. This follows from Lemma 3.5 and Theorem 3.10. □

We give a characterization of weak ghost cluster tilting subcategories.

Theorem 3.14. *Let $\mathscr{C}$ be a triangulated category with a cluster tilting subcategory $\mathcal{T}$, and $\mathscr{X}$ a subcategory of $\mathscr{C}$. Then $\mathscr{X}$ is a weak ghost cluster tilting subcategory if and only if $\mathscr{X}$ is a maximal ghost rigid subcategory such that $\mathcal{T} \subseteq \mathscr{X}[-1] * \mathscr{X}$.*

Proof. This follows from similar arguments as in the proof of Theorem 3.10. □

We conclude with a picture illustrating the relationships between ghost cluster tilting subcategories and related subcategories:

$$
\begin{tikzcd}
\boxed{\text{cluster tilting}} \arrow[d, shift left, "\text{Proposition 3.3}"] & \\
\boxed{\text{ghost cluster tilting}} \arrow[u, shift left, "\substack{\text{F-stable}\\ \text{Theorem 3.6}}"] \arrow[r, shift left, "\text{Corollary 3.11}"] \arrow[d, shift left, "\text{Proposition 3.7}"] & \boxed{\text{weak ghost cluster tilting}} \arrow[l, shift left, "\substack{\text{contravariantly finite}\\ \text{Definition 3.1}}"] \\
\boxed{\text{maximal ghost rigid}} \arrow[u, shift left, "\substack{\text{contravariantly finite}\\ \text{Theorem 3.10}}"] &
\end{tikzcd}
$$

A characterization of ghost cluster tilting objects. In this subsection, we always assume that $\mathscr{C}$ is a triangulated category with a Serre functor and a cluster tilting object T. We will prove that the $\mathsf{add}(T[1])$-cluster tilting objects are precisely the $T[1]$-cluster tilting objects introduced in [Yang and Zhu 2019], see Definition 2.4. Notice that the two objects have similar names but quite different definitions. To prove it, we need a lemma:

Lemma 3.15. (a) *Let T be a cluster tilting object and X a $T[1]$-cluster tilting object in $\mathscr{C}$. Let $T_0 \in \mathsf{add}\,T$, let $g : T_0 \longrightarrow X_0$ be a left $\mathsf{add}\,X$-approximation of T_0 and consider the associated triangle*:

$$M[-1] \xrightarrow{f} T_0 \xrightarrow{g} X_0 \xrightarrow{h} M.$$

Then $M \in \mathsf{add}\,X$.

(b) *Let T be a cluster tilting object and X a $T[1]$-cluster tilting object in $\mathscr{C}$. Let $T_0 \in \mathsf{add}\,T$, let $f : X_0[-1] \to T_0$ be a right $\mathsf{add}\,X[-1]$-approximation of T_0 and consider the associated triangle*:

$$X_0[-1] \xrightarrow{f} T_0 \xrightarrow{g} M \xrightarrow{h} X_0.$$

Then $M \in \mathsf{add}\,X$.

Proof. Using similar arguments as in the proof of Lemma 3.8 we conclude that

$$[T[1]](X \oplus M, (X \oplus M)[1]) = 0.$$

By Corollary 3.7(1) in [Yang and Zhu 2019], we know that the number of nonisomorphic indecomposable direct summands of any $T[1]$-rigid object is at most the number of nonisomorphic indecomposable direct summands of a cluster tilting object. Thus we have $|X \oplus M| \leq |T|$. Since $|X| = |T|$, we deduce that $M \in \mathsf{add}\,X$. $\square$

Our main result in this subsection is the following:

Theorem 3.16. *Let T be a cluster tilting object in a triangulated category $\mathscr{C}$ with a Serre functor. Let X be an object in $\mathscr{C}$. Then X is an $\mathsf{add}(T[1])$-cluster tilting object, that is to say,*

$$\mathsf{add}\,X = \{M \in \mathscr{C} \mid \mathsf{add}(T[1])(X, M[1]) = 0 \quad \textit{and} \quad \mathsf{add}(T[1])(M, X[1]) = 0\},$$

by Definition 3.1, if and only if X is a $T[1]$-cluster tilting object, that is to say,

$$T[1](X, X[1]) = 0 \quad \textit{and} \quad |X| = |T|,$$

by Definition 2.4.

Proof. (1) The "only if" part: Assume that X is an $\mathsf{add}\,T[1]$-cluster tilting object. Then X is $T[1]$-rigid. By [Yang and Zhu 2019, Corollary 3.7(2)], there exists an object $M \in \mathscr{C}$ such that $X \oplus M$ is a $T[1]$-cluster tilting object. That is to say, $X \oplus M$ is $T[1]$-rigid and $|X \oplus M| = |T|$. Since $X \oplus M$ is $T[1]$-rigid, we have

$$[T[1]](M, X[1]) = 0 = [T[1]](X, M[1]).$$

By Definition 3.1(iv), we have $M \in \mathsf{add}\,X$. It follows that $|X| = |X \oplus M| = |T|$. This shows that X is $T[1]$-cluster tilting.

(2) The "if" part: Assume X is a $T[1]$-cluster tilting object in Definition 2.4. Clearly,

$$\mathsf{add}\,X \subseteq \{M \in \mathscr{C} \mid [T[1]](X, M[1]) = 0 = [T[1]](M, X[1])\}.$$

Conversely, for any object $M \in \{M \in \mathscr{C} \mid [T[1]](X, M[1]) = 0 = [T[1]](M, X[1])\}$, since T is cluster tilting, we have a triangle

$$T_1 \xrightarrow{f} T_0 \xrightarrow{g} M \xrightarrow{h} T_1[1],$$

where $T_0, T_1 \in \mathsf{add}\,T$. For the object $T_0 \in \mathsf{add}\,T$, there is a left $\mathsf{add}\,X$-approximation l_1 of T_0, which can be extended to a triangle

$$X_1[-1] \xrightarrow{m} T_0 \xrightarrow{l_1} X_0 \to X_1.$$

By Lemma 3.15, we have $X_1 \in \mathsf{add}\,X$.

Let $l_2 = l_1 f$. It is easy to see that l_2 is a left $\mathsf{add}\,X$-approximation of T_1. Indeed, for any object $X' \in \mathsf{add}\,X$ and any map $a \in \mathrm{Hom}(T_1, X')$, we have that $ah[-1] \in [T](M[-1], X') = 0$. Then there exists $b : T_0 \longrightarrow X'$ such that $a = bf$. Because l_1 is a left $\mathsf{add}\,X$-approximation of T_0, there is a map $c : X_0 \longrightarrow X'$ such that $b = cl_1$. Therefore $a = bf = c(l_1 f) = cl_2$ and l_2 is a left $(\mathsf{add}\,X)$-approximation of T_1.

$$\begin{tikzcd} M[-1] \arrow[r, "h{[-1]}"] & T_1 \arrow[r, "f"] \arrow[d, "\forall a"'] & T_0 \arrow[r, "g"] \arrow[d, "l_1"] \arrow[dl, dotted, "b"] & M \arrow[r, "h"] & T_1[1] \\ & X' & X_0 \arrow[l, dotted, "c"] & & \end{tikzcd}$$

Using Lemma 3.15, we get a triangle

$$X_2[-1] \to T_1 \xrightarrow{l_2} X_0 \to X_2,$$

where $X_2 \in \operatorname{add} X$. Starting with $l_2 = l_1 f$, we get the following commutative diagram by the octahedral axiom.

$$\begin{tikzcd}
 & M[-1] \arrow[r, equal] \arrow[d] & M[-1] \arrow[d] & \\
X_0[-1] \arrow[r] \arrow[d, equal] & X_2[-1] \arrow[r] \arrow[d] & T_1 \arrow[r, "l_2"] \arrow[d, "f"] & X_0 \arrow[d, equal] \\
X_0[-1] \arrow[r] & X_1[-1] \arrow[r, "m"] \arrow[d, "n"] & T_0 \arrow[r, "l_1"] \arrow[d, "g"] & X_0 \\
 & M \arrow[r, equal] & M &
\end{tikzcd}$$

Since $n = gm \in [T](X_1[-1], M) = 0$, we get a split triangle and thus $M \in \operatorname{add} X$. This shows that X is an $\operatorname{add} T[1]$-cluster tilting object. □

Remark 3.17. Let $\mathscr{C}$ be a triangulated category with a cluster tilting object T. One may want to define $T[1]$-cluster tilting objects in the spirit of Definition 2.1 as one of the following two possibilities:

$$\operatorname{add} X = \{M \in \mathscr{C} \mid [T[1]](X, M[1]) = 0\}, \tag{3-7}$$

or

$$\operatorname{add} X = \{M \in \mathscr{C} \mid [T[1]](X, M[1]) = 0\} = \{M \in \mathscr{C} \mid [T[1]](M, X[1]) = 0\}. \tag{3-8}$$

However, neither one of these agrees with the description in Definition 2.4 of a $T[1]$-cluster tilting object which is a $T[1]$-rigid object with the same number of non-isomorphic indecomposable direct summands as $|T|$, as one can see in Example 3.18.

Example 3.18. Let Q be the quiver $1 \xrightarrow{\alpha} 2$ and τ_Q be the Auslander–Reiten translation in $D^b(kQ)$. We consider a triangulated category, named *repetitive cluster category* in [Zhu 2011], $\mathscr{C} = D^b(kQ)/\langle \tau_Q^{-2}[2] \rangle$, whose objects are the same in $D^b(kQ)$, and whose morphisms are given by

$$\operatorname{Hom}_{D^b(kQ)/\langle \tau_Q^{-2}[2] \rangle}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{D^b(kQ)}(X, (\tau_Q^{-2}[2])^i Y).$$

We depict the Auslander–Reiten quiver of $\mathscr{C}$ as follows.

$$\begin{tikzcd}[column sep=small]
& \substack{1\\2} \arrow[dr] & & 2[1] \arrow[ll, dotted] \arrow[dr] & & 1[1] \arrow[ll, dotted] \arrow[dr] & & \substack{1\\2}[2] \arrow[ll, dotted] \arrow[dr] & & 2[3] \arrow[ll, dotted] \arrow[dr] & & \substack{1\\2} \arrow[ll, dotted] \\
2 \arrow[ur] & & 1 \arrow[ll, dotted] \arrow[ur] & & \substack{1\\2}[1] \arrow[ll, dotted] \arrow[ur] & & 2[2] \arrow[ll, dotted] \arrow[ur] & & 1[2] \arrow[ll, dotted] \arrow[ur] & & 2 \arrow[ll, dotted] \arrow[ur] &
\end{tikzcd}$$

It is easy to check that the direct sum

$$T = 1 \oplus 2[1] \oplus \substack{1\\2}[2] \oplus 1[2]$$

of the encircled indecomposable objects is a cluster tilting object. Thus it is also a $T[1]$-cluster tilting object. Clearly,

$$\{M \in \mathscr{C} \mid [T[1]](T, M[1]) = 0\} = \mathscr{C} \neq \operatorname{add} T,$$

which means that (3-7) or (3-8) does not hold.

4. Connection with τ-tilting theory

Throughout this section, we assume that $\mathscr{C}$ is a k-linear, Hom-finite triangulated category with a cluster tilting subcategory $\mathcal{T}$. It is well known that the category $\operatorname{mod}\mathcal{T}$ of coherent $\mathcal{T}$-modules is abelian. By Theorem 2.8, we know that the restricted Yoneda functor $\mathbb{H} : \mathscr{C} \to \operatorname{mod}\mathcal{T}$ induces an equivalence

$$\mathscr{C}/[\mathcal{T}[1]] \xrightarrow{\sim} \operatorname{mod}\,\mathcal{T}.$$

We will investigate this relationship between $\mathscr{C}$ and $\operatorname{mod}\,\mathcal{T}$ via $\mathbb{H}$ more closely.

On the relationship between ghost cluster tilting and support τ-tilting. In this subsection, we give a direct connection between ghost cluster tilting subcategories of $\mathscr{C}$ and support τ-tilting pairs of $\operatorname{mod}\mathcal{T}$. We start with the following important observation.

Lemma 4.1. *Let $\mathscr{C}$ be a triangulated category with a cluster tilting subcategory $\mathcal{T}$ and $\mathscr{X}$ a subcategory of $\mathscr{C}$. For any object $X \in \mathscr{X}$, let*

$$T_1 \xrightarrow{f} T_0 \xrightarrow{g} X \xrightarrow{h} T_1[1] \tag{4-1}$$

be a triangle in $\mathscr{C}$ with $T_0, T_1 \in \mathcal{T}$. Then applying the functor $\mathbb{H}$ gives a projective presentation

$$P_1^{\mathbb{H}(X)} \xrightarrow{\pi^{\mathbb{H}(X)}} P_0^{\mathbb{H}(X)} \to \mathbb{H}(X) \to 0 \tag{4-2}$$

in $\operatorname{mod}\,\mathcal{T}$, and $\mathscr{X}$ is a $\mathcal{T}[1]$-rigid subcategory if and only if the class $\{\pi^{\mathbb{H}(X)} \mid X \in \mathscr{X}\}$ has Property (S).

Proof. Applying $\mathbb{H}$ to the triangle (4-1), we have the projective presentation (4-2). By Theorem 2.8(ii), for any object $X' \in \mathscr{X}$, we have the commutative diagram

$$\begin{tikzcd} \operatorname{Hom}_{\mathsf{mod}\,\mathcal{T}}(\mathbb{H}(T_0), \mathbb{H}(X')) \arrow[r, "\operatorname{Hom}_{\mathsf{mod}\,\mathcal{T}}(\pi^{\mathbb{H}(X)},\, \mathbb{H}(X'))"] \arrow[d, "\simeq"'] & \operatorname{Hom}_{\mathsf{mod}\,\mathcal{T}}(\mathbb{H}(T_1), \mathbb{H}(X')) \arrow[d, "\simeq"] \\ \operatorname{Hom}_{\mathscr{C}}(T_0, X') \arrow[r, "\operatorname{Hom}_{\mathscr{C}}(f,\ X')"] & \operatorname{Hom}_{\mathscr{C}}(T_1, X'). \end{tikzcd}$$

Thus the map $\operatorname{Hom}_{\mathsf{mod}\,\mathcal{T}}(\pi^{\mathbb{H}(X)}, \mathbb{H}(X'))$ is the same as

$$\operatorname{Hom}_{\mathscr{C}}(T_0, X') \xrightarrow{\operatorname{Hom}_{\mathscr{C}}(f,\ X')} \operatorname{Hom}_{\mathscr{C}}(T_1, X'). \tag{4-3}$$

So the class $\{\pi^{\mathbb{H}(X)} \mid X \in \mathscr{X}\}$ has Property (S) if and only if the morphism (4-3) is surjective for all $X, X' \in \mathscr{X}$.

Assume the class $\{\pi^{\mathbb{H}(X)} \mid X \in \mathscr{X}\}$ has Property (S). For any $a \in [\mathcal{T}[1]](\mathscr{X}, \mathscr{X}[1])$, we know that there exist two morphisms $a_1 : X \to T[1]$ and $a_2 : T[1] \to X'[1]$ such that $a = a_2 a_1$, where $X, X' \in \mathscr{X}$ and $T \in \mathcal{T}$. Since $\operatorname{Hom}_{\mathscr{C}}(T_0, T[1]) = 0$, there exists a morphism $b : T_1[1] \to T[1]$ such that $a_1 = bh$.

$$\begin{tikzcd} T_1 \arrow[r, "f"] & T_0 \arrow[r, "g"] & X \arrow[r, "h"] \arrow[d, "a_1"] & T_1[1] \arrow[dl, dotted, "b"] \\ & & T[1] \arrow[d, "a_2"] & \\ & & X'[1] & \end{tikzcd}$$

Since $\operatorname{Hom}_{\mathscr{C}}(f, X')$ is surjective, there exists a morphism $c : T_0 \to X'$ such that $a_2[-1] \circ b[-1] = cf$ and thus $a_2 b = c[1] \circ f[1]$. It follows that $a = a_2 a_1 = a_2 bh = c[1] \circ (f[1]h) = 0$. This shows that $[\mathcal{T}[1]](\mathscr{X}, \mathscr{X}[1]) = 0$. Hence $\mathscr{X}$ is a $\mathcal{T}[1]$-rigid subcategory.

Conversely, assume that $\mathscr{X}$ is a $\mathcal{T}[1]$-rigid subcategory. For any morphism $x : T_1 \to X'$, since $\mathscr{X}$ is $\mathcal{T}[1]$-rigid, we have $x \circ h[-1] = 0$. So there exists a morphism $y : T_0 \to X'$ such that $x = yf$.

$$\begin{tikzcd} X[-1] \arrow[r, "h[-1]"] & T_1 \arrow[r, "f"] \arrow[d, "x"] & T_0 \arrow[r, "g"] \arrow[dl, dotted, "y"] & X \arrow[r, "h"] & T_1[1] \\ & X' & & & \end{tikzcd}$$

This shows that $\operatorname{Hom}_{\mathscr{C}}(f, X') : \operatorname{Hom}_{\mathscr{C}}(T_0, X') \to \operatorname{Hom}_{\mathscr{C}}(T_1, X')$ is surjective. By the above discussion, we deduce that the class $\{\pi^{\mathbb{H}(X)} \mid X \in \mathscr{X}\}$ has Property (S). $\square$

The following result plays an important role in this paper:

Proposition 4.2. *Let $\mathscr{C}$ be a triangulated category with a cluster tilting subcategory $\mathcal{T}$. The functor $\mathbb{H} : \mathscr{C} \to \operatorname{Mod} \mathcal{T}$ induces a bijection between the sets of $\mathcal{T}[1]$-rigid subcategories of $\mathscr{C}$ and of τ-rigid pairs of $\operatorname{mod} \mathcal{T}$, given by*

$$\Phi : \mathscr{X} \longmapsto (\mathbb{H}(\mathscr{X}), \mathcal{T} \cap \mathscr{X}[-1]).$$

Proof. **Step 1**: The map Φ has values in τ-rigid pairs of $\operatorname{mod} \mathcal{T}$.

Assume that $\mathscr{X}$ is a $\mathcal{T}[1]$-rigid subcategory of $\mathscr{C}$. Since $\mathcal{T}$ is a cluster tilting subcategory, for any $X \in \mathscr{X}$, there exists a triangle in $\mathscr{C}$

$$T_1 \xrightarrow{f} T_0 \xrightarrow{g} X \xrightarrow{h} T_1[1],$$

where $T_0, T_1 \in \mathcal{T}$. By Lemma 4.1, we have that $\mathbb{H}$ sends the set of these triangles to a set of projective presentations (4-2) which has Property (S). It remains to show that for any $X \in \mathscr{X}$ and $X' \in \mathcal{T} \cap \mathscr{X}[-1]$, we have $\mathbb{H}(X)(X') = 0$. Indeed, since $\mathscr{X}$ is a $\mathcal{T}[1]$-rigid subcategory, we have $\mathbb{H}(X)(X') = \operatorname{Hom}_{\mathscr{C}}(X', X) = 0$.

$$\begin{tikzcd} X' \in \mathscr{X}[-1] \arrow[r] \arrow[dr, equal] & X \in \mathscr{X} \\ & X' \in \mathcal{T} \arrow[u] \end{tikzcd}$$

This shows that $(\mathbb{H}(\mathscr{X}), \mathcal{T} \cap \mathscr{X}[-1])$ is a τ-rigid pair of $\operatorname{mod} \mathcal{T}$.

Step 2: The map Φ is surjective.

Let $(\mathcal{M}, \mathcal{E})$ be a τ-rigid pair of $\operatorname{mod} \mathcal{T}$. For each $M \in \mathcal{M}$, take a projective presentation

$$P_1 \xrightarrow{\pi^M} P_0 \to M \to 0 \tag{4-4}$$

such that the class $\{\pi^M \mid M \in \mathcal{M}\}$ has Property (S). By Theorem 2.8(ii), there is a unique morphism $f_M : T_1 \to T_0$ in $\mathcal{T}$ such that $\mathbb{H}(f_M) = \pi^M$. Moreover, $\mathbb{H}(\operatorname{cone}(f_M)) \cong M$. Since (4-4) has Property (S), it follows from Lemma 4.1 that the category

$$\mathscr{X}_1 := \{\operatorname{cone}(f_M) \mid M \in \mathcal{M}\}$$

is a $\mathcal{T}[1]$-rigid subcategory.

Let $\mathscr{X} := \mathscr{X}_1 \vee \mathcal{E}[1]$. Now we show that $\mathscr{X}$ is a $T[1]$-rigid subcategory of $\mathscr{C}$. Let $E \in \mathcal{E} \subseteq \mathcal{T}$. Since $\mathcal{T}$ is cluster-tilting, we have

$$[\mathcal{T}[1]](\operatorname{cone}(f_M) \oplus E[1], E[2]) = 0.$$

Applying the functor $\operatorname{Hom}_{\mathscr{C}}(E, -)$ to the triangle $T_1 \xrightarrow{f_M} T_0 \to \operatorname{cone}(f_M) \to T_1[1]$, we have an exact sequence

$$\operatorname{Hom}_{\mathscr{C}}(E, T_1) \xrightarrow{f_M \circ} \operatorname{Hom}_{\mathscr{C}}(E, T_0) \to \operatorname{Hom}_{\mathscr{C}}(E, \operatorname{cone}(f_M)) \to 0,$$

which is isomorphic to

$$P_1(E) \xrightarrow{\pi^M} P_0(E) \to M(E) \to 0.$$

The condition $\mathcal{M}(\mathcal{E}) = 0$ implies that $\mathrm{Hom}_{\mathscr{C}}(E, \mathrm{cone}(f_M)) = 0$ and therefore

$$[\mathcal{T}[1]](E[1], \mathrm{cone}(f_M)[1]) = 0.$$

Thus the assertion follows.

Now we show that $\Phi(\mathscr{X}) = (\mathcal{M}, \mathcal{E})$.

It is straightforward to check that $\mathcal{T} \cap \mathscr{X}_1[-1] = 0$. For any object $X \in \mathcal{T} \cap \mathscr{X}[-1]$, we are able to write $X = X_1[-1] \oplus E \in \mathcal{T}$, where $X_1 \in \mathscr{X}_1$ and $E \in \mathcal{E}$. Since $X_1[-1] \in \mathcal{T} \cap \mathscr{X}_1[-1] = 0$, we have $X = E \in \mathcal{E}$. Thus we have $\mathcal{T} \cap \mathscr{X}[-1] \subseteq \mathcal{E}$. By the definition of τ-rigid pair, we have $\mathcal{E} \subseteq \mathcal{T}$. Noting that $\mathcal{E} \subseteq \mathscr{X}_1[-1] \vee \mathcal{E} = \mathscr{X}[-1]$, it follows that $\mathcal{E} \subseteq \mathcal{T} \cap \mathscr{X}[-1]$. Hence $\mathcal{T} \cap \mathscr{X}[-1] = \mathcal{E}$. It remains to show that $\mathbb{H}(\mathscr{X}) = \mathcal{M}$. Indeed, since $\mathcal{E} \subseteq \mathcal{T}$, we have

$$\mathbb{H}(\mathscr{X}) = \mathrm{Hom}_{\mathscr{C}}(\mathcal{T}, \mathscr{X}) = \mathrm{Hom}_{\mathscr{C}}(\mathcal{T}, \mathscr{X}_1) = \mathbb{H}(\mathscr{X}_1) = \mathcal{M}.$$

Step 3: The map Φ is injective.

Let $\mathscr{X}$ and $\mathscr{X}'$ be two $\mathcal{T}[1]$-rigid subcategories of $\mathscr{C}$ such that $\Phi(\mathscr{X}) = \Phi(\mathscr{X}')$. Let $\mathscr{X}_1$ and $\mathscr{X}_1'$ be respectively the full subcategories of $\mathscr{X}$ and $\mathscr{X}'$ consisting of objects without direct summands in $\mathcal{T}[1]$. Then $\mathscr{X} = \mathscr{X}_1 \vee (\mathscr{X} \cap \mathcal{T}[1])$ and $\mathscr{X}' = \mathscr{X}_1' \vee (\mathscr{X}' \cap \mathcal{T}[1])$. Since $\Phi(\mathscr{X}) = \Phi(\mathscr{X}')$, it follows that $\mathbb{H}(\mathscr{X}_1) = \mathbb{H}(\mathscr{X}_1')$ and $\mathscr{X} \cap \mathcal{T}[1] = \mathscr{X}' \cap \mathcal{T}[1]$.

For any object $X_1 \in \mathscr{X}_1$, there exists $X_1' \in \mathscr{X}_1'$ such that $\mathbb{H}(X_1) = \mathbb{H}(X_1')$. By Theorem 2.8(iii), there exists an isomorphism $X_1 \oplus Y[1] \simeq X_1' \oplus Z[1]$ for some $Y, Z \in \mathcal{T}$. Since $\mathscr{C}$ is Krull–Remak–Schmidt, we have $X_1 \simeq X_1'$. This implies that $\mathscr{X}_1 \subseteq \mathscr{X}_1'$. Similarly, we obtain $\mathscr{X}_1' \subseteq \mathscr{X}_1$ and then $\mathscr{X}_1 \simeq \mathscr{X}_1'$. Therefore $\mathscr{X} = \mathscr{X}'$. This shows that Φ is injective. □

Our main result in this subsection is the following:

Theorem 4.3. *Let $\mathscr{C}$ be a triangulated category with a cluster tilting subcategory $\mathcal{T}$. The functor $\mathbb{H}: \mathscr{C} \to \mathrm{Mod}\,\mathcal{T}$ induces a bijection between the sets of weak $\mathcal{T}[1]$-cluster tilting subcategories of $\mathscr{C}$ and of support τ-tilting pairs of* $\mathrm{mod}\,\mathcal{T}$, *given by*

$$\Phi : \mathscr{X} \longmapsto (\mathbb{H}(\mathscr{X}), \mathcal{T} \cap \mathscr{X}[-1]).$$

Proof. **Step 1**: The map Φ has values in support τ-tilting pairs of $\mathrm{mod}\,\mathcal{T}$.

Assume $\mathscr{X}$ is a weak $\mathcal{T}[1]$-cluster tilting subcategory of $\mathscr{C}$. By Proposition 4.2, we get that $\Phi(\mathscr{X})$ is a τ-rigid pair of $\mathrm{mod}\,\mathcal{T}$. Therefore $\mathcal{T} \cap \mathscr{X}[-1] \subseteq \mathrm{Ker}\ \mathbb{H}(\mathscr{X})$.

Let $T \in \mathcal{T}$ be an object of $\mathrm{Ker}\ \mathbb{H}(\mathscr{X})$, that is, $\mathrm{Hom}_{\mathscr{C}}(T, X) = 0$ for each $X \in \mathscr{X}$. This implies $[\mathcal{T}[1]](X \oplus T[1], \mathscr{X}[1]) = 0$. Note $[\mathcal{T}[1]](\mathscr{X}, (X \oplus T[1])[1]) = 0$. Since $\mathscr{X}$ is a $\mathcal{T}[1]$-cluster tilting subcategory, we have $X \oplus T[1] \in \mathscr{X}$ and thus

$T \subseteq \mathscr{X}[-1]$. Therefore $T \in \mathcal{T} \cap \mathscr{X}[-1]$. This shows that $\operatorname{Ker}\ \mathbb{H}(\mathscr{X}) \subseteq \mathcal{T} \cap \mathscr{X}[-1]$. Hence

$$\operatorname{Ker}\ \mathbb{H}(\mathscr{X}) = \mathcal{T} \cap \mathscr{X}[-1].$$

By the definition of weak $\mathcal{T}[1]$-cluster tilting subcategories, for any $T \in \mathcal{T}$, there exists a triangle

$$T \xrightarrow{f} X_1 \xrightarrow{g} X_2 \xrightarrow{h} T[1],$$

where $X_1, X_2 \in \mathscr{X}$. Applying the functor $\mathbb{H}$ to the above triangle, we obtain an exact sequence

$$\mathbb{H}(T) \xrightarrow{\mathbb{H}(f)} \mathbb{H}(X_1) \to \mathbb{H}(X_2) \to 0.$$

For any morphism $a : T \to X$, where $X \in \mathscr{X}$, since $\mathscr{X}$ is $\mathcal{T}[1]$-rigid, we have $ah[-1] = 0$. So there exists a morphism $b : X_1 \to X$ such that $a = bf$. This shows that $\operatorname{Hom}_{\mathscr{C}}(f, X)$ is surjective. Thus there exists the commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathscr{C}}(X_1, X) \arrow[r, "\operatorname{Hom}_{\mathscr{C}}(f,\ X)"] \arrow[d] & \operatorname{Hom}_{\mathscr{C}}(T, X) \arrow[r] \arrow[d] & 0 \\
\operatorname{Hom}_{\mathsf{mod}\ \mathcal{T}}(\mathbb{H}(X_1), \mathbb{H}(X)) \arrow[r, "\circ\mathbb{H}(f)"] & \operatorname{Hom}_{\mathsf{mod}\ \mathcal{T}}(\mathbb{H}(T), \mathbb{H}(X)). &
\end{tikzcd}$$

Using Theorem 2.8(ii), the right vertical map is an isomorphism. It follows that $\circ\mathbb{H}(f)$ is surjective, that is, $\mathbb{H}(f)$ is a left $\mathbb{H}(\mathscr{X})$-approximation. Altogether, we have shown that $\Phi(\mathscr{X})$ is a support τ-tilting pair of $\mathsf{mod}\ \mathcal{T}$.

Step 2: The map Φ is surjective.

Let $(\mathcal{M}, \mathcal{E})$ be a support τ-tilting pair of $\mathsf{mod}\ \mathcal{T}$ and let $\mathscr{X}$ be the preimage of $(\mathcal{M}, \mathcal{E})$ under Φ constructed in Proposition 4.2. Since $\mathbb{H}(\mathscr{X}) = \mathcal{M}$ is a support τ-tilting subcategory, for each $T \in \mathcal{T}$, there is an exact sequence

$$\mathbb{H}(T) \xrightarrow{\alpha} \mathbb{H}(X_3) \to \mathbb{H}(X_4) \to 0,$$

such that $X_3, X_4 \in \mathscr{X}$ and α is a left $\mathbb{H}(\mathscr{X})$-approximation. By Yoneda's lemma, there exists a unique morphism $\beta : T \to X_3$ such that $\mathbb{H}(\beta) = \alpha$. We complete this to a triangle

$$T \xrightarrow{\beta} X_3 \xrightarrow{\gamma} Y_T \xrightarrow{\delta} T[1]. \tag{4-5}$$

Let $\widetilde{\mathscr{X}} := \mathscr{X} \vee \mathsf{add}\{\, Y_T \mid T \in \mathcal{T}\,\}$ be the additive closure of $\mathscr{X}$ and $\{\, Y_T \mid T \in \mathcal{T}\,\}$. We claim $\widetilde{\mathscr{X}}$ is a weak $\mathcal{T}[1]$-cluster tilting subcategory of $\mathscr{C}$ such that $\Phi(\widetilde{\mathscr{X}}) = (\mathcal{M}, \mathcal{E})$.

It is clear that $\mathcal{T} \subseteq \mathscr{X}[-1] * \mathscr{X}$. It remains to show that

$$\widetilde{\mathscr{X}} = \{M \in \mathscr{C} \mid [\mathcal{T}[1]](M, \widetilde{\mathscr{X}}[1]) = 0 = [\mathcal{T}[1]](\widetilde{\mathscr{X}}, M[1])\}.$$

Applying the functor $\mathbb{H}$ to the triangle (4-5), we see that $\mathbb{H}(Y_T)$ and $\mathbb{H}(X_4)$ are isomorphic in $\mathsf{mod}\ \mathcal{T}$. For any object $X \in \mathscr{X}$, consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(X_3, X) \arrow[r, "\mathrm{Hom}_{\mathscr{C}}(\beta,\ X)"] \arrow[d, "\mathbb{H}(-)"'] & \mathrm{Hom}_{\mathscr{C}}(T, X) \arrow[d, "\simeq"] \\
\mathrm{Hom}_{\mathsf{mod}\ \mathcal{T}}(\mathbb{H}(X_3), \mathbb{H}(X)) \arrow[r, "\circ\alpha"] & \mathrm{Hom}_{\mathsf{mod}\ \mathcal{T}}(\mathbb{H}(T), \mathbb{H}(X)).
\end{tikzcd}$$

By Theorem 2.8, the map $\mathbb{H}(-)$ is surjective and the right vertical map is an isomorphism. Because α is a left $\mathbb{H}(\mathscr{X})$-approximation, $\circ\alpha$ is also surjective. Therefore $\mathrm{Hom}_{\mathscr{C}}(\beta, X)$ is surjective too.

For any morphism $a \in [\mathcal{T}[1]](Y_T, X[1])$, since $\mathscr{X}$ is $\mathcal{T}[1]$-rigid, we have $a\gamma = 0$. So there exists a morphism $b : T[1] \to X[1]$ such that $a = b\delta$.

$$\begin{tikzcd}
T \arrow[r, "\beta"] & X_3 \arrow[r, "\gamma"] & Y_T \arrow[r, "\delta"] \arrow[d, "a"] & T[1] \arrow[dl, dotted, "b"] \\
& & X[1] &
\end{tikzcd}$$

Since $\mathrm{Hom}_{\mathscr{C}}(\beta,\ X)$ is surjective, there exists a morphism $c : X_3 \to X$ such that $c\beta = b[-1]$ and thus $b = c[1] \circ \beta[1]$. It follows that $a = b\delta = c[1] \circ (\beta[1]\delta) = 0$. This shows that

$$[\mathcal{T}[1]](Y_T, \mathscr{X}[1]) = 0. \tag{4-6}$$

For any morphism $x \in [\mathcal{T}](X[-1], Y_T)$, we know that there exist two morphisms $x_1 : X[-1] \to T_1$ and $x_2 : T_1 \to Y_T$ such that $x = x_2x_1$, where $T_1 \in \mathcal{T}$. Since $\mathcal{T}$ is cluster tilting, we have $\delta x_2 = 0$. So there exists a morphism $y : T_1 \to X_3$ such that $x_2 = \gamma y$.

$$\begin{tikzcd}
& & X[-1] \arrow[d, "x_1"] & \\
& & T_1 \arrow[d, "x_2"] \arrow[dl, dotted, "y"'] & \\
T \arrow[r, "\beta"] & X_3 \arrow[r, "\gamma"] & Y_T \arrow[r, "\delta"] & T[1]
\end{tikzcd}$$

Since $\mathscr{X}$ is $\mathcal{T}[1]$-rigid, we have $x = x_2x_1 = \gamma(yx_1) = 0$. This shows that

$$[\mathcal{T}[1]](\mathscr{X}, Y_T[1]) = 0. \tag{4-7}$$

For any $T' \in \mathcal{T}$ and morphism $u \in [\mathcal{T}](Y_{T'}[-1], Y_T)$, we know that there exist two morphisms $u_1 : Y_{T'}[-1] \to T_2$ and $u_2 : T_2 \to Y_T$ such that $u = u_2u_1$, where $T_2 \in \mathcal{T}$. Since $\mathcal{T}$ is cluster tilting, we have $\delta u_2 = 0$. So there exists a morphism $v : T_2 \to X_3$ such that $u_2 = \gamma v$.

$$\begin{tikzcd}
 & & Y_{T'}[-1] \arrow[d, "u_1"] & \\
 & & T_2 \arrow[dl, dotted, "v"'] \arrow[d, "u_2"] & \\
T \arrow[r, "\beta"] & X_3 \arrow[r, "\gamma"] & Y_T \arrow[r, "\delta"] & T[1]
\end{tikzcd}$$

Since $[\mathcal{T}[1]](Y_T, \mathscr{X}[1]) = 0$, we have $vu_1 = 0$. It follows that $u = u_2u_1 = \gamma v u_1 = 0$. This shows that

$$[\mathcal{T}[1]](Y_{T'}, Y_T[1]) = 0. \tag{4-8}$$

Using equalities (4-6), (4-7) and (4-8), we deduce that $\widetilde{\mathscr{X}}$ is a $\mathcal{T}[1]$-rigid subcategory.

Now we show that $\{M \in \mathscr{C} \mid [\mathcal{T}[1]](M, \widetilde{\mathscr{X}}[1]) = 0 = [\mathcal{T}[1]](\widetilde{\mathscr{X}}, M[1])\} \subseteq \widetilde{\mathscr{X}}$.

For any object $M \in \mathscr{C}$, assume that $[\mathcal{T}[1]](M, \widetilde{\mathscr{X}}[1]) = 0 = [\mathcal{T}[1]](\widetilde{\mathscr{X}}, M[1])$. Since $\mathcal{T}$ is a cluster-tilting subcategory, there exists a triangle

$$T_5 \xrightarrow{f} T_6 \xrightarrow{g} M \xrightarrow{h} T_5[1],$$

where $T_5, T_6 \in \mathcal{T}$. By the above discussion, for an object $T_6 \in \mathcal{T}$, there exists a triangle

$$T_6 \xrightarrow{u} X_6 \xrightarrow{v} Y_{T_6} \xrightarrow{w} T_6[1],$$

where $X_6 \in \mathscr{X}$, $Y_{T_6} \in \widetilde{\mathscr{X}}$ and u is a left $\mathscr{X}$-approximation of T_6. For an object $T_5 \in \mathcal{T}$, there exists a triangle

$$T_5 \xrightarrow{u'} X_5 \xrightarrow{v'} Y_{T_5} \xrightarrow{w'} T_5[1],$$

where $X_5 \in \mathscr{X}$, $Y_{T_5} \in \widetilde{\mathscr{X}}$ and u' is a left $\mathscr{X}$-approximation of T_5. By the octahedral axiom, we have a commutative diagram

$$\begin{tikzcd}
T_5 \arrow[r, "f"] \arrow[d, equal] & T_6 \arrow[r, "g"] \arrow[d, "u"] & M \arrow[r, "h"] \arrow[d, "a"] & T_5[1] \arrow[d, equal] \\
T_5 \arrow[r, "x=uf"] & X_6 \arrow[r, "y"] \arrow[d, "v"] & N \arrow[r, "z"] \arrow[d, "b"] & T_5[1] \\
 & Y_{T_6} \arrow[r, equal] \arrow[d, "w"] & Y_{T_6} \arrow[d, "c"] & \\
 & T_6[1] \arrow[r, "{g[1]}"] & M[1] &
\end{tikzcd}$$

of triangles in $\mathscr{C}$. We claim that x is a left $\mathscr{X}$-approximation of T_5. Indeed, for any $d : T_5 \to X$, since $dh[-1] \in [\mathcal{T}](M[-1], \widetilde{\mathscr{X}}) = 0$, there exists a morphism

$e: T_6 \to X$ such that $d = ef$, where $X \in \mathscr{X}$.

$$\begin{tikzcd} M[-1] \arrow[r, "{h[-1]}"] & T_5 \arrow[r, "f"] \arrow[d, "d"] & T_6 \arrow[r, "g"] \arrow[dl, dotted, "e"] & M \arrow[r, "h"] & T_5[1] \\ & X & & & \end{tikzcd}$$

Since u is a left $\mathscr{X}$-approximation of T_6, there exists a morphism $k: X_6 \to X$ such that $ku = e$. It follows that $d = ef = kuf = kx$, as required.

Since x is a left $\mathscr{X}$-approximation of T_5, by Lemma 1.4.3 in [Neeman 2001], we have the commutative diagram

$$\begin{tikzcd} T_5 \arrow[r, "x"] \arrow[d, equal] & X_6 \arrow[r, "y"] \arrow[d, "\lambda"] & N \arrow[r, "z"] \arrow[d, dotted, "\varphi"] & T_5[1] \arrow[d, equal] \\ T_5 \arrow[r, "u'"] & X_5 \arrow[r, "v'"] & Y_{T_5} \arrow[r, "w'"] & T_5[1], \end{tikzcd}$$

where the middle square is homotopy cartesian and the differential $\partial = x[1] \circ w'$, that is, there exists a triangle

$$X_6 \xrightarrow{\binom{-y}{\lambda}} N \oplus X_5 \xrightarrow{(\varphi, v')} Y_{T_5} \xrightarrow{\partial} X_6[1].$$

Note that $\partial \in [\mathcal{T}[1]](\widetilde{\mathscr{X}}, \widetilde{\mathscr{X}}[1]) = 0$. Thus we have $N \oplus X_5 \simeq X_6 \oplus Y_{T_5} \in \widetilde{\mathscr{X}}$, which implies $N \in \widetilde{\mathscr{X}}$. Since $c = g[1]w \in [\mathcal{T}[1]](\widetilde{\mathscr{X}}, M[1]) = 0$, we deduce that the triangle

$$M \xrightarrow{a} N \xrightarrow{b} Y_{T_6} \xrightarrow{c} M[1]$$

splits. Hence M is a direct summand of N and thus $M \in \widetilde{\mathscr{X}}$.

This shows that

$$\widetilde{\mathscr{X}} = \{M \in \mathscr{C} \mid [\mathcal{T}[1]](M, \widetilde{\mathscr{X}}[1]) = 0 = [\mathcal{T}[1]](\widetilde{\mathscr{X}}, M[1])\}.$$

For any object $T \in \mathcal{T}$, $\mathbb{H}(Y_T) \simeq \mathbb{H}(X_4)$. Therefore

$$\mathbb{H}(\widetilde{\mathscr{X}}) \simeq \mathbb{H}(\mathscr{X}) \simeq \mathcal{M}.$$

Since $\mathcal{T} \cap \widetilde{\mathscr{X}}[-1] \supseteq \mathcal{T} \cap \mathscr{X}[-1] = \mathcal{E}$ and $\mathcal{T} \cap \widetilde{\mathscr{X}}[-1] \subseteq \mathrm{Ker}\ \mathbb{H}(\mathscr{X}) = \mathcal{E}$, we have

$$\mathcal{T} \cap \widetilde{\mathscr{X}}[-1] = \mathcal{E}.$$

This shows that Φ is surjective.

Step 3: The map Φ is injective.

This follows from Step 3 in Proposition 4.2. $\square$

For any support τ-tilting subcategory $\mathscr{Y}$ of $\operatorname{mod}\mathcal{T}$, by Theorem 4.3 there exists a unique weak $\mathcal{T}[1]$-cluster tilting subcategory $\mathscr{X}$ of $\mathscr{C}$ such that $\mathbb{H}(\mathscr{X}) = \mathscr{Y}$. Throughout this paper, we denote the preimage $\mathscr{X}$ by $\mathbb{H}^{-1}(\mathscr{Y})$ for simplicity. Consequently, we have the following result:

Theorem 4.4. *The bijection in Theorem 4.3 induces a bijection from the first of the following sets to the second*:

(I) $\mathcal{T}[1]$-*cluster tilting subcategories of* $\mathscr{C}$.

(II) *Support* τ*-tilting subcategories* $\mathscr{Y}$ *of* $\operatorname{mod}\mathcal{T}$ *such that* $\mathbb{H}^{-1}(\mathscr{Y})$ *is contravariantly finite in* $\mathscr{C}$.

Moreover, *if* $\mathscr{C}$ *admits a Serre functor* $\mathbb{S}$, *we get a bijection from the first to the second of the following sets.*

(1) *Cluster tilting subcategories of* $\mathscr{C}$.

(2) *Support* τ*-tilting subcategories* $\mathscr{Y}$ *of* $\operatorname{mod}\mathcal{T}$ *such that* $\mathbb{H}^{-1}(\mathscr{Y})$ *is contravariantly finite and* F*-stable in* $\mathscr{C}$.

Proof. The first bijection follows from the fact that $\mathcal{T}[1]$-cluster tilting subcategories of $\mathscr{C}$ are precisely contravariantly finite weak $\mathcal{T}[1]$-cluster tilting subcategories, and the second bijection follows from Theorem 3.6. □

τ-tilting subcategories and tilting subcategories. $\mathscr{C}$ and $\mathcal{T}$ are the same as above. By definition we know that the category $\operatorname{mod}\mathcal{T}$ is abelian and has enough projectives. Thus we can investigate the projective dimension of an object M in $\operatorname{mod}\mathcal{T}$, which we denote by $\operatorname{pd} M$. For a subcategory $\mathscr{D}$ of $\operatorname{mod}\mathcal{T}$, we say that the projective dimension of $\mathscr{D}$ is at most n, denoted by $\operatorname{pd}\mathscr{D} \le n$, if $\operatorname{pd} M \le n$ for any object $M \in \mathscr{D}$.

Let $X \in \mathscr{C}$, $\mathcal{I}_X(\mathcal{T}[1])$ be the ideal of $\mathcal{T}[1]$ formed by the morphisms between objects in $\mathcal{T}[1]$ factoring through the object X. For a subcategory $\mathscr{D}$ of $\mathscr{C}$, we define the *factorization ideal* of $\mathscr{D}$, denoted by $\mathcal{I}_{\mathscr{D}}(\mathcal{T}[1])$, as follows

$$\mathcal{I}_{\mathscr{D}}(\mathcal{T}[1]) := \{\mathcal{I}_X(\mathcal{T}[1]) \mid X \in \mathscr{D}\}.$$

Theorem 2.8 indicates that $\operatorname{mod}\mathcal{T}$ is Gorenstein of dimension at most one. Thus all objects in $\operatorname{mod}\mathcal{T}$ have projective dimension zero, one or infinity. The following result characterizes the objects in $\operatorname{mod}\mathcal{T}$ having finite projective dimension.

Theorem 4.5 [Beaudet et al. 2014; Lasnier 2011]. *Let* $\mathscr{C}$ *be a triangulated category with a cluster tilting subcategory* $\mathcal{T}$, *and* X *be an object in* $\mathscr{C}$ *having no direct summands in* $\mathcal{T}[1]$. *Then*

$$\operatorname{pd}\mathbb{H}(X) \le 1 \text{ if and only if } \mathcal{I}_X(\mathcal{T}[1]) = 0.$$

In this subsection, we introduce two important classes of subcategories of $\mathsf{mod}\,\mathcal{T}$ and give a connection with ghost cluster tilting subcategories and cluster tilting subcategories of $\mathscr{C}$. We start with the following definition.

Definition 4.6. Let $\mathscr{C}$ be a triangulated category with a cluster tilting subcategory $\mathcal{T}$.

(i) A subcategory $\mathcal{M}$ of $\mathsf{mod}\,\mathcal{T}$ is said to be *τ-tilting* if $(\mathcal{M}, 0)$ is a support τ-tilting pair of $\mathsf{mod}\,\mathcal{T}$.

(ii) [Beligiannis 2013] A subcategory $\mathcal{M}$ of $\mathsf{mod}\,\mathcal{T}$ is said to be *weak tilting* if the following three conditions are satisfied:

(T1) $\mathrm{Ext}^1_{\mathsf{mod}\,\mathcal{T}}(\mathcal{M}, \mathcal{M}) = 0$.

(T2) $\mathsf{pd}\,M \le 1$, for any $M \in \mathcal{M}$.

(T3) For any projective object P in $\mathsf{mod}\,\mathcal{T}$, there exists a short exact sequence

$$0 \to P \to M_0 \to M_1 \to 0,$$

where $M_0, M_1 \in \mathcal{M}$.

A weak tilting subcategory $\mathcal{M}$ is called a *tilting subcategory* if it also satisfies the following additional condition:

(T4) $\mathcal{M}$ is contravariantly finite in $\mathsf{mod}\,\mathcal{T}$.

Remark 4.7. Beligiannis [2010; 2013] indicates that a contravariantly finite subcategory $\mathcal{M}$ of $\mathsf{mod}\,\mathcal{T}$ is a tilting subcategory if and only if

$$\mathsf{Fac}(\mathcal{M}) = \{X \in \mathsf{mod}\,\mathcal{T} \mid \mathrm{Ext}^1_{\mathsf{mod}\,\mathcal{T}}(\mathcal{M}, X) = 0\},$$

where $\mathsf{Fac}(\mathcal{M})$ is the full subcategory of $\mathsf{mod}\,\mathcal{T}$ consisting of all factors of objects of $\mathcal{M}$.

Immediately, we have the following result:

Theorem 4.8. *Let $\mathscr{C}$ be a triangulated category with a cluster tilting subcategory $\mathcal{T}$. The functor $\mathbb{H} : \mathscr{C} \to \mathsf{Mod}\ \mathcal{T}$ induces a bijection*

$$\Phi : \mathscr{X} \longmapsto \mathbb{H}(\mathscr{X})$$

from the first of the following sets to the second:

(I) *Weak $\mathcal{T}[1]$-cluster tilting subcategories of $\mathscr{C}$ whose objects do not have nonzero direct summands in $\mathcal{T}[1]$.*

(II) *τ-tilting subcategories of* $\mathsf{mod}\ \mathcal{T}$.

It restricts to a bijection from the first to the second of the following sets.

(I) *$\mathcal{T}[1]$-cluster tilting subcategories of $\mathscr{C}$ whose objects do not have nonzero direct summands in $\mathcal{T}[1]$.*

(II) *τ-tilting subcategories $\mathscr{Y}$ of* $\mathsf{mod}\ \mathcal{T}$ *such that $\mathbb{H}^{-1}(\mathscr{Y})$ is contravariantly finite in $\mathscr{C}$.*

Moreover, if $\mathscr{C}$ admits a Serre functor $\mathbb{S}$, we get a bijection between the following sets.

(1) *Cluster tilting subcategories of $\mathscr{C}$ whose objects do not have nonzero direct summands in $\mathcal{T}[1]$.*

(2) *τ-tilting subcategories $\mathscr{Y}$ of $\operatorname{mod}\mathcal{T}$ such that $\mathbb{H}^{-1}(\mathscr{Y})$ is contravariantly finite and F-stable in $\mathscr{C}$.*

Proof. Note that objects in $\mathscr{X}$ do not have nonzero direct summands in $\mathcal{T}[1]$ if and only if $\mathcal{T}\cap\mathscr{X}[-1]=0$. This assertion follows from Theorems 4.3, 4.4 directly. □

Now we give a close relationship between τ-tilting subcategories and weak tilting subcategories.

Lemma 4.9. *Let $\mathscr{C}$ be a triangulated category with a cluster tilting subcategory $\mathcal{T}$. Then any weak tilting subcategory of $\operatorname{mod}\mathcal{T}$ is a τ-tilting subcategory.*

Proof. Let $\mathcal{M}$ be a weak tilting subcategory of $\operatorname{mod}\mathcal{T}$.

(1) We first show that $(\mathcal{M},0)$ is a τ-rigid pair of $\operatorname{mod}\mathcal{T}$. For any object $M\in\mathcal{M}$, since $\operatorname{pd}M\le 1$, we get a short exact sequence

$$0\to P_1\xrightarrow{\pi^M}P_0\to M\to 0.$$

Note that $P_1=0$ if $\operatorname{pd}M=0$. Applying the functor $\operatorname{Hom}_{\operatorname{mod}\mathcal{T}}(-,\mathcal{M})$ to it, we get an exact sequence

$$\operatorname{Hom}_{\operatorname{mod}\mathcal{T}}(P_0,\mathcal{M})\xrightarrow{\circ\pi^M}\operatorname{Hom}_{\operatorname{mod}\mathcal{T}}(P_1,\mathcal{M})\to\operatorname{Ext}^1_{\operatorname{mod}\mathcal{T}}(M,\mathcal{M})=0.$$

This means there is a class of projective presentations $\{P_1\xrightarrow{\pi^M}P_0\to M\to 0\mid M\in\mathcal{M}\}$ which has Property (S). Therefore $(\mathcal{M},0)$ is a τ-rigid pair of $\operatorname{mod}\mathcal{T}$ because $\mathcal{M}(0)=0$.

(2) We show that $(\mathcal{M},0)$ is a support τ-tilting pair of $\operatorname{mod}\mathcal{T}$. For each object $T\in\mathcal{T}$, $\mathcal{T}(-,T)$ is a projective object in $\operatorname{mod}\mathcal{T}$. Since $\mathcal{M}$ is weak tilting in $\operatorname{mod}\mathcal{T}$, there exists a short exact sequence

$$0\to\mathcal{T}(-,T)\xrightarrow{f}M_0,\to M_1\to 0$$

where $M_0,M_1\in\mathcal{M}$. Applying the functor $\operatorname{Hom}_{\operatorname{mod}\mathcal{T}}(-,\mathcal{M})$ to the above exact sequence, we have the following exact sequence:

$$\operatorname{Hom}_{\operatorname{mod}\mathcal{T}}(M_0,\mathcal{M})\xrightarrow{\circ f}\operatorname{Hom}_{\operatorname{mod}\mathcal{T}}(\mathcal{T}(-,T),\mathcal{M})\to\operatorname{Ext}^1_{\operatorname{mod}\mathcal{T}}(M_1,\mathcal{M})=0.$$

This shows that f is a left $\mathcal{M}$-approximation.

If $\mathcal{M}(E)=0$, where $E\in\mathcal{T}$, by the above discussion, there exists an exact sequence

$$0\to\mathcal{T}(-,E)\to M_0\to M_1\to 0$$

with $M^0, M^1 \in \mathcal{M}$. It follows that there exists an exact sequence

$$0 \to \mathcal{T}(E, E) \to M_0(E) \to M_1(E) \to 0$$

Since $M_0(E) = 0$, we have $\mathcal{T}(E, E) = 0$ and thus $E = 0$. Therefore $\operatorname{Ker}(\mathcal{M}) = 0$. This shows that $(\mathcal{M}, 0)$ is a support τ-tilting pair of $\operatorname{mod}\mathcal{T}$. □

The following result gives a criterion for a τ-tilting subcategory of $\operatorname{mod}\mathcal{T}$ to be a weak tilting subcategory.

Theorem 4.10. *Let $\mathscr{C}$ be a triangulated category with a cluster tilting subcategory $\mathcal{T}$. A τ-tilting subcategory of* $\operatorname{mod}\mathcal{T}$ *is a weak tilting subcategory if and only if its projective dimension is at most one.*

Proof. Let $\mathcal{M}$ be a τ-tilting subcategory of $\operatorname{mod}\mathcal{T}$ and $\operatorname{pd}\mathcal{M} \le 1$. By Theorem 4.8, there exists a weak $\mathcal{T}[1]$-tilting subcategory $\mathscr{X}$ of $\mathscr{C}$ whose objects do not have nonzero direct summands in $\mathcal{T}[1]$ such that $\mathbb{H}(\mathscr{X}) = \mathcal{M}$.

Step 1: We show $\operatorname{Ext}^1_{\operatorname{mod}\mathcal{T}}(\mathcal{M}, \mathcal{M}) = 0$. Namely, $\operatorname{Ext}^1_{\operatorname{mod}\mathcal{T}}(\mathbb{H}(\mathscr{X}), \mathbb{H}(\mathscr{X})) = 0$.

For any object $X_1 \in \mathscr{X}$, since $\mathcal{T}$ is cluster tilting, there exists a triangle

$$T_0 \xrightarrow{f} T_1 \xrightarrow{g} X_1 \xrightarrow{h} T_0[1], \tag{4-9}$$

where g is a minimal right $\mathcal{T}$-approximation of X_1 and $T_0, T_1 \in \mathcal{T}$. Since $\mathbb{H}(X_1) \in \mathcal{M}$, we have $\operatorname{pd}\mathbb{H}(X_1) \le 1$. Applying the functor $\mathbb{H}$ to the above triangle, we have a minimal projective presentation

$$0 \to H(T_0) \xrightarrow{f\circ} \mathbb{H}(T_1) \xrightarrow{g\circ} \mathbb{H}(X_1) \to 0$$

of $\mathbb{H}(X_1)$, since X_1 has no nonzero direct summands in $\mathcal{T}[1]$ and $\operatorname{pd}\mathbb{H}(X_1) \le 1$. Applying the functor $\operatorname{Hom}_{\operatorname{mod}\mathcal{T}}(-, \mathbb{H}(X_2))$, where $X_2 \in \mathscr{X}$, to the above exact sequence, we get an exact sequence:

$$\begin{aligned}\operatorname{Hom}(\mathbb{H}(T_1), \mathbb{H}(X_2)) &\to \operatorname{Hom}(\mathbb{H}(T_0), \mathbb{H}(X_2))\\ &\to \operatorname{Ext}^1(\mathbb{H}(X_1), \mathbb{H}(X_2)) \to \operatorname{Ext}^1(\mathbb{H}(T_1), \mathbb{H}(X_2)) = 0,\end{aligned}$$

where the Hom and Ext groups are taken over $\operatorname{mod}\mathcal{T}$. The last item vanishes because $\mathbb{H}(T_1)$ is projective in $\operatorname{mod}\mathcal{T}$. Note that the first map is isomorphic to

$$\operatorname{Hom}_{\mathscr{C}}(T_1, X_2) \xrightarrow{\operatorname{Hom}_{\mathscr{C}}(f, X_2)} \operatorname{Hom}_{\mathscr{C}}(T_0, X_2).$$

It follows that $\operatorname{Ext}^1_{\operatorname{mod}\mathcal{T}}(\mathbb{H}(X_1), \mathbb{H}(X_2))$ is isomorphic to $\operatorname{Coker}\operatorname{Hom}_{\mathscr{C}}(f, X_2)$.

Applying the functor $\operatorname{Hom}_{\mathscr{C}}(-, X_2)$ to the triangle (4-9), we have the following exact sequence:

$$\operatorname{Hom}_{\mathscr{C}}(T_1, X_2) \xrightarrow{\operatorname{Hom}_{\mathscr{C}}(f, X_2)} \operatorname{Hom}_{\mathscr{C}}(T_0, X_2) \xrightarrow{\operatorname{Hom}_{\mathscr{C}}(h[-1], X_2)} \operatorname{Hom}_{\mathscr{C}}(X_1[-1], X_2).$$

In particular, we have the following exact sequence:

$$\mathrm{Hom}_{\mathscr{C}}(T_1, X_2) \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(f, X_2)} \mathrm{Hom}_{\mathscr{C}}(T_0, X_2) \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(h[-1], X_2)} \mathrm{Im}\ \mathrm{Hom}_{\mathscr{C}}(h[-1],\ X_2) \to 0.$$

We claim that $\mathrm{Im}\,\mathrm{Hom}_{\mathscr{C}}(h[-1],\ X_2) = [\mathcal{T}](X_1[-1], X_2)$. Indeed,

$$\mathrm{Im}\,\mathrm{Hom}_{\mathscr{C}}(h[-1],\ X_2) \subseteq [\mathcal{T}](X_1[-1], X_2)$$

is clear. For any morphism $x \in [\mathcal{T}](X_1[-1], X_2)$, we have two morphisms $x_1 : X_1[-1] \to T$ and $x_2 : T \to X_2$, where $T \in \mathcal{T}$ such that $x = x_2 x_1$. Since $\mathrm{Hom}_{\mathscr{C}}(T_1[-1], T) = 0$, there exists a morphism $a : T_0 \to T$ such that $ah[-1] = x_1$.

$$\begin{tikzcd}
T_1[-1] \arrow[r, "{g[-1]}"] \arrow[dr, "0"'] & X_1[-1] \arrow[r, "{h[-1]}"] \arrow[d, "x_1"] & T_0 \arrow[r, "f"] \arrow[dl, dotted, "a"] & T_1 \arrow[r, "g"] & X_1 \arrow[r, "h"] & T_0[1] \\
& T & & & &
\end{tikzcd}$$

It follows that $x = x_2 x_1 = (x_2 a)h[-1] \in \mathrm{Im}\ \mathrm{Hom}_{\mathscr{C}}(h[-1],\ X_2)$, as required.

Since $\mathscr{X}$ is $\mathcal{T}[1]$-rigid, we have $[\mathcal{T}](X_1[-1], X_2) = 0$. Thus we obtain

$$\mathrm{Ext}^1_{\mathsf{mod}\,\mathcal{T}}(\mathbb{H}(X_1), \mathbb{H}(X_2)) \simeq \mathsf{Coker}\ \mathrm{Hom}_{\mathscr{C}}(f, X_2) = [\mathcal{T}](X_1[-1], X_2) = 0.$$

Step 2: We show that for any projective object P in $\mathsf{mod}\,\mathcal{T}$, there exists a short exact sequence

$$0 \to P \to M_0 \to M_1 \to 0,$$

where $M_0, M_1 \in \mathcal{M}$. We may assume $P = \mathcal{T}(-, T) = \mathbb{H}(T)$ in $\mathsf{mod}\,\mathcal{T}$, where $T \in \mathcal{T}$.

Since $T \in \mathcal{T} \subseteq \mathscr{X}[-1] * \mathscr{X}$, there exists a triangle

$$X_3[-1] \xrightarrow{u} T \xrightarrow{v} X_4 \xrightarrow{w} X_3,$$

where $X_3, X_4 \in \mathscr{X}$. Applying the functor $\mathbb{H}$ to the above triangle, we have the following exact sequence:

$$\mathbb{H}(X_3[-1]) \xrightarrow{\mathbb{H}(u)} \mathbb{H}(T) \to \mathbb{H}(X_4) \to \mathbb{H}(X_3) \to 0.$$

We claim that $\mathrm{Im}\,\mathbb{H}(u) = 0$. That is to say, for any morphism $y : T' \to X_3[-1]$, where $T' \in \mathcal{T}$, we have $uy = 0$. Indeed, since $\mathcal{T}$ is cluster tilting, there exists a triangle

$$T_2 \xrightarrow{\alpha} T_3 \xrightarrow{\beta} X_3 \xrightarrow{\gamma} T_2[1],$$

where β is a minimal right $\mathcal{T}$-approximation of X_3 and $T_2, T_3 \in \mathcal{T}$. Applying the functor $\mathbb{H}$ to the above triangle, we have a minimal projective presentation

$$\mathbb{H}(X_3[-1]) \xrightarrow{\mathbb{H}(\gamma[-1])} \mathbb{H}(T_2) \xrightarrow{\mathbb{H}(\alpha)} \mathbb{H}(T_3) \xrightarrow{\mathbb{H}(\beta)} \mathbb{H}(X_3) \to 0$$

of $\mathbb{H}(X_3)$, since X_1 has no nonzero direct summands in $\mathcal{T}[1]$. Since $\mathbb{H}(X_3) \in \mathcal{M}$, we have $\mathrm{pd}\,\mathbb{H}(X_3) \leq 1$. Thus we have Im $\mathbb{H}(\gamma[-1]) = 0$ and thus $\gamma[-1] \circ y = 0$. So there exists a morphism $b : T' \to T_3[-1]$ such that $y = \beta[-1] \circ b$.

$$\begin{tikzcd} & T' \arrow[dl, dotted, "b"'] \arrow[d, "y"] & & \\ T_3[-1] \arrow[r, "\beta[-1]"] & X_3[-1] \arrow[r, "\gamma[-1]"] & T_2 \arrow[r, "\alpha"] & T_3. \end{tikzcd}$$

It follows that $uy = (u\beta[-1])b = 0 \circ b = 0$, as required. Hence we have the following exact sequence:

$$0 \to \mathbb{H}(T) \to \mathbb{H}(X_4) \to \mathbb{H}(X_3) \to 0,$$

where $\mathbb{H}(X_4), \mathbb{H}(X_3) \in \mathcal{M}$.

This shows that $\mathcal{M}$ is a weak tilting subcategory of $\mathrm{mod}\,\mathcal{T}$. Combining this with Lemma 4.9, the assertion follows. □

Consequently, we have the following result:

Theorem 4.11. *Let $\mathscr{C}$ be a triangulated category with a cluster tilting subcategory $\mathcal{T}$. The functor $\mathbb{H} : \mathscr{C} \to \mathrm{Mod}\ \mathcal{T}$ induces a bijection*

$$\Phi : \mathscr{X} \longmapsto \mathbb{H}(\mathscr{X})$$

from the first of the following sets to the second:

(I) *Weak $\mathcal{T}[1]$-cluster tilting subcategories of $\mathscr{C}$ whose objects do not have nonzero direct summands in $\mathcal{T}[1]$ and whose factorization ideals vanish.*

(II) *Weak tilting subcategories of* $\mathrm{mod}\ \mathcal{T}$.

It restricts to a bijection from the first to the second of the following sets.

(1) *$\mathcal{T}[1]$-cluster tilting subcategories of $\mathscr{C}$ whose objects do not have nonzero direct summands in $\mathcal{T}[1]$ and whose factorization ideals vanish.*

(2) *Tilting subcategories $\mathscr{Y}$ of* $\mathrm{mod}\ \mathcal{T}$ *such that $\mathbb{H}^{-1}(\mathscr{Y})$ is contravariantly finite in $\mathscr{C}$.*

Moreover, if $\mathscr{C}$ admits a Serre functor $\mathbb{S}$, we get a bijection between the following sets.

(1) *Cluster tilting subcategories of $\mathscr{C}$ whose objects do not have nonzero direct summands in $\mathcal{T}[1]$ and whose factorization ideals vanish.*

(2) *Tilting subcategories $\mathscr{Y}$ of* $\mathrm{mod}\ \mathcal{T}$ *such that $\mathbb{H}^{-1}(\mathscr{Y})$ is contravariantly finite and F-stable in $\mathscr{C}$.*

Proof. This follows from Theorems 4.5, 4.8 and 4.10 directly. □

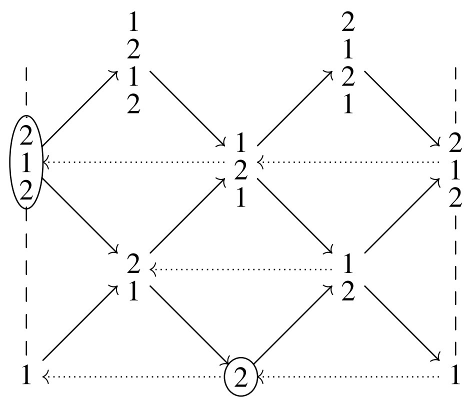
Remark 4.12. The above results generalize and improve several results in the literature. More precisely, Proposition 4.2, Theorems 4.3, 4.8 and 4.11 generalize a result in [Yang and Zhu 2019, Theorem 3.6], where analogous results were proved in the case where $\mathscr{C}$ is 2-Calabi–Yau and $\mathcal{T} = \mathsf{add}\, T$, see [Adachi et al. 2014, Theorem 4.1]. Theorem 4.11 generalizes a result of Beligiannis [2013, Theorem 6.6] in some cases, but we don't assume that $\mathsf{mod}\ \mathscr{X}$ has finite global dimension here.

We conclude this section with an example illustrating the bijections in Section 4:

Example 4.13. We revisit Example 3.4 presented in Section 3. Let $A = kQ/I$ be a self-injective algebra given by the quiver

$$Q:\ 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

and $I = \langle \alpha\beta\alpha\beta,\, \beta\alpha\beta\alpha \rangle$. The Auslander–Reiten quiver of $\mathsf{mod}\, A$ is



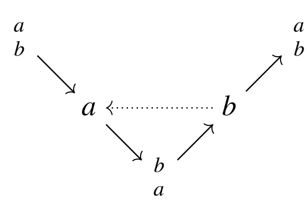
where the first and the last columns are identified. The stable module category

$$\mathscr{C} := \underline{\mathsf{mod}}\, A$$

is triangulated with a Serre functor. We get the Auslander–Reiten quiver of $\mathscr{C}$ by deleting the first row in above figure. By simple calculation, we obtain that

$$\mathcal{T} := \mathsf{add}\left(2 \oplus \begin{smallmatrix} 2 \\ 1 \\ 2 \end{smallmatrix}\right)$$

is a cluster tilting subcategory of $\mathscr{C}$. The Auslander–Reiten quiver of $\mathsf{mod}\, \mathcal{T}$ is



We illustrate the correspondences stated in this section as follows. In the table, weak $\mathcal{T}[1]$-cluster tilting subcategories of $\mathscr{C}$ and support τ-tilting pairs of $\mathsf{mod}\,\mathcal{T}$ are marked by ♣.

	$\mathcal{T}[1]$-rigid subcategories	τ-rigid pairs
	$\mathsf{add}\begin{pmatrix}2\\1\\2\end{pmatrix}$	$\big(\mathsf{add}\binom{a}{b}, 0\big)$
	$\mathsf{add}\begin{pmatrix}1\\2\\1\end{pmatrix}$	$\big(0, \mathsf{add}\binom{b}{a}\big)$
	$\mathsf{add}\binom{2}{1}$	$(a, 0)$
	$\mathsf{add}\binom{1}{2}$	$(b, 0)$
	$\mathsf{add}\,(2)$	$\big(\mathsf{add}\binom{b}{a}, 0\big)$
	$\mathsf{add}\,(1)$	$\big(0, \mathsf{add}\binom{a}{b}\big)$
♣	$\mathsf{add}\left(\begin{smallmatrix}2\\1\\2\end{smallmatrix}\oplus 2\right)$	$\big(\mathsf{add}\big(\begin{smallmatrix}a\\b\end{smallmatrix}\oplus\begin{smallmatrix}b\\a\end{smallmatrix}\big), 0\big)$
♣	$\mathsf{add}\big(\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 2\big)$	$\big(\mathsf{add}\big(\begin{smallmatrix}b\\a\end{smallmatrix}\oplus b\big), 0\big)$
♣	$\mathsf{add}\left(\begin{smallmatrix}2\\1\\2\end{smallmatrix}\oplus\begin{smallmatrix}2\\1\end{smallmatrix}\right)$	$\big(\mathsf{add}\big(\begin{smallmatrix}a\\b\end{smallmatrix}\oplus a\big), 0\big)$
♣	$\mathsf{add}\big(\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 1\big)$	$\big(b, \mathsf{add}\binom{a}{b}\big)$
♣	$\mathsf{add}\left(\begin{smallmatrix}2\\1\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\\1\end{smallmatrix}\right)$	$\big(a, \mathsf{add}\binom{b}{a}\big)$
♣	$\mathsf{add}\left(1\oplus\begin{smallmatrix}1\\2\\1\end{smallmatrix}\right)$	$\big(0, \mathsf{add}\big(\begin{smallmatrix}a\\b\end{smallmatrix}\oplus\begin{smallmatrix}b\\a\end{smallmatrix}\big)\big)$

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