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ON MASAS IN q-DEFORMED VON NEUMANN ALGEBRAS

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We study certain q-deformed analogues of the maximal abelian subalgebras of the group von Neumann algebras of free groups. The radial subalgebra is defined for Hecke deformed von Neumann algebras of the Coxeter group $(\mathbb{Z}/2\mathbb{Z})^{\star k}$ and shown to be a maximal abelian subalgebra which is singular and with Pukánszky invariant $\{\infty\}$. Further all nonequal generator masas in the q-deformed Gaussian von Neumann algebras are shown to be mutually nonintertwinable.

1. Introduction

Our aim is to investigate maximal abelian subalgebras in certain II_1-factors that can be viewed as deformations of $\mathrm{VN}(\mathbb{F}_n)$. Our particular interest lies in the analysis of counterparts of the radial masa A_r in $\mathrm{VN}(\mathbb{F}_n)$, studied for example in [Boca and Rădulescu 1992] and in [Cameron et al. 2010] (see also [Trenholme 1988]). The main open problem concerning the radial masa in $\mathrm{VN}(\mathbb{F}_n)$ is the question whether it is isomorphic to the generator masa(s); so far they share all the known properties, such as maximal injectivity, the same Pukánszky invariant, etc. They are also known not to be unitarily conjugate (see Proposition 3.1 of [Cameron et al. 2010]). More generally, radial masas have been studied for von Neumann algebras of groups of the type $(\mathbb{Z}/_{n\mathbb{Z}})^{\star k}$ in [Trenholme 1988] and [Boca and Rădulescu 1992].

Here we want to analyse the behaviour of counterparts of the radial/generator masa in some deformed versions of $\mathrm{VN}(\mathbb{F}_n)$ or $\mathrm{VN}((\mathbb{Z}/_{n\mathbb{Z}})^{\star k})$; more specifically in Hecke deformed von Neumann algebras of right-angled Coxeter groups $\mathrm{VN}_q(W)$ of Dymara [2006] (see also [Garncarek 2016] and [Caspers 2016a]) and in q-deformed Gaussian von Neumann algebras $\Gamma_q(\mathcal{H}_\mathbb{R})$ of Bożejko, Kümmerer and Speicher [Bożejko et al. 1997]. In the former case we can naturally define the radial subalgebra (and not the generator one), and in the latter the object that intuitively corresponds to the radial subalgebra is in fact obviously isomorphic to the generator one (as studied by Ricard [2005] and further by Wen [2017] and Parekh, Shimada

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and Wen [Parekh et al. 2018]). We show in Section 4 however that the different generator masas inside the $\Gamma_q(\mathcal{H}_\mathbb{R})$ are not unitarily conjugate.

Note that another example of a counterpart of the radial subalgebra in $\mathrm{VN}(\mathbb{F}_n)$ was studied and shown to be maximal abelian and singular in [Freslon and Vergnioux 2016]. It was a von Neumann subalgebra of the algebra $L^\infty(O_N^+)$, which shares many properties with $\mathrm{VN}(\mathbb{F}_n)$, although very recently the latter two were shown to be nonisomorphic [Brannan and Vergnioux 2018].

The plan of the paper is as follows: after finishing this section introducing certain notation, in Section 2 we define the radial subalgebra of the Hecke deformed von Neumann algebra $\mathrm{VN}_q(W)$ and show it to be maximal abelian. In Section 3, we compute its Pukánszky invariant and deduce its singularity. Finally Section 4 discusses the nonintertwinability of (a continuous family of) different generator masas in the q-deformed Gaussian von Neumann algebras.

Notation. Throughout this paper, by a *masa* we mean a maximal abelian von Neumann subalgebra of a given von Neumann algebra M. Let $U(\mathsf{M})$ be the group of unitaries in M. For a (unital) subalgebra $\mathsf{A} \subseteq \mathsf{M}$ we define the *normalizer* of A in M as

$$N_\mathsf{M}(\mathsf{A}) = \{u \in U(\mathsf{M}) \mid u\mathsf{A}u^* \subseteq \mathsf{A}\}.$$

A subalgebra $\mathsf{A} \subseteq \mathsf{M}$ is called *singular* if $N_\mathsf{M}(\mathsf{A}) \subseteq \mathsf{A}$. These notions were first introduced by Dixmier [1954].

$\mathbb{N}_0$ denotes the natural numbers including 0.

2. The radial Hecke masa

In this section we show that right-angled Hecke von Neumann algebras admit a radial algebra and prove that it is in fact a masa.

Let W denote a *right-angled Coxeter group*. Recall that this is the universal group generated by a finite set S of elements of order 2, with the relations forcing some of the distinct elements of S to commute, and some other to be free. This is formally encoded by a function

$$m : S \times S \setminus \{(s, s) : s \in S\} \to \{2, \infty\}$$

such that for all $s, t \in S$, $s \neq t$ we have

$$(st)^{m(s,t)} = e$$

(and $(st)^\infty = e$ means that s and t are free; necessarily $m(s, t) = m(t, s)$). We will always associate to W the length function $|\cdot| : W \to \mathbb{N}_0$ given by the generating set S. All the information about W is encoded by a graph Γ with a vertex set $V\Gamma = S$ and the edge set $E\Gamma = \{(s, t) \in S \times S : m(s, t) = 2\}$. Let $q \in (0, 1]$ and put $p = (q - 1)/q^{1/2}$ (note that our convention on q means that $p \leq 0$). The algebra $\mathbb{C}_q[W]$ is a *-algebra with a linear basis $\{T_w : w \in W\}$ satisfying the conditions ($s \in S$, $w \in W$)

$$T_s T_w = \begin{cases} T_{sw} & \text{if } |sw| > |w|, \\ T_{sw} + pT_w & \text{if } |sw| < |w|. \end{cases}$$

The algebra $\mathbb{C}_q[W]$ acts in a natural way (via bounded operators) on the space $\ell^2(W)$ and its von Neumann algebraic closure in $B(\ell^2(W))$ will be denoted by $\mathrm{VN}_q(W)$. The vector $\delta_e \in \ell^2(W)$ will sometimes be denoted by Ω; the corresponding vector state $\tau := \omega_\Omega$ on $\mathrm{VN}_q(W)$ is a faithful trace. More generally to any element $T \in \mathrm{VN}_q(W)$ we can associate its *symbol* $T\Omega$, and as Ω is a separating vector for $\mathrm{VN}_q(W)$ this correspondence is injective. Finally note that using the right action of the Hecke algebra on itself, we can define another von Neumann algebra acting on $\ell^2(W)$, say $\mathrm{VN}_q(W)^r$. It is obviously contained in the commutant of $\mathrm{VN}_q(W)$; in fact Proposition 19.2.1 of [Davis 2008] identifies it with $\mathrm{VN}_q(W)'$.

In what follows, we will write L to denote the cardinality of S.

Hecke von Neumann algebras were first considered in [Dymara 2006] and [Davis et al. 2007] in order to study weighted L^2-cohomology of Coxeter groups. In [Davis et al. 2007], the authors raised a natural question: how large is the centre of $\mathrm{VN}_q(W)$? A precise answer for the right-angled case was found in [Garncarek 2016], where the following result was shown.

Theorem 2.1. *Let $|S| \geq 3$ and assume that Γ is irreducible. Then for $q \in [\rho, 1]$ the right-angled Hecke von Neumann algebra $\mathbb{C}_q[W]$ is a II_1-factor and for $(0, \rho)$ we have that $\mathbb{C}_q[W]$ is a direct sum of a II_1-factor and $\mathbb{C}$. Here ρ is the radius of convergence of the fundamental power series $\sum_{k=0}^{\infty} \big|\{w \in W \mid |w| = k\}\big| z^k$.*

In particular $\mathrm{VN}_q(W)$ is diffuse if and only if $q \in [\rho, 1]$. Further structural results were obtained in [Caspers 2016a; 2016b; Caspers and Fima 2017] where for example noninjectivity, approximation properties, absence of Cartan subalgebras, the Connes embedding property and the existence of graph product decompositions were established for $\mathrm{VN}_q(W)$.

In this paper we consider the special case $W = (\mathbb{Z}_2)^{*L}$, i.e., the case where m is constantly equal to infinity. We assume also that $L \geq 3$. Here the main result of [Garncarek 2016] (see Theorem 2.1) says that $\mathrm{VN}_q(W)$ is a factor if and only if $q \in \left[\frac{1}{L-1}, 1\right]$, and results of [Dykema 1993] together with a calculation in Section 5 of [Garncarek 2016] show that for that range of q we have

$$\mathrm{VN}_q(W) \approx \mathrm{VN}(\mathbb{F}_{2Lq/(1+q)^2}),$$

where $\mathrm{VN}(\mathbb{F}_s)$ for $s \geq 1$ denote the interpolated free group factors of Dykema and Radulescu.

Definition 2.2. An element $T \in \mathrm{VN}_q(W)$ is said to be radial if for its symbol decomposition $T\Omega = \sum_{w\in W} c_w \delta_w$, where $c_w \in \mathbb{C}$, we have $c_w = c_v$ for every $v, w \in W$ with $l(v) = l(w)$. We say that T has radius (at most) n if the frequency support (i.e., the set of those $w \in W$ for which $c_w \neq 0$) of T_w is contained in the ball $\{w \in W : |w| \leq n\}$.

Define $h \in \mathbb{C}_q[W] \subset \mathrm{VN}_q(W)$ by the formula $h = \sum_{s\in S} T_s$ and put $\mathsf{B} := \{h\}''$.

Proposition 2.3. *The von Neumann algebra* B *coincides with the collection of all radial operators in* $\mathrm{VN}_q(W)$*. In particular the set of all radial operators forms an algebra.*

Proof. For each $n \in \mathbb{N}$ consider the radial operator $h_n := \sum_{w\in W,\, |w|=n} T_w \in \mathbb{C}_q[W]$ and put $h_0 := I$.

For each $n \in \mathbb{N}$, $n \geq 2$, we have

$$
\begin{aligned}
hh_n &= \sum_{s\in S} \sum_{\substack{|w|=n,\\ |sw|>|w|}} T_s T_w + \sum_{s\in S} \sum_{\substack{|w|=n,\\ |sw|<|w|}} T_s T_w \\
&= \sum_{s\in S} \sum_{\substack{|w|=n,\\ |sw|>|w|}} T_{sw} + \sum_{s\in S} \sum_{\substack{|w|=n,\\ |sw|<|w|}} T_{sw} + \sum_{s\in S} \sum_{\substack{|w|=n,\\ |sw|<|w|}} pT_w \\
&= h_{n+1} + (L-1)h_{n-1} + ph_n.
\end{aligned} \tag{2-1}
$$

We also have $h^2 = h_2 + ph + Lh_0$. This shows in particular that the algebra generated by h consists of radial operators. Moreover viewing the above as a recurrence formula we see that each h_n can be expressed as a polynomial in h and I, so that the subspace A generated by $\{h_n : n \in \mathbb{N}\}$ coincides with the unital *-algebra generated by h.

Further define the radial subspace

$$\ell^2(W)_r := \{(c_w)_{w\in W} \in \ell^2(W) : c_v = c_w \text{ for all } w, v \in W, |w| = |v|\}$$

and denote the orthogonal projection from $\ell^2(W)$ onto $\ell^2(W)_r$ by P_r. It is easy to see that $A\Omega$ is norm dense in $\ell^2(W)_r$. Thus the unique trace-preserving conditional expectation $\mathbb{E}$ onto $A'' \subset \mathrm{VN}_q(W)$ is given by the formula

$$\mathbb{E}(T)\Omega = P_r T\Omega, \quad T \in \mathrm{VN}_q(W).$$

This shows that the set of radial operators in $\mathrm{VN}_q(W)$ coincides with A'' and passing now to ultraweak closures we see that h generates the von Neumann algebra of all radial operators. □

Note that the above fact is not true (even for $p = 0$) for a general right-angled Coxeter group. Also note that formulae such as (2-1) (and the subsequent line in the proof) play a very relevant role in our proof of singularity in Section 3.

The first main theorem of this paper is based on the idea of Pytlik for the radial algebra in $\mathrm{VN}(\mathbb{F}_n)$ [1981]; see also [Sinclair and Smith 2008]. By $R_h \in \mathrm{VN}_q(W)^r$, we understand the operator on $\ell^2(W)$ given by the *right* action of $\sum_{s\in S} T_s$.

Lemma 2.4. *For every $v, w \in W$ with $|v| = |w|$ and for every $\varepsilon > 0$ there exists a vector $\eta \in \ell^2(W)$ such that*

$$\|e_v - e_w - (h\eta - R_h\eta)\|_2 < \varepsilon.$$

Proof. We first assume that $w = az$ and $v = zb$ for some word $z \in W$ with $|z| = |v| - 1$ and some letters $a, b \in S$. In the proof x and y will always be words in W and summations are always over x and y. Put for $k \in \mathbb{N}$

$$\psi_k = \sum_{\substack{|x|=|y|=k,\\ |xa|=|by|=k+1}} e_{xazby} \in \ell^2(W),$$

and define also $\psi_0 = e_{azb}$. Let $\delta > 0$. As for each $k \in \mathbb{N}$ there are $L(L-1)^{k-1}$ reduced words in W of length k,

$$\left\| \left(\frac{1-\delta}{L-1}\right)^k \psi_k \right\|_2^2 \le \left(\frac{1-\delta}{L-1}\right)^{2k} (L-1)^{2k-2} L^2 \le 4(1-\delta)^{2k}. \tag{2-2}$$

This means that we can define

$$\eta_\delta = \sum_{k=0}^{\infty} \left(\frac{1-\delta}{L-1}\right)^k \psi_k \in \ell^2(W).$$

We claim that the vector η_δ, for δ small enough (dependent on ε), satisfies the condition of the lemma. To show that we need to analyse the actions of h and R_h on ψ_k. For $k \ge 1$ we have (the bracket term included; the brackets are there in order to define further vectors in the remainder of the proof)

$$\begin{aligned} h\psi_k = &\sum_{s\in S} \sum_{\substack{|x|=|y|=k,\ |sx|=k+1\\ |xa|=|by|=k+1}} e_{sxazby} \\ &+ \sum_{s\in S} \sum_{\substack{|x|=|y|=k,\ |sx|=k-1\\ |xa|=|by|=k+1}} e_{sxazby} (+p e_{xazby}). \end{aligned} \tag{2-3}$$

and similarly, for $k \ge 1$,

$$\begin{aligned} R_h\psi_k = &\sum_{s\in S} \sum_{\substack{|x|=|y|=k,\ |ys|=k+1\\ |xa|=|by|=k+1}} e_{xazbys} \\ &+ \sum_{s\in S} \sum_{\substack{|x|=|y|=k,\ |ys|=k-1\\ |xa|=|by|=k+1}} e_{xazbys} (+p e_{xazby}). \end{aligned} \tag{2-4}$$

Finally

$$h\psi_0 = e_{zb} + pe_{azb} + \sum_{s\in S\setminus\{a\}} e_{sazb}, \qquad R_h\psi_0 = e_{az} + pe_{azb} + \sum_{s\in S\setminus\{b\}} e_{azbs}. \tag{2-5}$$

We now analyse the "commutators" $h\psi_k - R_h\psi_k$ and their sum. Note first that for each $k \in \mathbb{N}_0$ the summand in $h\psi_k$ given by pe_{xazby} also occurs in $R_h\psi_k$.

We define (compare to (2-5))

$$\phi_{1,0} = \sum_{s\in S\setminus\{a\}} e_{sazb}, \quad \phi_{2,0} = e_{zb}, \quad \chi_{1,0} = \sum_{s\in S\setminus\{b\}} e_{azbs}, \chi_{2,0} = e_{az}.$$

For $k \geq 1$ we set the following notation: let $\phi_{1,k}$ and $\phi_{2,k}$ be the two large sums on, respectively, the first and second line of (2-3), without the vectors between brackets. Similarly we define $\chi_{1,k}$ and $\chi_{2,k}$ to be the two large sums on, respectively, the first and second line of (2-4), without the vectors between brackets.

Then we have for all $k \in \mathbb{N}_0$

$$\phi_{1,k} = \frac{1}{L-1}\chi_{2,k+1}, \qquad \chi_{1,k} = \frac{1}{L-1}\phi_{2,k+1},$$

so that

$$\phi_{1,k} - \frac{1-\delta}{L-1}\chi_{2,k+1} = \delta\phi_{1,k}, \qquad \chi_{1,k} - \frac{1-\delta}{L-1}\phi_{2,k+1} = \delta\chi_{1,k}.$$

Thus a version of the telescopic argument yields the equality

$$\begin{aligned} h\eta_\delta - R_h\eta_\delta &= \sum_{k=0}^{\infty}\left(\frac{1-\delta}{L-1}\right)^k (\phi_{1,k} + \phi_{2,k} - \chi_{1,k} - \chi_{2,k}) \\ &= e_{zb} - e_{az} + \delta\left(\sum_{k=1}^{\infty}\left(\frac{1-\delta}{L-1}\right)^k (\phi_{1,k} - \chi_{1,k})\right). \end{aligned}$$

As $\delta \searrow 0$ this can be shown via a similar ℓ^2-counting estimate to that above to converge in norm to $e_{zb} - e_{az}$. From this we conclude the claim.

For general $v = v_1 \dots v_n$ and $w = w_1 \dots w_n$ with $v_n \neq w_1$ the proposition follows from a triangle inequality and an application of the argument in the first part of the proof to each pair $w_k \dots w_n v_1 \dots v_{k-1}$ and $w_{k+1} \dots w_n v_1 \dots v_k$. In the case where $v_n = w_1$ one can apply the above to the pairs $v_k \dots v_n b w_1 \dots w_{k-2}$ and $v_{k+1} \dots v_n b w_1 \dots w_{k-1}$ for some letter $b \neq v_n$. □

We are ready to formulate the first main result in this section.

Theorem 2.5. *The radial algebra* B *is a masa in* $\mathrm{VN}_q(W)$.

Proof. Suppose that $T \in \mathsf{B}' \cap \mathrm{VN}_q(W)$ and write $T\Omega = \sum_{u\in W} c_u e_u$. Let $v, w \in W$ with $|v| = |w|$, let $\varepsilon > 0$ and let η be as in Lemma 2.4. Note that as T commutes with h we have $\langle T\Omega, h\eta - R_h\eta\rangle = \langle (hT - R_hT)\Omega, \eta\rangle = \langle T(h - R_h)\Omega, \eta\rangle = 0$. Then we get

$$|\langle T\Omega, e_v - e_w\rangle| \leq |\langle T\Omega, e_v - e_w + h\eta - R_h\eta\rangle| \leq \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we see that $c_w = c_v$. Thus T is radial, which is equivalent to the fact that $T \in \mathsf{B}$ by Proposition 2.3. □

Remark 2.6. The recurrence formula (2-1) allows us to compute explicitly the distribution of h with respect to the canonical trace. As the formula (2-1) is valid only from $n = 2$ we first define "new" h_0 as $L/\widetilde{L}$, where $\widetilde{L} := L - 1$, so that with respect to the new variables it holds for all $n \in \mathbb{N}$. For simplicity assume that $q \in [1/\widetilde{L}, 1]$, so that $\mathrm{VN}_q(W)$ is a (finite) factor. Then the distribution of h is continuous (as B is diffuse) and the main result of [Cohen and Trenholme 1984] implies that the corresponding density is given (up to a normalising factor) by

$$\frac{\widetilde{L}\sqrt{4\widetilde{L} - (x-p)^2}}{\pi[-(x-p)^2 - p(2-L)(x-p) + p^2(L-1) + L^2]}dx.$$

Note that for $p = 0$ we obtain, as expected, the distribution of the radial element in the group $(\mathbb{Z}_2)^{*L}$ as computed in Theorem 4 of [Cohen and Trenholme 1984].

3. The Pukánszky invariant and singularity of the Hecke MASA

The Pukánszky invariant $\mathcal{P}(\mathsf{A})$ of a masa $\mathsf{A} \subseteq \mathsf{M}$ is determined by the von Neumann algebra generated by all A-A bimodule homomorphisms of $L^2(\mathsf{M})$. We refer to [Sinclair and Smith 2008] for further discussion of $\mathcal{P}(\mathsf{A})$. Popa [1985] showed that the Pukánszky invariant can be used to prove singularity of certain masas (and indeed this was successfully applied by Radulescu [1991] in order to obtain singularity of the radial masa in $\mathrm{VN}(\mathbb{F}_n)$). We will use this strategy in this section, following very closely the proof of [Rădulescu 1991], to show that the Hecke radial masa discussed in Section 2 is singular. In particular we determine its Pukánszky invariant.

We need some terminology. Let again $L \geq 3$, $W = (\mathbb{Z}_2)^{*L}$, $q \in \left[\frac{1}{L-1}, 1\right]$ and let B be the radial subalgebra of the factor $\mathrm{VN}_q(W)$ (shown to be a masa in Theorem 2.5).

Definition 3.1. The Pukánszky invariant of $\mathsf{B} \subseteq \mathrm{VN}_q(W)$ is defined as the type of the von Neumann algebra $\langle h, R_h\rangle' \subseteq \mathsf{B}(\ell^2(W))$, where h and R_h were defined in Section 2.

Next we introduce the necessary notation in order to determine the Pukánszky invariant of $\mathsf{B} \subseteq \mathrm{VN}_q(W)$. We need to construct certain bases, which are inspired by Radulescu's bases in free group factors (see [Rădulescu 1991]). For $l \in \mathbb{N}_0$

let $q_l : \mathbb{C}_q[W] \to \mathbb{C}_q[W]$ be the natural projection onto the span of $\{T_w, |w| = l\}$. Write $\mathbb{C}_q^l[W] = q_l(\mathbb{C}_q[W])$. As before set $h_l = \sum_{|w|=l} T_w$. We have for $m \geq 1$ (see (2-1) and its subsequent line)

$$h_1 h_m = h_m h_1 = h_{m+1} + p h_m + (L_m - 1) h_{m-1}, \tag{3-1}$$

where $L_m = L$ if $m \geq 2$ and $L_m = L + 1$ if $m = 1$. Let

$$S_l = \operatorname{span}\{q_l(h_1 x), q_l(x h_1) \mid x \in q_{l-1}(\mathbb{C}_q[W])\};$$

in particular $S_1 = \mathbb{C} h_1$. Further for $l \in \mathbb{N}$, $\gamma \in \mathbb{C}_q^l[W]$, set

$$\gamma_{m,n} = q_{m+n+l}(h_m \gamma h_n), \qquad m, n \in \mathbb{N}_0.$$

We also set $\gamma_{m,n} = 0$ in case $m < 0$ or $n < 0$. Finally for $l \in \mathbb{N}$ and $\gamma \in \mathbb{C}_q^l[W] \ominus S_l$ set

$$X_\gamma = \overline{\operatorname{span}}^{\|\ \|_2}\{\gamma_{m,n} \mid m, n \in \mathbb{N}_0\} \subset \ell^2(W).$$

Lemma 3.2 collects all computational results we need in what follows. As all the (rather easy) arguments are basically contained in [Rădulescu 1991, Lemma 1] we merely sketch the proof; all other proofs we give in this section will then be self-contained.

Lemma 3.2. (1) *For* $\gamma \in \mathbb{C}_q^l[W]$, $l \geq 1$, $m \geq 1$, $n \geq 0$, *we have*

$$h_1 \gamma_{m,n} = \gamma_{m+1,n} + p\gamma_{m,n} + (L-1)\gamma_{m-1,n}.$$

(2) *For* $\gamma \in \mathbb{C}_q^l[W] \ominus S_l$, $l \geq 2$, $m \geq 0$, $n \geq 0$, *we have*

$$h_1 \gamma_{m,n} = \gamma_{m+1,n} + p\gamma_{m,n} + (L-1)\gamma_{m-1,n}.$$

(*Note that only the case* $m = 0$ *was not already covered by* (1)).

(3) *For* $\beta \in \mathbb{C}_q^1[W] \ominus S_1$, $n \geq 0$, *we have*

$$h_1 \beta_{0,n} = \beta_{1,n} + p\beta_{0,n} - \beta_{0,n-1}.$$

(4) *For* $\gamma \in \mathbb{C}_q^l[W]$, $l \geq 1$ *we have*

$$q_{l+m+n+1}(h_1 h_m \gamma h_n) = q_{l+n+m+1}(h_1 q_{l+m+n}(h_m \gamma h_n)), \quad m, n \in \mathbb{N},$$
$$q_{l-m-n-1}(h_1 h_m \gamma h_n) = q_{l-m-n-1}(h_1 q_{l-m-n}(h_m \gamma h_n)), \quad 0 \leq m+n \leq l.$$

(5) *For* $\gamma \in \mathbb{C}_q^l[W]$, $l \geq 1$, *we have* $q_l(h_1 q_{l+1}(h_1 \gamma)) = (L-1)\gamma$.

(6) *For* $\beta \in \mathbb{C}_q^1[W] \ominus S_1$, *we have* $q_n(h_1 q_{n+1}(\beta h_n)) = -q_n(\beta h_{n-1})$.

(7) *For all* $\gamma \in \mathbb{C}_q^l[W] \ominus S_l$, $l \geq 2$, $n \in \mathbb{N}$, $m \geq 1$, *we have*

$$q_l(q_{m+n+l}(h_m \gamma h_n) h_{m+n}) = 0.$$

Proof. The proofs of (1)–(2) are easy consequences of (3-1); see also [Rădulescu 1991, Lemma 1 (a) and (b)]. The proof of (3) is essentially the same as [Rădulescu 1991, Lemma 1 (c)]. Statement (4) is a direct consequence of (3-1), and (5) and (6) follow from (1) and (3), respectively. Statement (7) follows from (1) and (2). □

The following theorem gives the cornerstone in our computation of the Pukánszky invariant. The idea is based on first showing that for suitable β and γ the mapping $T : X_\beta \to X_\gamma$ defined by the formula (3-2) is bounded and invertible. Then one uses a basis transition to the respective bases $\{h_m \beta h_n\}_{m,n\in\mathbb{N}}$ and $\{h_m \gamma h_n\}_{m,n\in\mathbb{N}}$ to show that T is actually a B-B bimodule map.

Theorem 3.3. *Let $l \in \mathbb{N}$, $l \geq 2$, let $\beta \in \mathbb{C}_q^1[W] \ominus S_1$ and let $\gamma \in \mathbb{C}_q^l[W](W) \ominus S_l$. Then the following hold*:

(1) *There exists a bounded invertible linear map $T : X_\beta \to X_\gamma$ determined by*

$$T : \beta_{m,n} \mapsto \gamma_{m,n} + \gamma_{m-1,n-1}, \qquad m, n \in \mathbb{N}_0. \tag{3-2}$$

(2) *We have $X_\beta = \overline{\mathsf{B}\beta\mathsf{B}}^{\|\,\|_2}$ and $X_\gamma = \overline{\mathsf{B}\gamma\mathsf{B}}^{\|\,\|_2}$. Moreover the map T defined by* (3-2) *agrees with the linear map*

$$T : h_m \beta h_n \mapsto h_m \gamma h_n, \qquad m, n \in \mathbb{N}_0. \tag{3-3}$$

The proof of Theorem 3.3 proceeds through a couple of lemmas, which we prove in two separate subsections.

Proof of Theorem 3.3 (1). The first statement of Theorem 3.3 is essentially a consequence of the following orthogonality property.

Lemma 3.4. *Let $l \in \mathbb{N}$, $l \geq 2$, and let $\beta, \beta' \in \mathbb{C}_q^1[W] \ominus S_1$, $\gamma \in \mathbb{C}_q^l[W] \ominus S_l$, $\gamma' \in \mathbb{C}_q^l[W]$, $l \geq 2$. We have then for each $m, n, m', n' \in \mathbb{N}_0$*

$$\langle \beta_{m,n}, \beta'_{m',n'} \rangle = \delta_{m+n,n'+m'} (L-1)^{m+n-|n-n'|} (-1)^{|n-n'|} \langle \beta, \beta' \rangle; \tag{3-4}$$

similarly,

$$\langle \gamma_{m,n}, \gamma'_{m',n'} \rangle = \delta_{m,m'} \delta_{n,n'} (L-1)^{m+n} \langle \gamma, \gamma' \rangle. \tag{3-5}$$

Proof. Let us first prove (3-5). Firstly, as $\gamma_{m,n}$ (resp. $\gamma'_{m',n'}$) is in the range of q_{m+n+l} (resp. $q_{m'+n'+l}$), we must have $m + n = m' + n'$ or else both sides of (3-5) are nonzero. We claim that

$$q_l(h_{m'} q_{m+n+l}(h_m \gamma h_n) h_{n'}) = \delta_{m,m'} \delta_{n,n'} (L-1)^{m+n} \gamma. \tag{3-6}$$

For $k := m + n = 0$ this is obvious. We proceed by induction on k and assume the assertion for $k - 1$. For $k \geq 1$ one of m and n is nonzero and we may assume without loss of generality that $m \neq 0$ (the proof for n can be done in the same way, or one can consider the adjoint of (3-6) which interchanges the roles of m and n).

If the left-hand side of (3-6) is nonzero, then we must have that m' is nonzero, because otherwise this expression reads $q_l(q_{m+n+l}(h_m\gamma h_n)h_{n+m})$ which is zero by Lemma 3.2 (7).

Using (3-1) together with the fact that $q_l(h_r q_{m+n+l}(x)h_n) = 0$ for every $r < m$ and $x \in \mathbb{C}_q[W]$ and $q_{m+n+l}(h_s\gamma h_n) = 0$ for $s < m$, we get

$$q_l(h_{m'}q_{m+n+l}(h_m\gamma h_n)h_{n'}) = q_l(h_{m'-1}h_1q_{m+n+l}(h_1h_{m-1}\gamma h_n)h_{n'}).$$

Using Lemma 3.2 (4) and (5) for the first two of the following equalities and then the induction hypothesis yields

$$\begin{aligned}(3\text{-}7)\quad q_l(h_{m'}q_{m+n+l}(h_m\gamma h_n)h_{n'}) &\\ &= q_l(h_{m'-1}q_{m+n+l-1}(h_1q_{m+n+l}(h_1q_{m+n+l-1}(h_{m-1}\gamma h_n))h_{n'})\\ &= (L-1)q_l(h_{m'-1}q_{m+n+l-1}(h_{m-1}\gamma h_n)h_{n'})\\ &= (L-1)(L-1)^{m+n-1}\delta_{m,m'}\delta_{n,n'}\gamma.\end{aligned}$$

This completes the proof of (3-6). Then using the fact that $h_{m'}$ and $h_{n'}$ are self-adjoint we get

$$\begin{aligned}(3\text{-}8)\qquad \langle\gamma_{m,n},\gamma'_{m',n'}\rangle &= \langle q_{m+n+l}(h_m\gamma h_n), q_{m'+n'+l}(h_{m'}\gamma' h_{n'})\rangle\\ &= \langle h_{m'}q_{m+n+l}(h_m\gamma h_n)h_{n'}, \gamma'\rangle\\ &= \langle q_l(h_{m'}q_{m+n+l}(h_m\gamma h_n)h_{n'}), \gamma'\rangle\\ &= (L-1)^{m+n}\delta_{m,m'}\delta_{n,n'}\langle\gamma,\gamma'\rangle.\end{aligned}$$

Next we sketch the proof of (3-4); it is largely the same as (3-5). The claim (3-6) gets replaced by

$$(3\text{-}9)\qquad q_l(h_{m'}q_{m+n+l}(h_m\beta h_n)h_{n'}) = (L-1)^{|m+n|-|n-n'|}(-1)^{|n-n'|}\delta_{m+n,m'+n'}\beta.$$

Again the proof proceeds by induction with respect to $k := m+n = m'+n'$. The case $k = 0$ is obvious so assume $k \geq 1$. First assume that both $m, m' \geq 1$. Similar to (3-7) and using the same results from Lemma 3.2 we find that

$$\begin{aligned}(3\text{-}10)\quad q_l(h_{m'}q_{m+n+l}(h_m\beta h_n)h_{n'}) &= q_l(h_{m'-1}h_1q_{m+n+l}(h_1h_{m-1}\beta h_n)h_{n'})\\ &= (L-1)q_l(h_{m'-1}q_{m+n+l-1}(h_{m-1}\beta h_n)h_{n'-1})\\ &= (L-1)^{m+n-|n-n'|}(-1)^{|n-n'|}\delta_{m+n,m'+n'}\langle\beta,\beta'\rangle.\end{aligned}$$

The proof of (3-10) (disregarding the intermediate steps) for the case $n, n' \geq 1$ proceeds in the same manner (or follows by taking adjoints of (3-10) which swaps the roles of m, m' and n, n'). The only case that remains is then $m = 0$ and $n' = 0$ (again the case $m' = 0$ and $n = 0$ follows by taking adjoints, or by symmetry).

Then $n \geq 1$ and $m' \geq 1$ and using Lemma 3.2 (6) for the second equality and then applying the induction hypothesis we obtain

$$\begin{aligned} q_1(h_{m'}q_{n+1}(\beta h_n)) &= q_1(h_{m'-1}q_n(h_1 q_{n+1}(\beta h_{n-1}h_1))) \\ &= -q_1(h_{m'-1}q_n(\beta h_{n-1})) \\ &= (L-1)^{m+n-|n-n'|}\delta_{m+n,m'+n'}(-1)^{|n-n'|}\langle \beta, \beta'\rangle. \end{aligned}$$

Then the lemma follows by replacing γ by β in (3-8). □

Recall the elementary fact (see [Rădulescu 1991, Lemma 5] for a proof) that for a real number a, $|a| < 1$, there exist constants $B_a > 0$ and $C_a > 0$ such that for any $k \in \mathbb{N}$, $\lambda_1, \dots, \lambda_k \in \mathbb{C}$, we have

$$B_a \sum_{i=1}^{k} |\lambda_i|^2 \leq \sum_{i=1}^{k} \lambda_i \overline{\lambda}_j a^{|i-j|} \leq C_a \sum_{i=1}^{k} |\lambda_i|^2. \tag{3-11}$$

Proof of Theorem 3.3 (1). By Lemma 3.4 and (3-11) we see that the assignment $\beta_{m,n} \mapsto \gamma_{m,n}$ extends to a bounded invertible linear mapping $T_0 : X_\beta \to X_\gamma$. By Lemma 3.4 we see that $S : X_\gamma \mapsto X_\gamma : \gamma_{m,n} \mapsto \gamma_{m-1,n-1}$ is bounded with norm $\|S\| \leq (L-1)^{-2}$. Therefore $\mathrm{Id}_{X_\gamma} + S$ is bounded and invertible. As the composition $(I+S) \circ T_0$ is bounded and invertible and agrees with (3-2) we are done. □

Proof of Theorem 3.3 (2). The following Lemma 3.5 is the crucial part of the proof of Theorem 3.3 (2).

Lemma 3.5. *Let $l \geq 2$, $\beta \in \mathbb{C}_q^1[W] \ominus S_1$, and let $\gamma \in \mathbb{C}_q^l[W] \ominus S_l$. For every $m, n \in \mathbb{N}_0$ there exist certain constants $b_{k,j}^{m,n}, c_{k,j}^{m,n} \in \mathbb{R}$, $k = 0, \dots, m$, $j = 0, \dots, n$, such that we have the expansions*

$$h_m \beta h_n = \sum_{k \leq m, j \leq n} b_{k,j}^{m,n} \beta_{k,j}, \qquad h_m \gamma h_n = \sum_{k \leq m, j \leq n} c_{k,j}^{m,n} \gamma_{k,j}. \tag{3-12}$$

Moreover, these constants satisfy

$$c_{k,j}^{m,n} = b_{k,j}^{m,n} + b_{k+1,j+1}^{m,n}, \quad m, n \in \mathbb{N},\ k = 0, \dots, m,\ j = 0, \dots, n, \tag{3-13}$$

where $b_{m+1,n+1}^{m,n} = 0$.

Proof. If $m = 0$ and $n \in \mathbb{N}$ arbitrary, then the existence of decompositions (3-12) is a consequence of Lemma 3.2. The relation (3-13) for $m = 0$ becomes $c_{k,j}^{0,n} = b_{k,j}^{0,n}$ which is a rather direct consequence of Lemma 3.2 as well.

The proof proceeds by induction on m. Let $L_k = L$ if $k > 1$ and let $L_1 = L+1$. We have by (3-1) and then Lemma 3.2 (1) and (3),

$$\begin{aligned}
h_m\beta h_n &= (h_1 - p)h_{m-1}\beta h_n - (L_{m-1}-1)h_{m-2}\beta h_n \\
&= (h_1-p)\sum_{k=0}^{m-1}\sum_{j=0}^{n} b_{k,j}^{m-1,n}\beta_{k,j} - (L_{m-1}-1)\sum_{k=0}^{m-2}\sum_{j=0}^{n} b_{k,j}^{m-2,n}\beta_{k,j} \\
&= \sum_{k=0}^{m-1}\sum_{j=0}^{n} b_{k,j}^{m-1,n}(\beta_{k+1,j} + (L-1)\beta_{k-1,j}) \\
&\qquad - \sum_{j=0}^{n} b_{0,j}^{m-1,n}\beta_{0,j-1} - (L_{m-1}-1)\sum_{k=0}^{m-2}\sum_{j=0}^{n} b_{k,j}^{m-2,n}\beta_{k,j} \\
&= \sum_{k=0}^{m}\sum_{j=0}^{n}(b_{k-1,j}^{m-1,n} + (L-1)b_{k+1,j}^{m-1,n})\beta_{k,j} \\
&\qquad - \sum_{j=0}^{n-1} b_{0,j+1}^{m-1,n}\beta_{0,j} - (L_{m-1}-1)\sum_{k=0}^{m-2}\sum_{j=0}^{n} b_{k,j}^{m-2,n}\beta_{k,j}.
\end{aligned} \tag{3-14}$$

This shows that for all $0 \le k \le m,\ 0 \le j \le n$, we obtain

$$b_{k,j}^{m,n} = b_{k-1,j}^{m-1,n} + (L-1)b_{k+1,j}^{m-1,n} - (L_{m-1}-1)b_{k,j}^{m-2,n} - \delta_{k,0}b_{0,j+1}^{m-1,n}.$$

Let $\delta_{k\ge 1}$ be 1 if $k \ge 1$ and 0 otherwise. We get then

$$\begin{aligned}
b_{k,j}^{m,n} + b_{k+1,j+1}^{m,n} = \delta_{k\ge1}(b_{k-1,j}^{m-1,n} + b_{k,j+1}^{m-1,n}) + (L-1)(b_{k+1,j}^{m-1,n} + b_{k+2,j+1}^{m,n+1}) \\
- (L_{m-1}-1)(b_{k,j}^{m-2,n} + b_{k+1,j+1}^{m-2,n}).
\end{aligned}$$

So by induction

$$\begin{aligned}
b_{k,j}^{m,n} + b_{k+1,j+1}^{m,n} &= \delta_{k\ge1}c_{k-1,j}^{m-1,n} + (L-1)c_{k+1,j}^{m-1,n} - (L_{m-1}-1)c_{k,j}^{m-2,n} \\
&= c_{k-1,j}^{m-1,n} + (L-1)c_{k+1,j}^{m-1,n} - (L_{m-1}-1)c_{k,j}^{m-2,n}.
\end{aligned} \tag{3-15}$$

Exactly as we computed (3-14) (with the difference that Lemma 3.2 (3) is replaced by Lemma 3.2 (2)) we get

$$h_m\gamma h_n = \sum_{k=0}^{m+1}\sum_{j=0}^{n}(c_{k-1,j}^{m-1,n} + (L-1)c_{k+1,j}^{m-1,n})\gamma_{k,j} - (L_{m-1}-1)\sum_{k=0}^{m-2}\sum_{j=0}^{n} c_{k,j}^{m-2,n}\gamma_{k,j}.$$

Thus

$$c_{k,j}^{m,n} = c_{k-1,j}^{m-1,n} + (L-1)c_{k+1,j}^{m-1,n} - (L_m-1)c_{k,j}^{m-2,n}.$$

Combining the above with (3-15) gives $c_{k,j}^{m,n} = b_{k,j}^{m,n} + b_{k+1,j+1}^{m,n}$ for all $0 \le k \le m$, $0 \le j \le n$. ☐

Proof of Theorem 3.3 (2). Lemma 3.5 shows that $\mathsf{B}\gamma\mathsf{B} \subseteq X_\gamma$ and $\mathsf{B}\beta\mathsf{B} \subseteq X_\beta$ and hence the inclusions hold also for the $\| \|_2$-closures. For the converse inclusion proceed by induction: take $h_n\gamma h_m \in \mathsf{B}\gamma\mathsf{B}$ and assume that all vectors $h_r\beta h_s$ with $r < n, s \le m$ are contained in X_γ (if $n = 0$ then assume that $r \le n, s < m$ and consider adjoints, or use a similar induction argument on m). By (3-1) we have

$$h_n\gamma h_m = (h_1 - p)h_{n-1}\gamma h_m - (L_n - 1)h_{n-2}\gamma h_m \in h_1 X_\gamma + X_\gamma.$$

Here again $L_n = L$ if $n \ge 2$ and $L_1 = L + 1$. So it suffices to show that $h_1 X_\gamma \subseteq X_\gamma$, but this is a consequence of Lemma 3.2 (2). The proof for β instead of γ is the same but uses Lemma 3.2 (1) and (3) for the latter argument.

The fact that (3-3) agrees with (3-2) is now a direct consequence of Lemma 3.5. Indeed,

$$\begin{aligned} T(h_m\beta h_n) &= T\bigg(\sum_{k\le m, j\le n} b^{m,n}_{k,j}\beta_{k,j}\bigg) = \sum_{k\le m, j\le n} b^{m,n}_{k,j}(\gamma_{k,j} + \gamma_{k-1,j-1}) \\ &= \sum_{k\le m, j\le n} (b^{m,n}_{k,j} + b^{m,n}_{k+1,j+1})\gamma_{k,j} = \sum_{k\le m, j\le n} c^{m,n}_{k,j}\gamma_{k,j} = h_m\gamma h_n. \end{aligned}$$ □

Consequences of Theorem 3.3. Let $\mathsf{B}_r = \langle R_h\rangle''$ (note that as $\mathrm{VN}_q(W)$ is in the standard form on $\ell^2(W)$, it is also equal to $J\mathsf{B}J$, where J is the antilinear Tomita–Takesaki modular conjugation $\delta_x \mapsto \delta_{x^{-1}}$). For a vector $\gamma \in \bigcup_{l\in\mathbb{N}_0}\mathbb{C}^l_q[W]$ we let p_γ be the central support in $(\mathsf{B}\cup\mathsf{B}_r)''$ of the vector state $\omega_{\gamma,\gamma}$. The operator p_γ is then given by the projection onto the closure of $\mathsf{B}\gamma\mathsf{B}$.

Lemma 3.6. *If vectors $\xi, \xi' \in \cup_{l\ge1}\mathbb{C}^l_q[W]\ominus S_l$ are orthogonal then p_ξ and $p_{\xi'}$ are orthogonal projections.*

Proof. Let $\xi \in \mathbb{C}^l_q[W]\ominus S_l$ and let $\xi' \in \mathbb{C}^{l'}_q[W]\ominus S_{l'}$ with $l, l' \ge 1$. If $l = l'$ then the lemma follows directly from Lemma 3.4. So assume that $l \ne l'$ and say that $l' \le l$. It suffices to show that

$$\xi'_{r,s} \perp \xi_{m,n} \quad \text{for every } r, s, m, n \in \mathbb{N}_0. \tag{3-16}$$

If $m+n+l \ne r+s+l'$ this is obvious as then the images of q_{m+n+l} and $q_{r+s+l'}$ are mutually orthogonal. We may then assume $m+n+l = r+s+l'$, so that $r+s \ge m+n$. If $m+n = 0$ then (3-16) is obvious, as $\xi \perp S_l$ whereas $\xi'_{r,s} \in S_l$. But then note that $\xi'_{r,s} = (\xi'_{a,b})_{r-a,s-b}$ for any $a = 0,\dots,r,\ b = 0,\dots s$ such that $l'+a+b = l$. As $\xi'_{a,b} \in S_l$ we see from Lemma 3.4 that $(\xi'_{a,b})_{r-a,s-b} \perp \xi_{m,n}$. □

We can now state and prove the main result of this section.

Theorem 3.7. *The von Neumann algebra $(\mathsf{B}\cup\mathsf{B}_r)'(1-p_\Omega)$ is homogeneous of type I_∞.*

Proof. Because $(\mathsf{B}\cup\mathsf{B}_r)''$ is abelian, the commutant $(\mathsf{B}\cup\mathsf{B}_r)'$ decomposes as a direct sum $\bigoplus_{n=1}^{\infty}\mathsf{A}_n\overline{\otimes}B(\mathcal{H}_n)$, where $\dim(\mathcal{H}_n)=n$ and the algebras A_n are abelian (see [Dixmier 1969]). Let $(\xi_i)_{i\in\mathbb{N}}$ be an orthonormal basis in $\bigcup_{l\geq 1}\mathbb{C}_q^l[W]\ominus S_l$. By Lemma 3.6 the projections $(p_{\xi_i})_{i\in\mathbb{N}}$ are mutually orthogonal and by Theorem 3.3 they have the same central support in $(\mathsf{B}\cup\mathsf{B}_r)'$. As by Lemma 3.6 we have $\sum_{i\in\mathbb{N}}p_{\xi_i}=1-p_\Omega$ and $1-p_\Omega$ is central in $(\mathsf{B}\cup\mathsf{B}_r)'$ (see [Popa 1985, Lemma 3.1]), we see that the central support of each p_{ξ_i} in $(\mathsf{B}\cup\mathsf{B}_r)'$ is $1-p_\Omega$, which is the unit in $(\mathsf{B}\cup\mathsf{B}_r)'(1-p_\Omega)$. Since we have a partition of unity formed by projections with the same central support, the above decomposition of $(\mathsf{B}\cup\mathsf{B}_r)'$ must in fact consist of only one element. As there are infinitely many orthogonal projections, this summand must correspond to $n=\infty$, so that we have $(\mathsf{B}\cup\mathsf{B}_r)'(1-p_\Omega)=A_\infty\overline{\otimes}B(\ell^2)$ □

Remark 3.8. Theorem 3.7 is phrased in the literature as follows: the Pukánszky invariant of B is $\{\infty\}$. This is because in the B-B-bimodule $(1-p_\Omega)L^2(\mathsf{M})$, the only factors occurring in the direct integral decomposition of the commutant of $\mathsf{B}\cup\mathsf{B}_r$ are infinite (and necessarily of type I).

Corollary 3.9. *The radial subalgebra* B *is a singular masa of* $\mathrm{VN}_q(W)$.

Proof. This follows from Theorem 3.7 by [Popa 1985, Remark 3.4]. □

4. Generator masas in q-deformed Gaussian von Neumann algebras

In this section we consider masas in a different deformation of the free group factors, i.e., so-called q-Gaussian algebras.

The starting point of the construction of q-Gaussian algebras is a real Hilbert space $\mathcal{H}_\mathbb{R}$. We complexify it, obtaining a complex Hilbert space $\mathcal{H}$, and form an algebraic direct sum $\bigoplus_{n\geqslant 0}\mathcal{H}^{\otimes n}$, where $\mathcal{H}^{\otimes 0}=\mathbb{C}$. Following [Bożejko et al. 1997] (see that paper for all facts stated below without proofs), we will define an inner product on this space using the parameter $q\in(-1,1)$. For each $n\in\mathbb{N}$ we define an operator $P_q^n:\mathcal{H}^{\otimes n}\to\mathcal{H}^{\otimes n}$ by the formula

$$P_q^n(e_1\otimes\cdots\otimes e_n)=\sum_{\pi\in S_n}q^{i(\pi)}e_{\pi(1)}\otimes\cdots\otimes e_{\pi(n)},$$

where $e_1,\dots,e_n\in\mathcal{H}$, S_n is the permutation group on n letters and $i(\pi)$ denotes the number of inversions in the permutation π. These operators are strictly positive, so they define an inner product on $\bigoplus_{n\geqslant 0}\mathcal{H}^{\otimes n}$ — the Hilbert space that we get after completion is called the q-Fock space and is denoted by $\mathcal{F}_q(\mathcal{H})$. The direct sum decomposition of the q-Fock space allows us to define shift-like operators.

Definition 4.1. Let $\xi\in\mathcal{H}$. We define the *creation operator* $a_q^*(\xi):\mathcal{F}_q(\mathcal{H})\to\mathcal{F}_q(\mathcal{H})$ by $a_q^*(\xi)(e_1\otimes\cdots\otimes e_n)=\xi\otimes e_1\otimes\cdots\otimes\cdots e_n$. The *annihilation operator* $a_q(\xi):\mathcal{F}_q(\mathcal{H})\to\mathcal{F}_q(\mathcal{H})$ is defined as the adjoint of $a_q^*(\xi)$. Using the definition of the q-deformed inner product we can find the formula for $a_q(\xi)$:

$$a_q(\xi)(e_1 \otimes \cdots \otimes e_n) = \sum_{i=1}^{n} q^{i-1} \langle \xi, e_i \rangle e_1 \otimes \cdots \widehat{e_i} \cdots \otimes e_n,$$

where $\widehat{e_i}$ means that the factor e_i is omitted. All the above operators extend to bounded operators on $\mathcal{F}_q(\mathcal{H})$.

Creation and annihilation operators will allow us to define q-Gaussian algebras.

Definition 4.2. Let $\mathcal{H}_\mathbb{R}$ be a real Hilbert space and let $\mathcal{H}$ be its complexification. The von Neumann subalgebra of $\mathrm{B}(\mathcal{F}_q(\mathcal{H}))$ generated by the set $\{a_q^*(\xi)+a_q(\xi) : \xi \in \mathcal{H}_\mathbb{R}\}$ is called the *q-Gaussian algebra* associated with $\mathcal{H}_\mathbb{R}$ and is denoted by $\Gamma_q(\mathcal{H}_\mathbb{R})$.

The vector $\Omega = 1 \in \mathbb{C} \subset \mathcal{H}^{\otimes 0} \subset \mathcal{F}_q(\mathcal{H})$ is called the *vacuum vector*. It is a cyclic and separating vector for $\Gamma_q(\mathcal{H}_\mathbb{R})$ and the associated vector state $\omega(x) := \langle \Omega, x\Omega \rangle$ is a normal faithful trace on $\Gamma_q(\mathcal{H}_\mathbb{R})$.

Remark 4.3. For $q = 0$ the assignment $\mathcal{H}_\mathbb{R} \mapsto \Gamma_q(\mathcal{H}_\mathbb{R})$ is precisely Voiculescu's free Gaussian functor. In particular $\Gamma_0(\mathcal{H}_\mathbb{R}) \simeq \mathrm{L}(\mathbb{F}_{\dim(\mathcal{H}_\mathbb{R})})$.

We will study problems pertaining to conjugacy of masas in the q-Gaussian algebras. It is a nice feature of these objects that the orthogonal operators on $\mathcal{H}_\mathbb{R}$ give rise to automorphisms of $\Gamma_q(\mathcal{H}_\mathbb{R})$. To introduce these automorphisms, we need to present the *first quantisation*.

Definition 4.4. Let $T : \mathcal{H} \to \mathcal{H}$ be a contraction. The assignment

$$\bigoplus_{k \geqslant 0} \mathcal{H}^{\otimes k} \ni e_1 \otimes \cdots \otimes e_n \mapsto Te_1 \otimes \cdots \otimes Te_n \in \bigoplus_{k \geqslant 0} \mathcal{H}^{\otimes k}$$

extends to a contraction $\mathcal{F}_q(T) : \mathcal{F}_q(\mathcal{H}) \to \mathcal{F}_q(\mathcal{H})$ and is called the first quantisation of T.

Remark 4.5. If $U : \mathcal{H} \to \mathcal{H}$ is a unitary then $\mathcal{F}_q(U)$ is also a unitary.

To work with $\Gamma_q(\mathcal{H}_\mathbb{R})$ we need a convenient notation for its generators. For any $\xi \in \mathcal{H}_\mathbb{R}$ we put $W(\xi) := a_q^*(\xi) + a_q(\xi)$. If $\eta = \xi_1 + i\xi_2 \in \mathcal{H}$ then we denote $W(\eta) = W(\xi_1) + iW(\xi_2)$; therefore $W(\eta)$ is complex-linear in η. Recall that the vacuum vector Ω is cyclic and separating. One can check that for any vectors $\eta_1, \ldots, \eta_n \in \mathcal{H}$ we have $\eta_1 \otimes \cdots \otimes \eta_n \in \Gamma_q(\mathcal{H}_\mathbb{R})\Omega$; the unique operator $W(\eta_1 \otimes \cdots \otimes \eta_n) \in \Gamma_q(\mathcal{H}_\mathbb{R})$ such that $W(\eta_1 \otimes \cdots \otimes \eta_n)\Omega = \eta_1 \otimes \cdots \otimes \eta_n$ is called a *Wick word*. The span of all such operators associated with finite simple tensors forms a strongly dense *-subalgebra of $\Gamma_q(\mathcal{H}_\mathbb{R})$, which we call the *algebra of Wick words*. Finally note that much as in Section 2 we can also consider the "right" version of $\Gamma_q(\mathcal{H}_\mathbb{R})$, generated by the combinations of right creation and annihilation operators, in particular containing the right Wick words, to be denoted $W_r(\xi)$. We are ready to introduce the *second quantisation*.

Definition 4.6. Let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space and let $\mathcal{H}$ be its complexification. Suppose that $T : \mathcal{H} \to \mathcal{H}$ is a contraction such that $T(\mathcal{H}_{\mathbb{R}}) \subset \mathcal{H}_{\mathbb{R}}$. Then the assignment $\Gamma_q(\mathcal{H}_{\mathbb{R}}) \ni W(\eta_1 \otimes \cdots \otimes \eta_n) \mapsto W(T\eta_1 \otimes \cdots \otimes T\eta_n) \in \Gamma_q(\mathcal{H}_{\mathbb{R}})$, where $\eta_1, \ldots, \eta_n \in \mathcal{H}$, may be extended to a normal, unital, completely positive map on $\Gamma_q(\mathcal{H}_{\mathbb{R}})$, denoted by $\Gamma_q(T)$.

Remark 4.7. Note that the condition $T(\mathcal{H}_{\mathbb{R}}) \subset \mathcal{H}_{\mathbb{R}}$ is essential, otherwise $\Gamma_q(T)$ would not even preserve the adjoint, let alone be completely positive.

We will only deal with automorphisms and, in this construction, they come from orthogonal operators on $\mathcal{H}_{\mathbb{R}}$. If $U : \mathcal{H}_{\mathbb{R}} \to \mathcal{H}_{\mathbb{R}}$ is orthogonal then $\Gamma_q(U)(x) = \mathcal{F}_q(U)x\mathcal{F}_q(U)^*$, where we still denote by U its canonical unitary extension to $\mathcal{H}$. It is easy to check that $\Gamma_q(U)W(\xi) = W(U\xi)$.

To find candidates for masas, we draw inspiration from the case $q = 0$, in which the most basic masas are the so-called generator masas. In our picture they correspond to subalgebras generated by a single element $W(\xi)$, where $\xi \in \mathcal{H}_{\mathbb{R}}$. Ricard [2005] proved they are also masas in the case of q-Gaussian algebras. As an application, he established factoriality of all q-Gaussian algebras $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ with $\dim(\mathcal{H}_{\mathbb{R}}) \geqslant 2$. Recently these generator masas were also shown to be singular [Wen 2017] and maximally injective [Parekh et al. 2018] (the latter for sufficiently small $|q|$).

Using the automorphisms produced by the second quantisation procedure, we can easily show that all these masas are conjugate by an outer automorphism. Indeed, consider masas generated by $W(\xi)$ and $W(\eta)$, where $\xi, \eta \in \mathcal{H}_{\mathbb{R}}$. By rescaling, we may assume that $\|\xi\| = \|\eta\| = 1$. Therefore one can find an orthogonal operator U such that $U\xi = \eta$; then $\Gamma_q(U)((W(\xi))'') = (W(\eta))''$. Our aim now is to show that they are never conjugate by a unitary.

Case of orthogonal vectors. We first want to deal with the case when $\mathsf{A} := (W(e_1))''$ and $\mathsf{B} := (W(e_2))''$ are masas in $\mathsf{M} := \Gamma_q(\mathcal{H}_{\mathbb{R}})$ coming from two orthogonal vectors. In the case $q = 0$ these masas correspond to two different generator masas of the free group factor. One can prove that these are not unitarily conjugate using Popa's notion of orthogonal pairs of subalgebras (see [Popa 1983, Corollary 4.3]). We will use another technique due to Popa giving a criterion for embedding A into B inside M (in a certain technical sense). We will actually only state the part of the theorem that is useful for us; for the full statement consult [Popa 2006, Theorem 2.1 and Corollary 2.3] or [Popa 2019, Theorem 1.3.1]. We call A and B *intertwinable* (inside M) if the intertwiner space $\mathcal{I}_{\mathsf{M}}(\mathsf{A}, \mathsf{B})$, defined in [Popa 2019, Subsection 1.3] is nontrivial.

Proposition 4.8 (Popa). *Let* A *and* B *be von Neumann subalgebras of a finite von Neumann algebra* (M, τ). *Suppose that there exists a sequence of unitaries* $(u_k)_{k\in\mathbb{N}} \subset \mathcal{U}(\mathsf{A})$ *such that for any* $x, y \in \mathsf{M}$ *we have* $\lim_{k\to\infty} \|\mathbb{E}_{\mathsf{B}}(xu_k y)\|_2 = 0$, *where* $\mathbb{E}_{\mathsf{B}}$ *is the unique* τ*-preserving conditional expectation from* M *onto* B*. Then* A *and* B *are nonintertwinable*; *in particular there does not exist a unitary* $u \in \mathsf{M}$ *such that* $u\mathsf{A}u^* = \mathsf{B}$.

Remark 4.9. Note that it suffices to check that $\lim_{k\to\infty} \|\mathbb{E}_{\mathsf{B}}(xu_k y)\|_2 = 0$ only for $x, y \in \widetilde{\mathsf{M}}$, where $\widetilde{\mathsf{M}}$ is a strongly dense *-subalgebra. It follows from Kaplansky's density theorem, because we can approximate in the strong operator topology (in particular in L^2) and control the norm of the approximants at the same time.

Proposition 4.10. *Let* $e_1, e_2 \in \mathcal{H}_{\mathbb{R}}$, $\|e_1\| = \|e_2\| = 1$, $e_1 \perp e_2$*. Set* $\mathsf{A} = (W(e_1))''$, $\mathsf{B} = (W(e_2))''$, *and* $\mathsf{M} = \Gamma_q(\mathcal{H}_{\mathbb{R}})$*. There exists a sequence of unitaries* $(u_k)_{k\in\mathbb{N}} \subset \mathcal{U}(\mathsf{A})$ *such that we have* $\lim_{k\to\infty} \|\mathbb{E}_{\mathsf{B}}(xu_k y)\|_2 = 0$ *for all* $x, y \in \widetilde{\mathsf{M}}$*, where* $\widetilde{\mathsf{M}}$ *is the algebra of Wick words.*

Proof. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}_{\mathbb{R}}$. Assume $x = W(e_{i_1} \otimes \cdots \otimes e_{i_n})$ and $y = W(e_{j_1} \otimes \cdots \otimes e_{j_m})$; it clearly suffices because the span of such elements is equal to $\widetilde{\mathsf{M}}$. By definition of the trace on $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ we have $\|\mathbb{E}_{\mathsf{B}}(xu_k y)\|_2 = \|(\mathbb{E}_{\mathsf{B}}(xu_k y))\Omega\|$. Since the conditional expectation on the level of the Fock space is just the orthogonal projection (denoted P) onto the closed linear span of the set $\{e_2^{\otimes n} : n \in \mathbb{N}\}$, we get $\|(\mathbb{E}_{\mathsf{B}}(xu_k y))\Omega\| = \|P(xu_k y\Omega)\|$. Note now that as the left and right actions of y on Ω produce the same result, $e_{j_1} \otimes \cdots \otimes e_{j_m}$, we can change y to its right version, $W_r(e_{j_1} \otimes \cdots \otimes e_{j_m})$, denoted now by $\tilde{y}$. Since $\tilde{y} \in \mathsf{M}'$, we get $\|P(xu_k y\Omega)\| = \|P(x\tilde{y}u_k\Omega)\|$. We now choose the sequence $(u_k)_{k\in\mathbb{N}}$ — it is an arbitrary sequence of unitaries in A such that the corresponding vectors $\eta_k := u_k\Omega$ converge weakly to zero (such a sequence exists, because A is diffuse). Let Q_l be the orthogonal projection from $\mathcal{F}_q(\mathbb{C}e_1)$ onto $\operatorname{span}\{e_1^{\otimes j} : j \leqslant l\}$. Then for any l the sequence $(Q_l\eta_k)_{k\in\mathbb{N}}$ converges to zero in norm. Therefore to check that $\lim_{k\to\infty} \|P(x\tilde{y}\eta_k)\| = 0$, it suffices to do so for η_k replaced by $(\mathbb{1} - Q_l)\eta_k$. We now choose $l = n + m$. Therefore any η_k consists solely of tensors $e_1^{\otimes d}$, where $d \geqslant n + m + 1$. Since x can be written as a sum of products of n (in total) creation and annihilation operators and y can be decomposed similarly into products of m creation and annihilation operators, any simple tensor appearing in $x\tilde{y}(\mathbb{1} - Q_{n+m})\eta_k$ will contain at least one e_1. But all such simple tensors are orthogonal to $\mathcal{F}_q(\mathbb{C}e_2)$, so they are killed by P. □

Corollary 4.11. *If the vectors* e_1 *and* e_2 *in* $\mathcal{H}_{\mathbb{R}}$ *are orthogonal, then masas* $(W(e_1))''$ *and* $(W(e_2))''$ *are not intertwinable inside* $\Gamma_q(\mathcal{H}_{\mathbb{R}})$.

General case. Let us check now if the method used for a pair of orthogonal vectors can be applied in a more general setting. Assume now that e_1 and v are two unit vectors and write $v = \alpha e_1 + \beta e_2$, where $e_2 \perp e_1$, $\alpha^2 + \beta^2 = 1$, and $\beta \neq 0$. We fix now an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $\mathcal{H}_{\mathbb{R}}$ (if $\mathcal{H}_{\mathbb{R}}$ is finite-dimensional then this should be a finite sequence).

Proposition 4.12. *The masas* $\mathsf{A} := W(v)''$ *and* $\mathsf{B} := (W(e_1))''$ *are not intertwinable* (*so in particular are not unitarily conjugate*).

Proof. We proceed exactly as in the proof of Proposition 4.10 and also use the same notation; note however that this time P will be the orthogonal projection onto $\overline{\operatorname{span}}\{e_1^{\otimes n} : n \geqslant 0\}$. The only problem is that now we do not have orthogonality. Write $\eta_k = \sum_{j\in\mathbb{N}} a_j^{(k)} v^{\otimes j}$. We have $\|v^{\otimes j}\| \simeq (1/\sqrt{1-q})^j$ (see the third displayed formula on page 660 of [Ricard 2005]). Let us compute $v^{\otimes j}$:

$$v^{\otimes j} = \sum_{k=0}^{j} \alpha^{j-k}\beta^k R_{j,k}(e_1^{\otimes(j-k)} \otimes e_2^{\otimes k}),$$

where $R_{j,k}(e_1^{\otimes(j-k)} \otimes e_2^{\otimes k})$ is equal to the sum of all simple tensors such that $j-k$ factors are equal to e_1 and k factors are equal to e_2; there are $\binom{j}{k}$ such simple tensors. Note now that if $k \geqslant n+m+1$ then after applying $x\tilde{y}$ at least one e_2 remains as a factor, so the orthogonal projection P kills it. We conclude that it suffices to perform the summation in the displayed formula above only up to $j \wedge (n+m)$; we call the resulting tensors $\tilde{v}^{\otimes j}$ and the corresponding η_k is dubbed $\widetilde{\eta}_k$. Since k is bounded, the number $\binom{j}{k}$ is polynomial in j, so if we get exponential decay of the norm of the individual factors in the sum, the factor $\binom{j}{k}$ does not affect the overall convergence. After neglecting the terms with $k > n+m$, we use the trivial estimate $\|P(x\tilde{y}\widetilde{\eta}_k)\| \leqslant C\|\widetilde{\eta}_k\|$. The proof will be completed if we show that $\|\widetilde{\eta}_k\|$ converges to 0. Note now that the square of the norm of $\widetilde{\eta}_k$ is equal to $\sum_{j\in\mathbb{N}} |a_j^{(k)}|^2 \cdot \|\tilde{v}^{\otimes j}\|^2$. Recall that $\|\eta_k\| \leqslant 1$ and $\|v^{\otimes j}\| \simeq (1/\sqrt{1-q})^j$, so the coefficients $a_j^{(k)}$ satisfy $\sum_{j\in\mathbb{N}} |a_j^{(k)}|^2 \big(\frac{1}{1-q}\big)^j \lesssim 1$. It therefore suffices to show that $\lim_{j\to\infty}(1-q)^j\|\tilde{v}^{\otimes j}\|^2 = 0$, remembering that the vectors η_k converge weakly to 0, so we only care about large j. We estimate the norm of $\tilde{v}^{\otimes j}$ by the triangle inequality:

$$\|\tilde{v}^{\otimes j}\| \leqslant \sum_{k=0}^{j\wedge(n+m)} |\alpha|^{j-k}|\beta|^k \binom{j}{k} \|e_1^{\otimes k} \otimes e_2^{j-k}\|.$$

Since k is bounded, one can easily get an estimate of the form

$$\|e_1^{\otimes k} \otimes e_2^{\otimes(j-k)}\| \leqslant C(1/\sqrt{1-q})^j$$

(see [Ricard 2005, Remark 2]). This yields $\|\tilde{v}^{\otimes j}\| \leqslant C(1/\sqrt{1-q})^j|\alpha|^j \cdot j^k$. This is the inequality that we wanted, i.e., we find out that $(1-q)^j\|\tilde{v}^{\otimes j}\|^2$ is bounded by $Cj^k|\alpha|^j$, which converges to zero very fast, as we assumed that $|\alpha| < 1$. This finishes the proof of the proposition. □

We can now use the result to prove that the second quantisation automorphisms are never inner, unless trivial; it extends a result of Houdayer and Shlyakhtenko in the free case [2011, Theorem 5.1].

Corollary 4.13. *Let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space and let $U : \mathcal{H}_{\mathbb{R}} \to \mathcal{H}_{\mathbb{R}}$ be an orthogonal transformation. If $\Gamma_q(U) : \Gamma_q(\mathcal{H}_{\mathbb{R}}) \to \Gamma_q(\mathcal{H}_{\mathbb{R}})$ is an inner automorphism then $U = \mathbb{1}$.*

Proof. If U is not a multiple of identity then there exists a vector $v \in \mathcal{H}_{\mathbb{R}}$ such that Uv is not a multiple of v. The masas $\mathsf{A} := W(v)''$ and $\mathsf{B} := (W(Uv))''$ are conjugate by the automorphism $\Gamma_q(U)$, but by Proposition 4.12 they are not conjugate by an inner automorphism.

The only remaining case is now $U = -\mathbb{1}$. We may assume that the dimension of $\mathcal{H}_{\mathbb{R}}$ is at least 2, because otherwise $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ is commutative and any nontrivial automorphism is outer. Pick two orthogonal vectors e_1 and e_2 and consider the masas $\mathsf{A} = (W(e_1))''$ and $\mathsf{B} = (W(e_2))''$. Assume now that the automorphism $x \mapsto \mathcal{F}_q(-\mathbb{1})x\mathcal{F}_q(-\mathbb{1})$ is inner, so there is a unitary $u \in \Gamma_q(\mathcal{H}_{\mathbb{R}})$ implementing it. Since $\mathcal{F}_q(-\mathbb{1})W(e_1)\mathcal{F}_q(-\mathbb{1}) = -W(e_1)$, this automorphism preserves A; it also preserves B. But the masas in question are singular, so $u \in \mathsf{A} \cap \mathsf{B}$. It follows that $u\Omega \in L^2(\mathsf{A}) \cap L^2(\mathsf{B}) = \mathbb{C}\Omega$, so u has to be a multiple of identity, which is a contradiction, because this would yield the trivial automorphism. □

Remark 4.14. The results above exhibit in particular explicitly a continuum of nonmutually intertwinable singular masas in $\Gamma_q(\mathcal{H}_{\mathbb{R}})$. Very recently Popa [2019] showed the existence of such uncountable families in every separable II_1-factor (see Corollary 2.2 of that paper).

Remark 4.15. Generator masas can be also studied for the so-called mixed q-Gaussians (see [Speicher 1993]). They are known to be masas by [Skalski and Wang 2018], and in fact an application of methods of that paper and general results of [Bikram and Mukherjee 2017] show that they are singular, as noted by Simeng Wang. There seems to be however nothing known about the "radial" subalgebra in this more general context. Is it a masa? Is it isomorphic to a generator one?

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