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**THE COMPACT PICTURE OF
SYMMETRY-BREAKING OPERATORS FOR
RANK-ONE ORTHOGONAL AND UNITARY GROUPS**

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We present a method to calculate intertwining operators between the underlying Harish-Chandra modules of degenerate principal series representations of a reductive Lie group G and a reductive subgroup G', and between their composition factors. Our method describes the restriction of these operators to the K'-isotypic components, $K' \subseteq G'$ a maximal compact subgroup, and reduces the representation-theoretic problem to an infinite system of scalar equations of a combinatorial nature. For rank-one orthogonal and unitary groups and spherical principal series representations we calculate these relations explicitly and use them to classify intertwining operators. We further show that in these cases automatic continuity holds; i.e., every intertwiner between the Harish-Chandra modules extends to an intertwiner between the Casselman–Wallach completions, verifying a conjecture by Kobayashi. Altogether, this establishes the compact picture of the recently studied symmetry-breaking operators for orthogonal groups by Kobayashi and Speh, gives new proofs of their main results, and extends them to unitary groups.

1. Introduction

Representation theory of reductive Lie groups consists to a large extent in the study of the structure of standard families of representations, for example principal series

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representations. Here intertwining operators, such as the classical Knapp–Stein operators, play an important role, and they also provide important examples of integral kernel operators appearing in classical harmonic analysis. Recently similar operators have been introduced in [Kobayashi 2015] in connection with branching laws, i.e., the study of how representations behave when restricted to a closed subgroup of the original group; see also [Kobayashi and Speh 2015; Möllers et al. 2016a]. Again these are integral kernel operators, now intertwining with respect to the subgroup, and they appear to be very natural objects, not only for the problem of restricting representations, see [Möllers and Oshima 2015], but also for questions in classical harmonic analysis and automorphic forms, see [Möllers and Ørsted 2017; Möllers et al. 2016c].

In this paper we shall give an alternative approach to this new class of symmetry-breaking operators, namely one based on the Harish-Chandra module, i.e., the K-finite vectors in the representation, in analogy with the idea of spectrum-generating operators [Branson et al. 1996]. This gives new proofs of the main results of [Kobayashi and Speh 2015] and generalizes these results to unitary groups. Moreover, our more algebraic framework provides an alternative proof of the discrete spectrum in certain unitary representations.

The approach is quite general and discussed in the first part of the paper, while in the second part we carry out all details for the real conformal case and the CR case.

1A. Symmetry-breaking operators.

Let G be a reductive Lie group with compact center and $G' \subseteq G$ a reductive subgroup also with compact center. For irreducible smooth representations π of G and τ of G' the space

$$\mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$$

of continuous G'-intertwining operators between π and τ and its dimension $m(\pi, \tau)$ have received considerable attention recently, in particular in connection with multiplicity-1 statements asserting that $m(\pi, \tau) \leq 1$ for certain pairs (G, G') of classical groups such as $(\mathrm{GL}(n, \mathbb{R}), \mathrm{GL}(n-1, \mathbb{R}))$, $(\mathrm{O}(p, q), \mathrm{O}(p, q-1))$ or $(\mathrm{U}(p, q), \mathrm{U}(p, q-1))$; see [Sun and Zhu 2012]. A more refined problem is to determine whether for given representations π and τ there exist nontrivial G'-intertwining operators $\pi|_{G'} \to \tau$, also called *symmetry-breaking operators* in [Kobayashi 2015], and to classify them. For the pair $(G, G') = (\mathrm{O}(1, n), \mathrm{O}(1, n-1))$ this question was completely answered in [Kobayashi and Speh 2015] in the case where π and τ are spherical principal series representations, and in joint work with Y. Oshima we generalized in [Möllers et al. 2016a] their construction of symmetry-breaking operators to a large class of symmetric pairs.

Instead of studying this problem in the smooth category we attempt to apply the "spectrum-generating method" by Branson, Ólafsson, and Ørsted [Branson et al. 1996] in the study of intertwining operators in the category of $(\mathfrak{g}', K')$-modules,

and verify a conjecture by Kobayashi on the automatic continuity of symmetry-breaking operators between Harish-Chandra modules. To given smooth admissible representations π of G and τ of G' one can associate the underlying Harish-Chandra modules π_{HC} and τ_{HC}. These are admissible $(\mathfrak{g}, K)$-modules, resp. $(\mathfrak{g}', K')$-modules, realized on the spaces of K-finite, resp. K'-finite, vectors of π, resp. τ, where $K \subseteq G$ and $K' \subseteq G'$ are maximal compact subgroups. We consider the space

$$\mathrm{Hom}_{(\mathfrak{g}', K')}(\pi_{HC}|_{(\mathfrak{g}', K')}, \tau_{HC})$$

of intertwining operators in the category of Harish-Chandra modules. The natural restriction map

(1-1) $$\mathrm{Hom}_{G'}(\pi|_{G'}, \tau) \to \mathrm{Hom}_{(\mathfrak{g}', K')}(\pi_{HC}|_{(\mathfrak{g}', K')}, \tau_{HC})$$

is injective but in general not surjective and hence there might be more intertwining operators in the category of Harish-Chandra modules than in the smooth category. According to [Kobayashi 2014, Remark 10.2 (4)] it is plausible that this map is surjective if the space $(G \times G')/\mathrm{diag}(G')$ is real spherical. (Note that for $G' = G$ the map is surjective by the Casselman–Wallach theorem.)

We remark that for $(G, G') = (\mathrm{GL}(2, \mathbb{F}) \times \mathrm{GL}(2, \mathbb{F}), \mathrm{GL}(2, \mathbb{F}))$, $\mathbb{F} = \mathbb{R}, \mathbb{C}$, and $(G, G') = (\mathrm{GL}(2, \mathbb{C}), \mathrm{GL}(2, \mathbb{R}))$ intertwining operators between Harish-Chandra modules were previously studied in [Loke 2001] using explicit computations.

1B. *Symmetry breaking of principal series.* In this paper we outline a method to classify symmetry-breaking operators between the Harish-Chandra modules of principal series representations induced from maximal parabolic subgroups, and their composition factors. Let $P = MAN \subseteq G$ be a maximal parabolic subgroup of G such that $P' = P \cap G' = M'A'N'$ is maximal parabolic in G' and write $\mathfrak{a}$ and $\mathfrak{a}'$ for the Lie algebras of A and A'. Fix $\nu \in \mathfrak{a}^*$ such that the roots of (P, A) are given by $\{\nu, 2\nu, \ldots, q\nu\}$ and do similarly for $\nu' \in (\mathfrak{a}')^*$. Consider the principal series representations (smooth normalized parabolic induction)

$$\pi_{\xi,r} = \mathrm{Ind}_P^G(\xi \otimes e^{r\nu} \otimes \mathbf{1}), \quad \tau_{\xi',r} = \mathrm{Ind}_{P'}^{G'}(\xi' \otimes e^{r'\nu'} \otimes \mathbf{1}),$$

where ξ and ξ' are finite-dimensional representations of M and M' and $r, r' \in \mathbb{C}$. Let $\xi' = \xi|_{M'}$ and assume that for all K-types α of $\pi_{\xi,r}$ and all K'-types α' of $\tau_{\xi',r'}$ the multiplicity-free properties

$$\dim \mathrm{Hom}_K(\alpha, \pi_{\xi,r}|_K) \leq 1,$$
$$\dim \mathrm{Hom}_{K'}(\alpha', \tau_{\xi',r'}|_{K'}) \leq 1,$$
$$\dim \mathrm{Hom}_{K'}(\alpha|_{K'}, \alpha') \leq 1.$$

hold; i.e., $\pi_{\xi,r}$ is K-multiplicity-free, $\tau_{\xi',r'}$ is K'-multiplicity-free, and every K-type in $\pi_{\xi,r}$ is K'-multiplicity-free.

Let $T : (\pi_{\xi,r})_{\text{HC}} \to (\tau_{\xi',r'})_{\text{HC}}$ be a $(\mathfrak{g}', K')$-intertwining operator; then T is in particular K'-intertwining. Consider a pair $(\alpha; \alpha')$ consisting of a K-type α in $\pi_{\xi,r}$ and a K'-type α' in $\tau_{\xi',r'}$ which also occurs in $\alpha|_{K'}$. By the multiplicity-free assumptions the restriction of T to the K'-type α' inside the K-type α in $\pi_{\xi,r}$ maps to the K'-type α' in $\tau_{\xi',r'}$ and is unique up to a scalar $t_{\alpha,\alpha'} \in \mathbb{C}$ (see Section 3A for the precise definition). This encodes every K'-intertwining operator $T : (\pi_{\xi,r})_{\text{HC}} \to (\tau_{\xi',r'})_{\text{HC}}$ into scalars $t_{\alpha,\alpha'}$. Using the method of spectrum-generating operators by Branson, Ólafsson, and Ørsted [Branson et al. 1996] we prove:

Theorem A (see Theorem 3.4 and Corollary 3.6). *Let $T : (\pi_{\xi,r})_{\text{HC}} \to (\tau_{\xi',r'})_{\text{HC}}$ be a K'-intertwining operator given by scalars $t_{\alpha,\alpha'}$. Then T is $(\mathfrak{g}', K')$-intertwining if and only if for all pairs $(\alpha; \alpha')$ and every K'-type β' the following relation holds*:

$$(1\text{-}2) \qquad \sum_{\substack{\beta \\ (\alpha;\alpha') \leftrightarrow (\beta;\beta')}} \lambda_{\alpha,\alpha'}^{\beta,\beta'} (\sigma_\beta - \sigma_\alpha + 2r) t_{\beta,\beta'} = (\sigma'_{\beta'} - \sigma'_{\alpha'} + 2r') t_{\alpha,\alpha'}.$$

Here we write $(\alpha; \alpha') \leftrightarrow (\beta; \beta')$ if the K'-type β' inside the K-type β in $\pi_{\xi,r}$ can be reached from α' inside α by a single application of $\pi_{\xi,r}(\mathfrak{g}')$ for generic $r \in \mathbb{C}$ (see Section 3B for details). Further, σ_α and $\sigma'_{\alpha'}$ as well as $\lambda_{\alpha,\alpha'}^{\beta,\beta'}$ are certain constants depending only on the representations ξ and ξ' (see Sections 2C and 3C for their definition).

We note that the relations characterizing intertwining operators depend linearly on the induction parameters r and r' and turn the representation-theoretic problem of classifying symmetry-breaking operators into a combinatorial problem. We also remark that Theorem A admits a slight modification characterizing also intertwining operators between any subquotients of $\pi_{\xi,r}$ and $\tau_{\xi',r'}$ (see Remark 3.5).

1C. *Examples.* For the two pairs $(G, G') = (\text{O}(1, n), \text{O}(1, n-1))$, $n \geq 3$, and $(\text{U}(1, n), \text{U}(1, n-1))$, $n \geq 2$, we explicitly write down the linear relations for the scalars $t_{\alpha,\alpha'}$ characterizing intertwining operators in the case where $\xi = \mathbf{1}$ is the trivial representation (see Theorems 4.1 and 5.1), and use these relations to compute multiplicities. For the statements we abbreviate $\pi_r = \pi_{\mathbf{1},r}$ and $\tau_{r'} = \tau_{\mathbf{1},r'}$. If $\mathcal{V}$ is a $(\mathfrak{g}, K)$-module and $\mathcal{W}$ a $(\mathfrak{g}', K')$-module we write

$$m(\mathcal{V}, \mathcal{W}) = \dim \text{Hom}_{(\mathfrak{g}',K')}(\mathcal{V}|_{(\mathfrak{g}',K')}, \mathcal{W}).$$

We note that much of the notation used here follows [Kobayashi and Speh 2015].

Theorem B (see Theorems 4.2(1) and 5.2(1)).
(1) *For $(G, G') = (\text{O}(1, n), \text{O}(1, n-1))$ we have*

$$m((\pi_r)_{\text{HC}}, (\tau_{r'})_{\text{HC}}) = \begin{cases} 1 & \text{for } (r, r') \in \mathbb{C}^2 \setminus L_{\text{even}}, \\ 2 & \text{for } (r, r') \in L_{\text{even}}, \end{cases}$$

where $L_{\text{even}} = \left\{ \left(-\frac{n-1}{2} - i, -\frac{n-2}{2} - j \right) : i, j \in \mathbb{N}, \ i - j \in 2\mathbb{N} \right\}$.

(2) *For* $(G, G') = (U(1, n), U(1, n - 1))$ *we have*

$$m((\pi_r)_{\mathrm{HC}}, (\tau_{r'})_{\mathrm{HC}}) = \begin{cases} 1 & \text{for } (r, r') \in \mathbb{C}^2 \setminus L, \\ 2 & \text{for } (r, r') \in L, \end{cases}$$

where $L = \{(-n - 2i, -(n - 1) - 2j) : i, j \in \mathbb{N}, \ j \leq i\}$.

Multiplicity 2 does not contradict the multiplicity-1 statements for the above pairs (G, G'), because for $(r, r') \in L_{\mathrm{even}}$, resp. L, both representations π_r and $\tau_{r'}$ are reducible. In the case $(G, G') = (O(1, n), O(1, n - 1))$ the representation $(\pi_r)_{\mathrm{HC}}$ is reducible if and only if $r = \pm\left(\frac{n-1}{2} + i\right)$, $i \in \mathbb{N}$, its composition factors consisting of a finite-dimensional subrepresentation $\mathcal{F}(i)$ and an infinite-dimensional unitarizable quotient $\mathcal{T}(i)$. Similarly, in the case $(G, G') = (U(1, n), U(1, n - 1))$ the representation $(\pi_r)_{\mathrm{HC}}$ is reducible if and only if $r = \pm(n + 2i)$, $i \in \mathbb{N}$, and its composition factors consist of a finite-dimensional subrepresentation $\mathcal{F}(i)$, two proper subquotients $\mathcal{T}_{\pm}(i)$, and a unitarizable quotient $\mathcal{T}(i)$. Write $\mathcal{F}'(j)$, $\mathcal{T}'_{\pm}(j)$ and $\mathcal{T}'(j)$ for the corresponding composition factors of $(\tau_{r'})_{\mathrm{HC}}$ at $r' = -\frac{n-2}{2} - j$, resp. $r' = -(n - 1) - 2j$.

Theorem C (see Theorems 4.2(2) and 5.2(2)).
(1) *For* $(G, G') = (O(1, n), O(1, n - 1))$, *the multiplicities* $m(\mathcal{V}, \mathcal{W})$ *are given by*

$\mathcal{V}{\downarrow}\ \ \mathcal{W}{\rightarrow}$	$\mathcal{F}'(j)$	$\mathcal{T}'(j)$
$\mathcal{F}(i)$	1	0
$\mathcal{T}(i)$	0	1

for $i - j \in 2\mathbb{N}$,

$\mathcal{V}{\downarrow}\ \ \mathcal{W}{\rightarrow}$	$\mathcal{F}'(j)$	$\mathcal{T}'(j)$
$\mathcal{F}(i)$	0	0
$\mathcal{T}(i)$	1	0

otherwise.

(2) *For* $(G, G') = (U(1, n), U(1, n - 1))$, *the multiplicities* $m(\mathcal{V}, \mathcal{W})$ *are given by*

$\mathcal{V}{\downarrow}\ \ \mathcal{W}{\rightarrow}$	$\mathcal{F}'(j)$	$\mathcal{T}'_+(j)$	$\mathcal{T}'_-(j)$	$\mathcal{T}'(j)$
$\mathcal{F}(i)$	1	0	0	0
$\mathcal{T}_+(i)$	0	1	0	0
$\mathcal{T}_-(i)$	0	0	1	0
$\mathcal{T}(i)$	0	0	0	1

for $j \leq i$,

$\mathcal{V}{\downarrow}\ \ \mathcal{W}{\rightarrow}$	$\mathcal{F}'(j)$	$\mathcal{T}'_+(j)$	$\mathcal{T}'_-(j)$	$\mathcal{T}'(j)$
$\mathcal{F}(i)$	0	0	0	0
$\mathcal{T}_+(i)$	0	0	0	0
$\mathcal{T}_-(i)$	0	0	0	0
$\mathcal{T}(i)$	1	0	0	0

otherwise.

We further construct a basis of $\mathrm{Hom}_{(\mathfrak{g}',K')}((\pi_r)_{\mathrm{HC}}|_{(\mathfrak{g}',K')}, (\tau_{r'})_{\mathrm{HC}})$ for all $r, r' \in \mathbb{C}$ by solving the relations (1-2) explicitly. More precisely, we find a family $(t_{\alpha,\alpha'}(r, r'))_{\alpha,\alpha'}$ consisting of rational functions in $r, r' \in \mathbb{C}$ that solve the relations (1-2). Renormalizing the functions $t_{\alpha,\alpha'}(r, r')$ gives a family $(t_{\alpha,\alpha'}^{(1)}(r, r'))_{\alpha,\alpha'}$ of holomorphic functions in $r, r' \in \mathbb{C}$ satisfying the relations (1-2). By Theorem A this constructs intertwining operators $T^{(1)}(r, r')$ depending holomorphically on $r, r' \in \mathbb{C}$. We show that

$$T^{(1)}(r, r') = 0 \quad \text{if and only if} \quad (r, r') \in L_{\mathrm{even}}, \text{ resp. } L.$$

For each $(r, r') \in L_{\mathrm{even}}$, resp. L, the holomorphic function $T^{(1)}(r, r')$ can be renormalized along two different affine complex lines through (r, r'), and one obtains two different nontrivial operators $T^{(2)}(r, r')$, $T^{(3)}(r, r')$ for every $(r, r') \in L_{\mathrm{even}}$, resp. L (see Propositions 4.6 and 5.6 for details).

Theorem D (see Theorems 4.9 and 5.8 and Remarks 4.10 and 5.9). *For the pair* $(G, G') = (\mathrm{O}(1, n), \mathrm{O}(1, n-1))$, *resp.* $(\mathrm{U}(1, n), \mathrm{U}(1, n-1))$, *we have*

$\mathrm{Hom}_{(\mathfrak{g}',K')}((\pi_r)_{\mathrm{HC}}|_{(\mathfrak{g}',K')}, (\tau_{r'})_{\mathrm{HC}})$
$$= \begin{cases} \mathbb{C}T^{(1)}(r, r') & \text{for } (r, r') \in \mathbb{C}^2 \setminus \mathcal{L}, \\ \mathbb{C}T^{(2)}(r, r') \oplus \mathbb{C}T^{(3)}(r, r') & \text{for } (r, r') \in \mathcal{L}, \end{cases}$$

where $\mathcal{L} = L_{\mathrm{even}}$, resp. L. Moreover, by composing $T^{(1)}(r, r')$ with embeddings and quotient maps for the composition factors of π_r and $\tau_{r'}$, and renormalizing along certain affine complex lines, one can obtain every intertwining operator between arbitrary composition factors of $(\pi_r)_{\mathrm{HC}}$ and $(\tau_{r'})_{\mathrm{HC}}$.

The previous theorem shows that basically all the information about intertwining operators between spherical principal series of G and G' and their composition factors is contained in the single holomorphic family $T^{(1)}(r, r')$ of intertwiners.

Finally we turn to the question of whether every intertwining operator between the Harish-Chandra modules $(\pi_r)_{\mathrm{HC}}$ and $(\tau_{r'})_{\mathrm{HC}}$ lifts to an intertwining operator between the smooth globalizations π_r and $\tau_{r'}$, i.e., the question of whether (1-1) is an isomorphism.

Theorem E (see Corollaries 4.12 and 5.11). *For the pairs* $(G, G') = (\mathrm{O}(1, n), \mathrm{O}(1, n-1))$ *and* $(\mathrm{U}(1, n), \mathrm{U}(1, n-1))$ *every intertwining operator between $(\pi_r)_{\mathrm{HC}}$ and $(\tau_{r'})_{\mathrm{HC}}$ (resp. any of their subquotients) extends to a continuous intertwining operator between π_r and $\tau_{r'}$ (resp. the Casselman–Wallach completions of the subquotients). In particular, the injective map (1-1) is surjective for all spherical principal series representations and their subquotients.*

This verifies Kobayashi's conjecture [2014, Remark 10.2 (4)] in the above cases.

For $(G, G') = (\mathrm{O}(1, n), \mathrm{O}(1, n-1))$ the analogues of Theorems B, C and D in the smooth category, i.e., for π_r and $\tau_{r'}$ instead of $(\pi_r)_{\mathrm{HC}}$ and $(\tau_{r'})_{\mathrm{HC}}$, were established

in [Kobayashi and Speh 2015] using analytic techniques. With Theorem E we obtain a new proof of their results as well as the corresponding results for $(G, G') = (\mathrm{U}(1, n), \mathrm{U}(1, n - 1))$.

1D. *Application.* For $(G, G') = (\mathrm{O}(1, n), \mathrm{O}(1, n - 1))$ we further present an application of the classification of symmetry-breaking operators. In Theorem 4.14 we use the explicit formula for the numbers $t_{\alpha, \alpha'}$ to construct discrete components in the restriction of certain unitary representations of G to G'. The representations in question are either spherical complementary series representations (i.e., those π_r which are unitarizable) or the unitarizable quotients $\mathcal{T}(i)$. This extends and gives new proofs of previous results of [Speh and Venkataramana 2011; Zhang 2015; Kobayashi and Speh 2015; Möllers and Oshima 2015] (see Remark 4.15). Analogous results hold for $(G, G') = (\mathrm{U}(1, n), \mathrm{U}(1, n - 1))$.

1E. *Structure of the paper.* In Section 2 we fix the notation for principal series representations and recall the method of spectrum-generating operators [Branson et al. 1996]. This method is applied in Section 3 to obtain an equivalent characterization of intertwining operators in the category of $(\mathfrak{g}', K')$-modules by means of scalar identities. After this quite general approach, we study in Section 4 the special case $(G, G') = (\mathrm{O}(1, n), \mathrm{O}(1, n - 1))$ in detail and give some applications. Finally, in Section 5 we repeat the same procedure for $(G, G') = (\mathrm{U}(1, n), \mathrm{U}(1, n - 1))$ providing a new classification of symmetry-breaking operators in this example. Appendix A contains some basic properties of Gegenbauer and Jacobi polynomials which are used in Appendix B to describe explicit branching laws for real and complex spherical harmonics.

Throughout we will use the notation $\mathbb{N} = \{0, 1, 2, \ldots\}$.

2. Preliminaries

We fix the necessary notation, discuss induced representations and the method of the spectrum-generating operator by Branson, Ólafsson, and Ørsted [Branson et al. 1996].

2A. *Compatible maximal parabolic subgroups.* Let G be a reductive Lie group with compact center and $G' \subseteq G$ a reductive subgroup also with compact center. Denote by $\mathfrak{g}$ and $\mathfrak{g}'$ the Lie algebras of G and G'. Choose a maximal parabolic subgroup $P \subseteq G$ with the property that $P' = P \cap G'$ is maximal parabolic in G' and write $P = MAN$ and $P' = M'A'N'$ for the Langlands decompositions of P and P'. We fix a Cartan involution θ of G which leaves G' and the Levi subgroups MA and $M'A'$ invariant. Write $K = G^\theta$ and $K' = (G')^\theta$ for the corresponding fixed point subgroups of G and G' which are maximal compact and denote by $\mathfrak{k}$ and $\mathfrak{k}'$ their Lie algebras. Let $\mathfrak{s}$ and $\mathfrak{s}'$ be the (-1)-eigenspaces of θ on $\mathfrak{g}$ and $\mathfrak{g}'$ so that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}, \quad \mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}'.$$

Example 2.1. (1) Let (G, G') be one of the pairs

$$(O(1, n), O(1, n-1)), \quad (U(1, n), U(1, n-1)),$$
$$(Sp(1, n), Sp(1, n-1)), \quad (F_{4(-20)}, Spin(8, 1)).$$

Then one can choose the minimal parabolic P such that $P' = P \cap G'$ is minimal parabolic in G'. Since G and G' are of rank 1, minimal parabolics are maximal and hence satisfy our assumptions.

(2) Let

$$(G, G') = (SL(n, \mathbb{R}), SL(n-1, \mathbb{R})),$$

with G' embedded in G as the upper-left block. Then all standard maximal parabolics $P = P_{p,q} = (S(GL(p, \mathbb{R}) \times GL(q, \mathbb{R}))) \ltimes \mathbb{R}^{p \times q}$ corresponding to the partition $n = p + q$ with $q > 1$ satisfy the assumptions. In this case $P' = P \cap G'$ is the standard maximal parabolic of G' corresponding to the partition $n - 1 = p + (q - 1)$.

2B. *Principal series representations.* For any finite-dimensional representation (ξ, V_ξ) of M and any $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, where $\mathfrak{a}$ denotes the Lie algebra of A, consider the induced representation $\operatorname{Ind}_P^G(\xi \otimes e^\nu \otimes \mathbf{1})$ (normalized smooth parabolic induction). This representation is realized on the space

$$\mathcal{E}(G; \xi, \nu) = \{F \in C^\infty(G, V_\xi) : F(gman) = a^{-\nu-\rho}\xi(m)^{-1}F(g)$$
$$\text{for all } g \in G, \, man \in MAN\},$$

where $\rho = \frac{1}{2}\operatorname{tr} \operatorname{ad}|_\mathfrak{n} \in \mathfrak{a}^*$. The group G acts on $\mathcal{E}(G; \xi, \nu)$ by the left-regular action. Since $G = KP$, restriction to K is an isomorphism $\mathcal{E}(G; \xi, \nu) \to \mathcal{E}(K; \xi|_{M \cap K})$, where

$$\mathcal{E}(K; \xi|_{M \cap K}) = \{F \in C^\infty(K, V_\xi) : F(km) = \xi(m)^{-1}F(k) \text{ for all } k \in K, \, m \in M \cap K\}.$$

Let $\pi_{\xi,\nu}$ denote the action of G on $\mathcal{E}(K; \xi|_{M \cap K})$ which makes this isomorphism G-equivariant. Then $(\pi_{\xi,\nu}, \mathcal{E}(K; \xi|_{M \cap K}))$ is a smooth admissible representation of G. The restriction of $\pi_{\xi,\nu}$ to K is simply the left-regular representation of K on $\mathcal{E}(K; \xi|_{M \cap K})$.

Corresponding to the smooth representation $\pi_{\xi,\nu}$ we consider its underlying $(\mathfrak{g}, K)$-module $(\pi_{\xi,\nu})_{\text{HC}}$ realized on the space $\mathcal{E} = \mathcal{E}(K; \xi|_{M \cap K})_K$ of K-finite vectors. Abusing notation we denote the action of the Lie algebra $\mathfrak{g}$ on $\mathcal{E}$ also by $\pi_{\xi,\nu}$. Then the restriction of $(\pi_{\xi,\nu})_{\text{HC}}$ to K decomposes as

$$\mathcal{E} = \bigoplus_{\alpha \in \widehat{K}} \mathcal{E}(\alpha),$$

with $\mathcal{E}(\alpha)$ being the α-isotypic component in $\mathcal{E}$. Note that $\mathcal{E}$ and hence its decomposition into K-isotypic components is independent of $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ and only depends on ξ.

Similarly we consider $\tau_{\xi',\nu'} = \mathrm{Ind}_{P'}^{G'}(\xi' \otimes e^{\nu'} \otimes \mathbf{1})$ for a finite-dimensional representation $(\xi', V_{\xi'})$ of M' and an element $\nu' \in (\mathfrak{a}')^*_{\mathbb{C}}$, and its underlying $(\mathfrak{g}', K')$-module $(\tau_{\xi',\nu'})_{\mathrm{HC}}$ realized on the space $\mathcal{E}' = \mathcal{E}(K'; \xi'|_{K' \cap M'})_{K'}$. As above we decompose the restriction of $(\tau_{\xi',\nu'})_{\mathrm{HC}}$ to K'

$$\mathcal{E}' = \bigoplus_{\alpha' \in \widehat{K'}} \mathcal{E}'(\alpha'),$$

with $\mathcal{E}'(\alpha')$ being the α'-isotypic component.

2C. *The spectrum-generating operator.* Since P is a maximal parabolic subgroup we have $\dim \mathfrak{a} = 1$ and we can choose $H \in \mathfrak{a}$ such that the eigenvalues of $\mathrm{ad}(H)$ on the Lie algebra $\mathfrak{n}$ of N are $1, \ldots, q$. Define $\nu \in \mathfrak{a}^*$ by $\nu(H) = 1$; then $\Sigma(\mathfrak{g}, \mathfrak{a}) = \{\pm\nu, \ldots, \pm q\nu\}$. We abbreviate $\pi_{\xi,r} = \pi_{\xi,r\nu}$ for $r \in \mathbb{C}$.

Let B be an invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$ normalized by $B(H, H) = 1$. For $1 \leq j \leq q$ let

$$\mathfrak{k}_j = \mathfrak{k} \cap (\mathfrak{g}_{j\nu} + \mathfrak{g}_{-j\nu}).$$

Choose a basis $(X_{j,k})_k$ of $\mathfrak{k}_j$, denote by $(X'_{j,k})_k$ the corresponding dual basis with respect to B and put

$$\mathrm{Cas}_j = \sum_k X_{j,k} X'_{j,k}.$$

Then Cas_j is an element of $\mathcal{U}(\mathfrak{k})$, the universal enveloping algebra of $\mathfrak{k}$. Clearly the elements $\mathrm{Cas}_j \in \mathcal{U}(\mathfrak{k})$ do not depend on the choice of the corresponding bases. Following [Branson et al. 1996] we define the *spectrum-generating operator* as the second-order element in $\mathcal{U}(\mathfrak{k})$ given by

$$\mathcal{P} = \sum_{j=1}^q j^{-1} \mathrm{Cas}_j.$$

We remark that even though the spaces $\mathfrak{k}_j$ do not form subalgebras the operator $\mathcal{P}$ can be written as a rational linear combination of Casimir elements of subalgebras of $\mathfrak{k}$; see [Branson et al. 1996, Remark 2.4]. Since the left-regular representation of K on $\mathcal{E}$ commutes with the right-action $\mathcal{R}_{\mathcal{P}}$ of $\mathcal{P}$ the restriction of $\mathcal{R}_{\mathcal{P}}$ to each isotypic component $\mathcal{E}(\alpha)$ is a linear transformation

$$\sigma_\alpha = \sigma_{\alpha,\xi|_{M \cap K}} \in \mathrm{End}\,\mathcal{E}(\alpha)$$

which only depends on ξ but not on ν.

Similarly we define $H' \in \mathfrak{a}'$, $\nu' \in (\mathfrak{a}')^*$ and choose an invariant nondegenerate symmetric bilinear form B' on $\mathfrak{g}'$ with $B'(H', H') = 1$. Let $\mathcal{P}'$ denote the spectrum-generating operator for G' and write $\sigma'_{\alpha'} \in \mathrm{End}\,\mathcal{E}'(\alpha')$ for the restriction of $\mathcal{R}_{\mathcal{P}'}$ to the isotypic component $\mathcal{E}'(\alpha')$.

2D. *Reduction to the cocycle.* For each $X \in \mathfrak{g}_{\mathbb{C}}$ we define a scalar-valued function $\omega(X)$ on K by

$$\omega(X)(k) = B(\mathrm{Ad}(k^{-1})X, H), \quad k \in K,$$

where we extend B to a symmetric $\mathbb{C}$-bilinear form on $\mathfrak{g}_{\mathbb{C}}$. This defines a K-equivariant map

$$\omega : \mathfrak{g}_{\mathbb{C}} \to \mathcal{E}(K; \mathbf{1}) \cong C^{\infty}(K/(M \cap K)),$$

where $\mathbf{1}$ is the trivial $M \cap K$-representation. The map ω is called a *cocycle*. Note that ω vanishes on $\mathfrak{k}_{\mathbb{C}}$. Let $m(\omega(X))$ denote the multiplication operator

$$\mathcal{E} \to \mathcal{E}, \quad \varphi \mapsto \omega(X)\varphi.$$

For $\alpha, \beta \in \widehat{K}$ with $\mathcal{E}(\alpha), \mathcal{E}(\beta) \neq 0$ we let

$$\omega_{\alpha}^{\beta}(X) = \mathrm{proj}_{\mathcal{E}(\beta)} \circ m(\omega(X))|_{\mathcal{E}(\alpha)}, \quad X \in \mathfrak{g}_{\mathbb{C}},$$

where $\mathrm{proj}_{\mathcal{E}(\beta)}$ denotes the projection from $\mathcal{E}$ onto $\mathcal{E}(\beta)$. We can now express the differential representation $\pi_{\xi,r}$ of $\mathfrak{g}$ on $\mathcal{E}$ in terms of the cocycle ω and the maps σ_{α}:

Theorem 2.2 [Branson et al. 1996, Corollary 2.6]. *For $X \in \mathfrak{s}_{\mathbb{C}}$ and any $\alpha, \beta \in \widehat{K}$ with $\mathcal{E}(\alpha), \mathcal{E}(\beta) \neq 0$ we have*

$$(2\text{-}1) \qquad \mathrm{proj}_{\mathcal{E}(\beta)} \circ \pi_{\xi,r}(X)|_{\mathcal{E}(\alpha)} = \tfrac{1}{2}(\sigma_{\beta}\omega_{\alpha}^{\beta}(X) - \omega_{\alpha}^{\beta}(X)\sigma_{\alpha} + 2r\omega_{\alpha}^{\beta}(X)).$$

Similarly we denote by $\omega'(X)$ the corresponding cocycle for G' and by $\omega_{\alpha'}^{\beta'}(X)$ the corresponding map from $\mathcal{E}'(\alpha')$ to $\mathcal{E}'(\beta')$. Then we obtain for $X \in \mathfrak{s}'_{\mathbb{C}}$ and any $\alpha', \beta' \in \widehat{K}'$ with $\mathcal{E}'(\alpha'), \mathcal{E}'(\beta') \neq 0$ the analogous identity

$$(2\text{-}2) \qquad \mathrm{proj}_{\mathcal{E}'(\beta')} \circ \tau_{\xi',r'}(X)|_{\mathcal{E}'(\alpha')} = \tfrac{1}{2}(\sigma'_{\beta'}\omega_{\alpha'}^{\beta'}(X) - \omega_{\alpha'}^{\beta'}(X)\sigma'_{\alpha'} + 2r'\omega_{\alpha'}^{\beta'}(X)).$$

3. The compact picture of symmetry-breaking operators

Consider the admissible $(\mathfrak{g}, K)$-module $(\pi_{\xi,r})_{\mathrm{HC}}$. Then its restriction $(\pi_{\xi,r})_{\mathrm{HC}}|_{(\mathfrak{g}',K')}$ is a $(\mathfrak{g}', K')$-module which is in general not admissible anymore. However, we can still study the space

$$\mathrm{Hom}_{(\mathfrak{g}',K')}((\pi_{\xi,r})_{\mathrm{HC}}|_{(\mathfrak{g}',K')}, (\tau_{\xi',r'})_{\mathrm{HC}})$$

of intertwining operators between the $(\mathfrak{g}', K')$-modules. In this section we use Theorem 2.2 to characterize these intertwining operators in terms of their action on the K'-isotypic components in the K-types $\mathcal{E}(\alpha)$.

3A. *Relating K-types and K'-types.* From now on we assume that both $\mathcal{E}$ and $\mathcal{E}'$ are multiplicity-free; i.e.,

(MF1) $\dim \operatorname{Hom}_K(\alpha, \mathcal{E}), \dim \operatorname{Hom}_{K'}(\alpha', \mathcal{E}') \leq 1$ for all $\alpha \in \widehat{K}$, $\alpha' \in \widehat{K}'$.

This implies by Schur's lemma that the maps σ_α and $\sigma'_{\alpha'}$ are scalars which we denote by the same symbols. We further assume that each K'-type $\mathcal{E}'(\alpha') \neq 0$ occurs at most once in each K-type $\mathcal{E}(\alpha) \neq 0$; i.e.,

(MF2) $\dim \operatorname{Hom}_{K'}(\mathcal{E}(\alpha), \mathcal{E}'(\alpha')) \leq 1$ for all $\mathcal{E}(\alpha), \mathcal{E}'(\alpha') \neq 0$.

Each K-isotypic component $\mathcal{E}(\alpha)$ decomposes under the action of $K' \subseteq K$ into

$$\mathcal{E}(\alpha) = \bigoplus_{\alpha' \in \widehat{K}'} \mathcal{E}(\alpha; \alpha'),$$

where $\mathcal{E}(\alpha; \alpha')$ is the α'-isotypic component in $\mathcal{E}(\alpha)$. Then our assumptions imply that if $\mathcal{E}(\alpha; \alpha'), \mathcal{E}'(\alpha') \neq 0$ then $\mathcal{E}(\alpha; \alpha') \cong \mathcal{E}'(\alpha')$. In all such cases we fix an isomorphism

$$R_{\alpha, \alpha'} : \mathcal{E}(\alpha; \alpha') \xrightarrow{\sim} \mathcal{E}'(\alpha').$$

To simplify notation, let $R_{\alpha, \alpha'} = 0$ whenever $\mathcal{E}'(\alpha') = 0$, so that we have surjective K'-equivariant maps $R_{\alpha, \alpha'} : \mathcal{E}(\alpha; \alpha') \to \mathcal{E}'(\alpha')$ for all $\mathcal{E}(\alpha; \alpha') \neq 0$.

In applications, it is often useful to choose a natural isomorphism $\mathcal{E}(\alpha; \alpha') \cong \mathcal{E}'(\alpha')$ relating K-types and K'-types. For this we study the restriction of functions from K to K'. Assume for simplicity that $\xi' = \xi|_{M'}$. In this case we can consider the restriction operator

$$\operatorname{rest} : \mathcal{E} \to \mathcal{E}', \quad \varphi \mapsto \varphi|_{K'}.$$

This operator is K'-equivariant and hence, if rest is nonzero on some K'-type $\mathcal{E}(\alpha; \alpha')$ in $\mathcal{E}$ then $\operatorname{rest}|_{\mathcal{E}(\alpha; \alpha')}$ is an isomorphism onto $\mathcal{E}'(\alpha')$ by Schur's lemma. However, rest might also vanish on some $\mathcal{E}(\alpha; \alpha')$ and therefore we need to combine the restriction with differentiation in the normal direction.

For this we write

$$\mathfrak{k} = (\mathfrak{m} \cap \mathfrak{k}) \oplus \mathfrak{q},$$

where $\mathfrak{q}$ is the orthogonal complement of $(\mathfrak{m} \cap \mathfrak{k})$ in $\mathfrak{k}$ with respect to the invariant form B. Note that $M \cap K$ acts on $\mathfrak{q}$. Similarly

$$\mathfrak{k}' = (\mathfrak{m}' \cap \mathfrak{k}') \oplus \mathfrak{q}'.$$

Let $\mathfrak{q}''$ denote the orthogonal complement of $\mathfrak{q}'$ in $\mathfrak{q}$; then

$$\mathfrak{q} = \mathfrak{q}' \oplus \mathfrak{q}''.$$

We note that since $M \cap K$ acts on $\mathfrak{q}$ and $M' \cap K'$ acts on $\mathfrak{q}'$, the group $M' \cap K'$ also acts on $\mathfrak{q}''$. We then have

$$\mathfrak{k}/(\mathfrak{m} \cap \mathfrak{k}) \cong \mathfrak{k}'/(\mathfrak{m}' \cap \mathfrak{k}') \oplus \mathfrak{q}'';$$

i.e., $\mathfrak{q}''$ identifies with the normal space of $K'/(M' \cap K')$ in $K/(M \cap K)$ at the base point. Denote by $S(\mathfrak{q}'')$ the symmetric algebra over $\mathfrak{q}''$ and by $S(\mathfrak{q}'')^{M' \cap K'}$ its $(M' \cap K')$-invariants. Note that $S(\mathfrak{q}'')^{M' \cap K'}$ acts naturally from the right by differential operators on functions defined on a small neighborhood of $K'/(M' \cap K')$ in $K/(M \cap K)$.

Lemma 3.1. *Let $(\alpha, \alpha') \in \widehat{K} \times \widehat{K}'$ with $\mathcal{E}(\alpha; \alpha') \neq 0$ and $D \in S(\mathfrak{q}'')^{M' \cap K'}$. Then the map* rest $\circ \, D : \mathcal{E} \to \mathcal{E}'$ *is K'-equivariant. In particular,*

$$(\text{rest} \circ D)|_{\mathcal{E}(\alpha; \alpha')} : \mathcal{E}(\alpha; \alpha') \to \mathcal{E}'(\alpha')$$

is an isomorphism whenever it is nonzero.

Remark 3.2. Of course one could as well consider other irreducible $M' \cap K'$-subrepresentations of $S(\mathfrak{q}'')$ than the trivial one. In fact, using an idea of [Ørsted and Vargas 2004] one can construct an injective K'-equivariant map

$$\mathcal{E} = C^\infty(K \times_{M \cap K} \xi)_K \to \bigoplus_{m=0}^{\infty} C^\infty(K' \times_{M' \cap K'} (\xi \otimes S^m(\mathfrak{q}'')))_{K'}$$

and use it to relate K-types and K'-types of the induced representations $\pi_{\xi, r}$ and $\tau_{\xi', r'}$ for $\xi'|_{M' \cap K'}$ any subrepresentation of $\xi|_{M' \cap K'} \otimes S(\mathfrak{q}'')$. Lemma 3.1 can then be viewed as the special case $\xi' = \xi \otimes \mathbb{C}D$, where $D \in S(\mathfrak{q}'')^{M' \cap K'}$ and hence $\mathbb{C}D$ is the trivial $M' \cap K'$-representation.

3B. *Intertwining operators between Harish-Chandra modules.* Let $\mathcal{V} \subseteq \mathcal{E}$ be a $(\mathfrak{g}', K')$-submodule of $(\pi_{\xi, r})_{\text{HC}}$; i.e., $\mathcal{V}$ is stable under $\pi_{\xi, r}(\mathfrak{g}')$ and stable under $\pi_{\xi, r}(K')$. A linear map $T : \mathcal{V} \to \mathcal{E}'$ is called an *intertwining operator for $\pi_{\xi, r}$ and $\tau_{\xi', r'}$* if for every $v \in \mathcal{V}$ we have

$$(3\text{-}1) \qquad (T \circ \pi_{\xi, r}(X))v = (\tau_{\xi', r'}(X) \circ T)v \quad \text{for all } X \in \mathfrak{g}',$$

$$(3\text{-}2) \qquad (T \circ \pi_{\xi, r}(k))v = (\tau_{\xi', r'}(k) \circ T)v \quad \text{for all } k \in K'.$$

In particular an intertwining operator commutes by (3-2) with the action of K' and hence restricts to a map $T_{\alpha, \alpha'} = T|_{\mathcal{E}(\alpha; \alpha')} : \mathcal{E}(\alpha; \alpha') \to \mathcal{E}'(\alpha')$ for all $\mathcal{E}(\alpha; \alpha') \subseteq \mathcal{V}$. If $\mathcal{E}'(\alpha') = 0$ then clearly $T_{\alpha, \alpha'} = 0$. Recall that we fixed in Section 3A K'-equivariant maps $R_{\alpha, \alpha'} : \mathcal{E}(\alpha; \alpha') \to \mathcal{E}'(\alpha')$; then by Schur's lemma $T_{\alpha, \alpha'}$ is a scalar multiple of $R_{\alpha, \alpha'}$. We write

$$(3\text{-}3) \qquad T_{\alpha, \alpha'} = t_{\alpha, \alpha'} \cdot R_{\alpha, \alpha'} \quad \text{for all } 0 \neq \mathcal{E}(\alpha; \alpha') \subseteq \mathcal{V},$$

with $t_{\alpha, \alpha'} \in \mathbb{C}$.

Restricting (3-1) to $\mathcal{E}(\alpha; \alpha')$ and composing with the projection $\text{proj}_{\mathcal{E}'(\beta')}$ we obtain

$$(3\text{-}4) \qquad \text{proj}_{\mathcal{E}'(\beta')} \circ T \circ \pi_{\xi,r}(X)|_{\mathcal{E}(\alpha;\alpha')} = \text{proj}_{\mathcal{E}'(\beta')} \circ \tau_{\xi',r'}(X) \circ T|_{\mathcal{E}(\alpha;\alpha')}.$$

To simplify both sides we let

$$\omega_{\alpha,\alpha'}^{\beta,\beta'} : \mathfrak{s}_{\mathbb{C}}' \otimes \mathcal{E}(\alpha; \alpha') \to \mathcal{E}(\beta; \beta'), \quad \omega_{\alpha,\alpha'}^{\beta,\beta'}(X) = \text{proj}_{\mathcal{E}(\beta;\beta')} \circ m(\omega(X))|_{\mathcal{E}(\alpha;\alpha')},$$

where we view $\omega_{\alpha,\alpha'}^{\beta,\beta'}(X)$, $X \in \mathfrak{s}'$, as a linear map $\mathcal{E}(\alpha; \alpha') \to \mathcal{E}(\beta; \beta')$. Write $(\alpha; \alpha') \to (\beta; \beta')$ if $\omega_{\alpha,\alpha'}^{\beta,\beta'} \neq 0$. The following lemma is proved along the same lines as [Branson et al. 1996, Lemma 4.4 (c)] and justifies the use of the notation $(\alpha; \alpha') \leftrightarrow (\beta; \beta')$ instead of $(\alpha; \alpha') \to (\beta; \beta')$:

Lemma 3.3. *For an orthonormal basis* $(X_k)_k \subseteq \mathfrak{s}'$ *put*

$$s_{\alpha,\alpha'}^{\beta,\beta'} = \sum_k \omega_{\beta,\beta'}^{\alpha,\alpha'}(X_k) \circ \omega_{\alpha,\alpha'}^{\beta,\beta}(X_k).$$

Then $s_{\alpha,\alpha'}^{\beta,\beta'}$ *is independent of the choice of* $(X_k)_k$ *and*

$$(\alpha; \alpha') \to (\beta; \beta') \quad \Longleftrightarrow \quad s_{\alpha,\alpha'}^{\beta,\beta'} \neq 0 \quad \Longleftrightarrow \quad (\beta; \beta') \to (\alpha; \alpha').$$

Now, on the left-hand side of the identity (3-4) we can express $\pi_{\xi,r}(X)|_{\mathcal{E}(\alpha;\alpha')}$ in terms of the cocycle using (2-1):

$$\text{proj}_{\mathcal{E}'(\beta')} \circ T \circ \pi_{\xi,r}(X)|_{\mathcal{E}(\alpha;\alpha')} = \sum_{\substack{\beta \\ (\alpha;\alpha') \leftrightarrow (\beta;\beta')}} T \circ \text{proj}_{\mathcal{E}'(\beta;\beta')} \circ \pi_{\xi,r}(X)|_{\mathcal{E}(\alpha;\alpha')}$$

$$= \frac{1}{2} \sum_{\substack{\beta \\ (\alpha;\alpha') \leftrightarrow (\beta;\beta')}} (\sigma_\beta - \sigma_\alpha + 2r) \cdot (T \circ \omega_{\alpha,\alpha'}^{\beta,\beta'}(X))$$

$$= \frac{1}{2} \sum_{\substack{\beta \\ (\alpha;\alpha') \leftrightarrow (\beta;\beta')}} (\sigma_\beta - \sigma_\alpha + 2r) t_{\beta,\beta'} \cdot (R_{\beta,\beta'} \circ \omega_{\alpha,\alpha'}^{\beta,\beta'}(X)).$$

Similarly we use (2-2) to obtain for the right-hand side

$$\text{proj}_{\mathcal{E}'(\beta')} \circ \tau_{\xi',r'}(X) \circ T|_{\mathcal{E}(\alpha;\alpha')} = \tfrac{1}{2}(\sigma_{\beta'}' - \sigma_{\alpha'}' + 2r') t_{\alpha,\alpha'} \cdot (\omega_{\alpha'}^{\beta'}(X) \circ R_{\alpha,\alpha'}).$$

Inserting both expressions into the initial equation (3-4) we obtain:

Theorem 3.4. *Assume* (MF1) *and* (MF2) *and fix* $R_{\alpha,\alpha'} : \mathcal{E}(\alpha; \alpha') \to \mathcal{E}'(\alpha')$ *as in Section 3A. Let* $\mathcal{V} \subseteq \mathcal{E}$ *be a* $(\mathfrak{g}', K')$-*submodule of* $(\pi_{\xi,r})_{\text{HC}}$. *A linear map* $T : \mathcal{V} \to \mathcal{E}'$ *is an intertwining operator for* $\pi_{\xi,r}$ *and* $\tau_{\xi',r'}$ *if and only if*

$$T|_{\mathcal{E}(\alpha;\alpha')} = t_{\alpha,\alpha'} \cdot R_{\alpha,\alpha'} \quad \textit{for all } 0 \neq \mathcal{E}(\alpha; \alpha') \subseteq \mathcal{V},$$

and for all $0 \neq \mathcal{E}(\alpha; \alpha') \subseteq \mathcal{V}$ and $\mathcal{E}'(\beta') \neq 0$ we have

$$(3\text{-}5) \qquad \sum_{\substack{\beta \\ (\alpha;\alpha') \leftrightarrow (\beta;\beta')}} (\sigma_\beta - \sigma_\alpha + 2r) t_{\beta,\beta'} \cdot (R_{\beta,\beta'} \circ \omega_{\alpha,\alpha'}^{\beta,\beta'})$$
$$= (\sigma_{\beta'}' - \sigma_{\alpha'}' + 2r') t_{\alpha,\alpha'} \cdot (\omega_{\alpha'}^{\beta'} \circ R_{\alpha,\alpha'}).$$

Remark 3.5. Through the formulation of Theorem 3.4 for any submodule $\mathcal{V}$ of $(\pi_{\xi,r})_{\mathrm{HC}}$ one can also use (3-5) to describe intertwining operators from subquotients of $(\pi_{\xi,r})_{\mathrm{HC}}$ to $(\tau_{\xi',r'})_{\mathrm{HC}}$. In fact, if $\mathcal{V}' \subseteq \mathcal{V} \subseteq \mathcal{E}$ are $(\mathfrak{g}, K)$-submodules of $(\pi_{\xi,r})_{\mathrm{HC}}$ then any intertwining operator $\mathcal{V}/\mathcal{V}' \to \mathcal{E}'$ for the actions $\pi_{\xi,r}$ and $\tau_{\xi',r'}$ is given by an intertwining operator $\mathcal{V} \to \mathcal{E}'$ which vanishes on $\mathcal{V}'$.

A little more complicated is the description of intertwining operators into subquotients of $(\tau_{\xi',r'})_{\mathrm{HC}}$. Let $\mathcal{W}' \subseteq \mathcal{W} \subseteq \mathcal{E}'$ be $(\mathfrak{g}', K')$-submodules of $(\tau_{\xi',r'})_{\mathrm{HC}}$ and decompose $\mathcal{W} = \mathcal{W}' \oplus \mathcal{W}''$ as K'-modules. Then a close examination of the arguments above shows that any operator $\mathcal{V} \to \mathcal{W}/\mathcal{W}'$ which intertwines the actions of $\pi_{\xi,r}$ and $\tau_{\xi',r'}$ is given by a K'-intertwining linear map $T : \mathcal{V} \to \mathcal{W}''$ with $T|_{\mathcal{E}(\alpha;\alpha')} = t_{\alpha,\alpha'} \cdot R_{\alpha,\alpha'}$ such that the relations (3-5) hold for any $0 \neq \mathcal{E}(\alpha; \alpha') \subseteq \mathcal{V}$ and $0 \neq \mathcal{E}'(\beta') \subseteq \mathcal{W}''$. Note that $t_{\alpha,\alpha'} = 0$ whenever $\mathcal{E}'(\alpha') \nsubseteq \mathcal{W}''$.

3C. *Scalar identities.* To extract from (3-5) information on the constants $t_{\alpha,\alpha'}$ we have to transform it into a scalar identity. For this we assume additionally that

$$(\mathrm{MF3}) \qquad \dim \mathrm{Hom}_{K'}(\mathfrak{s}'_{\mathbb{C}} \otimes \alpha', \beta') \leq 1 \quad \text{for all } 0 \neq \mathcal{E}(\alpha; \alpha') \subseteq \mathcal{V}, \ \mathcal{E}(\beta') \neq 0.$$

This implies that the maps

$$\eta_{\alpha,\alpha'}^{\beta,\beta'} = R_{\beta,\beta'} \circ \omega_{\alpha,\alpha'}^{\beta,\beta'} : \mathfrak{s}_{\mathbb{C}} \otimes \mathcal{E}'(\alpha, \alpha') \to \mathcal{E}'(\beta')$$

are proportional to each other. If further the map

$$\eta_{\alpha,\alpha'}^{\beta'} = \omega_{\alpha'}^{\beta'} \circ R_{\alpha,\alpha'} : \mathfrak{s}'_{\mathbb{C}} \otimes \mathcal{E}'(\alpha; \alpha') \to \mathcal{E}'(\beta')$$

is nonzero then there exist constants $\lambda_{\alpha,\alpha'}^{\beta,\beta'} \neq 0$ such that

$$\eta_{\alpha,\alpha'}^{\beta,\beta'} = \lambda_{\alpha,\alpha'}^{\beta,\beta'} \eta_{\alpha,\alpha'}^{\beta'}.$$

We call $\lambda_{\alpha,\alpha'}^{\beta,\beta'}$ the *proportionality constants*. In this case (3-5) simplifies:

Corollary 3.6. *Under the multiplicity-freeness assumption* (MF3) *the identity* (3-5) *is equivalent to*

$$(3\text{-}6) \qquad \sum_{\substack{\beta \\ (\alpha;\alpha') \leftrightarrow (\beta;\beta')}} \lambda_{\alpha,\alpha'}^{\beta,\beta'} (\sigma_\beta - \sigma_\alpha + 2r) t_{\beta,\beta'} = (\sigma_{\beta'}' - \sigma_{\alpha'}' + 2r') t_{\alpha,\alpha'}.$$

Whereas the constants σ_α and $\sigma_{\alpha'}'$ are easy to calculate using the highest weights of α and α', see [Branson et al. 1996], we do not have a general method to find the

constants $\lambda^{\beta,\beta'}_{\alpha,\alpha'}$. Of course one can always try to compute the action of the cocycle on explicit K-finite vectors and decompose the result, but this turns out to be quite involved already in low-rank cases. However, in some special cases the following information is enough to determine $\lambda^{\beta,\beta'}_{\alpha,\alpha'}$:

Lemma 3.7. *Assume that the elements $H \in \mathfrak{a}$ and $H' \in \mathfrak{a}'$ coincide. Let $\mathcal{E}(\alpha;\alpha') \neq 0$ and $\mathcal{E}'(\beta') \neq 0$ and assume that $R_{\alpha,\alpha'} = R_{\beta,\beta'} = \mathrm{rest}$ for all β with $(\alpha;\alpha') \leftrightarrow (\beta;\beta')$. Then*

$$\sum_{\substack{\beta \\ (\alpha;\alpha') \leftrightarrow (\beta;\beta')}} \lambda^{\beta,\beta'}_{\alpha,\alpha'} = 1, \qquad \sum_{\substack{\beta \\ (\alpha;\alpha') \leftrightarrow (\beta;\beta')}} (\sigma_\beta - \sigma_\alpha)\lambda^{\beta,\beta'}_{\alpha,\alpha'} = \sigma'_{\beta'} - \sigma'_{\alpha'} + 2(\rho - \rho').$$

Here ρ and ρ' are identified with the numbers $\rho(H)$ and $\rho'(H')$.

Proof. For the first identity we note that $H = H'$ implies $\omega(X)|_{K'} = \omega'(X)$ for all $X \in \mathfrak{s}'$. Hence

$$R_{\beta,\beta'} \circ \omega(X) = \omega'(X) \circ R_{\alpha,\alpha'} \quad \text{for all } X \in \mathfrak{g}',$$

which implies

$$\eta^{\beta'}_{\alpha,\alpha'} = \sum_{\substack{\beta \\ (\alpha;\alpha') \leftrightarrow (\beta;\beta')}} \eta^{\beta,\beta'}_{\alpha,\alpha'}$$

and the claimed identity follows. For the second identity note that for $r + \rho = r' + \rho'$ the restriction operator $\mathrm{rest} : \mathcal{E} \to \mathcal{E}'$ is intertwining for $\pi_{\xi,r}$ and $\tau_{\xi',r'}$. Hence the identity (3-6) is satisfied with $t_{\alpha,\alpha'} = 1$ for all $\mathcal{E}(\alpha;\alpha') \neq 0$. Eliminating r and r' gives the desired formula. $\qquad\square$

Remark 3.8. The knowledge of any intertwining operator $T : (\pi_{\xi,r})_{\mathrm{HC}} \to (\tau_{\xi',r'})_{\mathrm{HC}}$ and the corresponding numbers $t_{\alpha,\alpha'}$ provides an additional identity for the constants $\lambda^{\beta,\beta'}_{\alpha,\alpha'}$ just as in the proof of Lemma 3.7 for the restriction operator $T = \mathrm{rest}$ with $r + \rho = r' + \rho'$ and $t_{\alpha,\alpha'} = 1$.

3D. *Automatic continuity.* In this section we study the question of whether $(\mathfrak{g}', K')$-intertwining operators $(\pi_{\xi,r})_{\mathrm{HC}} \to (\tau_{\xi',r'})_{\mathrm{HC}}$ between Harish-Chandra modules extend to G'-intertwining operators $\pi_{\xi,r} \to \tau_{\xi',r'}$ between the smooth representations, i.e., whether the natural injective map

$$\mathrm{Hom}_{G'}(\pi_{\xi,r}|_{G'}, \tau_{\xi',r'}) \to \mathrm{Hom}_{(\mathfrak{g}',K')}((\pi_{\xi,r})_{\mathrm{HC}}|_{(\mathfrak{g}',K')}, (\tau_{\xi',r'})_{\mathrm{HC}})$$

is an isomorphism. It is expected, see [Kobayashi 2014, Remark 10.2 (4)], that this is true if the space $(G \times G')/\mathrm{diag}(G')$ is real spherical. Statements of this type are also known as "automatic continuity theorems" since they imply continuity with respect to the smooth topologies of every intertwining operator between the

algebraic Harish-Chandra modules. We provide a criterion to show automatic continuity in the context of this paper.

Fix a Haar measure dk' on K'. Then the nondegenerate bilinear pairing

$$\mathcal{E}(K'; \xi'|_{M' \cap K'}) \times \mathcal{E}(K'; \xi'^{\vee}|_{M' \cap K'}) \to \mathbb{C}, \ (f_1, f_2) \mapsto \int_{K'} \langle f_1(k'), f_2(k') \rangle \, dk'$$

is invariant under $\tau_{\xi',r'} \otimes \tau_{\xi'^{\vee},-r'}$ for any $r' \in \mathbb{C}$, where $\xi'^{\vee}$ denotes the contragredient representation of ξ' on the dual space $V_{\xi'}^{\vee}$. Using this pairing we identify $\tau_{\xi',r'}$ with a subrepresentation of the contragredient representation $\tau_{\xi'^{\vee},-r'}^{\vee}$ of $\tau_{\xi'^{\vee},-r'}$, which is realized on the topological dual space $\mathcal{E}(K'; \xi'^{\vee}|_{M' \cap K'})^{\vee}$ carrying the weak-$\star$ topology.

Lemma 3.9. *Every continuous G'-intertwining operator $T : \pi_{\xi,r} \to \tau_{\xi'^{\vee},-r'}^{\vee}$ maps into $\tau_{\xi',r'}$ and defines a continuous G'-intertwining operator $T : \pi_{\xi,r} \to \tau_{\xi',r'}$.*

Proof. Let $T : \mathcal{E}(K; \xi|_{M \cap K}) \to \mathcal{E}(K'; \xi'^{\vee}|_{M' \cap K'})^{\vee}$ be a continuous linear operator which is G'-intertwining for $\pi_{\xi,r}$ and $\tau_{\xi'^{\vee},-r'}^{\vee}$. Then T induces a continuous linear functional

$$\overline{T} : \mathcal{E}(K; \xi|_{M \cap K}) \widehat{\otimes} \mathcal{E}(K'; \xi'^{\vee}|_{M' \cap K'}) \to \mathbb{C},$$

which is invariant under the diagonal action of K'. The left-hand side is naturally isomorphic to $\mathcal{E}(K \times K'; (\xi \otimes \xi'^{\vee})|_{(M \cap K) \times (M' \cap K')})$. Composing with the surjective continuous linear operator

$$\flat : C^{\infty}(K \times K'; V_{\xi} \otimes V_{\xi'}^{\vee}) \to \mathcal{E}(K \times K'; (\xi \otimes \xi'^{\vee})|_{(M \cap K) \times (M' \cap K')}),$$

$$\flat F(k, k') = \int_{M \cap K} \int_{M' \cap K'} (\xi(m) \otimes \xi'(m')^{\vee}) F(km, k'm') \, dm' \, dm,$$

we obtain a functional

$$K_T := \overline{T} \circ \flat : C^{\infty}(K \times K'; V_{\xi} \otimes V_{\xi'}^{\vee}) \to \mathbb{C},$$

i.e., a distribution on $K \times K'$ with values in $V_{\xi} \otimes V_{\xi'}^{\vee}$. (This is basically the Schwartz kernel of the operator T, avoiding distribution sections of vector bundles.) The distribution K_T is invariant under the diagonal action of K' from the left and equivariant under the action of $(M \cap K) \times (M' \cap K')$ from the right. We define a distribution $\widetilde{K}_T$ on K with values in $V_{\xi} \otimes V_{\xi'}^{\vee}$, i.e., a continuous linear functional on $C^{\infty}(K; V_{\xi} \otimes V_{\xi'}^{\vee})$, by

$$\langle \widetilde{K}_T, \phi \rangle := \langle K_T(x, x'), \phi(x'^{-1}x) \rangle.$$

Then for $\phi \in \mathcal{E}(K; \xi|_{M\cap K})$ and $\psi \in \mathcal{E}(K'; \xi'^{\vee}|_{M'\cap K'})$ we have

$$\langle T\phi, \psi \rangle = \langle K_T, \phi \otimes \psi \rangle = \int_{K'} \langle K_T(x, x'), \phi(k'x) \otimes \psi(k'x') \rangle \, dk'$$

$$= \left\langle K_T(x, x'), \int_{K'} \phi(k'x) \otimes \psi(k'x') \, dk' \right\rangle$$

$$= \left\langle K_T(x, x'), \int_{K'} \phi(k'x'^{-1}x) \otimes \psi(k') \, dk' \right\rangle$$

$$= \int_{K'} \langle K_T(x, x'), \phi(k'x'^{-1}x) \otimes \psi(k') \rangle \, dk' = \int_{K'} \langle \widetilde{K}_T, \phi(k' \cdot) \otimes \psi(k') \rangle \, dk'.$$

This implies that for any $\lambda \in V_{\xi'}^{\vee}$

$$\langle \lambda, T\phi(k') \rangle = \langle \widetilde{K}_T, \phi(k' \cdot) \otimes \lambda \rangle,$$

which shows that $T\phi \in C^{\infty}(K'; V_{\xi'})$. That $T\phi \in \mathcal{E}(K'; \xi'|_{M'\cap K'})$ easily follows from the equivariance property of $\widetilde{K}_T$ with respect to $M \cap K$ and $M' \cap K'$. Finally, continuity of the thus defined operator $T : \pi_{\xi,r} \to \tau_{\xi',r'}$ follows from the continuity of the functional $\widetilde{K}_T$ on $C^{\infty}(K; V_{\xi} \otimes V_{\xi'}^{\vee})$ and the proof is complete. $\qquad\square$

Fix invariant inner products on the representation $\xi|_{M\cap K}$, resp. $\xi'|_{M'\cap K'}$, and let $\|\cdot\|$, resp. $\|\cdot\|'$, denote the corresponding L^2-norm on $L^2(K \times_{M\cap K} \xi)$, resp. $L^2(K' \times_{M'\cap K'} \xi')$. These norms induce norms on each K'-type $\mathcal{E}(\alpha; \alpha')$ resp. $\mathcal{E}'(\alpha')$. Write $\|R_{\alpha,\alpha'}\|_{L^2 \to L^2}$ for the operator norm of $R_{\alpha,\alpha'} : \mathcal{E}(\alpha; \alpha') \to \mathcal{E}'(\alpha')$ with respect to the L^2-norms.

For any $F \in L^2(K \times_{M\cap K} \xi)$ write

$$F = \sum_{\alpha \in \widehat{K}} F_\alpha,$$

with $F_\alpha \in \mathcal{E}(\alpha)$. Then the sequence $\{\|F_\alpha\|\}_\alpha$ belongs to $\ell^2(\widehat{K})$, the space of square-summable sequences. We identify the set $\widehat{K}$, resp. $\widehat{K}'$, with the corresponding weight lattice so that it becomes a subset of a finite-dimensional vector space. Denote by $|\cdot|$, resp. $|\cdot|'$, a norm on this finite-dimensional vector space. It is known that $F \in \mathcal{E}(K; \xi_{M\cap K})$ if and only if the sequence $\{\|F_\alpha\|\}_\alpha$ belongs to $s(\widehat{K})$, the space of rapidly decreasing sequences, i.e., those that are still bounded if multiplied with any power $|\alpha|^N$. Moreover, $\mathcal{E}(K; \xi^{\vee}|_{M\cap K})^{\vee}$ is identified with all formal expansions $F = \sum_\alpha F_\alpha$, where $\{\|F_\alpha\|\}_\alpha$ belongs to $s'(\widehat{K})$, the space of tempered sequences, i.e., those that grow at most at the rate of $|\alpha|^N$ for some $N \in \mathbb{N}$.

Proposition 3.10. *A $(\mathfrak{g}', K')$-intertwining operator $T : (\pi_{\xi,r})_{\mathrm{HC}} \to (\tau_{\xi',r'})_{\mathrm{HC}}$ with $T|_{\mathcal{E}(\alpha;\alpha')} = t_{\alpha,\alpha'} \cdot R_{\alpha,\alpha'}$ extends to a continuous G'-intertwining operator $\pi_{\xi,r} \to \tau_{\xi',r'}$ if both $t_{\alpha,\alpha'}$ and $\|R_{\alpha,\alpha'}\|_{L^2 \to L^2}$ are of at most polynomial growth in α and α'.*

Proof. By Lemma 3.9 it suffices to show that T extends to a continuous G'-intertwining operator $\pi_{\xi,r} \to (\tau_{\xi'^\vee,-r'})'$. Let $F \in \pi_{\xi,r}$; then $F = \sum_\alpha F_\alpha$ with $\{\|F_\alpha\|\}_\alpha$ a sequence in $s(\widehat{K})$. We have $TF = \sum_{\alpha'}(TF)_{\alpha'}$ with

$$(TF)_{\alpha'} = \sum_\alpha t_{\alpha,\alpha'} \cdot R_{\alpha,\alpha'} F_\alpha.$$

By the assumptions

$$|t_{\alpha,\alpha'}| \leq C_1(1 + |\alpha| + |\alpha'|)^{N_1},$$

$$\|R_{\alpha,\alpha'}\|_{L^2 \to L^2} \leq C_2(1 + |\alpha| + |\alpha'|)^{N_2}$$

for some $C_1, C_2 > 0$ and $N_1, N_2 \in \mathbb{N}$. Further, since $\|F_\alpha\| \in s(\widehat{K})$, for every $N \in \mathbb{N}$ there exists $C > 0$ such that $\|F_\alpha\| \leq C(1 + |\alpha|)^{-N}$. Hence, we have for any α'

$$\|(TF)_{\alpha'}\|' \leq CC_1C_2 \sum_\alpha (1 + |\alpha| + |\alpha'|)^{N_1+N_2}(1 + |\alpha|)^{-N}.$$

Choosing N large enough, this is uniformly bounded by a constant times $(1 + |\alpha'|)^{N_1+N_2}$, and hence $\|(TF)_{\alpha'}\|' \in s'(\widehat{K'})$ so that $TF \in \mathcal{E}(K'; \xi'^\vee|_{M'\cap K'})^\vee$. This shows that T extends to a G'-intertwining operator $\pi_{\xi,r} \to \tau^\vee_{\xi'^\vee,-r'}$. Continuity of this operator also follows by the above estimates. $\qquad\square$

4. Rank-one orthogonal groups

In this section we apply our method to classify symmetry-breaking operators for rank-one orthogonal groups. Let $n \geq 3$ and consider the indefinite orthogonal group $G = \mathrm{O}(1, n)$ of $(n+1) \times (n+1)$ real matrices leaving the standard bilinear form on $\mathbb{R}^{n+1}$ of signature $(1, n)$ invariant. The subgroup $G' \subseteq G$ of matrices fixing the last standard basis vector e_{n+1} is isomorphic to $\mathrm{O}(1, n-1)$.

4A. *K-types.* We fix $K = \mathrm{O}(1) \times \mathrm{O}(n)$ and choose

$$H = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & \mathbf{0}_{n-1} \end{pmatrix}$$

so that $P = MAN$, with $M = \Delta\,\mathrm{O}(1) \times \mathrm{O}(n-1)$, where $\Delta\,\mathrm{O}(1) = \{\mathrm{diag}(x, x) : x \in \mathrm{O}(1)\}$. Note that $\rho = \frac{n-1}{2}$. Then K acts transitively on S^{n-1} via $\mathrm{diag}(\varepsilon, k) \cdot x = \varepsilon k x$, $\varepsilon \in \mathrm{O}(1)$, $k \in \mathrm{O}(n)$, $x \in S^{n-1}$, and M is the stabilizer subgroup of the first standard basis vector $e_1 \in S^{n-1}$, whence $K/M \cong S^{n-1}$. The subgroup $G' = \mathrm{O}(1, n-1)$ is embedded into G such that $K' = \mathrm{O}(1) \times \mathrm{O}(n-1)$ and $P' = G' \cap P = M'A'N'$, with $A' = A$ and $M' = \Delta\,\mathrm{O}(1) \times \mathrm{O}(n-2)$. Then $K'/M' = S^{n-2}$, viewed as the equator in $K/M = S^{n-1} \subseteq \mathbb{R}^n$ given by $x_n = 0$. Further we have $\nu = \nu'$ and $\rho' = \frac{n-2}{2}$.
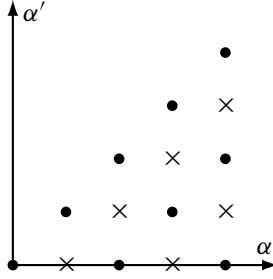
Legend: • K'-types $\mathcal{E}(\alpha; \alpha')$ with $\alpha - \alpha' \in 2\mathbb{Z}$
         × K'-types $\mathcal{E}(\alpha; \alpha')$ with $\alpha - \alpha' \in 2\mathbb{Z} + 1$

Figure 1

Let $\xi = \mathbf{1}$, $\xi' = \mathbf{1}$ be the trivial representations of M and M' and abbreviate $\pi_r = \pi_{\xi,r}$ and $\tau_{r'} = \tau_{\xi',r'}$. As K-modules, resp. K'-modules, we have

$$\mathcal{E} = \bigoplus_{\alpha=0}^{\infty} \underbrace{\mathrm{sgn}^\alpha \boxtimes \mathcal{H}^\alpha(\mathbb{R}^n)}_{\mathcal{E}(\alpha)}, \quad \mathcal{E}' = \bigoplus_{\alpha'=0}^{\infty} \underbrace{\mathrm{sgn}^{\alpha'} \boxtimes \mathcal{H}^{\alpha'}(\mathbb{R}^{n-1})}_{\mathcal{E}'(\alpha')},$$

so that (MF1) is satisfied. Further, each K-type decomposes by (B-1) into K'-types as

$$(\mathrm{sgn}^\alpha \boxtimes \mathcal{H}^\alpha(\mathbb{R}^n))|_{K'} \simeq \bigoplus_{0 \leq \alpha' \leq \alpha} (\mathrm{sgn}^\alpha \boxtimes \mathcal{H}^{\alpha'}(\mathbb{R}^{n-1})),$$

and hence (MF2) holds. Comparing the sign representations of the O(1)-factor of K' we find that $\mathrm{Hom}_{K'}(\mathcal{E}(\alpha)|_{K'}, \mathcal{E}'(\alpha')) \neq 0$ if and only if $\alpha - \alpha' \in 2\mathbb{Z}$. In this case formulas (B-2) and (A-2) show that the restriction operator

$$R_{\alpha,\alpha'} = \mathrm{rest}\,|_{\mathcal{E}(\alpha;\alpha')} : \mathcal{E}(\alpha; \alpha') \to \mathcal{E}'(\alpha')$$

is an isomorphism. Hence the restriction $T_{\alpha,\alpha'} = T|_{\mathcal{E}(\alpha;\alpha')}$ of a K'-intertwining operator $T : \mathcal{E} \to \mathcal{E}'$ is given by $T_{\alpha,\alpha'} = t_{\alpha,\alpha'} R_{\alpha,\alpha'}$ for $\alpha - \alpha' \in 2\mathbb{N}$ and $T_{\alpha,\alpha'} = 0$ else. The K- and K'-types are illustrated in Figure 1.

4B. *Proportionality constants.* The eigenvalues of the spectrum-generating operator on the K-types are simply the eigenvalues of the Laplacian on S^{n-1} and given by, see [Branson et al. 1996, Section 3.a],

$$\sigma_\alpha = \alpha(\alpha + n - 2), \quad \sigma'_{\alpha'} = \alpha'(\alpha' + n - 3).$$

We identify $\mathfrak{s} \cong \mathbb{R}^n$ via

$$\mathbb{R}^n \to \mathfrak{s}, \quad y \mapsto X_y = \begin{pmatrix} 0 & y^t \\ y & \mathbf{0}_n \end{pmatrix}.$$

Then $\mathfrak{s}' \cong \mathbb{R}^{n-1}$, embedded in $\mathbb{R}^n$ as the first $n-1$ coordinates. Since $\mathfrak{s}'_{\mathbb{C}} \simeq \mathbb{C}^{n-1}$ is a weight multiplicity-free K'-module, (MF3) holds and we can use Corollary 3.6. To compute the proportionality constants $\lambda_{\alpha,\alpha'}^{\beta,\beta'}$ we use Lemma 3.7, which applies to this situation because $H = H'$ and $R_{\alpha,\alpha'} = \text{rest}$. The cocycle ω is given by

$$\omega(X_y)(x) = y^t x, \quad x \in S^{n-1}, \ y \in \mathbb{R}^n.$$

Using (B-4) it is easy to see that for fixed $0 \le \alpha' \le \alpha$

$$(\alpha; \alpha') \leftrightarrow (\beta; \beta') \quad \Longleftrightarrow \quad |\alpha - \beta| = |\alpha' - \beta'| = 1.$$

By Lemma 3.7 we have the following equations for $\lambda_{\alpha,\alpha'}^{\beta,\beta'}$: for $\beta' = \alpha' + 1$ we obtain

$$\lambda_{\alpha,\alpha'}^{\alpha+1,\alpha'+1} + \lambda_{\alpha,\alpha'}^{\alpha-1,\alpha'+1} = 1,$$
$$(2\alpha + n - 1)\lambda_{\alpha,\alpha'}^{\alpha+1,\alpha'+1} - (2\alpha + n - 3)\lambda_{\alpha,\alpha'}^{\alpha-1,\alpha'+1} = 2\alpha' + n - 1,$$

which gives

$$\lambda_{\alpha,\alpha'}^{\alpha+1,\alpha'+1} = \frac{\alpha + \alpha' + n - 2}{2\alpha + n - 2}, \quad \lambda_{\alpha,\alpha'}^{\alpha-1,\alpha'+1} = \frac{\alpha - \alpha'}{2\alpha + n - 2},$$

and for $\beta' = \alpha' - 1$ we get

$$\lambda_{\alpha,\alpha'}^{\alpha+1,\alpha'-1} + \lambda_{\alpha,\alpha'}^{\alpha-1,\alpha'-1} = 1,$$
$$(2\alpha + n - 1)\lambda_{\alpha,\alpha'}^{\alpha+1,\alpha'-1} - (2\alpha + n - 3)\lambda_{\alpha,\alpha'}^{\alpha-1,\alpha'-1} = -2\alpha' - n + 5,$$

implying

$$\lambda_{\alpha,\alpha'}^{\alpha+1,\alpha'-1} = \frac{\alpha - \alpha' + 1}{2\alpha + n - 2}, \quad \lambda_{\alpha,\alpha'}^{\alpha-1,\alpha'-1} = \frac{\alpha + \alpha' + n - 3}{2\alpha + n - 2}.$$

We remark that the constants $\lambda_{\alpha,\alpha'}^{\beta,\beta'}$ can in this case also be obtained by computing the action of $\omega(X)$ on explicit K-finite vectors using (B-2) and recurrence relations for the Gegenbauer polynomials. With the explicit form of the constants $\lambda_{\alpha,\alpha'}^{\beta,\beta'}$ Corollary 3.6 now provides the following characterization of symmetry-breaking operators:

Theorem 4.1. *An operator $T : \mathcal{E} \to \mathcal{E}'$ is intertwining for π_r and $\tau_{r'}$ if and only if*

$$T|_{\mathcal{E}(\alpha;\alpha')} = \begin{cases} t_{\alpha,\alpha'} \cdot \text{rest}\,|_{\mathcal{E}(\alpha;\alpha')} & \text{for } \alpha - \alpha' \in 2\mathbb{Z}, \\ 0 & \text{for } \alpha - \alpha' \in 2\mathbb{Z} + 1, \end{cases}$$

with numbers $t_{\alpha,\alpha'}$ satisfying

(4-1) $(2\alpha+n-2)(2r'+2\alpha'+n-2)t_{\alpha,\alpha'} = (\alpha+\alpha'+n-2)(2r+2\alpha+n-1)t_{\alpha+1,\alpha'+1}$

$$+ (\alpha-\alpha')(2r-2\alpha-n+3)t_{\alpha-1,\alpha'+1}$$
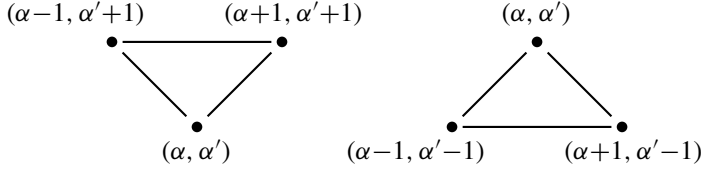
Figure 2. The relations (4-1) and (4-2).

and

$$(4\text{-}2) \quad (2\alpha+n-2)(2r'-2\alpha'-n+4)t_{\alpha,\alpha'} = (\alpha-\alpha'+1)(2r+2\alpha+n-1)t_{\alpha+1,\alpha'-1}$$
$$+(\alpha+\alpha'+n-3)(2r-2\alpha-n+3)t_{\alpha-1,\alpha'-1}.$$

We view these two relations as triangles connecting three vertices in the K-type picture (see Figure 2).

Note that if $r \notin -\rho-\mathbb{N}$ then $2r+2\alpha+n-1 \neq 0$ for all α and hence one can define $t_{\alpha+1,\alpha'+1}$ in terms of $t_{\alpha,\alpha'}$ and $t_{\alpha-1,\alpha'+1}$ using (4-1) and do similarly for $t_{\alpha+1,\alpha'-1}$ using (4-2). If $r = -\rho-i \in -\rho-\mathbb{N}$ and $\alpha = i$ the coefficient $(2r+2\alpha+n-1)$ vanishes and (4-1) and (4-2) reduce to identities involving only two terms. We indicate this by drawing a vertical line between i and $i+1$ indicating that one cannot "step" from the left-hand side to the right-hand side (see Figure 3). Similarly we have that if $r' \notin -\rho'-\mathbb{N}$ then $2r'+2\alpha'+n-2 \neq 0$ for all α' and we can define $t_{\alpha,\alpha'}$ in terms of $t_{\alpha\pm1,\alpha'+1}$ using (4-1). If $r' = -\rho'-j \in -\rho'-\mathbb{N}$ and $\alpha' = j$ we obtain a horizontal line between j and $j+1$ as barrier, indicating that we cannot step from the part above this line to the part below. Note that if there is a vertical, resp. horizontal, barrier like this the coefficient $(2r-2\alpha-n+3)$, resp. $(2r'-2\alpha'-n+4)$, never vanishes and one can step in the other direction, namely from right to left, resp. from the part below the line to the part above.

4C. *Multiplicities.* The $(\mathfrak{g}, K)$-module $(\pi_r)_{\mathrm{HC}}$ is reducible if and only if $r \in \pm(\rho+\mathbb{N})$. More precisely, for $r = -\rho-i$ the module $(\pi_r)_{\mathrm{HC}}$ contains a unique nontrivial finite-dimensional $(\mathfrak{g}, K)$-submodule $\mathcal{F}(i) \subseteq \mathcal{E}$ with K-types $\mathcal{E}(\alpha)$, $0 \leq \alpha \leq i$. Its quotient $\mathcal{T}(i) = \mathcal{E}/\mathcal{F}(i)$ is irreducible and can be identified with the unique nontrivial $(\mathfrak{g}, K)$-submodule of $(\pi_{-r})_{\mathrm{HC}}$. Similarly we denote for $r' = -\rho'-j$ by
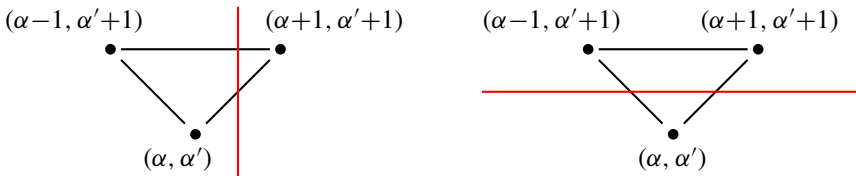

Figure 3. Barriers for $r = -\rho-i$ and $r' = -\rho'-j$.

$\mathcal{F}'(j)$ the unique finite-dimensional $(\mathfrak{g}', K')$-submodule of $(\tau_{r'})_{\mathrm{HC}}$ and by $\mathcal{T}'(j)$ its irreducible quotient. Let

$$L_{\mathrm{even}} = \{(r, r') : r = -\rho - i, \ r' = -\rho' - j, \ i - j \in 2\mathbb{N}\},$$

$$L_{\mathrm{odd}} = \{(r, r') : r = -\rho - i, \ r' = -\rho' - j, \ i - j \in 2\mathbb{N} + 1\}.$$

This notation agrees with the notation used in [Kobayashi and Speh 2015].

Theorem 4.2. (1) *The multiplicities between spherical principal series of G and G' are given by*

$$m((\pi_r)_{\mathrm{HC}}, (\tau_{r'})_{\mathrm{HC}}) = \begin{cases} 1 & \text{for } (r, r') \in \mathbb{C}^2 \setminus L_{\mathrm{even}}, \\ 2 & \text{for } (r, r') \in L_{\mathrm{even}}. \end{cases}$$

(2) *For $i, j \in \mathbb{N}$ the multiplicities $m(\mathcal{V}, \mathcal{W})$ between subquotients are given by*

$\mathcal{V}\downarrow \ \mathcal{W}\to$	$\mathcal{F}'(j)$	$\mathcal{T}'(j)$
$\mathcal{F}(i)$	1	0
$\mathcal{T}(i)$	0	1

for $i - j \in 2\mathbb{N}$,

$\mathcal{V}\downarrow \ \mathcal{W}\to$	$\mathcal{F}'(j)$	$\mathcal{T}'(j)$
$\mathcal{F}(i)$	0	0
$\mathcal{T}(i)$	1	0

otherwise.

To prove Theorem 4.2 we study how the relations (4-1) and (4-2) determine the numbers $t_{\alpha,\alpha'}$. We first consider the diagonal $\alpha = \alpha'$. Relation (4-1) then simplifies to

(4-3) $$(2r' + 2\alpha + n - 2)t_{\alpha,\alpha} = (2r + 2\alpha + n - 1)t_{\alpha+1,\alpha+1}.$$

This immediately yields:

Lemma 4.3. (1) *For $(r, r') \in \mathbb{C}^2 \setminus (L_{\mathrm{even}} \cup L_{\mathrm{odd}})$ the space of diagonal sequences $(t_{\alpha,\alpha})_\alpha$ satisfying (4-3) has dimension 1. Any generator $(t_{\alpha,\alpha})_\alpha$ satisfies:*

(a) *For $r \notin -\rho - \mathbb{N}$, $r' \notin -\rho' - \mathbb{N}$,*

$$t_{\alpha,\alpha} \neq 0 \quad \text{for all } \alpha \in \mathbb{N}.$$

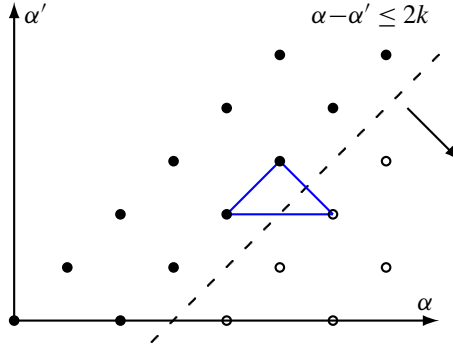(b) *For $r = -\rho - i \in -\rho - \mathbb{N}$, $r' \notin -\rho' - \mathbb{N}$,*

$$t_{\alpha,\alpha} = 0 \quad \text{for all } \alpha \leq i \qquad \text{and} \qquad t_{\alpha,\alpha} \neq 0 \quad \text{for all } \alpha > i.$$

(c) *For $r \notin -\rho - \mathbb{N}$, $r' = -\rho' - j \in -\rho' - \mathbb{N}$,*

$$t_{\alpha,\alpha} \neq 0 \quad \text{for all } \alpha \leq j \qquad \text{and} \qquad t_{\alpha,\alpha} = 0 \quad \text{for all } \alpha > j.$$

(d) *For $r = -\rho - i \in -\rho - \mathbb{N}$, $r' = -\rho' - j \in -\rho' - \mathbb{N}$, with $i < j$,*

$$t_{\alpha,\alpha} \neq 0 \quad \text{for all } i < \alpha \leq j \qquad \text{and} \qquad t_{\alpha,\alpha} = 0 \quad \text{else.}$$

Legend: • K'-types $\mathcal{E}(\alpha; \alpha')$ with $\alpha - \alpha' \le 2k$ ($t_{\alpha,\alpha'}$ already defined)
       ◦ K'-types $\mathcal{E}(\alpha; \alpha')$ with $\alpha - \alpha' > 2k$ ($t_{\alpha,\alpha'}$ yet to define)

Figure 4

(2) *For* $(r, r') = (-\rho - i, -\rho' - j) \in (L_{\text{even}} \cup L_{\text{odd}})$, *the space of diagonal sequences* $(t_{\alpha,\alpha})_\alpha$ *satisfying* (4-3) *has dimension* 2. *It has a basis* $(t'_{\alpha,\alpha})_\alpha$, $(t''_{\alpha,\alpha})_\alpha$ *with the properties*

$$t'_{\alpha,\alpha} \ne 0 \quad \text{for all } \alpha \le j, \qquad t'_{\alpha,\alpha} = 0 \quad \text{for all } \alpha > j,$$
$$t''_{\alpha,\alpha} = 0 \quad \text{for all } \alpha \le i, \qquad t''_{\alpha,\alpha} \ne 0 \quad \text{for all } \alpha > i.$$

Next we investigate how a diagonal sequence $(t_{\alpha,\alpha})_\alpha$ satisfying (4-3) can be extended to a sequence $(t_{\alpha,\alpha'})_{(\alpha,\alpha')}$ satisfying (4-1) and (4-2).

Lemma 4.4. *Let* $(r, r') \in \mathbb{C}^2 \setminus (L_{\text{even}} \cup L_{\text{odd}})$. *Then every diagonal sequence* $(t_{\alpha,\alpha})_\alpha$ *satisfying* (4-3) *has a unique extension to a sequence* $(t_{\alpha,\alpha'})_{(\alpha,\alpha')}$ *satisfying* (4-1) *and* (4-2).

Proof. Step 1. We first treat the case $r \notin -\rho - \mathbb{N}$. In this case the coefficients $(2r + 2\alpha + n - 1)$ in (4-1) and (4-2) never vanish. We now extend the diagonal sequence $(t_{\alpha,\alpha})_\alpha$ inductively to a sequence $(t_{\alpha,\alpha'})_{\alpha-\alpha' \le 2k}$ with $k \in \mathbb{N}$ which satisfies (4-1) for (α, α') with $\alpha - \alpha' \le 2k$ and (4-2) for (α, α') with $\alpha - \alpha' \le 2k - 2$ as visualized in Figure 4 (i.e., the two relations hold whenever the corresponding triangles in Figure 2 are contained in the region $\alpha - \alpha' \le 2k$). For $k = 0$ the diagonal sequence we start with satisfies these assumptions. For the induction step $k \to k + 1$ let $\alpha - \alpha' = 2k$ and define $t_{\alpha+1,\alpha'-1}$ and $t_{\alpha+2,\alpha'}$ using (4-2) (the blue triangles in Figure 5) in terms of $t_{\alpha-1,\alpha'-1}$, $t_{\alpha,\alpha'}$ and $t_{\alpha+1,\alpha'+1}$. This is possible, because $2r + 2\alpha + n - 1 \ne 0$ for all α and hence the corresponding coefficients in (4-2) are nonzero. Now we have to prove that (4-1) holds for $(\alpha + 1, \alpha' - 1)$ (the red triangle). This can be done by an elementary calculation using the blue triangles that are by definition valid as well as the green triangles that are valid by the induction assumption. Hence this extends the diagonal sequence $(t_{\alpha,\alpha})_\alpha$ to a
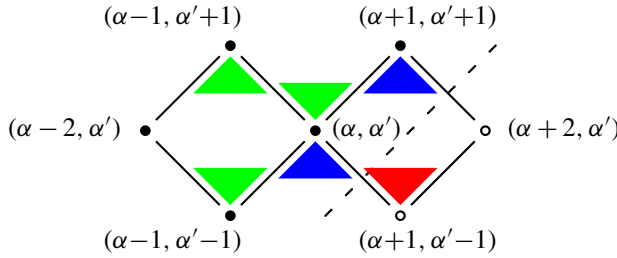
$(\alpha-1, \alpha'+1)$ $(\alpha+1, \alpha'+1)$

$(\alpha-2, \alpha')$ (α, α') $(\alpha+2, \alpha')$

$(\alpha-1, \alpha'-1)$ $(\alpha+1, \alpha'-1)$

Figure 5

sequence $(t_{\alpha,\alpha'})_{0\le\alpha'\le\alpha}$ satisfying (4-1) and (4-2). Since the relations were used to extend the diagonal sequence this extension is unique.

<u>Step 2</u>. Next assume $r = -\rho - i \in -\rho - \mathbb{N}$ and $r' \notin -\rho' - \mathbb{N}$. Then the coefficient $(2r + 2\alpha + n - 1)$ vanishes if and only if $\alpha = i$. We can therefore use the technique in Step 1 to extend the upper part $(t_{\alpha,\alpha})_{\alpha>i}$ of the diagonal sequence to a sequence $(t_{\alpha,\alpha'})_{i<\alpha'\le\alpha}$ in the region $\alpha' > i$. Next we extend the sequence $(t_{\alpha,\alpha})_{\alpha'>i}$ inductively to a sequence $(t_{\alpha,\alpha'})_{\alpha'>i-k}$ with $k = 0, \ldots, i+1$ which satisfies (4-1) for (α, α') with $\alpha' > i - k$ and (4-2) for (α, α') with $\alpha' > i - k + 1$ as visualized in Figure 6 (i.e., the two relations hold whenever the corresponding triangles in Figure 2 are contained in the region $\alpha' > i - k$). For $k = 0$ the sequence we obtained using Step 1 satisfies these assumptions by Step 1. For the induction step $k \to k+1$ let $\alpha' = i - k + 1$ and define $t_{\alpha-1,\alpha'-1}$ and $t_{\alpha+1,\alpha'-1}$ using (4-1) (the blue triangles in Figure 7) in terms of $t_{\alpha-2,\alpha'}$, $t_{\alpha,\alpha'}$ and $t_{\alpha+2,\alpha'}$. This is possible, because $r' \notin -\rho' - \mathbb{N}$ and hence the corresponding coefficient $(2r' + 2\alpha' + n - 2)$ in (4-1) never vanishes. Now we have to prove that (4-2) holds for (α, α') (the red triangle) which is done in a similar fashion as in Step 1 using the green triangle. This finishes Step 2.
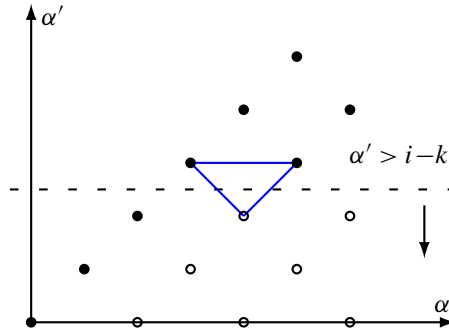
Legend: • K'-types $\mathcal{E}(\alpha; \alpha')$ with $\alpha' > i - k$ ($t_{\alpha,\alpha'}$ already defined)
          ○ K'-types $\mathcal{E}(\alpha; \alpha')$ with $\alpha' \le i - k$ ($t_{\alpha,\alpha'}$ yet to define)

Figure 6
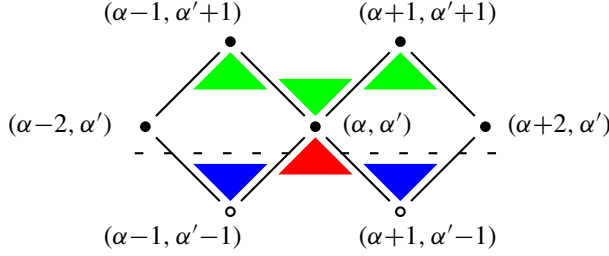
Figure 7

<u>Step 3</u>. Now let $r = -\rho - i \in -\rho - \mathbb{N}$ and $r' = -\rho' - j \in -\rho' - \mathbb{N}$ with $i, j \in \mathbb{N}$, $j > i$. Note that to carry out Step 2 we only need that $(2r' + 2\alpha' + n - 2) \neq 0$ for $\alpha' \leq i$. This is satisfied since

$$2r' + 2\alpha' + n - 2 = 2(\alpha' - j) < 2(\alpha' - i) \leq 0$$

by assumption. Hence the technique in Step 2 carries over to this case. $\square$

Lemma 4.5. *Let* $(r, r') = (-\rho - i, -\rho' - j)$, $i, j \in \mathbb{N}$:

(1) *For* $(r, r') \in L_{\text{even}}$ *every diagonal sequence* $(t_{\alpha,\alpha})_\alpha$ *satisfying* (4-3) *has a unique extension to a sequence* $(t_{\alpha,\alpha'})_{(\alpha,\alpha')}$ *satisfying* (4-1) *and* (4-2).

(2) *For* $(r, r') \in L_{\text{odd}}$ *any sequence* $(t_{\alpha,\alpha'})_{\alpha,\alpha'}$ *satisfying* (4-1) *and* (4-2) *has the property* $t_{\alpha,\alpha'} = 0$ *for* $\alpha \leq i$ *or* $\alpha' > j$. *Conversely, for any choice of* $t_{i+1,j} \in \mathbb{C}$ *there exists a unique extension to a sequence* $(t_{\alpha,\alpha'})_{(\alpha,\alpha')}$ *satisfying* (4-1) *and* (4-2).

Proof. (1) First Steps 1 and 2 in the proof of Lemma 4.4 extend a diagonal sequence $(t_{\alpha,\alpha})_\alpha$ uniquely to the range $\{(\alpha, \alpha') : \alpha \leq i \text{ or } \alpha' > j\}$. This extension satisfies $t_{\alpha,\alpha'} = 0$ whenever $j < \alpha' \leq \alpha \leq i$. Next one can use (4-2) for $(\alpha, \alpha') = (i+1, j+1)$ to define $t_{i+2,j}$ in terms of $t_{i,j}$ and $t_{i+1,j+1}$ (the blue triangle in Figure 8). Inductively, using (4-2) for $(\alpha, \alpha') = (i + 2k + 1, j + 1)$, $k = 0, 1, 2, \ldots$, the values of $t_{i+2k+2,j}$ are determined for all k. In the next step the technique from Step 2 in the proof of Lemma 4.4 is used to inductively define $t_{\alpha,\alpha'}$ for $\alpha > i$ and $\alpha' = j - k$, $k = 0, \ldots, j$ (the red triangle). That all relations (4-1) and (4-2) are satisfied within the four quadrants in Figure 8 is clear from the arguments in Steps 1 and 2 in the proof of Lemma 4.4. That these relations are also satisfied at the edges between the quadrants holds either by definition or since all terms vanish.

(2) Let $(t_{\alpha,\alpha'})_{\alpha,\alpha'}$ be a sequence satisfying (4-1) and (4-2). Note that Lemma 4.3(2) already implies $t_{\alpha,\alpha} = 0$ for $j < \alpha \leq i$. Then by Step 1 in the proof of Lemma 4.4 we have $t_{\alpha,\alpha'} = 0$ whenever $j < \alpha' \leq \alpha \leq i$ (the black zeroes in Figure 9). We first show inductively that $t_{i-2k-1,j} = 0$ for $k = 0, \ldots, \frac{i-j-1}{2}$ (the red zeroes). To show the statement for $k = 0$ consider the relation (4-2) for $(\alpha, \alpha') = (i, j+1)$. By the previous considerations $t_{\alpha,\alpha'} = 0$ and further the coefficient $(2r + 2\alpha + n - 1)$ of $t_{\alpha+1,\alpha'-1}$ vanishes. Hence $t_{\alpha-1,\alpha'-1} = t_{i-1,j} = 0$. For the induction step assume $t_{i-2k-1,j} = 0$ and

Legend: • K'-types $\mathcal{E}(\alpha; \alpha')$ with $t_{\alpha,\alpha'}$ already defined
          ∘ K'-types $\mathcal{E}(\alpha; \alpha')$ with $t_{\alpha,\alpha'}$ yet to define
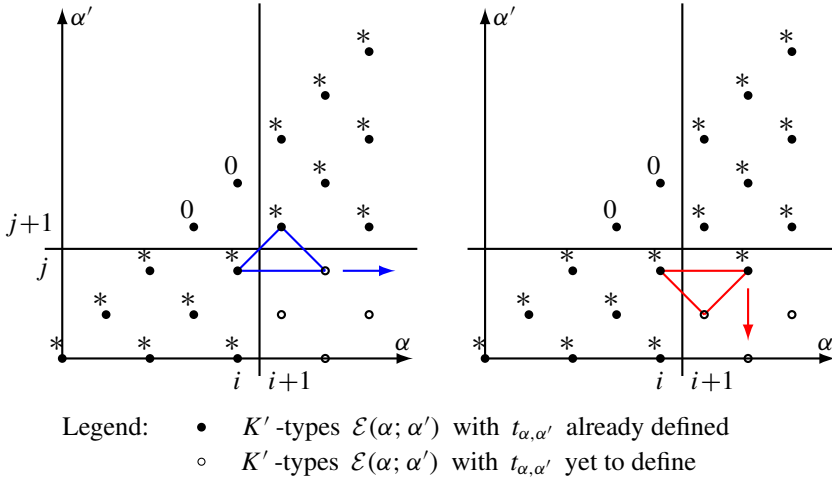
Figure 8

consider the relation (4-2) for $(\alpha, \alpha') = (i - 2k - 2, j + 1)$. Then $t_{\alpha,\alpha'} = t_{\alpha+1,\alpha'-1} = 0$ and therefore $t_{\alpha-1,\alpha'-1} = t_{i-2(k+1)-1,j} = 0$. Thus we have showed $t_{j,j} = 0$. But in view of (4-3) this yields $t_{\alpha,\alpha} = 0$ for $\alpha \leq j$. In a similar way one uses (4-1) and (4-2) for $(\alpha, \alpha') = (i + 1, j + 2k)$, $k = 0, \ldots, \frac{i-j+1}{2}$, to show that $t_{i+1,i+1} = 0$ and hence $t_{\alpha,\alpha} = 0$ for all $\alpha > i$. From the vanishing of the diagonal the techniques in Steps 1 and 2 in the proof of Lemma 4.4 yield $t_{\alpha,\alpha'} = 0$ whenever $\alpha \leq i$ or $\alpha' > j$.

Now let $t_{i+1,j} \in \mathbb{C}$ be given and put $t_{\alpha,\alpha'} = 0$ whenever $\alpha \leq i$ or $\alpha' > j$. Then (4-1) and (4-2) are trivially satisfied whenever all three terms are defined. Further, using Steps 1 and 2 it is again easy to see that this sequence has a unique extension $(t_{\alpha,\alpha'})_{\alpha,\alpha'}$ with the required properties.   □

Proof of Theorem 4.2. (1) Let first $(r, r') \in \mathbb{C}^2 \setminus (L_{\text{even}} \cup L_{\text{odd}})$. Then by Lemma 4.3 the space of diagonal sequences satisfying (4-3) is one-dimensional and each such
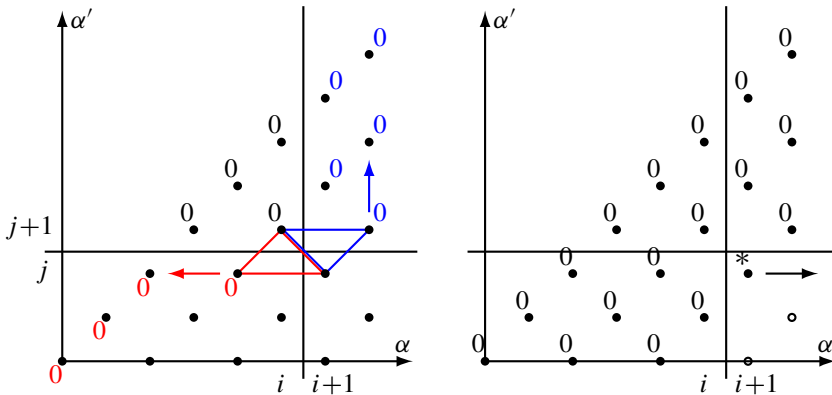
Figure 9

sequence gives by Lemma 4.4 rise to a unique extension $(t_{\alpha,\alpha'})_{(\alpha,\alpha')}$ satisfying
(4-1) and (4-2). Hence, by Theorem 4.1 the multiplicity is 1. Similarly we
obtain multiplicity 2 for $(r, r') \in L_{\text{even}}$ using Lemma 4.5(1). For $(r, r') \in L_{\text{odd}}$
the multiplicity statement is contained in Lemma 4.5(2).

(2) We first consider the case $\mathcal{V} = \mathcal{F}(i)$ and $\mathcal{W} = \mathcal{F}(j)$. Then any intertwining
operator in $\text{Hom}_{(\mathfrak{g}',K')}(\mathcal{V}|_{(\mathfrak{g}',K')}, \mathcal{W})$ corresponds to an intertwining operator T :
$(\pi_r)_{\text{HC}} \to (\tau_{r'})_{\text{HC}}$ for $r = \rho + i$ and $r' = -\rho' - j$ such that $T|_{\mathcal{E}(\alpha)} = 0$ for all
$\alpha > i$ and $T(\mathcal{E}) \subseteq \mathcal{F}'(j)$. This implies that T is given by a sequence $(t_{\alpha,\alpha'})_{\alpha,\alpha'}$ with
$t_{\alpha,\alpha'} = 0$ if either $\alpha > i$ or $\alpha' > j$. By part (1) the space of intertwining operators
$T : (\pi_r)_{\text{HC}} \to (\tau_{r'})_{\text{HC}}$ is one-dimensional, and using Lemma 4.3(1c) and Step 1 in
the proof of Lemma 4.4 it is easy to see that this operator satisfies the conditions on
$t_{\alpha,\alpha'}$ if and only if $i - j \in 2\mathbb{N}$. Hence $m(\mathcal{F}(i), \mathcal{F}'(j)) = 1$ for $i - j \in 2\mathbb{N}$ and $= 0$ else.
Similar considerations for $r = -\rho - i$ and $r' = \rho' + j$ show that $m(\mathcal{T}(i), \mathcal{T}'(j)) = 1$
for $i - j \in 2\mathbb{N}$ and $= 0$ else.

Now let $\mathcal{V} = \mathcal{T}(i)$ and $\mathcal{W} = \mathcal{F}'(j)$. Then $m(\mathcal{V}, \mathcal{W}) \neq 0$ if and only if there
exists a nontrivial sequence $(t_{\alpha,\alpha'})_{\alpha,\alpha'}$ satisfying (4-1) and (4-2) for $r = -\rho - i$ and
$r' = -\rho' - j$ such that $t_{\alpha,\alpha'} = 0$ whenever $\alpha \leq i$ or $\alpha' > j$. First assume $j > i$, then
by part (1) there exists a unique sequence $(t_{\alpha,\alpha'})_{\alpha,\alpha'}$, and by Lemma 4.3(1d) and
Step 3 in the proof of Lemma 4.4 it is easy to see that for this sequence $t_{\alpha,\alpha'} = 0$
if either $\alpha \leq i$ or $\alpha' > j$. Hence $m(\mathcal{T}(i), \mathcal{F}'(j)) = 1$ in this case. Next assume
$j \leq i$; then by Lemmas 4.3(2) and 4.5 there can only exist a sequence $(t_{\alpha,\alpha'})_{\alpha,\alpha'}$
with the above properties if $i - j \in 2\mathbb{N} + 1$. This shows the claimed formulas for
$m(\mathcal{T}(i), \mathcal{F}'(j))$. That $m(\mathcal{F}(i), \mathcal{T}'(j)) = 0$ for any i, j follows easily by similar
considerations. $\square$

4D. *Explicit formula for the spectral function.* From the relations (4-1) and (4-2)
one can deduce an explicit *spectral function* $(t_{\alpha,\alpha'}(r, r'))_{0 \leq \alpha' \leq \alpha}$, i.e., a set of solutions
to the relations for all $r, r' \in \mathbb{C}$ depending meromorphically on r and r':

Proposition 4.6. *For* $(\alpha, \alpha') \in \mathbb{N}$ *with* $\alpha - \alpha' \in 2\mathbb{Z}$ *the numbers*

$$(4\text{-}4) \quad t_{\alpha,\alpha'}(r, r') = \sum_{k=0}^{\infty} \frac{2^{4k}\Gamma\left(\frac{\alpha+\alpha'+n-2}{2}+k\right)\Gamma\left(\frac{\alpha-\alpha'+2}{2}\right)}{(2k)!\,\Gamma\left(\frac{\alpha+\alpha'+n-2}{2}\right)\Gamma\left(\frac{\alpha-\alpha'+2}{2}-k\right)}$$
$$\times \frac{\Gamma(r+\rho)\Gamma(r'+\rho'+\alpha')\Gamma\left(\frac{2r'+2r+1}{4}+k\right)\Gamma\left(\frac{2r'-2r+3}{4}\right)}{\Gamma(r+\rho+\alpha'+2k)\Gamma(r'+\rho')\Gamma\left(\frac{2r'+2r+1}{4}\right)\Gamma\left(\frac{2r'-2r+3}{4}-k\right)}$$

are rational functions in r *and* r' *satisfying* (4-1) *and* (4-2). *They are normalized to*
$t_{0,0} \equiv 1$.

Proof. First note that since $\alpha - \alpha' \in 2\mathbb{Z}$ the number $\frac{\alpha-\alpha'+2}{2} - k$ is a negative integer
for $k \gg 0$ and hence the sum is actually finite for each fixed pair (α, α'). It is also

easy to see that each summand is a rational function in r and r'. A short calculation shows that for each $k \in \mathbb{N}$ the term

$$\frac{\Gamma\left(\frac{\alpha+\alpha'+n-2}{2}+k\right)\Gamma\left(\frac{\alpha-\alpha'+2}{2}\right)\Gamma(r'+\rho'+\alpha')}{\Gamma\left(\frac{\alpha+\alpha'+n-2}{2}\right)\Gamma\left(\frac{\alpha-\alpha'+2}{2}-k\right)\Gamma(r+\rho+\alpha'+2k)}$$

solves (4-1). If we further make the ansatz

$$t_{\alpha,\alpha'} = \sum_{k=0}^{\infty} b_k \frac{\Gamma\left(\frac{\alpha+\alpha'+n-2}{2}+k\right)\Gamma\left(\frac{\alpha-\alpha'+2}{2}\right)\Gamma(r'+\rho'+\alpha')}{\Gamma\left(\frac{\alpha+\alpha'+n-2}{2}\right)\Gamma\left(\frac{\alpha-\alpha'+2}{2}-k\right)\Gamma(r+\rho+\alpha'+2k)},$$

with $b_k = b_k(r, r')$ not depending on α and α' then we find that (4-2) holds if and only if

$$\sum_{k=0}^{\infty} b_k \frac{\Gamma\left(\frac{\alpha+\alpha'+n-2}{2}+k-1\right)\Gamma\left(\frac{\alpha-\alpha'+2}{2}\right)\Gamma(r'+\rho'+\alpha'-1)}{\Gamma\left(\frac{\alpha+\alpha'+n-2}{2}\right)\Gamma\left(\frac{\alpha-\alpha'+2}{2}-k+1\right)\Gamma(r+\rho+\alpha'+2k)}$$
$$\times\left[(2r'+2r+4k+1)(2r'-2r-4k-1)\left(\tfrac{\alpha-\alpha'+2}{2}-k\right)\left(\tfrac{\alpha+\alpha'+n-2}{2}+k-1\right)\right.$$
$$\left. -2k(2k-1)(r+\rho+\alpha'+2k-1)(r+\rho+\alpha'+2k-2)\right]=0.$$

Substituting $k-1$ for k in the first summand in the brackets gives the condition

$$\sum_{k=1}^{\infty} \frac{\Gamma\left(\frac{\alpha+\alpha'+n-2}{2}+k-1\right)\Gamma\left(\frac{\alpha-\alpha'+2}{2}\right)\Gamma(r'+\rho'+\alpha'-1)}{\Gamma\left(\frac{\alpha+\alpha'+n-2}{2}\right)\Gamma\left(\frac{\alpha-\alpha'+2}{2}-k+1\right)\Gamma(r+\rho+\alpha'+2k-2)}$$
$$\times\left[(2r'+2r+4k-3)(2r'-2r-4k+3)b_{k-1}-2k(2k-1)b_k\right]=0,$$

which holds if

$$2k(2k-1)b_k = (2r'+2r+4k-3)(2r'-2r-4k+3)b_{k-1}.$$

This recurrence relation has the solution

$$b_k = c \cdot \frac{2^{4k}\Gamma\left(\frac{2r'+2r+1}{4}+k\right)}{(2k)!\,\Gamma\left(\frac{2r'-2r+3}{4}-k\right)},$$

with $c = c(r, r')$ not depending on k. Finally $t_{0,0} \equiv 1$ implies

$$c = \frac{\Gamma(r+\rho)\Gamma\left(\frac{2r'-2r+3}{4}\right)}{\Gamma(r'+\rho')\Gamma\left(\frac{2r'+2r+1}{4}\right)}. \qquad \qquad \square$$

Corollary 4.7. (1) *The renormalized numbers*

$$t_{\alpha,\alpha'}^{(1)}(r, r') = \frac{1}{\Gamma(r+\rho)} t_{\alpha,\alpha'}(r, r')$$

are holomorphic in $(r, r') \in \mathbb{C}^2$ *for every* $\alpha, \alpha' \in \mathbb{N}$, $\alpha - \alpha' \in 2\mathbb{N}$. *Further,* $t_{\alpha,\alpha'}^{(1)}(r, r') = 0$ *for all* α, α' *if and only if* $(r, r') \in L_{\text{even}}$.

(2) *Fix $r' = -\rho' - j$, $j \in \mathbb{N}$; then the renormalized numbers*

$$t^{(2)}_{\alpha,\alpha'}(r, r') = \frac{\Gamma\left(\frac{(r+\rho)-(r'+\rho')}{2}\right)}{\Gamma(r+\rho)} t_{\alpha,\alpha'}(r, r')$$

are holomorphic in $r \in \mathbb{C}$ for every $\alpha, \alpha' \in \mathbb{N}$, $\alpha - \alpha' \in 2\mathbb{N}$. We have $t^{(2)}_{\alpha,\alpha'}(r, r') \equiv 0$ for $\alpha' > j$. Further, for every $r \in \mathbb{C}$ there exists a pair (α, α') with $t^{(2)}_{\alpha,\alpha'}(r, r') \neq 0$.

(3) *Fix $N \in \mathbb{N}$ and let $r' + \rho' = r + \rho + 2N$; then the renormalized numbers*

$$t^{(3)}_{\alpha,\alpha'}(r, r') = \frac{\Gamma(r'+\rho')}{\Gamma(r+\rho)} t_{\alpha,\alpha'}(r, r')$$

are holomorphic in $r \in \mathbb{C}$ for every $\alpha, \alpha' \in \mathbb{N}$, $\alpha - \alpha' \in 2\mathbb{N}$. Further, for every $r \in \mathbb{C}$ there exists $\alpha_0 \in \mathbb{N}$ such that $t^{(3)}_{\alpha,\alpha}(r, r') \neq 0$ for $\alpha \geq \alpha_0$.

Proof. (1) We can write

$$t^{(1)}_{\alpha,\alpha'}(r, r') = (r'+\rho')_{\alpha'} \sum_{k=0}^{\frac{\alpha-\alpha'}{2}} \frac{2^{4k}\left(\frac{\alpha+\alpha'+n-2}{2}\right)_k\left(-\frac{\alpha-\alpha'}{2}\right)_k\left(\frac{2r'+2r+1}{4}\right)_k\left(\frac{2r-2r'+1}{4}\right)_k}{(2k)!\,\Gamma(r+\rho+\alpha'+2k)},$$

where $(\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1)$ denotes the Pochhammer symbol. This expression is obviously holomorphic in $(r, r') \in \mathbb{C}^2$. Now assume $t^{(1)}_{\alpha,\alpha'}(r, r') = 0$ for all α, α'. For $\alpha = \alpha'$ we have $\left(-\frac{\alpha-\alpha'}{2}\right)_k = 0$ for $k > 0$ and hence

$$t^{(1)}_{\alpha,\alpha}(r, r') = \frac{(r'+\rho')_\alpha}{\Gamma(r+\rho+\alpha)},$$

which vanishes for all $\alpha \in \mathbb{N}$ if and only if $r + \rho = -i$ and $r' + \rho' = -j$ with $j \leq i$. We claim that $i - j \in 2\mathbb{N}$. In fact, if $i - j \in 2\mathbb{N}+1$ then for $(\alpha, \alpha') = (i+1, j)$ only the summand for $k = \frac{i-j+1}{2}$ is nonzero and hence $t^{(1)}_{\alpha,\alpha'}(r, r') \neq 0$, a contradiction. Therefore $i - j \in 2\mathbb{N}$ which means $(r, r') \in L_{\text{even}}$.

Conversely assume $r + \rho = -i$, $r' + \rho' = -j$, with $i - j \in 2\mathbb{N}$. Then in each summand at least one of the three factors

$$\left(\tfrac{2r-2r'+1}{4}\right)_k = \left(-\tfrac{i-j}{2}\right)_k, \quad (r'+\rho')_{\alpha'} = (-j)_{\alpha'}, \quad \tfrac{1}{\Gamma(r+\rho+\alpha'+2k)} = \tfrac{1}{\Gamma(-i+\alpha'+2k)}$$

vanishes and hence $t^{(1)}_{\alpha,\alpha'}(r, r') = 0$ for all α, α'.

(2) We can write

$$t^{(2)}_{\alpha,\alpha'}(r, r') = (-j)_{\alpha'} \sum_{k=0}^{\frac{\alpha-\alpha'}{2}} \frac{2^{4k}\left(\frac{\alpha+\alpha'+n-2}{2}\right)_k\left(-\frac{\alpha-\alpha'}{2}\right)_k\left(\frac{2r-2j-n+3}{4}\right)_k \Gamma\left(\frac{2r+2j+n-1}{4}+k\right)}{(2k)!\,\Gamma(r+\rho+\alpha'+2k)}$$

as a meromorphic function of r. Then $t^{(2)}_{\alpha,\alpha'}(r, r') \equiv 0$ for $\alpha' > j$. Further, for $\alpha' \leq j$ each pole r of the factor $\Gamma\left(\frac{2r+2j+n-1}{4}+k\right)$ is simple and also a pole of

the denominator $\Gamma(r + \rho + \alpha' + 2k)$, whence $t^{(2)}_{\alpha,\alpha'}(r, r')$ is holomorphic in $r \in \mathbb{C}$. Now assume $t^{(2)}_{\alpha,\alpha'}(r, r') = 0$ for all α, α'. Then

$$0 = t^{(2)}_{j,j}(r, r') = (-j)_j \frac{\Gamma\left(\frac{2r+2j+n-1}{4}\right)}{\Gamma(r + \rho + j)}$$

and hence r has to be a pole of the denominator while it is a regular point for the numerator. This means $r + \rho = -i \in -\mathbb{N}$ with $i \geq j$ and $\frac{2r+2j+n-1}{4} = \frac{j-i}{2} \notin -\mathbb{N}$, i.e., $i - j \in 2\mathbb{N} + 1$. But for $(\alpha, \alpha') = (i + 1, j)$ only the summand for $k = \frac{i-j+1}{2}$ is nonzero and hence $t^{(2)}_{\alpha,\alpha'}(r, r') \neq 0$, a contradiction.

(3) Note that $\left(\frac{2r-2r'+1}{4}\right)_k = (-N)_k = 0$ for $k > N$ and hence we can write

$$(4\text{-}5) \quad t^{(3)}_{\alpha,\alpha'}(r, r') = \sum_{k=0}^{N} \frac{2^{4k}(-N)_k\left(\frac{\alpha+\alpha'+n-2}{2}\right)_k\left(-\frac{\alpha-\alpha'}{2}\right)_k}{(2k)!}$$
$$\times \left(r+N+\tfrac{1}{2}\right)_k (r+\rho+\alpha'+2k)_{2N-2k},$$

which is clearly holomorphic in $r \in \mathbb{C}$. Further, $t^{(3)}_{\alpha,\alpha}(r, r') = (r + \rho + \alpha)_{2N}$, which is nonzero for $\alpha > -(r + \rho)$. $\qquad\square$

Remark 4.8. After a few modifications we find that

$$t_{\alpha,\alpha'}(r, r') = \frac{(r' + \rho')_{\alpha'}}{(r + \rho)_{\alpha'}} {}_4F_3\left(-\frac{\alpha-\alpha'}{2}, \frac{\alpha+\alpha'+n-2}{2}, \frac{2r+2r'+1}{4}, \frac{2r-2r'+1}{4}; \right.$$
$$\left. \frac{1}{2}, \frac{r+\rho+\alpha'}{2}, \frac{r+\rho+\alpha'+1}{2}; 1\right).$$

Note that the generalized hypergeometric function ${}_4F_3(a_1, a_2, a_3, a_4; b_1, b_2, b_3; z)$ occurring here is balanced, i.e., $a_1 + a_2 + a_3 + a_4 + 1 = b_1 + b_2 + b_3$. However, there does not exist an explicit formula for its special value at $z = 1$ in the literature. Also, we could not find estimates for special values of such hypergeometric functions for large/small parameters, and therefore were not able to show that $t_{\alpha,\alpha'}(r, r')$ grows at most polynomially in $\alpha, \alpha' \geq 0$ for fixed $(r, r') \in \mathbb{C}^2$. This is what is needed to apply Proposition 3.10 in order to show automatic continuity of intertwining operators. We will therefore first describe all intertwining operators in terms of the holomorphic family $T^{(1)}(r, r')$ (see Theorem 4.9) and then show automatic continuity using the corresponding holomorphic family in the smooth category obtained in joint work with Y. Oshima [Möllers et al. 2016a]. This is done in Corollary 4.12.

Theorem 4.9. *For $i = 1, 2, 3$ we let $T^{(i)}(r, r')$ be the intertwining operators $(\pi_r)_{\mathrm{HC}} \to (\tau_{r'})_{\mathrm{HC}}$ corresponding to the numbers $t^{(i)}_{\alpha,\alpha'}(r, r')$ in Corollary 4.7. Then the operator $T^{(1)}(r, r')$ is defined for $(r, r') \in \mathbb{C}^2$, the operator $T^{(2)}(r, r')$ is defined for $r' \in -\rho' - \mathbb{N}$ and the operator $T^{(3)}(r, r')$ is defined for $(r+\rho) - (r'+\rho') \in -2\mathbb{N}$.*

We have

$$\mathrm{Hom}_{(\mathfrak{g}',K')}((\pi_r)_{\mathrm{HC}}|_{(\mathfrak{g}',K')}, (\tau_{r'})_{\mathrm{HC}})$$
$$= \begin{cases} \mathbb{C}T^{(1)}(r,r') & \text{for } (r,r') \in \mathbb{C}^2 \setminus L_{\mathrm{even}}, \\ \mathbb{C}T^{(2)}(r,r') \oplus \mathbb{C}T^{(3)}(r,r') & \text{for } (r,r') \in L_{\mathrm{even}}. \end{cases}$$

Remark 4.10. By the proof of Theorem 4.2(2) every intertwining operator between the subquotients $\mathcal{F}(i), \mathcal{T}(i)$ and $\mathcal{F}'(j), \mathcal{T}'(j)$ can be constructed by composing an intertwining operator $(\pi_r)_{\mathrm{HC}} \to (\tau_{r'})_{\mathrm{HC}}$ for particular r, r' with embeddings and/or quotient maps for the subquotients. Hence, also every intertwining operator between subquotients is given by an operator in one of the three families $T^{(i)}(r, r')$. Therefore, all information about intertwining operators between $(\pi_r)_{\mathrm{HC}}$ and $(\tau_{r'})_{\mathrm{HC}}$ and any of their subquotients is contained in the holomorphic family $T^{(1)}(r, r')$.

Remark 4.11. The family of operators $T^{(3)}$ is (up to a constant) equal to Juhl's family of conformally invariant differential restriction operators $D_{2N}(r): C^\infty(S^{n-1}) \to C^\infty(S^{n-2})$; see [Juhl 2009; Kobayashi and Speh 2015]. The constants $t^{(3)}_{\alpha,\alpha'}$ then give the "spectrum" of Juhl's operators in the sense that they describe how the operators are acting on explicit K-finite vectors. Note that by (4-5) the number of summands for $t^{(3)}_{\alpha,\alpha'}(r, r')$ is at most $N + 1$.

Corollary 4.12. *For* $(G, G') = (\mathrm{O}(1, n), \mathrm{O}(1, n-1))$ *the natural injective map*

$$(4\text{-}6) \qquad \mathrm{Hom}_{G'}(\pi|_{G'}, \tau) \to \mathrm{Hom}_{(\mathfrak{g}',K')}(\pi_{\mathrm{HC}}|_{(\mathfrak{g}',K')}, \tau_{\mathrm{HC}})$$

is an isomorphism for all spherical principal series π of G and τ of G' and their subquotients.

Proof. By Remark 4.10 all intertwining operators between subquotients arise by composing with quotient maps and embeddings. It therefore suffices to show that (4-6) is an isomorphism for $\pi = \pi_r$ and $\tau = \tau_{r'}$ for all $(r, r') \in \mathbb{C}^2$. In [Möllers et al. 2016a] a holomorphic family $A(r, r') \in \mathrm{Hom}_{G'}(\pi_r|_{G'}, \tau_{r'})$ was constructed in the smooth category using singular integral operators (see Section 4F for details). Denote by $\bar{A}(r, r') \in \mathrm{Hom}_{(\mathfrak{g}',K')}((\pi_r)_{\mathrm{HC}}|_{(\mathfrak{g}',K')}, (\tau_{r'})_{\mathrm{HC}})$ its image under the map (4-6). By Theorem 4.9 this space is generically spanned by $T^{(1)}(r, r')$, and since both $\bar{A}(r, r')$ and $T^{(1)}(r, r')$ depend holomorphically on $(r, r') \in \mathbb{C}^2$ there exists a meromorphic function $\phi(r, r')$ such that

$$\bar{A}(r, r') = \phi(r, r') \cdot T^{(1)}(r, r').$$

Replacing $A(r, r')$ and $\bar{A}(r, r')$ by $\phi(r, r')^{-1}A(r, r')$ and $\phi(r, r')^{-1}\bar{A}(r, r')$ we may assume that

$$\bar{A}(r, r') = T^{(1)}(r, r').$$

This already implies that for $(r, r') \in \mathbb{C}^2 \setminus L_{\text{even}}$ every intertwining operator in the space $\text{Hom}_{(\mathfrak{g}', K')}((\pi_r)_{\text{HC}}|_{(\mathfrak{g}', K')}, (\tau_{r'})_{\text{HC}})$ extends to the smooth globalization. Further, for $(r, r') \in L_{\text{even}}$ we may restrict $(r, r') \mapsto T^{(1)}(r, r')$ to an affine complex line and renormalize to obtain all intertwining operators in $\text{Hom}_{(\mathfrak{g}', K')}((\pi_r)_{\text{HC}}|_{(\mathfrak{g}', K')}, (\tau_{r'})_{\text{HC}})$ by Theorem 4.9. The same restriction and renormalization can be applied to $(r, r') \mapsto A(r, r')$, and in this way one obtains extensions of all operators in $\text{Hom}_{(\mathfrak{g}', K')}((\pi_r)_{\text{HC}}|_{(\mathfrak{g}', K')}, (\tau_{r'})_{\text{HC}})$ to the smooth globalization. Note that renormalization of $A(r, r')$ preserves continuity of the operators. This shows that the map (4-6) is surjective, hence an isomorphism for all $(r, r') \in \mathbb{C}^2$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 4.13. The operators $T^{(i)}(r, r')$ are related to the operators $\widetilde{\mathbb{A}}_{\lambda, \nu}$, $\widetilde{\widetilde{\mathbb{A}}}_{\lambda, \nu}$ and $\widetilde{\mathbb{C}}_{\lambda, \nu}$ studied in [Kobayashi and Speh 2015] for $\lambda = r + \rho$, $\nu = r' + \rho'$. In fact, using their notation we have

$$T^{(1)}(r, r') = \pi^{-\frac{n-2}{2}} \widetilde{\mathbb{A}}_{\lambda, \nu}, \quad T^{(2)}(r, r') = \pi^{-\frac{n-2}{2}} \widetilde{\widetilde{\mathbb{A}}}_{\lambda, \nu}, \quad T^{(3)}(r, r') = \frac{(-1)^N N!}{2^{2N}} \widetilde{\mathbb{C}}_{\lambda, \nu},$$

where for $i = 3$ we write $r' + \rho' = r + \rho + 2N$ with $N \in \mathbb{N}$.

4E. *Discrete components in the restriction of unitary representations.* We apply our results to branching problems for unitary representations. The $(\mathfrak{g}, K)$-modules $(\pi_r)_{\text{HC}}$ are unitarizable if and only if $r \in i\mathbb{R} \cup (-\rho, \rho)$ and we denote by $\hat{\pi}_r$ their unitary completions. For $r \in i\mathbb{R}$ these representations form the unitary principal series and for $r \in (-\rho, \rho)$ they belong to the complementary series. Further, all irreducible quotients $\mathcal{T}(i)$ are unitarizable and their unitary completions will be denoted by $\hat{\pi}_{-\rho-i}$. We note that for $r \in -(\rho + \mathbb{Z})$, $r < 0$, each representation $\hat{\pi}_r$ is isomorphic to some Zuckerman derived functor module $A_{\mathfrak{q}}(\lambda)$ and occurs discretely in the decomposition of the regular representation on $L^2(G/G')$.

Similarly we denote by $\hat{\tau}_{r'}$, $r' \in i\mathbb{R} \cup (-\rho', \rho')$, the unitary completions of $\tau_{r'}$ and by $\hat{\tau}_{-\rho'-j}$, $j \in \mathbb{N}$, the unitary completions of $\mathcal{T}'(j)$.

For $r \in \mathbb{R}$ we define the finite set

$$D(r) = \left(r + \tfrac{1}{2} + 2\mathbb{N}\right) \cap (-\infty, 0)$$

and note that for $r \in (-\rho, 0) \cup (-\rho - \mathbb{N})$ and $r' \in D(r)$ we have $r' \in (-\rho', 0) \cup (-\rho' - \mathbb{N})$; i.e., $\hat{\tau}_{r'}$ is a unitary representation.

Theorem 4.14. *Let $r \in (-\rho, 0) \cup (-\rho - \mathbb{N})$. Then for every $r' \in D(r)$ the representation $\hat{\tau}_{r'}$ occurs discretely with multiplicity 1 in the restriction of $\hat{\pi}_r$ to G'.*

We note that for a complementary series representation $\hat{\pi}_r$, $r \in (-\rho, 0)$, all representations $\hat{\tau}_{r'}$, $r' \in D(r)$, are complementary series representations. If $\hat{\pi}_r$ is an $A_{\mathfrak{q}}(\lambda)$-module, $r \in -\rho + \mathbb{Z}$, $r < 0$, then so are the representations $\hat{\tau}_{r'}$, $r' \in D(r)$.

The restriction of the $A_{\mathfrak{q}}(\lambda)$-modules $\hat{\pi}_r$ to G' decomposes with both discrete and continuous spectrum and is therefore hard to study by purely algebraic methods.

Remark 4.15. For the special case $r' = r + \frac{1}{2}$, i.e., $N = 0$, the occurrence of $\hat{\tau}_{r'}$ in $\hat{\pi}_r|_{G'}$ was first proved in [Speh and Venkataramana 2011] for $r \in \left[-\rho, -\frac{1}{2}\right)$ and generalized in [Zhang 2015] to the case $r \in \left(-\rho, -\frac{1}{2}\right) \cup (-\rho - \mathbb{N})$. Later Kobayashi and Speh [2015, Theorem 1.4] proved Theorem 4.14 for the case $r \in (-\rho, 0)$. The full decomposition of $\hat{\pi}_r|_{G'}$ for $r \in (-\rho, 0) \cup (-\rho - \mathbb{N})$ including the continuous spectrum was given in [Möllers and Oshima 2015].

We first describe the invariant norms on the unitarizable constituents for $r \in \mathbb{R}$. For this we fix the L^2-norm $\| \cdot \|_{L^2(S^{n-1})}$ on $L^2(K/M) = L^2(S^{n-1})$ corresponding to the standard Euclidean measure on S^{n-1}. For $r \in (-\rho, \rho)$ the norm $\| \cdot \|_r$ on $\mathcal{E}$ given by

$$\|v\|_r^2 = \sum_{\alpha=0}^{\infty} b_\alpha(r) \|v_\alpha\|_{L^2(S^{n-1})}^2 \quad \text{for } v = \sum_{\alpha=0}^{\infty} v_\alpha \in \bigoplus_{\alpha=0}^{\infty} \mathcal{E}(\alpha),$$

with

$$b_\alpha = \frac{\Gamma(\rho - r + \alpha)}{\Gamma(\rho + r + \alpha)} \sim (1 + \alpha)^{-2r}$$

turns $(\pi_r)_{\mathrm{HC}}$ into a unitary $(\mathfrak{g}, K)$-module. Further, for $r = -\rho - i$ the seminorm $\| \cdot \|_r$ on $\mathcal{E}$ has kernel $\mathcal{F}(i)$ and turns the quotient $\mathcal{T}(i) = \mathcal{E}/\mathcal{F}(i)$ into a unitary $(\mathfrak{g}, K)$-module.

Similarly we denote by $\| \cdot \|_{r'}'$ the $\tau_{r'}$-invariant norm on $\mathcal{E}'$, respectively $\mathcal{T}'(j)$, given by

$$\|w\|_{r'}'^2 = \sum_{\alpha=0}^{\infty} b_{\alpha'}'(r') \|w_{\alpha'}\|_{L^2(S^{n-2})}^2 \quad \text{for } w = \sum_{\alpha'=0}^{\infty} w_{\alpha'} \in \bigoplus_{\alpha'=0}^{\infty} \mathcal{E}'(\alpha'),$$

with

$$b_{\alpha'}' = \frac{\Gamma(\rho' - r' + \alpha')}{\Gamma(\rho' + r' + \alpha')} \sim (1 + \alpha')^{-2r'}.$$

We need the following two basic results; see, e.g., [Zhang 2015, Lemmas 3.2 and 3.5]:

Lemma 4.16. *Let $\mathcal{V} \subseteq \mathcal{E}$ be a K-invariant subspace and $\mathcal{W} \subseteq \mathcal{E}'$ a K'-invariant subspace and assume that $\mathcal{V}$ and $\mathcal{W}$ are endowed with pre-Hilbert space structures with respect to which the groups K and K' act unitarily. A linear map $T : \mathcal{V} \to \mathcal{W}$ is bounded if and only if there exists a constant $C > 0$ such that*

$$\sum_{\substack{\alpha \\ \mathcal{E}(\alpha;\alpha') \subseteq \mathcal{V}}} \|T|_{\mathcal{E}(\alpha;\alpha')}\|_{\mathcal{V} \to \mathcal{W}}^2 \le C \quad \text{for all } \alpha',$$

where $\| \cdot \|_{\mathcal{V} \to \mathcal{W}}$ denotes the operator norm with respect to the given pre-Hilbert space structures.

Lemma 4.17. *Suppose that $\alpha > -1$, $\beta \geq 0$, and $\beta - \alpha > 1$. Then there exists a constant $C > 0$ such that*

$$\sum_{p=0}^{\infty} \frac{(1+p)^{\alpha}}{(1+p+q)^{\beta}} \leq \frac{C}{(1+q)^{\beta-\alpha-1}} \quad \text{for all } q \geq 0.$$

Proof of Theorem 4.14. For $r \in (-\rho, 0)$ let $\mathcal{V} = \mathcal{E}$ and for $r = -\rho - i \in -\rho - \mathbb{N}$ let $\mathcal{V} = \bigoplus_{\alpha=i+1}^{\infty} \mathcal{E}(\alpha)$. Let $r' \in D(r)$; then similarly we put $\mathcal{W} = \mathcal{E}'$ for $r' \in (-\rho', 0)$ and $\mathcal{W} = \bigoplus_{\alpha'=j+1}^{\infty} \mathcal{E}'(\alpha')$ for $r' = -\rho' - j \in -\rho' - \mathbb{N}$. By Theorem 4.2 there exists (up to scalar) a unique nonzero intertwining operator $T : (\pi_r)_{\text{HC}} \to (\tau_{r'})_{\text{HC}}$ with $T(\mathcal{V}) \subseteq \mathcal{W}$ and if $r = -\rho - i$ additionally $T|_{\mathcal{F}(i)} = 0$. In our notation

$$T|_{\mathcal{E}(\alpha;\alpha')} = t_{\alpha,\alpha'} \cdot \text{rest}\,|_{\mathcal{E}(\alpha;\alpha')},$$

with $t_{\alpha,\alpha'} = t_{\alpha,\alpha'}^{(3)}$ for $\alpha' > j$ and $t_{\alpha,\alpha'} = 0$ else (see Corollary 4.7 for the definition of $t_{\alpha,\alpha'}^{(3)}$). We show that T is bounded if we endow $\mathcal{V}$ with the norm $\|\cdot\|_r$ and $\mathcal{W}$ with the norm $\|\cdot\|_{r'}$. To apply Lemma 4.16 we calculate

$$\|T|_{\mathcal{E}(\alpha;\alpha')}\|_{\mathcal{V}\to\mathcal{W}}^2 = t_{\alpha,\alpha'}^2 \| \text{rest}\,|_{\mathcal{E}(\alpha;\alpha')}\|_{\mathcal{E}(\alpha;\alpha')\to\mathcal{E}'(\alpha')}^2 \frac{b'_{\alpha'}(r')}{b_{\alpha}(r)},$$

where $\|\cdot\|_{\mathcal{E}(\alpha;\alpha')\to\mathcal{E}'(\alpha')}$ denotes the operator norm with respect to the L^2-inner products on $\mathcal{E}(\alpha;\alpha') \subseteq L^2(S^{n-1})$ and $\mathcal{E}'(\alpha') \subseteq L^2(S^{n-2})$. Using (A-2), (B-2) and (B-3) it is easy to see that for $\alpha = \alpha' + 2\ell$ we have

$$\| \text{rest}\,|_{\mathcal{E}(\alpha;\alpha')}\|_{\mathcal{E}(\alpha;\alpha')\to\mathcal{E}'(\alpha')}^2 = \frac{2^{2\alpha'+n-3}\left(\alpha'+2\ell+\frac{n-2}{2}\right)(2\ell)!\,\Gamma\left(\alpha'+\ell+\frac{n-2}{2}\right)^2}{\pi(\ell!)^2\Gamma(2\alpha'+2\ell+n-2)}$$

$$= \frac{\left(\alpha'+2\ell+\frac{n-2}{2}\right)\Gamma\left(\ell+\frac{1}{2}\right)\Gamma\left(\alpha'+\ell+\frac{n-2}{2}\right)}{\pi\,\Gamma(\ell+1)\Gamma\left(\alpha'+\ell+\frac{n-1}{2}\right)} \sim \frac{(1+\alpha'+\ell)^{\frac{1}{2}}}{(1+\ell)^{\frac{1}{2}}}.$$

Then Lemma 4.16 translates into

$$\sum_{\ell=0}^{\infty} t_{\alpha'+2\ell,\alpha'}^2 \frac{(1+\alpha'+\ell)^{\frac{1}{2}+2r}}{(1+\ell)^{\frac{1}{2}}} \leq C(1+\alpha')^{2r'}.$$

It is enough to check this for each of the $N+1$ summands of $t_{\alpha'+2\ell,\alpha'}$ in (4-5) where $r' + \rho' = r + \rho + 2N$. The k-th summand grows of order

$$\sim (1+\alpha')^{(r'+\rho')-(r+\rho+2k)}(1+\ell)^k(1+\alpha'+\ell)^k$$

and hence the claim follows by Lemma 4.17. Altogether this shows that T induces a bounded G'-intertwining operator $\widetilde{T} : \hat{\pi}_r|_{G'} \to \hat{\tau}_{r'}$ whose adjoint $\widetilde{T}^* : \hat{\tau}_{r'} \to \hat{\pi}_r|_{G'}$ embeds $\hat{\tau}_{r'}$ isometrically as a subrepresentation of $\hat{\pi}_r$ by Schur's lemma. Multiplicity 1 follows from the fact that any G'-equivariant embedding $S : \hat{\tau}_{r'} \to \hat{\pi}_r|_{G'}$ induces an intertwiner $S^* : (\pi_r)_{\text{HC}} \to (\tau_{r'})_{\text{HC}}$ between the Harish-Chandra

modules by taking the adjoint operator and then passing to K-finite vectors. Such an operator is unique (up to scalars) by Theorem 4.2 and since K-finite vectors are dense in $\hat{\pi}_r$ the embedding S is unique (up to scalars). $\qquad\square$

4F. *Comparison with singular integral operators.* In [Kobayashi and Speh 2015; Möllers et al. 2016a] a meromorphic family of intertwining operators $A(r, r')$: $u_{1,rv}|_{G'} \to u'_{1,r'v}$ in the smooth category is constructed as family of singular integral operators. In the compact picture this family is (up to scalars) given by

$$A(r, r') : C^\infty(S^{n-1}) \to C^\infty(S^{n-2}),$$

$$A(r, r')f(y) = \int_{S^{n-1}} (|x' - y|^2 + x_n^2)^{-(r'+\rho')} |x_n|^{(r-\rho)+(r'+\rho')} f(x)\, dx,$$

where dx denotes the Euclidean measure on S^{n-1}.

Theorem 4.18. *Let* $T(r, r') : C^\infty(S^{n-1}) \to C^\infty(S^{n-2})$ *denote the intertwining operator with spectrum given by the numbers* $t_{\alpha,\alpha'}(r, r')$ *in* (4-4). *Then*

$$A(r, r') = \frac{2^{r-r'+\frac{1}{2}} \pi^{\frac{n-2}{2}} \Gamma\left(\frac{2r+2r'+1}{4}\right) \Gamma\left(\frac{2r-2r'+1}{4}\right)}{\Gamma\left(r + \frac{n-1}{2}\right)} \cdot T(r, r').$$

Proof. Since by Theorem 4.2(1) and Corollary 4.12 we generically have

$$\dim \operatorname{Hom}_{G'}(\pi_r|_{G'}, \tau_{r'}) = 1$$

and both $A(r, r')$ and $T(r, r')$ are meromorphic in $r, r' \in \mathbb{C}$ there exists a scalar meromorphic function $c(r, r')$ with $A(r, r') = c(r, r')T(r, r')$. To determine $c(r, r')$ we put $f \equiv 1$:

$$c(r, r') = \int_{S^{n-1}} (|x' - y|^2 + x_n^2)^{-(r'+\frac{n-2}{2})} |x_n|^{r+r'-\frac{1}{2}}\, dx.$$

Using the stereographic projection

$$x = \left(\frac{1 - |z|^2}{1 + |z|^2}, \frac{2z}{1 + |z|^2} \right), \quad z \in \mathbb{R}^{n-1},$$

the measure transforms by $dx = 2^{n-1}(1 + |z|^2)^{-(n-1)}\, dz$, where dz is the standard Lebesgue measure on $\mathbb{R}^{n-1}$. Writing

$$y = \left(\frac{1 - |w|^2}{1 + |w|^2}, \frac{2w}{1 + |w|^2} \right), \quad w \in \mathbb{R}^{n-2},$$

we find

$$c(r, r') = 2^{r-r'+\frac{1}{2}} (1 + |w|^2)^{r'+\frac{n-2}{2}} \int_{\mathbb{R}^{n-1}} (|z' - w|^2 + z_{n-1}^2)^{-(r'+\frac{n-2}{2})}$$
$$\times |z_{n-1}|^{r+r'-\frac{1}{2}} (1 + |z|^2)^{-(r+\frac{n-1}{2})}\, dz,$$

where we have written $z = (z', z_{n-1})$. This integral is evaluated in [Kobayashi and Speh 2015, Proposition 7.4] and we obtain

$$c(r, r') = \frac{2^{r-r'+\frac{1}{2}} \pi^{\frac{n-2}{2}} \Gamma\left(\frac{2r+2r'+1}{4}\right) \Gamma\left(\frac{2r-2r'+1}{4}\right)}{\Gamma\left(r + \frac{n-1}{2}\right)},$$

which shows the claim. $\square$

Remark 4.19. The special value of the intertwiners $A(r, r')$ at the spherical vector $f \equiv 1$ was also calculated in [Möllers and Ørsted 2017] by a different method.

Remark 4.20. The action of $A(r, r')$ on K-finite vectors was also computed in [Kobayashi and Speh 2015, Lemma 7.7]. However, their parametrization of K-finite vectors differs from our parametrization by (α, α'), and therefore it is nontrivial to see the equivalence of their identity and our identity (4-4).

5. Rank-one unitary groups

We indicate in this section how the calculations in Section 4 can be generalized to rank-one unitary groups and state the corresponding results. Let $n \geq 2$ and consider the indefinite unitary group $G = \mathrm{U}(1, n)$ of $(n+1) \times (n+1)$ complex matrices leaving the standard Hermitian form on $\mathbb{C}^{n+1}$ of signature $(1, n)$ invariant. The subgroup $G' \subseteq G$ of matrices fixing the last standard basis vector e_{n+1} is isomorphic to $\mathrm{U}(1, n-1)$.

5A. K-types. We fix $K = \mathrm{U}(1) \times \mathrm{U}(n)$ and choose

$$H = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & \mathbf{0}_{n-1} \end{pmatrix}$$

so that $P = MAN$ with $M = \Delta \mathrm{U}(1) \times \mathrm{U}(n-1)$, where $\Delta \mathrm{U}(1) = \{\mathrm{diag}(x, x) : x \in \mathrm{U}(1)\}$. Note that $\rho = n$. Then K acts transitively on the unit sphere $S^{2n-1} \subseteq \mathbb{C}^n$ via $\mathrm{diag}(\lambda, k) \cdot z = \lambda^{-1} k z$, $\lambda \in \mathrm{U}(1)$, $k \in \mathrm{U}(n)$, $z \in S^{2n-1}$, and M is the stabilizer subgroup of the first standard basis vector e_1, whence $K/M \cong S^{2n-1}$. The subgroup $G' = \mathrm{U}(1, n-1)$ is embedded into G such that $K' = \mathrm{U}(1) \times \mathrm{U}(n-1)$ and $P' = G' \cap P = M'A'N'$ with $A' = A$ and $M' = \Delta \mathrm{U}(1) \times \mathrm{U}(n-2)$. Then $K'/M' = S^{2n-3} \subseteq \mathbb{C}^{n-1}$, viewed as the codimension 2 submanifold in $K/M = S^{2n-1} \subseteq \mathbb{C}^n$ given by $z_n = 0$. Further we have $\rho' = n-1$.

Let $\xi = \mathbf{1}$, $\xi' = \mathbf{1}$ be the trivial representations of M and M' and abbreviate $\pi_r = \pi_{\xi,r}$ and $\tau_{r'} = \tau_{\xi',r'}$. Then as K-modules, resp. K'-modules, we have

$$\mathcal{E} = \bigoplus_{\alpha_1, \alpha_2 = 0}^{\infty} \underbrace{e^{i(\alpha_1 - \alpha_2)\theta} \boxtimes \mathcal{H}^{\alpha_1, \alpha_2}(\mathbb{C}^n)}_{\mathcal{E}(\alpha)}, \quad \mathcal{E}' = \bigoplus_{\alpha_1', \alpha_2' = 0}^{\infty} \underbrace{e^{i(\alpha_1' - \alpha_2')\theta} \boxtimes \mathcal{H}^{\alpha_1', \alpha_2'}(\mathbb{C}^{n-1})}_{\mathcal{E}'(\alpha')},$$

where we abbreviate $\alpha = (\alpha_1, \alpha_2)$ and $\alpha' = (\alpha'_1, \alpha'_2)$. Hence, (MF1) is satisfied. Further, each K-type decomposes by (B-5) into K'-types as

$$(e^{i(\alpha_1-\alpha_2)\theta} \boxtimes \mathcal{H}^{\alpha_1,\alpha_2}(\mathbb{C}^n))|_{K'} = \bigoplus_{\substack{0 \leq \alpha'_1 \leq \alpha_1 \\ 0 \leq \alpha'_2 \leq \alpha_2}} (e^{i(\alpha_1-\alpha_2)\theta} \boxtimes \mathcal{H}^{\alpha'_1,\alpha'_2}(\mathbb{C}^{n-1})),$$

so that (MF2) holds. Comparing the characters of the U(1)-factor of K' we find that $\mathrm{Hom}_{K'}(\mathcal{E}(\alpha)|_{K'}, \mathcal{E}'(\alpha')) \neq 0$ if and only if $\alpha_1 - \alpha_2 = \alpha'_1 - \alpha'_2$. In this case formulas (B-6) and (A-3) show that the restriction operator

$$R_{\alpha,\alpha'} = \mathrm{rest}\,|_{\mathcal{E}(\alpha;\alpha')} : \mathcal{E}(\alpha; \alpha') \to \mathcal{E}'(\alpha')$$

is an isomorphism. Hence the restriction $T_{\alpha,\alpha'} = T|_{\mathcal{E}(\alpha;\alpha')}$ of a K'-intertwining operator $T : \mathcal{E} \to \mathcal{E}'$ is given by $T_{\alpha,\alpha'} = t_{\alpha,\alpha'} R_{\alpha,\alpha'}$ for $\alpha_1 - \alpha_2 = \alpha'_1 - \alpha'_2$ and $T_{\alpha,\alpha'} = 0$ else.

5B. *Proportionality constants.* The eigenvalues of the spectrum-generating operator on the K-types are given by (see [Branson et al. 1996, Section 3.b])

$$\sigma_{(\alpha_1,\alpha_2)} = 2\alpha_1(\alpha_1 + n - 1) + 2\alpha_2(\alpha_2 + n - 1),$$
$$\sigma'_{(\alpha'_1,\alpha'_2)} = 2\alpha'_1(\alpha'_1 + n - 2) + 2\alpha'_2(\alpha'_2 + n - 2).$$

We write $\mathfrak{s}_{\mathbb{C}} = \mathfrak{s} + J\mathfrak{s} = \mathfrak{s}_+ + \mathfrak{s}_-$ and identify $\mathfrak{s}_\pm \cong \mathbb{C}^n$ via

$$\mathbb{C}^n \to \mathfrak{s}_\pm, \quad w \mapsto X_{w,\pm} = \begin{pmatrix} 0 & w^* \mp Jiw^* \\ w \pm Jiw & \mathbf{0}_n \end{pmatrix}.$$

Then $\mathfrak{s}'_\pm \simeq \mathbb{C}^{n-1}$, embedded in $\mathbb{C}^n$ as the first $n - 1$ coordinates. Since both $\mathfrak{s}'_\pm$ are multiplicity-free K'-modules, (MF3) holds (with $\mathfrak{s}'_{\mathbb{C}}$ replaced by $\mathfrak{s}'_\pm$) and we can use Corollary 3.6. The cocycle ω is given by

$$\omega(X_{w,+})(z) = w^* z, \quad w \in \mathfrak{s}_+, \qquad \omega(X_{w,-})(z) = z^* w, \quad w \in \mathfrak{s}_-,$$

where $z \in S^{2n-1} \subseteq \mathbb{C}^n$.

We note by (B-7) that if $X \in \mathfrak{s}_+$ then the multiplication map $m(\omega(X))$ maps the K-type $\mathcal{E}(\alpha_1, \alpha_2)$ into the K-types $\mathcal{E}(\alpha_1 + 1, \alpha_2)$ and $\mathcal{E}(\alpha_1, \alpha_2 - 1)$ and if $X \in \mathfrak{s}_-$ into the K-types $\mathcal{E}(\alpha_1, \alpha_2 + 1)$ and $\mathcal{E}(\alpha_1 - 1, \alpha_2)$. Because of similar considerations for $\mathfrak{s}'_+$ and $\mathfrak{s}'_-$ the equivalence relation $(\alpha, \alpha') \leftrightarrow (\beta, \beta')$ is given by

$$((\alpha_1, \alpha_2); (\alpha'_1, \alpha'_2)) \leftrightarrow (\beta; (\alpha'_1+1, \alpha'_2)) \quad \Longleftrightarrow \quad \beta \in \{(\alpha_1+1, \alpha_2), (\alpha_1, \alpha_2-1)\},$$
$$((\alpha_1, \alpha_2); (\alpha'_1, \alpha'_2)) \leftrightarrow (\beta; (\alpha'_1-1, \alpha'_2)) \quad \Longleftrightarrow \quad \beta \in \{(\alpha_1-1, \alpha_2), (\alpha_1, \alpha_2+1)\},$$
$$((\alpha_1, \alpha_2); (\alpha'_1, \alpha'_2)) \leftrightarrow (\beta; (\alpha'_1, \alpha'_2+1)) \quad \Longleftrightarrow \quad \beta \in \{(\alpha_1-1, \alpha_2), (\alpha_1, \alpha_2+1)\},$$
$$((\alpha_1, \alpha_2); (\alpha'_1, \alpha'_2)) \leftrightarrow (\beta; (\alpha'_1, \alpha'_2-1)) \quad \Longleftrightarrow \quad \beta \in \{(\alpha_1+1, \alpha_2), (\alpha_1, \alpha_2-1)\}.$$

Now, Lemma 3.7 yields the following equations for $\lambda_{\alpha,\alpha'}^{\beta,\beta'}$: for $\beta' = (\alpha_1' + 1, \alpha_2')$ we obtain

$$\lambda_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}^{(\alpha_1+1,\alpha_2),(\alpha_1'+1,\alpha_2')} + \lambda_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}^{(\alpha_1,\alpha_2-1),(\alpha_1'+1,\alpha_2')} = 1,$$

$$(2\alpha_1 + n)\lambda_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}^{(\alpha_1+1,\alpha_2),(\alpha_1'+1,\alpha_2')} - (2\alpha_2 + n - 2)\lambda_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}^{(\alpha_1,\alpha_2-1),(\alpha_1'+1,\alpha_2')} = 2\alpha_1' + n,$$

which gives

$$\lambda_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}^{(\alpha_1+1,\alpha_2),(\alpha_1'+1,\alpha_2')} = \frac{\alpha_1' + \alpha_2 + n - 1}{\alpha_1 + \alpha_2 + n - 1}, \quad \lambda_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}^{(\alpha_1,\alpha_2-1),(\alpha_1'+1,\alpha_2')} = \frac{\alpha_1 - \alpha_1'}{\alpha_1 + \alpha_2 + n - 1},$$

for $\beta' = (\alpha_1' - 1, \alpha_2')$ we get

$$\lambda_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}^{(\alpha_1-1,\alpha_2),(\alpha_1'-1,\alpha_2')} + \lambda_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}^{(\alpha_1,\alpha_2+1),(\alpha_1'-1,\alpha_2')} = 1,$$

$$(2\alpha_1 + n - 2)\lambda_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}^{(\alpha_1-1,\alpha_2),(\alpha_1'-1,\alpha_2')} - (2\alpha_2 + n)\lambda_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}^{(\alpha_1,\alpha_2+1),(\alpha_1'-1,\alpha_2')} = 2\alpha_1' + n - 4,$$

implying

$$\lambda_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}^{(\alpha_1-1,\alpha_2),(\alpha_1'-1,\alpha_2')} = \frac{\alpha_1' + \alpha_2 + n - 2}{\alpha_1 + \alpha_2 + n - 1}, \quad \lambda_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}^{(\alpha_1,\alpha_2+1),(\alpha_1'-1,\alpha_2')} = \frac{\alpha_1 - \alpha_1' + 1}{\alpha_1 + \alpha_2 + n - 1},$$

and similarly we find

$$\lambda_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}^{(\alpha_1,\alpha_2+1),(\alpha_1',\alpha_2'+1)} = \frac{\alpha_1 + \alpha_2' + n - 1}{\alpha_1 + \alpha_2 + n - 1}, \quad \lambda_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}^{(\alpha_1-1,\alpha_2),(\alpha_1',\alpha_2'+1)} = \frac{\alpha_2 - \alpha_2'}{\alpha_1 + \alpha_2 + n - 1},$$

$$\lambda_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}^{(\alpha_1,\alpha_2-1),(\alpha_1',\alpha_2'-1)} = \frac{\alpha_1 + \alpha_2' + n - 2}{\alpha_1 + \alpha_2 + n - 1}, \quad \lambda_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}^{(\alpha_1+1,\alpha_2),(\alpha_1',\alpha_2'-1)} = \frac{\alpha_2 - \alpha_2' + 1}{\alpha_1 + \alpha_2 + n - 1}.$$

We remark that the constants $\lambda_{\alpha,\alpha'}^{\beta,\beta'}$ can in this case also be obtained by computing the action of $\omega(X)$ on explicit K-finite vectors using (B-6) and recurrence relations for the Jacobi polynomials. With the explicit form of the constants $\lambda_{\alpha,\alpha'}^{\beta,\beta'}$ Corollary 3.6 now provides the following characterization of symmetry-breaking operators:

Theorem 5.1. *An operator $T : \mathcal{E} \to \mathcal{E}'$ is intertwining for π_r and $\tau_{r'}$ if and only if*

$$T|_{\mathcal{E}(\alpha;\alpha')} = \begin{cases} t_{\alpha,\alpha'} \cdot \mathrm{rest}\,|_{\mathcal{E}(\alpha;\alpha')} & \text{for } \alpha_1 - \alpha_2 = \alpha_1' - \alpha_2', \\ 0 & \text{else,} \end{cases}$$

with numbers $t_{\alpha,\alpha'}$ satisfying the following four relations:

$$(5\text{-}1) \quad (\alpha_1 + \alpha_2 + n - 1)(r' + 2\alpha_1' + n - 1)t_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}$$
$$= (\alpha_1' + \alpha_2 + n - 1)(r + 2\alpha_1 + n)t_{(\alpha_1+1,\alpha_2),(\alpha_1'+1,\alpha_2')}$$
$$+ (\alpha_1 - \alpha_1')(r - 2\alpha_2 - n + 2)t_{(\alpha_1,\alpha_2-1),(\alpha_1'+1,\alpha_2')},$$

$$(5\text{-}2) \quad (\alpha_1 + \alpha_2 + n - 1)(r' - 2\alpha_1' - n + 3)t_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}$$
$$= (\alpha_1' + \alpha_2 + n - 2)(r - 2\alpha_1 - n + 2)t_{(\alpha_1-1,\alpha_2),(\alpha_1'-1,\alpha_2')}$$
$$+ (\alpha_1 - \alpha_1' + 1)(r + 2\alpha_2 + n)t_{(\alpha_1,\alpha_2+1),(\alpha_1'-1,\alpha_2')},$$

$$(5\text{-}3) \quad (\alpha_1 + \alpha_2 + n - 1)(r' + 2\alpha_2' + n - 1)t_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}$$
$$= (\alpha_1 + \alpha_2' + n - 1)(r + 2\alpha_2 + n)t_{(\alpha_1,\alpha_2+1),(\alpha_1',\alpha_2'+1)}$$
$$+ (\alpha_2 - \alpha_2')(r - 2\alpha_1 - n + 2)t_{(\alpha_1-1,\alpha_2),(\alpha_1',\alpha_2'+1)},$$

$$(5\text{-}4) \quad (\alpha_1 + \alpha_2 + n - 1)(r' - 2\alpha_2' - n + 3)t_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}$$
$$= (\alpha_1 + \alpha_2' + n - 2)(r - 2\alpha_2 - n + 2)t_{(\alpha_1,\alpha_2-1),(\alpha_1',\alpha_2'-1)}$$
$$+ (\alpha_2 - \alpha_2' + 1)(r + 2\alpha_1 + n)t_{(\alpha_1+1,\alpha_2),(\alpha_1',\alpha_2'-1)}.$$

5C. *Multiplicities.* The $(\mathfrak{g}, K)$-module $(\pi_r)_{\mathrm{HC}}$ is reducible if and only if $r \in \pm(\rho + 2\mathbb{N})$. More precisely, for $r = -\rho - 2i$ the module $(\pi_r)_{\mathrm{HC}}$ contains a unique nontrivial finite-dimensional $(\mathfrak{g}, K)$-submodule

$$\mathcal{F}(i) = \bigoplus_{\alpha_1,\alpha_2=0}^{i} \mathcal{E}(\alpha_1, \alpha_2)$$

as well as the two nontrivial infinite-dimensional submodules

$$\mathcal{F}_+(i) = \bigoplus_{\alpha_1=0}^{\infty} \bigoplus_{\alpha_2=0}^{i} \mathcal{E}(\alpha_1, \alpha_2), \quad \mathcal{F}_-(i) = \bigoplus_{\alpha_1=0}^{i} \bigoplus_{\alpha_2=0}^{\infty} \mathcal{E}(\alpha_1, \alpha_2).$$

Then the composition series of $(\pi_r)_{\mathrm{HC}}$ is given by

$$\{0\} \subseteq \mathcal{F}(i) \subseteq \mathcal{F}_+(i) \subseteq (\mathcal{F}_+(i) + \mathcal{F}_-(i)) \subseteq \mathcal{E}$$

(or equivalently with $\mathcal{F}_+$ and $\mathcal{F}_-$ switched). Hence the quotients

$$\mathcal{T}(i) = \mathcal{E}/(\mathcal{F}_+(i) + \mathcal{F}_-(i)) \quad \text{and} \quad \mathcal{T}_\pm(i) = \mathcal{F}_\pm(i)/\mathcal{F}(i)$$

are irreducible and infinite-dimensional. Similarly we denote by $\mathcal{F}'(j)$, $\mathcal{F}'_\pm(j)$ and $\mathcal{T}'(j)$, $\mathcal{T}'_\pm(j)$ the corresponding composition factors of $(\tau_{r'})_{\mathrm{HC}}$ for $r' = -\rho' - 2j$, $j \in \mathbb{N}$.

Define

$$L = \{(r, r') \in \mathbb{C}^2 : r = -\rho - 2i, \ r' = -\rho' - 2j, \ 0 \le j \le i\}.$$

Theorem 5.2. (1) *The multiplicities between spherical principal series of G and G' are given by*

$$m((\pi_r)_{\mathrm{HC}}, (\tau_{r'})_{\mathrm{HC}}) = \begin{cases} 1 & \text{for } (r, r') \in \mathbb{C}^2 \setminus L, \\ 2 & \text{for } (r, r') \in L. \end{cases}$$

(2) *For $i, j \in \mathbb{N}$ the multiplicities $m(\mathcal{V}, \mathcal{W})$ between subquotients are given by*

$\mathcal{V}\downarrow \ \mathcal{W}\rightarrow$	$\mathcal{F}'(j)$	$\mathcal{T}'_+(j)$	$\mathcal{T}'_-(j)$	$\mathcal{T}'(j)$	
$\mathcal{F}(i)$	1	0	0	0	
$\mathcal{T}_+(i)$	0	1	0	0	*for $j \leq i$,*
$\mathcal{T}_-(i)$	0	0	1	0	
$\mathcal{T}(i)$	0	0	0	1	

$\mathcal{V}\downarrow \ \mathcal{W}\rightarrow$	$\mathcal{F}'(j)$	$\mathcal{T}'_+(j)$	$\mathcal{T}'_-(j)$	$\mathcal{T}'(j)$	
$\mathcal{F}(i)$	0	0	0	0	
$\mathcal{T}_+(i)$	0	0	0	0	*otherwise.*
$\mathcal{T}_-(i)$	0	0	0	0	
$\mathcal{T}(i)$	1	0	0	0	

To prove Theorem 5.2 we proceed approximately as in Section 4C. For this we first reduce the four relations (5-1)–(5-4) in the four parameters $\alpha_1, \alpha_2, \alpha'_1, \alpha'_2$ with $\alpha_1 - \alpha_2 = \alpha'_1 - \alpha'_2$ to two pairs of two relations with only two parameters.

Put

$$p = \alpha_1 + \alpha_2, \quad q_1 = \alpha'_1, \quad q_2 = \alpha'_2.$$

Then

$$\alpha_1 = \tfrac{p+q_1-q_2}{2}, \quad \alpha_2 = \tfrac{p-q_1+q_2}{2}, \quad \alpha'_1 = q_1, \quad \alpha'_2 = q_2.$$

Then $0 \leq \alpha'_1 \leq \alpha_1$, $0 \leq \alpha'_2 \leq \alpha_2$, and $\alpha_1 - \alpha_2 = \alpha'_1 - \alpha'_2$ if and only if $p, q_1, q_2 \in \mathbb{N}$ with $p - q_1 - q_2 \in 2\mathbb{N}$. With this reparametrization, the parameter q_2 is constant in the identities (5-1) and (5-2) and the parameter q_1 is constant in (5-3) and (5-4). Abusing notation and writing t_{p,q_1,q_2} for $t_{(\alpha_1,\alpha_2),(\alpha'_1,\alpha'_2)}$ the relations (5-1)–(5-4) become

$$(5\text{-}5) \quad (p+n-1)(r'+2q_1+n-1)t_{p,q_1,q_2}$$
$$= \left(\tfrac{p+q_1+q_2}{2}+n-1\right)(r+p+q_1-q_2+n)t_{p+1,q_1+1,q_2}$$
$$+ \left(\tfrac{p-q_1-q_2}{2}\right)(r-p+q_1-q_2-n+2)t_{p-1,q_1+1,q_2},$$

$$(5\text{-}6) \quad (p+n-1)(r'-2q_1-n+3)t_{p,q_1,q_2}$$
$$= \left(\tfrac{p+q_1+q_2}{2}+n-2\right)(r-p-q_1+q_2-n+2)t_{p-1,q_1-1,q_2}$$
$$+ \left(\tfrac{p-q_1-q_2}{2}+1\right)(r+p-q_1+q_2+n)t_{p+1,q_1-1,q_2},$$

$$(5\text{-}7) \quad (p+n-1)(r'+2q_2+n-1)t_{p,q_1,q_2}$$
$$= \left(\tfrac{p+q_1+q_2}{2}+n-1\right)(r+p-q_1+q_2+n)t_{p+1,q_1,q_2+1}$$
$$+ \left(\tfrac{p-q_1-q_2}{2}\right)(r-p-q_1+q_2-n+2)t_{p-1,q_1,q_2+1},$$

$$(5\text{-}8) \quad (p+n-1)(r'-2q_2-n+3)t_{p,q_1,q_2}$$
$$= \left(\tfrac{p+q_1+q_2}{2}+n-2\right)(r-p+q_1-q_2-n+2)t_{p-1,q_1,q_2-1}$$
$$+ \left(\tfrac{p-q_1-q_2}{2}+1\right)(r+p+q_1-q_2+n)t_{p+1,q_1,q_2-1}.$$

Note that q_2 is fixed in (5-5) and (5-6), and these relations hold for $p, q_1 \in \mathbb{N}$ with $p - q_1 \in q_2 + 2\mathbb{N}$. The obvious similar statement holds for (5-7) and (5-8).

We first consider the diagonal $p = q_1 + q_2$; then relations (5-5) and (5-7) simplify to

$$(5\text{-}9) \qquad (r' + 2q_1 + n - 1)t_{q_1+q_2,q_1,q_2} = (r + 2q_1 + n)t_{q_1+q_2+1,q_1+1,q_2},$$

$$(5\text{-}10) \qquad (r' + 2q_2 + n - 1)t_{q_1+q_2,q_1,q_2} = (r + 2q_2 + n)t_{q_1+q_2+1,q_1,q_2+1}.$$

This immediately yields:

Lemma 5.3. (1) *For $(r, r') \in \mathbb{C}^2 \setminus L$ the space of diagonal sequences $(t_{q_1+q_2,q_1,q_2})_{q_1,q_2}$ satisfying (5-9) and (5-10) has dimension 1. Any generator $(t_{q_1+q_2,q_1,q_2})_{q_1,q_2}$ satisfies*:

(a) *For $r \notin -\rho - 2\mathbb{N}$, $r' \notin -\rho' - 2\mathbb{N}$,*

$$t_{q_1+q_2,q_1,q_2} \neq 0 \quad \text{for all } q_1, q_2 \in \mathbb{N}.$$

(b) *For $r = -\rho - 2i \in -\rho - 2\mathbb{N}$, $r' \notin -\rho' - 2\mathbb{N}$,*

$$t_{q_1+q_2,q_1,q_2} = 0 \quad \text{for all } q_1 \leq i \text{ or } q_2 \leq i \qquad \text{and} \qquad t_{q_1+q_2,q_1,q_2} \neq 0 \quad \text{for all } q_1, q_2 > i.$$

(c) *For $r \notin -\rho - 2\mathbb{N}$, $r' = -\rho' - 2j \in -\rho' - 2\mathbb{N}$,*

$$t_{q_1+q_2,q_1,q_2} \neq 0 \quad \text{for all } q_1, q_2 \leq j \qquad \text{and} \qquad t_{q_1+q_2,q_1,q_2} = 0 \quad \text{for all } q_1 > j \text{ or } q_2 > j.$$

(d) *For $r = -\rho - 2i \in -\rho - 2\mathbb{N}$, $r' = -\rho' - 2j \in -\rho' - 2\mathbb{N}$ with $i < j$,*

$$t_{q_1+q_2,q_1,q_2} \neq 0 \quad \text{for all } i < q_1, q_2 \leq j \qquad \text{and} \qquad t_{q_1+q_2,q_1,q_2} = 0 \quad \text{else.}$$

(2) *For $(r, r') = (-\rho - 2i, -\rho' - 2j) \in L$ the space of diagonal sequences $(t_{q_1+q_2,q_1,q_2})_{q_1,q_2}$ satisfying (5-9) and (5-10) has dimension 4.*

Next we investigate how a diagonal sequence $(t_{q_1+q_2,q_1,q_2})_{q_1,q_2}$ satisfying (5-9) and (5-10) can be extended to a sequence $(t_{p,q_1,q_2})_{p,q_1,q_2}$ satisfying (5-5) and (5-6) and the corresponding relations in q_2. For this note that if we fix, say, q_2, and put $p' = p - q_2$, then the relations (5-5) and (5-6) read

$$(5\text{-}11) \quad (p' + q_2 + n - 1)(r' + 2q_1 + n - 1)t_{p',q_1}$$
$$= \left(\tfrac{p'+q_1}{2} + q_2 + n - 1\right)(r + p' + q_1 + n)t_{p'+1,q_1+1}$$
$$+ \left(\tfrac{p'-q_1}{2}\right)(r - p' + q_1 - 2q_2 - n + 2)t_{p'-1,q_1+1},$$

$$(5\text{-}12) \quad (p' + q_2 + n - 1)(r' - 2q_1 - n + 3)t_{p',q_1}$$
$$= \left(\tfrac{p'+q_1}{2} + q_2 + n - 2\right)(r - p' - q_1 - n + 2)t_{p'-1,q_1-1}$$
$$+ \left(\tfrac{p'-q_1}{2} + 1\right)(r + p' - q_1 + 2q_2 + n)t_{p'+1,q_1-1},$$

where we again abuse notation and write t_{p',q_1} for t_{p,q_1,q_2}. Similar relations hold if q_1 is fixed. We note that (5-11) and (5-12) have to be satisfied for all $p', q_1 \in \mathbb{N}$ with
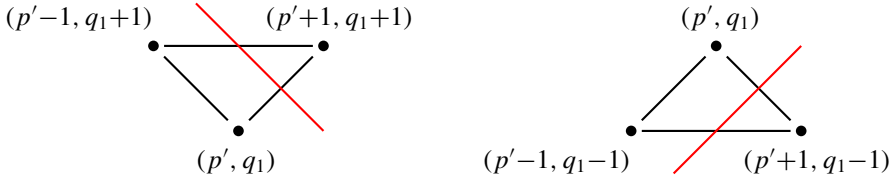
Figure 10. Barriers for $r = -\rho - 2i$.

$p' - q_1 \in 2\mathbb{N}$, just as in the case of orthogonal groups, see Figure 1. Thus, many arguments used in the orthogonal situation can be translated to this context. There are, however, differences to the orthogonal situation. If $r = -\rho - 2i \in -\rho - 2\mathbb{N}$ then the coefficient $(r + p' + q_1 + n)$ in (5-11) vanishes for $p' + q_1 = 2i$ and the coefficient $(r + p' - q_1 + 2q_2 + n)$ in (5-12) vanishes for $p' - q_1 = 2(i - q_2)$, which we indicate by diagonal lines as in Figure 10. Further, if $r' = -\rho' - 2j \in -\rho' - 2\mathbb{N}$ then the coefficient $(r' + 2q_1 + n - 1)$ in (5-11) vanishes for $q_1 = j$, which we indicate by a vertical line as in Figure 11.

Lemma 5.4. *Let* $(r, r') \in \mathbb{C}^2 \setminus L$. *Then every diagonal sequence* $(t_{q_1+q_2,q_1,q_2})_{q_1,q_2}$ *satisfying* (5-9) *and* (5-10) *has a unique extension to a sequence* $(t_{p,q_1,q_2})_{p,q_1,q_2}$ *satisfying* (5-5)–(5-8).

Proof. The proof is similar to the proof of Lemma 4.4 and we only indicate the relevant steps.

Step 1. We first treat the case $r \notin -\rho - 2\mathbb{N}$. We fix q_2; then the diagonal sequence determines t_{p',q_1} for $p' = q_1$. Since $r \notin -\rho - 2\mathbb{N}$ the coefficient $(r + p' - q_1 + 2q_2 + n)$ in (5-12) never vanishes. Hence, (5-12) can be used to express $t_{p'+1,q_1-1}$ in terms of t_{p',q_1} and $t_{p'-1,q_1-1}$. As in the proof of Lemma 4.4, Step 1, this uniquely determines all numbers t_{p',q_1}. Since q_2 was arbitrary this determines all numbers t_{p,q_1,q_2}.

Step 2. Next assume $r = -\rho - 2i \in -\rho - 2\mathbb{N}$ and $r' \notin -\rho' - 2\mathbb{N}$. Then the coefficient $(r + p' - q_1 + 2q_2 + n)$ vanishes if and only if $p' - q_1 = 2(i - q_2)$. In particular, it does not vanish for $q_2 > i$. We can therefore use the technique in Step 1 to extend the diagonal sequence to t_{p,q_1,q_2} for $q_2 > i$ and all p, q_1. Fixing q_1 instead of q_2 we are in the situation that t_{p',q_2} is given on the diagonal $p' = q_2$ and in the region $q_2 > i$. Since $r' \notin -\rho' - 2\mathbb{N}$ the coefficient $(r' + 2q_2 + n - 1)$ in (5-11) (with q_1 and
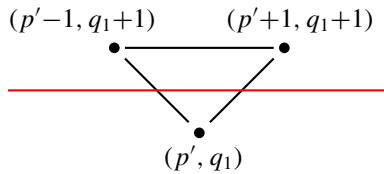
Figure 11. Barrier for $r' = -\rho' - 2j$.

q_2 interchanged) never vanishes, so we can use (5-11) (with q_1 and q_2 interchanged) to extend t_{p',q_2} to all p', q_2 as in the proof of Lemma 4.4 Step 2. Since q_1 was arbitrary this determines all numbers t_{p,q_1,q_2}.

Step 3. Now let $r = -\rho - 2i \in -\rho - 2\mathbb{N}$ and $r' = -\rho' - 2j \in -\rho' - 2\mathbb{N}$, with $i, j \in \mathbb{N}$, $j > i$. Note that to carry out Step 2 we only need that $r' + 2q_2 + n - 1 \neq 0$ for $q_2 \leq i$. This is satisfied since

$$r' + 2q_2 + n - 1 = 2(q_2 - j) < 2(q_2 - i) \leq 0$$

by assumption. Hence the technique in Step 2 carries over to this case. $\square$

The case $(r, r') \in L$ has to be handled a little differently from the orthogonal situation.

Lemma 5.5. *Let* $(r, r') = (-\rho - 2i, -\rho' - 2j) \in L$. *Then every choice of* $t_{0,0,0}$ *and* $t_{2i+2,0,0}$ *determines a unique sequence* $(t_{p,q_1,q_2})_{p,q_1,q_2}$ *satisfying* (5-5)–(5-8).

Proof. Fix $q_2 = 0$, $p' = p - q_2 = p$; then by the assumption t_{p',q_1} is known for $(p', q_1) = (0, 0)$ and $(2i+2, 0)$. This is illustrated in Figure 12, where the barriers are as in Figures 10 and 11. Then the techniques from the proof of Lemma 5.4 extend $t_{0,0}$ uniquely to the region $p' + q_1 \leq 2i$; see also Figure 12. To overcome the barrier given by $p' + q_1 = 2i$ we use (5-12) for $p' - q_1 = 2i$ in which the coefficient $(r + p' - q_1 + 2q_2 + n)$ vanishes. Hence, this relation can be applied to extend along the diagonal line $p' - q_1 = 2i$ as indicated in Figure 12. It may also be applied anywhere above the diagonal $p' - q_1 = 2i$ so that we actually extend to the area $p' - q_1 \leq 2i$; see Figure 13. Next we need to overcome the barrier $p' - q_1 = 2i$, which we do by using (5-11) for $q_1 = j$. In this relation the coefficient $(r' + 2q_1 + n - 1)$ vanishes, and hence we can extend along the line $q_1 = j + 1$. Using again (5-12) even extends to the whole region
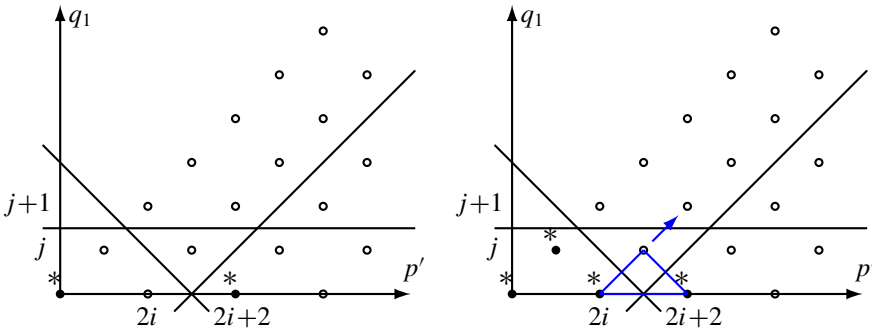
Legend: • K'-types with t_{p',q_1} already defined
          ∘ K'-types with t_{p',q_1} yet to define

Figure 12

Legend: • K'-types with t_{p',q_1} already defined
          ∘ K'-types with t_{p',q_1} yet to define
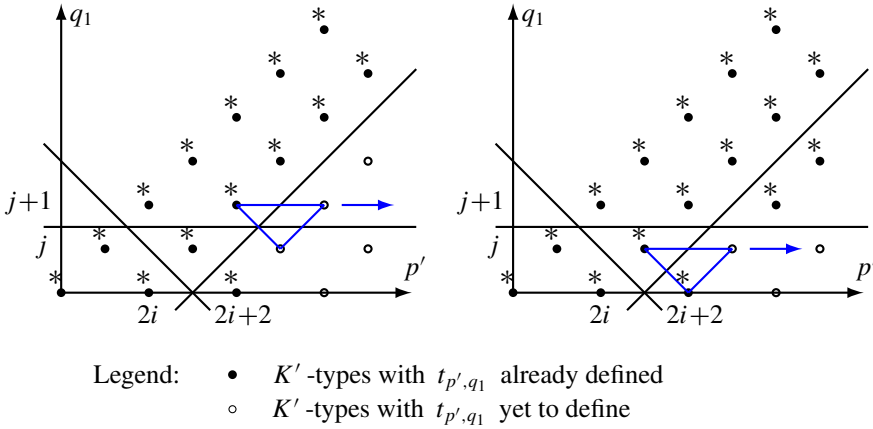
Figure 13

$q_1 > j$; see Figure 13. We note that up to this point we have not yet made use of $t_{2i+2,0}$. This is needed now to extend into the region $\{(p', q_1) : p' - q_1 > 2i,\ q_1 \leq j\}$; see Figure 13. Here both relations (5-11) and (5-12) are needed. Summarizing, we have extended $t_{0,0,0}$ and $t_{2i+2,0,0}$ uniquely to a sequence $(t_{p,q_1,0})_{p,q_1}$. Next fix q_1 and let $p' = p - q_1$. Then t_{p',q_2} is already determined for $(p', q_2) = (p', 0)$ with p' arbitrary; see Figure 14. Note that in relation (5-12) (with q_1 and q_2 interchanged) the coefficient $(r' - 2q_2 - n + 3)$ never vanishes, and hence this relation can be used to extend $(t_{p',0})_{p'}$ uniquely to $(t_{p',q_2})_{p',q_2}$; see Figure 14. Since q_1 was arbitrary this finally yields t_{p,q_1,q_2} for any p, q_1, q_2 and finishes the proof.   □

Proof of Theorem 5.2. (1) This statement is contained in Lemmas 5.3, 5.4, and 5.5.

(2) Composing with embeddings and quotient maps most of the multiplicity statements can be reduced to statements about the (non-)existence of intertwining operators $T : (\pi_r)_{\mathrm{HC}} \to (\tau_{r'})_{\mathrm{HC}}$ for particular r and r' such that the numbers

Legend: • K'-types with t_{p',q_1} already defined
          ∘ K'-types with t_{p',q_1} yet to define
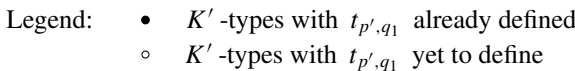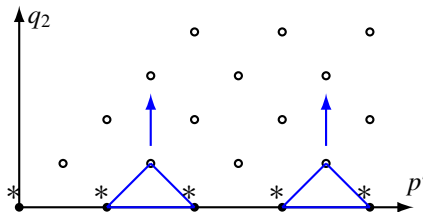
Figure 14

$t_{(\alpha_1,\alpha_2),(\alpha'_1,\alpha'_2)}$ vanish in certain regions. These statements can be checked using the techniques used in Lemmas 5.3, 5.4 and 5.5. This does not work if either $\mathcal{V} = \mathcal{T}_{\pm}(i)$ or $\mathcal{W} = \mathcal{T}'_{\pm}(j)$. We therefore show the multiplicity statements for $m(\mathcal{T}_+(i), \mathcal{T}_+(j))$ in detail, using Remark 3.5. Similar considerations can then be applied to the remaining cases.

Let first $\mathcal{V} = \mathcal{T}_+(i)$ and $\mathcal{W} = \mathcal{T}'_+(j)$. Then, due to Remark 3.5, an intertwining operator $\mathcal{T}_+(i) \to \mathcal{T}_+(j)$ is given by an operator

$$T : \mathcal{F}_+(i) \to \bigoplus_{\alpha'_1=j+1}^{\infty} \bigoplus_{\alpha'_2=0}^{j} \mathcal{E}'(\alpha'_1, \alpha'_2), \quad T|_{\mathcal{E}(\alpha;\alpha')} = t_{\alpha,\alpha'} \cdot R_{\alpha,\alpha'}$$

such that $T|_{\mathcal{F}(i)} = 0$, and the numbers $t_{\alpha,\alpha'}$ solve the relations (5-1)–(5-4) whenever the two terms $t_{(\beta_1,\beta_2),(\beta'_1,\beta'_2)}$ on the right-hand sides of (5-1)–(5-4) satisfy $\beta'_1 > j$, $\beta'_2 \le j$ (i.e., the two upper, resp. lower, vertices of the corresponding triangles are contained in the region $\{(\beta'_1, \beta'_2) : \beta'_1 > j, \ \beta'_2 \le j\}$).

Assume first that $j > i$. Then for any fixed $q_2 \le j$ and $p' = p - q_2$ we are looking for numbers t_{p',q_1} which vanish if either $q_1 \le j$ (i.e., $\alpha'_1 \le j$, the region below the horizontal line in Figure 15) or $p' - q_1 > 2(i - q_2)$ (i.e., $\alpha_2 > i$, the region below the diagonal line going into the upper right corner in Figure 15). As indicated in Figure 15, relation (5-11) can be used along the diagonal to obtain $t_{q_1,q_1} = 0$ for $q_1 > j$. Then using (5-12) yields $t_{p',q_1} = 0$ for all p', q_1, so that $m(\mathcal{T}_+(i), \mathcal{T}'_+(j)) = 0$. Next assume $j \le i$. Then for fixed $q_2 \le j$ and $p' = p - q_2$ we have to find numbers as indicated in Figure 16. Here the relations (5-11) and (5-12) don't force any of the numbers in the region $\{(p', q_1) : p' - q_1 \le 2(i - q_2), \ p' + q_1 > 2i\}$ to vanish and hence the choice of one t_{p',q_1} determines the remaining numbers. We note that in this case $t_{p',q_1} = 0$ for $p' + q_1 \le 2i$ and $q_1 > j$ as desired. Similarly, if we fix
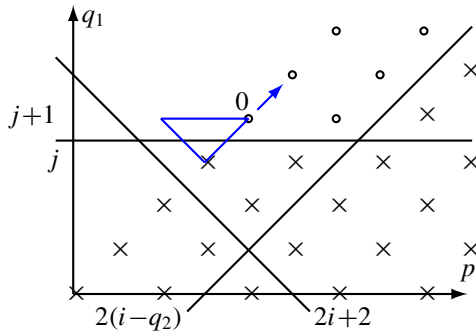
Legend: $\circ$ K'-types with t_{p',q_1} to be determined

           $\times$ K'-types with $t_{p',q_1} = 0$ by formal reasons

Figure 15
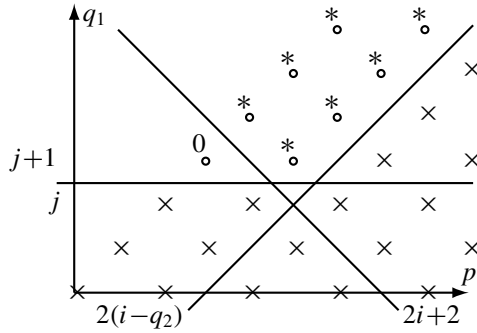
Legend: ∘ K'-types with t_{p',q_1} to be determined
           × K'-types with $t_{p',q_1} = 0$ by formal reasons

Figure 16

$q_1 > j$ and let $p' = p - q_1$ we are in the situation of Figure 17. More precisely, we need to find numbers t_{p',q_2} satisfying the relations (5-11) and (5-12) (with q_1 and q_2 interchanged) in the region $\{(p', q_2) : q_2 \le j,\ p' + q_2 \le 2i\}$ such that $t_{p',q_2} = 0$ for $p' - q_2 \le 2i$. Again the relations do not force any number in the nontrivial region to vanish (indicated by stars in Figure 17). Within this region, the choice of one of the numbers uniquely determines the rest. Together with the previous observation for the case of $q_2 \le j$ fixed we obtain $m(\mathcal{T}_+(i), \mathcal{T}_+(j)) = 1$.   □

5D. *Explicit formula for the spectral function.* As in Section 4D we also find the generic solution to the relations (5-1)–(5-4) as a meromorphic function in $r, r' \in \mathbb{C}$.
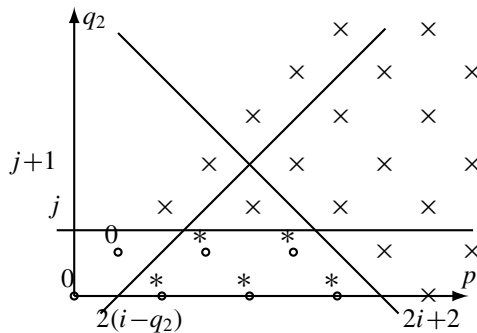
Legend: ∘ K'-types with t_{p',q_1} to be determined
           × K'-types with $t_{p',q_1} = 0$ by formal reasons

Figure 17

Proposition 5.6. *For $\alpha_1, \alpha_2 \in \mathbb{N}$ and $0 \leq \alpha_1' \leq \alpha_1$, $0 \leq \alpha_2' \leq \alpha_2$, with $\alpha_1 - \alpha_2 = \alpha_1' - \alpha_2'$, the numbers*

$$t_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}(r,r')$$

$$= \sum_{k=0}^{\infty} \frac{2^k \Gamma\left(\frac{\alpha_1+\alpha_2-\alpha_1'-\alpha_2'+2}{2}\right)\Gamma\left(\frac{\alpha_1+\alpha_2+\alpha_1'+\alpha_2'}{2}+n-1+k\right)}{(k!)^2 \Gamma\left(\frac{\alpha_1+\alpha_2-\alpha_1'-\alpha_2'+2}{2}-k\right)\Gamma\left(\frac{\alpha_1+\alpha_2+\alpha_1'+\alpha_2'}{2}+n-1\right)}$$

$$\times \frac{\Gamma\left(\frac{r+n}{2}\right)^2\Gamma\left(\frac{r'+n-1}{2}+\alpha_1'\right)\Gamma\left(\frac{r'+n-1}{2}+\alpha_2'\right)\Gamma\left(\frac{r'-r+1}{2}\right)\Gamma\left(\frac{r'+r+1}{2}+k\right)}{\Gamma\left(\frac{r'+n-1}{2}\right)^2\Gamma\left(\frac{r+n}{2}+\alpha_1'+k\right)\Gamma\left(\frac{r+n}{2}+\alpha_2'+k\right)\Gamma\left(\frac{r'+r+1}{2}\right)\Gamma\left(\frac{r'-r+1}{2}-k\right)}$$

are rational functions in r and r' satisfying the relations (5-1)–(5-4). They are normalized to $t_{(0,0),(0,0)} \equiv 1$.

Proof. The proof is similar to the proof of Proposition 4.6 and we omit some of the details. For simplicity we use the reparametrization (p, q_1, q_2) instead of (α_1, α_2) and (α_1', α_2'). Fix q_2 and let $p' = p - q_2$; then it is easy to see that for every $k \in \mathbb{N}$ the expression

$$\frac{\Gamma\left(\frac{p'-q_1+2}{2}\right)\Gamma\left(\frac{p'+q_1}{2}+q_2+n-1+k\right)\Gamma\left(\frac{r'+2q_1+n-1}{2}\right)}{\Gamma\left(\frac{p'-q_1+2}{2}-k\right)\Gamma\left(\frac{p'+q_1}{2}+q_2+n-1\right)\Gamma\left(\frac{r+2q_1+n}{2}+k\right)}$$

satisfies (5-11). Further, the series

$$\sum_{k=0}^{\infty} b_k \frac{\Gamma\left(\frac{p'-q_1+2}{2}\right)\Gamma\left(\frac{p'+q_1}{2}+q_2+n-1+k\right)\Gamma\left(\frac{r'+2q_1+n-1}{2}\right)}{\Gamma\left(\frac{p'-q_1+2}{2}-k\right)\Gamma\left(\frac{p'+q_1}{2}+q_2+n-1\right)\Gamma\left(\frac{r+2q_1+n}{2}+k\right)}$$

satisfies (5-12) if and only if

$$b_k = c \frac{2^k \Gamma\left(\frac{r'+r+1}{2}+k\right)}{(k!)^2 \Gamma\left(\frac{r+2q_2+n}{2}+k\right)\Gamma\left(\frac{r'-r+1}{2}-k\right)}$$

for some constant $c = c(r, r', q_2)$ which does not depend on p', q_1 and k. Plugging in $p' = p - q_2$, using the symmetry of the relations (5-5)–(5-8) in q_1 and q_2, and normalizing to $t_{0,0,0} \equiv 0$ yields

$$(5\text{-}13) \quad t_{p,q_1,q_2}(r,r')$$

$$= \sum_{k=0}^{\infty} \frac{2^k \Gamma\left(\frac{p-q_1-q_2+2}{2}\right)\Gamma\left(\frac{p+q_1+q_2}{2}+n-1+k\right)}{(k!)^2 \Gamma\left(\frac{p-q_1-q_2+2}{2}-k\right)\Gamma\left(\frac{p+q_1+q_2}{2}+n-1\right)}$$

$$\times \frac{\Gamma\left(\frac{r+n}{2}\right)^2\Gamma\left(\frac{r'+n-1}{2}+q_1\right)\Gamma\left(\frac{r'+n-1}{2}+q_2\right)\Gamma\left(\frac{r'-r+1}{2}\right)\Gamma\left(\frac{r'+r+1}{2}+k\right)}{\Gamma\left(\frac{r'+n-1}{2}\right)^2\Gamma\left(\frac{r+n}{2}+q_1+k\right)\Gamma\left(\frac{r+n}{2}+q_2+k\right)\Gamma\left(\frac{r'+r+1}{2}\right)\Gamma\left(\frac{r'-r+1}{2}-k\right)}$$

Reparametrizing p, q_1, q_2 to $\alpha_1, \alpha_2, \alpha_1', \alpha_2'$ shows the claimed formula. Rewriting (5-13) as

$$(5\text{-}14) \quad t_{p,q_1,q_2}(r, r')$$

$$= \sum_{k=0}^{\frac{p-q_1-q_2}{2}} \frac{1}{(k!)^2 \left(\frac{r+n}{2}\right)_{q_1+k} \left(\frac{r+n}{2}\right)_{q_2+k}} 2^k \left(-\tfrac{p-q_1-q_2}{2}\right)_k \left(\tfrac{p+q_1+q_2}{2} + n - 1\right)_k$$

$$\times \left(\tfrac{r'+n-1}{2}\right)_{q_1} \left(\tfrac{r'+n-1}{2}\right)_{q_2} \left(\tfrac{r-r'+1}{2}\right)_k \left(\tfrac{r'+r+1}{2}\right)_k$$

further shows that this is a rational function in r and r'. $\qquad\square$

Also the next two results are proven along the same lines as Corollary 4.7 and Theorem 4.9.

Corollary 5.7. (1) *The renormalized numbers*

$$t^{(1)}_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}(r, r') = \frac{1}{\Gamma\left(\frac{r+\rho}{2}\right)^2} t_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}(r, r')$$

are holomorphic in $(r, r') \in \mathbb{C}^2$ *for all* $(\alpha_1, \alpha_2), (\alpha_1', \alpha_2')$. *Further,*

$$t^{(1)}_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}(r, r') = 0$$

for all $(\alpha_1, \alpha_2), (\alpha_1', \alpha_2')$ *if and only if* $(r, r') \in L$.

(2) *Fix* $r' = -\rho' - 2j$, $j \in \mathbb{N}$; *then the renormalized numbers*

$$t^{(2)}_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}(r, r') = \frac{\Gamma\left(\frac{(r+\rho)-(r'+\rho')}{2}\right)}{\Gamma\left(\frac{r+\rho}{2}\right)^2} t_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}(r, r')$$

are holomorphic in $r \in \mathbb{C}$ *for all* $(\alpha_1, \alpha_2), (\alpha_1', \alpha_2')$. *We have* $t^{(2)}_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}(r, r') \equiv 0$ *whenever* $\alpha_1' > j$ *or* $\alpha_2' > j$. *Further, for every* $r \in \mathbb{C}$ *there exist* $(\alpha_1, \alpha_2), (\alpha_1', \alpha_2')$ *with* $t^{(2)}_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}(r, r') \neq 0$.

(3) *Fix* $N \in \mathbb{N}$ *and let* $r' + \rho' = r + \rho + 2N$; *then the renormalized numbers*

$$t^{(3)}_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}(r, r') = \frac{\Gamma\left(\frac{r'+\rho'}{2}\right)^2}{\Gamma\left(\frac{r+\rho}{2}\right)^2} t_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}(r, r')$$

are holomorphic in $r \in \mathbb{C}$ *for all* $(\alpha_1, \alpha_2), (\alpha_1', \alpha_2')$. *Further, for every* $r \in \mathbb{C}$ *there exists* $\alpha_0 \in \mathbb{N}$ *such that* $t^{(3)}_{(\alpha_1,\alpha_2),(\alpha_1,\alpha_2)}(r, r') \neq 0$ *for* $\alpha_1, \alpha_2 \geq \alpha_0$.

Theorem 5.8. *For* $i = 1, 2, 3$ *we let* $T^{(i)}(r, r')$ *be the intertwining operators* $(\pi_r)_{\mathrm{HC}} \to (\tau_{r'})_{\mathrm{HC}}$ *corresponding to the numbers* $t^{(i)}_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}(r, r')$ *in Corollary 5.7. Then the operator* $T^{(1)}(r, r')$ *is defined for* $(r, r') \in \mathbb{C}^2$, *the operator* $T^{(2)}(r, r')$ *is*

defined for $r' \in -\rho' - 2\mathbb{N}$ and the operator $T^{(3)}(r, r')$ is defined for $(r + \rho) - (r' + \rho') \in -2\mathbb{N}$. We have

$$\mathrm{Hom}_{(\mathfrak{g}', K')}((\pi_r)_{\mathrm{HC}}|_{(\mathfrak{g}', K')}, (\tau_{r'})_{\mathrm{HC}})$$
$$= \begin{cases} \mathbb{C}T^{(1)}(r, r') & \textit{for } (r, r') \in \mathbb{C}^2 \setminus L, \\ \mathbb{C}T^{(2)}(r, r') \oplus \mathbb{C}T^{(3)}(r, r') & \textit{for } (r, r') \in L. \end{cases}$$

Remark 5.9. We remark that also every intertwining operator between subquotients $\mathcal{V} = \mathcal{F}(i), \mathcal{T}_\pm(i), \mathcal{T}(i)$ and $\mathcal{W} = \mathcal{F}'(j), \mathcal{T}'_\pm(j), \mathcal{T}'(j)$ can be obtained from the holomorphic family $T^{(i)}(r, r')$ by restricting and renormalizing. More precisely, if $\mathcal{V}$ is a quotient of $(\pi_r)_{\mathrm{HC}}$ and $\mathcal{W}$ is a subrepresentation of $(\tau_{r'})_{\mathrm{HC}}$ then any intertwining operator $T : \mathcal{V} \to \mathcal{W}$ gives rise to an intertwining operator $(\pi_r)_{\mathrm{HC}} \to (\tau_{r'})_{\mathrm{HC}}$ and is hence of the form $T^{(i)}(r, r')$ for some $i = 1, 2, 3$. This constructs all except the intertwiners $\mathcal{T}_\pm(i) \to \mathcal{T}'_\pm(j)$ for $0 \le j \le i$. These can be obtained from $T^{(1)}(r, r')$ as follows:

We first construct an intertwining operator $T^+ : \mathcal{F}_+(i) \to (\tau_{r'})_{\mathrm{HC}}$ for $r' = -\rho' - 2j$ such that $T^+(\mathcal{F}_+(i)) \subseteq \mathcal{F}_+(j)$. Since $\mathcal{F}_+(i)$ consists of all K-type $\mathcal{E}(\alpha_1, \alpha_2)$ with $\alpha_2 \le i$ it is given by a sequence $(t^+_{(\alpha_1, \alpha_2), (\alpha'_1, \alpha'_2)})_{\alpha_2 \le i}$. Reparametrizing to p, q_1, q_2 this means that we have to find a sequence $(t^+_{p, q_1, q_2})_{p - q_1 + q_2 \le 2i}$ satisfying the necessary relations. Let $r' + \rho' = r + \rho + 2N$, with $N = i - j \in \mathbb{N}$, and define

$$t^+_{p, q_1, q_2}(r, r') := \frac{\Gamma\left(\frac{r' + \rho'}{2}\right)}{\Gamma\left(\frac{r + \rho}{2}\right)} t_{p, q_1, q_2}(r, r'), \quad p - q_1 + q_2 \le 2i.$$

Then by (5-14) we have

$$t^+_{p, q_1, q_2}(r, r') = \sum_k \frac{1}{(k!)^2 \left(\frac{r+n}{2}\right)_{q_2+k}} 2^k \left(-\frac{p - q_1 - q_2}{2}\right)_k \left(\frac{p + q_1 + q_2}{2} + n - 1\right)_k$$
$$\times \left(\frac{r+n}{2} + k + q_1\right)_{N-k} \left(\frac{r+n}{2} + N\right)_{q_2} (-N)_k (r + N + 1)_k.$$

In the sum all terms for $k > \frac{p - q_1 - q_2}{2}$ vanish, so that $k \le \frac{p - q_1 - q_2}{2} = \frac{p - q_1 + q_2}{2} - q_2 \le i - q_2$. This implies that the denominator does not vanish at $r = -\rho - 2i$. Therefore $t^+_{p, q_1, q_2}(r, r')$ is holomorphic in $r = -\rho - 2i$ and evaluation there yields

$$t^+_{p, q_1, q_2} = t^+_{p, q_1, q_2}(-\rho - 2i, -\rho' - 2j)$$
$$= \sum_{k=0}^{i-j} \frac{1}{(k!)^2 (-i)_{q_2+k}} 2^k \left(-\frac{p - q_1 - q_2}{2}\right)_k$$
$$\times \left(\frac{p + q_1 + q_2}{2} + n - 1\right)_k (k + q_1 - i)_{i-j-k} (-j)_{q_2} (j - i)_k (1 - n - i - j)_k.$$

The sequence t^+_{p, q_1, q_2} clearly satisfies the necessary relations since it is simply a renormalization of the sequence t_{p, q_1, q_2}, and hence it defines an intertwining operator $T^+ : \mathcal{F}_+(i) \to (\tau_{r'})_{\mathrm{HC}}$. We note that for $q_2 > j$ the term $(-j)_{q_2}$ vanishes

so that $t_{p,q_1,q_2}^+ = 0$. Therefore $T^+(\mathcal{F}_+(i)) \subseteq \mathcal{F}'_+(j)$. Composing with the quotient map $\mathcal{F}'_+(j) \to \mathcal{T}'_+(j)$ yields an intertwiner $T^+ : \mathcal{F}_+(i) \to \mathcal{T}'_+(j)$. We claim that this intertwiner vanishes on $\mathcal{F}(i)$ and hence factorizes through $\mathcal{T}_+(i)$. In fact, for $\alpha_2 = \frac{p+q_1-q_2}{2} \leq i$ and $q_1 > j$ we have $\frac{p-q_1-q_2}{2} \leq i - q_1$ so that we may take the sum over all $k \leq i - q_1$. But then $q_1 + k - i \leq 0$ and therefore $(q_1 + k - i)_{i-j-k} = 0$, whence $t_{p,q_1,q_2}^+ = 0$. This implies that $T^+ : \mathcal{F}_+(i) \to \mathcal{T}'_+(j)$ factorizes to an intertwiner $T^+ : \mathcal{T}_+(i) \to \mathcal{T}_+(j)$. To finally see that this intertwiner is nontrivial we note that for all $q_1 > i$, $q_2 \leq j$, and $p = q_1 + q_2$ we have

$$t_{p,q_1,q_2}^+ = \frac{(q_1 - i)_{i-j}(-j)_{q_2}}{(-i)_{q_2}} \neq 0.$$

Remark 5.10. The operators $T^{(1)}(r, r')$ are related to the meromorphic family of singular integral operators constructed in [Möllers et al. 2016a]. Further, the family $T^{(3)}$ is (up to a constant) equal to the differential restriction operators on the Heisenberg group constructed in [Möllers et al. 2016b]. They can be viewed as a generalization of Juhl's conformally invariant operators (see Remark 4.11). It would be interesting to carry out a detailed investigation of all operators $T^{(i)}(r, r')$, $i = 1, 2, 3$, in the noncompact picture as in [Kobayashi and Speh 2015].

As in the real case, we can prove automatic continuity using the full classification in Theorem 5.8 in terms of the holomorphic family $T^{(1)}(r, r')$. Note that the corresponding holomorphic family of intertwining operators in the smooth category was also constructed in [Möllers et al. 2016a].

Corollary 5.11. *For $(G, G') = (\mathrm{U}(1, n), \mathrm{U}(1, n-1))$ the natural injective map*

$$\mathrm{Hom}_{G'}(\pi|_{G'}, \tau) \to \mathrm{Hom}_{(\mathfrak{g}', K')}(\pi_{\mathrm{HC}}|_{(\mathfrak{g}', K')}, \tau_{\mathrm{HC}})$$

is an isomorphism for all spherical principal series π of G and τ of G' and their subquotients.

Appendix A: Orthogonal polynomials

Gegenbauer polynomials. The classical Gegenbauer polynomials $C_n^\lambda(z)$ can be defined by, see [Erdélyi et al. 1953, 10.9, equation (18)],

$$C_n^\lambda(z) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m (\lambda)_{n-m}}{m! \, (n-2m)!} (2z)^{n-2m}.$$

They obviously satisfy the parity condition, see [Erdélyi et al. 1953, 10.9, equation (16)],

(A-1) $$C_n^\lambda(-z) = (-1)^n C_n^\lambda(z).$$

The special value at $z = 0$ can be written as

(A-2)
$$C_n^\lambda(0) = \frac{2^n \sqrt{\pi} \Gamma\left(\lambda + \frac{n}{2}\right)}{n! \, \Gamma\left(\frac{1-n}{2}\right) \Gamma(\lambda)} \overset{(n=2k)}{=} \frac{(-1)^k \Gamma(\lambda + k)}{k! \, \Gamma(\lambda)}.$$

Jacobi polynomials. The classical Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ can be defined by, see [Erdélyi et al. 1953, 10.8, equation (12)],

$$P_n^{(\alpha,\beta)}(z) = 2^{-n} \sum_{m=0}^{n} \binom{n+\alpha}{m} \binom{n+\beta}{n-m} (x-1)^{n-m}(x+1)^m.$$

The special value at $z = 1$ is given by

(A-3)
$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}.$$

Appendix B: Spherical harmonics

Real spherical harmonics. Let $\mathcal{H}^\alpha(\mathbb{R}^n)$ denote the space of harmonic homogeneous polynomials of degree α on $\mathbb{R}^n$. Endowed with the natural action of $O(n)$, the space $\mathcal{H}^\alpha(\mathbb{R}^n)$ is an irreducible representation. It is unitary with respect to the norm on $\mathcal{H}^\alpha(\mathbb{R}^n)$ given by

$$\|\phi\|_{L^2(S^{n-1})}^2 = \int_{S^{n-1}} |\phi(x)|^2 \, dx,$$

where dx denotes the Euclidean measure on S^{n-1}. Upon restriction to the subgroup $O(n-1)$ the representation $\mathcal{H}^\alpha(\mathbb{R}^n)$ decomposes into

(B-1)
$$\mathcal{H}^\alpha(\mathbb{R}^n) \simeq \bigoplus_{0 \le \alpha' \le \alpha} \mathcal{H}^{\alpha'}(\mathbb{R}^{n-1}).$$

Explicit $O(n-1)$-equivariant embeddings of the direct summands are given by, see [Kobayashi and Mano 2011, Fact 7.5.1],

(B-2) $\quad I_{\alpha'\to\alpha}^n : \mathcal{H}^{\alpha'}(\mathbb{R}^{n-1}) \to \mathcal{H}^\alpha(\mathbb{R}^n), \quad I_{\alpha'\to\alpha}^n(\phi)(x', x_n) = \phi(x') C_{\alpha-\alpha'}^{\frac{n-2}{2}+\alpha'}(x_n),$

where $x = (x', x_n) \in S^{n-1}$. The following Plancherel formula holds for $\phi \in \mathcal{H}^{\alpha'}(\mathbb{R}^{n-1})$ (see [Kobayashi and Mano 2011, Fact 7.5.1(3)], note the different normalization of the Gegenbauer polynomials):

(B-3) $\quad \|I_{\alpha'\to\alpha}^n(\phi)\|_{L^2(S^{n-1})}^2 = \dfrac{2^{3-n-2\alpha'} \pi \, \Gamma(n-2+\alpha+\alpha')}{(\alpha-\alpha')! \left(\alpha+\frac{n-2}{2}\right) \Gamma\left(\alpha'+\frac{n-2}{2}\right)^2} \|\phi\|_{L^2(S^{n-2})}^2.$

For $\phi \in \mathcal{H}^\alpha(\mathbb{R}^n)$ we have

(B-4)
$$x_j\phi = \phi_j^+ + |x|^2 \phi_j^-,$$

with $\phi_j^{\pm} \in \mathcal{H}^{\alpha \pm 1}(\mathbb{R}^n)$ given by

$$\phi_j^+ = x_j \phi - \frac{|x|^2}{n + 2\alpha - 2} \frac{\partial \phi}{\partial x_j}, \quad \phi_j^- = \frac{1}{n + 2\alpha - 2} \frac{\partial \phi}{\partial x_j}.$$

Complex spherical harmonics. Identifying $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ we embed $\mathrm{U}(n)$ into $\mathrm{O}(2n)$. Then the restriction of the irreducible representation $\mathcal{H}^\alpha(\mathbb{R}^{2n})$ of $\mathrm{O}(2n)$ to the subgroup $\mathrm{U}(n)$ decomposes into

$$\mathcal{H}^\alpha(\mathbb{R}^{2n}) = \bigoplus_{\alpha_1 + \alpha_2 = \alpha} \mathcal{H}^{\alpha_1, \alpha_2}(\mathbb{C}^n),$$

where $\mathcal{H}^{\alpha_1, \alpha_2}(\mathbb{C}^n)$ denotes the space of harmonic polynomials on $\mathbb{C}^n$ which are holomorphic of degree α_1 and antiholomorphic of degree α_2. Endowed with the natural action of $\mathrm{U}(n)$ the space $\mathcal{H}^{\alpha_1, \alpha_2}(\mathbb{C}^n)$ is an irreducible representation. It is unitary with respect to the norm $\|\cdot\|_{L^2(S^{2n-1})}$, where we view S^{2n-1} as the unit sphere in $\mathbb{C}^n$. Upon restriction to the subgroup $\mathrm{U}(n-1)$ the representation $\mathcal{H}^{\alpha_1, \alpha_2}(\mathbb{C}^n)$ decomposes into

$$(\text{B-5}) \qquad \mathcal{H}^{\alpha_1, \alpha_2}(\mathbb{C}^n) = \bigoplus_{\substack{0 \leq \alpha_1' \leq \alpha_1 \\ 0 \leq \alpha_2' \leq \alpha_2}} \mathcal{H}^{\alpha_1', \alpha_2'}(\mathbb{C}^{n-1}).$$

Explicit $\mathrm{U}(n-1)$-equivariant embeddings

$$I^n_{(\alpha_1', \alpha_2') \to (\alpha_1, \alpha_2)} : \mathcal{H}^{\alpha_1', \alpha_2'}(\mathbb{C}^{n-1}) \to \mathcal{H}^{\alpha_1, \alpha_2}(\mathbb{C}^n)$$

are given by

$$(\text{B-6}) \quad I^n_{(\alpha_1', \alpha_2') \to (\alpha_1, \alpha_2)}(\phi)(z', z_n)$$

$$= \phi(z') \begin{cases} z_n^{(\alpha_1 - \alpha_2) - (\alpha_1' - \alpha_2')} P_{\alpha_2 - \alpha_2'}^{((\alpha_1 - \alpha_2) - (\alpha_1' - \alpha_2'), \alpha_1' + \alpha_2' + n - 2)}(1 - 2|z_n|^2) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } \alpha_1 - \alpha_2 \geq \alpha_1' - \alpha_2', \\ \bar{z}_n^{(\alpha_1' - \alpha_2') - (\alpha_1 - \alpha_2)} P_{\alpha_1 - \alpha_1'}^{((\alpha_1' - \alpha_2') - (\alpha_1 - \alpha_2), \alpha_1' + \alpha_2' + n - 2)}(1 - 2|z_n|^2) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } \alpha_1 - \alpha_2 \leq \alpha_1' - \alpha_2', \end{cases}$$

where $z = (z', z_n) \in S^{2n-1}$. For $\phi \in \mathcal{H}^{\alpha_1, \alpha_2}(\mathbb{C}^n)$ we have

$$(\text{B-7}) \qquad z_j \phi = \phi_j^{+, \text{hol}} + |z|^2 \phi_j^{-, \text{ahol}}, \quad \bar{z}_j \phi = \phi_j^{+, \text{ahol}} + |z|^2 \phi_j^{-, \text{hol}},$$

with $\phi_j^{\pm, \text{hol}} \in \mathcal{H}^{\alpha_1 \pm 1, \alpha_2}(\mathbb{C}^n)$ and $\phi_j^{\pm, \text{ahol}} \in \mathcal{H}^{\alpha_1, \alpha_2 \pm 1}(\mathbb{C}^n)$ given by

$$\phi_j^{+, \text{hol}} = z_j \phi - \frac{|z|^2}{\alpha_1 + \alpha_2 + n - 1} \frac{\partial \phi}{\partial \bar{z}_j}, \quad \phi_j^{-, \text{hol}} = \frac{1}{\alpha_1 + \alpha_2 + n - 1} \frac{\partial \phi}{\partial z_j},$$

$$\phi_j^{+, \text{ahol}} = \bar{z}_j \phi - \frac{|z|^2}{\alpha_1 + \alpha_2 + n - 1} \frac{\partial \phi}{\partial z_j}, \quad \phi_j^{-, \text{ahol}} = \frac{1}{\alpha_1 + \alpha_2 + n - 1} \frac{\partial \phi}{\partial \bar{z}_j}.$$

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