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In the Chow ring of symplectic/odd orthogonal Grassmann bundles the degeneracy loci classes can be expressed as a sum of Schur–Pfaffians. An analogous Schur–Pfaffian formula was obtained for K-theory by the authors together with T. Ikeda and M. Naruse. Here we generalize this explicit formula of degeneracy loci classes to algebraic cobordism, which is universal among all oriented cohomology theories.

1. Introduction

The r-th *degeneracy locus* for a morphism of vector bundles $\varphi : E \to F$ over a smooth quasi-projective scheme M is the subvariety X_r of M consisting of all the points at which the rank of φ is at most r. Assuming φ to be sufficiently general, the classical Giambelli–Thom–Porteous formula describes the Chow ring fundamental class $[X_r]$ as a Schur-determinant in the Chern classes of E and F. Similarly, one can consider more restrictive settings in which φ is either *skewsymmetric* or *symmetric*. In both cases $[X_r]$ is given as a Schur-Pfaffian instead of a Schur-determinant. A more general family of degeneracy loci can be constructed by considering flags of subbundles of E and F and imposing multiple rank conditions.

Fundamental examples of these loci are the Schubert varieties of *isotropic* Grassmannians. The *isotropic* Grassmannian consists of subspaces on which a given symplectic or odd orthogonal form vanishes identically. Inside this ambient space, the degeneracy loci correspond to the Schubert varieties indexed by the combinatorial objects known as k-*strict* partitions.

Pragacz [1991] considered the maximal isotropic case and showed that the Chow ring fundamental classes of Schubert varieties can be expressed through a *Schur-Pfaffian* formula. Kazarian [2000] generalised Pragacz's formula to general degeneracy loci (compare [Ikeda 2007]). Buch, Kresh and Tamvakis [Buch et al. 2017] obtained a *theta polynomial* formula for the non-maximal isotropic Grassmannians, which can also be written as a sum of Schur–Pfaffian. Wilson [2010] conjectured an analogous formula for general degeneracy loci, which was proved

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in [Ikeda and Matsumura 2015] (compare [Tamvakis and Wilson 2016; Anderson and Fulton 2018]).

In recent years, following the trend of generalised Schubert calculus, there has been an attempt to lift results as the ones above from the Chow ring CH^* to other functors like *connective K-theory* CK^* and *algebraic cobordism* Ω^*, highlighting the role played in the formulas by the associated formal group law F and formal inverse χ. In [Hudson et al. 2017], together with T. Ikeda and H. Naruse, we generalized aforementioned results for CH^* to CK^*, and established a Pfaffian-sum formula describing the degeneracy loci classes of symplectic and odd orthogonal Grassmann bundles in CK^*. The goal of this paper is to further extend these formulas to Ω^*.

We begin by explaining our results in the symplectic case. Let $E \to X$ be a vector bundle of rank $2n$ with a nowhere vanishing skewsymmetric form and fix a nonnegative integer $k \leq n$. Consider the *symplectic Grassmann bundle* $SG^k(E)$ whose fiber at $x \in X$ is the Grassmannian of $(n-k)$-dimensional isotropic subspaces of E_x. For each k-strict partition λ, there is the degeneracy locus $X_\lambda \subset SG^k(E)$. Following [Kazarian 2000], we can construct a resolution of singularities $\varpi : Y_\lambda \to X_\lambda$ inside of a certain flag bundle over $SG^k(E)$. In CH^* or CK^*, the fundamental class of X_λ is well-defined and it coincides with the pushforward $\varpi_*[Y_\lambda]$ of the fundamental class of Y_λ along ϖ. *However, in algebraic cobordism, not all degeneracy loci have a well-defined notion of fundamental class. Hence we consider* $\varpi_*[Y_\lambda]$ *as a replacement of* $[X_\lambda]$. As in [Hudson et al. 2017], the fundamental class $[Y_\lambda]$ can be expressed as a product of top Chern classes of certain bundles. In our previous paper [Hudson and Matsumura 2019], we developed a technique to compute the pushforward of such classes along a flag bundle in terms of *relative Segre classes* of vector bundles. With that method at our disposal, we are able to obtain the following description of the class $\varpi_*[Y_\lambda]$ as our main result. The tautological isotropic subbundle of $SG^k(E)$ is denoted by U and the subbundles

$$0 = F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset F^0 \subset F^{-1} \subset \cdots \subset F^{-n} = E,$$

form the reference flag used to define the degeneracy loci. In $\Omega^*(SG^k(E))$, we consider the relative Segre classes

$$\mathscr{C}_m^{(\ell)} := \mathscr{S}_m\left(U^\vee - (E/F^\ell)^\vee\right) \qquad (\forall m \in \mathbb{Z}, -n \leq \forall \ell \leq n).$$

Main Theorem (Theorem 3.9). *Let* $\lambda = (\lambda_1, \ldots, \lambda_r)$ *be a k-strict partition such that* $r \leq n-k$ *and* $\lambda_1 \leq n+k$, *and let* $\boldsymbol{\chi} = (\chi_1, \ldots, \chi_r)$ *be its characteristic index* (see (3-1)). *In* $\Omega^*(SG^k(E))$, *we have*

$$[Y_\lambda \to SG^k(E)] := \varpi_*[Y_\lambda] = \sum_{\mathbf{s}=(s_1,\ldots,s_r)\in\mathbb{Z}^r} f_{\mathbf{s}}^\lambda \mathscr{C}_{\lambda_1+s_1}^{(\chi_1)} \cdots \mathscr{C}_{\lambda_r+s_r}^{(\chi_r)}. \tag{1-1}$$

Here $c_{\mathbf{s}}^{\lambda} \in \mathbb{L}$ are the coefficients of the Laurent series expansion

$$\frac{\prod_{1\le i<j\le r}(1-t_i/t_j)P(t_j,t_i)}{\prod_{(i,j)\in C(\lambda)}(1-\chi(t_i)/t_j)P(t_j,\chi(t_i))} = \sum_{\mathbf{s}=(s_1,\ldots,s_r)\in\mathbb{Z}^r} f_{\mathbf{s}}^{\lambda}\cdot t_1^{s_1}\cdots t_r^{s_r}, \tag{1-2}$$

where $C(\lambda) := \{(i,j) \mid 1 \le i < j \le r,\ \ \chi_i + \chi_j \ge 0\}$ and $P(u,v)$ is the unique power series satisfying $F(u,\chi(v)) = (u-v)P(u,v)$.

Now consider the odd orthogonal Grassmann bundle $OG^k(E)$, with E of rank $2n+1$ and each fiber being an orthogonal Grassmannian of $(n-k)$-dimensional isotropic subspaces. The essential difference with the previous situation is that it is far more complex to deal with the case of quadric bundles $OG^{n-1}(E) = Q(E)$, the orthogonal analogue of projective bundles. Let the reference flag be denoted by

$$0 = F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset F^0 \subset (F^0)^{\perp} \subset F^{-1} \subset \cdots \subset F^{-n} = E.$$

The fundamental classes of the Schubert varieties $X_{(\lambda_1)}$ are actually well-defined in $\Omega^*(Q(E))$ and, as elements of $\Omega^*(Q(E)) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$, they are given by

$$[X_{\lambda_1} \to Q(E)] = \mathscr{B}_{\lambda_1}^{(\chi_1)} := \begin{cases} \mathscr{S}_{\lambda_1}\big(U^{\vee} - (E/F^{\chi_1})^{\vee}\big) & (0 \le \lambda_1 < n), \\ \dfrac{1}{F^{(2)}\big(c_1(U^{\vee})\big)}\mathscr{S}_{\lambda_1}\big(U^{\vee} - (E/F^{\chi_1})^{\vee}\big) & (n \le \lambda_1 < 2n), \end{cases}$$

where $F^{(2)}(u)$ is the power series defined by the equation $F(u,u) = u \cdot F^{(2)}(u)$. More generally, the pushforward classes $[Y_\lambda \to OG^k(E)]$ are obtained from (1-1) by replacing $\mathscr{C}_m^{(i)}$ with $\mathscr{B}_m^{(i)}$ (see Theorem 4.12).

A key aspect of algebraic cobordism, which was established in [Levine and Morel 2007], is its universality. In particular, this means that formulas which hold for Ω^* can be specialised to any other oriented cohomology theory. An easy example of this phenomenon is illustrated by the behaviour of the first Chern class of line bundles. In CH^* one has

$$c_1(L \otimes M) = c_1(L) + c_1(M) \ \text{ and } \ c_1(L^{\vee}) = -c_1(L),$$

while in CK^* these equalities become

$$c_1(L \otimes M) = c_1(L) + c_1(M) - \beta c_1(L)c_1(M) \ \text{ and } \ c_1(L^{\vee}) = \frac{-c_1(L)}{1-\beta c_1(L)},$$

where $\beta \in CK^*(\operatorname{Spec}\mathbf{k})$ is the pushforward of the fundamental class of $\mathbb{P}^1$ to the point. If we set $\beta = 0$, we recover the identities for CH^*. In algebraic cobordism Ω^*, the expressions describing $c_1(L \otimes M)$ and $c_1(L^{\vee})$ are respectively given by the *universal formal group law* $F(u,v)$ and the *universal formal inverse* $\chi(u)$ which are certain power series with coefficients in $\Omega(\operatorname{Spec}\mathbf{k})$. The universality of Ω^* implies that in any other oriented cohomology theory A^*, $c_1(L \otimes M)$ and $c_1(L^{\vee})$ can be obtained by specializing the coefficients of $F(u,v)$ and $\chi(u)$ to $A^*(\operatorname{Spec}\mathbf{k})$.

In particular, in CK^* we have $P(u, v) = \frac{1}{1-\beta v}$ and $\chi(u) = \frac{-u}{1-\beta u}$ and the Laurent series (1-2) can be expressed as a sum of Pfaffians ([Hudson et al. 2017, Lemma 5.18]). As a consequence (1-1) reduces to the Pfaffian sum formula describing the K-theoretic degeneracy loci classes [Hudson et al. 2017, Theorem 5.20].

Our choice of resolutions Y_λ has the advantage of being stable: the class $\varpi_*[Y_\lambda]$ doesn't change when $n \to \infty$. On the other hand, there are different resolutions for X_λ, such as *Bott–Samelson resolutions*. These resolutions are well-studied in the context of generalized Schubert calculus. On the one hand the advantage of Bott–Samelson classes is represented by their compatibility with divided difference operators, however this comes at the cost of not being stable along the limit $n \to \infty$. See, for example, [Hornbostel and Kiritchenko 2011; Kiritchenko and Krishna 2013; Hornbostel and Perrin 2018]. The classes related to other resolutions are also studied in [Nakagawa and Naruse 2016; 2018], a Hall–Littlewood type formula in Ω^* is derived. All of these resolution classes coincide with honest Schubert classes if one works in CK^*, while they form different families of classes in Ω^*. As an application of our explicit formulas, it would be interesting to compare those different classes which replace Schubert classes in algebraic cobordism. To this aim it would be advisable to first consider functors more suitable for computations like the infinitesimal theories used in [Hudson and Matsumura 2018].

Anderson [2019] extended the results of [Hudson et al. 2017] to more general degeneracy loci including those arising from even orthogonal Grassmann bundles. His work is based on the approach he and Fulton employed in their study of the Chow ring fundamental classes of degeneracy loci for all types [Anderson and Fulton 2012; 2018]. In our future work we would like to lift Anderson's results to Ω^* so to cover the even orthogonal case as well.

The organisation of the paper is as follows. In section 2 we recall some basic facts about Borel–Moore homology theories and we translate into this setting the results on Segre classes presented in [Hudson and Matsumura 2019]. This becomes necessary because the resolutions are not smooth in general. In section 3 we prove the main theorem for symplectic Grassmann bundles, while in section 4 we first deal with the special case of quadric bundles and then establish the main theorem for odd orthogonal Grassmann bundles.

Notations and conventions. Throughout this paper $\mathbf{k}$ will be a field of characteristic 0. By $\mathbf{Sch_k}$ we will denote the category of separated schemes of finite type over $\mathbf{k}$ and $\mathbf{Lci_k}$ will stand for its full subcategory constituted by the objects whose structural morphism is a local complete intersection. For a given category C we will write C' to refer to its subcategory given by allowing only projective morphisms. $\mathbf{Ab}_*$ represents the category of graded abelian groups.

2. Preliminaries

The goal of this section is to collect some basic properties of Borel–Moore homology theories and to translate in this context some of the results on generalised Segre classes presented in [Hudson and Matsumura 2019].

Borel–Moore homology theories. An oriented Borel–Moore (BM) homology theory on $\mathbf{Sch_k}$ (or *mutatis mutandis* on $\mathbf{Lci_k}$) is given by a covariant functor $A_* : \mathbf{Sch}'_{\mathbf{k}} \to \mathbf{Ab}_*$, by a family of pullback maps $\{f^* : A_*(Y) \to A_*(X)\}$ associated to *l.c.i.* morphism and by an external product $A_*(X) \otimes A_*(Y) \to A_*(X \times_{\operatorname{Spec} k} Y)$. Let us remind the reader that a morphism is a *local complete intersection* if and only if it can be factored as the composition of a regular embedding and a smooth morphism. A detailed description of the properties that these three components have to satisfy would force us to take a significant detour, so we will focus only on the aspects that are more relevant to our work and refer the reader to [Levine and Morel 2007, Definition 5.1.3] for the precise definition.

For us the most relevant feature of oriented BM homology theories is that they satisfy the projective bundle formula. Roughly speaking it states that for every vector bundle E of rank e with $X \in \mathbf{Sch_k}$, the evaluation of A_* on the associated dual projective bundle $\mathbb{P}^*(E) \xrightarrow{q} X$ can be described in terms of $A_*(X)$. More precisely for $i \in \{0, 1, \ldots, e-1\}$ one has operations

$$\xi^{(i)} : A_{*+i-e+1}(X) \longrightarrow A_*(\mathbb{P}^*(E))$$

given by $\xi^{(i)} := \tilde{c}_1(\mathcal{Q})^i \circ q^*$, where $\mathcal{Q} \to \mathbb{P}^*(E)$ is the tautological line bundle and $\tilde{c}_1(\mathcal{Q}) := s^* \circ s_*$, for any section $s : \mathbb{P}^*(E) \to \mathcal{Q}$. Altogether these yield the following isomorphism

$$\Psi : \bigoplus_{i=0}^{e-1} A_{*+i-e+1}(X) \xrightarrow{\sum_{i=0}^{e-1} \xi^{(i)}} A_*(\mathbb{P}^*(E)).$$

A very important consequence of this is that every oriented BM homology theory admits a theory of Chern class operators: to E one associates $\{\tilde{c}_i^A(E) : A_*(X) \to A_{*-i}(X)\}_{0 \le i \le e}$. These are defined by setting $\tilde{c}_0^A(E) = \mathrm{id}_{A_*(X)}$ and, up to a sign, by considering the different components of $\Psi^{-1} \circ \xi^{(e)}$, so that one obtains the relation

$$\sum_{i=0}^{e} (-1)^i \xi^{(e-i)} \circ \tilde{c}_i^A(E) = 0.$$

These operators can be collected in the so-called *Chern polynomial* $\tilde{c}^A(E; u) := \sum_{i=0}^{e} \tilde{c}_i^A(E) u^i$ and it is worth mentioning that, in view of the Whitney formula, its definition can be extended to the Grothendieck group of vector bundles by setting

$$\tilde{c}^A(E-F;u) := \frac{\tilde{c}^A(E;u)}{\tilde{c}^A(F;u)}.$$

Beside being extremely useful for computations, Chern classes allow one to get some insight on how a general oriented BM homology theory A_* differs from the Chow group CH_*, probably the most commonly known example. Let us consider, as an example, the behaviour of the first Chern class with respect to the tensor product of two line bundles L and M. While in CH_* one has

$$\tilde{c}_1^{CH}(L\otimes M) = \tilde{c}_1^{CH}(L) + \tilde{c}_1^{CH}(M),$$

in general the relation between the three Chern class operators is described by a *formal group law* $\big(A_*(\operatorname{Spec} k), F_A\big)$, where $F_A(u,v)$ is a special power series with coefficients in the coefficient ring of the theory $A_*(\operatorname{Spec} k)$. The precise relation is given by

$$\tilde{c}_1^A(L\otimes M) = F_A(\tilde{c}_1^A(L), \tilde{c}_1^A(M)).$$

In a similar fashion, whereas in CH_* one simply has $\tilde{c}_1^{CH}(L^\vee) = -\tilde{c}_1^{CH}(L)$, in general one needs to introduce the *formal inverse* χ_A, a power series in one variable satisfying both

$$\tilde{c}_1^A(L^\vee) = \chi_A(\tilde{c}_1(L)) \quad \text{and} \quad F_A(u, \chi_A(u)) = 0.$$

In some case we will denote the formal inverse $\chi_A(u)$ simply by $\bar{u}$.

All our computations will take place in the algebraic cobordism of Levine–Morel Ω_* and our choice is motivated by the following fundamental result.

Theorem 2.1 [Levine and Morel 2007, Theorems 7.1.3 and 4.3.7]. *The algebraic cobordism Ω_* is universal among oriented BM homology theories on $\mathbf{Lci_k}$. That is, for any other oriented BM homology theory A_* there exists a unique morphism*

$$\vartheta_A : \Omega_* \to A_*$$

of oriented BM homology theories. Furthermore, its associated formal group law $\big(\Omega_*(\operatorname{Spec} \mathbf{k}), F_\Omega\big)$ is isomorphic to the universal one defined on the Lazard ring $(\mathbb{L}, F)$.

One consequence of this universality is that all the formulas obtained for Ω_* can be specialised to every other oriented BM homology theory A_*. In other words, algebraic cobordism allows one to work with all theories at once. Since we will only work with algebraic cobordism, in the remainder of the paper we will remove the subscript $_\Omega$ from the notation.

Let us conclude our general discussion by briefly mentioning the construction of fundamental classes and some results which can be used to compute them. To every $X \in \mathbf{Sch_k}$ whose structural morphism π_X is l.c.i. we associate its fundamental class by setting $1_X := \pi_X^*(1)$. Notice that here 1 stands for the multiplicative unit in $A_*(\operatorname{Spec}\mathbf{k})$. In the special case of the zero scheme of a bundle, the fundamental class can be computed via the following lemma.

Lemma 2.2 [Levine and Morel 2007, Lemma 6.6.7]. *Let E be a vector bundle of rank e over $X \in \mathbf{Sch_k}$. Suppose that E has a section $s : X \to E$ such that the zero scheme of s, $i : Z \to X$ is a regularly embedded closed subscheme of codimension e. Then we have*

$$\tilde{c}_e(E) = i_* \circ i^*.$$

In particular, if X is an l.c.i. scheme, we have

$$\tilde{c}_e(E)(1_X) = i_*(1_Z).$$

Finally, as it will play an important role in our computations, we would like to make more explicit the case of the fundamental class of a nonreduced divisor. For this we will require a bit of notation. For every integer $n \geq 2$, let $n \cdot_{F_A} u$ be the *formal multiplication* by n, that is, the power series obtained by adding n times the variable u using the formal group law F_A. Since F_A is a formal group law, one has

$$n \cdot_{F_A} u = u \cdot F_A^{(n)}(u) \tag{2-1}$$

for some degree 0 power series $F_A^{(n)}(u)$ whose costant term is n. We are now able to restate [Levine and Morel 2007, Proposition 7.2.2] for the particular case we will need.

Lemma 2.3. *Let W be a smooth scheme and D a smooth prime divisor of W. For any integer $n \geq 2$, let $|E|$ be the closed subscheme associated to the divisor $E = nD$. If L is the line bundle corresponding to D and $\iota : D \to |E|$ is the natural morphism, then in $A_*(|E|)$ we have*

$$1_{|E|} = \iota_*\big(F_A^{(n)}(\tilde{c}_1^A(L_{|D}))(1_D)\big),$$

where $L_{|D}$ is the restriction of L to D.

Segre class operators. In [Hudson and Matsumura 2019], in order to be able to describe the pushforwards along projective bundles over a smooth scheme, we generalised to algebraic cobordism the classical definition of Segre classes given in [Fulton 1998]. As in this paper we deal with the resolutions of symplectic or orthogonal degeneracy loci, it becomes necessary to extend such description to the case of projective bundles over non-smooth schemes. Therefore, we will now introduce Segre class operators for oriented BM homology theories, since these can be defined for more general schemes.

Following [Hudson et al. 2017, §4], we define the relative Segre operators in terms of generating functions. Let $X \in \mathbf{Sch_k}$.

Definition 2.4. Let $\tilde{x}_1, \dots, \tilde{x}_e$ be Chern root operators of a vector bundle E over X so that $\tilde{c}(E; u) = \prod_{i=1}^{e}(1+\tilde{x}_i u)$. We define

$$\widetilde{w}(E; u) = \sum_{s\geq 0}^{\infty} \widetilde{w}_{-s}(E)u^{-s} = \prod_{i=1}^{e} P(u^{-1}, \tilde{x}_i),$$

where $P(u, v)$ is defined by $F(u, \chi(v)) = (u - v)P(u, v)$ (compare [Hudson and Matsumura 2019, Lemma 4.1]). Since the right-hand side is symmetric in the $\tilde{x}_i$, this definition of $\widetilde{w}_{-s}(E)$ is independent of the choice of Chern root operators of E. It should be noticed that $\widetilde{w}_0(E)$ has constant term 1 and as a consequence $\widetilde{w}(E; u)$ is an invertible power series in u^{-1}. One can also define $\widetilde{w}(E - F; u)$ for a virtual bundle $[E - F]$, where E and F are vector bundles over X, by setting

$$\widetilde{w}(E - F; u) = \sum_{s\geq 0}^{\infty} \widetilde{w}_{-s}(E - F)u^{-s} = \frac{\widetilde{w}(E; u)}{\widetilde{w}(F; u)}.$$

Definition 2.5. Let E be a vector bundle of rank e over X and n a nonnegative integer. Consider the dual projective bundle $\pi : \mathbb{P}^*(E \oplus O_X^{\oplus n}) \to X$ where O_X is the trivial line bundle over X. For every integer $m \geq -e - n + 1$, define the degree *m Segre class operator* $\widetilde{\mathscr{S}}_m(E)$ of E by setting

$$\widetilde{\mathscr{S}}_m(E) = \pi_* \circ \tilde{c}_1(\mathcal{Q})^{m+e+n-1} \circ \pi^*, \tag{2-2}$$

where $\mathcal{Q}$ is the tautological quotient line bundle of $\mathbb{P}^*(E \oplus O_X^{\oplus n})$. It is easy to verify (see [Hudson and Matsumura 2019, Remark 4.4]) that this assignment is independent of n. Finally, we set

$$\widetilde{\mathscr{S}}(E; u) := \sum_{m\in\mathbb{Z}} \widetilde{\mathscr{S}}_m(E)u^m.$$

Proposition 2.6. *Let $E \to X$ be a vector bundle of rank e over $X \in \mathbf{Sch_k}$. Then we have the following equality of power series*:

$$\widetilde{\mathscr{S}}(E; u) = \frac{\widetilde{\mathscr{P}}(u)}{\tilde{c}(E; -u)\widetilde{w}(E; u)}.$$

Here $\widetilde{\mathscr{P}}(u) := \sum_{i=0}^{\infty} \widetilde{[\mathbb{P}^i]}u^{-i}$ is the power series collecting the operators given by external multiplication with the pushforwards classes of projective spaces $[\mathbb{P}^i] := [\mathbb{P}^i \to \operatorname{Spec} k] \in \mathbb{L}^{-i}$.

Proof. Once one has translated in the language of operators the proof given in [Hudson and Matsumura 2019, Theorem 4.6], the only thing left to check is that for every trivial dual projective bundle $(\mathbb{P}^n_X)^* \xrightarrow{\pi} X$ the composition $\pi_* \circ \pi^*$ coincides with external multiplication by $[\mathbb{P}^n]$. This can be verified directly at the level of cobordism cycles by making use of the definitions of pushforward and pullback morphisms and of the external product. □

Remark 2.7. It is worth mentioning that, provided one restricts to the case $X \in \mathbf{Sm_k}$, Proposition 2.6 can be derived from the analogue of Quillen's formula for algebraic cobordism established in [Vishik 2007, Theorem 5.35]. The same formula can be used to express the classes $[\mathbb{P}^i]$ in terms of the generators of the Lazard ring and, as a consequence, of the coefficients of the formal group law. On the other hand, an easy computation shows that Quillen's formula can be recovered from Proposition 2.6, provided one knows the expression for the classes of projective spaces. In this sense our approach allows us to extend the validity of Vishik's result from $\mathbf{Sm_k}$ to $\mathbf{Sch_k}$.

In view of the last proposition, we are now able to extend to virtual bundles the definition of Segre classes.

Definition 2.8. For vector bundles E and F over X, define the *relative Segre class operators* $\widetilde{\mathscr{S}}_m(E-F)$ on $\Omega_*(X)$ as

$$\widetilde{\mathscr{S}}(E-F;u) := \sum_{m\in\mathbb{Z}} \widetilde{\mathscr{S}}_m(E-F)u^m = \widetilde{\mathscr{P}}(u)\frac{\tilde{c}(F;-u)\widetilde{w}(F;u)}{\tilde{c}(E;-u)\widetilde{w}(E;u)}. \tag{2-3}$$

Remark 2.9. If the rank of F is f, then we have

$$\widetilde{\mathscr{S}}_m(E-F) = \sum_{p=0}^{f}\sum_{q=0}^{\infty}(-1)^p\tilde{c}_p(F)\circ\widetilde{w}_{-q}(F)\circ\widetilde{\mathscr{S}}_{m-p+q}(E).$$

Even if F itself is a virtual bundle, this equation holds by replacing f with ∞.

We conclude this section by providing a description of relative Segre classes in terms of pushforwards of Chern classes. This should be seen as an analogue of [Hudson and Matsumura 2019, Theorem 4.9].

Theorem 2.10. *Let $X \in \mathbf{Sch_k}$ and let E and F be two vector bundles over X, respectively of rank e and f. Let $\pi : \mathbb{P}^*(E) \to X$ be the dual projective bundle of E and $\mathcal{Q}$ its universal quotient line bundle. As operators over $\Omega_*(X)$, we have*

$$\pi_* \circ \tilde{c}_1(\mathcal{Q})^s \circ \tilde{c}_f(\mathcal{Q}\otimes F^\vee)\circ\pi^* = \widetilde{\mathscr{S}}_{s+f-e+1}(E-F). \tag{2-4}$$

In particular if $X \in \mathbf{Lci_k}$, then one has

$$\pi_* \circ \tilde{c}_1(\mathcal{Q})^s \circ \tilde{c}_f(\mathcal{Q}\otimes F^\vee)(1_{\mathbb{P}^*(E)}) = \widetilde{\mathscr{S}}_{s+f-e+1}(E-F)(1_X).$$

Proof. Let us begin by observing that an easy Chern roots computation analogue to [Hudson and Matsumura 2019, formula (4.1)] gives us

$$\tilde{c}_f(\mathcal{Q}\otimes F^\vee)=\sum_{p=0}^{f}\sum_{q=0}^{\infty}(-1)^p\tilde{c}_p(F)\circ\widetilde{w}_{-q}(F)\circ\tilde{c}_1(\mathcal{Q})^{f-p+q}.$$

Thus the left-hand side of (2-4) can be rewritten as

$$\sum_{p=0}^{f}\sum_{q=0}^{\infty}(-1)^p\tilde{c}_p(F)\circ\widetilde{w}_{-q}(F)\circ\pi_*\circ\tilde{c}_1(\mathcal{Q})^{s+f-p+q}\circ\pi^*. \tag{2-5}$$

By (2-2), we find that (2-5) equals to

$$\sum_{p=0}^{f}\sum_{q=0}^{\infty}(-1)^p\tilde{c}_p(F)\circ\widetilde{w}_{-q}(F)\circ\widetilde{\mathscr{S}}_{s+f-e+1-p+q}(E),$$

which coincides with the right-hand side of (2-4) in view of Remark 2.9. The second statement follows immediately by applying both sides of (2-4) to the fundamental class 1_X. □

Remark 2.11. If E is a line bundle, then one has $\pi=\mathrm{id}_X$ and $\mathcal{Q}=E$. As a consequence we have

$$\tilde{c}_1(\mathcal{Q})^s\circ\tilde{c}_f(\mathcal{Q}\otimes F^\vee)(1_X)=\widetilde{\mathscr{S}}_{f-e+1+s}(E-F)(1_X).$$

3. Symplectic degeneracy loci

For this section we fix a nonnegative integer k.

k-strict partitions and characteristic indices. A k-strict partition λ is a weakly decreasing infinite sequence $(\lambda_1,\lambda_2,\dots)$ of nonnegative integers such that the number of nonzero parts is finite, and if $\lambda_i>k$, then $\lambda_i>\lambda_{i+1}$. The length of λ is the number of nonzero parts of λ. Let $\mathcal{SP}^k$ be the set of all k-strict partitions. Let $\mathcal{SP}^k_r$ be the set of all k-strict partitions with the length at most r. If $\lambda\in\mathcal{SP}^k_r$, then we often write $\lambda=(\lambda_1,\dots,\lambda_r)$. Let $\mathcal{SP}^k(n)$ be the set of all k-strict partitions such that $\lambda_1\le n+k$ and the length of λ is at most $n-k$.

Let W_∞ be the infinite hyperoctahedral group which can be identified with the group of all signed permutations (permutations w of $\mathbb{Z}\backslash\{0\}$ such that $w(i)\neq i$ for only finitely many $i\in\mathbb{Z}\backslash\{0\}$, and $\overline{w(i)}=w(\bar{i})$ for all i where $\bar{i}:=-i$). A signed permutation w is determined by the sequence $(w(1),w(2),\dots)$ which we call one line notation. An element $w\in W_\infty$ is called k-Grassmannian if

$$0<w(1)<\cdots<w(k),\quad w(k+1)<w(k+2)<\cdots.$$

The set of all k-Grassmannian elements in W_∞ is denoted by $W_\infty^{(k)}$.

Between $W_\infty^{(k)}$ and $\mathcal{SP}^k$, there is a bijection defined as follows. For each $w \in W_\infty^{(k)}$, the corresponding k-strict partition is given by

$$\lambda_i := \begin{cases} w(k+i) & \text{if } w(k+i) < 0, \\ \#\{j \le k \mid w(j) > w(k+i)\} & \text{if } w(k+i) > 0. \end{cases}$$

For each $\lambda \in \mathcal{SP}^k$ (with the corresponding $w \in W_\infty^{(k)}$), we define its characteristic index $\boldsymbol{\chi} = (\chi_1, \chi_2, \dots)$ by

$$\chi_i := \begin{cases} -w(k+i) - 1 & \text{if } w(k+i) < 0, \\ -w(k+i) & \text{if } w(k+i) > 0. \end{cases} \tag{3-1}$$

Moreover, the following notations are necessary for our formulas of Grassmannian degeneracy loci in type C and B: for each $\lambda \in \mathcal{SP}^k$ and the corresponding characteristic index $\boldsymbol{\chi}$, define

$$C(\lambda) := \{(i, j) \mid 1 \le i < j, \quad \chi_i + \chi_j \ge 0\},$$
$$\gamma_j := \sharp\{i \mid 1 \le i < j, \quad \chi_i + \chi_j \ge 0\} \quad \text{for each } j > 0.$$

Symplectic degeneracy loci and the class κ_λ^C. Let E be a symplectic vector bundle over a smooth scheme X of rank $2n$, i.e., we are given a nowhere degenerating section of $\bigwedge^2 E$. For a subbundle F of E, we denote by $F^\perp$ the orthogonal complement of F with respect to the symplectic form. Fix a reference flag $F^\bullet$ of subbundles of E,

$$0 = F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset F^0 \subset F^{-1} \subset \cdots \subset F^{-n} = E,$$

where $\operatorname{rk} F^i = n - i$ and $(F^i)^\perp = F^{-i}$ for all i with $-n \le i \le n$. Let $SG^k(E) \to X$ be the Grassmannian bundle over X consisting of pairs (x, U_x) where $x \in X$ and U_x is an $n-k$ dimensional isotropic subspace of E_x. Let U be the tautological bundle of $SG^k(E)$.

For each $\lambda \in \mathcal{SP}^k(n)$ of length r, let X_λ^C be the symplectic degeneracy locus in $SG^k(E)$ defined by

$$X_\lambda^C = \big\{(x, U_x) \in SG^k(E) \mid \dim(U_x \cap F_x^{\chi_i}) \ge i, \quad i = 1, \dots, r\big\},$$

where $\boldsymbol{\chi} = (\chi_1, \chi_2, \dots)$ is the characteristic index for λ.

Let $Fl_r(U) \to SG^k(E)$ be the r-step flag bundle of U where the fiber at $(x, U_x) \in SG^k(E)$ consists of the flag $(D_\bullet)_x = \{(D_1)_x \subset \cdots \subset (D_r)_x\}$ of subspaces of U_x with $\dim(D_i)_x = i$. Let $D_1 \subset \cdots \subset D_r$ be the flag of tautological bundles of $Fl_r(U)$. We set $D_0 = 0$. The bundle $Fl_r(U)$ can be constructed as a tower of projective bundles

$$\pi : Fl_r(U) = \mathbb{P}(U/D_{r-1}) \xrightarrow{\pi_r} \mathbb{P}(U/D_{r-2}) \xrightarrow{\pi_{r-1}} \cdots \xrightarrow{\pi_3} \mathbb{P}(U/D_1) \xrightarrow{\pi_2} \mathbb{P}(U) \xrightarrow{\pi_1} SG^k(E). \tag{3-2}$$

The quotient line bundle D_j/D_{j-1} is regarded as the tautological line bundle of $\mathbb{P}(U/D_{j-1})$ and we set $\widetilde{\tau}_j := \tilde{c}_1((D_j/D_{j-1})^\vee)$.

We are now able to define the resolution of singularities of the degeneracy loci.

Definition 3.1. For each $j = 1, \dots, r$, we define a subvariety Y_j of $\mathbb{P}(U/D_{j-1})$ by

$$Y_j := \big\{(x, U_x, (D_1)_x, \dots, (D_j)_x) \in \mathbb{P}(U/D_{j-1}) \mid (D_i)_x \subset F_x^{\chi_i},\ i = 1, \dots, j\big\}.$$

We set $Y_0 := SG_r^k(U)$ and $Y_\lambda^C := Y_r$. Let $P_{j-1} := \pi_j^{-1}(Y_{j-1})$, $\pi'_j : P_{j-1} \to Y_{j-1}$ the projection and $\iota_j : Y_j \to P_{j-1}$ the obvious inclusion. Let $\varpi_j := \pi'_j \circ \iota_j$. We have the commutative diagram

$$\begin{array}{ccc}
\mathbb{P}(U/D_{j-1}) & \xrightarrow[\pi_j]{} & \mathbb{P}(U/D_{j-2}) \\
\uparrow & & \uparrow \\
P_{j-1} & \xrightarrow[\pi'_j]{} & Y_{j-1} \\
\uparrow{\scriptstyle \iota_j} & \nearrow_{\varpi_j} & \\
Y_j & &
\end{array}$$

Definition 3.2. Let $\varpi := \varpi_1 \circ \cdots \circ \varpi_r : Y_\lambda^C \to SG^k(E)$. Define the class $\kappa_\lambda^C \in \Omega_*(SG^k(E))$ by

$$\kappa_\lambda^C = [Y_\lambda^C \to SG^k(E)] := \varpi_*(1_{Y_\lambda^C}).$$

Remark 3.3. It is also known that Y_λ^C is irreducible and has at worst rational singularities. Furthermore Y_λ^C is birational to X_λ^C through the projection π (see [Hudson et al. 2017], for example). Therefore in K-theory and Chow ring of $SG^k(E)$ the class κ_λ^C coincides with the fundamental class of the degeneracy loci X_λ^C. Note that in a general oriented cohomology theory, the fundamental class of X_λ^C is not defined since X_λ^C may not be an l.c.i. scheme.

Computing κ_λ^C. In this section, we establish an explicit formula of the class κ_λ^C in $\Omega_*(SG^k(E))$ in terms of a power series in relative Segre classes. The key ingredients for the computation are twofold: one is the formula that computes pushforwards along each ϖ_j and the other is so-called *umbral calculus* which is a computational technique to combine the pushforwards along all the ϖ_j.

We begin by the following lemma which was proved in [Hudson et al. 2017] for CK_*. One can easily check that the proof works for an arbitrary oriented BM homology and in particular for Ω_*.

Lemma 3.4. *For each $j = 1, \dots, r$, the variety Y_j is regularly embedded in P_{j-1} and P_{j-1} is regularly embedded in $\mathbb{P}(U/D_{j-1})$. Furthermore, in $\Omega_*(P_{j-1})$, we have*

$$\iota_{j*}(1_{Y_j}) = \tilde{c}_{\lambda_j+n-k-j}\big((D_j/D_{j-1})^\vee \otimes (D_{\gamma_j}^\perp/F^{\chi_j})\big)(1_{P_{j-1}}).$$

Based on this lemma together with Theorem 2.10, we have the next pushforward formula for ϖ_j. For simplicity, let us introduce the following notation: for each $m \in \mathbb{Z}$ and $-n \leq \ell \leq n$, let

$$\widetilde{\mathscr{C}}_m^{(\ell)} := \widetilde{\mathscr{S}}_m\big(U^\vee - (E/F^\ell)^\vee\big).$$

In $\Omega^*(SG^k(E))$, we set $\mathscr{C}_m^{(\ell)} := \widetilde{\mathscr{C}}_m^{(\ell)}(1_{SG^k(E)})$.

Lemma 3.5. *In $\Omega_*(Y_{j-1})$, we have*

$$\varpi_{j*} \circ \tilde{\tau}_j^s(1_{Y_j}) = \sum_{p=0}^{j-1}\sum_{q=0}^{\infty}(-1)^p \tilde{c}_p(D_{j-1}^\vee - D_{\gamma_j}) \circ w_{-q}(D_{j-1}^\vee - D_{\gamma_j}) \circ \widetilde{\mathscr{C}}_{\lambda_j+s-p+q}^{(\chi_j)}(1_{Y_{j-1}}).$$

Proof. By Lemma 3.4, we have

$$\varpi_{j*} \circ \tilde{\tau}_j^s(1_{Y_j}) = \pi'_{j*} \circ \iota_{j*} \circ \tilde{\tau}_j^s(1_{Y_j}) = \pi'_{j*} \circ \tilde{\tau}_j^s \circ \iota_{j*}(1_{Y_j}) = \pi'_{j*} \circ \tilde{\tau}_j^s \circ \tilde{\alpha}_j(1_{P_{j-1}}),$$

where $\tilde{\alpha}_j := \tilde{c}_{\lambda_j+n-k-j}((D_j/D_{j-1})^\vee \otimes (D_{\gamma_j}^\perp/F^{\chi_j}))$. By Theorem 2.10, we have

$$\begin{aligned}\pi'_{j*} \circ \tilde{\tau}_j^s \circ \tilde{\alpha}_j(1_{P_{j-1}}) &= \widetilde{\mathscr{S}}_{s+\lambda_j}\big((U/D_{j-1})^\vee - (D_{\gamma_j}^\perp/F^{\chi_j})^\vee\big)(1_{Y_{j-1}})\\ &= \widetilde{\mathscr{S}}_{s+\lambda_j}\big(U^\vee - (E/F^{\chi_j})^\vee - (D_{j-1} - D_{\gamma_j}^\vee)^\vee\big)(1_{Y_{j-1}}),\end{aligned}$$

where we have used $D_{\gamma_j}^\perp = E - D_{\gamma_j}^\vee$. Now the claim follows from Remark 2.9. □

For the umbral calculus mentioned above, we need to establish some notation. Let $R = \Omega^*(Gr_d(E))$, viewed as a graded algebra over $\mathbb{L}$, and let $t_1, \dots, t_r$ be indeterminates of degree 1. We use the multi-index notation $t^{\mathbf{s}} := t_1^{s_1} \cdots t_r^{s_r}$ for $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{Z}^r$. A formal Laurent series $f(t_1, \dots, t_r) = \sum_{\mathbf{s}\in\mathbb{Z}^r} a_{\mathbf{s}} t^{\mathbf{s}}$ is *homogeneous of degree $m \in \mathbb{Z}$* if $a_{\mathbf{s}}$ is zero unless $a_{\mathbf{s}} \in R_{m-|\mathbf{s}|}$ with $|\mathbf{s}| = \sum_{i=1}^r s_i$. Let $\operatorname{supp} f = \{\mathbf{s} \in \mathbb{Z}^r \mid a_{\mathbf{s}} \neq 0\}$. For each $m \in \mathbb{Z}$, define $\mathcal{L}_m^R$ to be the space of all formal Laurent series of homogeneous degree m such that there exists $\mathsf{n} \in \mathbb{Z}^r$ for which $\mathsf{n} + \operatorname{supp} f$ is contained in the cone in $\mathbb{Z}^r$ defined by $s_1 \geq 0,\ s_1 + s_2 \geq 0,\ \cdots,\ s_1 + \cdots + s_r \geq 0$. Then $\mathcal{L}^R := \bigoplus_{m\in\mathbb{Z}} \mathcal{L}_m^R$ is a graded ring over R with the obvious product. For each $i = 1, \dots, r$, let $\mathcal{L}^{R,i}$ be the R-subring of $\mathcal{L}^R$ consisting of series that do not contain any negative powers of $t_1, \dots, t_{i-1}$. In particular, $\mathcal{L}^{R,1} = \mathcal{L}^R$. A series $f(t_1, \dots, t_r)$ is a *power series* if it doesn't contain any negative powers of $t_1, \dots, t_r$. Let $R[\![t_1, \dots, t_r]\!]_m$ denote the set of all power series in $t_1, \dots, t_r$ of degree $m \in \mathbb{Z}$. We set $R[\![t_1, \dots, t_r]\!]_{\mathrm{gr}} := \bigoplus_{m\in\mathbb{Z}} R[\![t_1, \dots, t_r]\!]_m$.

Definition 3.6. Define a graded R-module homomorphism $\phi_1 : \mathcal{L}^R \to \Omega_*(SG^k(E))$ as

$$\phi_1^C(t_1^{s_1} \cdots t_r^{s_r}) = \widetilde{\mathscr{C}}_{s_1}^{(\chi_1)} \circ \cdots \circ \widetilde{\mathscr{C}}_{s_r}^{(\chi_r)}(1_{SG^k(E)}).$$

Similarly, for each $j = 2, \dots, d$, define a graded R-module homomorphism $\phi_j^C : \mathcal{L}^{R,j} \to \Omega_*(Y_{j-1})$ by setting

$$\phi_j^C(t_1^{s_1} \cdots t_r^{s_r}) = \tilde{\tau}_1^{s_1} \circ \cdots \circ \tilde{\tau}_{j-1}^{s_{j-1}} \circ \widetilde{\mathscr{C}}_{s_j}^{(\chi_j)} \circ \cdots \circ \widetilde{\mathscr{C}}_{s_r}^{(\chi_r)}(1_{Y_{j-1}}).$$

Remark 3.7. By regarding $\Omega^*(SG^k(E)) = \Omega_{\dim SG^k(E)-*}(SG^k(E))$, we have

$$\phi_1^C(t_1^{s_1} \cdots t_r^{s_r}) = \mathscr{C}_{s_1}^{(\chi_1)} \cdots \mathscr{C}_{s_r}^{(\chi_r)}.$$

Using ϕ_j^C, we can restate Lemma 3.5 as follows.

Lemma 3.8. *One has*

$$\varpi_{j*} \circ \tilde{\tau}_j^s(1_{Y_j}) = \phi_j^C\left(t_j^{\lambda_j+s} \frac{\prod_{i=1}^{j-1}(1 - t_i/t_j)P(t_j, t_i)}{\prod_{i=1}^{\gamma_j}(1 - \bar{t}_i/t_j)P(t_j, \bar{t}_i)}\right).$$

Proof. Consider the functions of $t_1, \dots, t_{j-1}$ defined by the following generating functions:

$$\sum_{p=0}^{\infty} H_p^\lambda(t_1, \dots, t_{j-1})u^p := \frac{e(t_1, \dots, t_{j-1}; u)}{e(\bar{t}_1, \dots, \bar{t}_{\gamma_j}; u)} = \frac{\prod_{i=1}^{j-1}(1 + t_i u)}{\prod_{i=1}^{\gamma_j}(1 + \bar{t}_i u)},$$

$$\sum_{q=0}^{\infty} W_{-q}^\lambda(t_1, \dots, t_{j-1})u^{-q} := \frac{w(t_1, \dots, t_{j-1}; u)}{w(\bar{t}_1, \dots, \bar{t}_{\gamma_j}; u)} = \frac{\prod_{i=1}^{j-1} P(u^{-1}, t_i)}{\prod_{i=1}^{\gamma_j} P(u^{-1}, \bar{t}_i)}.$$

Then we have

$$H_p^\lambda(\tilde{\tau}_1, \dots, \tilde{\tau}_{j-1}) = \tilde{c}_p(D_{j-1}^\vee - D_{\gamma_j}), \quad W_{-q}^\lambda(\tilde{\tau}_1, \dots, \tilde{\tau}_{j-1}) = \tilde{w}_{-q}(D_{j-1}^\vee - D_{\gamma_j}).$$

Thus, by Lemma 3.5 and the definition of ϕ_j^C, we have

$$\begin{aligned}
&\varpi_{j*} \circ \tilde{\tau}_j^s(1_{Y_j}) \\
&\quad = \phi_j^C\left(\sum_{p=0}^{j-1}\sum_{q=0}^{\infty}(-1)^p H_p^\lambda(t_1, \dots, t_{j-1}) W_{-q}^\lambda(t_1, \dots, t_{j-1}) t_j^{\lambda_j+s-p+q}\right) \\
&\quad = \phi_j^C\left(t_j^{\lambda_j+s}\left(\sum_{p=0}^{j-1}(-1)^p H_p^\lambda(t_1, \dots, t_{j-1}) t_j^{-p}\right)\left(\sum_{q=0}^{\infty} W_{-q}^\lambda(t_1, \dots, t_{j-1}) t_j^q\right)\right).
\end{aligned}$$

The claim follows from the definitions of H_p^λ and W_{-q}^λ in terms of the generating functions. ☐

Finally, we are able to prove the main theorem in the case of symplectic Grassmann bundles.

Theorem 3.9. *For a strict partition $\lambda \in \mathcal{SP}^k(n)$, the associated class κ_λ^C is given by*

$$\kappa_\lambda^C = \sum_{\mathbf{s}=(s_1,\dots,s_r)\in\mathbb{Z}^r} f_{\mathbf{s}}^\lambda \mathscr{C}_{s_1+\lambda_1}^{(\chi_1)} \cdots \mathscr{C}_{s_r+\lambda_r}^{(\chi_r)},$$

where $f_{\mathbf{s}}^\lambda \in \mathbb{L}$ are the coefficients of the Laurent series

$$\frac{\prod_{1\le i<j\le r}(1-t_i/t_j)P(t_j,t_i)}{\prod_{(i,j)\in C(\lambda)}(1-\bar{t}_i/t_j)P(t_j,\bar{t}_i)} = \sum_{\mathbf{s}=(s_1,\dots,s_r)\in\mathbb{Z}^r} f_{\mathbf{s}}^\lambda \cdot t_1^{s_1}\cdots t_r^{s_r} \tag{3-3}$$

as an element of $\mathcal{L}^{\mathbb{L}}$.

Proof. By Definition 3.2, it follows from successive applications of Lemma 3.8 (compare [Hudson et al. 2017]) that

$$\kappa_\lambda^C = \phi_1^C\left(t_1^{\lambda_1}\cdots t_r^{\lambda_r}\frac{\prod_{1\le i<j\le r}(1-t_i/t_j)P(t_j,t_i)}{\prod_{(i,j)\in C(\lambda)}(1-\bar{t}_i/t_j)P(t_j,\bar{t}_i)}\right).$$

Then, in view of the definition of the coefficients $f_{\mathbf{s}}$, it suffices to apply ϕ_1^C to obtain the claim. □

4. Odd orthogonal degeneracy loci

For this section we fix a nonnegative integer k.

Orthogonal degeneracy loci. Consider the vector bundle E of rank $2n+1$ over X with a symmetric non-degenerate bilinear form $\langle\ ,\ \rangle : E\otimes E \to O_X$ where O_X is the trivial line bundle over X. Let $\xi : OG^k(E) \to X$ be the Grassmann bundle consisting of pairs (x, U_x) where $x\in X$ and U_x is an $n-k$ dimensional isotropic subspace of E_x. Note that the bilinear form $\langle\ ,\ \rangle$ on E induces an isomorphism $F^\perp/F \otimes F^\perp/F \cong O_X$ for any maximal isotropic subbundle F of E where $F^\perp$ is the orthogonal complement of F with respect to $\langle\ ,\ \rangle$. This implies that $c_1(F^\perp/F)=0$ in $\Omega^*(X)\otimes_{\mathbb{Z}}\mathbb{Z}[1/2]$.

Fix a reference flag

$$0 = F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset F^0 \subset (F^0)^\perp \subset F^{-1} \subset \cdots \subset F^{-n} = E,$$

such that $\operatorname{rk} F^i = n-i$ for $i\ge 0$ and $(F^i)^\perp = F^{-i}$ for all $i\ge 1$. For each $\lambda\in\mathcal{SP}^k(n)$ of length r, we define the associated degeneracy loci X_λ^B in $OG^k(E)$ is defined by

$$X_\lambda^B = \left\{(x,U_x)\in OG^k(E) \mid \dim(U_x\cap F^{\chi_i}) \ge i,\ i=1,\dots,r\right\},$$

where $\boldsymbol{\chi}$ is the characteristic index associated to λ.

Quadric bundle. The bundle $OG^{n-1}(E)$ is also known as the quadric bundle and we denote it by $Q(E)$. In this section, we do not assume that X is smooth as long as it is regularly embedded in a quasi-projective smooth variety. Let S be the tautological line bundle of $Q(E)$. In this particular case the Schubert varieties of $Q(E)$ are indexed by a single integer λ_1 and can be explicitly described as follows:

$$X^B_{\lambda_1} = \begin{cases} Q(E) \cap \mathbb{P}(F^{\lambda_1 - n}) & (0 \le \lambda_1 < n), \\ \mathbb{P}(F^{\lambda_1 - n}) & (n \le \lambda_1 < 2n). \end{cases} \tag{4-1}$$

It is worth noting that λ_1 represents the codimension of $X^B_{\lambda_1}$ in $Q(E)$.

Lemma 4.1. *The fundamental class of the subvariety $X^B_{\lambda_1}$ in $\Omega_*(Q(E))$ for $\lambda_1 < n$ is given by*

$$[X^B_{\lambda_1} \to Q(E)] = \tilde{c}_{\lambda_1}(S^\vee \otimes E/F^{\lambda_1 - n})(1_{Q(E)}). \tag{4-2}$$

Moreover the fundamental class of $X^B_{\lambda_1}$ in $\Omega_*(Q(E))$ for $\lambda_1 \ge n$ satisfies the identity

$$\begin{aligned} F^{(2)}\big(\tilde{c}_1(S^\vee \otimes (F^0)^\perp/F^0)\big)\big([X^B_{\lambda_1} \to Q(E)]\big) \\ = \tilde{c}_{\lambda_1}\big(S^\vee \otimes (E/(F^0)^\perp \oplus F^0/F^{\lambda_1 - n})\big)\big(1_{Q(E)}\big), \end{aligned} \tag{4-3}$$

where $F^{(2)}$ is a special case of the power series defined in (2-1).

Proof. The formula (4-2) follows from Lemma 2.2. For (4-3), first we show the case for $\lambda_1 = n$, by computing the class $[X^B_n \to Q(E)]$ in $\Omega^*(Q(E))$ in two different ways. The variety X^B_n is a divisor in $\mathbb{P}((F^0)^\perp)$, corresponding to the line bundle $S^\vee \otimes (F^0)^\perp/F^0$. Moreover, the scheme theoretic intersection $Q(E) \cap \mathbb{P}((F^0)^\perp)$ defines the Weil divisor $2X^B_n$ on $\mathbb{P}((F^0)^\perp)$ and in view of Lemma 2.3 we have

$$1_{Q(E)\cap\mathbb{P}((F^0)^\perp)} = \iota_*\big(F^{(2)}(\tilde{c}_1(S^\vee \otimes (F^0)^\perp/F^0))(1_{X^B_n})\big),$$

where $\iota : X^B_n \to Q(E) \cap \mathbb{P}((F^0)^\perp)$ is the obvious inclusion. Thus, by pushing forward this identity to $Q(E)$, we obtain the following identity in $\Omega_*(Q(E))$:

$$[Q(E) \cap \mathbb{P}((F^0)^\perp) \to Q(E)] = F^{(2)}\big(\tilde{c}_1(S^\vee \otimes (F^0)^\perp/F^0)\big)\big([X^B_n \to Q(E)]\big).$$

On the other hand, Lemma 2.2 implies that

$$[Q(E) \cap \mathbb{P}((F^0)^\perp) \to Q(E)] = \tilde{c}_n\big(S^\vee \otimes E/(F^0)^\perp\big)(1_{Q(E)}).$$

This proves (4-3) for $\lambda_1 = n$.

If $\lambda_1 > n$, again by Lemma 2.2 we have $[X^B_{\lambda_1} \to X^B_n] = \tilde{c}_i(S^\vee \otimes F^0/F^i)(1_{X^B_n})$ in $\Omega_*(X^B_n)$. Thus we have

$$\begin{aligned} F^{(2)}\big(\tilde{c}_1(S^\vee \otimes (F^0)^\perp/F^0)\big) = \big([X^B_{\lambda_1} \to X^B_n]\big) \\ = F^{(2)}\big(\tilde{c}_1(S^\vee \otimes (F^0)^\perp/F^0)\big) \circ \tilde{c}_i(S^\vee \otimes F^0/F^i)(1_{X^B_n}). \end{aligned}$$

By pushing it forward to $\Omega^*(Q(E))$ and applying (4-3) for $\lambda_1 = n$, we obtain

$$\begin{aligned} F^{(2)}\big(\tilde{c}_1(S^\vee \otimes (F^0)^\perp/F^0)\big)\big([X^B_{\lambda_1} \to Q(E)]\big) \\ &= \tilde{c}_n\big(S^\vee \otimes E/(F^0)^\perp\big) \circ \tilde{c}_{\lambda_1 - n}(S^\vee \otimes F^0/F^{\lambda_1 - n})(1_{Q(E)}) \\ &= \tilde{c}_{\lambda_1}\big(S^\vee \otimes (E/(F^0)^\perp \oplus F^0/F^i)\big)(1_{Q(E)}). \end{aligned}$$

This proves (4-3) for $\lambda_1 > n$. □

As mentioned above, we have $\tilde{c}_1((F^0)^\perp/F^0) = 0$ in $\Omega_*(Q(E)) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$ so that $\tilde{c}_1(S^\vee \otimes (F^0)^\perp/F^0) = \tilde{c}_1(S^\vee)$. Therefore we have

$$F^{(2)}\big(\tilde{c}_1(S^\vee \otimes (F^0)^\perp/F^0)\big) = F^{(2)}\big(\tilde{c}_1(S^\vee)\big).$$

Notice that, since it is homogeneous of degree 0 with constant term 2, the series $F^{(2)}(u)$ is invertible in $\mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$. Thus we have the following corollary.

Corollary 4.2. *In* $\Omega_*(Q(E)) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$, *we have*

$$[X^B_{\lambda_1} \to Q(E)] = \begin{cases} \tilde{c}_{\lambda_1}(S^\vee \otimes E/F^{\lambda_1 - n})(1_{Q(E)}) & (0 \le \lambda_1 < n), \\ \dfrac{1}{F^{(2)}\big(\tilde{c}_1(S^\vee)\big)} \circ \tilde{c}_{\lambda_1}(S^\vee \otimes E/F^{\lambda_1 - n})(1_{Q(E)}) & (n \le \lambda_1 < 2n). \end{cases}$$

Remark 4.3. As mentioned in Remark 2.11, we have

$$[X^B_{\lambda_1} \to Q(E)] = \begin{cases} \widetilde{\mathscr{P}}_{\lambda_1}\big(S^\vee - (E/F^{\lambda_1 - n})^\vee\big)(1_{Q(E)}) & (0 \le \lambda_1 < n), \\ \dfrac{1}{F^{(2)}\big(c_1(S^\vee)\big)} \widetilde{\mathscr{P}}_{\lambda_1}\big(S^\vee - (E/F^{\lambda_1 - n})^\vee\big)(1_{Q(E)}) & (n \le \lambda_1 \le 2n). \end{cases}$$

Resolution of singularities and the class κ^B_λ. Consider the r-step flag bundle $\pi : Fl_r(U) \to OG^k(E)$ as before. We let $D_1 \subset \cdots \subset D_r$ be the tautological flag. Recall that $Fl_r(U)$ can be constructed as the tower of projective bundles

(4-4) $$\pi : Fl_r(U) = \mathbb{P}(U/D_{r-1}) \xrightarrow{\pi_r} \cdots \xrightarrow{\pi_3} \mathbb{P}(U/D_1) \xrightarrow{\pi_2} \mathbb{P}(U) \xrightarrow{\pi_1} OG^k(E)$$

We regard D_j/D_{j-1} as the tautological line bundle of $\mathbb{P}(U/D_{j-1})$ where we let $D_0 = 0$. For each $j = 1, \dots, r$, let $\tilde{\tau}_j := \tilde{c}_1((D_j/D_{j-1})^\vee)$ be the first Chern class operator of $(D_j/D_{j-1})^\vee$ on $\Omega_*(\mathbb{P}(U/D_{j-1}))$.

Definition 4.4. Let $\lambda \in \mathcal{SP}^k(n)$ be of length r. For each $j = 1, \dots, r$, we define a subvariety Y_j of $\mathbb{P}(U/D_{j-1})$ by setting

$$Y_j := \big\{(x, U_x, (D_1)_x, \dots, (D_j)_x) \in \mathbb{P}(U/D_{j-1}) \mid (D_i)_x \subset F^{\chi_i}_x,\ i = 1, \dots, j\big\}.$$

We set $Y_0 := SG^k_r(U)$ and $Y^B_\lambda := Y_r$. Let $P_{j-1} := \pi_j^{-1}(Y_{j-1})$, $\pi'_j : P_{j-1} \to Y_{j-1}$ the projection and $\iota_j : Y_j \to P_{j-1}$ the obvious inclusion. Let $\varpi_j := \pi'_j \circ \iota_j$. We have the commutative diagram

$$
\begin{array}{ccc}
\mathbb{P}(U/D_{j-1}) & \xrightarrow[\pi_j]{} & \mathbb{P}(U/D_{j-2}) \\
\uparrow & & \uparrow \\
P_{j-1} & \xrightarrow[\pi'_j]{} & Y_{j-1} \\
\uparrow{\scriptstyle \iota_j} & \nearrow_{\varpi_j} & \\
Y_j & &
\end{array}
$$

As in the symplectic case we set $\varpi := \varpi_1 \circ \cdots \circ \varpi_r : Y_\lambda^B \to OG^k(E)$ and define

$$\kappa_\lambda^B := \varpi_*(1_{Y_\lambda^B}).$$

Computing κ_λ^B. The following lemma is known from [Hudson et al. 2017], where the computation of the fundamental class of Y_j in P_{j-1} is done in connective K-theory CK_*. However, the proof is valid in an arbitrary oriented BM homology and in particular in Ω_*.

Lemma 4.5. *For each $j = 1, \dots, r$, the variety Y_j is regularly embedded in P_{j-1} and P_{j-1} is regularly embedded in $\mathbb{P}(U/D_{j-1})$, in particular they both belong to* $\mathbf{Lci_k}$. *Moreover we have*

$$\iota_{j*}(1_{Y_j}) = \tilde{\alpha}_j(1_{P_{j-1}})$$

in $\Omega_*(P_{j-1})$, where

$$\tilde{\alpha}_j = \begin{cases} \tilde{c}_{\lambda_j+n-k-j}\big((D_j/D_{j-1})^\vee \otimes (D_{\gamma_j}^\perp/F^{\chi_j})\big) & (-n \le \chi_j < 0), \\ \dfrac{1}{F^{(2)}(c_1((D_j/D_{j-1})^\vee))}\, \tilde{c}_{\lambda_j+n-k-j}\big((D_j/D_{j-1})^\vee \otimes (D_{\gamma_j}^\perp/F^{\chi_j})\big) & \\ & (0 \le \chi_j < n). \end{cases}$$

Definition 4.6. Let $-n \le \ell < n$. For each $m \in \mathbb{Z}$, we define the operators $\widetilde{\mathscr{B}}_m^{(\ell)}$ for $\Omega_*(OG^k(E)) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$ by means of the following generating function

$$\sum_{m\in\mathbb{Z}} \widetilde{\mathscr{B}}_m^{(\ell)} u^m = \begin{cases} \widetilde{\mathscr{S}}\big(U^\vee - (E/F^\ell)^\vee; u\big) & (-n \le \ell < 0), \\ \dfrac{1}{F^{(2)}(u^{-1})} \widetilde{\mathscr{S}}\big(U^\vee - (E/F^\ell)^\vee; u\big) & (0 \le \ell < n). \end{cases}$$

If $\frac{1}{F^{(2)}(u^{-1})} = \sum_{s\ge 0} f_s u^{-s}$ with $f_s \in \mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$, then we have

$$\widetilde{\mathscr{B}}_m^{(\ell)} = \sum_{s\ge 0} f_s \widetilde{\mathscr{S}}_{m+s}\big(U^\vee - (E/F^\ell)^\vee\big) \quad (0 \le \ell < n).$$

Remark 4.7. If $\lambda = (\lambda_1) \in \mathcal{SP}^k(n)$, we have $\kappa_\lambda^B = \mathscr{B}_{\lambda_1}^{(\chi_1)}$.

Lemma 4.8. *For each $s \geq 0$, we have*

$$\varpi_{j*} \circ \tilde{\tau}_j^s(1_{Y_j}) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^p \tilde{c}_p(D_{j-1}^{\vee} - D_{\gamma_j}) \circ \widetilde{w}_{-q}(D_{j-1}^{\vee} - D_{\gamma_j}) \circ \widetilde{\mathscr{B}}_{\lambda_j+s-p+q}^{(\chi_j)}(1_{Y_{j-1}}).$$

Proof. By Lemma 4.5, we have

$$\varpi_{j*}\circ\tilde{\tau}_j^s(1_{Y_j}) = \pi'_{j*}\circ\iota_{j*}\circ\tilde{\tau}_j^s(1_{Y_j}) = \pi'_{j*}\circ\tilde{\tau}_j^s\circ\iota_{j*}(1_{Y_j}) = \pi'_{j*}\circ\tilde{\tau}_j^s\circ\tilde{\alpha}_j(1_{P_{j-1}}). \tag{4-5}$$

Suppose that $\chi_j < 0$. By Theorem 2.10, the right-hand side of (4-5) equals

$$\widetilde{\mathscr{S}}_{\lambda_j+s}\big((U/D_{j-1}-D_{\gamma_j}^{\perp}/F^{\chi_j})^{\vee}\big)(1_{Y_{j-1}}) = \widetilde{\mathscr{S}}_{\lambda_j+s}\big((U-E/F^{\chi_j}-D_{j-1}+D_{\gamma_j}^{\vee})^{\vee}\big)(1_{Y_{j-1}}),$$

where $D_{\gamma_j}^{\perp} = E - D_{\gamma_j}^{\vee}$. Then the claim follows from Remark 2.9. Similarly, if $0 \leq \chi_j$, Theorem 2.10 implies that the right-hand side of (4-5) equals

$$\sum_{s'=0}^{\infty} f_{s'}\widetilde{\mathscr{S}}_{\lambda_j+s+s'}\big((U/D_{j-1})^{\vee} - (D_{\gamma_j}^{\perp}/F^{\chi_j})^{\vee}\big)(1_{Y_{j-1}}),$$

where we set $F^{(2)}(u^{-1})^{-1} = \sum_{s'\geq 0} f_{s'}u^{-s'}$ with $f_{s'} \in \mathbb{L}\otimes_{\mathbb{Z}}[1/2]$ as above. Again, we use the identity $D_{\gamma_j}^{\perp} = E - D_{\gamma_j}^{\vee}$ and then the claim follows from Remark 2.9. □

Set $R := \Omega^*(OG^k(E))\otimes_{\mathbb{Z}}\mathbb{Z}[1/2]$ and let $\mathcal{L}^R$ be the ring of formal Laurent series with indeterminates $t_1, \ldots, t_r$ defined in the previous section.

Definition 4.9. Define a graded R-module homomorphism

$$\phi_1^B : \mathcal{L}^R \otimes_{\mathbb{Z}} \mathbb{Z}[1/2] \to \Omega_*(OG^k(E))\otimes_{\mathbb{Z}}\mathbb{Z}[1/2]$$

by

$$\phi_1^B(t_1^{s_1}\cdots t_r^{s_r}) = \widetilde{\mathscr{S}}_{s_1}\big(U^{\vee} - (E/F^{\chi_1})^{\vee}\big)\circ\cdots\circ\widetilde{\mathscr{S}}_{s_r}\big(U^{\vee} - (E/F^{\chi_r})^{\vee}\big)(1_{OG^k(E)}).$$

Similarly, for each $j = 2, \ldots, r$, define a graded R-module homomorphism

$$\phi_j^B : \mathcal{L}^{R,j}\otimes_{\mathbb{Z}}\mathbb{Z}[1/2] \to \Omega_*(Y_{j-1})\otimes_{\mathbb{Z}}\mathbb{Z}[1/2]$$

by

$$\phi_j^B(t_1^{s_1}\cdots t_r^{s_r}) = \tilde{\tau}_1^{s_1}\circ\cdots\circ\tilde{\tau}_{j-1}^{s_j}\circ\widetilde{\mathscr{S}}_{s_j}\big(U^{\vee} - (E/F^{\chi_j})^{\vee}\big)\circ\cdots\circ\widetilde{\mathscr{S}}_{s_r}\big(U^{\vee} - (E/F^{\chi_r})^{\vee}\big)(1_{Y_{j-1}}).$$

Remark 4.10. Note that ϕ_j^B replaces $\frac{t_i^m}{F^{(2)}(t_i)}$ by $\widetilde{\mathscr{B}}_m^{(\chi_i)}(1_{Y_{j-1}})$ for each i such that $j \leq i \leq r$ and $\chi_i \geq 0$, and $m \in \mathbb{Z}$.

As with Lemma 3.8, by making use of Lemma 4.8 we can prove the following lemma.

Lemma 4.11. *We have*

$$\varpi_{j*}\circ\tilde{\tau}_j^s(1_{Y_j})=\begin{cases}\phi_j^B\left(t_j^{\lambda_j+s}\dfrac{\prod_{i=1}^{j-1}(1-t_i/t_j)P(t_j,t_i)}{\prod_{i=1}^{\gamma_j}(1-\bar{t}_i/t_j)P(t_j,\bar{t}_i)}\right) & (\chi_j<0),\\ \phi_j^B\left(\dfrac{t_j^{\lambda_j+s}}{F^{(2)}(t_j)}\dfrac{\prod_{i=1}^{j-1}(1-t_i/t_j)P(t_j,t_i)}{\prod_{i=1}^{\gamma_j}(1-\bar{t}_i/t_j)P(t_j,\bar{t}_i)}\right) & (0\le\chi_j),\end{cases}$$

for all $s\geq 0$.

A repeated application of Lemma 4.11 to the definition of κ_λ^B, together with Remark 4.10, allows us to obtain the main theorem for odd orthogonal Grassmannians.

Theorem 4.12. *We have*

$$\kappa_\lambda^B=\sum_{\mathbf{s}=(s_1,\dots,s_r)\in\mathbb{Z}^r} f_{\mathbf{s}}^\lambda\mathscr{B}_{\lambda_1+s_1}^{(\chi_1)}\cdots\mathscr{B}_{\lambda_r+s_r}^{(\chi_r)},$$

where the $f_{\mathbf{s}}^\lambda\in\mathbb{L}$ *are the coefficients of the Laurent series*

$$\frac{\prod_{1\le i<j\le r}(1-t_i/t_j)P(t_j,t_i)}{\prod_{(i,j)\in C(\lambda)}(1-\bar{t}_i/t_j)P(t_j,\bar{t}_i)}=\sum_{\mathbf{s}=(s_1,\dots,s_r)\in\mathbb{Z}^r} f_{\mathbf{s}}^\lambda\cdot t_1^{s_1}\cdots t_r^{s_r}\tag{4-6}$$

viewed as an element of $\mathcal{L}^{\mathbb{L}}$.

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