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Let (M, g) be a closed Riemannian manifold of dimension $5 \leq n \leq 7$. Assume that (M, g) is not conformally equivalent to the round sphere. If the scalar curvature R_g is greater than or equal to 0 and the Q-curvature Q_g is greater than or equal to 0 on M with $Q_g(p) > 0$ for some point $p \in M$, we prove that the set of metrics in the conformal class of g with prescribed constant positive Q-curvature is compact in $C^{4,\alpha}$ for any $0 < \alpha < 1$.

1. Introduction

On a manifold (M^n, g) of dimension $n \geq 5$, the Q-curvature of [1985] is defined by

$$Q_g = -\frac{2}{(n-2)^2}|\mathrm{Ric}_g|^2 + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} R_g^2 - \frac{1}{2(n-1)}\Delta_g R_g,$$

where Ric_g is the Ricci curvature of g, R_g is the scalar curvature of g and Δ_g is the Laplacian operator with negative eigenvalues. The Paneitz operator [1983], which is the linear operator in the conformal transformation formula of the Q-curvature, is defined as

$$P_g = \Delta_g^2 - \mathrm{div}_g(a_n R_g g - b_n \mathrm{Ric}_g)\nabla_g + \frac{n-4}{2} Q_g, \tag{1-1}$$

with

$$a_n = \frac{(n-2)^2 + 4}{2(n-1)(n-2)} \quad \text{and} \quad b_n = \frac{4}{n-2}.$$

In fact, under the conformal change $\tilde{g} = u^{4/(n-4)} g$, the transformation formula of the Q-curvature is given by

$$P_g u = \frac{n-4}{2} Q_{\tilde{g}} u^{\frac{n+4}{n-4}}.$$

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In comparison, for $n \geq 3$ the change of scalar curvature under the conformal change $\tilde{g} = u^{4/(n-2)} g$ satisfies

$$L_g u \equiv -\frac{4(n-1)}{(n-2)})\Delta_g u + R_g u = R_{\tilde{g}}\, u^{\frac{n+2}{n-2}}.$$

Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 5$. For existence of solutions u to the prescribed constant positive Q-curvature equation

$$P_g u = \frac{n-4}{2} \overline{Q} u^{\frac{n+4}{n-4}}, \tag{1-2}$$

with $\overline{Q} = \frac{1}{8} n(n^2-4)$, one may refer to [Esposito and Robert 2002; Qing and Raske 2006b; Hebey and Robert 2004; Gursky and Malchiodi 2015; Hang and Yang 2016a; 2016b; Gursky et al. 2016]. Recently, based on a version of maximum principle, Gursky and Malchiodi proved the following:

Theorem 1.1 [Gursky and Malchiodi 2015]. *For a closed Riemannian manifold (M^n, g) of dimension $n \geq 5$, if $R_g \geq 0$ and $Q_g \geq 0$ on M with Q_g not identically zero, then there is a conformal metric $h = u^{4/(n-4)} g$ with positive scalar curvature and constant Q-curvature $Q_h = \overline{Q}$.*

Moreover, they showed positivity of the Green's function of the Paneitz operator. Also, for $n = 5, 6, 7$, they proved a version of the positive mass theorem (see Theorem 2.1), which is important in proving compactness of the set of positive solutions to the prescribed constant Q-curvature problem in $C^{4,\alpha}(M)$ with $0 < \alpha < 1$. Note that when the pointwise condition in Theorem 1.1 is replaced by the requirement that the Yamabe constant $Y(M, [g])$ be greater than 0 and $Q_g \geq 0$, existence of solutions to (1-2) is proved in [Hang and Yang 2016b].

For compactness results of solutions to the prescribed constant Q-curvature equation under different conditions; see [Djadli et al. 2000; Hebey and Robert 2004; Humbert and Raulot 2009; Qing and Raske 2006a]. Djadli, Hebey and Ledoux [2000] studied the optimal Sobolev constant in the embedding $W^{2,2} \hookrightarrow L^{2n/(n-4)}$ when P_g has constant coefficients when g is an Einstein metric and also when P_g is replaced by a more general Paneitz-type operator. With some additional assumptions, they studied compactness of solutions to the related equations with $W^{2,2}$ bound and obtained existence of positive solutions for the corresponding equations. Under the assumption that the Paneitz operator is of strong positive type, Hebey and Robert [2004] considered compactness of positive solutions to (1-2) with $W^{2,2}$ bound in locally conformally flat manifolds with positive scalar curvature. They showed that under these conditions, when the Green's function of P_g satisfies a positive mass theorem, the compactness of solutions to (1-2) holds. Later, Humbert and Raulot [2009] showed that the positive mass theorem holds automatically under the assumption in [Hebey and Robert 2004]. Qing and Raske [2006a], with the use of the developing map and moving plane method, they showed an L^∞ bound of solutions to (1-2), for locally conformally flat manifolds with positive scalar curvature and an upper bound of the so-called Poincaré exponent (see [Chang et al. 2004]).

In this article we want to study compactness of solutions to (1-2) under the hypotheses in Theorem 1.1, following Schoen's outline of the proof of compactness of solutions to the prescribed scalar curvature problem. It is known that nonuniqueness of solutions to the prescribed scalar curvature problem (the Yamabe problem) could happen when the Yamabe constant of (M, g) is positive ([Schoen 1989; Pollack 1993]). In the conformal class of the round sphere metric, the solutions to the Yamabe problem are not uniformly bounded. Compactness of solutions to the Yamabe problem with positive Yamabe constant are well studied when g is not conformally equivalent to the round sphere metric. Following Schoen's original outline, one has the compactness of the solutions when (M^n, g) is locally conformally flat, or when $n \leq 24$ and the positive mass theorem holds on (M, g); see [Schoen 1991; Schoen and Zhang 1996; Li and Zhu 1999; Druet 2004; Chen and Lin 1998; Li and Zhang 2005; 2007; Marques 2005; Khuri et al. 2009]. It is interesting that when $n \geq 25$, there are conformal classes (which are not the round sphere metrics) with infinitely many solutions to the Yamabe problem which are not uniformly bounded; see [Brendle 2008; Brendle and Marques 2009]. In comparison, Wei and Zhao [2013] showed noncompactness of solutions to the positive constant Q-curvature equations for $n \geq 25$ in some conformal class different from that of the round sphere. For the compactness argument for the Nirenberg problem for a more general type conformal equation on the round sphere, see [Jin et al. 2017]. More precisely, we follow the approach in [Li and Zhu 1999] and [Marques 2005] for compactness of the set of solutions to the prescribed constant Q-curvature problem in dimension $5 \leq n \leq 7$ under the hypotheses of Theorem 1.1.

Our main theorem is the following:

Theorem 1.2. *Let (M^n, g) be a closed Riemannian manifold of dimension $5 \leq n \leq 7$ with $R_g \geq 0$, and also $Q_g \geq 0$ with $Q_g(p) > 0$ for some point $p \in M$. Assume that (M, g) is not conformally equivalent to the round sphere. Then there exists $C > 0$ depending on M and g such that for any positive solution u to* (1-2), *we have that*

$$C^{-1} \leq u \leq C,$$

and for any $0 < \alpha < 1$, there exists $C' > 0$ depending on M, g, and α such that

$$\|u\|_{C^{4,\alpha}} \leq C'.$$

We use a contradiction argument based on local information derived from a Pohozaev type identity for constant Q-curvature metrics and global information derived from the positive mass theorem of Gursky and Malchiodi [2015] (see Theorem 2.1). In comparison, for compactness of the Yamabe problem, the application of the positive mass theorem by Schoen and Yau [1979] (see also [Eichmair 2013; Eichmair et al. 2016; Witten 1981]) is crucial.

We extend the maximum principle in [Gursky and Malchiodi 2015] to manifolds with boundary under a Dirichlet-type condition and a scalar curvature condition restricted on the boundary; see Lemma 3.2. It turns out to be very useful and performs a role of a comparison theorem in the proof of the lower bound of the solutions away from the isolated blowup points (see Theorem 3.3) and in estimating upper bounds of solutions near blowup points (see Lemma 5.4). The Green's function is used as a comparison function in the uniform lower bound estimate Theorem 3.3. Note that Theorem 3.3 is important in the proof of the remark on page 138, Proposition 5.3 and Proposition 6.1. Since the main term of order $O(d_g^{-n})$ vanishes in $P_g d_g^{4-n}$, there is no comparison function to give the upper bound estimate in Proposition 5.3 directly. For that, the upper bound estimates of a sequence of blowup solutions near isolated simple blowup points are decomposed to a series of lemmas, following the approach in [Li and Zhu 1999] and in [Marques 2005]; see Section 5. We are able to prove a Harnack type inequality near the isolated blowup points for $5 \leq n \leq 9$; see Lemma 5.1. Besides the prescribed Q-curvature equation, nonnegativity of the scalar curvature is also important in the analysis of the blowing-up argument. With the aid of the Pohozaev type identity, we get a nice expansion of the limit of the blowing-up sequence near the blowup point, see Proposition 5.9, and using this we then show that in dimension $5 \leq n \leq 7$, each isolated blowup point is in fact an isolated simple blowup point. For the proof of Proposition 5.9, as in [Marques 2005], we need to estimate the speed of convergence of the rescaled functions to the limit, and for that, in Lemma 5.7 we need to classify bounded solutions to a linear fourth order elliptic equation on the Euclidean space which vanish uniformly at infinity, for $5 \leq n \leq 7$. The main difficulty for the classification problem in the Euclidean space is that the fourth order linear equation lacks the maximum principle, which is overcome by a combination of a comparison theorem for an initial value problem of ODEs, Kelvin transformation and an energy estimate; see Appendix B. After that, the proof of Theorem 1.2 is more or less standard, except that for the fourth order equation, more is involved for the blowing-up limit in ruling out the bubble accumulations; see Proposition 7.3. The Pohozaev type identity and the positive mass theorem in [Gursky and Malchiodi 2015] finally derive a contradiction on the sign of the constant term of the expansion of the singular limit function at the singular point in the proof of the main theorem. In Appendix A, we analyze the singular solutions to a linear fourth order elliptic equation near an isolated singular point, which is needed in Lemma 5.5 when finding the upper bound estimates of the solutions near the isolated simple blowup points. It is interesting to point out that in comparison with the proof of compactness of solutions to the Yamabe problem, here for compactness of positive constant Q-curvature metrics, no argument on vanishing of the Weyl tensor is needed for dimension $5 \le n \le 7$.

For $n \geq 8$, the Weyl tensor and its covariant derivatives are involved in the expansion of the Green's function and a vanishing argument of the Weyl tensor at the blowup points is needed (for instance, in Corollary 5.8 and Proposition 5.9), and yet a positive mass theorem for the Paneitz operator for cases which are not locally conformally flat in these dimensions is lacking. In this paper, for technical reasons, the Harnack inequality in Lemma 5.1 is only proved for $n \le 9$, the decay at infinity of the limit function $w(x)$ in Lemma 5.7 is only proved for $n \le 8$ due to the estimate (5-46), and the classification theorem (Corollary B.5) of solutions to the linear problem in Appendix B is given for $n \le 8$. But we believe that Lemma 5.1 and Corollary B.5 can be extended to high dimensions with some more discussion.

Remark. Let $Y(M, [g])$ be the Yamabe constant of (M, g) so that

$$Y(M,[g]) = \inf_{u\in C^\infty(M),\, u>0} \frac{\int_M \frac{4(n-1)}{n-2}|\nabla u|^2 + R_g u^2\, dV_g}{\left(\int_M u^{2n/(n-2)}\, dV_g\right)^{(n-2)/n}}.$$

Also, for $\alpha = \frac{4}{n-4}$ define

$$Y_4^*(M,[g]) = \inf_{u\in C^\infty(M),\, u>0,\, R_{u^\alpha g}>0} \frac{\int_M u\, P_g u\, dV_g}{\|u\|^2_{L^{2n/(n-4)}(M,g)}}.$$

From [Gursky et al. 2016], the following three statements are equivalent for dimension $n \geq 6$:

(1) $Y(M^n, [g]) > 0,\ \ P_g > 0$.

(2) $Y(M, [g]) > 0,\ \ Y_4^*(M, [g]) > 0$.

(3) There exists a metric $g_1 \in [g]$ such that $R_{g_1} > 0$ and $Q_{g_1} > 0$ on M.

As a corollary of Theorem 1.2, compactness of solutions to (1-2) holds for these conformal classes different from that of the round sphere for dimension $n = 6,\ 7$.

Remark. Recently, Li and Xiong [2019] proved compactness of prescribed constant Q metrics in a more general setting independently, by using the integration method developed from [Jin et al. 2017]. We follow the classical approach of [Li and Zhu 1999] and [Marques 2005].

To end the introduction, we introduce the definition of isolated blowup points and isolated simple blowup points.

Definition 1.3. Let g_j be a sequence of Riemannian metrics on a domain $\Omega \subseteq M$ with a uniform lower bound of injectivity radius $\bar{\delta} > 0$. Let $\{u_j\}_j$ be a sequence of positive solutions to (1-2) under the background metrics g_j in Ω. We call a point $\bar{x} \in \Omega$ an *isolated blowup point* of $\{u_j\}$ if there exist $\overline{C} > 0$, $0 < \delta < \min\left\{\frac{\bar{\delta}}{2}, \mathrm{dist}_{g_j}(\bar{x}, \partial\Omega)\right\}$ and $x_j \to \bar{x}$ as a local maximum of u_j with $u_j(x_j) \to \infty$ satisfying

(1-3) $B_\delta^{g_j}(\bar{x}),\ B_\delta^{g_j}(x_j) \subseteq \Omega$;

(1-4) $(B_\delta^{g_j}(x_j), x_j, g_j) \to (B_\delta^{g}(\bar{x}), \bar{x}, g)$ in $C^{k,\alpha}$ in the pointed Cheeger–Gromov sense, for $k > 0$ large and $0 < \alpha < 1$ and a smooth Riemannian metric g;

(1-5) $u_j(x) \le \overline{C} d_{g_j}(x, x_j)^{(4-n)/2}$ for $d_{g_j}(x, x_j) \le \delta$,

where $B_\delta^{g_j}$ is the δ-geodesic ball with respect to the metric g_j, and $d_{g_j}(x, x_j)$ is the geodesic distance between x and x_j with respect to the metric g_j.

In this paper, the sequence of metrics $\{g_j\}_j$ in the definition of the isolated blowup points are either a fixed metric on M, or the rescaled metrics $\{T_j g\}_j$ of g with a sequence of numbers $T_j \to \infty$, which converge to the flat metric as $j \to \infty$. Both these two cases satisfy the condition (1-4). For an isolated blowup point $x_j \to \bar{x}$ of u_j, we define

$$\bar{u}_j(r) = \frac{1}{|\partial B_r^{g_j}(x_j)|} \int_{\partial B_r^{g_j}(x_j)} u_j \, ds_{g_j}, \quad 0 < r < \delta,$$

and

$$\hat{u}_j(r) = r^{\frac{n-4}{2}} \bar{u}_j(r), \quad 0 < r < \delta, \tag{1-6}$$

with $B_r^{g_j}(x_j)$ that r-geodesic ball centered at x_j, ds_{g_j} the area element and $|\partial B_r^{g_j}(x_j)|$ the volume of $\partial B_r^{g_j}(x_j)$.

Definition 1.4. We call $\bar{x}$ an isolated simple blowup point if it is an isolated blowup point and there exists $0 < \delta_1 < \delta$ independent of j such that $\hat{u}_j$ has precisely one critical point in $(0, \delta_1)$, for j large.

2. The Green's representation

In this section, we assume that (M^n, g) is a closed Riemannian manifold of dimension $n \ge 5$ with $R_g \ge 0$, and also $Q_g \ge 0$ with $Q_g(p) > 0$ for some point $p \in M$.

Theorem 2.1 [Gursky and Malchiodi 2015]. *For a closed Riemannian manifold (M^n, g) of dimension $n \ge 5$, if $R_g \ge 0$, $Q_g \ge 0$ on M and also $Q_g(p) > 0$ for some point $p \in M$, then*:

- *The scalar curvature R_g is greater than* 0 *in M.*
- *The Paneitz operator P_g is in fact positive and the Green's function G of P_g is positive where $G : M \times M - \{(q, q), q \in M\} \to \mathbb{R}$. Also, if $u \in C^4(M)$ and $P_g u \ge 0$ on M, then either $u \equiv 0$ or $u > 0$ on M.*

- *For any metric g_1 in the conformal class of g, if $Q_{g_1} \geq 0$, then $R_{g_1} > 0$.*
- *For any distinct points $q_1, q_2 \in M$,*

$$G(q_1,q_2) = G(q_2,q_1) = c_n d_g(q_1,q_2)^{4-n}(1+f(q_1,q_2)), \tag{2-1}$$

  with $c_n = \frac{1}{(n-2)(n-4)\omega_{n-1}}$, $\omega_{n-1} = |S^{n-1}|$, and $d_g(q_1,q_2)$ the distance between q_1 and q_2. Here f is bounded and $f \to 0$ as $d_g(q_1,q_2) \to 0$ and

$$|\nabla^j f| \leq C_j d_g(q_1,q_2)^{1-j} \tag{2-2}$$

  for $1 \leq j \leq 4$.
- (*positive mass theorem*) *For $5 \leq n \leq 7$, or when (M,g) is locally conformally flat with dimension $n \geq 5$, for any point $q_1 \in M$, let $x = (x^1, \dots, x^n)$ be the conformal normal coordinates constructed in* [Lee and Parker 1987] *centered at q_1 and h be the corresponding conformal metric. For q_2 close to q_1, the Green's function $G_h(q_2,q_1)$ of the Paneitz operator P_h has the expansion*

$$G_h(q_2,q_1) = c_n d_h(q_2,q_1)^{4-n} + \alpha + f(q_2)$$

  with a constant $\alpha \geq 0$ and f satisfying (2-2) *and $f(q_2) \to 0$ as $q_2 \to q_1$; moreover, $\alpha = 0$ if and only if (M^n, g) is conformally equivalent to the round sphere.*

Let $u \in C^{4,\alpha}(M)$ be a solution to the equation

$$P_g u = f \geq 0.$$

Then we have the Green's representation

$$u(x) = \int_M G(x,y) f(y)\, dV_g(y)$$

for $x \in M$.

Now let $u > 0$ be a solution to the constant Q-curvature equation (1-2). Using the Green's representation

$$u(x) = \frac{n-4}{2} \overline{Q} \int_M G(x,y)\, u^{\frac{n+4}{n-4}}(y)\, dV_g(y),$$

we first show some basic estimates on the solution u.

Lemma 2.2. *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 5$ with $R_g > 0$, $Q_g \geq 0$ on M and $Q_g(p) > 0$ for some point $p \in M$. Then there exist C_1, $C_2 > 0$ depending on (M,g), so that for any solution u to* (1-2), *we have*

$$\inf_M u \leq C_1, \quad \sup_M u \geq C_2.$$

Proof. Let $u(q) = \inf_M u$. Then by the Green's representation,

$$\begin{aligned} u(q) &= \frac{n-4}{2}\bar{Q}\int_M G(q,y)\,u(y)^{\frac{n+4}{n-4}}\,dV_g(y) \\ &\ge u(q)^{\frac{n+4}{n-4}} \times \frac{n-4}{2}\bar{Q}\int_M G(q,y)\,dV_g(y) \ge C_1^{-\frac{8}{n-4}} u(q)^{\frac{n+4}{n-4}} \end{aligned}$$

with C_1 independent of the solution u and the point q, and the last inequality follows from (2-1). Therefore, the upper bound of $\inf_M u$ is established. A similar argument leads to the lower bound of $\sup_M u$. □

Next we give an integral type inequality, which shows that if u is bounded from above, then we get the lower bound of u.

Lemma 2.3. *Let (M^n, g) be a closed Riemannian manifold with dimension $n \ge 5$, $R_g > 0$, and also $Q_g \ge 0$ with $Q_g(p) > 0$ for some point $p \in M$. Then we have the inequality*

$$\inf_M u \ge C\left(\int_M G(z,y)^p\,u(y)^{\frac{8}{n-4}\alpha p}\,dV_g(y)\right)^{-\frac{q}{p}}$$

where $p = \frac{n+4}{n-4} - a$, $\frac{1}{p}+\frac{1}{q} = 1$, and $\alpha = \frac{(n-4)a}{8p}$, for any fixed number $\frac{4}{n-4} < a < \frac{8}{n-4}$, and z is the maximum point of u and $C = C(a, g) > 0$ is a constant. In particular, a uniform upper bound of u implies a uniform lower bound of u.

Proof. Let $u(x) = \inf_M u$ and $u(z) = \sup_M u$.

By the expansion formula (2-1), there exist two constants $C_3, C_4 > 0$ such that

$$0 < C_3 < \frac{1}{C_4} d_g(z_1,z_2)^{4-n} \le G(z_1,z_2) \le C_4 d_g(z_1,z_2)^{4-n} \tag{2-3}$$

for any two distinct points $z_1, z_2 \in M$.

By the Green's representation at the maximum point z we choose, we have

$$\begin{aligned} u(z) &= \frac{n-4}{2}\bar{Q}\int_M G(z,y)\,u(y)^{\frac{n+4}{n-4}}\,dV_g(y) \\ &\le \frac{n-4}{2}\bar{Q}u(z)\int_M G(z,y)\,u(y)^{\frac{8}{n-4}}\,dV_g(y) \end{aligned}$$

so that

$$\begin{aligned} 1 &\le \frac{(n-4)}{2}\bar{Q}\int_M G(z,y)\,u(y)^{\frac{8}{n-4}(\alpha+(1-\alpha))}\,dV_g(y) \\ &\le \frac{(n-4)}{2}\bar{Q}\left(\int_M G(z,y)^p\,u(y)^{\frac{8}{n-4}\alpha p}\,dV_g(y)\right)^{\frac{1}{p}}\left(\int_M u(y)^{\frac{8}{(n-4)}(1-\alpha)q}\,dv_g(y)\right)^{\frac{1}{q}} \\ &= \frac{(n-4)}{2}\bar{Q}\left(\int_M G(z,y)^p\,u(y)^{\frac{8}{n-4}\alpha p}\,dV_g(y)\right)^{\frac{1}{p}}\left(\int_M u(y)^{\frac{n+4}{n-4}}\,dv_g(y)\right)^{\frac{1}{q}}, \end{aligned}$$

with α, p, q chosen in the statement of the lemma. Here the second inequality is by Hölder's inequality. The range of a in the lemma keeps $0 < \alpha < 1$, $p > 1$ and $q > 1$, and also $p(4-n) > -n$ so that G^p is integrable.

Therefore, combining this with (2-3) we have

$$\begin{aligned}\inf_M u = u(x) &= \frac{n-4}{2}\overline{Q}\int_M G(x,y)u(y)^{\frac{n+4}{n-4}}\,dV_g(y)\\ &\geq C'\int_M u(y)^{\frac{n+4}{n-4}}\,dV_g(y) \geq C\biggl(\int_M G(z,y)^p\,u(y)^{\frac{8}{n-4}\alpha p}\,dV_g(y)\biggr)^{-\frac{q}{p}},\end{aligned}$$

where C', $C > 0$ are uniform constants independent of u, z and x. □

3. A maximum principle

In this section we prove a maximum principle for smooth domains with boundary in the manifold (M, g) defined in Lemma 2.2, which is a modification of the maximum principle given by Gursky and Malchiodi; see Lemma 3.2. As an application, we give a lower bound estimate of the blowing-up sequence.

Lemma 3.1. *Let $(\overline{\Omega}, g)$ be a compact Riemannian manifold of dimension $n \geq 5$ with boundary $\partial\Omega$. Let Ω be the interior of $\overline{\Omega}$. Assume the scalar curvature R_g is greater than or equal to 0 in $\overline{\Omega}$ and $R_g > 0$ at points on the boundary, and also $Q_g \geq 0$ in $\overline{\Omega}$. Then $R_g > 0$ in $\overline{\Omega}$.*

Proof. The proof is similar to that for closed manifolds. The Q-curvature is expressed as

$$Q_g = -\frac{1}{2(n-1)}\Delta_g R_g + c_1(n)R_g^2 - c_2(n)|\mathrm{Ric}|_g^2$$

with $c_1(n)$, $c_2(n)$ positive. By the nonnegativity of Q_g,

$$\frac{1}{2(n-1)}\Delta_g R_g \leq c_1(n)R_g^2.$$

By the strong maximum principle and the boundary condition, $R_g > 0$ in $\overline{\Omega}$. □

Lemma 3.2. *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 5$ with $R_g \geq 0$, and $Q_g \geq 0$. Let $\Omega \subseteq M$ be an open domain with smooth boundary $\partial\Omega$ so that $\overline{\Omega} = \Omega \cup \partial\Omega$. Assume that $u \in C^4(\overline{\Omega})$ with $u > 0$ on $\partial\Omega$ satisfies*

$$P_g u \geq 0 \quad \text{in}\ \Omega. \tag{3-1}$$

Let $\tilde{g} = u^{4/(n-4)}g$ be the conformal metric in a neighborhood $\mathcal{U}$ of $\partial\Omega$ where $u > 0$. If the scalar curvature of $(\mathcal{U}, \tilde{g})$ satisfies $R_{\tilde{g}}(p) > 0$ for all points $p \in \partial\Omega$, then $u > 0$ in Ω.

Proof. Our conditions on the boundary guarantee that all the arguments are focused on the interior and then the argument is the same as in the proof of the maximum principle by Gursky and Malchiodi. For completeness, we present the proof.

We define the function

$$u_\lambda = (1-\lambda) + \lambda u$$

for $\lambda \in [0, 1]$, so that $u_0 = 1$ and $u_1 = u$. We assume

$$\min_{\overline{\Omega}} u \le 0.$$

Then there exists $\lambda_0 \in (0, 1]$ so that

$$\lambda_0 = \min\{\lambda \in (0, 1],\ \min_{\overline{\Omega}} u_\lambda = 0\}.$$

By definition, for $0 < \lambda < \lambda_0$, $u_\lambda > 0$. For the metric

$$g_\lambda = u_\lambda^{\frac{4}{n-4}} g,$$

the Q-curvature satisfies

$$Q_{g_\lambda} \ge 0 \quad \text{in } \Omega,$$

for $0 < \lambda < \lambda_0$. That follows from the conformal transformation formula

$$\begin{aligned} Q_{g_\lambda} &= \frac{2}{n-4} u_\lambda^{-\frac{n+4}{n-4}} P_g u_\lambda = \frac{2}{n-4} u_\lambda^{-\frac{n+4}{n-4}} ((1-\lambda) P_g(1) + \lambda P_g u) \\ &= \frac{2}{n-4} u_\lambda^{-\frac{n+4}{n-4}} ((1-\lambda) \frac{n-4}{2} Q_g + \lambda P_g u) \ge (1-\lambda) Q_g u_\lambda^{-\frac{n+4}{n-4}} \ge 0. \end{aligned}$$

Under the conformal transformation, the scalar curvature of g_λ satisfies

$$\begin{aligned} R_{g_\lambda} &= u_\lambda^{-\frac{n}{n-4}} \left(-\frac{4(n-1)}{n-4} \Delta_g u_\lambda - \frac{8(n-1)}{(n-4)^2} \frac{|\nabla_g u_\lambda|^2}{u_\lambda} + R_g u_\lambda \right) \\ &= u_\lambda^{-\frac{n}{n-4}} \left(-\frac{4(n-1)}{n-4} \lambda \Delta_g u - \frac{8(n-1)}{(n-4)^2} \frac{\lambda^2 |\nabla_g u|^2}{(1-\lambda) + \lambda u} + R_g u_\lambda \right) \\ &\ge u_\lambda^{-\frac{n}{n-4}} \left(-\frac{4(n-1)}{n-4} \lambda \Delta_g u - \frac{8(n-1)}{(n-4)^2} \frac{\lambda |\nabla_g u|^2}{u} + \lambda R_g u \right) \\ &= \lambda \left(\frac{u}{u_\lambda} \right)^{\frac{n}{n-4}} R_{\tilde{g}} > 0 \end{aligned}$$

on $\partial\Omega$ for $0 < \lambda < \lambda_0$. Then by Lemma 3.1,

$$R_{g_\lambda} > 0 \quad \text{in } \Omega,$$

for $0 < \lambda < \lambda_0$. Again by the conformal transformation formula of scalar curvature,

$$\Delta_g u_\lambda \le \frac{n-4}{4(n-1)} R_g u_\lambda \quad \text{in } \Omega.$$

By taking limit $\lambda \nearrow \lambda_0$, this also holds at $\lambda = \lambda_0$. But

$$u_\lambda = (1-\lambda) + \lambda u > 0$$

on $\partial\Omega$ for $0 \le \lambda \le 1$. By the strong maximum principle, $u_{\lambda_0} > 0$ in $\overline{\Omega}$, contradicting our choice of λ_0. Therefore, for all $0 \le \lambda \le 1$,

$$u_\lambda > 0 \quad \text{in } \Omega.$$

In particular, $u > 0$ in Ω. □

Theorem 3.3. *Let (M^n, g) be a closed Riemannian manifold of dimension $n \ge 5$ with $R_g \ge 0$, and also $Q_g \ge 0$ with $Q_g(p_0) > 0$ for some point $p_0 \in M$. There exists $C > 0$ such that if there exists a sequence of positive solutions $\{u_j\}_{j=1}^\infty$ of* (1-2) *such that*

$$M_j = u_j(x_j) = \sup_M u_j \to \infty$$

as $j \to \infty$, then

$$u_j(p) \ge C M_j^{-1} d_g^{4-n}(p, x_j) \tag{3-2}$$

for any $p \in M$ such that $d_g(p, x_j) \ge M_j^{-2/(n-4)}$.

Proof. To prove the theorem, we only need to show that there exists $C > 0$ such that for any blowing-up sequence, there exists a subsequence such that (3-2) holds.

For each j, let $x = (x^1, \dots, x^n)$ be the corresponding normal coordinates in a small geodesic ball centered at x_j with radius $\delta > 0$ and x_j the origin. Let $y = M_j^{2/(n-4)} x$ and the metric h_j be given by $(h_j)_{pq}(y) = g_{pq}(M_j^{-2/(n-4)} y)$. Let

$$v_j(y) = M_j^{-1} u_j(\exp_{x_j}(M_j^{-\frac{2}{n-4}} y)) \quad \text{for } |y| \le \delta M_j^{\frac{2}{n-4}}.$$

Then,

$$0 < v_j(y) \le v_j(0) = 1,$$
$$P_{h_j} v_j(y) = \frac{n-4}{2} \overline{Q} v_j(y)^{\frac{n+4}{n-4}} \quad \text{for } |y| \le \delta M_j^{2/(n-4)}.$$

Here h_j converges to the Euclidean metric on $\mathbb{R}^n$ in C^k norm for any $k \ge 0$. By ellipticity, we have, after passing to a subsequence (still denoted as $\{v_j\}$), $v_j \to v$ in $C^4_{\text{loc}}(\mathbb{R}^n)$, and v satisfies

$$\begin{aligned} &0 \le v(y) \le v(0) = 1 \quad \text{in } \mathbb{R}^n, \\ &\Delta^2 v(y) = \frac{n-4}{2} \overline{Q} v(y)^{\frac{n+4}{n-4}} \quad \text{in } \mathbb{R}^n. \end{aligned} \tag{3-3}$$

Also, since $R_{h_j} > 0$ and $R_{u_j^{4/(n-4)} g} > 0$ (by Theorem 2.1) on M, by the conformal transformation formula of scalar curvature,

$$\Delta_{h_j} v_j \le \frac{n-4}{4(n-1)} R_{h_j} v_j.$$

Passing to the limit we have

$$\Delta v(y) \le 0 \quad \text{in } \mathbb{R}^n.$$

By the strong maximum principle, since $v(0) = 1$, we have that $v(y) > 0$ in $\mathbb{R}^n$. Then by the classification theorem of C.S. Lin [1998], we have

$$v(y) = \left(\frac{1}{1+4^{-1}|y|^2}\right)^{\frac{n-4}{2}} \quad \text{in } \mathbb{R}^n.$$

We will abuse the notation with $v(|y|) = v(y)$. Thus, for fixed $R > 0$, for j large,

$$\tfrac{1}{2}\left(\frac{1}{1+4^{-1}R^2}\right)^{\frac{n-4}{2}} M_j \le u_j(\exp_{x_j}(x)) \le M_j \quad \text{for } |x| \le RM_j^{-\frac{2}{n-4}}.$$

For any $\epsilon > 0$, there exists $j_0 > 0$ such that, for $j > j_0$,

$$\|v_j - v\|_{C^4} \le \epsilon \quad \text{for } |y| \le 2.$$

We define $\phi_j : M - \{x_j\} \to \mathbb{R}$ as

$$\phi_j(p) = u_j(p) - \tau M_j^{-1} G_{x_j}(p),$$

with $G_{x_j}(p) = G(x_j, p)$ the Green's function of the Paneitz operator and $\tau > 0$ a small constant to be chosen. We will use the maximum principle to show that for $\epsilon, \tau > 0$ small,

$$\phi_j > 0 \ \text{ in } M - B_{M_j^{-2/(n-4)}}(x_j) \quad \text{for } j > j_0.$$

Here, we denote by $B_{M_j^{-2/(n-4)}}(x_j)$ the geodesic $M_j^{-2/(n-4)}$-ball centered at x_j in (M, g). If this holds, we will choose $\{u_j\}_{j>j_0}$ as the subsequence and the theorem is proved.

It is clear that

$$P_g\phi_j = P_g u_j = \frac{n-4}{2}\bar{Q} u_j^{\frac{n+4}{n-4}} > 0 \quad \text{in } M - B_{M_j^{-2/(n-4)}}(x_j).$$

To apply the maximum principle, we only need to verify the sign of ϕ_j and the related scalar curvature on $\partial B_{M_j^{-2/(n-4)}}(x_j)$.

First, for $|x| = M_j^{-\frac{2}{n-4}}$, we choose ϵ small so that for $j > j_0$,

$$u_j(\exp_{x_j}(x)) = M_j v_j(M_j^{\frac{2}{n-4}} x) \ge \tfrac{1}{2} v(1) M_j;$$

while by (2-3),

$$M_j^{-1} G_{x_j}(\exp_{x_j}(x)) \le C_4 M_j.$$

We take $\tau < v(1)/(4C_4)$. Then

$$\phi_j > 0 \quad \text{on } \partial B_{M_j^{-2/(n-4)}}(x_j) \quad \text{for } j > j_0.$$

Now let $\tilde{g}_j = \phi_j^{4/(n-4)} g_j$ in small neighborhood of $\partial B_{M_j^{-2/(n-4)}}(x_j)$ where $\phi_j > 0$. By conformal transformation,

$$R_{\tilde{g}_j} = \phi_j^{-\frac{n}{n-4}} \left(-\frac{4(n-1)}{n-4} \Delta_g \phi_j - \frac{8(n-1)}{(n-4)^2} \frac{|\nabla_g \phi_j|^2}{\phi_j} + R_g \phi_j \right).$$

Note that $R_g \phi_j > 0$ on $\partial B_{M_j^{-2/(n-4)}}(x_j)$. We only need to show that

$$-\frac{4(n-1)}{n-4} \left(\Delta_g \phi_j + \frac{2}{n-4} \frac{|\nabla_g \phi_j|^2}{\phi_j} \right) > 0 \quad \text{on } \partial B_{M_j^{-2/(n-4)}}(x_j) \quad \text{for } j > j_0. \tag{3-4}$$

Recall that

$$\left(\Delta_g u_j + \frac{2}{n-4} \frac{|\nabla_g u_j|^2}{u_j} \right) = M_j^{1+\frac{4}{n-4}} \left(\Delta_{h_j} v_j + \frac{2}{n-4} \frac{|\nabla_{h_j} v_j|^2}{v_j} \right).$$

Also,

$$\begin{aligned}
&\left(\Delta_{h_j} v_j + \frac{2}{n-4} \frac{|\nabla_{h_j} v_j|^2}{v_j} \right) \\
&\qquad \to \left(\Delta v + \frac{2}{n-4} \frac{|\nabla v|^2}{v} \right) \\
&\qquad = 2(4-n)(|y|^2+4)^{-\frac{n}{2}}(|y|^2+2n) \ + \frac{2}{n-4} \frac{(4-n)^2(|y|^2+4)^{2-n}|y|^2}{(|y|^2+4)^{(4-n)/2}} \\
&\qquad = 2(4-n)(|y|^2+4)^{-\frac{n}{2}}(|y|^2+2n) \ + 2(n-4)(|y|^2+4)^{-\frac{n}{2}}|y|^2 \\
&\qquad = 4n(4-n)(|y|^2+4)^{-\frac{n}{2}} < 0 \quad \text{at } |y| = 1.
\end{aligned}$$

Then we can choose $\epsilon < |v|_{C^4(B_1(0))}/100^n$. Combining this with the fact that

$$|D_g^k G_p(q)| \le C_k d_g^{4-n-k}(p,q) \quad \text{for } 0 \le k \le 4,$$

for any distinct points $p, q \in M$ with constants $C_k > 0$ independent of p and q, we have that there exists $\tau > 0$ only depending on C_k and ϵ so that

$$\begin{aligned}
\tau M_j^{-1} |\Delta_g G_{x_j}(\exp_{x_j}(M_j^{-\frac{2}{n-4}} y))| < -M_j^{1+\frac{4}{n-4}} \frac{\Delta v}{4(2n+1)}, \quad &\text{and} \\
\frac{|\nabla_g \phi_j|^2}{\phi_j} \le \tfrac{5}{4} M_j^{1+\frac{4}{n-4}} \frac{|\nabla v|^2}{v} \quad &\text{at } |y| = 1, \ \text{for } j > j_0.
\end{aligned}$$

Therefore, (3-4) holds for $j > j_0$, which implies

$$R_{\tilde{g}_j} > 0 \quad \text{on } \partial B_{M_j^{-2/(n-4)}}(x_j).$$

By Lemma 3.2, $\phi_j > 0$ in $M - B_{M_j^{-2/(n-4)}}(x_j)$. Recall that ϵ and τ are chosen independent of choice of the sequence. This completes the proof of the theorem. □

4. A Pohozaev type identity

In this section we introduce a Pohozaev type identity related to the constant Q-curvature equation. It will provide local information on the solutions in later use.

Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 5$ with $R_g \geq 0$, and $Q_g \geq 0$ with $Q_g(p_0) > 0$ for some point $p_0 \in M$. Let u be a positive solution to (1-2). For any geodesic ball $\Omega = B_\delta(q)$ in M with 2δ less than the injectivity radius of (M, g), we let

$$x = (x^1, \dots, x^n)$$

be the geodesic normal coordinates centered at q so that $g_{ij}(0) = \delta_{ij}$ and the Christoffel symbols $\Gamma_{ij}^k(0) = 0$. In this section, the gradient ∇, Laplacian Δ, divergence div, volume element dx, area element ds, σ-ball B_σ and

$$|x|^2 = (x^1)^2 + \dots + (x^n)^2$$

are all with respect to the Euclidean metric. Define

$$\begin{aligned}
\mathcal{P}(u) &\equiv \int_\Omega \Big(x \cdot \nabla u + \frac{n-4}{2} u\Big) \Delta^2 u \, dx \\
&= \int_\Omega \Big[\frac{n-4}{2} \operatorname{div}(u \nabla(\Delta u) - \Delta u \nabla u) \\
&\qquad\qquad + \operatorname{div}((x \cdot \nabla u)\nabla(\Delta u) - \nabla(x \cdot \nabla u)\Delta u + \tfrac{1}{2}(\Delta u)^2 x)\Big] \, dx \\
&= \int_{\partial\Omega} \frac{n-4}{2} \Big(u \frac{\partial}{\partial \nu}(\Delta u) - \Delta u \frac{\partial}{\partial \nu} u\Big) \\
&\qquad\qquad + \Big((x \cdot \nabla u) \frac{\partial}{\partial \nu}(\Delta u) - \frac{\partial}{\partial \nu}(x \cdot \nabla u)\Delta u + \tfrac{1}{2}(\Delta u)^2 x \cdot \nu\Big) \, ds,
\end{aligned}$$

where ν is the outward-pointing normal vector of $\partial\Omega$ in the Euclidean metric. Then using (1-2), we have

$$\begin{aligned}
\mathcal{P}(u) &= \int_\Omega \Big(x \cdot \nabla u + \frac{n-4}{2} u\Big)(\Delta^2 - P_g)u + \Big(x \cdot \nabla u + \frac{n-4}{2} u\Big) P_g u \, dx \\
&= \int_\Omega \Big(x \cdot \nabla u + \frac{n-4}{2} u\Big)(\Delta^2 - P_g)u + \frac{n-4}{2} \bar{Q} \Big(x \cdot \nabla u + \frac{n-4}{2} u\Big) u^{\frac{n+4}{n-4}} \, dx \\
&= \int_\Omega \Big(x \cdot \nabla u + \frac{n-4}{2} u\Big)(\Delta^2 - P_g)u + \frac{(n-4)^2}{4n} \bar{Q} \operatorname{div}(u^{\frac{2n}{n-4}} x) \, dx \\
&= \int_\Omega \Big(x \cdot \nabla u + \frac{n-4}{2} u\Big)(\Delta^2 - P_g)u \, dx + \frac{(n-4)^2}{4n} \bar{Q} \int_{\partial\Omega} (x \cdot \nu) u^{\frac{2n}{n-4}} \, dx.
\end{aligned}$$

Using (1-1), we have

$$(\Delta^2 - P_g)u = (\Delta^2 - \Delta_g^2)u + \operatorname{div}_g(a_n R_g g - b_n \mathrm{Ric}_g)\nabla_g u - \frac{n-4}{2} Q_g u.$$

Since $\Gamma^k_{ij}(0) = 0$ and $g_{ij}(0) = \delta_{ij}$,

$$\begin{aligned}
&(\Delta^2 - \Delta_g^2)u \\
&\quad = (\delta^{pq}\delta^{ij}\nabla_p\nabla_q\nabla_i\nabla_j - g^{pq}g^{ij}\nabla^g_p\nabla^g_q\nabla^g_i\nabla^g_j)u \\
&\quad = (\delta^{pq}\delta^{ij} - g^{pq}g^{ij})\nabla_p\nabla_q\nabla_i\nabla_j u + O(|x|)|D^3u| + O(1)|D^2u| + O(1)|Du| \\
&\quad = O(|x|^2)|D^4u| + O(|x|)|D^3u| + O(1)|D^2u| + O(1)|Du|.
\end{aligned}$$

It follows that there exists $C > 0$ which depends on $|Rm_g|_{L^\infty(\Omega)}$, $|Q_g|_{C(\Omega)}$ and $|\mathrm{Ric}_g|_{C^1(\Omega)}$ such that

$$|(\Delta^2 - P_g)u| \le C(|x|^2|D^4u| + |x|\,|D^3u| + |D^2u| + |Du| + u). \tag{4-1}$$

5. Upper bound estimates near isolated simple blowup points

In this section we perform a parallel approach of [Li and Zhu 1999] to show the upper bound estimates of the solutions to (1-2) near an isolated simple blowup point; see Proposition 5.3. We start with a Harnack type inequality near an isolated blowup point.

Lemma 5.1. *Let (M^n, g) be a closed Riemannian manifold of dimension $5 \le n \le 9$ with $R_g \ge 0$, and also $Q_g \ge 0$ with $Q_g(p_0) > 0$ for some point $p_0 \in M$. Let $\{u_j\}$ be a sequence of positive solutions to* (1-2) *and $x_j \to \bar{x}$ be an isolated blowup point. Then there exists a constant $C > 0$ such that for any $0 < r < \frac{\delta}{3}$ and $j > 0$, we have*

$$\max_{q\in B_{2r}(x_j)-B_{r/2}(x_j)} u_j(q) \le C \min_{q\in B_{2r}(x_j)-B_{r/2}(x_j)} u_j(q). \tag{5-1}$$

Proof. Let $x = (x^1, \dots, x^n)$ be the geodesic normal coordinates centered at x_j. Here $\delta > 0$ (see Definition 1.3) and 2δ is less than the injectivity radius. Let $y = r^{-1}x$. Define

$$v_j(y) = r^{\frac{n-4}{2}} u_j(\exp_{x_j}(ry)) \quad \text{for } |y| < 3.$$

Then by (1-5),

$$\begin{aligned}
v_j(y) &\le \overline{C}|y|^{-\frac{n-4}{2}} && \text{for } |y| < 3, \\
v_j(y) &\le 3^{\frac{n-4}{2}}\overline{C} && \text{for } \tfrac{1}{3} < |y| < 3.
\end{aligned}$$

We denote

$$\Omega_r = B_{3r}(x_j) - B_{\frac{r}{3}}(x_j).$$

By the Green's representation,

$$v_j(y) = r^{\frac{n-4}{2}} u_j(\exp_{x_j}(ry)) = \frac{(n-4)\overline{Q}}{2} r^{\frac{n-4}{2}} \int_M G(\exp_{x_j}(ry), q) u_j(q)^{\frac{n+4}{n-4}}\, dV_g(q)$$

$$= \frac{(n-4)\bar{Q}}{2} r^{\frac{n-4}{2}} \Big(\int_{\Omega_r} G(\exp_{x_j}(ry), q) u_j(q)^{\frac{n+4}{n-4}} \, dV_g(q) + \int_{M-\Omega_r} G(\exp_{x_j}(ry), q) u_j(q)^{\frac{n+4}{n-4}} \, dV_g(q) \Big).$$

We claim that for $\frac{5}{12} \le |y| \le \frac{12}{5}$, if

$$v_j(y) \ge 2 \times \frac{(n-4)\bar{Q}}{2} r^{\frac{n-4}{2}} \int_{\Omega_r} G(\exp_{x_j}(ry), q) u_j(q)^{\frac{n+4}{n-4}} \, dV_g(q), \tag{5-2}$$

then there exists $C > 0$ independent of j, x_j, r and y, such that for any $\frac{5}{12} \le |z| \le \frac{12}{5}$,

$$v_j(z) \ge C v_j(y). \tag{5-3}$$

In fact, by (2-3), there exists $C > 0$, such that

$$G(\exp_{x_j}(ry), q) \le C G(\exp_{x_j}(rz), q)$$

for $q \in M - \Omega_r$. Therefore,

$$\begin{aligned} \tfrac{1}{2} v_j(y) &\le \frac{(n-4)\bar{Q}}{2} r^{\frac{n-4}{2}} \int_{M-\Omega_r} G(\exp_{x_j}(ry), q) u_j(q)^{\frac{n+4}{n-4}} \, dV_g(q) \\ &\le C r^{\frac{n-4}{2}} \int_{M-\Omega_r} G(\exp_{x_j}(rz), q) u_j(q)^{\frac{n+4}{n-4}} \, dV_g(q) \\ &\le C v_j(z). \end{aligned}$$

This proves the claim.

We denote

$$\mathcal{C} = \Big\{ y \in \mathbb{R}^n,\ \tfrac{5}{12} \le |y| \le \tfrac{12}{5},\ \text{so that (5-2) fails for } y \Big\}.$$

We choose $\frac{5}{12} \le |y| \le \frac{12}{5}$ with

$$v_j(y) \ge \tfrac{1}{2} \sup_{5/12 \le |z| \le 12/5} v_j(z).$$

If $y \notin \mathcal{C}$, then using the claim, we are done. If $y \in \mathcal{C}$, we will prove that the Harnack inequality (5-1) still holds.

By Hölder's inequality,

$$u_j(\exp_{x_j}(ry)) \le 2 \times \frac{(n-4)\bar{Q}}{2} \int_{\Omega_r} G(\exp_{x_j}(ry), q) u_j(q)^{\frac{n+4}{n-4}} \, dV_g(q)$$

$$
\begin{aligned}
&\leq (n-4)\overline{Q}\Big(\int_{\Omega_r} G(\exp_{x_j}(ry),q)^{\alpha}\,dV_g(q)\Big)^{\frac{1}{\alpha}} \\
&\qquad\qquad \times\Big(\int_{\Omega_r} u_j(q)^{\frac{n+4}{n-4}\beta}\,dV_g(q)\Big)^{\frac{1}{\beta}} \\
&\leq C(\alpha) r^{4-n+\frac{n}{\alpha}}\Big(\int_{\Omega_r} u_j(q)^{\frac{n+4}{n-4}\beta}\,dV_g(q)\Big)^{\frac{1}{\beta}} \\
&\leq C(\alpha) r^{4-n+\frac{n}{\alpha}}\,(\overline{C}3^{\frac{n-4}{2}}r^{\frac{4-n}{2}})^{\frac{n+4}{n-4}\left(1-\frac{1}{\beta}\right)}\Big(\int_{\Omega_r} u_j(q)^{\frac{n+4}{n-4}}\,dV_g(q)\Big)^{\frac{1}{\beta}} \\
&\leq C(\alpha) r^{4-n+\frac{n}{\alpha}}\,(\overline{C}3^{\frac{n-4}{2}}r^{\frac{4-n}{2}})^{\frac{n+4}{n-4}\left(1-\frac{1}{\beta}\right)} \\
&\qquad\qquad \times\Big(\int_{\Omega_r} C_4(4r)^{n-4}G(\exp_{x_j}(rz),q)u_j(q)^{\frac{n+4}{n-4}}\,dV_g(q)\Big)^{\frac{1}{\beta}} \\
&\leq C(\alpha) r^{4-n+\frac{n}{\alpha}}(\overline{C}3^{\frac{n-4}{2}}r^{\frac{4-n}{2}})^{\frac{n+4}{n-4}\left(1-\frac{1}{\beta}\right)} r^{\frac{n-4}{\beta}} u_j(\exp_{x_j}(rz))^{\frac{1}{\beta}} \\
&= C(\alpha,\overline{C},n) r^{\left(2-\frac{n}{2}\right)\left(1-\frac{1}{\beta}\right)} u_j(\exp_{x_j}(rz))^{\frac{1}{\beta}}
\end{aligned}
$$

for any $\frac{1}{3} \leq |z| \leq 3$, where $1 < \alpha < \frac{n}{n-4}$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ such that $\beta > \frac{n}{4}$. Here we have used (1-5) and (2-3).

Since

$$
\frac{n+4}{n-4} > \frac{n}{4}
$$

for $5 \leq n \leq 9$, we set $\beta = \frac{n+4}{n-4}$ and obtain

$$
u_j(\exp_{x_j}(rz)) \geq C(\overline{C},n)\, r^4\, u_j(\exp_{x_j}(ry))^{\frac{n+4}{n-4}} \tag{5-4}
$$

$$
\geq C(\overline{C},n)\, r^4\, (2^{-1}u_j(q))^{\frac{n+4}{n-4}}, \tag{5-5}
$$

for all $q \in B_{12r/5}(x_j) - B_{5r/12}(x_j)$ and $\frac{1}{2} \leq |z| \leq 2$, where $5 \leq n \leq 9$.

For any $\frac{1}{2} \leq |z| \leq 2$,

$$
\begin{aligned}
&|\nabla_g u_j|(\exp_{x_j}(rz)) \\
&\quad\leq \frac{n-4}{2}\overline{Q}\int_{B_{12r/5}(x_j)-B_{5r/12}(x_j)} |\nabla_g G(\exp_{x_j}(rz),q)|\,u_j(q)^{\frac{n+4}{n-4}}\,dV_g(q) \\
&\qquad + \frac{n-4}{2}\overline{Q}\int_{M-(B_{12r/5}(x_j)-B_{5r/12}(x_j))} |\nabla_g G(\exp_{x_j}(rz),q)|\,u_j(q)^{\frac{n+4}{n-4}}\,dV_g(q).
\end{aligned} \tag{5-6}
$$

Note that for $\frac{1}{2} \le |z| \le 2$,

$$
\begin{aligned}
\text{(5-7)} \quad & u_j(\exp_{x_j}(rz)) \\
&\ge \frac{n-4}{2}\overline{Q} \int_{M-(B_{12r/5}(x_j)-B_{5r/12}(x_j))} G(\exp_{x_j}(rz), q)\, u_j(q)^{\frac{n+4}{n-4}}\, dV_g(q) \\
&\ge Cr \int_{M-(B_{12r/5}(x_j)-B_{5r/12}(x_j))} |\nabla_g G(\exp_{x_j}(rz), q)|\, u_j(q)^{\frac{n+4}{n-4}}\, dV_g(q),
\end{aligned}
$$

for a uniform constant C independent of j and the choice of points, where for the last inequality we have used (2-1).

Combining (5-4), (5-7) and (5-6), for $\frac{1}{2} \le |z| \le 2$ we have the gradient estimate

$$
\begin{aligned}
&|\nabla_g \log(u_j(\exp_{x_j}(rz)))| \\
&\quad = \frac{|\nabla_g u_j(\exp_{x_j}(rz))|}{u_j(\exp_{x_j}(rz))} \\
&\quad \le \frac{1}{u_j(\exp_{x_j}(rz))} \frac{n-4}{2}\overline{Q} \int_{B_{12r/5}(x_j)-B_{5r/12}(x_j)} |\nabla_g G(\exp_{x_j}(rz), q)|\, u_j(q)^{\frac{n+4}{n-4}}\, dV_g(q) \\
&\qquad + \frac{1}{u_j(\exp_{x_j}(rz))} \frac{n-4}{2}\overline{Q} \\
&\qquad\qquad \times \int_{M-(B_{12r/5}(x_j)-B_{5r/12}(x_j))} |\nabla_g G(\exp_{x_j}(rz), q)|\, u_j(q)^{\frac{n+4}{n-4}}\, dV_g(q) \\
&\quad \le \frac{n-4}{2}\overline{Q} \int_{B_{12r/5}(x_j)-B_{5r/12}(x_j)} |\nabla_g G(\exp_{x_j}(rz), q)|\, C(\overline{C}, n)^{-1} r^{-4} 2^{-\frac{n+4}{n-4}}\, dV_g(q) \\
&\qquad + C^{-1} r^{-1} \\
&\quad \le C(\overline{C}, n)(r^3 r^{-4} + r^{-1}) \\
&\quad = C(\overline{C}, n) r^{-1},
\end{aligned}
$$

where $C(\overline{C}, n)$ is some uniform constant depending on $\overline{C}$, the manifold and n. For any two points $p,\ q \in B_{2r}(x_j) - B_{r/2}(x_j)$, by the gradient estimate,

$$
\frac{u_j(p)}{u_j(q)} \le e^{C(\overline{C}, n) r^{-1} d_g(p,q)} \le e^{4nC(\overline{C}, n)}.
$$

This completes the proof of the Harnack inequality. □

Next we show that near an isolated blowup point, after rescaling the functions u_j converge to a standard solution to (3-3) in $\mathbb{R}^n$.

Lemma 5.2. *Let (M^n, g) be a closed Riemannian manifold of dimension $5 \le n \le 9$ with $R_g \ge 0$, and also $Q_g \ge 0$ with $Q_g(p_0) > 0$ for some point $p_0 \in M$. Let $\{u_j\}$ be a sequence of positive solutions to (1-2) and $x_j \to \bar{x}$ be an isolated blowup point. Let $M_j = u_j(x_j)$. Assume $\{T_j\}_j$ and $\{\epsilon_j\}_j$ are any sequences of positive numbers* *such that* $T_j \to +\infty$ *and* $\epsilon_j \to 0$ *as* $j \to \infty$. *Then after possibly passing to a subsequence* u_{k_j} *and* x_{k_j} *(still denoted as* u_j *and* x_j*),*

$$\|M_j^{-1} u_j(\exp_{x_j}(M_j^{-\frac{2}{n-4}} y)) - (1+4^{-1}|y|^2)^{-\frac{n-4}{2}}\|_{C^4(B_{2T_j})} + \|M_j^{-1} u_j(\exp_{x_j}(M_j^{-\frac{2}{n-4}} y)) - (1+4^{-1}|y|^2)^{-\frac{n-4}{2}}\|_{H^4(B_{2T_j})} \le \epsilon_j, \tag{5-8}$$

and

$$\frac{T_j}{\log(M_j)} \to 0 \quad \textit{as } j \to \infty. \tag{5-9}$$

Proof. Let $x = (x^1, \dots, x^n)$ be geodesic normal coordinates centered at x_j, $y = r^{-1}x$ and the metric $h = r^{-2}g$ be the rescaled metric such that $(h_j)_{pq}(y) = (g_j)_{pq}(ry)$ in normal coordinates. Define

$$v_j(y) = M_j^{-1} u_j(\exp_{x_j}(M_j^{-\frac{2}{n-4}} y)) \quad \text{for } |y| < \delta\, M_j^{\frac{2}{n-4}}.$$

Then v_j satisfies

$$P_{h_j} v_j(y) = \frac{n-4}{2} \bar{Q} v_j(y)^{\frac{n+4}{n-4}} \quad \text{for } |y| \le \delta M_j^{\frac{2}{n-4}}, \tag{5-10}$$

$$v_j(0) = 1, \quad \nabla_{h_j} v_j(0) = 0, \tag{5-11}$$

$$0 < v_j(y) \le \bar{C} |y|^{-\frac{n-4}{2}} \quad \text{for } |y| \le \delta M_j^{\frac{2}{n-4}}. \tag{5-12}$$

We next show that v_j is uniformly bounded. Since $R_{h_j} > 0$ and $R_{u_j^{4/(n-4)} g} > 0$ on M, by the conformal transformation formula of the scalar curvature,

$$\Delta_{h_j} v_j \le \frac{n-4}{4(n-1)} R_{h_j} v_j, \tag{5-13}$$

where $R_{h_j} \to 0$ uniformly in $|y| \le 2$ as $j \to \infty$. Then the function $\eta_j(y) = (1+|y|^2)^{-1} v_j(y)$ satisfies

$$\Delta_{h_j} \eta_j + \sum_{k=1}^{n} b_k(y) \partial_k \eta_j(y) \le 0,$$

in $|y| \le 2$ with some function $b_k(y)$. By the maximum principle,

$$\eta_j(0) \ge \inf_{|y|=r} \eta_j(y) \quad \text{for } 0 < r \le 1. \tag{5-14}$$

By the Harnack inequality (5-1) in Lemma 5.1,

$$\max_{|y|=r} v_j(y) \le C \min_{|y|=r} v_j(y) \quad \text{for } 0 < r \le 1, \tag{5-15}$$

where C is independent of r and j. The inequalities (5-14) and (5-15) immediately lead to

$$\max_{|y|=r} v_j(y) \le C \min_{|y|=r} v_j(y) \le C v_j(0) = C \quad \text{for } 0 < r \le 1.$$

Combining this with (5-12), we have for $|y| \le \delta M_j^{2/(n-4)}$,

$$v_j(y) \le C,$$

with C independent of j, y and r.

Standard elliptic estimates of v_j imply that, after possibly passing to a subsequence, $v_j \to v$ in C^4_{loc} in $\mathbb{R}^n$ where, by (5-11) and (5-13), v satisfies

$$\Delta^2 v(y) = \frac{n-4}{2} \bar{Q} v^{\frac{n+4}{n-4}}, \quad \Delta v(y) \le 0, \quad v(y) \ge 0, \quad \text{for } y \in \mathbb{R}^n,$$
$$v(0) = 1, \ \nabla v(0) = 0.$$

By the strong maximum principle, $v(y) > 0$ in $\mathbb{R}^n$. Then the classification theorem in [Lin 1998] gives

$$v(y) = (1 + 4^{-1}|y|^2)^{-\frac{n-4}{2}}. \qquad \square$$

Remark. From Lemma 5.2, we can see that the proof of Theorem 3.3 still works at the isolated blowup point $x_j \to \bar{x}$. Therefore, there exists $C > 0$ independent of $j > 0$ such that for any isolated blowup point $x_j \to \bar{x}$,

$$u_j(q) \ge C u_j(x_j)^{-1} d_g^{4-n}(q, x_j)$$

for any $q \in M$ such that $d_g(q, x_j) \ge u_j(x_j)^{-2/(n-4)}$.

We now state the upper bound estimate of u_j near the isolated simple blowup points.

Proposition 5.3. *Let (M^n, g) be a closed Riemannian manifold of dimension $5 \le n \le 9$ with $R_g \ge 0$, and also $Q_g \ge 0$ with $Q_g(p_0) > 0$ for some point $p_0 \in M$. Let $\{u_j\}$ be a sequence of positive solutions to* (1-2) *and $x_j \to \bar{x}$ be an isolated simple blowup point. Let δ_1 and $\bar{C}$ be the constants defined in Definition 1.4 and* (1-5). *Then there exists a constant C depending only on δ_1, $\bar{C}$, $\|R_g\|_{C^1(B_{\delta_1}(\bar{x}))}$ and $\|Q_g\|_{C^1(B_{\delta_1}(\bar{x}))}$, such that*

$$u_j(p) \le C u_j(x_j)^{-1} d_g(p, x_j)^{4-n} \quad \textit{for } d_g(p, x_j) \le \frac{\delta_1}{2}, \tag{5-16}$$

for $\delta_1 > 0$ small. Moreover, up to a subsequence,

$$u_j(x_j) u_j(p) \to a G(\bar{x}, p) + b(p) \quad \textit{in } C^4_{\text{loc}}(B_{\delta_1}(\bar{x}) - \{\bar{x}\}), \tag{5-17}$$

where G is the Green's function of the Paneitz operator P_g, $a > 0$ is a constant and $b(p) \in C^4(B_{\delta_1/2}(\bar{x}))$ satisfies $P_g b = 0$ in $B_{\delta_1/2}(\bar{x})$.

The proof of the proposition follows after a series of lemmas.

We first give a rough estimate on the upper bound of u_j near the isolated simple blowup points.

Lemma 5.4. *Under the condition in Proposition 5.3, assume $T_j \to \infty$ and $0 < \epsilon_j < e^{-T_j}$ satisfy* (5-8) *and* (5-9). *Denote $M_j = u_j(x_j)$. Then for any small number $0 < \sigma < \frac{1}{100}$, there exists $0 < \delta_2 < \delta_1$ and $C > 0$ independent of j such that*

$$M_j^{\lambda} u_j(p) \le C d_g(p, x_j)^{4-n+\sigma}, \tag{5-18}$$

$$M_j^{\lambda} |\nabla_g^k u_j(p)| \le C d_g(p, x_j)^{4-n-k+\sigma}, \tag{5-19}$$

for any p in $T_j M_j^{-2/(n-4)} \le d_g(p, x_j) \le \delta_2$ and $1 \le k \le 4$, where $\lambda = 1 - \frac{2}{n-4}\sigma$.

Proof. The outline of the proof is from [Li and Zhu 1999], while the use of our maximum principle here is more subtle. Let $x = (x^1, \dots, x^n)$ be the geodesic normal coordinates centered at x_j for $d_g(p, x_j) \le \delta$. Let $r = |x|$. For any $\delta_2 \in (0, \delta_1)$ to be chosen, let

$$\Omega_j = \{p \in M,\ T_j M_j^{-\frac{2}{n-4}} \le d_g(p, x_j) \le \delta_2\}.$$

We want to use the maximum principle to get the upper bound of u_j. Before the construction of the barrier function on Ω_j, we first go through some properties of u_j.

From Lemma 5.2, we know that

$$u_j(p) \le C T_j^{4-n} M_j \quad \text{for } d_g(p, x_j) = T_j M_j^{-\frac{2}{n-4}}, \tag{5-20}$$

and there exists a critical point r_0 of $\hat{u}_j(r)$ defined in (1-6) in $0 < r < T_j M_j^{-2/(n-4)}$; moreover, for $r > r_0$, $\hat{u}_j(r)$ is decreasing. Using the assumption that $\bar{x}$ is an isolated simple blowup point, $\hat{u}_j$ is strictly decreasing for $T_j M_j^{-2/(n-4)} < r < \delta_1$. Therefore, combined with the Harnack inequality (5-1), for $p \in \Omega_j$ we have

$$\begin{aligned} d_g(p, x_j)^{\frac{n-4}{2}} u_j(p) &\le C \bar{u}_j(d_g(p, x_j)) \\ &\le C T_j^{\frac{n-4}{2}} M_j^{-1} \bar{u}_j(T_j M_j^{-\frac{2}{n-4}}) \\ &\le C T_j^{\frac{n-4}{2}} M_j^{-1} T_j^{4-n} M_j \\ &= C T_j^{-\frac{n-4}{2}}. \end{aligned}$$

This leads to

$$u_j(p)^{\frac{8}{n-4}} \le C T_j^{-4} d_g(p, x_j)^{-4} \quad \text{for } T_j M_j^{-\frac{2}{n-4}} < r < \delta_1. \tag{5-21}$$

We now define a linear elliptic operator on Ω_j,

$$L_j \phi = P_g \phi - \frac{n-4}{2} \bar{Q} u_j^{\frac{8}{n-4}} \phi \quad \text{for } \phi \in C^4(\Omega_j).$$

Therefore

$$L_j u_j = 0 \quad \text{in } \Omega_j.$$

Set

$$\varphi_j(p) = B\overline{M}_j\delta_2^{\sigma} d_g(p, x_j)^{-\sigma} + AM_j^{-1+\frac{2}{n-4}\sigma} d_g(p, x_j)^{-n+4+\sigma}, \;\; p \in \Omega_j,$$

where $A, B > 0$ are constants to be determined, $0 < \sigma < \frac{1}{100}$ and

$$\overline{M}_j = \sup_{d_g(p,x_j)=\delta_2} u_j \leq \overline{C}\delta_2^{-\frac{n-4}{2}}.$$

There exists $C > 0$ such that for $m > 0,\; 1 \leq k \leq 4$, and any $p \in M$ fixed and $q \in M$ with $d_g(p, q) < \delta_2$ and δ_2 less than the injectivity radius, we have

$$|D_g^k d_g(p, q)^{-m}| \leq Cm^k d_g(p, q)^{-m-k}. \tag{5-22}$$

It is easy to check that there exists $\delta_2 > 0$ independent of j so that in Ω_j,

$$|(P_g - \Delta_0^2)|x|^{-\sigma}| \leq 100^{-1}|P_g(|x|^{-\sigma})|,$$
$$|(P_g - \Delta_0^2)|x|^{-n+4+\sigma}| \leq 100^{-1}|P_g(|x|^{-n+4+\sigma})|,$$

where $|x| = d_g(p, x_j)$ and Δ_0 is the Euclidean Laplacian in the normal coordinates. It is easy to check that for $0 < m < n-4$ and $0 < r < \delta_2$,

$$-\Delta_0 r^{-m} = -m(m+2-n)r^{-m-2} > 0, \tag{5-23}$$

$$\Delta_0^2 r^{-m} = m(m+2-n)(m+2)(m+4-n)r^{-m-4} > 0. \tag{5-24}$$

But for $p \in \Omega_j$, by (5-21),

$$\frac{n-4}{2}\overline{Q}u_j(p)^{\frac{8}{n-4}}r^{-m} \leq \frac{n-4}{2}\overline{Q}CT_j^{-4}r^{-m-4}.$$

Therefore,

$$L_j\varphi_j \geq 0 \quad \text{in } \Omega_j,$$

for j large. By (5-20), for $A > 1$,

$$u_j(p) < \varphi_j(p) \quad \text{for } d_g(p, x_j) = T_j M_j^{-\frac{2}{n-4}}. \tag{5-25}$$

Also, for $B > 1$,

$$u_j(p) < \varphi_j(p) \quad \text{for } d_g(p, x_j) = \delta_2. \tag{5-26}$$

We now want to check the sign of the scalar curvature $R_{(\varphi_j - u_j)^{4/(n-4)}g}$ near $\partial\Omega_j$. By the conformal transformation formula, it has the same sign as

$$-\frac{4(n-1)}{n-4}\Delta_g(\varphi_j - u_j) - \frac{8(n-1)}{(n-4)^2}\frac{|\nabla_g(\varphi_j - u_j)|^2}{(\varphi_j - u_j)} + R_g(\varphi_j - u_j).$$

Combining (1-5) and the standard interior estimate of (1-2), we have, for $k = 1,\ 2$,

$$|D^k_g u_j(p)| \le C d_g(p, x_j)^{-\frac{n-4}{2}-k} \tag{5-27}$$

for some constant C independent of j and any $p \in \Omega_j$. It is easy to check that for $0 < m < n-4$,

$$\begin{aligned}\Delta_0|x|^{-m} + \frac{2}{n-4}\frac{|\nabla_0|x|^{-m}|^2}{|x|^{-m}} &= \left(m(m+2-n) + \frac{2m^2}{n-4}\right)|x|^{-m-2} \\ &= \frac{m(n-2)(m-(n-4))}{n-4}|x|^{-m-2} < 0.\end{aligned} \tag{5-28}$$

Also, note that for any positive functions $\phi_1,\ \phi_2 \in C^2$,

$$\begin{aligned}&\Delta_0(\phi_1+\phi_2) + \frac{2}{n-4}\frac{|\nabla_0(\phi_1+\phi_2)|^2}{\phi_1+\phi_2} \\ &\qquad \le \left(\Delta_0\phi_1 + \frac{2}{n-4}\frac{|\nabla_0(\phi_1)|^2}{\phi_1}\right) + \left(\Delta_0\phi_2 + \frac{2}{n-4}\frac{|\nabla_0(\phi_2)|^2}{\phi_2}\right).\end{aligned} \tag{5-29}$$

Here we have used the fact that for any four positive numbers $a, b, c, d > 0$, we have

$$\frac{2cd}{a+b} \le \frac{bc^2}{a(a+b)} + \frac{ad^2}{b(a+b)}$$

so that

$$\frac{(c+d)^2}{a+b} = \frac{c^2+2c\,d+d^2}{a+b} \le \frac{c^2}{a} + \frac{d^2}{b}.$$

Using (5-25)–(5-29), we can choose $A, B > 100^n(1+C)$ independent of j and t with $C > 0$ in (5-27) so that

$$\begin{aligned}&-\frac{4(n-1)}{n-4}\Delta_g(t\varphi_j - u_j) \\ &\qquad -\frac{8(n-1)}{(n-4)^2}\frac{|\nabla_g(t\varphi_j-u_j)|^2}{(t\varphi_j-u_j)} + R_g(t\varphi_j-u_j) > 0 \quad \text{on } \partial\Omega_j,\end{aligned} \tag{5-30}$$

for all $t \ge 1$. Now for $t \ge 1$, we define

$$\phi^t_j(p) = t\varphi_j(p) - u_j(p), \qquad p \in \Omega_j.$$

Then

$$0 \le L_j\phi^t_j = P_g\phi^t_j - \frac{n-4}{2}\overline{Q}\phi^t_j \quad \text{in } \Omega_j. \tag{5-31}$$

If

$$\phi^1_j = \varphi_j - u_j \ge 0 \quad \text{in } \Omega_j, \tag{5-32}$$

then we are done. Otherwise, since Ω_j is compact, we pick the smallest number $t_j > 1$

so that $\phi_j^{t_j} \geq 0$. Therefore, by (5-31)

$$P_g \phi_j^{t_j} \geq \frac{n-4}{2} \bar{Q} \phi_j^{t_j} \geq 0. \tag{5-33}$$

Combining (5-25), (5-26), (5-30) and (5-33), the maximum principle in Lemma 3.2 implies

$$\phi_j^{t_j} > 0 \quad \text{in } \Omega_j,$$

contradicting the choice of t_j. Therefore, (5-32) holds. Now for $p \in \Omega_j$, we use Lemma 5.1, monotonicity of $\hat{u}_j$, and apply (5-32) at p to obtain

$$\begin{aligned}\delta_2^{\frac{n-4}{2}} \overline{M}_j &\leq C \hat{u}_j(\delta_2) \leq C \hat{u}_j(d_g(p, x_j)) \\ &\leq C d_g(p, x_j)^{\frac{n-4}{2}} (B \overline{M}_j \delta_2^{\sigma} d_g(p, x_j)^{-\sigma} + A M_j^{-\lambda} d_g(p, x_j)^{4-n+\delta}).\end{aligned}$$

Here $\frac{n-4}{2} > \sigma$. We choose p with $d_g(p, x_j)$ a small fixed number depending on n, σ, δ_2 to obtain

$$\overline{M}_j \leq C(n, \sigma, \delta_2) M_j^{-\lambda}.$$

The inequality (5-18) is then established from (5-32), and by the standard interior estimates for derivatives of u_j, the lemma is proved. □

Lemma 5.5. *Under the assumption in Proposition 5.3, for any* $0 < \rho \leq \delta_2/2$ *there exists a constant* $C(\rho) > 0$ *such that*

$$\limsup_{j \to \infty} \max_{p \in \partial B_\rho(x_j)} u_j(p) M_j \leq C(\rho),$$

where $M_j = u_j(x_j)$.

Proof. By Lemma 5.1, it suffices to show the inequality for some fixed small constant $\rho > 0$.

For any $p_\rho \in \partial B_\rho(x_j)$, we denote $\xi_j(p) = u_j(p_\rho)^{-1} u_j(p)$. Then ξ_j satisfies

$$P_g \xi_j(p) = \frac{n-4}{2} \bar{Q} u_j(p_\rho)^{\frac{8}{n-4}} \xi_j(p)^{\frac{n+4}{n-4}}.$$

For any compact subset $K \subseteq B_{\delta_2/2}(\bar{x}) - \{\bar{x}\}$, there exists $C(K) > 0$ such that for j large,

$$C(K)^{-1} \leq \xi_j \leq C(K) \quad \text{in } K.$$

Moreover, by Lemma 5.1, there exists $C > 0$ independent of $0 < r < \delta_2$ and j such that

$$\max_{B_r(x_j) - B_{r/2}(x_j)} u_j \leq C \inf_{B_r(x_j) - B_{r/2}(x_j)} u_j. \tag{5-34}$$

By the estimate (5-18), $u_j(p_\rho) \to 0$ as $j \to \infty$. Therefore, by the interior estimates of ξ_j, up to a subsequence,

$$\xi_j \to \xi \quad \text{in } C^4_{\text{loc}}(B_{\delta_2/2}(\bar{x}) - \{\bar{x}\}),$$

with $\xi > 0$ such that

$$P_g \xi = 0 \quad \text{in } B_{\delta_2/2}(\bar{x}) - \{\bar{x}\},$$

and ξ satisfies (5-34) for $0 < r < \delta_2/2$. Moreover, for $0 < r < \rho$ and $\bar{\xi}(r) = |\partial B_r|^{-1} \int_{\partial B_r(\bar{x})} \xi \, ds_g$,

$$\lim_{j\to\infty} u_j(p_\rho)^{-1} r^{\frac{n-4}{2}} \bar{u}_j(r) = r^{\frac{n-4}{2}} \bar{\xi}(r).$$

Since $x_j \to \bar{x}$ is an isolated simple blowup point, $r^{(n-4)/2}\bar{\xi}(r)$ is nonincreasing in $0 < r < \rho$. Therefore, $\bar{x}$ is not a regular point of ξ.

Recall that

$$-\frac{4(n-1)}{n-2} \Delta_g u_j^{\frac{n-2}{n-4}} + R_g u_j^{\frac{n-2}{n-4}} = R_{u_j^{4/(n-4)} g} u_j^{\frac{n+2}{n-4}} \geq 0.$$

Passing to the limit, we have

$$-\frac{4(n-1)}{n-2} \Delta_g \xi^{\frac{n-2}{n-4}} + R_g \xi^{\frac{n-2}{n-4}} \geq 0, \tag{5-35}$$

in $B_{\delta_2/2}(\bar{x}) - \{\bar{x}\}$.

By Corollary A.5, for $\rho > 0$ small, there exists $m > 0$ independent of j such that for j large,

$$\begin{aligned} \int_{B_\rho(x_j)} \Big(P_g\xi_j - \frac{n-4}{2} Q_g \xi_j\Big) \, dV_g &= \int_{\partial B_\rho(x_j)} \Big(\frac{\partial}{\partial \nu}\Delta_g \xi_j - \Big(a_n R_g \frac{\partial}{\partial\nu}\xi_j - b_n \mathrm{Ric}_g(\nabla_g \xi_j, \nu)\Big)\Big) \, ds_g \\ &= \int_{\partial B_\rho(x_j)} \Big(\frac{\partial}{\partial \nu}\Delta_g \xi - \Big(a_n R_g \frac{\partial}{\partial\nu}\xi - b_n \mathrm{Ric}_g(\nabla_g \xi, \nu)\Big)\Big) \, ds_g + o(1) > m. \end{aligned} \tag{5-36}$$

On the other hand, nonnegativity of Q_g implies

$$\begin{aligned} \int_{B_\rho(x_j)} \Big(P_g\xi_j - \frac{n-4}{2} Q_g \xi_j\Big) \, dV_g &= \int_{B_\rho(x_j)} \Big(\frac{n-4}{2} \bar{Q} u_j(p_\rho)^{-1} u_j(p)^{\frac{n+4}{n-4}} - \frac{n-4}{2} Q_g \xi_j\Big) \, dV_g \\ &\leq \frac{n-4}{2} \bar{Q} \int_{B_\rho(x_j)} u_j(p_\rho)^{-1} u_j(p)^{\frac{n+4}{n-4}} \, dV_g. \end{aligned} \tag{5-37}$$

Using (5-8) and $\epsilon_j \le e^{-T_j}$, we have

$$\int_{B_{T_j M_j^{-2/(n-4)}}(x_j)} u_j^{\frac{n+4}{n-4}}\, dV_g \le C M_j^{-1},$$

while by (5-18) we have

$$\begin{aligned}\int_{B_\rho(x_j)-B_{T_j M_j^{-2/(n-4)}}(x_j)} u_j^{\frac{n+4}{n-4}}\, dV_g &\le C \int_{B_\rho(x_j)-B_{T_j M_j^{-2/(n-4)}}(x_j)} (M_j^{-\lambda} d_g(p,x_j)^{4-n+\sigma})^{\frac{n+4}{n-4}} \\ &\le C(T_j M_j^{-\frac{2}{n-4}})^{-4+\frac{n+4}{n-4}\sigma} M_j^{-\lambda\frac{n+4}{n-4}} \\ &= T_j^{-4+\frac{n+4}{n-4}\sigma} M_j^{-1} = o(1) M_j^{-1}.\end{aligned}$$

Therefore,

$$\int_{B_\rho(x_j)} u_j^{\frac{n+4}{n-4}}\, dV_g \le C M_j^{-1}. \tag{5-38}$$

Lemma 5.5 follows from (5-36)–(5-38). □

Proof of Proposition 5.3. Suppose (5-16) fails. Let $M_j = u_j(x_j)$. Then there exists a subsequence u_j and $\{p_j\}$ with $d_g(p_j, x_j) \le \delta_2/2$ with δ_2 in Lemma 5.4 such that

$$u_j(p_j) M_j d_g(p_j, x_j)^{n-4} \to \infty. \tag{5-39}$$

By Lemma 5.2 and $0 < \epsilon_j \le e^{-T_j}$,

$$T_j M_j^{-\frac{2}{n-4}} \le d_g(p_j, x_j) \le \frac{\delta_2}{2}.$$

For each j, let $x = (x^1, \dots, x^n)$ be the geodesic normal coordinates centered at x_j. Denote $y = d_j^{-1} x$ where $d_j = d_g(p_j, x_j)$. We rescale:

$$v_j(y) = d_j^{\frac{n-4}{2}} u_j(\exp_{x_j}(d_j y)), \quad |y| \le 2.$$

Then v_j satisfies

$$P_{h_j} v_j(y) = \frac{n-4}{2} \bar{Q} v_j(y)^{\frac{n+4}{n-4}}, \quad |y| \le 2,$$

where $h_j = d_j^{-2} g$ so that $(h_j)_{pq}(y) = (g)_{pq}(d_j y)$. The metrics h_j depend on j. But since d_j has a uniform upper bound, the sequence of metrics stays in compact sets of $C^{k,\alpha}$ with $k > 4$ large and all the results in Lemma 5.5 hold uniformly for j. Also, the conclusion of Lemma 5.4 is scaling invariant. Note that the metrics h_j converge to a metric h in $C^{k,\alpha}$ with $k > 4$, and hence the Green's functions of Paneitz operators P_{h_j} converge to the Green's functions of Paneitz operators P_h uniformly away from the singularity. In particular, if $d_j \to 0$ then h_j converges to a flat metric on $B_2(0)$ so that in the proof of Proposition A.4, $G(p,\bar{x})$ will be replaced by $c_n|y|^{4-n}$ in Euclidean balls with c_n in (2-1). Therefore, Lemma 5.5 holds for v_j, and hence

$$\max_{|x|=1} v_j(0)v_j(x) \le C,$$

which shows that

$$M_j u_j(p_j) d_g(p_j,x_j)^{4-n} \le C,$$

contradicting (5-39). We have proved (5-16) in $B_{\delta_2/2}(\bar{x})$. By Lemma 5.1, the inequality (5-16) holds in $B_{\delta_1}(\bar{x})$.

The same properties for ξ_j in Lemma 5.5 now hold for $M_j u_j$ in $B_{\delta_2/2}(\bar{x})$. Up to a subsequence

$$M_j u_j \to v \quad \text{in } C^4_{\text{loc}}(B_{\delta_2/2}(\bar{x})),$$

and

$$P_g v = 0 \quad \text{in } B_{\delta_2/2}(\bar{x}).$$

By the remark on page 138, $v > 0$ in $B_{\delta_2/2}(\bar{x})$. Since $\bar{x}$ is an isolated simple blowup point, the same argument in Lemma 5.5 shows that $r^{(n-4)/2}\bar{v}(r)$ is nonincreasing for $0 < r < \delta_2/2$, where $\bar{v}(r) = |\partial B_r(\bar{x})|^{-1}\int_{\partial B_r(\bar{x})} v\, ds_g$. Combined with the Harnack inequality, it implies that v is not regular at $\bar{x}$. Also, v satisfies the condition in Proposition A.4. By Proposition A.4, we obtain (5-17). This completes the proof of Proposition 5.3. □

As an easy consequence of Proposition 5.3 and by the standard interior estimates of the elliptic equation (1-2), we have the following corollary:

Corollary 5.6. *Under the condition in Lemma 5.4, there exists $\delta_2 > 0$ independent of j such that for $T_j M_j^{-2/(n-4)} \le d_g(p,x_j) \le \delta_2$,*

$$|\nabla_g^k u_j(p)| \le C M_j^{-1} d_g(p,x_j)^{4-n-k} \quad \textit{for } 0 \le k \le 4, \tag{5-40}$$

where $M_j = u_j(x_j)$, and C is a constant independent of j. For each j, let x be the geodesic normal coordinates of (Ω, g) centered at x_j. Then there exists $C > 0$ depending on $|g|_{C^3(\Omega)}$ such that for any fixed $r \le \delta_2$,

$$\left|\int_{d_g(p,x_j)\le r}\left(x\cdot\nabla u_j + \frac{n-4}{2}u_j\right)(\Delta^2 - P_g)u_j\,dx\right| \le C M_j^{-\frac{4}{n-4}+o(1)} \tag{5-41}$$

where $o(1) \to 0$ as $j \to \infty$.

Proof. Inequality (5-40) is a direct consequence of Proposition 5.3 and standard interior estimates of the elliptic equation (1-2). We will next establish (5-41). Note that $0 < \epsilon_j \le e^{-T_j}$. Using the estimates (5-40), (5-8) and (5-9), and recalling the error bound (4-1), we have

$$\begin{aligned}
&\int_{|x|\le T_j M_j^{-2/(n-4)}} \left| \left(x\cdot\nabla u_j + \frac{n-4}{2} u_j \right)(\Delta^2 - P_g) u_j \right| dx \\
&\qquad \le \int_{|x|\le T_j M_j^{-2/(n-4)}} C(|x|\,|Du_j(x)| + u_j(x)) \\
&\qquad\qquad \times \big(|x|^2|D^4 u_j(x)| + |x|\,|D^3 u_j(x)| + |D^2 u_j(x)| + |Du_j(x)| + u_j(x)\big)\,dx \\
&\qquad \le C \int_{|y|\le T_j} M_j(1+4^{-1}|y|^2)^{-\frac{n-4}{2}} M_j (1+4^{-1}|y|^2)^{-\frac{n-4}{2}-1} M_j^{\frac{4}{n-4}} M_j^{-\frac{2n}{n-4}}\,dy \\
&\qquad = C M_j^{-\frac{4}{n-4}} \int_{|y|\le T_j} (1+4^{-1}|y|^2)^{3-n}\,dy = C M_j^{-\frac{4}{n-4}+o(1)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{T_j M_j^{-2/(n-4)}\le |x|\le r} \left| \left(x\cdot\nabla u_j + \frac{n-4}{2} u_j \right)(\Delta^2 - P_g) u_j \right| dx \\
&\qquad \le \int_{T_j M_j^{-2/(n-4)}\le |x|\le r} C(|x|\,|Du_j(x)| + u_j(x)) \\
&\qquad\qquad \times \big(|x|^2|D^4 u_j(x)| + |x|\,|D^3 u_j(x)| + |D^2 u_j(x)| + |Du_j(x)| + u_j(x)\big)\,dx \\
&\qquad \le C \int_{T_j M_j^{-2/(n-4)}\le |x|\le r} M_j^{-2} |x|^{6-2n}\,dx \\
&\qquad \le C M_j^{-\frac{4}{n-4}+o(1)},
\end{aligned}$$

where $o(1)\to 0$ as $j\to\infty$ and $C>0$ is a constant depending on $|g|_{C^3(\Omega)}$. Therefore,

$$\int_{|x|\le r} \left| \left(x\cdot\nabla u_j + \frac{n-4}{2} u_j \right)(\Delta^2 - P_g) u_j \right| dx \le C M_j^{-\frac{4}{n-4}+o(1)} \quad \text{for } T_j M_j^{-\frac{2}{n-4}} \le r,$$

where $C>0$ is a constant independent of j and $o(1)\to 0$ as $j\to\infty$. □

For $n\ge 6$, a better estimate is needed in order to cancel the error terms in the Pohozaev identity. By (5-8),

$$u_j(\exp_{x_j}(x)) \le 2M_j(1+4^{-1}M_j^{\frac{4}{n-4}}|x|^2)^{-\frac{n-4}{2}} \quad \text{for } |x| \le T_j M_j^{-\frac{2}{n-4}}.$$

Combining this with Proposition 5.3, we have

$$\begin{aligned}
u_j(\exp_{x_j}(x)) &\le C\min\{M_j(1+4^{-1}M_j^{\frac{4}{n-4}}|x|^2)^{-\frac{n-4}{2}},\ C M_j^{-1}|x|^{4-n}\} \\
&\le C\, M_j(1+4^{-1}M_j^{\frac{4}{n-4}}|x|^2)^{-\frac{n-4}{2}} \quad \text{for } |x| \le \delta_2.
\end{aligned}$$

For $n=6$ and $T_j M_j^{-2/(n-4)} \le r$,

$$\int_{|x|\le r}\left|\left(x\cdot\nabla u_j+\frac{n-4}{2}u_j\right)(\Delta^2-P_g)u_j\right|dx \le C\int_1^{M_j^{2/(n-4)}r} M_j^{-2}M_j^{\frac{2(n-6)}{n-4}}|y|^{5-n}\,d|y|$$
$$\le CM_j^{-\frac{4}{n-4}}\ln(M_j^{\frac{2}{n-4}}r).$$

For $n\ge 7$ and $T_j M_j^{-2/(n-4)} \le r$,

$$\int_{|x|\le r}\left|\left(x\cdot\nabla u_j+\frac{n-4}{2}u_j\right)(\Delta^2-P_g)u_j\right|dx \le C\int_1^{M_j^{2/(n-4)}r} M_j^{-2}M_j^{\frac{2(n-6)}{n-4}}|y|^{5-n}\,d|y|$$
$$\le CM_j^{-\frac{4}{n-4}}.$$

For the term $M_j^2\int_{|x|\le r}|Q_g|\,(u_j^2+|x|\,|Du_j|\,u_j)\,dx$ with $r>0$ fixed,

$$M_j^2\int_{|x|\le r}|Q_g|\,(u_j^2+|x|\,|Du_j|\,u_j)\,dx \le C\,M_j^2\int_0^{rM_j^{2/(n-4)}} M_j^2(1+|y|)^{8-2n}M_j^{-\frac{2n}{n-4}}|y|^{n-1}\,d|y|$$
$$\le C\,M_j^{2-\frac{8}{n-4}}\int_0^{rM_j^{2/(n-4)}}(1+|y|)^{7-n}\,d|y|.$$

For $n=6$,

$$M_j^2\int_{|x|\le r}|Q_g|\,(u_j^2+|x|\,|Du_j|\,u_j)\,dx \le Cr^2.$$

For $n=7$,

$$M_j^2\int_{|x|\le r}|Q_g|\,(u_j^2+|x|\,|Du_j|\,u_j)\,dx \le Cr.$$

These are good terms. For later use, estimates on the error term

$$M_j^2\int_{|x|\le r}\left|\left(x\cdot\nabla u_j+\frac{n-4}{2}u_j\right)(\Delta^2-P_g)u_j\right|dx$$

are needed for $n\ge 6$.

For manifolds (M^n, g) of dimension $5\le n\le 7$, to estimate the error terms and to analyze the expansion of the limit function of $M_j u_j$ at the singular point, we have to work with the conformal normal coordinates. Let u_j be a sequence of positive solutions to (1-2) with isolated blowup points $x_j\to\bar{x}$. For each j, let $x=(x^1,\dots,x^n)$ be the conformal normal coordinates centered at x_j with the corresponding conformal metrics $g_j=\rho_j^{4/(n-4)}g$ constructed in [Lee and Parker 1987] such that

$$\det((g_j)_{pq}(x)) = 1+O(|x|^N),$$

with some large number N, say $N=10n$. We define $g_j=\rho_j^{4/(n-4)}g$ globally on M

by replacing the coefficient $\rho_j^{4/(n-4)}$ with $(\eta\rho_j+(1-\eta))^{4/(n-4)}$ which is still denoted as $\rho_j^{4/(n-4)}$ for simplicity, where η is a cut-off function supported in $B_{\delta_2}(x_j)$ under the metric g and $\eta = 1$ in $B_{\delta_2/(2)}(x_j)$. Recall that $\rho_j(x) = 1 + O(|x|^2)$ for $|x|$ small. Since $x_j \to \bar{x}$, by the construction of the conformal normal coordinates, $\rho_j(x) \to \rho(x)$ in $C^N(M)$ with $g_0 = \rho^{4/(n-4)}g$ the conformal metric corresponding to the conformal normal coordinates centered at $\bar{x}$. Let $\check{u}_j = \rho_j^{-1}u_j$. Then $\check{u}_j$ satisfies the equation

$$P_{g_j}\check{u}_j = \frac{n-4}{2}\bar{Q}\check{u}_j \quad \text{on } M.$$

Let

$$\hat{M}_j = \check{u}_j(x_j) = u_j(x_j)\rho_j(x_j)^{-1}.$$

We define the scaled coordinates $y = \hat{M}_j^{2/(n-4)}x$. Let $h_j = \hat{M}_j^{4/(n-4)}g_j$ and $v_j(y) = \hat{M}_j^{-1}\check{u}_j(\hat{M}_j^{-2/(n-4)}y)$. Denote

$$U_0(y) = (1+4^{-1}|y|^2)^{-\frac{n-4}{2}}, \quad y \in \mathbb{R}^n.$$

By the same argument as in Lemma 5.2, v_j converges to U_0 locally uniformly with the control as in (5-8) and (5-9). We will use this notation in Lemma 5.7.

Lemma 5.7. *Let (M^n, g) be a closed Riemannian manifold of dimension $5 \le n \le 7$ with $R_g \ge 0$, and also $Q_g \ge 0$ with $Q_g(p) > 0$ for some point $p \in M$. Let $\{u_j\}$ be a sequence of positive solutions to* (1-2) *and $x_j \to \bar{x}$ be an isolated simple blowup point. For each j, let $x = (x^1, \dots, x^n)$ be the conformal normal coordinates at x_j with the corresponding conformal metric g_j. Denote $y = \hat{M}_j^{2/(n-4)}x$. Then there exist $\delta_2 > 0$ and $C > 0$ independent of j such that for $|y| \le \delta_2\hat{M}_j^{2/(n-4)}$,*

$$|v_j(y) - U_0(y)| \le C\hat{M}_j^{-2}, \tag{5-42}$$

where $\hat{M}_j = \check{u}_j(x_j)$.

Proof. The proof is a modification of Lemma 5.1 in [Marques 2005].

Let $s_j = \delta_2\hat{M}_j^{2/(n-4)}$ and

$$\Lambda_j = \max_{|y|\le s_j}|v_j - U_0| = |v_j(y_j) - U_0(y_j)|,$$

for some $|y_j| \le s_j$.

We claim that if there exists $c > 0$ such that $|y_j| \ge c\hat{M}_j^{2/(n-4)}$, there exists $C > 0$ such that (5-42) holds. To see this, observe that for $|y_j| \ge c\hat{M}_j^{2/(n-4)}$, by (5-16),

$$v_j(y_j) \le C|y_j|^{4-n} \le C\hat{M}_j^{-2},$$

and therefore

$$\Lambda_j \le C\hat{M}_j^{-2}.$$

This proves the claim.

Now assume $|y_j|\hat{M}_j^{-2/(n-4)} \to 0$ as $j \to \infty$. Then for $j > 0$ large, $|y_j| \le s_j/2$. Let

$$w_j(y) = \Lambda_j^{-1}(v_j(y) - U_0(y)).$$

Then $w_j(0) = 0$ and $Dw_j(0) = 0$.

We will argue by contradiction. If (5-42) fails, then, as $j \to \infty$,

$$\Lambda_j^{-1}\hat{M}_j^{-2} \to 0.$$

Let $h_j = \hat{M}_j^{4/(n-4)} g_j$. Then w_j satisfies the equation

$$P_{h_j} w_j - b_j w_j = H_j, \quad \text{for } |y| \le \delta_2 \hat{M}_j^{\frac{2}{n-4}},$$

where

$$b_j = \frac{(n-4)\overline{Q}(v_j^{(n+4)/(n-4)} - U_0^{(n+4)/(n-4)})}{2(v_j - U_0)} \ge 0,$$

and

$$\begin{aligned}
H_j(y) &= \Lambda_j^{-1}\Big(-P_{h_j}U_0 + \frac{n-4}{2}\overline{Q}U_0^{\frac{n+4}{n-4}}\Big) = \Lambda_j^{-1}(-P_{h_j} + \Delta_0^2)U_0(y)\\
&= \Lambda_j^{-1}\Big(\hat{M}_j^{-\frac{8}{n-4}} Q_{g_j}(\hat{M}_j^{-\frac{2}{n-4}} y)U_0(y) + \hat{M}_j^{-\frac{2}{n-4}N} O(|y|^N)(1+4^{-1}|y|^2)^{-\frac{n}{2}}\\
&\qquad + \hat{M}_j^{-\frac{2}{n-4}(1+N)} O(|y|^N)|y|(1+4^{-1}|y|^2)^{-\frac{n}{2}}\\
&\qquad + \hat{M}_j^{-\frac{2}{n-4}(2+N)} O(|y|^N)(1+4^{-1}|y|^2)^{1-\frac{n}{2}}\\
&\qquad + \hat{M}_j^{-\frac{2}{n-4}(3+N)} O(|y|^N)|y|(1+4^{-1}|y|^2)^{1-\frac{n}{2}}\Big)\\
&= \Lambda_j^{-1}(\hat{M}_j^{-\frac{8}{n-4}} Q_{g_j}(\hat{M}_j^{-\frac{2}{n-4}} y)U_0(y) + \hat{M}_j^{-\frac{2}{n-4}N} O(|y|^N)(1+4^{-1}|y|^2)^{-\frac{n}{2}}),
\end{aligned}$$

with $N = 10n$. By (5-16), for $|y| \le s_j$,

$$v_j(y) \le CU_0(y) \quad \text{and} \quad b_j(y) \le c\overline{Q}(1+4^{-1}|y|^2)^{-4} \quad \text{for some constant } c > 0.$$

By the interior estimates of the equation

$$P_{g_j} w_j = \hat{M}_j^{\frac{8}{n-4}} P_{h_j} w_j = \hat{M}_j^{\frac{8}{n-4}}(b_j w_j + H_j),$$

we have

$$\begin{aligned}
&|\nabla^k w_j(y)|_{h_j}\\
&\le C\hat{M}_j^{-\frac{2k}{n-4}}\big(\sup\nolimits_{B_{\frac12(\delta_2)2\hat{M}_j^{2/(n-4)}}(y)} |w_j| + \hat{M}_j^{\frac{8}{n-4}} \sup\nolimits_{B_{\frac12(\delta_2)2\hat{M}_j^{2/(n-4)}}(y)} |b_j w_j + H_j|\big)\\
&\le C(\hat{M}_j^{-\frac{2k}{n-4}} + \hat{M}_j^{\frac{8-2k}{n-4}}(1+|y|^2)^{-4}) \min\{1, \Lambda_j^{-1}(1+|y|^2)^{\frac{4-n}{2}}\} + C\hat{M}_j^{\frac{8-2k}{n-4}}\Lambda_j^{-1}\\
&\qquad \times \big(\hat{M}_j^{-\frac{8}{n-4}} Q_{g_j}(\hat{M}_j^{-\frac{2}{n-4}} y)U_0(y) + \hat{M}_j^{-\frac{2}{n-4}N} O(|y|^N)(1+4^{-1}|y|^2)^{-\frac{n}{2}}\big),
\end{aligned}$$

for $|\hat{M}_j^{-2/(n-4)} y| \le \delta_2$ and $1 \le k \le 3$.

For $\frac{1}{2}(\delta_2)\hat{M}_j^{2/(n-4)} \le |y| \le \delta_2\hat{M}_j^{2/(n-4)}$, we have that $|w_j(y)| \le C\hat{M}_j^{-2}\Lambda_j^{-1}$, and then by a bootstrapping argument we get the estimate

$$|\nabla^k w_j(y)|_{h_j} \le C\hat{M}_j^{-\frac{2k}{n-4}}\hat{M}_j^{-2}\Lambda_j^{-1}, \tag{5-43}$$

for $1 \le k \le 5$.

Since $|w_j| \le 1$, by the interior estimates of the equation

$$P_{h_j}w_j = (b_j w_j + H_j),$$

we have that

$$|\nabla^k w_j(y)|_{h_j} \le C$$

where $|y| \le \delta_2\hat{M}_j^{2/(n-4)}$ and $1 \le k \le 5$. Therefore, up to a subsequence, $w_j \to w$ in $C^4_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, $H_j(y) \to 0$ and w satisfies

$$\Delta^2 w(y) = \frac{n+4}{2}\bar{Q}U_0(y)^{\frac{8}{n-4}}w(y), \qquad y \in \mathbb{R}^n. \tag{5-44}$$

For any fixed $y \in \mathbb{R}^n$, by the Green's representation, for j large,

$$\begin{aligned}
w_j(y) &= \int_\Omega G_{h_j}(y,z)P_{h_j}w_j(z)\,dV_{h_j}(z)\\
&\quad - \int_{\partial\Omega} G_{h_j}(y,z)\Big[\frac{\partial}{\partial\nu}\Delta_{h_j}w_j - a_n\mathrm{Ric}_{h_j}(\nu,\nabla w_j) + b_n R_{h_j}\frac{\partial}{\partial\nu}w_j\Big]\,dS_{h_j}\\
&\quad - \int_{\partial\Omega}\Big[-\frac{\partial}{\partial\nu}G_{h_j}(y,z)\Delta_{h_j}w_j\\
&\qquad\qquad + a_n\mathrm{Ric}_{h_j}(\nu,\nabla G_{h_j}(y,z))w_j - b_n R_{h_j}w_j\frac{\partial}{\partial\nu}G_{h_j}(y,z)\Big]\,dS_{h_j}\\
&\quad - \int_{\partial\Omega}\Big[\Delta_{h_j}G_{h_j}(y,z)\frac{\partial}{\partial\nu}w_j - \frac{\partial}{\partial\nu}\Delta_{h_j}G_{h_j}(y,z)w_j\Big]\,dS_{h_j}\\
&= \int_\Omega G_{h_j}(y,z)P_{h_j}w_j(z)\,dV_{h_j}(z) + O(1)M_j^{-2}\Lambda_j^{-1},
\end{aligned}$$

as $j \to \infty$, where $\Omega = \{|z| \le \delta_2\hat{M}_j^{2/(n-4)}\}$ and the last equation is by (5-43). But for any $\delta > 0$, there exists $R(\delta) > |y| + 1 > 0$ independent of j such that

$$\begin{aligned}
&\int_{\Omega\cap\{|z|\ge R(\delta)\}} G_{h_j}(y,z)|P_{h_j}w_j(z)|\,dV_{h_j}(z)\\
&= \int_{\Omega\cap\{|z|\ge R(\delta)\}} G_{h_j}(y,z)|b_j w_j(z) + H_j(z)|\,dV_{h_j}(z)\\
&\le C(y)\int_R^{\delta_2\hat{M}_j^{2/(n-4)}} |z|^{4-n} \times \Big|\big(1+\tfrac{1}{4}|z|^2\big)^{-4}w_j + \Lambda_j^{-1}\hat{M}_j^{-\frac{8}{n-4}}|z|^{4-n}\\
&\qquad\qquad + \Lambda_j^{-1}\hat{M}_j^{-\frac{2N}{n-4}}|z|^N(1+|z|^2)^{-\frac{n}{2}}\Big| \times |z|^{n-1}\,d|z|
\end{aligned}$$

$$
\begin{aligned}
&\le C(y)\int_R^{\delta_2\hat{M}_j^{2/(n-4)}} |z|^3(|z|^{-8}|w_j| + \Lambda_j^{-1}\hat{M}_j^{-2}\hat{M}_j^{\frac{-16+2n}{n-4}}|z|^{4-n} \\
&\qquad\qquad + \Lambda_j^{-1}\hat{M}_j^{-2}(\hat{M}_j^{-\frac{2}{n-4}}|z|)^{N-n+4}|z|^{-4})\,d|z| \\
&\le C(y)\int_R^{\delta_2\hat{M}_j^{2/(n-4)}} (|z|^{-5} + \Lambda_j^{-1}\hat{M}_j^{-2}\hat{M}_j^{\frac{-16+2n}{n-4}}|z|^{7-n} \\
&\qquad\qquad + \Lambda_j^{-1}\hat{M}_j^{-2}(\hat{M}_j^{-\frac{2}{n-4}}|z|)^{N-n+4}|z|^{-1})\,d|z| \\
&\le C(y)(R^{-4} + \Lambda_j^{-1}\hat{M}_j^{-2}) \le \delta
\end{aligned}
$$

for j large and $5 \le n \le 7$.

Therefore,

$$
w(y)=c_n\int_{\mathbb{R}^n}|y-z|^{4-n}\Delta_0^2 w(z)\,dz=\frac{n+4}{2}c_n\int_{\mathbb{R}^n}|y-z|^{4-n}U_0(z)^{\frac{8}{n-4}}w(z)\,dz. \tag{5-45}
$$

Also, for $|y| \le \frac{1}{2}\delta_2\hat{M}_j^{2/(n-4)}$, since $|w_j| \le 1$, we have

$$
\begin{aligned}
|w_j(y)| &= \left|\int_\Omega G_{h_j}(y,z)P_{h_j}w_j(z)\,dV_{h_j}(z) + O(1)\hat{M}_j^{-2}\Lambda_j^{-1}\right| \\
&= \left|\int_\Omega G_{h_j}(y,z)(b_j w_j + H_j)\,dV_{h_j}(z) + O(1)\hat{M}_j^{-2}\Lambda_j^{-1}\right| \\
&\le C\Big[(1+|y|)^{-4} + (1+|y|)^{4-n} + \Lambda_j^{-1}\hat{M}_j^{-2}(\hat{M}_j^{\frac{2n-16}{n-4}}(1+|y|)^{8-n} \\
&\qquad\qquad + (\hat{M}_j^{-\frac{2}{n-4}}|y|)^{N-n+4} + 1) + \Lambda_j^{-1}\hat{M}_j^{-2}\Big],
\end{aligned} \tag{5-46}
$$

with $N = 10n$. Therefore, for $5 \le n \le 7$, there exists $C > 0$ such that for $y \in \mathbb{R}^n$,

$$
|w(y)| \le C\,[\,(1+|y|)^{-4} + (1+|y|)^{4-n}].
$$

Since $v_j(0) = 1$ and $Dv_j(0) = 0$, we also have that $w(0) = 0$ and $Dw(0) = 0$.

Now by Corollary B.5, $w(y) = 0$ for $y \in \mathbb{R}^n$. Therefore, $y_j \to \infty$ as $j \to \infty$. But then by (5-46), $w_j(y_j) \to 0$ as $j \to \infty$, which is a contradiction with $w_j(y_j) = 1$ for $j \ge 1$. This completes the proof of Lemma 5.7. □

Remark. Using (5-42) and the equation satisfied by $(v_j - U_j)$ instead of that of w_j in the proof of Lemma 5.7, there exists a constant $C > 0$ independent of j such that

$$
|\nabla^k(v_j - U_j)| \le C\hat{M}_j^{-2}(1+|y|)^{-k},
$$

for $|y| \le \delta_2\hat{M}_j^{2/(n-4)}$ and $1 \le k \le 4$.

Corollary 5.8. *Under the condition in Lemma 5.4, for each j let $x = (x^1, \ldots, x^n)$ be the conformal normal coordinates of (Ω, g) centered at x_j constructed in* [*Lee* *and Parker 1987*], *and we denote g_j as the corresponding conformal metrics so that*

$$\det(g_j) = 1 + O(r^N),$$

where $N = 10n$. Then there exists $C > 0$ such that for any fixed $r \le \delta_2$,

$$\lim_{j\to\infty} \hat{M}_j^2 \left| \int_{d_{g_j}(p,x_j)\le r} \left(x \cdot \nabla \check{u}_j + \frac{n-4}{2} \check{u}_j \right) (\Delta^2 - P_{g_j}) \check{u}_j \, dx \right| \le Cr \tag{5-47}$$

for $5 \le n \le 7$, where $\check{u}_j = u_j \rho_j^{-1}$ and $\hat{M}_j = \check{u}_j(x_j)$ are defined as in the paragraph preceding Lemma 5.7, $N = 10n$ and $g_j = \rho_j^{4/(n-4)} g$.

Proof. Let

$$\tilde{u}_j(x) = \hat{M}_j^{-1} (|x|^2 + \hat{M}_j^{-\frac{4}{n-4}})^{\frac{4-n}{2}}.$$

We denote

$$\Lambda_j(r) = \hat{M}_j^2 \int_{d_{g_j}(p,x_j)\le r} \left(x \cdot \nabla \check{u}_j + \frac{n-4}{2} \check{u}_j \right) (\Delta^2 - P_{g_j}) \check{u}_j \, dx,$$

and

$$\tilde{\Lambda}_j(r) = \hat{M}_j^2 \int_{d_{g_j}(p,x_j)\le r} \left(x \cdot \nabla \tilde{u}_j + \frac{n-4}{2} \tilde{u}_j \right) \left(\Delta^2 - P_{g_j} + \frac{n-4}{2} Q_{g_j} \right) \tilde{u}_j \, dx$$

for $r < \delta_2$.

As in the discussion below Corollary 5.6, there exists a constant $C > 0$ independent of j such that

$$\hat{M}_j^2 \left| \int_{d_{g_j}(p,x_j)\le r} \left(x \cdot \nabla \check{u}_j + \frac{n-4}{2} \check{u}_j \right) Q_{g_j} \check{u}_j \, dx \right| \le Cr^{8-n}$$

for $5 \le n \le 7$. Therefore,

$$\begin{aligned} &|\Lambda_j(r) - \tilde{\Lambda}_j(r)| \\ &\le \hat{M}_j^2 \left| \int_{|x|\le r} \left[\left(x \cdot \nabla \check{u}_j + \frac{n-4}{2} \check{u}_j \right) \left(\Delta^2 - \Delta_{g_j}^2 + \operatorname{div}_{g_j}(a_n R_{g_j} g_j - b_n \operatorname{Ric}_{g_j}) \nabla_{g_j} \right) \check{u}_j \right.\right. \\ &\qquad \left.\left. - \left(x \cdot \nabla \tilde{u}_j + \frac{n-4}{2} \tilde{u}_j \right) \left(\Delta^2 - \Delta_{g_j}^2 + \operatorname{div}_{g_j}(a_n R_{g_j} g_j - b_n \operatorname{Ric}_{g_j}) \nabla_{g_j} \right) \tilde{u}_j \right] dx \right| + Cr^{8-n} \end{aligned}$$

for some constant $C > 0$ independent of j. The change of variables $y = \hat{M}_j^{2/(n-4)} x$ yields

$$\begin{aligned} \int_{|x|\le r} \Big\{ &\left(x \cdot \nabla \check{u}_j + \frac{n-4}{2} \check{u}_j \right) \left(\Delta^2 - \Delta_{g_j}^2 + \operatorname{div}_{g_j}(a_n R_{g_j} g_j - b_n \operatorname{Ric}_{g_j}) \nabla_{g_j} \right) \check{u}_j \\ &- \left(x \cdot \nabla \tilde{u}_j + \frac{n-4}{2} \tilde{u}_j \right) \left(\Delta^2 - \Delta_{g_j}^2 + \operatorname{div}_{g_j}(a_n R_{g_j} g_j - b_n \operatorname{Ric}_{g_j}) \nabla_{g_j} \right) \tilde{u}_j \Big\} \, dx \end{aligned}$$

$$
\begin{aligned}
&= \int_{|y|\le M_j^{2/(n-4)}r} \Bigg\{ \hat M_j \left(y^k \partial_{y^k} v_j + \frac{n-4}{2} v_j \right) \hat M_j^{\frac{8}{n-4}+1} \times \Big[\delta^{ab}\delta^{cd} \partial_{y^a}\partial_{y^b}\partial_{y^c}\partial_{y^d} v_j \\
&\quad - \big(g_j^{ab}(x)\partial_{y^a}\partial_{y^b} + \big(\partial_{y^a} g_j^{ap}(x) - \tfrac12 g_j^{ab} g_j^{ps} \partial_{y^s}(g_j)_{ab}\big)\partial_{y^p}\big) \\
&\qquad \times \big(g_j^{cd}\partial_{y^c}\partial_{y^d} + \big(\partial_{y^c} g_j^{cq} - \tfrac12 g_j^{cd} g_j^{qk}\partial_{y^k}(g_j)_{cd}\big)\partial_{y^q}\big) v_j \\
&\quad + (a_n - \tfrac12 b_n) \hat M_j^{-\frac{4}{n-4}} g_j^{pq}(x)\partial_{y^p} R_g(x) \partial_{y^q} v_j(y) \\
&\quad + a_n \hat M_j^{-\frac{4}{n-4}} R_{g_j}(x) \big(g_j^{pq}\partial_{y^p}\partial_{y^q} v_j + \big(\partial_{y^c} g_j^{cq} - \tfrac12 g_j^{cd} g_j^{qk} \partial_{y^k}(g_j)_{cd}\big)\partial_{y^q} v_j\big) \\
&\quad - b_n \hat M_j^{-\frac{4}{n-4}} \mathrm{Ric}_{g_j}^{pq}(x) \\
&\qquad \times \big(\partial_{y^p}\partial_{y^q} v_j - \tfrac12 g_j^{sk} (\partial_{y^p}(g_j)_{qk} + \partial_{y^q}(g_j)_{pk} - \partial_{y^k}(g_j)_{pq})\partial_{y^s} v_j \big) \Big] \\
&\quad - \hat M_j \left(y^k \partial_{y^k} U_0(y) + \frac{n-4}{2} U_0 \right) \hat M_j^{\frac{8}{n-4}+1} \times \Big[\delta^{ab}\delta^{cd} \partial_{y^a}\partial_{y^b}\partial_{y^c}\partial_{y^d} U_0 \\
&\quad - \big(g_j^{ab}(x)\partial_{y^a}\partial_{y^b} + \big(\partial_{y^a} g_j^{ap}(x) - \tfrac12 g_j^{ab} g_j^{ps} \partial_{y^s}(g_j)_{ab}\big)\partial_{y^p}\big) \\
&\qquad \times \big(g_j^{cd}\partial_{y^c}\partial_{y^d} + \big(\partial_{y^c} g_j^{cq} - \tfrac12 g_j^{cd} g_j^{qk}\partial_{y^k}(g_j)_{cd}\big)\partial_{y^q}\big) U_0 \\
&\quad + \big(a_n - \tfrac12 b_n\big) \hat M_j^{-\frac{4}{n-4}} g_j^{pq}(x)\partial_{y^p} R_g(x) \partial_{y^q} U_0(y) \\
&\quad + a_n \hat M_j^{-\frac{4}{n-4}} R_{g_j}(x) \big(g_j^{pq}\partial_{y^p}\partial_{y^q} U_0(y) + \big(\partial_{y^c} g_j^{cq} - \tfrac12 g_j^{cd} g_j^{qk} \partial_{y^k}(g_j)_{cd}\big)\partial_{y^q} U_0\big) \\
&\quad - b_n \hat M_j^{-\frac{4}{n-4}} \mathrm{Ric}_{g_j}^{pq}(x) \\
&\qquad \times \big(\partial_{y^p}\partial_{y^q} U_0 - \tfrac12 g_j^{sk} \big(\partial_{y^p}(g_j)_{qk} + \partial_{y^q}(g_j)_{pk} - \partial_{y^k}(g_j)_{pq}\big)\partial_{y^s} U_0 \big) \Big] \Bigg\} \hat M_j^{-\frac{2n}{n-4}} \, dy.
\end{aligned}
$$

Then by Lemma 5.7, one can check that

$$
\begin{aligned}
&|\Lambda_j(r) - \tilde\Lambda_j(r)| \\
&\quad \le c \hat M_j^2 \int_{|y|\le \hat M_j^{2/(n-4)}r} \Big[\big(|v_j(y) - U_0(y)| + |y|\,|D_y(v_j - U_0)| \big) \\
&\qquad\qquad\qquad\qquad \times \big(\hat M_j^{-\frac{2}{n-4}} (1+|y|)^{1-n} + \hat M_j^{-\frac{6}{n-4}} (1+|y|)^{3-n} \big) \\
&\qquad\quad + |D_y(v_j - U_0)|\, \hat M_j^{-\frac{6}{n-4}} (1+|y|)^{4-n} + |D_y^2(v_j - U_0)| \hat M_j^{-\frac{4}{n-4}} (1+|y|)^{4-n} \\
&\qquad\quad + |D_y^3(v_j - U_0)| \hat M_j^{-\frac{6}{n-4}} (1+|y|)^{4-n} \Big] \, dy + C r^{8-n} \\
&\quad \le cr + Cr^{8-n} \le Cr.
\end{aligned}
$$

Also, by the construction of conformal normal coordinates,

$$
\begin{aligned}
&|\tilde\Lambda_j(r)| \\
&\quad = \hat M_j^2 \int_{|x|\le r} \left| \left(x\cdot\nabla \tilde u_j + \frac{n-4}{2} \tilde u_j \right) \big(\Delta^2 - \Delta_{g_j}^2 + \mathrm{div}_{g_j}(a_n R_{g_j} g_j - b_n \mathrm{Ric}_{g_j}) \nabla_{g_j} \big) \tilde u_j \, dx \right|
\end{aligned}
$$

$$\leq c\hat{M}_j^2 \int_{|y|\leq M_j^{2/(n-4)}r} \hat{M}_j(1+|y|)^{4-n}\hat{M}_j^{\frac{8}{n-4}+1} \times \Big[\hat{M}_j^{-\frac{6}{n-4}}|x|^{N-3}(1+|y|)^{3-n} + \hat{M}_j^{-\frac{4}{n-4}}|x|^{N-2}(1+|y|)^{2-n} + \hat{M}_j^{-\frac{2}{n-4}}|x|^{N-1}(1+|y|)^{1-n}\Big]\hat{M}_j^{-\frac{2n}{n-4}}\,dy$$

$$\leq C(r^{N+4-n} + \hat{M}_j^{2-\frac{2N}{n-4}}).$$

Therefore, (5-47) holds for $5 \leq n \leq 7$. □

Proposition 5.9. *Let (M^n, g) be a closed Riemannian manifold of dimension $5 \leq n \leq 7$ with $R_g \geq 0$, and also $Q_g \geq 0$ with $Q_g(p) > 0$ for some point $p \in M$. Let $\{u_j\}$ be a sequence of positive solutions to* (1-2) *and $x_j \to \bar{x}$ be an isolated simple blowup point so that*

$$u_j(x_j)u_j(p) \to H(p) \quad \text{in } C^{4,\alpha}_{\text{loc}}(B_{\delta_2}(\bar{x}) - \{\bar{x}\}),$$

for some $0 < \alpha < 1$. Assume that for some constants $a > 0$ and A,

$$H(p) = \frac{a}{d_g(p,\bar{x})^{n-4}} + A + o(1) \quad \text{as } d_g(p,\bar{x}) \to 0, \tag{5-48}$$

for $n = 5$, or

$$\hat{H}(p) \equiv \rho^{-1}(\bar{x})\rho^{-1}(p)H(p) = \frac{a}{d_{g_0}(p,\bar{x})^{n-4}} + A + o(1) \quad \text{as } d_{g_0}(p,\bar{x}) \to 0, \tag{5-49}$$

for $5 \leq n \leq 7$, where $g_0 = \rho^{4/(n-4)}g$ is the conformal metric corresponding to the conformal normal coordinates centered at $\bar{x}$. Then $A = 0$.

Proof. Let us first consider $n = 5$ under the condition (5-48).

Let $x = (x^1, \dots, x^n)$ be the geodesic normal coordinates at x_j for each j. Denote $\Omega_{\gamma,j} = B_\gamma(x_j)$ for $\gamma < \delta_2/(2)$. Then $\Omega_{\gamma,j} \to \Omega_\gamma = B_\gamma(\bar{x})$. By the Pohozaev identity,

$$\int_{\partial\Omega_{\gamma,j}} \frac{n-4}{2}\Big(u_j\frac{\partial}{\partial\nu}(\Delta u_j) - \Delta u_j\frac{\partial}{\partial\nu}u_j\Big) + \Big((x\cdot\nabla u_j)\frac{\partial}{\partial\nu}(\Delta u_j) - \frac{\partial}{\partial\nu}(x\cdot\nabla u_j)\Delta u_j + \tfrac{1}{2}(\Delta u_j)^2 x\cdot\nu\Big)\,ds$$
$$= \int_{\Omega_{\gamma,j}}\Big(x\cdot\nabla u_j + \frac{n-4}{2}u_j\Big)(\Delta^2 - P_g)u_j\,dx + \frac{(n-4)^2}{4n}\overline{Q}\int_{\partial\Omega_{\gamma,j}}(x\cdot\nu)u_j^{\frac{2n}{n-4}}\,dx.$$

Multiplying $M_j^2 = u_j(x_j)^2$ on both sides and taking $\lim_{\gamma\to 0^+}\limsup_{j\to\infty}$ on both sides, we have that by Corollary 5.6,

$$\lim_{\gamma\to 0}\limsup_{j\to\infty} M_j^2\int_{\Omega_{\gamma,j}}\Big(x\cdot\nabla u_j + \frac{n-4}{2}u_j\Big)(\Delta^2 - P_g)u_j\,dx = 0,$$

and

$$\begin{aligned}
&\lim_{\gamma\to 0}\biggl[\int_{\partial\Omega_\gamma} \frac{n-4}{2}\Bigl(H\frac{\partial}{\partial\nu}(\Delta H)-\Delta H\frac{\partial}{\partial\nu}H\Bigr)\\
&\qquad\qquad +\Bigl((x\cdot\nabla H)\frac{\partial}{\partial\nu}(\Delta H)-\frac{\partial}{\partial\nu}(x\cdot\nabla H)\Delta H+\tfrac{1}{2}(\Delta H)^2x\cdot\nu\Bigr)\,ds\biggr]\\
&\quad=\lim_{\gamma\to 0}\limsup_{j\to\infty} M_j^2\int_{\partial\Omega_{\gamma,j}}\biggl[\frac{n-4}{2}\Bigl(u_j\frac{\partial}{\partial\nu}(\Delta u_j)-\Delta u_j\frac{\partial}{\partial\nu}u_j\Bigr)\\
&\qquad\qquad +\Bigl((x\cdot\nabla u_j)\frac{\partial}{\partial\nu}(\Delta u_j)-\frac{\partial}{\partial\nu}(x\cdot\nabla u_j)\Delta u_j+\tfrac{1}{2}(\Delta u_j)^2x\cdot\nu\Bigr)\biggr]\,ds\\
&\quad=\lim_{\gamma\to 0}\limsup_{j\to\infty} M_j^{-\frac{8}{n-4}}\int_{\partial\Omega_{\gamma,j}}(x\cdot\nu)(M_j\,u_j)^{\frac{2n}{n-4}}\,dx=0.
\end{aligned}$$

By assumption,

$$\begin{aligned}
&\lim_{\gamma\to 0}\biggl[\int_{\partial\Omega_\gamma} \frac{n-4}{2}\Bigl(H\frac{\partial}{\partial\nu}(\Delta H)-\Delta H\frac{\partial}{\partial\nu}H\Bigr)\\
&\qquad\qquad +\Bigl((x\cdot\nabla H)\frac{\partial}{\partial\nu}(\Delta H)-\frac{\partial}{\partial\nu}(x\cdot\nabla H)\Delta H+\tfrac{1}{2}(\Delta H)^2x\cdot\nu\Bigr)\,ds\biggr]\\
&\quad=\lim_{\gamma\to 0}\int_{\partial\Omega_\gamma}(n-4)^2(n-2)aA|x|^{1-n}\,ds\\
&\quad=(n-4)^2(n-2)aA|\mathbb{S}^{n-1}|,
\end{aligned}$$

where $|\mathbb{S}^{n-1}|$ is the area of an $(n-1)$-dimensional round sphere. Therefore,

$$A=0.$$

For $5\le n\le 7$ under the condition (5-49), for each j, let $x=(x^1,\dots,x^n)$ be the conformal normal coordinates of (Ω,g) centered at x_j and $g_j=\rho_j^{4/(n-4)}g$ the corresponding conformal metrics defined as in the paragraph preceding Lemma 5.7. Denote $\Omega_{\gamma,j}=B_\gamma(x_j)$ with respect to the metric g_j, for $\gamma<\delta_2/2$. Then

$$\Omega_{\gamma,j}\to\Omega_\gamma=B_\gamma(\bar{x}).$$

By the Pohozaev identity,

$$\begin{aligned}
&\int_{\partial\Omega_{\gamma,j}}\frac{n-4}{2}\Bigl(\check{u}_j\frac{\partial}{\partial\nu}(\Delta\check{u}_j)-\Delta\check{u}_j\frac{\partial}{\partial\nu}\check{u}_j\Bigr)\\
&\qquad\qquad +\Bigl((x\cdot\nabla\check{u}_j)\frac{\partial}{\partial\nu}(\Delta\check{u}_j)-\frac{\partial}{\partial\nu}(x\cdot\nabla\check{u}_j)\Delta\check{u}_j+\tfrac{1}{2}(\Delta\check{u}_j)^2x\cdot\nu\Bigr)\,ds\\
&\quad=\int_{\Omega_{\gamma,j}}\Bigl(x\cdot\nabla\check{u}_j+\frac{n-4}{2}\check{u}_j\Bigr)(\Delta^2-P_{g_j})\check{u}_j\,dx+\frac{(n-4)^2}{4n}\,\overline{Q}\int_{\partial\Omega_{\gamma,j}}(x\cdot\nu)\check{u}_j^{\frac{2n}{n-4}}\,dx,
\end{aligned}$$

where $\check{u}_j = u_j \rho_j^{-1}$. Note that

$$\check{u}_j(p)\check{u}_j(x_j) \to H(p)\rho(\bar{x})^{-1}\rho(p)^{-1} = \hat{H}(p),$$

in

$$C^{4,\alpha}_{\text{loc}}(B_{\delta_2/2}(\bar{x}) - \{\bar{x}\}).$$

Multiplying $\hat{M}_j^2 = \check{u}_j(x_j)^2$ on both sides of the identity and taking the limit $\lim_{\gamma\to 0^+} \limsup_{j\to\infty}$ on both sides, we have that by Corollary 5.8,

$$\lim_{\gamma\to 0} \limsup_{j\to\infty} \hat{M}_j^2 \int_{\Omega_{\gamma,j}} \left(x\cdot\nabla\check{u}_j + \frac{n-4}{2}\check{u}_j \right)(\Delta^2 - P_{g_j})\check{u}_j \, dx = 0,$$

and

$$\begin{aligned}
\lim_{\gamma\to 0}\Bigg[\int_{\partial\Omega_\gamma} &\frac{n-4}{2}\left(\hat{H}\frac{\partial}{\partial\nu}(\Delta\hat{H}) - \Delta\hat{H}\frac{\partial}{\partial\nu}\hat{H}\right) \\
&+\left((x\cdot\nabla\hat{H})\frac{\partial}{\partial\nu}(\Delta\hat{H}) - \frac{\partial}{\partial\nu}(x\cdot\nabla\hat{H})\Delta\hat{H} + \tfrac{1}{2}(\Delta\hat{H})^2 x\cdot\nu\right) ds\Bigg] \\
= \lim_{\gamma\to 0}\limsup_{j\to\infty} \hat{M}_j^2 &\int_{\partial\Omega_{\gamma,j}} \Bigg[\frac{n-4}{2}\left(\check{u}_j\frac{\partial}{\partial\nu}(\Delta\check{u}_j) - \Delta\check{u}_j\frac{\partial}{\partial\nu}\check{u}_j\right) \\
&+\left((x\cdot\nabla\check{u}_j)\frac{\partial}{\partial\nu}(\Delta\check{u}_j) - \frac{\partial}{\partial\nu}(x\cdot\nabla\check{u}_j)\Delta\check{u}_j + \tfrac{1}{2}(\Delta\check{u}_j)^2 x\cdot\nu\right)\Bigg] ds \\
= \lim_{\gamma\to 0}\limsup_{j\to\infty} \hat{M}_j^{-\frac{8}{n-4}} &\int_{\partial\Omega_{\gamma,j}} (x\cdot\nu)(\hat{M}_j\,\check{u}_j)^{\frac{2n}{n-4}} \, dx = 0.
\end{aligned}$$

By assumption,

$$\begin{aligned}
\lim_{\gamma\to 0}\Bigg[\int_{\partial\Omega_\gamma} &\frac{n-4}{2}\left(\hat{H}\frac{\partial}{\partial\nu}(\Delta\hat{H}) - \Delta\hat{H}\frac{\partial}{\partial\nu}\hat{H}\right) + \\
&\left((x\cdot\nabla\hat{H})\frac{\partial}{\partial\nu}(\Delta\hat{H}) - \frac{\partial}{\partial\nu}(x\cdot\nabla\hat{H})\Delta\hat{H} + \tfrac{1}{2}(\Delta\hat{H})^2 x\cdot\nu\right) ds\Bigg] \\
&= \lim_{\gamma\to 0}\int_{\partial\Omega_\gamma} (n-4)^2(n-2)aA|x|^{1-n}\, ds \\
&= (n-4)^2(n-2)aA|\mathbb{S}^{n-1}|,
\end{aligned}$$

where $|\mathbb{S}^{n-1}|$ is the area of an $(n-1)$-dimensional round sphere. Therefore,

$$A = 0. \qquad \square$$

Remark. It is easy to check that all conclusions in this section hold for an isolated (respectively, simple) blowup point $x_j \to \bar{x}$ of a sequence of solutions $\{v_j\}_j$ to (1-2), with the background metric g replaced by a sequence of rescaled metrics $g_j = T_j g$ corresponding to a sequence of positive numbers $T_j \to \infty$ as $j \to \infty$. In this situation, $\rho \equiv 1$ in (5-49) in Proposition 5.9.

6. From isolated blowup points to isolated simple blowup points

In this section we show that an isolated blowup point is an isolated simple blowup point.

Proposition 6.1. *Let* (M^n, g) *be a closed Riemannian manifold of dimension* $5 \leq n \leq 7$ *with* $R_g \geq 0$, *and also* $Q_g \geq 0$ *with* $Q_g(p_0) > 0$ *for some point* $p_0 \in M$. *Let* $\{u_j\}$ *be a sequence of positive solutions to* (1-2) *and* $x_j \to \bar{x}$ *be an isolated blowup point. Let* $M_j = u_j(x_j)$. *Then* $\bar{x}$ *is an isolated simple blowup point.*

Proof. We prove the proposition by a contradiction argument. Assume that $\bar{x}$ is not an isolated simple blowup point. Then there exist two critical points of $r^{(n-4)/2}\bar{u}_j(r)$ in $(0, \bar{\mu}_j)$ with some $\bar{\mu}_j \to 0$ up to a subsequence as $j \to \infty$. By Lemma 5.2 with $0 < \epsilon_j < e^{-T_j}$, we have $r^{(n-4)/2}\bar{u}_j(r)$ has precisely one critical point in $(0, T_j M_j^{-2/(n-4)})$. We choose μ_j to be the second critical point of $r^{(n-4)/2}\bar{u}_j(r)$ so that $\mu_j \geq T_j M_j^{-2/(n-4)}$ and by assumption $\mu_j \to 0$.

For each j let $x = (x^1, \dots, x^n)$ be the geodesic normal coordinates centered at x_j, and let $y = \mu_j^{-1}x$. For ease of notation, we assume $\delta_2 = 1$. We define the scaled metric $\tilde{g}_j = \mu_j^{-2} g$ so that $(\tilde{g}_j)_{pq}(\mu_j^{-1}x)dx^p dx^q = g_{pq}(x)dx^p dx^q$, and

$$\xi_j(y) = \mu_j^{\frac{n-4}{2}} u_j(\exp_{x_j}(\mu_j y)) \quad \text{for } |y| < \mu_j^{-1}.$$

We denote $\bar{\xi}_j$ as the spherical average of ξ_j. Then we have:

(6-1) $P_{\tilde{g}_j}\xi_j(y) = \frac{n-4}{2}\bar{Q}\xi_j(y)^{(n+4)/(n-4)}$, where $|y| < \mu_j^{-1}$,

(6-2) $|y|^{(n-4)/2}\xi_j(y) \leq C$, where $|y| < \mu_j^{-1}$.

(6-3) $\lim_{j\to\infty} \xi_j(0) = \infty$.

(6-4) $-\frac{4(n-1)}{n-2}\Delta_{\tilde{g}_j}\xi_j^{(n-2)/(n-4)} + R_{\tilde{g}_j}\xi_j^{(n-2)/(n-4)} \geq 0$, where $|y| < \mu_j^{-1}$.

(6-5) $r^{(n-4)/2}\bar{\xi}_j(r)$ has precisely one critical point in $0 < r < 1$.

(6-6) $\frac{d}{dr}(r^{(n-4)/2}\bar{\xi}_j(r)) = 0$ at $r = 1$.

Therefore $\{0\}$ is an isolated simple blowup point of the sequence $\{\xi_j\}$. Note that the remark on page 138 holds for u_j so

$$\xi_j(0)\xi_j(y) \geq C|y|^{4-n} \quad \text{for } |y| \geq \mu_j^{-1}T_j M_j^{-\frac{2}{n-4}}, \tag{6-7}$$

where $\mu_j^{-1}T_j M_j^{-2/(n-4)} \leq 1$. By Lemma 5.1, there exists $C > 0$ independent of j and k so that for any $k \in \mathbb{R}$,

$$\max_{2^k \leq |y| \leq 2^{k+1}} \xi_j(0)\xi_j(y) \leq C \min_{2^k \leq |y| \leq 2^{k+1}} \xi_j(0)\xi_j(y), \text{ when } 2^{k+1} < \mu_j^{-1}\frac{\delta_2}{3}. \tag{6-8}$$

Note that $Q_{\tilde{g}_j} \geq 0$ and $R_{\tilde{g}_j} > 0$ in M. Also the rescaled metrics $\tilde{g}_j$ are all well controlled in $|y| \leq 1$. In the proof of Lemma 5.4 the maximum principle holds for $\tilde{g}_j$ and the coefficients of the test function are still uniformly chosen for $\tilde{g}_j$ so that the estimate in Lemma 5.4 holds for each ξ_j in $|y| \le \tilde{\delta}_2$ for some $\tilde{\delta}_2 < 1$ independent of j. Hence Proposition 5.3 holds for ξ_j in $|y| \le \tilde{\delta}_2$. This combined with (6-7) and (6-8) implies

$$C(K)^{-1} \le \xi_j(0)\xi_j(y) \le C(K)$$

for $K \Subset \mathbb{R}^n - \{0\}$ when j is large; moreover, $\tilde{g}_j$ converges to the flat metric and there exists $a > 0$ such that $\xi_j(0)\xi_j(y)$ converges to

$$H(y) = a|y|^{4-n} + b(y) \quad \text{in } C^4_{\text{loc}}(\mathbb{R}^n - \{0\}),$$

where $b(y) \in C^4(\mathbb{R}^n)$ satisfies

$$\Delta^2 b = 0$$

in $\mathbb{R}^n$. Here $H > 0$ in $\mathbb{R}^n - \{0\}$. Also,

$$-\Delta H(y)^{\frac{n-2}{n-4}} \ge 0, \quad |y| > 0. \tag{6-9}$$

Moreover, for a fixed point y_0 in $|y| = 1$, by (6-8),

$$H(y) \le |y|^{2+\frac{\ln C}{\ln 2}} H(y_0)$$

for $|y| \ge 1$. Since $H > 0$ for $|y| > 0$, it follows that $b(y)$ is a polyharmonic function of polynomial growth on $\mathbb{R}^n$. Therefore, $b(y)$ must be a polynomial in $\mathbb{R}^n$; see [Armitage 1973]. Nonnegativity of H near infinity implies that $b(y)$ is of even order. Then either $b(y)$ is a nonnegative constant or $b(y)$ is a polynomial of even order with order at least two and $b(y)$ is nonnegative at infinity. The later case contradicts (6-9) for y near infinity. Therefore, $b(y)$ must be a nonnegative constant on $\mathbb{R}^n$ and

$$H(y) = a|y|^{4-n} + b$$

with a constant $a > 0$ and a constant b.

By (6-6),

$$\frac{d}{dr}\left(r^{\frac{n-4}{2}} H(r)\right) = 0 \quad \text{at } r = 1.$$

We then have $b = a > 0$, which contradicts Proposition 5.9. In fact, Proposition 5.9 applies to isolated simple blowup points with respect to the sequence of rescaled metrics $\{\tilde{g}_j\}$ with uniform curvature bound and uniform bound of injectivity radius with the property that $Q_{\tilde{g}_j} > 0$ and $R_{\tilde{g}_j} > 0$ (see the proof of Proposition 5.9). Here $\hat{H} = H$ in the condition (5-49). Indeed, for $n = 6, 7$, after rescaling, the conformal metric $g_j = \rho_j^{4/(n-4)} g$ corresponding to the conformal normal coordinates centered at x_j becomes $\hat{g}_j(y) = \mu_j^{-2}\rho_j(\mu_j y)^{4/(n-4)} g(\mu_j y)$ and the functions $\hat{\rho}_j(y) = \rho_j(\mu_j y) \to \rho(y) \equiv 1$ locally uniformly in C^N as $j \to +\infty$. This completes the proof of Proposition 6.1. $\square$

Remark. It is easy to check the proof of Proposition 6.1 shows that an isolated blowup point $x_j \to \bar{x}$ of a sequence of solutions $\{v_j\}_j$ to (1-2), with the background metric g replaced by a sequence of rescaled metrics $g_j = T_j g$ corresponding to a sequence of positive numbers $T_j \to \infty$ as $j \to \infty$, is in fact an isolated simple blowup point.

7. Compactness of solutions to the constant Q-curvature equations

Based on Propositions 5.3 and 6.1, the proof of compactness of the solutions is more or less standard; see, for example, [Li and Zhu 1999]. But again we need to deal with the limit of the blowup argument carefully, which satisfies a fourth order elliptic equation; see Lemma 7.1 and Proposition 7.3.

We first show that there are no bubble accumulations.

Lemma 7.1. *Let (M^n, g) be a closed Riemannian manifold of dimension $5 \le n \le 9$ with $R_g \ge 0$, and also $Q_g \ge 0$ with $Q_g(p_0) > 0$ for some point $p_0 \in M$. For any given $\epsilon > 0$ and large constant $T > 1$, there exists some constant $C_1 > 0$ depending on $M,\ g,\ \epsilon,\ T,\ \|Q_g\|_{C^1(M)}$ such that for any solution u to (1-2) and any compact subset $K \subset M$ satisfying*

$$\max_{p \in M-K} d(p, K)^{\frac{n-4}{2}} u(p) \ge C_1 \quad \textit{if } K \neq \varnothing$$

and

$$\max_{p \in M} u(p) \ge C_1 \quad \textit{if } K = \varnothing,$$

we have that there exists some local maximum point p' of u in $M - K$ with $B_{T\, u(p')^{-2/(n-4)}}(p') \subset M - K$ satisfying

$$\|u(p')^{-1} u(\exp_{p'}(u(p')^{-\frac{2}{n-4}} y)) - (1 + 4^{-1}|y|^2)^{-\frac{n-4}{2}}\|_{C^4(|y| \le 2T)} < \epsilon. \tag{7-1}$$

Proof. We argue by contradiction. That is to say, there exists a sequence of compact subsets K_j and a sequence of solutions u_j to (1-2) on M such that

$$\max_{p \in M-K_j} d(p, K_j)^{\frac{n-4}{2}} u(p) \ge j,$$

but no point satisfies (7-1) (here $d(p, K_j) = 1$ when $K_j = \varnothing$). We choose $x_j \in M - K_j$ satisfying

$$d_g(x_j, K_j)^{\frac{n-4}{2}} u_j(x_j) = \max_{p \in M-K_j} d_g(p, K_j)^{\frac{n-4}{2}} u_j(p).$$

Denote $T_j \equiv \frac{1}{4} u_j(x_j)^{2/(n-4)} d_g(x_j, K_j)$. We then define

$$v_j(y) = u_j(x_j)^{-1} u_j(\exp_{x_j}(u_j(x_j)^{-\frac{2}{n-4}} y)) \quad \text{for } |y| \le T_j.$$

Let $h_j = u_j(x_j)^{4/(n-4)} g$. The rescaled function v_j satisfies

$$P_{h_j} v_j = \frac{n-4}{2} \overline{Q} v_j^{\frac{n+4}{n-4}}, \tag{7-2}$$

and by Theorem 2.1,

$$\Delta_{h_j} v_j \le \frac{n-4}{4(n-1)} R_{h_j} v_j. \tag{7-3}$$

We will analyze the limit of the sequence $\{v_j\}$ as in Theorem 3.3 and conclude that (7-1) indeed holds. By assumption,

$$T_j \equiv \tfrac{1}{4} u_j(x_j)^{\frac{2}{n-4}} d_g(y_j, K_j) \ge \tfrac{1}{4} j^{\frac{2}{n-4}},$$

and

$$d_g(\exp_{x_j}(u_j(x_j)^{-\frac{2}{n-4}} y), K_j) \ge \tfrac{1}{2} d_g(x_j, K_j) \quad \text{for } |y| \le T_j.$$

It follows that

$$\begin{aligned} 0 < v_j(y) &= u_j(x_j)^{-1} u_j(\exp_{x_j}(u_j(x_j)^{-\frac{2}{n-4}} y)) \\ &\le u_j(x_j)^{-1} d_g(\exp_{x_j}(u_j(x_j)^{-\frac{2}{n-4}} y), K_j)^{-\frac{n-4}{2}} d_g(x_j, K_j)^{\frac{n-4}{2}} u_j(x_j) \\ &\le 2^{\frac{n-4}{2}} \quad \text{for } |y| \le T_j. \end{aligned}$$

Standard elliptic estimates imply that up to a subsequence,

$$v_j \to v \quad \text{in } C^4_{\text{loc}}(\mathbb{R}^n),$$

with v satisfying

$$\begin{aligned} \Delta^2 v &= \frac{n-4}{2} \overline{Q} v^{\frac{n+4}{n-4}} \quad \text{in } \mathbb{R}^n, \\ v(0) &= 1, \quad 0 \le v \le 2^{\frac{n-4}{2}} \quad \text{in } \mathbb{R}^n, \\ \Delta v &\le 0, \quad \text{in } \mathbb{R}^n. \end{aligned}$$

By the strong maximum principle, $v > 0$ in $\mathbb{R}^n$. Then by the classification theorem of C.S. Lin [1998]),

$$v(y) = \left(\frac{\lambda}{1 + 4^{-1} \lambda^2 |y - \bar{y}|^2} \right)^{\frac{n-4}{2}} \quad \text{in } \mathbb{R}^n,$$

with $v(0) = 1$ and $v(y) \le \lambda^{(n-4)/2} \le 2^{(n-4)/2}$. Therefore, $|\bar{y}| \le C(n)$ with $C(n) > 0$ only depending on n. We choose y_j to be the local maximum point of v_j converging to $\bar{y}$. Then $p_j = \exp_{x_j}(u_j(x_j)^{-2/(n-4)} y_j) \in M - K_j$ is a local maximum point of u_j. We now repeat the blowup argument with x_j replaced by p_j and $u_j(x_j)$ replaced by $u_j(p_j)$ and obtain the limit

$$v(y) = (1 + 4^{-1} |y|^2)^{-\frac{n-4}{2}} \quad \text{in } \mathbb{R}^n.$$

Therefore, for large j, there exists $p_j \in M - K_j$ such that (7-1) holds. This contradicts the assumption. Therefore, the proof of the lemma is completed. □

Lemma 7.2. *Let (M^n, g) be a closed Riemannian manifold of dimension $5 \le n \le 9$ with $R_g \ge 0$, and also $Q_g \ge 0$ with $Q_g(p_0) > 0$ for some point $p_0 \in M$. For any given $\epsilon > 0$ and a large constant $T > 1$, there exist some constants $C_1 > 0$ and $C_2 > 0$ depending on M, g, ϵ, T, $\|Q_g\|_{C^1(M)}$ such that for any solution u to* (1-2) *with*

$$\max_{p \in M} u(p) > C_1,$$

there exists some integer $N = N(u)$ depending on u and N local maximum points $\{p_1, \ldots, p_N\}$ of u such that:

(i) *For $i \neq j$,*

$$\overline{B_{\gamma_i}(p_i)} \cap \overline{B_{\gamma_j}(p_j)} = \varnothing,$$

with $\gamma_j = T u(p_j)^{-2/(n-4)}$ and $B_{\gamma_j}(p_j)$ the geodesic γ_j-ball centered at p_j, and

$$\|u(p_j)^{-1} u(\exp_{p_j}(u(p_j)^{-\frac{2}{n-4}} y)) - (1 + 4^{-1}|y|^2)^{-\frac{n-4}{2}}\|_{C^4(|y| \le 2R)} < \epsilon, \tag{7-4}$$

where $y = u(p_j)^{2/(n-4)} x$, with x geodesic normal coordinates centered at p_j, and $|y| = \sqrt{(y^1)^2 + \cdots + (y^n)^2}$.

(ii) *For $i < j$, we have $d_g(p_i, p_j)^{(n-4)/2} u(p_j) \ge C_1$, while for $p \in M$,*

$$d_g(p, \{p_1, \ldots, p_n\})^{\frac{n-4}{2}} u(p) \le C_2.$$

Proof. We will use Lemma 7.1 and prove the lemma by induction. To start, we apply Lemma 7.1 with $K = \varnothing$. We choose p_1 to be a maximum point of u and thus (7-4) holds. Next we let $K = \overline{B_{\gamma_1}(p_1)}$.

Assume that for some $i_0 \ge 1$, (i) holds for $1 \le j \le i_0$ and $1 \le i < j$, and also $d_g(p_i, p_j)^{(n-4)/2} u(p_j) \ge C_1$ with p_j chosen as in Lemma 7.1 by induction (this holds for $i_0 = 1$). Then we let $K = \bigcup_{j=1}^{i_0} \overline{B_{\gamma_j}(p_j)}$. It follows that for $\epsilon > 0$ small, for any p such that $d_g(p, p_j) \le 2\gamma_j$ with $1 \le j \le i_0$, we have

$$\begin{aligned} d_g(p, \{p_1, \ldots, p_{i_0}\})^{\frac{n-4}{2}} u(p) &\le d_g(p, p_j)^{\frac{n-4}{2}} u(p) \le 2 d_g(p, p_j)^{\frac{n-4}{2}} u(p_j) \\ &\le 2(2T u(p_j)^{-\frac{2}{n-4}})^{\frac{n-4}{2}} u(p_j) = 2^{\frac{n-2}{2}} T^{\frac{n-4}{2}}, \end{aligned}$$

and therefore, for $p \in \bigcup_{j=1}^{i_0} \overline{B_{2\gamma_j}(p_j)}$,

$$d_g(p, \{p_1, \ldots, p_{i_0}\})^{\frac{n-4}{2}} u(p) \le 2^{\frac{n-2}{2}} T^{\frac{n-4}{2}}. \tag{7-5}$$

If, for all $p \in M$, the inequality

$$d_g(p, \{p_1, \ldots, p_{i_0}\})^{\frac{n-4}{2}} u(p) \le C_1,$$

holds then the induction stops. Otherwise, we apply Lemma 7.1, and we denote p_{i_0+1} as the local maximum point y_0 obtained in Lemma 7.1 so that

$$B_{T\,u(p_{i_0+1})^{-2/(n-4)}}(p_{i_0+1}) \subset M - K.$$

Thus, (i) holds for $i_0 + 1$. Also, by assumption, $d_g(p_j, p_{i_0+1})^{(n-4)/2} u(p_{i_0+1}) > C_1$. By the same argument, (7-5) holds for i_0 replaced by $i_0 + 1$. The induction must stop in a finite time $N = N(u)$, since $\int_M u^{2n/(n-4)}\, dV_g$ is bounded and

$$\int_{B_{\gamma_j}(p_j)} u^{\frac{2n}{n-4}}\, dV_g$$

is bounded below by a uniform positive constant. It is clear now that for $p \in M - \bigcup_{j=1}^{N} B_{\gamma_j}(p_j)$,

$$d(p, \{p_1, \dots, p_N\})^{\frac{n-4}{2}} u(p) \le 2^{\frac{n-4}{2}} d\left(p, \bigcup_{j=1}^{N} B_{\gamma_j}(p_j)\right)^{\frac{n-4}{2}} u(p) \le 2^{\frac{n-4}{2}} C_1.$$

By induction, (7-5) holds for i_0 replaced by N. We set

$$C_2 = 2^{\frac{n-2}{2}} T^{\frac{n-4}{2}} + 2^{\frac{n-4}{2}} C_1. \qquad \square$$

The next proposition rules out the bubble accumulations.

Proposition 7.3. *Let (M^n, g) be a closed Riemannian manifold of dimension $5 \le n \le 7$ with $R_g \ge 0$, and also $Q_g \ge 0$ with $Q_g(p_0) > 0$ for some point $p_0 \in M$. For $\epsilon > 0$ small enough and a constant $T > 1$ large enough, there exists $\gamma > 0$ depending on M, g, ϵ, T, $\|R_g\|_{C^1(M)}$ and $\|Q_g\|_{C^1(M)}$ such that for any solution u to* (1-2) *with* $\max_{p\in M} u(p) > C_1$, *we have*

$$d(p_i, p_j) \ge \gamma,$$

for $1 \le i,\ j \le N$ and $i \ne j$, where $N = N(u)$, $p_j = p_j(u)$, $p_i = p_i(u)$ and C_1 are defined in Lemma 7.2.

Proof. Suppose the proposition fails, which implies that there exist $\epsilon > 0$ small and $T > 0$ large and a sequence of solutions u_j to (1-2) such that $\max_{p\in M} u_j(p) > C_1$ and

$$\lim_{j\to\infty} \min_{i\ne k} d(p_i(u_j), p_k(u_j)) = 0.$$

We denote $p_{1,j}$ and $p_{2,j}$ to be the two points realizing the minimum distance in $\{p_1(u_j), \dots, p_N(u_j)\}$ of u_j constructed in Lemma 7.2. Let $\bar{\gamma}_j = d_g(p_{1,j}, p_{2,j})$. Since

$$B_{Tu_j(p_{1,j})^{-2/(n-4)}}(p_{1,j}) \cap B_{Tu_j(p_{2,j})^{-2/(n-4)}}(p_{2,j}) = \varnothing,$$

we have that $u_j(p_{1,j}) \to \infty$ and $u_j(p_{2,j}) \to \infty$.

For each j, let $x=(x^1,\dots,x^n)$ be the geodesic normal coordinates centered at $p_{1,j}$, $y=\bar{\gamma}_j^{-1}x$, and $\exp_{p_{1,j}}(x)$ be exponential map under the metric g. We define the scaled metric $h_j=\bar{\gamma}_j^{-2}g$, and the rescaled function

$$v_j(y)=\bar{\gamma}_j^{\frac{n-4}{2}}u_j(\exp_{p_{1,j}}(\bar{\gamma}_j y)).$$

It follows that v_j satisfies $v_j>0$ in $|y|\le\bar{\gamma}_j^{-1}r_0$ and that

$$P_{h_j}v_j(y)=\frac{n-4}{2}\bar{Q}v_j(y)^{\frac{n+4}{n-4}}\quad\text{for } |y|\le\bar{\gamma}_j^{-1}r_0, \tag{7-6}$$

$$\Delta_{h_j}v_j\le\frac{(n-4)}{4(n-1)}R_{h_j}v_j\qquad\text{for } |y|\le\bar{\gamma}_j^{-1}r_0, \tag{7-7}$$

where r_0 is half of the injectivity radius of (M,g). We define $y_k=y_k(u_j)\in\mathbb{R}^n$ such that $\exp_{p_{1,j}}(\bar{\gamma}_j y_k)=p_k$ for the points $p_k(u_j)$. It follows that for $p_k\neq p_{1,j}$,

$$|y_k|\ge 1+o(1)$$

with $o(1)\to 0$ as $j\to\infty$. Let $y_{2,j}\in\mathbb{R}^n$ be such that $p_{2,j}=\exp_{p_{1,j}}(\bar{\gamma}_j y_{2,j})$. Then

$$|y_{2,j}|\to 1\quad\text{as } j\to\infty.$$

It follows that there exists $\bar{y}\in\mathbb{R}^n$ with $|\bar{y}|=1$ such that up to a subsequence,

$$\bar{y}=\lim_{j\to\infty}y_{2,j}.$$

By Lemma 7.2,

$$\bar{\gamma}_j\ge C\max\{Tu_j(p_{1,j})^{-\frac{2}{n-4}},\ Tu_j(p_{2,j})^{-\frac{2}{n-4}}\}.$$

Thus, $v_j(0)\ge C_3$, $v_j(y_{2,j})\ge C_3$ for some $C_3>0$ independent of j, y_k is a local maximum point of v_j for all $1\le k\le N(u_j)$, and $\min_{1\le k\le N(u_j)}|y-y_k|^{(n-4)/2}v_j(y)\le C_2$ for all $|y|\le\bar{\gamma}_j^{-1}$.

We claim that either

$$v_j(0)\to\infty\quad\text{and}\quad v_j(y_{2,j})\to\infty, \tag{7-8}$$

or both of these two sequences are uniformly bounded. To see this, we first assume that one of them tends to infinity up to a subsequence, say $v_j(0)\to\infty$ for instance. It is clear that 0 is an isolated blowup point, and by Proposition 6.1 it is an isolated simple blowup point. Then $v_j(y_{2,j})\to\infty$ in this subsequence since otherwise, by the control (7-4) at $p_{2,j}$ in Lemma 7.2 and the rescaling, the upper bound of v_j in the $\frac{1}{2}$-geodesic ball centered at $y_{2,j}$ under h_j is controlled by the lower bound of v_j in it up to a uniform multiplier, and thus by the Harnack inequality (5-1) in $B_{4/5}(0)-B_{1/5}(0)$ and Proposition 5.3, $v_j\to 0$ in $B_{1/2}(p_{2,j})$, contradicting $v_j(y_{2,j})\ge C_3$. The claim is established. If v_j are uniformly bounded on any fixed compact subset of $\mathbb{R}^n$, then as discussed in Lemma 7.1, $v_j \to v$ in $C^4_{\text{loc}}(\mathbb{R}^n)$ with $v > 0$ and

$$\Delta^2 v = \frac{n-4}{2}\bar{Q}v^{\frac{n+4}{n-4}}$$

in $\mathbb{R}^n$. Also, 0 and $\bar{y}$ are local maximum points of v. That contradicts the classification theorem in [Lin 1998]. Therefore, the set (denoted as K_0) of isolated blowup points of $\{v_j\}$ is nonempty. Hence v_j is uniformly bounded on any compact subset in $\mathbb{R}^n - K_0$. By a similar argument as the claim, there are at least two points in K_0 and for any two distinct points $y, z \in K$, $|y - z| \geq 1$. Also, by Proposition 6.1 (see also the remark on page 159), K_0 is a set of isolated simple blowup points.

Choose any two blowup points $y_{m,j} \to y_m$ and $y_{k,j} \to y_k \in K_0$. For j large, we pick a point p on the $\frac{1}{2}$-geodesic sphere of $y_{k,j}$. Now we apply Theorem 3.3 (see also the remark on page 138) about the blowup point y_m of v_j at p and Proposition 5.3 about the blowup point y_k of v_j at p; then we have that there exists a constant $C > 0$ independent of j such that

$$v_j(y_{m,j}) \geq C v_j(y_{k,j}).$$

Similarly, there exists a constant $C' > 0$ independent of j such that

$$v_j(y_{k,j}) \geq C' v_j(y_{m,j}).$$

For any point $y \in \mathbb{R}^n - K_0$, let y_k be one of the nearest points to y in K_0. Let Ω be the convex hull of $B_{1/2}(y) \cup B_{1/2}(y_k)$. The argument in Lemma 5.1 still holds with $B_{2r}(x_j)$ and $B_{2r}(x_j) - B_{r/2}(x_j)$ replaced by Ω and any compact subset of $\Omega - \{y_{k,j}\}$ containing y, and therefore the Harnack inequality holds uniformly for v_j on each compact subset of $\mathbb{R}^n - K_0$ when j is large. Therefore, by Proposition 5.3, for a given blowup point $y_{k,j} \to y_k \in K_0$, $v_j(y_{k,j})v_j$ is uniformly bounded in any fixed compact subset of $\mathbb{R}^n - K_0$. Multiplying $v_j(y_{k,j})$ on both sides of (7-6) and (7-7), we have that, up to a subsequence,

$$\lim_{j\to\infty} v_j(y_{k,j})v_j = F \geq 0 \quad \text{in } C^4_{\text{loc}}(\mathbb{R}^n - K_0),$$

such that

$$\Delta^2 F = 0 \quad \text{in } \mathbb{R}^n - K_0, \tag{7-9}$$

$$\Delta F \leq 0 \quad \text{in } \mathbb{R}^n - K_0. \tag{7-10}$$

Pick a point $y_m \in K_0 - \{y_k\}$. Since all the blowup points in K_0 are isolated simple blowup points, by Proposition 5.3,

$$F(y) = a_1|y - y_k|^{4-n} + \Phi_1(y) = a_1|y - y_k|^{4-n} + a_2|y - y_m|^{4-n} + \Phi_2(y)$$

for $y \in \mathbb{R}^n - K_0$ with the constants $a_1, a_2 > 0$. Moreover,

$$\Phi_2 \in C^4(\mathbb{R}^n - (K_0 - \{y_k, y_m\}))$$

and Φ_2 satisfies (7-9) in $\mathbb{R}^n - (K_0 - \{y_k, y_m\})$. Define $\xi = \Delta\Phi_1$ in $\mathbb{R}^n - (K_0 - \{y_k\})$. By (7-10), $F > 0$ in $\mathbb{R}^n - K_0$. Therefore,

$$\liminf_{|y|\to\infty} \Phi_1(y) = \liminf_{|y|\to\infty} (F(y) - a_1|y - y_k|^{4-n}) \geq 0, \tag{7-11}$$

$$\limsup_{|y|\to\infty} \xi(y) = \limsup_{|y|\to\infty} \Delta(F(y) - a_1|y - y_k|^{4-n}) \leq 0, \tag{7-12}$$

where for (7-12) we have used (7-10). Moreover, $\xi < 0$ near any isolated singular point in $K_0 - \{y_k\}$ by Proposition 5.3. Applying the strong maximum principle to ξ and the equation

$$\Delta\xi = \Delta^2(F - a_1|y - y_k|^{4-n}) = 0$$

in $\mathbb{R}^n - (K_0 - \{y_k\})$,

$$\xi = \Delta\Phi_1 < 0$$

in $\mathbb{R}^n - (K_0 - \{y_k\})$. Since $\Phi_1 > 0$ near any isolated singular point in $K_0 - \{y_k\}$ by Proposition 5.3, and also (7-11) holds, applying the strong maximum principle to Φ_1 and $\Delta\Phi_1 < 0$ in $\mathbb{R}^n - (K_0 - \{y_k\})$, we have $\Phi_1 > 0$ in $\mathbb{R}^n - (K_0 - \{y_k\})$. It follows that

$$F(y) = a_1|y - y_k|^{4-n} + \Phi_1(0) + O(|y - y_k|) \text{ with } \Phi_1(y_k) > 0 \text{ near } y = y_k,$$

contradicting Proposition 5.9 (It is easy to check that Proposition 5.9 applies for the scaled metrics h_j instead of g.). Here in the statement of Proposition 5.9, $H = \hat{H} = F$. Indeed, for $5 \leq n \leq 7$, after rescaling, for each j the conformal metric $g_j = \rho_j^{4/(n-4)} g$ corresponding to the conformal normal coordinates centered at x_j becomes

$$\hat{g}_j(y) = \bar{\gamma}_j^{-2} \rho_j(\bar{\gamma}_j y)^{4/(n-4)} g(\bar{\gamma}_j y)$$

and the functions $\hat{\rho}_j(y) = \rho_j(\bar{\gamma}_j y) \to \rho(y) \equiv 1$ locally uniformly in C^N as $j \to +\infty$. Proposition 7.3 is then established. $\square$

We are now ready to prove the compactness theorem of positive solutions to (1-2).

Proof of Theorem 1.2. By Lemma 2.3 and the ellipticity of (1-2), we only need to show that there is a constant $C > 0$ depending on M and g such that

$$u \leq C.$$

Suppose the contrary, then there exists a sequence of positive solutions u_j to (1-2) such that

$$\max_{p\in M} u_j \to \infty$$

as $j \to \infty$. By Proposition 7.3, after passing to a subsequence, there exist N distinct isolated simple blowup points $p_{1,j} \to p_1, \dots, p_{N,j} \to p_N$ with $N \geq 1$ independent of j. Applying Proposition 5.3, we have that up to a subsequence,

$$u_j(p_{1,j})u_j(p) \to F(p) = \sum_{k=1}^{N} a_k G_g(p_k, p) + b(p) \quad \text{in } C^4_{\mathrm{loc}}(M - \{p_1, \dots, p_N\}),$$

where $a_1 > 0,\ a_2 \geq 0, \dots,\ a_N \geq 0$ are some constants, G_g is the Green's function of P_g under the metric g and $b(p) \in C^4(M)$ satisfying

$$P_g b = 0$$

on M. Since $Q_g \geq 0$ on M with $Q_g > 0$ at some point, by the strong maximum principle of P_g, $b \geq 0$ in M. We know that $G_g(p_k, p) > 0$ for $1 \leq k \leq N$ by Theorem 2.1. Let $x = (x^1, \dots, x^n)$ be the conformal normal coordinates centered at $p_{1,j}$ for each j (respectively, p_1) constructed in [Lee and Parker 1987] with respect to the conformal metric $h_j = \rho_j^{-4/(n-4)} g$ (respectively, $h = \rho^{-4/(n-4)} g$) such that

$$\det(h_{ij}) = 1 + O(|x|^{10n}).$$

Then there exists $C_1 > 0$ independent of j such that

$$C_1^{-1} \leq \rho_j \leq C_1,$$

and

$$\|\rho_j - \rho\|_{C^N(M)} \to 0 \quad \text{as } j \to \infty.$$

As shown in Theorem 2.1, under the conformal normal coordinates $x = (x^1, \dots, x^n)$ centered at p_1, the Green's function under metric h satisfies

$$G_h(p_1, p) = \rho^2(p) G_g(p_1, p) = d_h(p_1, p)^{4-n} + A + o(1)$$

near p_1 with the constant $A > 0$ and $o(1) \to 0$ as $p \to p_1$. Therefore,

$$\rho(p)^2 F(p) = a_1 d_h(p_1, p)^{4-n} + B + o(1)$$

$B = a_1 A + \sum_{k=2}^{N} a_k \rho(p_1)^2 G_g(p_k, p_1) + b(p_1) > 0$ and $o(1) \to 0$ as $p \to p_1$. That contradicts Proposition 5.9 with $\hat{H} = F$ in (5-49). Therefore, Theorem 1.2 is established. □

Appendix A: Positive solutions of certain linear fourth order elliptic equations in punctured balls

Assume $B_\delta(\bar{x})$ is a geodesic δ-ball on a complete Riemannian manifold (M^n, g) with 2δ less than the injectivity radius. For application, for $5 \leq n \leq 9$, (M, g) could either be the closed manifold in Proposition 5.3, or the Euclidean space.

Lemma A.1. *Let $u \in C^4(B_\delta(\bar{x}) - \{\bar{x}\})$ be a solution to*

$$P_g u = 0 \quad \textit{in } B_\delta(\bar{x}) - \{\bar{x}\}. \tag{A-1}$$

If $u(p) = o(d_g(p, \bar{x})^{4-n})$ as $p \to \bar{x}$, then $u \in C^{4,\alpha}_{\mathrm{loc}}(B_\delta(\bar{x}))$ for $0 < \alpha < 1$.

Proof. **Step 1.** We show that (A-1) holds in $B_\delta(\bar{x})$ in the distribution sense.

To see this, given any small $\epsilon > 0$, we define the cutoff function η_ϵ on $B_\delta(\bar{x})$ with $0 \le \eta_\epsilon \le 1$ so that

$$\begin{aligned} \eta_\epsilon(p) &= 1 && \text{for } d_g(p, \bar{x}) \le \epsilon, \\ \eta_\epsilon(p) &= 0 && \text{for } d_g(p, \bar{x}) \ge 2\epsilon, \\ |\nabla \eta_\epsilon(p)| &\le C\epsilon^{-1} && \text{for } \epsilon \le d_g(p, \bar{x}) \le 2\epsilon. \end{aligned}$$

For any given $\phi \in C^\infty_c(B_\delta(\bar{x}))$ we multiply by $\phi(1 - \eta_\epsilon)$ on both sides of (A-1) and do integration by parts,

$$\int_{B_\delta(\bar{x})} P_g(\phi(1 - \eta_\epsilon)) u \, dV_g = 0.$$

Let $\epsilon \to 0$, then

$$\int_{B_\delta(\bar{x})} (1 - \eta_\epsilon) u \, P_g \phi \, dV_g = O(1)\left(C\epsilon^{-4} \int_{B_{2\epsilon}(\bar{x}) - B_\epsilon(\bar{x})} |u|\right) + C \int_{B_\epsilon(\bar{x})} |u| \to 0,$$

where in the last step we have used $u(p) = o(d_g(p, \bar{x})^{4-n})$. Therefore, Step 1 is established.

Step 2. The assumption of u near $\bar{x}$ implies that $u \in L^p_{\mathrm{loc}}(B_\delta(\bar{x}))$ for any $1 < p < \frac{n}{n-4}$. By $W^{4,p}$ estimates of the elliptic equation we obtain that $u \in W^{4,p}_{\mathrm{loc}}(B_\delta(\bar{x}))$; see [Agmon 1959] for instance. The standard bootstrap argument gives $u \in C^{4,\alpha}_{\mathrm{loc}}(B_\delta(\bar{x}))$. $\square$

For later use, we now present Lemma 9.2 from [Li and Zhu 1999] without proof.

Lemma A.2. *There exists some constant $0 < \delta_0 \le \delta$ depending on n, $\|g_{ij}\|_{C^2(B_\delta(\bar{x}))}$ and $\|R_g\|_{L^\infty(B_\delta(\bar{x}))}$ such that the maximum principle for $-\frac{4(n-1)}{n-2}\Delta_g + R_g$ holds on $B_{\delta_0}(\bar{x})$, and there exists a unique $G_1(p) \in C^2(B_{\delta_0}(\bar{x}) - \{\bar{x}\})$ satisfying*

$$\begin{aligned} -\frac{4(n-1)}{n-2}\Delta_g G_1 + R_g G_1 &= 0 \quad \textit{in } B_{\delta_0}(\bar{x}) - \{\bar{x}\}, \\ G_1 &= 0 \quad \textit{on } \partial B_{\delta_0}(\bar{x}), \\ \lim_{p \to \bar{x}} d_g(p, \bar{x})^{n-2} G_1(p) &= 1. \end{aligned}$$

Furthermore, $G_1(p) = d_g(p, \bar{x})^{2-n} + \mathcal{R}(p)$ where, for all $0 < \epsilon < 1$, $\mathcal{R}(p)$ satisfies

$$d_g(p, \bar{x})^{n-4+\epsilon}|\mathcal{R}(p)| + d_g(p, \bar{x})^{n-3+\epsilon}|\nabla \mathcal{R}(p)| \le C(\epsilon), \quad p \in B_{\delta_0}(\bar{x}), \ n \ge 4,$$

where $C(\epsilon)$ depends on ϵ, n, $\|g_{ij}\|_{C^2(B_\delta(\bar{x}))}$ and $\|R_g\|_{L^\infty(B_\delta(\bar{x}))}$.

Lemma A.3. *Suppose a positive function $u \in C^4(B_\delta(\bar{x}) - \{\bar{x}\})$ satisfies* (A-1) *in $B_\delta(\bar{x}) - \{\bar{x}\}$, and assume that there exists a constant $C > 0$ such that for $0 < r < \delta$, the Harnack inequality holds*:

$$\max_{d_g(p,\bar{x})=r} u(p) \le C \min_{d_g(p,\bar{x})=r} u(p).$$

If moreover,

$$-\frac{4(n-1)}{n-2}\Delta_g u^{\frac{n-2}{n-4}} + R_g u^{\frac{n-2}{n-4}} \ge 0 \quad \textit{in } B_\delta(\bar{x}) - \{\bar{x}\},$$

then

$$a = \limsup_{p\to\bar{x}} d_g(p,\bar{x})^{n-4} u(p) < +\infty.$$

Proof. If the lemma is not true, then for any $A > 0$, there exists $r_i \to 0^+$ satisfying

$$u(p) > A\, r_i^{4-n} \quad \text{for all } d_g(p,\bar{x}) = r_i.$$

Let $v_A = \frac{1}{2}A^{(n-2)/(n-4)} G_1$ with G_1 in Lemma A.2. For i large, by the maximum principle,

$$u(p)^{\frac{n-2}{n-4}} \ge v_A(p) \quad \text{for } r_i < d_g(p,\bar{x}) < \delta_0.$$

As $i \to \infty$,

$$u(p)^{\frac{n-2}{n-4}} \ge v_A(p) \quad \text{for } 0 < d_g(p,\bar{x}) < \delta_0.$$

Since A can be arbitrarily large, $u(p) = \infty$ in $0 < d_g(p,\bar{x}) < \delta_0$, which is a contradiction. □

Proposition A.4. *Let u be as in Lemma A.3. Then there exists a constant $b \ge 0$ such that*

$$u(p) = bG(p,\bar{x}) + E(p) \quad \textit{for } p \in B_{\delta_0}(\bar{x}) - \{\bar{x}\}, \tag{A-2}$$

where G is the Green's function of P_g (for the existence of the Green's function in our application, G is the limit of the Green's function of the Paneitz operator of a sequence of metrics on M restricted to certain domains, and when g is the flat metric, let $G(x,y) = c_n|x-y|^{4-n}$), and δ_0 is defined in Lemma A.2. Here $E \in C^4(B_{\delta_0}(\bar{x}))$ satisfies $P_g E = 0$ in $B_{\delta_0}(\bar{x})$.

Proof. We rewrite (A-1) as

$$\Delta_g(\Delta_g u) = \mathrm{div}_g(a_n R_g g - b_n \mathrm{Ric}_g)\nabla_g u - \frac{n-4}{2} Q_g u.$$

By Lemma A.3, $0 < u(p) \le a_1 G(p,\bar{x})$ with some constant $a_1 > a$ in $B_{\delta_0}(\bar{x}) - \{\bar{x}\}$ with $\delta_0 > 0$ in Lemma A.2. Combining this with the interior estimates, there exists a constant $C > 0$ such that

$$\left|\mathrm{div}_g(a_n R_g g - b_n \mathrm{Ric}_g)\nabla_g u - \frac{n-4}{2} Q_g u\right| \le C d_g^{2-n}(p, \bar{x}), \tag{A-3}$$

$$|\Delta_g u(p)| \le C\, d_g^{2-n}(p, \bar{x}), \tag{A-4}$$

for $p \in \bar{B}_{\delta_0}(\bar{x}) - \{0\}$. We define G_2 to be a Green's function of Δ_g on $\bar{B}_{\delta_0}(\bar{x})$ such that

$$0 < G_2(p, q) \le C d_g(p, q)^{2-n}, \tag{A-5}$$

for some constant $C > 0$ and any two distinct points p and q in $B_{\delta_0}(\bar{x})$. Then

$$\phi_1(p) = \int_{B_{\delta_0}(\bar{x})} G_2(p, q)\Big(\mathrm{div}_g(a_n R_g g - b_n \mathrm{Ric}_g)\nabla_g u(q) - \frac{n-4}{2} Q_g u(q)\Big)\, dV_g(q)$$

is a special solution to the equation

$$\Delta_g \phi = \mathrm{div}_g(a_n R_g g - b_n \mathrm{Ric}_g)\nabla_g u - \frac{n-4}{2} Q_g u \quad \text{in } B_{\delta_0}(\bar{x}) - \{\bar{x}\}.$$

Combining (A-3) and (A-5), we have that there exists a constant $C > 0$ such that

$$|\phi_1(p)| \le C d_g(p, \bar{x})^{4-n}$$

for $p \in B_{\delta_0}(\bar{x}) - \{\bar{x}\}$. Therefore,

$$\Delta_g(\Delta_g u - \phi_1) = 0 \quad \text{in } B_{\delta_0}(\bar{x}) - \{\bar{x}\}.$$

Since we also have (A-4), the proof of Proposition 9.1 in [Li and Zhu 1999] applies and there exists a constant $-C \le b_2 \le C$ such that

$$(\Delta_g u(p) - \phi_1(p)) = b_2 G_1(p) + \varphi_1(p) \quad \text{in } B_{\delta_0}(\bar{x}) - \{\bar{x}\},$$

with G_1 as in Lemma A.2 and φ_1 a harmonic function on $\bar{B}_{\delta_0}(\bar{x})$. Therefore,

$$\Delta_g u(p) = b_2 G_1(p) + \phi_1(p) + \varphi_1(p) \quad \text{in } B_{\delta_0}(\bar{x}) - \{\bar{x}\}.$$

By the same argument, there exists $b_3 \in \mathbb{R}$ such that

$$\begin{aligned} u(p) &= b_3 G_1(p) + \varphi_2(p) + \int_{B_{\delta_0}(\bar{x})} G_2(p, q)[b_2 G_1(q) + \phi_1(q) + \varphi_1(q)]\, dV_g(q) \\ &= b_3 G_1(p) + \varphi_2(p) + O(d_g(p, \bar{x})^{4-n}) \end{aligned}$$

in $B_{\delta_0}(\bar{x}) - \{\bar{x}\}$, with φ_2 a harmonic function on $B_{\delta_0}(\bar{x})$. But since $0 < u(p) \le a_1 G(p, \bar{x})$, we have $b_3 = 0$ and

$$u(p) = b_2 \int_{B_{\delta_0}(\bar{x})} G_2(p, q) G_1(q)\, dV_g(q) + o(d_g(p, \bar{x})^{4-n})$$

in $B_{\delta_0}(\bar{x})-\{\bar{x}\}$. Therefore, there exists a constant $b \geq 0$ such that

$$\begin{aligned} u(p) &= b d_g(p,\bar{x})^{4-n} + o(d_g(p,\bar{x})^{4-n}) \\ &= bG(p,\bar{x}) + o(d_g(p,\bar{x})^{4-n}). \end{aligned}$$

Then by Lemma A.1, there exists a function $E \in C^4(B_{\delta_0}(\bar{x}))$ satisfying (A-1) and

$$u(p) = bG(p,\bar{x}) + E(p)$$

for $p \in B_{\delta_0}(\bar{x})-\{\bar{x}\}$.

This completes the proof of the proposition. □

Using Proposition A.4, we immediately conclude the following corollary.

Corollary A.5. *For $n \geq 5$, assume that $u \in C^4(B_{\delta_0}(\bar{x})-\{\bar{x}\})$ is a positive solution of* (A-1) *with $\bar{x}$ a singular point, and also that the assumptions in Lemma A.3 hold for u. Then*

$$\begin{aligned} &\lim_{r\to 0}\int_{B_r(\bar{x})}\left(P_g u - \frac{n-4}{2}\bar{Q}u\right)dV_g \\ &\quad = \lim_{r\to 0}\int_{\partial B_r(\bar{x})}\left(\frac{\partial}{\partial\nu}\Delta_g u - (a_n R_g \frac{\partial}{\partial\nu}u - b_n \mathrm{Ric}_g(\nabla_g u,\nu))\right)ds_g \\ &\quad = b\lim_{r\to 0}\int_{\partial B_r(\bar{x})}\frac{\partial}{\partial\nu}\Delta_g G(p,\bar{x})\,ds_g(p) = 2(n-2)(n-4)|\mathbb{S}^{n-1}|\,b > 0, \end{aligned}$$

where ν is the outer unit normal and $b > 0$ is as in (A-2).

Appendix B: Classification of solutions with decay at infinity for a fourth order linear equation

Let $n \geq 5$. It is easy to check that $U_0 = (1+4^{-1}|x|^2)^{-(n-4)/2}$ is a solution to the Q-curvature equation

$$\Delta^2 U_0 = \frac{n-4}{2}\bar{Q}U_0^{\frac{n+4}{n-4}}$$

on $\mathbb{R}^n$ with $\bar{Q} = \frac{1}{8}n(n^2-4)$.

We now consider bounded solutions to the linearized equation

$$\Delta^2\phi(x) = \frac{n+4}{2}\bar{Q}U_0^{\frac{8}{n-4}}\phi(x), \quad x \in \mathbb{R}^n. \tag{B-1}$$

Chen and Lin [1998] classified bounded solutions to the linearized equation of the Yamabe equation in $\mathbb{R}^n$ with certain decay near infinity. Similarly, we want to show that if a solution ϕ to (B-1) has the decay $\phi \to 0$ uniformly as $|x| \to \infty$, then

$$\phi = c_0\left(x\cdot\nabla U_0 + \frac{n-4}{2}U_0\right) + \sum_{j=1}^n c_j \partial_{x_j} U_0.$$

Let $\{\xi_{k,m}\}_m$ be the eigenfunctions of the Laplacian on $\mathbb{S}^{n-1}$, with respect to the eigenvalue $\lambda_k = k(n+k-2)$. Let $x = r\theta$ with $r = |x|$. Then we have the decomposition

$$\phi(r\theta) = \sum_{k=0}^{\infty} \sum_m \phi_{k,m}(r)\xi_{k,m}(\theta),$$

which converges locally uniformly, with $\phi_{k,m}(r) = \int_{S^{n-1}} \phi(r\theta)\xi_{k,m}(\theta)\,dS$. Let $u_{k,m}(r\theta) = \phi_{k,m}(r)\xi_{k,m}(\theta)$. Then $u_{k,m}$ satisfies the equation

$$\Delta^2 u_{k,m}(x) = \frac{n+4}{2}\bar{Q}U_0(x)^{\frac{8}{n-4}} u_{k,m}(x), \quad x \in \mathbb{R}^n, \tag{B-2}$$

and $\phi_{k,m}$ satisfies

$$\Big(\partial_r^2 + \frac{n-1}{r}\partial_r - \frac{\lambda_k}{r^2}\Big)\Big(\partial_r^2 + \frac{n-1}{r}\partial_r - \frac{\lambda_k}{r^2}\Big)\phi_{k,m} = \frac{n+4}{2}\bar{Q}U_0^{\frac{8}{n-4}}\phi_{k,m}, \quad r > 0, \tag{B-3}$$

with $\phi_{k,m}(0) = 0$ and $\phi'_{k,m}(0) = 0$. Equivalently, $\phi_{k,m}$ is a solution to the equation

$$\Big(\Delta - \frac{\lambda_k}{r^2}\Big)\Big(\Delta - \frac{\lambda_k}{r^2}\Big)\phi_{k,m} = \frac{n+4}{2}\bar{Q}U_0^{\frac{8}{n-4}}\phi_{k,m}. \tag{B-4}$$

Denote

$$v_{k,m}(r) = \Big(\partial_r^2 + \frac{n-1}{r}\partial_r - \frac{\lambda_k}{r^2}\Big)\phi_{k,m}.$$

Then

$$\Big(\partial_r^2 + \frac{n-1}{r}\partial_r - \frac{\lambda_k}{r^2}\Big)\phi_{k,m} = v_{k,m}, \tag{B-5}$$

$$\Big(\partial_r^2 + \frac{n-1}{r}\partial_r - \frac{\lambda_k}{r^2}\Big)v_{k,m} = \frac{n+4}{2}\bar{Q}U_0^{\frac{8}{n-4}}\phi_{k,m}, \tag{B-6}$$

where

$$\phi_{k,m}(0) = 0, \quad \phi'_{k,m}(0) = 0, \quad v_{k,m}(0) = 0 \quad \text{and} \quad v'_{k,m}(0) = 0. \tag{B-7}$$

By (B-2), we know that $u_{k,m}$ is analytic locally in $\mathbb{R}^n$. Then the solutions $\phi_{k,m}$ to (B-3) and (B-7) are generated linearly by the two solutions

$$\phi_{1,k,m}(r) = r^k + E_1 r^{k+4} + \sum_{j=2}^{\infty} E_j r^{k+2+2j},$$

$$\phi_{2,k,m}(r) = r^{k+2} + C_1 r^{k+6} + \sum_{j=2}^{\infty} C_j r^{k+4+2j},$$

with $E_1 > 0$ and $C_1 > 0$. The constants E_i and C_j can be calculated inductively using (B-3). It is easy to check that the radius of convergence of $\phi_{i,k,m}$ is positive for $i = 1, 2$ and $k \geq 1$. Therefore,

$$\phi_{k,m} = C\phi_{1,k,m}(r) + C'\phi_{2,k,m}(r),$$

with C and C' constants.

Now we employ a useful comparison theorem motivated by [Grunau et al. 2008]; see also [McKenna and Reichel 2003] and [Choi and Xu 2009].

Theorem B.1. *Let ϕ and v be a solution to* (B-5) *and* (B-6) *in $r > 0$. If it holds that for some $r_1 > 0$,*

$$\phi(r_1) \geq 0, \quad \phi'(r_1) \geq 0, \quad v(r_1) \geq 0 \quad \textit{and} \quad v'(r_1) \geq 0,$$

with one of them nonzero, then

$$\phi(r) > 0, \quad \phi'(r) > 0, \quad v(r) > 0 \quad \textit{and} \quad v'(r) > 0 \tag{B-8}$$

for $r > r_1$, and there exists a constant $C > 0$ such that $\phi(r) \geq C(r - r_1 - 1)^2$ for $r > r_1 + 1$. Moreover, there exists a positive constant $C' = C'(k)$ such that $\phi(r) \leq C'(r^{n+k+2} + 1)$. In particular, $\phi(r)$ is positive and exists for all $r > r_1$.

Proof. By the equations (B-5) and (B-6),

$$\partial_r(r^{n-1}\partial_r\phi) = r^{n-1}v + \frac{\lambda_k}{r^2}\phi r^{n-1},$$
$$\partial_r(r^{n-1}\partial_r v) = \frac{n+4}{2}\bar{Q}U_0^{\frac{8}{n-4}}\phi r^{n-1} + \frac{\lambda_k}{r^2}v r^{n-1}.$$

Using integration,

$$r^{n-1}\phi'(r) = r_1^{n-1}\phi'(r_1) + \int_{r_1}^{r} r^{n-1}v + \frac{\lambda_k}{r^2}\phi r^{n-1}\,dr,$$
$$r^{n-1}v'(r) = r_1^{n-1}v'(r_1) + \int_{r_1}^{r} \frac{n+4}{2}\bar{Q}U_0^{\frac{8}{n-4}}\phi r^{n-1} + \frac{\lambda_k}{r^2}v r^{n-1}\,dr.$$

Then it is easy to see that (B-8) holds for $r > r_1$. Also, for $r > r_1 + 1$,

$$\begin{aligned} r^{n-1}\phi'(r) &= (r_1+1)^{n-1}\phi'(r_1+1) + \int_{r_1+1}^{r} r^{n-1}v + \frac{\lambda_k}{r^2}\phi r^{n-1}\,dr \\ &\geq (r_1+1)^{n-1}\phi'(r_1+1) + \int_{r_1+1}^{r} r^{n-1}v(r_1+1)\,dr \\ &\geq v(r_1+1)\Big(\frac{1}{n}r^n - \frac{1}{n}(r_1+1)^n\Big), \end{aligned}$$

with $v(r_1+1) > 0$. Therefore, for $r > r_1 + 1$,

$$\phi'(r) \geq \frac{1}{n}v(r_1+1)r - \frac{1}{n}(r_1+1)v(r_1+1).$$

Therefore, ϕ grows at least quadratically.

Now let's see the upper bound of growth of ϕ. It is easy to check that

$$\left(\Delta - \frac{\lambda_k}{r^2}\right)\left(\Delta - \frac{\lambda_k}{r^2}\right) r^{n+k+2} \geq \frac{n+4}{2}\bar{Q}U_0^{\frac{8}{n-4}} r^{n+k+2}, \quad r > 0.$$

Also,

$$\frac{d}{dr} r^{n+k+2} > 0, \quad \left(\Delta - \frac{\lambda_k}{r^2}\right) r^{n+k+2} > 0, \text{ and } \frac{d}{dr}\left(\Delta - \frac{\lambda_k}{r^2}\right) r^{n+k+2} > 0 \text{ for } r > 0.$$

Therefore, for any $r_1 > 0$, there exists a constant $\delta = \delta(r_1) > 0$ such that the function $\varphi(r) = r^{n+k+2} - \delta\phi(r)$ satisfies (B-8) at $r = r_1$. Note that

$$\left(\Delta - \frac{\lambda_k}{r^2}\right)\left(\Delta - \frac{\lambda_k}{r^2}\right)\varphi(r) \geq \frac{n+4}{2}\bar{Q}U_0^{\frac{8}{n-4}}\varphi(r), \quad r > 0.$$

Denote

$$\tilde{v}(r) = \left(\Delta - \frac{\lambda_k}{r^2}\right)\varphi(r)$$

so that

$$\left(\Delta - \frac{\lambda_k}{r^2}\right)\tilde{v}(r) \geq \frac{n+4}{2}\bar{Q}U_0^{\frac{8}{n-4}}\varphi(r), \quad r > 0.$$

Using the same integration argument starting from $r = r_1$, we obtain that $\varphi(r) > 0$ for $r \geq r_1$. This completes the proof of Theorem B.1. □

Now we consider the behavior of $\phi_{1,k,m}$ and $\phi_{2,k,m}$.

Let $v_{1,k,m}$ and $v_{2,k,m}$ be defined as above with respect to $\phi_{1,k,m}$ and $\phi_{2,k,m}$:

$$v_{1,k,m}(r) = \left(\partial_r^2 + \frac{n-1}{r}\partial_r - \frac{\lambda_k}{r^2}\right)\phi_{1,k,m},$$

$$v_{2,k,m}(r) = \left(\partial_r^2 + \frac{n-1}{r}\partial_r - \frac{\lambda_k}{r^2}\right)\phi_{2,k,m}.$$

By the Taylor expansion, for $r > 0$ close to 0, $\phi_{1,k,m}(r) > 0$, $\phi'_{1,k,m}(r) > 0$, $v_{1,k,m}(r) > 0$ and $v'_{1,k,m}(r) > 0$. Then by Theorem B.1, $\phi_{1,k,m}(r)$ keeps increasing at least quadratically as r increases. Also, for any $\epsilon > 0$, there exists $C = C(\epsilon, k)$ such that $\phi_{1,k,m}(r)$ is bounded from above by Cr^{n+k+2} with some constant C for $r > \epsilon$. In particular, $\phi_{1,k,m}(r)$ is positive and exists for any $r > 0$. The same holds for $\phi_{2,k,m}$.

For any $r_1 > 0$, we know that $\phi_{i,k,m}$ satisfies (B-8) at $r = r_1$, for $i = 1, 2$ and $k \geq 1$. Then there exists $C > 0$ such that both $(\phi_{1,k,m} - C^{-1}\phi_{2,k,m})$ and $(\phi_{2,k,m} - C^{-1}\phi_{1,k,m})$ satisfy (B-8) at $r = r_1$. Then by Theorem B.1, for $r > r_1$,

$$\phi_{1,k,m}(r) - C^{-1}\phi_{2,k,m}(r) > 0 \quad \text{and} \quad \phi_{2,k,m}(r) - C^{-1}\phi_{1,k,m}(r) > 0.$$

That is to say, $\phi_{1,k,m}$ and $\phi_{2,k,m}$ are both positive on $(0,\infty)$ and they go to infinity as $r\to\infty$ in the same order. This leads to the following corollary.

Corollary B.2. *For any $k\geq 1$, there is at most one constant $C>0$ such that $\phi_{1,k,m}-C\phi_{2,k,m}$ is bounded on $r\in(0,+\infty)$.*

Now we consider the asymptotic behavior of bounded solutions to (B-3) and (B-7) which vanish at infinity.

Lemma B.3. *Let $\phi_{k,m}=\phi_{1,k,m}-C\phi_{2,k,m}$ be a bounded solution to the initial value problem* (B-3) *and* (B-7) *such that $\phi_{k,m}(r)=o(1)$ as $r\to\infty$. Then $\phi_{k,m}(r)=O(r^{2-k-n})$ as $r\to+\infty$.*

Proof. We introduce

$$\phi^*_{k,m}(r)=r^{4-n}\phi_{k,m}(r^{-1}),\quad r>0,$$

to be the Kelvin transformation of $\phi_{k,m}$ and

$$v^*_{k,m}(r)=\left(\Delta-\frac{\lambda_k}{r^2}\right)\phi^*_{k,m}(r),\quad r>0.$$

Also, for $u_{k,m}(r\theta)=\phi_{k,m}(r)\xi_{k,m}(\theta)$, we denote

$$u^*_{k,m}(x)=|x|^{4-n}u_{k,m}\left(\frac{x}{|x|^2}\right),\quad x\in\mathbb{R}^n,$$

to be the Kelvin transformation of $u_{k,m}$. Then it is easy to check that $\phi^*_{k,m}$ is a solution to (B-3) and equivalently a solution to (B-4) in $(0,+\infty)$ and $u^*_{k,m}$ is a solution to (B-2) in $\mathbb{R}^n-\{0\}$. By our assumption on the decay of ϕ_k near infinity,

$$u^*_{k,m}(x)=o(|x|^{4-n})$$

as $x\to 0$. Then using the proof of Lemma A.1 in Appendix A we have that 0 is a removable singularity of $u^*_{k,m}$ and $u^*_{k,m}(x)=\phi^*_{k,m}(r)\xi_{k,m}(\theta)$ is a solution to (B-2) in $\mathbb{R}^n$. Therefore, $\phi^*_{k,m}$ and $v^*_{k,m}$ satisfy

$$\phi^*_{k,m}(0)=0,\quad (\phi^*_{k,m})'(0)=0,\quad v^*_{k,m}(0)=0,\quad (v^*_{k,m})'(0)=0.$$

Also, by the definition,

$$\phi^*_{k,m}(r)=r^{4-n}\phi_{k,m}(r^{-1})=O(r^{4-k-n})\quad\text{as } r\to+\infty.$$

Recall that $\phi_{1,k,m}$ and $\phi_{2,k,m}$ form a basis of the solution space to the problem (B-3) and (B-7). Since $\phi_{k,m}$ and $\phi^*_{k,m}$ are both bounded solutions to (B-3) and (B-7), by Corollary B.2 there exists a constant $a\in(-\infty,+\infty)$ such that $\phi^*_{k,m}(r)=a\phi_{k,m}(r)$ for $r>0$. Note that $\phi^*_{k,m}(1)=\phi_{k,m}(1)$. If $\phi_{k,m}(1)\neq 0$, then $a=1$. Otherwise, if also $\phi'_{k,m}(1)\neq 0$, then by L'Hospital's Rule, $a=-1$; else, if also $\phi'_{k,m}(1)=0$ but $v_{k,m}(1)\neq 0$, then by L'Hospital's rule, $a=1$; else, if also $\phi'_{k,m}(1)=0$,

$v_{k,m}(1) = 0$ but $v'_{k,m}(1) \neq 0$, then by L'Hospital's rule, $a = -1$ (In fact, by the comparison theorem Theorem B.1, since $\phi_{k,m}$ is bounded in $(0, +\infty)$, this could not happen). Since $\phi_{k,m}$ is assumed not to be identically zero, it is not possible that all the four data vanishes at $r = 1$. Therefore, a is either 1 or -1. Therefore,

$$\begin{aligned} \phi_{k,m}(r) &= r^k + O(r^{k+2}) && \text{as } r \to 0, \\ \phi_{k,m}(r) &= \pm r^{4-k-n} + O(r^{2-n-k}) && \text{as } r \to +\infty. \end{aligned} \qquad \square$$

Let ϕ be a solution to (B-1) with the decay $\phi \to 0$ uniformly as $|x| \to \infty$. Let $u_{k,m}(r\theta) = \phi_{k,m}(r)\xi_{k,m}(\theta) = \int_{S^{n-1}} \phi(r)\xi_{k,m}(\theta)\, dS\, \xi_{k,m}(\theta),\ k \geq 1$. Then $\phi_{k,m}(r) = o(1)$ as $r \to \infty$. Using the energy method, in the following theorem we show that for $5 \leq n \leq 8,\ \phi_{k,m} = 0$ for $k \geq 2$.

Theorem B.4. *Let $\phi_{k,m}$ with $k \geq 2$ be a bounded solution to the initial value problem* (B-3) *and* (B-7) *for $5 \leq n \leq 8$ such that $\phi_{k,m}(r) = o(1)$ as $r \to \infty$. Then $\phi_{k,m} = 0$.*

Proof. By Lemma B.3, it is easy to check that $\phi_{k,m} \in H^2(\mathbb{R}^n)$, for $k \geq 2$.

By (B-4), for any $\epsilon > 0$,

$$\int_{\mathbb{R}^n - B_\epsilon(0)} \phi_{k,m}\Big(\Delta - \frac{\lambda_k}{r^2}\Big)\Big(\Delta - \frac{\lambda_k}{r^2}\Big)\phi_{k,m}\, dx = \int_{\mathbb{R}^n - B_\epsilon(0)} \frac{n+4}{2} \bar{Q} U_0^{\frac{8}{n-4}} \phi_{k,m}^2\, dx.$$

Using integration by parts and letting $\epsilon \to 0$, we have that

$$\int_{\mathbb{R}^n} \Big(\Delta - \frac{\lambda_k}{r^2}\Big)\phi_{k,m}\Big(\Delta - \frac{\lambda_k}{r^2}\Big)\phi_{k,m}\, dx = \int_{\mathbb{R}^n} \frac{n+4}{2} \bar{Q} U_0^{\frac{8}{n-4}} \phi_{k,m}^2\, dx. \tag{B-9}$$

Note that

$$\begin{aligned} &\int_{\mathbb{R}^n} \Big(\Delta - \frac{\lambda_k}{r^2}\Big)\phi_{k,m}\Big(\Delta - \frac{\lambda_k}{r^2}\Big)\phi_{k,m}\, dx \\ &\qquad = \int_{\mathbb{R}^n} \big[(\Delta\phi_{k,m})^2 - 2\lambda_k r^{-2}\phi_{k,m}\Delta\phi_{k,m} + \lambda_k^2 r^{-4}\phi_{k,m}^2\big]\, dx, \end{aligned}$$

where by integration by parts,

$$\begin{aligned} \int_{\mathbb{R}^n} -2\lambda_k r^{-2}\phi_{k,m}\Delta\phi_{k,m}\, dx &= \int_{\mathbb{R}^n} 2\lambda_k r^{-2}|\nabla\phi_{k,m}|^2\, dx + \int_{\mathbb{R}^n} 2\lambda_k \phi_{k,m}\nabla\phi_{k,m}\cdot\nabla r^{-2}\, dx \\ &= \int_{\mathbb{R}^n} 2\lambda_k r^{-2}|\nabla\phi_{k,m}|^2\, dx + \int_{\mathbb{R}^n} \lambda_k \nabla(\phi_{k,m}^2)\cdot\nabla r^{-2}\, dx \\ &= \int_{\mathbb{R}^n} 2\lambda_k r^{-2}|\nabla\phi_{k,m}|^2\, dx - \int_{\mathbb{R}^n} \lambda_k \phi_{k,m}^2 \Delta r^{-2}\, dx \\ &= \int_{\mathbb{R}^n} 2\lambda_k r^{-2}|\nabla\phi_{k,m}|^2\, dx + (2n-8)\int_{\mathbb{R}^n} \lambda_k r^{-4}\phi_{k,m}^2\, dx \end{aligned}$$

for $n \geq 6$. Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Big(\Delta - \frac{\lambda_k}{r^2}\Big)\phi_{k,m}\Big(\Delta - \frac{\lambda_k}{r^2}\Big)\phi_{k,m}\,dx \\
&\quad = \int_{\mathbb{R}^n} (\Delta\phi_{k,m})^2\,dx + \int_{\mathbb{R}^n} 2\lambda_k r^{-2}|\nabla\phi_{k,m}|^2\,dx + (2n\lambda_k - 8\lambda_k + \lambda_k^2)\int_{\mathbb{R}^n} r^{-4}\phi_{k,m}^2\,dx \\
&\quad \geq (2n\lambda_k - 8\lambda_k + \lambda_k^2)\int_{\mathbb{R}^n} r^{-4}\phi_{k,m}^2\,dx.
\end{aligned}$$

Since $(1 + 4^{-1}r^2)^{-1} \leq r^{-1}$ for $r > 0$ and, for $k \geq 2$ and $5 \leq n \leq 8$,

$$2n\lambda_k - 8\lambda_k + \lambda_k^2 = (2n-8)k(n+k-2) + k^2(n+k-2)^2 > \frac{n+4}{2} \times \bar{Q} = \frac{n+4}{2} \times \frac{n(n^2-4)}{8},$$

we have that

$$\int_{\mathbb{R}^n} \Big(\Delta - \frac{\lambda_k}{r^2}\Big)\phi_{k,m}\Big(\Delta - \frac{\lambda_k}{r^2}\Big)\phi_{k,m}\,dx > \int_{\mathbb{R}^n} \frac{n+4}{2}\bar{Q}U_0^{\frac{8}{n-4}}\phi_{k,m}^2\,dx,$$

which contradicts (B-9) for $k \geq 2$ and $5 \leq n \leq 8$. Therefore, there exists no nontrivial bounded solution ϕ_k to (B-3) such that $\phi_k(r) = o(1)$ as $r \to +\infty$ for $k \geq 2$ and $5 \leq n \leq 8$. □

It is easy to check that

$$u_0 + \sum_m u_{1,m} = c_0\Big(x \cdot \nabla U_0 + \frac{n-4}{2}U_0\Big) + \sum_{j=1}^{n} c_j \partial_{x_j} U_0$$

with $c_0, \ldots, c_n$ some constants. As a direct corollary of Theorem B.4, we have:

Corollary B.5. *Let ϕ be a solution to* (B-1) *with the decay $\phi \to 0$ uniform as $|x| \to \infty$. Then for $5 \leq n \leq 8$, we have that*

$$\phi = c_0\Big(x \cdot \nabla U_0 + \frac{n-4}{2}U_0\Big) + \sum_{j=1}^{n} c_j \partial_{x_j} U_0$$

for some constants $c_0, c_1, \ldots, c_n$.

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