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CINDY (SIN YI) TSANG AND STANLEY YAO XIAO

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BINARY QUARTIC FORMS WITH BOUNDED INVARIANTS AND SMALL GALOIS GROUPS

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We consider integral and irreducible binary quartic forms whose Galois group is isomorphic to a subgroup of the dihedral group of order eight. We first show that the set of all such forms is a union of families indexed by integral binary quadratic forms $f(x, y)$ of nonzero discriminant. Then, we shall enumerate the $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes of all such forms associated to a fixed $f(x, y)$.

1. Introduction

The problem of enumerating $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes of integral and irreducible binary forms of a fixed degree has a long history. The quadratic and cubic cases were solved in [Gauss 1801; Siegel 1944] and [Davenport 1951b; 1951c], respectively, where the forms are ordered by the natural height, namely the discriminant $\Delta(-)$. The quartic case turns out to be more challenging. This is because the ring of polynomial invariants of quartic forms have two independent generators, usually denoted $I(-)$ and $J(-)$. For

$$F(x, y) = a_4x^4 + a_3x^3y + a_2x^2y^2 + a_1xy^3 + a_0y^4, \tag{1-1}$$

they are given by the explicit formulae

$$I(F) = 12a_4a_0 - 3a_3a_1 + a_2^2,$$
$$J(F) = 72a_4a_2a_0 + 9a_3a_2a_1 - 27a_4a_1^2 - 27a_3^2a_0 - 2a_2^3,$$

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which are of degrees two and three, respectively. Bhargava and Shankar [2015], instead of using the discriminant, introduced the height function

$$H_{\mathrm{BS}}(F) = \max\{|I(F)|^3, J(F)^2/4\}. \tag{1-2}$$

For $X > 0$, let us define

$$N_{\mathbb{Z}}(X) = \#\big\{[F] : \text{integral and irreducible binary quartic forms } F \text{ such that } H_{\mathrm{BS}}(F) \le X\big\},$$

where $[-]$ denotes $\mathrm{GL}_2(\mathbb{Z})$-equivalence class. In [loc. cit.], they proved that

$$N_{\mathbb{Z}}(X) = \frac{44\zeta(2)}{135} X^{\frac{5}{6}} + O_\epsilon(X^{\frac{3}{4}+\epsilon}) \quad \text{for any } \epsilon > 0. \tag{1-3}$$

This is the first result ever obtained, and as far as we know, the only known result in the literature, for the quartic case.

1A. ***Set-up and notation.*** In this paper, we shall also be interested in the quartic case, but only the integral and irreducible binary quartic forms F with *small* Galois group $\mathrm{Gal}(F)$, which is defined to be the Galois group of the splitting field of $F(x, 1)$ over $\mathbb{Q}$. We know that $\mathrm{Gal}(F)$ is isomorphic to one of the following:

$$\begin{aligned} S_4 &= \text{the symmetric group on four letters,}\\ A_4 &= \text{the alternating group on four letters,}\\ D_4 &= \text{the dihedral group of order eight,}\\ C_4 &= \text{the cyclic group of order four,}\\ V_4 &= \text{the Klein-four group.} \end{aligned}$$

We shall say that $\mathrm{Gal}(F)$ is *small* if it is isomorphic to D_4, C_4, or V_4. Recall that the *cubic resolvent of* F is defined by

$$\mathcal{Q}_F(x) = x^3 - 3I(F)x + J(F).$$

Then, equivalently, we have the classical characterization that for irreducible F

$$\mathrm{Gal}(F) \text{ is small if and only if } \mathcal{Q}_F(x) \text{ is reducible.}$$

It turns out that whether $\mathrm{Gal}(F)$ is small or not may also be characterized in terms of binary quadratic forms and the following so-called *twisted action* of $\mathrm{GL}_2(\mathbb{R})$.

Given a complex binary form $\xi(x, y)$, let $\mathrm{GL}_2(\mathbb{R})$ act on it via

$$\xi_T(x, y) = \frac{1}{\det(T)^{\deg\xi/2}} \xi(t_1x + t_2y, t_3x + t_4y) \quad \text{for } T = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix}.$$

Observe that this is only an action up to sign when $\deg\xi$ is odd, in the sense that for $T_1, T_2 \in \mathrm{GL}_2(\mathbb{R})$, we only have $\xi_{T_1T_2} = \pm(\xi_{T_1})_{T_2}$ in general. Now, given a real binary quadratic form $f(x, y) = \alpha x^2 + \beta xy + \gamma y^2$ with $\Delta(f) \neq 0$, write

$$M_f = \begin{pmatrix} \beta & 2\gamma \\ -2\alpha & -\beta \end{pmatrix}$$

for its associated matrix in $\mathrm{GL}_2(\mathbb{R})$. Its action on binary quartic forms clearly remains unchanged if we scale $f(x, y)$ by a constant in $\mathbb{R}^\times$. The second author, Xiao, proved [2019] that for any real binary quartic form F with $\Delta(F) \neq 0$, elements of

$$\{T \in \mathrm{GL}_2(\mathbb{R}) : T \text{ is not a scalar multiple of } I_{2\times 2} \text{ and } F_T = F\}$$

all arise from binary quadratic forms in this way; see Proposition 2.1. Recall that an integral binary quadratic form is called *primitive* if its coefficients are coprime. Using this result from [Xiao 2019], in Section 2, we shall first show:

Theorem 1.1. *Let F be an integral binary quartic form with $\Delta(F) \neq 0$. Then, the following are equivalent.*

(1) *$\mathcal{Q}_F(x)$ is reducible.*

(2) *$F_T = F$ for some $T \in \mathrm{GL}_2(\mathbb{Q})$ which is not a scalar multiple of $I_{2\times 2}$.*

(3) *$F_{M_f} = F$ for an integral and primitive binary quadratic form f with $\Delta(f) \neq 0$.*

Moreover, in the case that $\mathcal{Q}_F(x)$ is reducible:

(a) *If $\Delta(F) \neq \square$, then there is a unique such f up to sign.*

(b) *If $\Delta(F) = \square$, then there are exactly three such f up to sign, among which one is definite and two are indefinite.*

Given a real binary quadratic form $f(x, y)$ with $\Delta(f) \neq 0$, let us further make the following definitions. First put

$$V_{\mathbb{R},f} = \{\text{real binary quartic forms } F \text{ such that } F_{M_f} = F\},$$

$$V_{\mathbb{Z},f} = \{\text{integral binary quartic forms } F \text{ such that } F_{M_f} = F\}.$$

Clearly $V_{\mathbb{R},f}$ is a vector space over $\mathbb{R}$ and $V_{\mathbb{Z},f}$ a lattice over $\mathbb{Z}$. A straightforward calculation shows that $\dim_{\mathbb{R}} V_{\mathbb{R},f}$ is three; see (3-1) and (3-2) below. Also, put

$$V^0_{\mathbb{R},f} = \{F \in V_{\mathbb{R},f} : \Delta(F) \neq 0\} \quad \text{and} \quad V^0_{\mathbb{Z},f} = \{F \in V_{\mathbb{Z},f} : \Delta(F) \neq 0\}.$$

For $F \in V^0_{\mathbb{R},f}$, we shall define two new invariants as follows. As we shall see in (2-3), there is a unique root $\omega_f(F)$ of $\mathcal{Q}_F(x)$ corresponding to f. Let $\omega'_f(F), \omega''_f(F)$ denote the other two roots of $\mathcal{Q}_F(x)$ and define

$$L_f(F) = \omega_f(F) \quad \text{and} \quad K_f(F) = -\omega'_f(F)\omega''_f(F). \tag{1-4}$$

By Proposition 3.2 below, they have degrees one and two, respectively, in the coefficients of F. Following (1-2), let us define the *height of F associated to f* by

$$H_f(F) = \max\{L_f(F)^2, |K_f(F)|\}.$$

This is comparable to the height (1-2) because by comparing coefficients in

$$x^3 - I(F)x + J(F) = (x - \omega_f(F))(x - \omega'_f(F))(x - \omega''_f(F)),$$

we easily deduce the relations

$$3I(F) = L_f(F)^2 + K_f(F) \quad \text{and} \quad J(F) = L_f(F)K_f(F), \tag{1-5}$$

which in turn imply that

$$(H_f(F)/10)^3 \le H_{\mathrm{BS}}(F) \le H_f(F)^3. \tag{1-6}$$

Let us note that

$$\begin{aligned}\Delta(F) &= \frac{4I(F)^3 - J(F)^2}{27} \\ &= \left(\frac{L_f(F)^2 + 4K_f(F)}{9}\right)\left(\frac{2L_f(F)^2 - K_f(F)}{9}\right)^2,\end{aligned} \tag{1-7}$$

where the first equality is well-known, and the second equality holds by (1-5). Also, our height $H_f(-)$ is an invariant in the sense that for any $T \in \mathrm{GL}_2(\mathbb{R})$, we have

$$H_{f_T}(F_T) = H_f(F),$$

as shown in Proposition 3.1 below. This implies that the map

$$V_{\mathbb{R},f} \to V_{\mathbb{R},f_T}, \quad F \mapsto F_T, \tag{1-8}$$

which is a well-defined bijection because $M_{f_T} = T^{-1}M_fT$, is height-preserving when restricted to the forms of nonzero discriminant.

Now, let us return to the integral and irreducible binary quartic forms with small Galois group. Write $V_{\mathbb{Z}}^{\mathrm{sm}}$ for the set of all such forms and set

$$V_{\mathbb{Z}}^{\mathrm{sm},\dagger} = \{F \in V_{\mathbb{Z}}^{\mathrm{sm}} : \mathrm{Gal}(F) \not\simeq V_4\}.$$

By Theorem 1.1, we know that

$$\begin{aligned}V_{\mathbb{Z}}^{\mathrm{sm}} &= \bigcup_{f \in \mathfrak{F}^*} \{F \in V_{\mathbb{Z},f}^0 : F \text{ is irreducible}\}, \\ V_{\mathbb{Z}}^{\mathrm{sm},\dagger} &= \bigsqcup_{f \in \mathfrak{F}^*} \{F \in V_{\mathbb{Z},f}^0 : F \text{ is irreducible and } \mathrm{Gal}(F) \not\simeq V_4\},\end{aligned} \tag{1-9}$$

where $\mathfrak{F}^*$ denotes the set of all integral and primitive binary quadratic forms of nonzero discriminant, up to sign. In particular, given $F \in V_{\mathbb{Z}}^{\mathrm{sm},\dagger}$, there is a unique $f \in \mathfrak{F}^*$ such that $F \in V_{\mathbb{Z},f}^0$, and we may define the *height of* F by setting

$$H(F) = H_f(F).$$

For $X > 0$, let us define

$$N_{\mathbb{Z}}^{\dagger}(X) = \#\{[F] : F \in V_{\mathbb{Z}}^{\mathrm{sm},\dagger} \text{ such that } H(F) \le X\},$$
$$N_{\mathbb{Z},f}^{\dagger}(X) = \#\{[F] : F \in V_{\mathbb{Z}}^{\mathrm{sm},\dagger} \cap V_{\mathbb{Z},f}^0 \text{ such that } H(F) \le X\}.$$

Then, by (1-8) and (1-9), we have

$$N_{\mathbb{Z}}^{\dagger}(X) = \sum_{f \in \mathfrak{F}} N_{\mathbb{Z},f}^{\dagger}(X),$$

where $\mathfrak{F}$ denotes a set of representatives of the $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes on $\mathfrak{F}^*$. In Theorem 1.2, which is our main result, for $f \in \mathfrak{F}^*$, we shall determine the asymptotic formula for $N_{\mathbb{Z},f}^{\dagger}(X)$. In fact, we shall consider the finer counts

$$N_{\mathbb{Z},f}^{(D_4)}(X) = \#\{[F] : F \in V_{\mathbb{Z}}^{\mathrm{sm}} \cap V_{\mathbb{Z},f}^0 \text{ such that } \mathrm{Gal}(F) \simeq D_4 \text{ and } H(F) \le X\},$$
$$N_{\mathbb{Z},f}^{(C_4)}(X) = \#\{[F] : F \in V_{\mathbb{Z}}^{\mathrm{sm}} \cap V_{\mathbb{Z},f}^0 \text{ such that } \mathrm{Gal}(F) \simeq C_4 \text{ and } H(F) \le X\},$$
$$N_{\mathbb{Z},f}^{(V_4)}(X) = \#\{[F] : F \in V_{\mathbb{Z}}^{\mathrm{sm}} \cap V_{\mathbb{Z},f}^0 \text{ such that } \mathrm{Gal}(F) \simeq V_4 \text{ and } H_f(F) \le X\},$$

and show that the latter two are negligible compared to $N_{\mathbb{Z},f}^{(D_4)}(X)$. This means that most of the forms in $V_{\mathbb{Z}}^{\mathrm{sm}} \cap V_{\mathbb{Z},f}^0$ have Galois group isomorphic to D_4. However, all of our error estimates depend upon f. Currently, we do not know how to control them in a uniform way, and so we are unable to obtain an asymptotic formula for $N_{\mathbb{Z}}^{\dagger}(X)$ by summing over $f \in \mathfrak{F}$.

Finally, let us explain, for each $f \in \mathfrak{F}^*$, how counting forms in $V_{\mathbb{Z}}^{\mathrm{sm}} \cap V_{\mathbb{Z},f}^0$ may be reduced to counting lattice points. Write $f(x, y) = \alpha x^2 + \beta xy + \gamma y^2$ with $\alpha, \beta, \gamma \in \mathbb{Z}$. By (3-1) and (3-2), the set $V_{\mathbb{R},f}$ is a vector space isomorphic to $\mathbb{R}^3$ via

$$\Theta_1 : a_4x^4 + a_3x^3y + a_2x^2y^2 + a_1xy^3 + a_0y^4 \mapsto (a_4, a_3, a_2) \quad \text{if } \alpha \neq 0,$$
$$\Theta_2 : a_4x^4 + a_3x^3y + a_2x^2y^2 + a_1xy^3 + a_0y^4 \mapsto (a_4, a_2, a_0) \quad \text{if } \beta, \beta^2 + 4\alpha\gamma \neq 0.$$

Recall that the subset $V_{\mathbb{Z},f}$ has the structure of a rank-three $\mathbb{Z}$-lattice, which may be identified with the lattices

$$\Lambda_{f,1} = \Theta_1(V_{\mathbb{Z},f}) \quad \text{and} \quad \Lambda_{f,2} = \Theta_2(V_{\mathbb{Z},f}) \tag{1-10}$$

in $\mathbb{Z}^3$. Let us mention here that we shall use the isomorphism

$$\Theta_{w(f)}, \quad \text{where } w(f) = \begin{cases} 1 & \text{if } f \text{ is irreducible,} \\ 2 & \text{if } f \text{ is reducible.} \end{cases}$$

Thus, the problem is reduced to counting points in $\Lambda_{f,1}$ or $\Lambda_{f,2}$, and then sieving out those which come from reducible forms. In turn, counting lattice points amounts to computing certain volumes by a result of Davenport [1951a]; see Proposition 5.1.

1B. *Statement of the main theorem.* It is clear that we may choose the set $\mathfrak{F}$ of representatives to be such that for all $f \in \mathfrak{F}$, the x^2-coefficient is positive, and

$$f(x, y) = \alpha x^2 + \beta xy, \quad \text{where } \gcd(\alpha, \beta) = 1 \text{ and } 0 < \alpha \le \beta \tag{1-11}$$

when f is reducible. Let $\sim$ denote $\mathrm{GL}_2(\mathbb{Z})$-equivalence. Then, our main result is:

Theorem 1.2. *Let $f(x, y)$ be an integral and primitive binary quadratic form of nonzero discriminant and with positive x^2-coefficient. Write $D_f = |\Delta(f)|$, and put*

$$s_f = \begin{cases} 8 & \text{if } D_f \text{ is odd,} \\ 1 & \text{if } D_f \text{ is even.} \end{cases}$$

(a) *Suppose that f is positive definite. Then, we have*

$$N_{\mathbb{Z},f}^{(D_4)}(X) = \frac{1}{s_f r_f} \frac{13\pi}{27 D_f^{3/2}} X^{3/2} + O_f(X^{1+\epsilon}) \quad \text{for any } \epsilon > 0,$$

where

$$r_f = \begin{cases} 6 & \text{if } f(x, y) \sim x^2 + xy + y^2, \\ 2 & \text{if } f(x, y) \sim ax^2 + cy^2 \text{ or } f(x, y) \sim ax^2 + bxy + ay^2 \text{ with } a \neq b, \\ 1 & \text{otherwise.} \end{cases}$$

(b) *Suppose that f is reducible and that f has the shape* (1-11). *Then, we have*

$$N_{\mathbb{Z},f}^{(D_4)}(X) = \frac{1}{s_f r_f} \frac{8}{9\beta^{3/2}} X^{3/2} \log X + O_f(X^{3/2}),$$

where

$$r_f = \begin{cases} 1 & \text{if } \beta \nmid \alpha^2 + 1 \text{ and } \beta \nmid \alpha^2 - 1, \\ 2 & \text{otherwise.} \end{cases}$$

(c) *Suppose that f is indefinite and irreducible. Define $t_{D_f} \in \mathbb{R}$ to be such that $e^{t_{D_f}}$ is the fundamental unit of the quadratic order $\mathbb{Z}[(D_f + \sqrt{D_f})/2]$, or equivalently*

$$t_{D_f} = \log((u_{D_f} + v_{D_f}\sqrt{D_f})/2),$$

where $(u_{D_f}, v_{D_f}) \in \mathbb{N}^2$ is the least solution to $x^2 - D_f y^2 = \pm 4$. Then, we have

$$N_{\mathbb{Z},f}^{(D_4)}(X) = \frac{1}{s_f r_f} \frac{32 t_{D_f}}{9 D_f^{3/2}} X^{3/2} + O_f(X^{1+\epsilon}) \quad \text{for any } \epsilon > 0,$$

where

$$r_f = \begin{cases} 2 & \text{if } f(x, y) \sim ax^2 + bxy - ay^2 \\ & \quad \text{or } f(x, y) \sim ax^2 + bxy + cy^2 \text{ with } a \mid b, \\ 1 & \text{otherwise.} \end{cases}$$

(d) *In all three cases, for any* $\epsilon > 0$, *we have*

$$N^{(V_4)}_{\mathbb{Z},f}(X) = O_{f,\epsilon}(X^{1+\epsilon}),$$

and also

$$N^{(C_4)}_{\mathbb{Z},f}(X) = \begin{cases} O_{f,\epsilon}(X^{1/2+\epsilon}) & \text{if } -\Delta(f) \neq \square, \\ O_f(X) & \text{if } -\Delta(f) = \square. \end{cases}$$

Notice that the error terms in Theorem 1.2 depend upon f. Hence, we are unable to obtain an asymptotic formula for $N^{\dagger}_{\mathbb{Z}}(X)$ by summing over $f \in \mathfrak{F}$. However, there are only three $f \in \mathfrak{F}$ that need to be considered if we restrict to the forms in

$$V^{\mathrm{sm},*}_{\mathbb{Z}} = \{F \in V^{\mathrm{sm}}_{\mathbb{Z}} : F_T = F \text{ for some } T \in \mathrm{GL}_2(\mathbb{Z}) \setminus \{\pm I_{2\times 2}\}\}.$$

This is because by Proposition 2.1 below, such a matrix T must be of the shape M_f or $M_f/2$ up to sign, where $f \in \mathfrak{F}^*$. From (1-9), we then deduce that

$$V^{\mathrm{sm},*}_{\mathbb{Z}} = \bigcup_{\substack{f \in \mathfrak{F}^* \\ \Delta(f) \in \{-4,1,4\}}} \{F \in V^0_{\mathbb{Z},f} : F \text{ is irreducible}\},$$

$$V^{\mathrm{sm},*,\dagger}_{\mathbb{Z}} = \bigsqcup_{\substack{f \in \mathfrak{F}^* \\ \Delta(f) \in \{-4,1,4\}}} \{F \in V^0_{\mathbb{Z},f} : F \text{ is irreducible and } \mathrm{Gal}(F) \not\simeq V_4\}.$$

For $X > 0$, let us put

$$N^{*,\dagger}_{\mathbb{Z}}(X) = \#\{[F] : F \in V^{\mathrm{sm},*,\dagger}_{\mathbb{Z}} \text{ such that } H(F) \leq X\}.$$

Then, by (1-8) and the above discussion, we have

$$N^{*,\dagger}_{\mathbb{Z}}(X) = N^{*,\dagger}_{\mathbb{Z},f^{(1)}}(X) + N^{*,\dagger}_{\mathbb{Z},f^{(2)}}(X) + N^{*,\dagger}_{\mathbb{Z},f^{(3)}}(X),$$

where we may take

$$f^{(1)}(x,y) = x^2 + y^2, \quad f^{(2)}(x,y) = x^2 + xy, \quad f^{(3)}(x,y) = x^2 + 2xy,$$

whose discriminants are -4, 1, and 4, respectively. We have:

Corollary 1.3. *We have*

$$N^{*,\dagger}_{\mathbb{Z}}(X) = \tfrac{1}{9} X^{3/2} \log X + O(X^{3/2}).$$

Proof. Theorem 1.2 implies that

$$N^{\dagger}_{\mathbb{Z},f^{(1)}}(X) = O(X^{3/2}) \quad \text{and} \quad N^{\dagger}_{\mathbb{Z},f^{(i)}}(X) = \tfrac{1}{18} X^{3/2} \log X + O(X^{3/2}) \quad \text{for } i = 2, 3.$$

Summing these terms up then yields the claim. □

Finally, as a consequence of the proof of Theorem 1.2, we also have:

Theorem 1.4. *Let $D=\beta^2+4\alpha^2$, where $\alpha, \beta \in \mathbb{N}$ are coprime and D is not a square. Then, the negative Pell's equation $x^2-Dy^2=-4$ has integer solutions if and only if the integral binary quadratic form $\alpha x^2+\beta xy-\alpha y^2$ is $\mathrm{GL}_2(\mathbb{Z})$-equivalent to a form of the shape $ax^2+bxy+cy^2$ with a dividing b.*

We now discuss some potential applications of our Theorem 1.2 and Corollary 1.3.

First, it is natural to ask whether the asymptotic formula (1-3), which was proven using Proposition 5.1, admits a secondary main term. From the arguments in [Bhargava and Shankar 2015], we see that the error term arising from volumes of the lower dimensional projections in Proposition 5.1 is only of order $O(X^{3/4})$. Thus, possibly $X^{3/4}$ is the order of a second main term, but it is dominated by another error term coming from

$$N^*_{\mathbb{Z},\mathrm{BS}}(X)=\#\{[F]:F\in V_{\mathbb{Z}}^{\mathrm{sm},*} \text{ such that } H_{\mathrm{BS}}(F)\leq X\}.$$

In particular, it was shown in [Bhargava and Shankar 2015, Lemma 2.4] that

$$N^*_{\mathbb{Z},\mathrm{BS}}(X)=O_\epsilon(X^{3/4+\epsilon}) \quad \text{for any } \epsilon>0.$$

Our Corollary 1.3 removes this obstacle, because

$$N_{\mathbb{Z}}^{*,\dagger}(X^{1/3})\leq N^*_{\mathbb{Z},\mathrm{BS}}(X)\leq N_{\mathbb{Z}}^{*,\dagger}(10X^{1/3})+O_\epsilon(X^{1/3+\epsilon})$$

by (1-6) and Theorem 1.2(d), whence we have

$$N^*_{\mathbb{Z},\mathrm{BS}}(X)\asymp X^{1/2}\log X.$$

This improvement potentially allows one to prove a secondary main term for (1-3) by using similar methods from [Bhargava et al. 2013], where it was shown that the counting theorem in [Davenport and Heilbronn 1971] for cubic fields has a secondary main term of order $X^{5/6}$; this latter fact was proven independently in [Taniguchi and Thorne 2013] as well.

Next, integral binary quartic forms are closely related to quartic orders, and maximal irreducible quartic orders may be regarded as quartic fields. More generally, by the construction of Birch and Merriman [1972] or Nakagawa [1989], any integral binary form F gives rise to a $\mathbb{Z}$-order Q_F whose rank is the degree of F, where $\mathrm{GL}_2(\mathbb{Z})$-equivalence class of F corresponds to isomorphism class of Q_F. By [Delone and Faddeev 1964], it is well-known that all cubic orders come from integral binary cubic forms, which enabled the enumeration of cubic orders having a nontrivial automorphism as well as cubic fields by their discriminant; see [Bhargava and Shnidman 2014] and [Davenport and Heilbronn 1971], respectively. But this is not true for orders of higher rank. Parametrizations of quartic and quintic orders were given by Bhargava in his seminal work [2004; 2008]. Wood [2012] further showed that the quartic orders arising from integral binary quartic forms are exactly those having a monogenic *cubic resolvent*; see [Bhargava 2004] for the definition. This implies that the forms in

$$V_{\mathbb{Z}}^{\mathrm{sm},\star} = \{F \in V_{\mathbb{Z}}^{\mathrm{sm}} : Q_F \text{ is maximal}\}$$

correspond to quartic D_4-, C_4-, and V_4-fields whose ring of integers has a monogenic cubic resolvent. In our upcoming paper [Tsang and Xiao 2017], we shall enumerate $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes of forms in $V_{\mathbb{Z}}^{\mathrm{sm},\star}$ with respect to a height corresponding to the conductor of fields, as motivated by [Altuğ et al. 2017]. In fact, we shall that show that

$$\text{for all } f \in \mathfrak{F}^* : \quad F \in V_{\mathbb{Z}}^{\mathrm{sm},\star} \cap V_{\mathbb{Z},f}^{0} \neq \varnothing \quad \text{if and only if} \quad \Delta(f) \in \{-4, 1, 4\}.$$

Thus, our counting theorem in [Tsang and Xiao 2017] may be regarded as a refinement and an extension of Corollary 1.3 above.

Last but not least, binary quartic forms are connected to elliptic curves as well. In particular, any integral binary quartic form F gives rise to an elliptic curve

$$E_F : y^2 = x^3 - \frac{I(F)}{3}x - \frac{J(F)}{27}$$

defined over $\mathbb{Q}$. Bhargava and Shankar [2015] applied (1-3) as well as a parametrization of 2-Selmer groups due to Birch and Swinnerton-Dyer to show that the average rank of elliptic curves over $\mathbb{Q}$, when ordered by a *naive* height analogous to (1-2), is at most $\frac{3}{2}$. This result is remarkable in that it is the first to show, unconditional on the BSD-conjecture and the Grand Riemann Hypothesis, boundedness of the average rank of large families of elliptic curves over $\mathbb{Q}$. Conditional bounds were obtained by Brumer [1992], Heath-Brown [2004], and Young [2006] previously. Now, the relations in (1-5) imply that for $F \in V_{\mathbb{Z}}^{\mathrm{sm}} \cap V_{\mathbb{Z},f}^{0}$ with $f \in \mathfrak{F}^*$, we have

$$E_F : y^2 = \left(x + \frac{L_f(F)}{3}\right)\left(x^2 - \frac{L_f(F)}{3}x - \frac{K_f(F)}{9}\right),$$

which has a rational 2-torsion point. Hence, our Theorem 1.2 potentially allows one to study arithmetic properties of elliptic curves with 2-torsion over $\mathbb{Q}$. Let us remark that unlike a *large* family of elliptic curves over $\mathbb{Q}$, in the sense of [Bhargava and Shankar 2015, Section 3], the family consisting of those curves with a rational 2-torsion exhibits a rather peculiar behavior. Indeed, Klagsbrun and Lemke Oliver [2014] proved that the average size of the 2-Selmer groups in this family is unbounded, and they conjectured an asymptotic growth rate. One might be able to obtain such an asymptotic growth rate using our Theorem 1.2 and a sieve that detects local solubility; this line of inquiry is pursued in an upcoming paper due to D. Kane and Z. Klagsbrun.

2. Characterization of forms with small Galois groups

2A. ***Cremona covariants.*** Let F be a real binary quartic form with $\Delta(F) \neq 0$. As Cremona defined [1999], we have three quadratic covariants $\mathfrak{C}_{F,\omega}(x, y)$, each of which is associated to a root ω of $\mathcal{Q}_F(x)$; see [Xiao 2019, Subsection 4.2] for the explicit definition. They satisfy the syzygy

$$\mathfrak{C}_{F,\omega}(x, y)^2 = \tfrac{1}{3}(F_4(x, y) + 4\omega F(x, y)), \tag{2-1}$$

where F_4 is the *Hessian covariant of* F and is given by

$$\begin{aligned} F_4(x, y) = 3(a_3^2 - 8a_4a_2)x^4 + 4(a_3a_2 - 6a_4a_1)x^3y + 2(2a_2^2 - 24a_4a_0 - 3a_3a_1)x^2y^2 \\ + 4(a_2a_1 - 6a_3a_0)xy^3 + (3a_1^2 - 8a_2a_0)y^4. \end{aligned}$$

We shall label the roots $\omega_1(F), \omega_2(F), \omega_3(F)$ of $\mathcal{Q}_F(x)$ such that

$$\mathfrak{C}_{F,\omega_i(F)}(x, y) = \mathfrak{C}_{F,i}(x, y) \quad \text{for all } i = 1, 2, 3,$$

where $\mathfrak{C}_{F,i}(x, y)$ is defined as in [Xiao 2019, (4.6)]. Then, from (2-1) and the explicit expressions for $\mathfrak{C}_{F,\omega}(x, y)$ given in [Xiao 2019], we have the following observations:

(1) For $\omega = \omega_1(F)$, the binary quadratic form $\mathfrak{C}_{F,\omega}(x, y)$ has real coefficients.

(2) For $\omega = \omega_2(F), \omega_3(F)$, we have:

- If $\Delta(F) > 0$, then $\lambda_\omega \cdot \mathfrak{C}_{F,\omega}(x, y)$ has real coefficients for some $\lambda_\omega \in \{1, \sqrt{-1}\}$.
- If $\Delta(F) < 0$, then $\lambda \cdot \mathfrak{C}_{F,\omega}(x, y)$ does not have real coefficients for all $\lambda \in \mathbb{C}^\times$.

Also, it is easy to check that

$$\Delta(\mathfrak{C}_{F,\omega_1(F)}), \Delta(\mathfrak{C}_{F,\omega_3(F)}) > 0 \quad \text{and} \quad \Delta(\mathfrak{C}_{F,\omega_2(F)}) < 0. \tag{2-2}$$

We shall require the following result by Xiao [2019].

Proposition 2.1. *Let F be a real binary quartic form with $\Delta(F) \neq 0$. Then, a set of representatives for the quotient group*

$$\{T \in \mathrm{GL}_2(\mathbb{R}) : F_T = F\}/\{\lambda \cdot I_{2\times 2} : \lambda \in \mathbb{R}^\times\}$$

is given by

$$\begin{cases} \{I_{2\times 2}, M_f : f \in \{\mathfrak{C}_{F,\omega_1(F)}, \lambda_{\omega_2(F)} \cdot \mathfrak{C}_{F,\omega_2(F)}, \lambda_{\omega_3(F)} \cdot \mathfrak{C}_{F,\omega_3(F)}\}\} & \text{if } \Delta(F) > 0, \\ \{I_{2\times 2}, M_f : f \in \{\mathfrak{C}_{F,\omega_1(F)}\}\} & \text{if } \Delta(F) < 0. \end{cases}$$

Furthermore, the quadratic forms $\mathfrak{C}_{F,\omega_1(F)}(x, y)$, $\mathfrak{C}_{F,\omega_2(F)}(x, y)$, and $\mathfrak{C}_{F,\omega_3(F)}(x, y)$, are pairwise nonproportional over $\mathbb{C}^\times$.

Proof. For the first statement, see [Xiao 2019, Proposition 4.6]. As for the second statement, since $\mathfrak{C}_{F,\omega_i(F)}(x, y)$ are covariants, replacing F by a $\mathrm{GL}_2(\mathbb{R})$-translate if necessary, we may assume that

$$F(x, y) = a_4x^4 + a_2x^2y^2 \pm a_4y^4.$$

In this special case, it is not hard to verify the claim using the explicit expressions for $\mathfrak{C}_{F,\omega_i(F)}(x, y)$ in [Xiao 2019, (4.6)]. □

Let F be a real binary quartic form with $\Delta(F) \neq 0$. Proposition 2.1 implies that for any real binary quadratic form f with $\Delta(f) \neq 0$, we have $F \in V_{\mathbb{R},f}$ if and only if

$$f(x, y) \text{ is proportional to } \mathfrak{C}_{F,\omega}(x, y) \text{ for a root } \omega \text{ of } \mathcal{Q}_F(x). \tag{2-3}$$

Moreover, this root ω is unique, and we shall denote it by $\omega_f(F)$. This was required in order to define the L_f- and K_f-invariants in (1-4).

2B. ***Proof of Theorem 1.1.*** The key is the following lemma.

Lemma 2.2. *Let F be an integral binary quartic form with $\Delta(F) \neq 0$ and let ω be a root of $\mathcal{Q}_F(x)$. Then, the quadratic form $\mathfrak{C}_{F,\omega}(x, y)$ is proportional over $\mathbb{C}^\times$ to a form with integer coefficients if and only if $\omega \in \mathbb{Z}$.*

Proof. If $\omega \in \mathbb{Z}$, then we easily see from (2-1) that $\lambda \cdot \mathfrak{C}_{F,\omega}(x, y)$ has integer coefficients for some $\lambda \in \mathbb{C}^\times$. Conversely, if $\lambda \cdot \mathfrak{C}_{F,\omega}(x, y)$ has integer coefficients for some $\lambda \in \mathbb{C}^\times$, then consider the action of an element $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, where $\bar{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$. It is clear from the definition of $\mathfrak{C}_{F,\omega}(x, y)$ that $\lambda \in \bar{\mathbb{Q}}$. From (2-1), we have

$$\tfrac{4}{3}(\omega - \sigma(\omega))F(x, y) = \mathfrak{C}_{F,\omega}(x, y)^2 - \sigma(\mathfrak{C}_{F,\omega}(x, y)^2) = \left(1 - \frac{\lambda^2}{\sigma(\lambda)^2}\right)\mathfrak{C}_{F,\omega}(x, y)^2,$$

and this last binary quartic form has zero discriminant. This shows that $\omega - \sigma(\omega) = 0$ for all $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Thus, we have $\omega \in \mathbb{Q}$, and so $\omega \in \mathbb{Z}$ since $\mathcal{Q}_F(x)$ is monic. □

The first claim in Theorem 1.1 now follows from Proposition 2.1, Lemma 2.2, and (2-3). Note that

$$\Delta(F) = 27^2\Delta(\mathcal{Q}_F),$$

which means that $\mathcal{Q}_F(x)$ has three integer roots if and only if $\mathcal{Q}_F(x)$ is reducible and $\Delta(F) = \square$. The second claim then follows from this fact and (2-2).

3. Basic properties of forms in $V_{\mathbb{R},f}$ of nonzero discriminant

Throughout this section, let $f(x, y) = \alpha x^2 + \beta xy + \gamma y^2$ be a real binary quadratic form with $\Delta(f) \neq 0$. It is not hard to check, by a direct calculation, that

$$V_{\mathbb{R},f} = \left\{ Ax^4 + Bx^3y + Cx^2y^2 + \left(\frac{4\beta\gamma A - (\beta^2 + 2\alpha\gamma)B + 2\alpha\beta C}{2\alpha^2} \right) xy^3 + \left(\frac{4\gamma(\beta^2+2\alpha\gamma)A - \beta(\beta^2+4\alpha\gamma)B + 2\alpha\beta^2 C}{8\alpha^3} \right) y^4 : A, B, C \in \mathbb{R} \right\} \tag{3-1}$$

if $\alpha \neq 0$, and similarly that

$$V_{\mathbb{R},f} = \left\{ Ax^4 + \left(\frac{\gamma(4\beta^2 + 8\alpha\gamma)A + 2\alpha\beta^2 B - 8\alpha^3 C}{\beta(\beta^2 + 4\alpha\gamma)} \right) x^3y + Bx^2y^2 - \left(\frac{8\gamma^3 A - 2\beta^2\gamma B - \alpha(4\beta^2+8\alpha\gamma)C}{\beta(\beta^2+4\alpha\gamma)} \right) xy^3 + Cy^4 : A, B, C \in \mathbb{R} \right\} \tag{3-2}$$

if $\beta, \beta^2 + 4\alpha\gamma \neq 0$. Below, we shall give some basic properties of $V^0_{\mathbb{R},f}$ and $V^0_{\mathbb{Z},f}$.

3A. *The two new invariants.* Recall the definitions of the L_f- and K_f-invariants given in (1-4). First, we shall show that they are indeed invariants under the twisted action of $\mathrm{GL}_2(\mathbb{R})$ in the following sense.

Proposition 3.1. *For all $F \in V^0_{\mathbb{R},f}$ and $T \in \mathrm{GL}_2(\mathbb{R})$, we have*

$$L_{f_T}(F_T) = L_f(F) \quad \text{and} \quad K_{f_T}(F_T) = K_f(F).$$

Proof. Notice that $\mathcal{Q}_F(x) = \mathcal{Q}_{F_T}(x)$. For any root ω of $\mathcal{Q}_F(x)$, because $\mathfrak{C}_{F,\omega}(x, y)$ is a covariant up to sign by (2-1), if $\mathfrak{C}_{F,\omega}(x, y)$ is proportional to $f(x, y)$, then $\mathfrak{C}_{F_T,\omega}(x, y)$ is proportional to $f_T(x, y)$. It then follows from the definition that $L_{f_T}(F_T) = L_f(F)$. Since $I(F_T) = I(F)$, we also have $K_{f_T}(F_T) = K_f(F)$ by the first equality in (1-5). □

We shall give explicit formulae for $L_f(-)$ and $K_f(-)$ in two special cases.

Proposition 3.2. *The following holds.*

(a) *Assume that $\alpha \neq 0$. Then, for all $F \in V^0_{\mathbb{R},f}$ as in* (3-1), *we have*

$$L_f(F) = -(12\gamma A - 3\beta B + 2\alpha C)/(2\alpha),$$
$$K_f(F) = (72\beta^2\gamma A^2 + 9\alpha(\beta^2 + 4\alpha\gamma)B^2 + 8\alpha^3C^2 - 18\beta(\beta^2 + 4\alpha\gamma)AB + 12\alpha(3\beta^2 - 4\alpha\gamma)AC - 24\alpha^2\beta BC)/(4\alpha^3).$$

Moreover, we have

$$\frac{4(L_f(F)^2+4K_f(F))}{9}=\frac{L_{f,1}(F)^2-\Delta(f)L_{f,2}(F)^2}{\alpha^4},$$

where

$$L_{f,1}(F)=4(\beta^2-\alpha\gamma)A-3\alpha\beta B+2\alpha^2C \quad \text{and} \quad L_{f,2}(F)=2(2\beta A-\alpha B).$$

(b) *Assume that* $\gamma=0$. *Then, for all* $F\in V^0_{\mathbb{R},f}$ *as in* (3-2), *we have*

$$\begin{aligned}L_f(F)&=(2\beta^2B-12\alpha^2C)/\beta^2,\\ K_f(F)&=(-\beta^4B^2+144\alpha^4C^2+36\beta^4AC-24\alpha^2\beta^2BC)/\beta^4.\end{aligned}$$

Moreover, we have

$$\frac{4(L_f(F)^2+4K_f(F))}{9}=\frac{8C}{\beta^2}\Big(8\beta^2A-8\alpha^2B+\frac{40\alpha^4}{\beta^2}C\Big).$$

Proof. This may be verified by explicit computation. □

We shall also need the following observation.

Proposition 3.3. *Assume that f is integral. Then, for all $F\in V^0_{\mathbb{Z},f}$, we have*

$$L_f(F),\ K_f(F),\ (L_f(F)^2+4K_f(F))/9,\ (2L_f(F)^2-K_f(F))/9\in\mathbb{Z}.$$

Moreover, when f is primitive in addition, we have

$$4(2L_f(F)^2-K_f(F))/(9\Delta(f))\in\mathbb{Z}.$$

Proof. We have $L_f(F)\in\mathbb{Z}$ by Lemma 2.2. Since $I(F)\in\mathbb{Z}$, we deduce from the first equality in (1-5) that $K_f(F)\in\mathbb{Z}$ holds as well. Observe that

$$\begin{aligned}I(F)+K_f(F)&=(L_f(F)^2+4K_f(F))/3,\\ 2I(F)-K_f(F)&=(2L_f(F)^2-K_f(F))/3,\end{aligned}$$

both of which are integers. Since $\Delta(F)\in\mathbb{Z}$, we deduce from (1-7) that at least one of the above expressions is divisible by 3. But again by (1-5), we have

$$3I(F)=(L_f(F)^2+4K_f(F))/3+(2L_f(F)^2-K_f(F))/3,$$

so in fact both expressions are divisible by 3. This proves the first claim.

Next, assume that f is primitive in addition. In view of Proposition 3.1, by applying a $\mathrm{GL}_2(\mathbb{Z})$-action on f if necessary, we may assume that $\alpha\neq 0$ and that α is coprime to $\Delta(f)$. Using Proposition 3.2(a), we then compute that

$$\frac{4(2L_f(F)^2-K_f(F))}{9}=\Delta(f)\Big(\frac{\alpha(B^2-4AC)+2A(\beta B-4\gamma A)}{\alpha^3}\Big).$$

This expression is an integer by the first claim, and hence must be divisible by $\Delta(f)$, because α is taken to be coprime to $\Delta(f)$. This proves the second claim. $\square$

3B. *Determinants of the two lattices.* In this subsection, assume that f is integral and primitive. Let $\Lambda_{f,1}$ and $\Lambda_{f,2}$ denote the lattices defined in (1-10). Below, we shall compute their determinants in terms of the number s_f as in Theorem 1.2.

Proposition 3.4. *We have* $\det(\Lambda_{f,1}) = s_f|\alpha|^3$ *and* $\det(\Lambda_{f,2}) = s_f|\beta(\beta^2+4\alpha\gamma)|/8$.

Proof. Observe that the linear transformation defined by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ * & -\mathcal{B} & * \end{pmatrix}, \quad \text{where } \mathcal{B} = \frac{\beta(\beta^2+4\alpha\gamma)}{8\alpha^3},$$

has determinant $\mathcal{B}$, and it sends $\Lambda_{f,1}$ to $\Lambda_{f,2}$. Thus, it suffices to prove the first claim. Recall from (3-1) that $\Lambda_{f,1}$ is the set of tuples $(A, B, C) \in \mathbb{Z}^3$ satisfying

$$4\beta\gamma A - (\beta^2+2\alpha\gamma)B + 2\alpha\beta C \equiv 0 \pmod{2\alpha^2},$$
$$4\gamma(\beta^2+2\alpha\gamma)A - \beta(\beta^2+4\alpha\gamma)B + 2\alpha\beta^2 C \equiv 0 \pmod{8\alpha^3}.$$

If $\beta\gamma = 0$, then it is easy to check that $\det(\Lambda_{f,1}) = s_f|\alpha|^3$. If $\beta\gamma \neq 0$, then we shall use the fact that

$$\det(\Lambda_{f,1}) = \prod_p \det(\Lambda_{f,1}^{(p)}) = \prod_{p \mid 2\alpha} \det(\Lambda_{f,1}^{(p)}), \quad \text{where } \Lambda_{f,1}^{(p)} = \mathbb{Z}_p \otimes_{\mathbb{Z}} \Lambda_{f,1},$$

and so $\det(\Lambda_{f,1}) = s_f|\alpha|^3$ indeed holds by Lemma 3.5 below. $\square$

Lemma 3.5. *Let p be a prime dividing 2α and let $p^k \| \alpha$. Then, we have*

$$\det(\Lambda_{f,1}^{(p)}) = s_f^{\epsilon_p} p^{3k}, \quad \textit{where } \epsilon_p = \begin{cases} 1 & \textit{if } p = 2, \\ 0 & \textit{if } p \geq 3. \end{cases}$$

Proof. For brevity, write

$$\alpha = p^k a \quad \text{and} \quad \beta = p^\ell b, \quad \text{where } k, \ell, a, b \in \mathbb{Z} \text{ with } k, \ell \geq 0 \text{ and } p \nmid a, b.$$

Then, the claim may be restated as

$$\det(\Lambda_{f,1}^{(p)}) = \begin{cases} p^{3k+3\epsilon_p} & \text{if } \ell = 0, \\ p^{3k} & \text{if } \ell \geq 1. \end{cases}$$

By definition, the lattice $\Lambda_{f,1}^{(p)}$ is the set $(A, B, C) \in \mathbb{Z}_p^3$ of tuples satisfying

$$\mathcal{T}_1(A, B, C) \equiv 0 \pmod{p^{2k+\epsilon_p}} \quad \text{and} \quad \mathcal{T}_2(A, B, C) \equiv 0 \pmod{p^{3k+3\epsilon_p}},$$

where

$$\mathcal{T}_1(A, B, C) = p^\ell b(4\gamma A - p^\ell bB) - 2p^k a\gamma B + 2p^{k+\ell} abC,$$
$$\mathcal{T}_2(A, B, C) = (p^{2\ell}b^2 + 4p^k a\gamma)(4\gamma A - p^\ell bB) - 8p^k a\gamma^2 A + 2p^{k+2\ell} ab^2 C.$$

Observe that we have the relation

$$\mathcal{T}_2(A, B, C) - p^\ell b \mathcal{T}_1(A, B, C) = 2p^k a\gamma(4\gamma A - p^\ell bB). \tag{3-3}$$

For $\ell = 0$, we deduce from (3-3) that $\Lambda_{f,1}^{(p)}$ is defined solely by

$$\mathcal{T}_2(A, B, C) \equiv 0 \pmod{p^{3k+3\epsilon_p}}.$$

For $\ell \geq 1$ and $\ell \geq k + 2\epsilon_p$, it is easy to see that $\Lambda_{f,1}^{(p)}$ is in fact defined by

$$A \equiv 0 \pmod{p^{2k}} \quad \text{and} \quad B \equiv 0 \pmod{p^k}.$$

For $\ell \geq 1$ and $\ell \leq k + \epsilon_p$, we shall first show that $\Lambda_{f,1}^{(p)}$ is also defined by

$$\begin{cases} A \equiv 0 & \pmod{p^{2\ell-2\epsilon_p}}, \\ B \equiv 0 & \pmod{p^{\ell-\epsilon_p}}, \\ (4\gamma A - p^\ell bB)/p^{2\ell-\epsilon_p} \equiv 0 & \pmod{p^{k-\ell+\epsilon_p}}, \\ \mathcal{T}_2(A, B, C)/p^{k+2\ell+\epsilon_p} \equiv 0 & \pmod{p^{2k-2\ell+2\epsilon_p}}. \end{cases} \tag{3-4}$$

If (3-4) is satisfied, then from (3-3), it is easy to see that $(A, B, C) \in \Lambda_{f,1}^{(p)}$. Conversely, if $(A, B, C) \in \Lambda_{f,1}^{(p)}$, then the assumption $\ell \leq k + \epsilon_p$ implies that

$$\mathcal{T}_1(A, B, C) \equiv 0 \pmod{p^{k+\ell}} \quad \text{and} \quad \mathcal{T}_2(A, B, C) \equiv 0 \pmod{p^{k+2\ell+\epsilon_p}},$$

while reducing (3-3) mod $p^{2k+\ell+\epsilon_p}$ also yields

$$4\gamma A - p^\ell bB \equiv 0 \pmod{p^{k+\ell}}.$$

From these three congruence equations, it follows that (3-4) is indeed satisfied. In all cases, we then see that $\det(\Lambda_{f,1}^{(p)})$ is as claimed. $\square$

3C. ***Forms with abelian Galois groups.*** In this subsection, assume that f is integral. Consider an irreducible form $F \in V_{\mathbb{Z},f}^0$. By Theorem 1.1, we have $\mathrm{Gal}(F) \simeq D_4$, C_4, or V_4. To distinguish among these three possibilities, note that the *cubic resolvent polynomial of* F, defined by

$$R_F(x) = a_4^3 X^3 - a_4^2 a_2 X^2 + a_4(a_3 a_1 - 4a_4 a_0)X - (a_3^2 a_0 + a_4 a_1^2 - 4a_4 a_2 a_0)$$

when F has the shape (1-1), is reducible since $\mathrm{Gal}(F)$ is small. Also, it has a unique root $r_F \in \mathbb{Q}$ precisely when $\Delta(F) \neq \square$, in which case we define

$$\theta_1(F) = (a_3^2 - 4a_4(a_2 - r_F a_4))\Delta(F) \quad \text{and} \quad \theta_2(F) = a_4(r_F^2 a_4 - 4a_0)\Delta(F).$$

Then, we have the well-known criterion

$$\mathrm{Gal}(F) \simeq V_4 \iff \Delta(F) = \square,$$

$$\mathrm{Gal}(F) \simeq C_4 \iff \Delta(F) \neq \square \text{ and } \theta_1(F), \theta_2(F) = \square \text{ in } \mathbb{Q}.$$

See [Conrad 2012], for example. We then deduce:

Proposition 3.6. *Let $F \in V^0_{\mathbb{Z},f}$ be an irreducible form. Then, we have*

$$\mathrm{Gal}(F) \simeq V_4 \iff L_f(F)^2 + 4K_f(F) = \square,$$

as well as

$$\mathrm{Gal}(F) \simeq C_4 \iff \begin{cases} L_f(F)^2 + 4K_f(F) \neq \square, \\ (L_f(F)^2 + 4K_f(F))(2L_f(F)^2 - K_f(F))/\Delta(f) = \square. \end{cases}$$

Proof. Observe that by (1-7), we have

$$\Delta(F) = \square \quad \text{if and only if} \quad L_f(F)^2 + 4K_f(F) = \square.$$

The first claim is then clear. Next, suppose that $\Delta(F) \neq \square$. By Proposition 3.1, we may assume that $\alpha \neq 0$. For F in the shape as in (3-1), a direct computation yields

$$r_F = (-4\gamma A + \beta B)/(2\alpha A).$$

Using Proposition 3.2 (a), we further compute that

$$\theta_1(F) = 4\alpha^2(2L_f(F)^2 - K_f(F))\Delta(F)/(9\Delta(f)),$$
$$\theta_2(F) = \beta^2(2L_f(F)^2 - K_f(F))\Delta(F)/(9\Delta(f)).$$

By (1-7) and the criterion above, it follows that $\theta_1(F), \theta_2(F)$ are squares if and only if $(L_f(F)^2 + 4K_f(F))(2L_f(F)^2 - K_f(F))/\Delta(f)$ is a square, as desired. $\square$

3D. *Reducible forms.* In this subsection, assume that f is integral. We shall study the reducible forms in $V^0_{\mathbb{Z},f}$. Let us first make a definition and an observation.

Definition 3.7. Let $F \in V^0_{\mathbb{Z},f}$ be a reducible form.

(1) We say that F is of *type* 1 if $F = m \cdot pp_{M_f}$ for some $m \in \mathbb{Q}^\times$ and integral binary quadratic form p.

(2) We say that F is of *type* 2 if $F = pq$ for some integral binary quadratic forms p and q satisfying $p_{M_f} = -p$ and $q_{M_f} = -q$.

Lemma 3.8. *For all reducible forms $F \in V^0_{\mathbb{Z},f}$ of type* 1, *we have*

$$L_f(F)^2 + 4K_f(F) = \square.$$

Proof. This may be verified by a direct computation. $\square$

Below, we shall show that the two reducibility types in Definition 3.7 are in fact the only possibilities. We shall require two further lemmas.

Lemma 3.9. *Let $\ell(x, y) = \ell_1 x + \ell_0 y$ be a nonzero complex binary linear form, and suppose that $\ell_{M_f} = \lambda \cdot \ell$ for some $\lambda \in \mathbb{C}^\times$. Then, we have $\lambda = \pm\sqrt{-1}$, with*

$$\lambda = \begin{cases} -\sqrt{-1} & \text{if and only if } \ell_0 = (\beta + \sqrt{\Delta(f)})\ell_1/(2\alpha), \\ \sqrt{-1} & \text{if and only if } \ell_0 = (\beta - \sqrt{\Delta(f)})\ell_1/(2\alpha), \end{cases}$$

in the case that $\alpha \neq 0$.

Proof. The hypothesis implies that

$$\frac{1}{\sqrt{-\Delta(f)}} \begin{pmatrix} \beta & -2\alpha \\ 2\gamma & -\beta \end{pmatrix} \begin{pmatrix} \ell_1 \\ \ell_0 \end{pmatrix} = \lambda \begin{pmatrix} \ell_1 \\ \ell_0 \end{pmatrix}.$$

Then, by computing the eigenvalues and eigenspaces of the 2×2 matrix above, we see that the claim holds. □

Lemma 3.10. *Let $p(x, y) = p_2 x^2 + p_1 xy + p_0 y^2$ be a nonzero complex binary quadratic form, and suppose that $p_{M_f} = \lambda \cdot p$ for some $\lambda \in \mathbb{C}^\times$. Then, we have $\lambda = \pm 1$, with*

$$\lambda = \begin{cases} -1 & \text{if and only if } p_0 = (\beta p_1 - 2\gamma p_2)/(2\alpha), \\ 1 & \text{if and only if } p = (p_2/\alpha) f, \end{cases}$$

in the case that $\alpha \neq 0$.

Proof. The hypothesis implies that

$$\frac{1}{-\Delta(f)} \begin{pmatrix} \beta^2 & -2\alpha & 4\alpha^2 \\ 4\beta\gamma & -(\beta^2 + 4\alpha\gamma) & 4\alpha\beta \\ 4\gamma^2 & -2\beta\gamma & \beta^2 \end{pmatrix} \begin{pmatrix} p_2 \\ p_1 \\ p_0 \end{pmatrix} = \lambda \begin{pmatrix} p_2 \\ p_1 \\ p_0 \end{pmatrix}.$$

Then, by computing the eigenvalues and eigenspaces of the 3×3 matrix above, it is not hard to check that the claim holds. □

Proposition 3.11. *Any reducible form $F \in V^0_{\mathbb{Z},f}$ is either of type 1 or of type 2.*

Proof. Write $F = g^{(1)} g^{(2)} g^{(3)} g^{(4)}$, where the $g^{(k)}$ are complex binary linear forms, and are pairwise nonproportional because $\Delta(F) \neq 0$. Since F is reducible, by renumbering if necessary, we may assume that

$$\begin{cases} g^{(1)},\, g^{(2)} g^{(3)} g^{(4)} & \text{when } F \text{ has exactly one rational linear factor}, \\ g^{(1)},\, g^{(2)},\, g^{(3)} g^{(4)} & \text{when } F \text{ has exactly two rational linear factors}, \\ g^{(1)} g^{(2)},\, g^{(3)} g^{(4)} & \text{when } F \text{ has no rational linear factor}, \\ g^{(1)},\, g^{(2)},\, g^{(3)},\, g^{(4)} & \text{when } F \text{ has four rational linear factors}, \end{cases}$$

have integer coefficients and are irreducible. We have $M_f^2 = \Delta(f) \cdot I_{2\times 2}$ and $F_{M_f} = F$ by definition. Hence, up to scaling, the matrix M_f acts on the $g^{(k)}$ via a permutation σ on four letters of order dividing two. This has two consequences.

By (1-8), without loss of generality, we may assume that $\alpha \neq 0$. First, the form F cannot have exactly one rational linear factor, for otherwise

$$\sigma(1) = 1 \quad \text{and} \quad \sigma(k_0) = k_0 \text{ for at least one } k_0 \in \{2, 3, 4\}.$$

From Lemma 3.9, it would follow that $\Delta(f)$ is a square and that $g^{(k_0)}$ is proportional to a form with integer coefficients, which is a contradiction. Second, when F has four rational linear factors, by further renumbering if necessary, we may assume that

$$\sigma \in \{(1), (12), (12)(34)\}.$$

Now, in all three of the possible cases for the factorization of F, define

$$p = g^{(1)} g^{(2)} \quad \text{and} \quad q = g^{(3)} g^{(4)},$$

which are integral binary quadratic forms by definition. We then deduce that

$$(p_{M_f}, q_{M_f}) = (\lambda \cdot q, \lambda^{-1} \cdot p) \quad \text{or} \quad (p_{M_f}, q_{M_f}) = (\lambda \cdot p, \lambda^{-1} \cdot q)$$

for some $\lambda \in \mathbb{Q}^\times$. In the former case, it is clear that F is of type 1. In the latter case, we have $\lambda = -1$ by Lemma 3.10 and the fact that $\Delta(F) \neq 0$, so F is of type 2. $\square$

4. Parametrizing forms in $V_{\mathbb{R},f}$ of nonzero discriminant

Throughout this section, let $f(x, y) = \alpha x^2 + \beta xy + \gamma y^2$ be a real binary quadratic form with $\Delta(f) \neq 0$ and $\alpha > 0$. We shall give an alternative parametrization of $V^0_{\mathbb{R},f}$, different from (3-1) and (3-2), in terms of the regions

$$\begin{cases} \Omega^0 = \{(L, K) \in \mathbb{R}^2 : L^2 + 4K \neq 0 \text{ and } 2L^2 - K \neq 0\}, \\ \Omega^+ = \{(L, K) \in \mathbb{R}^2 : L^2 + 4K > 0 \text{ and } 2L^2 - K \neq 0\}, \\ \Omega^- = \{(L, K) \in \mathbb{R}^2 : L^2 + 4K < 0 \text{ and } 2L^2 - K > 0\}, \end{cases} \tag{4-1}$$

corresponding to the L_f- and K_f-invariants, as well as a parameter $t \in \mathbb{R}$ arising from the *orthogonal group of* f, defined by

$$O_f(\mathbb{R}) = \{T \in \mathrm{GL}_2(\mathbb{R}) : \det(T) = \pm 1 \text{ and } f_T = \pm f\}.$$

Note that by (1-7), for any $F \in V^0_{\mathbb{R},f}$, we have

$$(L_f(F), K_f(F)) \in \Omega^+ \iff \Delta(F) > 0,$$
$$(L_f(F), K_f(F)) \in \Omega^- \iff \Delta(F) < 0.$$

First, we shall show that it suffices to consider $x^2 + y^2$ and $x^2 - y^2$. It shall be helpful to recall (1-8) as well as the isomorphisms Θ_1 and Θ_2 defined in Section 1A.

Lemma 4.1. *Define a matrix*

$$T_f = \begin{pmatrix} \delta_f^{-1/4} & 0 \\ 0 & \delta_f^{1/4} \end{pmatrix} \cdot \frac{1}{2\sqrt{\alpha}} \begin{pmatrix} 2\alpha & \beta \\ 0 & 2 \end{pmatrix}, \quad \text{where } \delta_f = \frac{|\Delta(f)|}{4}.$$

Then, we have a well-defined bijective linear map

$$\begin{cases} \Psi_f : V_{\mathbb{R},x^2+y^2} \to V_{\mathbb{R},f}, & \Psi_f(F) = F_{T_f} \quad \text{if } f \text{ is positive definite}, \\ \Psi_f : V_{\mathbb{R},x^2-y^2} \to V_{\mathbb{R},f}, & \Psi_f(F) = F_{T_f} \quad \text{if } f \text{ is indefinite}, \end{cases}$$

and we have $\det(\Psi_f) = 8\alpha^3|\Delta(f)|^{-3/2}$.

Proof. The first claim holds by (1-8) and the fact

$$\delta_f^{-1/2} \cdot f = \begin{cases} (x^2+y^2)_{T_f} & \text{if } f \text{ is positive definite}, \\ (x^2-y^2)_{T_f} & \text{if } f \text{ is indefinite}. \end{cases}$$

Identifying $V_{\mathbb{R},x^2\pm y^2}$ and $V_{\mathbb{R},f}$ with $\mathbb{R}^3$ via Θ_1, we see from (3-1) that

$$\Psi_f : \begin{pmatrix} a_4 \\ a_3 \\ a_2 \end{pmatrix} \mapsto \begin{pmatrix} \alpha^2/\delta_f & 0 & 0 \\ 2\alpha\beta/\delta_f & \alpha/\sqrt{\delta_f} & 0 \\ 3\beta^2/2\delta_f & 3\beta/(2\sqrt{\delta_f}) & 1 \end{pmatrix} \begin{pmatrix} a_4 \\ a_3 \\ a_2 \end{pmatrix}, \tag{4-2}$$

from which the second claim follows. □

In the subsequent subsections, we shall prove the following propositions.

Proposition 4.2. *There exists an explicit bijection*

$$\Phi : \Omega^+ \times [-\pi/4, \pi/4) \to V^0_{\mathbb{R},x^2+y^2},$$

defined as in (4-4), *such that*

(a) *we have* $L_{x^2+y^2}(\Phi(L, K, t)) = L$ *and* $K_{x^2+y^2}(\Phi(L, K, t)) = K$,

(b) *the Jacobian matrix of* $\Theta_1 \circ \Phi$ *has determinant* $-\frac{1}{18}$.

Proposition 4.3. *There exist explicit injections*

$$\Phi^{(1)}, \Phi^{(2)} : \Omega^+ \times \mathbb{R} \to V^0_{\mathbb{R},x^2-y^2} \quad \text{and} \quad \Phi^{(3)}, \Phi^{(4)} : \Omega^- \times \mathbb{R} \to V^0_{\mathbb{R},x^2-y^2},$$

defined as in (4-6), *with*

$$V^0_{\mathbb{R},x^2-y^2} = \Phi^{(1)}(\Omega^+ \times \mathbb{R}) \sqcup \Phi^{(2)}(\Omega^+ \times \mathbb{R}) \sqcup \Phi^{(3)}(\Omega^- \times \mathbb{R}) \sqcup \Phi^{(4)}(\Omega^- \times \mathbb{R})$$

such that, for all $i = 1, 2, 3, 4$,

(a) *we have* $L_{x^2-y^2}(\Phi^{(i)}(L, K, t)) = L$ *and* $K_{x^2-y^2}(\Phi^{(i)}(L, K, t)) = K$,

(b) *the Jacobian matrix of* $\Theta_1 \circ \Phi^{(i)}$ *has determinant* $-\frac{1}{18}$.

In view of (1-11), we shall give another parametrization of $V_{\mathbb{R},f}$ when $\gamma = 0$, which does not require reducing to the form $x^2 - y^2$ via Lemma 4.1.

Proposition 4.4. *Suppose that $\gamma = 0$. Then, there exist explicit injections*

$$\Phi_f^{(1)}, \Phi_f^{(2)} : \Omega^0 \times \mathbb{R} \to V_{\mathbb{R},f}^0,$$

defined as in (4-9), *with*

$$V_{\mathbb{R},f}^0 = \Phi_f^{(1)}(\Omega^0 \times \mathbb{R}) \sqcup \Phi_f^{(2)}(\Omega^0 \times \mathbb{R})$$

such that, for both $i = 1, 2$,

(a) *we have $L_f(\Phi^{(i)}(L, K, t)) = L$ and $K_f(\Phi^{(i)}(L, K, t)) = K$,*

(b) *the Jacobian matrix of $\Theta_2 \circ \Phi_f^{(i)}$ has determinant $-\frac{1}{18}$.*

For $t \in \mathbb{R}$, we shall use the notation

$$T^+(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \quad \text{and} \quad T^-(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, \tag{4-3}$$

which is an element of $O_{x^2+y^2}(\mathbb{R})$ and $O_{x^2-y^2}(\mathbb{R})$, respectively.

4A. *Positive definite case.* Define

$$\Phi : \Omega^+ \times [-\pi/4, \pi/4) \to V_{\mathbb{R},x^2+y^2}^0, \quad \Phi(L, K, t) = (F_{(L,K)})_{T^+(t)}, \tag{4-4}$$

where

$$F_{(L,K)}(x, y) = \frac{-3L + \sqrt{L^2+4K}}{24}x^4 + \frac{-L - \sqrt{L^2+4K}}{4}x^2y^2 + \frac{-3L + \sqrt{L^2+4K}}{24}y^4.$$

The image of Φ lies in $V_{\mathbb{R},x^2+y^2}$ by (3-1) and (1-8). Using Propositions 3.1 and 3.2(a), it is easy to check that Proposition 4.2(a) holds.

Now, by (3-1), an arbitrary $F \in V_{\mathbb{R},x^2+y^2}^0$ has the shape

$$F(x, y) = a_4x^4 + a_3x^3y + a_2x^2y^2 - a_3xy^3 + a_4y^4.$$

Write $L = L_{x^2+y^2}(F)$ and $K = K_{x^2+y^2}(F)$. Note that $(L, K) \in \Omega^+$ because $\Delta(F) > 0$ by (1-7). For $t \in \mathbb{R}$, a direct computation yields

$$F_{T^+(t)}(x, y) = A(t)x^4 + B(t)x^3y + C(t)x^2y^2 - B(t)xy^3 + A(t)y^4,$$

where

$$\begin{cases} A(t) = \dfrac{6a_4+a_2}{8} + \dfrac{2a_4-a_2}{8}\cos(4t) - \dfrac{a_3}{4}\sin(4t), \\ B(t) = a_3\cos(4t) + \dfrac{2a_4-a_2}{2}\sin(4t), \\ C(t) = \dfrac{6a_4+a_2}{4} - \dfrac{3(2a_4-a_2)}{4}\cos(4t) + \dfrac{3a_3}{2}\sin(4t). \end{cases}$$

It is not hard to show that there exists a unique $t_0 \in (-\pi/4, \pi/4]$ such that $B(t_0) = 0$ and $2A(t_0) - C(t_0) > 0$. Put $(A, C) = (A(t_0), C(t_0))$. Then, we have

$$(L, K) = \big(L_{x^2+y^2}(F_{T^+(t_0)}), K_{x^2+y^2}(F_{T^+(t_0)})\big) = (-6A - C, -2C(6A - C))$$

by Propositions 3.1 and 3.2(a). We solve that $F_{T^+(t_0)} = F_{(L,K)}$, or equivalently

$$F = (F_{(L,K)})_{T^+(-t_0)} = \Phi(L, K, -t_0).$$

Since $-t_0 \in [-\pi/4, \pi/4)$ is uniquely determined by F, this shows that Φ is a bijection.

Finally, the above calculation also yields

$$(\Theta_1 \circ \Phi)(L, K, t) = \big(\Phi_1(L, K, t), \Phi_2(L, K, t), \Phi_3(L, K, t)\big),$$

where

$$\begin{cases} \Phi_1(L, K, t) = -\dfrac{L}{8} + \dfrac{\sqrt{L^2+4K}}{24}\cos(4t), \\ \Phi_2(L, K, t) = \dfrac{\sqrt{L^2+4K}}{6}\sin(4t), \\ \Phi_3(L, K, t) = -\dfrac{L}{4} - \dfrac{\sqrt{L^2+4K}}{4}\cos(4t). \end{cases} \tag{4-5}$$

By a direct computation, we then see that Proposition 4.2(b) holds.

4B. *Indefinite case.* Define

$$\begin{cases} \Phi^{(i)} : \Omega^+ \times \mathbb{R} \to V^0_{\mathbb{R},x^2-y^2}, & \Phi^{(i)}(L, K, t) = (F^{(i)}_{(L,K)})_{T^-(t)} \quad \text{for } i = 1, 2, \\ \Phi^{(i)} : \Omega^- \times \mathbb{R} \to V^0_{\mathbb{R},x^2-y^2}, & \Phi^{(i)}(L, K, t) = (F^{(i)}_{(L,K)})_{T^-(t)} \quad \text{for } i = 3, 4, \end{cases} \tag{4-6}$$

where

$$F^{(i)}_{(L,K)}(x, y) = \frac{3L + (-1)^i\sqrt{L^2+4K}}{24}x^4 + \frac{-L + (-1)^i\sqrt{L^2+4K}}{4}x^2y^2 + \frac{3L + (-1)^i\sqrt{L^2+4K}}{24}y^4$$

for $i = 1, 2$, and

$$F^{(i)}_{(L,K)}(x, y) = \frac{(-1)^i\sqrt{2L^2-K}}{3}x^3y - Lx^2y^2 + \frac{(-1)^i\sqrt{2L^2-K}}{3}xy^3$$

for $i = 3, 4$. The images of $\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}, \Phi^{(4)}$ lie in $V_{\mathbb{R},x^2-y^2}$ by (3-1) and (1-8). Using Propositions 3.1 and 3.2(a), it is easy to check that Proposition 4.3(a) holds.

Now, by (3-1), an arbitrary $F \in V^0_{\mathbb{R},x^2-y^2}$ has the shape

$$F(x, y) = a_4x^4 + a_3x^3y + a_2x^2y^2 + a_3xy^3 + a_4y^4.$$

Write $L = L_{x^2-y^2}(F)$ and $K = K_{x^2-y^2}(F)$. For $t \in \mathbb{R}$, a direct computation yields

$$F_{T^-(t)}(x, y) = A(t)x^4 + B(t)x^3y + C(t)x^2y^2 + B(t)xy^3 + A(t)y^4,$$

where

$$\begin{cases} A(t) = \dfrac{6a_4 - a_2}{8} + \dfrac{2a_4 + a_2}{8}\cosh(4t) + \dfrac{a_3}{4}\sinh(4t), \\ B(t) = a_3\cosh(4t) + \dfrac{2a_4 + a_2}{2}\sinh(4t), \\ C(t) = -\dfrac{6a_4 - a_2}{4} + \dfrac{3(2a_4 + a_2)}{4}\cosh(4t) + \dfrac{3a_3}{2}\sinh(4t). \end{cases}$$

Note that $\frac{d}{dt}A(t) = \frac{1}{2}B(t)$. It is not hard to check that:

- If $\Delta(F) > 0$, then there is a unique $t_0 \in \mathbb{R}$ such that $B(t_0) = 0$.
- If $\Delta(F) < 0$, then $B(t) \neq 0$ for all $t \in \mathbb{R}$, and there is a unique $t_0 \in \mathbb{R}$ such that $A(t_0) = 0$.

Put $(A, B, C) = (A(t_0), B(t_0), C(t_0))$. Then, we have

$$\begin{aligned}(L, K) &= (L_{x^2-y^2}(F_{T^-(t_0)}), K_{x^2-y^2}(F_{T^-(t_0)})) \\ &= \begin{cases} (6A - C, 2C(6A + C)) & \text{if } \Delta(F) > 0, \\ (-C, -9B^2 + 2C^2) & \text{if } \Delta(F) < 0. \end{cases}\end{aligned}$$

by Propositions 3.1 and 3.2(a). We solve that $F_{T^-(t_0)} = F^{(i)}_{(L,K)}$, or equivalently

$$F = (F^{(i)}_{(L,K)})_{T^-(-t_0)} = \Phi^{(i)}(L, K, -t_0), \quad \text{for exactly one } i \in \{1, 2, 3, 4\}.$$

Since t_0 is uniquely determined by F, this shows that $\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}, \Phi^{(4)}$ are all injections, and that the stated disjoint union holds.

Finally, the above calculation also yields

$$(\Theta_1 \circ \Phi^{(i)})(L, K, t) = \big(\Phi^{(i)}_1(L, K, t), \Phi^{(i)}_2(L, K, t), \Phi^{(i)}_3(L, K, t)\big),$$

where

$$\begin{cases} \Phi^{(i)}_1(L, K, t) = \dfrac{L}{8} + \dfrac{(-1)^i\sqrt{L^2+4K}}{24}\cosh(4t), \\ \Phi^{(i)}_2(L, K, t) = \dfrac{(-1)^i\sqrt{L^2+4K}}{6}\sinh(4t), \\ \Phi^{(i)}_3(L, K, t) = -\dfrac{L}{4} + \dfrac{(-1)^i\sqrt{L^2+4K}}{4}\cosh(4t), \end{cases} \tag{4-7}$$

for $i = 1, 2$, and

$$
\text{(4-8)} \quad \begin{cases} \Phi_1^{(i)}(L, K, t) = \dfrac{L}{8} - \dfrac{L}{8}\cosh(4t) + \dfrac{(-1)^i\sqrt{2L^2-K}}{12}\sinh(4t), \\ \Phi_2^{(i)}(L, K, t) = \dfrac{(-1)^i\sqrt{2L^2-K}}{3}\cosh(4t) - \dfrac{L}{2}\sinh(4t), \\ \Phi_3^{(i)}(L, K, t) = -\dfrac{L}{4} - \dfrac{3L}{4}\cosh(4t) + \dfrac{(-1)^i\sqrt{2L^2-K}}{2}\sinh(4t), \end{cases}
$$

for $i = 3, 4$. By a direct computation, we then see that Proposition 4.3(b) holds.

4C. ***Reducible case.*** Suppose $\gamma = 0$. For $t \in \mathbb{R}$, put

$$
T(t) = \begin{pmatrix} e^{-t} & 0 \\ \dfrac{2\alpha \sinh t}{\beta} & e^t \end{pmatrix},
$$

which is an element of $O_f(\mathbb{R})$. Define

$$
\text{(4-9)} \quad \Phi_f^{(i)} : \Omega^0 \times \mathbb{R} \to V^0_{\mathbb{R},f}, \quad \Phi_f^{(i)}(L, K, t) = (F^{(i)}_{f,(L,K)})_{T(t)} \quad \text{for } i = 1, 2,
$$

where

$$
F^{(i)}_{f,(L,K)}(x, y) = \left(\frac{L^2 + (-1)^i 72\alpha^2 L + 4K + 144\alpha^4}{(-1)^i 144\beta^2}\right)x^4 + \left(\frac{\alpha L + (-1)^i 4\alpha^3}{\beta}\right)x^3 y + \left(\frac{L + (-1)^i 12\alpha^2}{2}\right)x^2y^2 + (-1)^i 4\alpha\beta xy^3 + (-1)^i\beta^2 y^4.
$$

The images of $\Phi_f^{(1)}$, $\Phi_f^{(2)}$ lie in $V_{\mathbb{R},f}$ by (3-2) and (1-8). Using Propositions 3.1 and 3.2(b), it is easy to check that Proposition 4.4(a) holds.

Now, by (3-2), an arbitrary $F \in V^0_{\mathbb{R},f}$ has the shape

$$
\text{(4-10)} \quad F(x, y) = a_4x^4 + \left(\frac{2\alpha(\beta^2 a_2 - 4\alpha^2 a_0)}{\beta^3}\right)x^3y + a_2x^2y^2 + \left(\frac{4\alpha a_0}{\beta}\right)xy^3 + a_0y^4.
$$

Write $L = L_f(F)$ and $K = K_f(F)$. For $t \in \mathbb{R}$, a direct computation yields

$$
F_{T(t)}(x, y) = A(t)x^4 + (*)x^3y + B(t)x^2y^2 + (*)xy^3 + C(t)y^4,
$$

where

$$
\begin{cases} A(t) = e^{-4t}a_4 + \dfrac{\alpha^2}{\beta^2}(e^{4t} - 1)e^{-4t}a_2 + \dfrac{\alpha^4}{\beta^4}(e^{4t} - 1)(e^{4t} - 5)e^{-4t}a_0, \\ B(t) = a_2 + \dfrac{6\alpha^2}{\beta^2}(e^{4t} - 1)a_0, \\ C(t) = e^{4t}a_0. \end{cases}
$$

Since $\Delta(F)\neq 0$, we have $(-1)^i a_0>0$ for a unique $i\in\{1,2\}$, and there is a unique $t_0\in\mathbb{R}$ such that $C(t_0)=(-1)^i\beta^2$. Put $(A,B)=(A(t_0),B(t_0))$. Then, we have

$$\begin{aligned}(L,K)&=(L_f(F_{T(t_0)}),K_f(F_{T(t_0)}))\\&=\bigl(2B-(-1)^i 12\alpha^2,\,-B^2+(-1)^i 36\beta^2A-(-1)^i 24\alpha^2B+144\alpha^4\bigr),\end{aligned}$$

by Propositions 3.1 and 3.2(b). We solve that $F_{T(t_0)}=F^{(i)}_{f,(L,K)}$, or equivalently

$$F=(F^{(i)}_{f,(L,K)})_{T(-t_0)}=\Phi^{(i)}_f(L,K,-t_0).$$

Since t_0 and i are uniquely determined by F, this shows that $\Phi^{(1)}_f$ and $\Phi^{(2)}_f$ are both injections, and that the stated disjoint union holds.

Finally, the above calculation also yields

$$(\Theta_2\circ\Phi^{(i)}_f)(L,K,t)=\bigl(\Phi^{(i)}_{f,1}(L,K,t),\,\Phi^{(i)}_{f,2}(L,K,t),\,\Phi^{(i)}_{f,3}(L,K,t)\bigr),$$

where

$$\begin{cases}\Phi^{(i)}_{f,1}(L,K,t)=\dfrac{(-1)^ie^{-4t}}{144\beta^2}(L^2+4K)+\dfrac{\alpha^2}{2\beta^2}L+\dfrac{(-1)^i\alpha^4e^{4t}}{\beta^2},\\ \Phi^{(i)}_{f,2}(L,K,t)=\dfrac{L}{2}+(-1)^i6\alpha^2e^{4t},\\ \Phi^{(i)}_{f,3}(L,K,t)=(-1)^i\beta^2e^{4t}.\end{cases}\tag{4-11}$$

By a direct computation, we then see that Proposition 4.4(b) holds.

5. Definition of a bounded semialgebraic set

Throughout this section, let $f(x,y)=\alpha x^2+\beta xy+\gamma y^2$ be an integral and primitive binary quadratic form with $\Delta(f)\neq 0$ and $\alpha>0$, in the shape (1-11) whenever f is reducible. As we have already explained in Section 1A, the proof of Theorem 1.2 is reduced to counting points in the lattices in (1-10), which in turn amounts to certain volume computations, by the result below.

Proposition 5.1 (Davenport's lemma). *Let $\mathcal{R}$ be a bounded semialgebraic multiset in $\mathbb{R}^n$ having maximum multiplicity m and which is defined by at most k polynomial inequalities, each having degree at most ℓ. Then, the number of integral lattice points (counted with multiplicity) contained in the region $\mathcal{R}$ is*

$$\mathrm{Vol}(\mathcal{R})+O(\max\{\mathrm{Vol}(\bar{\mathcal{R}}),1\}),$$

where $\mathrm{Vol}(\bar{\mathcal{R}})$ denotes the greatest d-dimensional volume of any projection of $\mathcal{R}$ onto a coordinate subspace by equating $n-d$ coordinates to zero, with $1\le d\le n-1$. The implied constant in the second summand depends only on n,m,k,ℓ.

Proof. This is a result of Davenport [1951a], and the above formulation is due to Bhargava and Shankar [2015, Proposition 2.6]. □

For $X > 0$, define

$$V^0_{\mathbb{R},f}(X) = \{F \in V^0_{\mathbb{R},f} : H_f(F) \le X\} \quad \text{and} \quad V^0_{\mathbb{Z},f}(X) = \{F \in V^0_{\mathbb{Z},f} : H_f(F) \le X\}.$$

However, to prove Theorem 1.2, we cannot apply Proposition 5.1 directly to

$$\Theta_{w(f)}(V^0_{\mathbb{R},f}(X)), \quad \text{where } w(f) = \begin{cases} 1 & \text{if } f \text{ is irreducible,} \\ 2 & \text{if } f \text{ is reducible,} \end{cases}$$

as in Section 1A, to count the lattice points in $\Theta_{w(f)}(V^0_{\mathbb{Z},f}(X)) \subset \Lambda_{f,w(f)}$ because

(1) the set $\Theta_{w(f)}(V^0_{\mathbb{R},f}(X))$ is unbounded when f is indefinite,

(2) distinct forms in $V^0_{\mathbb{Z},f}(X)$ might be $\mathrm{GL}_2(\mathbb{Z})$-equivalent.

Recall (4-1) and define

$$\Omega^*(X) = \{(L, K) \in \Omega^* : \max\{L^2, |K|\} \le X\} \quad \text{for } * \in \{0, +, -\}.$$

In the notation of Lemma 4.1 as well as Propositions 4.2, 4.3, and 4.4, we have

$$V^0_{\mathbb{R},f}(X) = \begin{cases} (\Psi_f \circ \Phi)(\Omega^+(X) \times [-\pi/4, \pi/4)), \\ \bigsqcup_{i=1}^{2} (\Psi_f \circ \Phi^{(i)})(\Omega^+(X) \times \mathbb{R}) \sqcup \bigsqcup_{i=3}^{4} (\Psi_f \circ \Phi^{(i)})(\Omega^-(X) \times \mathbb{R}), \\ \bigsqcup_{i=1}^{2} \Phi_f^{(i)}(\Omega^0(X) \times \mathbb{R}), \end{cases} \tag{5-1}$$

respectively, if f is positive definite, indefinite, and reducible. We shall overcome the two issues above by restricting the values for $t \in \mathbb{R}$.

For brevity, in this section, write

$$D_f = |\Delta(f)| \quad \text{and} \quad \delta_f = D_f/4,$$

as in Theorem 1.2 and Lemma 4.1, respectively.

Definition 5.2. If f is positive definite, define

$$\mathcal{S}_f(X) = (\Psi_f \circ \Phi)(\Omega^+(X) \times [-\pi/4, \pi/4)).$$

If f is reducible, define

$$\mathcal{S}_f(X) = \bigsqcup_{i=1}^{2} \Phi_f^{(i)}(\Omega^0(X) \times [t_{f,1}, t_{f,2}]) \quad \text{for } t_{f,1} = -\frac{\log 8}{4} \text{ and } t_{f,2} = \frac{\log(5X/18)}{4}.$$

If f is indefinite and irreducible, define

$$\mathcal{S}_f(X) = \bigsqcup_{i=1}^{2} (\Psi_f \circ \Phi^{(i)})(\Omega^+(X) \times [0, t_{D_f})) \sqcup \bigsqcup_{i=3}^{4} (\Psi_f \circ \Phi^{(i)})(\Omega^-(X) \times [0, t_{D_f})),$$

where t_{D_f} is defined as in Theorem 1.2(c).

The goal of this section to prove the following preliminary results and estimates:

Proposition 5.3. *The set $\Theta_{w(f)}(\mathcal{S}_f(X))$ is bounded, semialgebraic, and definable by an absolutely bounded number of polynomial inequalities whose degrees are absolutely bounded.*

Proposition 5.4. *The following statements hold.*

(a) *A form in $V^0_{\mathbb{Z},f}(X)$ is $\mathrm{GL}_2(\mathbb{Z})$-equivalent to at least one form in $\mathcal{S}_f(X)$.*

(b) *A form in $V^0_{\mathbb{Z},f}(X)$ for which $\Delta(F) \neq \square$ is $\mathrm{GL}_2(\mathbb{Z})$-equivalent to exactly r_f forms in $\mathcal{S}_f(X)$, where r_f is defined as in Theorem 1.2.*

5A. ***Alternative description.*** First, we shall give an alternative description of the set $\mathcal{S}_f(X)$ in terms of the coefficients of the forms in $V^0_{\mathbb{R},f}(X)$.

Lemma 5.5. *If f is positive definite, then $\mathcal{S}_f(X) = V^0_{\mathbb{R},f}(X)$.*

Proof. This is clear from (5-1). □

Lemma 5.6. *If f is reducible, then*

$$\mathcal{S}_f(X) = \{F \in V^0_{\mathbb{R},f}(X) : \beta^2/8 \le |C_F| \le 5\beta^2 X/18\},$$

where C_F denotes the y^4-coefficient of F.

Proof. For $i = 1, 2$ and for any $F = \Phi^{(i)}_f(L, K, t)$, we have $C_F = (-1)^i \beta^2 e^{4t}$ by (4-11), and the claim is then clear from (5-1). □

Lemma 5.7. *If f is an indefinite and irreducible, then*

$$\mathcal{S}_f(X) = \{F \in V^0_{\mathbb{R},f}(X) : 1 \le E_{f,1}(F) Z_f(F)/E_{f,2}(F) < e^{8t_{D_f}}\},$$

where in the notation of Proposition 3.2(a), we define

$$E_{f,1}(F) = L_{f,1}(F) - \sqrt{D_f} L_{f,2}(F) \quad \textit{and} \quad E_{f,2}(F) = L_{f,1}(F) + \sqrt{D_f} L_{f,2}(F),$$

and for F in the image of $\Psi_f \circ \Phi^{(i)}$, we define

$$Z_f(F) = \begin{cases} 1 & \textit{for } i = 1, 2, \\ \dfrac{L_f(F)^2 + 4K_f(F)}{(4L_f(F) - (-1)^i 2\sqrt{2L_f(F)^2 - K_f(F)})^2} & \textit{for } i = 3, 4. \end{cases}$$

Proof. For $i = 1, 2, 3, 4$, consider $F = (\Psi_f \circ \Phi^{(i)})(L, K, t)$. For $k = 1, 2$, we have

$$E_{f,k}(F) = \begin{cases} (-1)^i 2\alpha^2 \sqrt{L_f(F)^2 + 4K_f(F)} e^{(-1)^{k+1} 4t}/3 & \text{if } i = 1, 2, \\ -2\alpha^2 (3L_f(F) + (-1)^{k+i} 2\sqrt{2L_f(F)^2 - K_f(F)}) e^{(-1)^{k+1} 4t}/3 & \text{if } i = 3, 4, \end{cases}$$

by a direct computation using (4-2), (4-7), and (4-8). We then see that

$$E_{f,1}(F) Z_f(F)/E_{f,2}(F) = e^{8t},$$

from which the claim follows. □

5B. ***Proof of Proposition 5.3.*** From (4-5), (4-7), (4-8), and (4-11), it is clear that the set $\mathcal{S}_f(X)$ is bounded. Thus, it remains to show that $\mathcal{S}_f(X)$ is a semialgebraic set definable by an absolutely bounded number of polynomial inequalities whose degrees are absolutely bounded.

5B1. *The case when f is positive definite or reducible.* The claim follows immediately from Lemmas 5.5 and 5.6 as well as Proposition 3.2.

5B2. *The case when f is indefinite and irreducible.* The only problem is that, for F in the image of $\Psi_f \circ \Phi^{(i)}$ for $i = 3, 4$, the expression $Z_f(F)$ is not a polynomial in the x^4, x^3y, and x^2y^2-coefficients of F. We shall resolve this issue in Lemma 5.8 below. The claim then follows from Lemma 5.7 and Proposition 3.2.

Lemma 5.8. *For $i = 3, 4$, let $F \in (\Psi_f \circ \Phi^{(i)})(\Omega^- \times \mathbb{R})$. Then, the condition*

$$1 \leq E_{f,1}(F)Z_f(F)/E_{f,2}(F) < e^{8t_{D_f}}$$

is equivalent to an absolutely bounded number of polynomial inequalities in the variables $L_f(F)$, $K_f(F)$, $E_{f,1}(F)$, $E_{f,2}(F)$ whose degrees are absolutely bounded.

Proof. For brevity, define

$$Y_{f,1}(F) = -E_{f,1}(F)(L_f(F)^2+4K(F))+E_{f,2}(F)(17L_f(F)^2-4K_f(F)),$$
$$Y_{f,2}(F) = -E_{f,1}(F)(L_f(F)^2+4K_f(F))+e^{8t_{D_f}}E_{f,2}(F)(17L_f(F)^2-4K_f(F)),$$

as well as write

$$\begin{aligned}(L, K, E_1, E_2, Z, Y_1, Y_2)\\ &= (L_f(F), K_f(F), E_{f,1}(F), E_{f,2}(F), Z_f(F), Y_{f,1}(F), Y_{f,2}(F)).\end{aligned}$$

Note that $L^2+4K < 0$ by (1-7) because $\Delta(F) < 0$. This implies that $Z < 0$ and so the stated condition may be rewritten as

$$\begin{cases} E_2 \leq E_1 Z < e^{8t_{D_f}} E_2 & \text{if } E_2 > 0\text{, which is equivalent to } i = 3,\\ E_2 \geq E_1 Z > e^{8t_{D_f}} E_2 & \text{if } E_2 < 0\text{, which is equivalent to } i = 4.\end{cases}$$

By rearranging, we may further rewrite the above as

$$12E_2L\sqrt{2L^2-K} \leq (-1)^i Y_1 \quad \text{and} \quad 12e^{8t_{D_f}}E_2L\sqrt{2L^2-K} > (-1)^i Y_2.$$

From here, we shall consider the different possibilities for the signs of E_2, L, Y_1, Y_2. For example, when $E_2 > 0$ and $L \geq 0$, the above is equivalent to $Y_1 \leq 0$ and

$$\begin{cases} (12E_2L)^2(2L^2-K) \leq Y_1^2 & \text{if } Y_2 > 0,\\ (12E_2L)^2(2L^2-K) \leq Y_1^2 \text{ and } (12e^{8t_{D_f}}E_2L)^2(2L^2-K) > Y_2^2 & \text{if } Y_2 \leq 0.\end{cases}$$

The other cases are analogous. We then see that the claim holds. $\square$

5C. *Integral orthogonal groups.* We shall require an explicit description of

$$O_f(\mathbb{Z}) = O_f(\mathbb{R}) \cap \mathrm{GL}_2(\mathbb{Z}).$$

In the notation of Lemma 4.1, observe that

$$O_f(\mathbb{R}) = \begin{cases} T_f^{-1}(O_{x^2+y^2}(\mathbb{R}))T_f & \text{if } f \text{ is positive definite,} \\ T_f^{-1}(O_{x^2-y^2}(\mathbb{R}))T_f & \text{if } f \text{ is indefinite.} \end{cases} \tag{5-2}$$

Moreover, it is well-known that

$$\begin{aligned} O_{x^2+y^2}(\mathbb{R}) &= \{J_k T^+(t) : k \in \{1,4\} \text{ and } t \in \mathbb{R}\}, \\ O_{x^2-y^2}(\mathbb{R}) &= \{\pm J_k T^-(t) : k \in \{1,2,3,4\} \text{ and } t \in \mathbb{R}\}, \end{aligned}$$

where $T^+(t)$ and $T^-(t)$ are defined as in (4-3), and

$$J_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5-3}$$

We shall need the following lemma.

Lemma 5.9. *Suppose that $T \in O_f(\mathbb{Z}) \setminus \{\pm I_{2\times 2}\}$ has finite order. Then, the form f is $\mathrm{GL}_2(\mathbb{Z})$-equivalent to a form of the shape*

$$\begin{cases} x^2+y^2,\ x^2+xy+y^2,\ \text{or } ax^2+bxy-ay^2 & \text{if } \det(T)=1, \\ xy,\ x^2-y^2,\ ax^2+cy^2,\ \text{or } ax^2+bxy+ay^2 & \text{if } \det(T)=-1, \end{cases}$$

for some integers a, b, and c.

Proof. By [Newman 1972, Chapter IX], for example, a finite cyclic subgroup of $\mathrm{GL}_2(\mathbb{Z})$ not contained in $\{\pm I_{2\times 2}\}$ is conjugate to the subgroup generated by one of the following:

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We then deduce that there exists $P \in \mathrm{GL}_2(\mathbb{Z})$ such that $Q = P^{-1}TP$ is equal to one of the following matrices up to sign:

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since f is primitive with $\alpha > 0$ by assumption and $(f_P)_Q = \pm f_P$, we then check that f_P must have one of the stated shapes. □

Proposition 5.10. *Suppose that f is positive definite. Then, we have*

$$O_f(\mathbb{Z}) = \{\pm I_{2\times 2}\}$$

if f is not $\mathrm{GL}_2(\mathbb{Z})$*-equivalent to the forms below, and the group* $O_f(\mathbb{Z})$ *is equal to*

$$\begin{cases} \{\pm I_{2\times 2}, \pm\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \pm\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\} & \text{if } f(x,y) = x^2+y^2, \\ \{\pm I_{2\times 2}, \pm\left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right), \pm\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right), \pm\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right), \pm\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \pm\left(\begin{smallmatrix} -1 & 0 \\ 1 & 1 \end{smallmatrix}\right)\} & \\ & \text{if } f(x,y) = x^2+xy+y^2, \\ \{\pm I_{2\times 2}, \pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\} & \text{if } f(x,y) = \alpha x^2+\gamma y^2 \text{ for } \alpha\neq\gamma, \\ \{\pm I_{2\times 2}, \pm\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\} & \text{if } f(x,y) = \alpha x^2+\beta xy+\alpha y^2 \text{ for } \beta\notin\{0,\alpha\}. \end{cases}$$

Proof. Elements in $O_f(\mathbb{Z})$ have finite order by (5-2) and so the first claim follows from Lemma 5.9. Using (5-2), we compute that elements in $O_f(\mathbb{R})$ are of the forms

$$\begin{pmatrix} \phi_t + \dfrac{\beta\psi_t}{2\sqrt{\delta_f}} & \dfrac{\gamma\psi_t}{\sqrt{\delta_f}} \\ -\dfrac{\alpha\psi_t}{\sqrt{\delta_f}} & \phi_t - \dfrac{\beta\psi_t}{2\sqrt{\delta_f}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \phi_t - \dfrac{\beta\psi_t}{2\sqrt{\delta_f}} & \dfrac{\beta}{\alpha}\left(\phi_t - \dfrac{\beta\psi_t}{2\sqrt{\delta_f}}\right) + \dfrac{\gamma\psi_t}{\sqrt{\delta_f}} \\ \dfrac{\alpha\psi_t}{\sqrt{\delta_f}} & -\phi_t - \dfrac{\beta\psi_t}{2\sqrt{\delta_f}} \end{pmatrix},$$

where $t\in\mathbb{R}$ and $(\phi_t,\psi_t) = (\cos t, \sin t)$. With the help of the proof of Lemma 5.9, it is not hard to check that $O_f(\mathbb{Z})$ is as claimed. □

Proposition 5.11. *Suppose that f is reducible. Then, the group $O_f(\mathbb{Z})$ is equal to*

$$\begin{cases} \{\pm I_{2\times 2}\} & \text{if } \beta\nmid\alpha^2+1 \text{ and } \beta\nmid\alpha^2-1, \\ \{\pm I_{2\times 2}, \pm\left(\begin{smallmatrix} \alpha & \beta \\ -(\alpha^2+1)/\beta & -\alpha \end{smallmatrix}\right)\} & \text{if } \beta\mid\alpha^2+1 \text{ and } \beta\nmid\alpha^2-1, \\ \{\pm I_{2\times 2}, \pm\left(\begin{smallmatrix} \alpha & \beta \\ -(\alpha^2-1)/\beta & -\alpha \end{smallmatrix}\right)\} & \text{if } \beta\nmid\alpha^2+1 \text{ and } \beta\mid\alpha^2-1, \\ \{\pm I_{2\times 2}, \pm\left(\begin{smallmatrix} -1 & 0 \\ 2 & 1 \end{smallmatrix}\right), \pm\left(\begin{smallmatrix} 1 & 1 \\ -2 & -1 \end{smallmatrix}\right), \pm\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)\} & \text{if } f(x,y) = x^2+xy, \\ \{\pm I_{2\times 2}, \pm\left(\begin{smallmatrix} -1 & 0 \\ 1 & 1 \end{smallmatrix}\right), \pm\left(\begin{smallmatrix} 1 & 2 \\ -1 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 \\ 0 & -1 \end{smallmatrix}\right)\} & \text{if } f(x,y) = x^2+2xy. \end{cases}$$

Proof. Using (5-2), we compute that elements in $O_f(\mathbb{R})$ are of the forms

$$\pm\begin{pmatrix} \phi_t - \psi_t & 0 \\ 2\alpha\psi_t/\beta & \phi_t+\psi_t \end{pmatrix} \quad \text{and} \quad \pm\begin{pmatrix} \phi_t+\psi_t & (\beta/\alpha)(\phi_t+\psi_t) \\ -2\alpha\psi_t/\beta & -\phi_t-\psi_t \end{pmatrix},$$

where $t\in\mathbb{R}$ and $(\phi_t,\psi_t)\in\{(\cosh t,\sinh t),(\sinh t,\cosh t)\}$. For the matrix on the left to have integer entries, necessarily

$$2\cosh t, 2\sinh t\in\mathbb{Z}, \quad \text{so} \quad (2\cosh t, 2\sinh t) = (2,0).$$

Similarly, for the matrix on the right to have integer entries, necessarily

$$2\alpha\cosh t, 2\alpha\sinh t, (\cosh t+\sinh t)/\alpha\in\mathbb{Z}, \qquad \text{so} \quad (2\alpha\cosh t, 2\alpha\sinh t) = (\alpha^2+1, \alpha^2-1).$$

We then deduce that

$$O_f(\mathbb{Z}) = \left\{\pm I_{2\times 2}, \pm\left(\begin{smallmatrix} -1 & 0 \\ 2\alpha/\beta & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha & \beta \\ -(\alpha^2\pm 1)/\beta & -\alpha \end{smallmatrix}\right)\right\} \cap \mathrm{GL}_2(\mathbb{Z}).$$

Since f has the shape (1-11) by assumption, we have

$$\beta \mid \alpha^2+1 \text{ and } \beta \mid \alpha^2-1 \Longleftrightarrow \alpha = 1 \text{ and } \beta \in \{1, 2\},$$

and we see that the claim indeed holds. □

Proposition 5.12. *Suppose that f is indefinite and irreducible. Define*

$$G_f(\mathbb{Z}) = \{\pm T_{D_f}^n : n \in \mathbb{Z}\}, \quad \textit{where } T_{D_f} = \begin{pmatrix} \frac{1}{2}(u_{D_f} - \beta v_{D_f}) & -\gamma v_{D_f} \\ \alpha v_{D_f} & \frac{1}{2}(u_{D_f} + \beta v_{D_f}) \end{pmatrix}$$

and $(u_{D_f}, v_{D_f}) \in \mathbb{N}^2$ is the least solution to $x^2 - D_f y^2 = \pm 4$. Then, we have

$$O_f(\mathbb{Z}) = G_f(\mathbb{Z})$$

if f is not $\mathrm{GL}_2(\mathbb{Z})$-equivalent to the forms below, and the group $O_f(\mathbb{Z})$ is equal to

$$\begin{cases} G_f(\mathbb{Z}) \sqcup G_f(\mathbb{Z})\left(\begin{smallmatrix} 1 & \beta/\alpha \\ 0 & -1 \end{smallmatrix}\right) & \textit{if } f(x,y) = \alpha x^2 + \beta xy + \gamma y^2 \textit{ with } \alpha \mid \beta, \\ G_f(\mathbb{Z}) \sqcup G_f(\mathbb{Z})\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) & \textit{if } f(x,y) = \alpha x^2 + \beta xy - \alpha y^2. \end{cases}$$

Proof. By (5-2), elements in $O_f(\mathbb{R})$ of infinite order are of the shape

$$\pm\begin{pmatrix} \phi_t - \beta\psi_t/(2\sqrt{\delta_f}) & -\gamma\psi_t/\sqrt{\delta_f} \\ \alpha\psi_t/\sqrt{\delta_f} & \phi_t + \beta\psi_t/(2\sqrt{\delta_f}) \end{pmatrix},$$

where $t \in \mathbb{R}$ and $(\phi_t, \psi_t) \in \{(\cosh t, \sinh t), (\sinh t, \cosh t)\}$. We then see that

$$G_f(\mathbb{Z}) = \{\pm I_{2\times 2}\} \sqcup \{T \in O_f(\mathbb{Z}) : T \text{ has infinite order}\}.$$

Hence, the first claim follows from Lemma 5.9 and the fact that $ax^2 + bxy + ay^2$ is $\mathrm{GL}_2(\mathbb{Z})$-equivalent to the form

$$(2a-b)x^2 + (2a-b)xy + ay^2 \quad \text{via } \left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right). \tag{5-4}$$

Now, again by (5-2), elements in $O_f(\mathbb{R})$ of finite order have the shape

$$\begin{pmatrix} \dfrac{-\beta}{\sqrt{D_f}} & -\dfrac{2\gamma}{\sqrt{D_f}} \\ \dfrac{2\alpha}{\sqrt{D_f}} & \dfrac{\beta}{\sqrt{D_f}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \phi_t + \dfrac{\beta\psi_t}{2\sqrt{\delta_f}} & \dfrac{\beta}{\alpha}\left(\phi_t + \dfrac{\beta\psi_t}{2\sqrt{\delta_f}}\right) - \dfrac{\gamma\psi_t}{\sqrt{\delta_f}} \\ -\dfrac{\alpha\psi_t}{\sqrt{\delta_f}} & -\phi_t - \dfrac{\beta\psi_t}{2\sqrt{\delta_f}}, \end{pmatrix} \tag{5-5}$$

where $t \in \mathbb{R}$ and $(\phi_t, \psi_t) \in \{(\cosh t, \sinh t), (\sinh t, \cosh t)\}$. Notice that the matrix on the left cannot lie in $\mathrm{GL}_2(\mathbb{Z})$ because D_f is not square when f is irreducible. Using the description of $O_{x^2-y^2}(\mathbb{R})$, it is then not hard to check that $[O_f(\mathbb{Z}) : G_f(\mathbb{Z})] \le 2$, from which the second claim follows. □

5D. ***Proof of Theorem 1.4.*** Suppose that $f(x, y) = \alpha x^2 + \beta xy - \alpha y^2$ and that D_f is not a square. In the notation of Proposition 5.12, we have

$$x^2 - D_f y^2 = -4 \quad \text{has integer solutions if and only if} \quad \det(T_{D_f}) = -1$$

by definition. But Proposition 5.12 also implies that $\det(T_{D_f}) = -1$ is equivalent to

$$O_f(\mathbb{Z}) \text{ has an element of finite order and negative determinant.}$$

The theorem now follows from Lemma 5.9 and (5-4).

5E. ***Proof of Proposition 5.4.*** We shall need the following lemma.

Lemma 5.13. *For all $F \in V^0_{\mathbb{Z},f}$ with $\Delta(F) \neq \square$ and $T \in \mathrm{GL}_2(\mathbb{Z}) \setminus \{\pm I_{2\times 2}\}$, we have*

(a) $$F_T \in V^0_{\mathbb{Z},f} \quad \textit{if and only if} \quad T \in O_f(\mathbb{Z}),$$

(b) $$F_T = F \quad \textit{if and only if} \quad T = \pm D_f^{-1/2} M_f.$$

Proof. Note that $F_T \in V^0_{\mathbb{Z},f_T}$ by (1-8). By Theorem 1.1(a), we then have $F_T \in V^0_{\mathbb{Z},f}$ if and only if $f_T = \pm f$, whence part (a) holds. By Theorem 1.1(a) and Proposition 2.1, we have $F_T = F$ if and only if T is proportional to M_f, from which part (b) follows since $\det(T) = \pm 1$. $\square$

5E1. *The case when f is positive definite or reducible.* Let us first observe that:

Lemma 5.14. *We have $V^0_{\mathbb{Z},f}(X) \subset \mathcal{S}_f(X)$.*

Proof. Let $F \in V^0_{\mathbb{Z},f}(X)$ be given. If f is positive definite, then clearly $F \in \mathcal{S}_f(X)$ by Lemma 5.5. If f is reducible, then recall Lemma 5.6, and we have $F \in \mathcal{S}_f(X)$ since

$$\frac{8C_F}{\beta^2} \in \mathbb{Z} \quad \text{and} \quad \left|\frac{8C_F}{\beta^2}\right| \leq \left|\frac{4(L_f(F)^2 + 4K_f(F))}{9}\right| \leq \frac{20X}{9}$$

by (4-10) and Proposition 3.2(b), respectively. $\square$

Lemma 5.14 implies that part (a) holds. Together with Lemma 5.13(a), it further implies that for $F \in V^0_{\mathbb{Z},f}(X)$ with $\Delta(F) \neq \square$, the number of forms in $\mathcal{S}_f(X)$ which are $\mathrm{GL}_2(\mathbb{Z})$-equivalent to F is equal to

$$[O_f(\mathbb{Z}) : \mathrm{Stab}_{O_f(\mathbb{Z})}(F)].$$

By Lemma 5.13(b), we in turn have

$$[O_f(\mathbb{Z}) : \mathrm{Stab}_{O_f(\mathbb{Z})}(F)] = [O_f(\mathbb{Z}) : O_f(\mathbb{Z}) \cap \{\pm I_{2\times 2}, \pm D_f^{-1/2} M_f\}],$$

which may be verified to be equal to r_f using Propositions 5.10 and 5.11.

5E2. *The case when f is indefinite and irreducible.* We shall use the notation from Lemma 4.1, Proposition 5.12, (4-3), and (5-3). Then, by definition, we have

$$T_{D_f} = T_f^{-1} J_{k(f)} T^-(t_{D_f}) T_f, \quad \text{where } k(f) = \begin{cases} 1 & \text{if } u_{D_f}^2 - D_f v_{D_f}^2 = -4, \\ 2 & \text{if } u_{D_f}^2 - D_f v_{D_f}^2 = 4. \end{cases}$$

Now, by (5-1) and (4-6), a form in $V^0_{\mathbb{Z},f}(X)$ is of the shape

$$F = (F^{(i)}_{(L,K)})_{T^-(t)T_f}, \quad \text{where } (L, K, t) \in \Omega^0(X) \times \mathbb{R} \text{ and } i \in \{1, 2, 3, 4\}.$$

Observe that J_1 and J_2 commute with $T^-(t)$ as well as fix the forms in $V_{\mathbb{R},x^2-y^2}$. For any $n \in \mathbb{Z}$, we then deduce that

$$F_{T^n_{D_f}} = (F^{(i)}_{(L,K)})_{T^-(t)J^n_{k(f)}T^-(nt_{D_f})T_f} = (F^{(i)}_{(L,K)})_{T^-(t+nt_{D_f})T_f}.$$

Let $n_1 \in \mathbb{Z}$ be the unique integer such that $0 \le t + n_1 t_{D_f} < t_{D_f}$. The existence of n_1 then implies part (a).

Next, suppose that $\Delta(F) \neq \square$, in which case

$$\text{for } T \in \mathrm{GL}_2(\mathbb{Z}) : \qquad F_T \in V^0_{\mathbb{Z},f} \quad \text{if and only if} \quad T \in O_f(\mathbb{Z})$$

by Lemma 5.13(a). If $O_f(\mathbb{Z}) = G_f(\mathbb{Z})$, then part (b) holds by the uniqueness of n_1. If $O_f(\mathbb{Z}) \neq G_f(\mathbb{Z})$, then recall from Proposition 5.12 that

$$O_f(\mathbb{Z}) = G_f(\mathbb{Z}) \sqcup G_f(\mathbb{Z})M, \quad \text{where } M \text{ has finite order.}$$

From (5-2), we see that

$$M = \pm T_f^{-1} J_{k_0} T^-(t_0) T_f, \quad \text{where } t_0 \in \mathbb{R} \text{ and } k_0 \in \{3, 4\}.$$

Then, for any $n \in \mathbb{Z}$, it is straightforward to verify that

$$\begin{aligned} F_{T^n_{D_f}M} &= (F^{(i)}_{(L,K)})_{T^-(t+nt_{D_f})J_{k_0}T^-(t_0)T_f} \\ &= \begin{cases} (F^{(i)}_{(L,K)})_{T^-(-(t+nt_{D_f})+t_0)T_f} & \text{for } i \in \{1, 2\}, \\ (F^{(j)}_{(L,K)})_{T^-(-(t+nt_{D_f})+t_0)T_f} & \text{for } i \in \{3, 4\}, \text{ where } j \in \{3, 4\} \setminus \{i\}. \end{cases} \end{aligned}$$

There is a unique $n_2 \in \mathbb{Z}$ such that $0 \le -(t + n_2 t_{D_f}) + t_0 < t_{D_f}$. Observe that

$$F_{T^{n_1}_{D_f}} = F_{T^{n_2}_{D_f}M} \quad \text{would imply} \quad F_{T^{n_1}_{D_f}} = (F_{T^{n_1}_{D_f}})_{T^{n_2-n_1}_{D_f}M}.$$

But $T^{n_2-n_1}_{D_f}M$ has finite order, and so it cannot proportional to M_f by (5-5), which is a contradiction by Lemma 5.13(b). Then, we conclude from Proposition 5.12 that part (b) indeed holds.

6. Error estimates and the main theorem

Throughout this section, let $f(x, y) = \alpha x^2 + \beta xy + \gamma y^2$ be an integral and primitive binary quadratic form with $\Delta(f) \neq 0$ and $\alpha > 0$, in the shape (1-11) whenever f is reducible. Let D_f, r_f and s_f be as in Theorem 1.2.

In Subsections 6A and 6B, respectively, we shall first prove:

Proposition 6.1. *For any $\epsilon > 0$, we have*

$$\#\big\{F \in \mathcal{S}_f(X) \cap V^0_{\mathbb{Z},f} : L_f(F)^2 + 4K_f(F) = \square\big\} = O_{f,\epsilon}(X^{1+\epsilon}),$$

and

$$\#\Big\{F \in \mathcal{S}_f(X) \cap V^0_{\mathbb{Z},f} : (L_f(F)^2 + 4K_f(F))(2L_f(F) - K_f(F))/\Delta(f) = \square \text{ and } L_f(F) \neq 0\Big\} = O_f(X^{1/2+\epsilon}).$$

Further, the number

$$\#\big\{F \in \mathcal{S}_f(X) \cap V^0_{\mathbb{Z},f} : -4K_f(F)^2/\Delta(f) = \square \text{ and } L_f(F) = 0\big\}$$

is equal to zero if $-\Delta(f) \neq \square$, and is bounded by $O_f(X)$ otherwise.

Propositions 6.1, 3.6, and 5.4 then imply part (d) of Theorem 1.2.

Proposition 6.2. *We have*

$$\#\{F \in \mathcal{S}_f(X) \cap V^0_{\mathbb{Z},f} : F \text{ is reducible}\} = \begin{cases} O_f(X(\log X)^2) & \text{if } f \text{ is irreducible,} \\ O_f(X(\log X)^3) & \text{if } f \text{ is reducible.} \end{cases}$$

Now, from Propositions 5.4, 6.1, and 6.2, we also easily see that

$$N^{(D_4)}_{\mathbb{Z},f}(X) = \frac{1}{r_f}\#(\mathcal{S}_f(X) \cap V^0_{\mathbb{Z},f}) + O_{f,\epsilon}(X^{1+\epsilon}) \quad \text{for any } \epsilon > 0. \tag{6-1}$$

Let $\mathcal{L}_{f,w(f)}$ be a linear transformation on $\mathbb{R}^3$ which takes $\Lambda_{f,w(f)}$ to $\mathbb{Z}^3$, and define

$$\mathcal{R}_f(X) = (\mathcal{L}_{f,w(f)} \circ \Theta_{w(f)})(\mathcal{S}_f(X)), \quad \text{where } w(f) = \begin{cases} 1 & \text{if } f \text{ is irreducible,} \\ 2 & \text{if } f \text{ is reducible,} \end{cases}$$

as before. Observe that then

$$\#(\mathcal{S}_f(X) \cap V^0_{\mathbb{Z},f}) = \#(\Theta_{w(f)}(\mathcal{S}_f(X)) \cap \Lambda_{f,w(f)}) = \#(\mathcal{R}_f(X) \cap \mathbb{Z}^3).$$

By Proposition 5.3, we may apply Proposition 5.1 to obtain

$$
\begin{aligned}
(6\text{-}2)\quad & \#(S_f(X) \cap V^0_{\mathbb{Z},f}) \\
&= \operatorname{Vol}(\mathcal{R}_f(X)) + O(\max\{\operatorname{Vol}(\overline{\mathcal{R}_f(X)}), 1\}) \\
&= \frac{1}{\det(\Lambda_{f,w(f)})} \operatorname{Vol}(\Theta_{w(f)}(\mathcal{S}_f(X))) + O_f(\max\{\operatorname{Vol}(\overline{\Theta_{w(f)}(\mathcal{S}_f(X))}), 1\}),
\end{aligned}
$$

where by Proposition 3.4, we know that

$$
\det(\Lambda_{f,w(f)}) = \begin{cases} s_f\alpha^3 & \text{if } f \text{ is irreducible,} \\ s_f\beta^3/8 & \text{if } f \text{ is reducible.} \end{cases}
$$

Hence, it remains to compute the above volumes, which we shall do in Section 6C.

6A. ***Proof of Proposition 6.1.*** Recall the notation from Proposition 3.2. By definition and Proposition 3.3, we then have a well-defined map

$$
\iota : V^0_{\mathbb{Z},f} \to \mathbb{Z}^3, \quad \iota(F) = (L_f(F), L_{f,1}(F), L_{f,2}(F)).
$$

Using Proposition 3.2, it is easy to verify that ι is in fact injective. We shall also need the following result.

Lemma 6.3 [Heath-Brown 2002, Corollary 2]. *Let $\xi(x_1, x_2, x_3)$ be a ternary quadratic form such that its corresponding matrix M_ξ has nonzero determinant. For $B_1, B_2, B_3 > 0$, let $N_\xi(B_1, B_2, B_3)$ denote the number of tuples $(x_1, x_2, x_3) \in \mathbb{Z}^3$ such that*

$$
|x_1| \le B_1, \quad |x_2| \le B_2, \quad |x_3| \le B_3, \quad \gcd(x_1, x_2, x_3) = 1, \quad \xi(x_1, x_2, x_3) = 0.
$$

Then, we have

$$
N_\xi(B_1, B_2, B_3) \ll_\epsilon \left(1 + \left(B_1 B_2 B_3 \cdot \frac{\det_0(M_\xi)^2}{|\det(M_\xi)|}\right)^{1/3+\epsilon}\right) d_3(|\det(M_\xi)|),
$$

where $\det_0(M_\xi)$ denotes the greatest common divisor of the 2×2 minors of M_ξ, and $d_3(|\det(M_\xi)|)$ is the number of ways to write $|\det(M_\xi)|$ as a product of three positive integers.

In what follows, consider $F \in \mathcal{S}_f(X) \cap V^0_{\mathbb{Z},f}$, and for brevity, write

$$
(L, K, L_1, L_2) = (L_f(F), K_f(F), L_{f,1}(F), L_{f,2}(F)).
$$

Since ι is injective, it is enough to estimate the number of choices for (L, L_1, L_2). To that end, let us put $\mathcal{D}_f = \Delta(f)$. Recall from Propositions 3.2 and 3.3 that

$$
L, K, L_1, L_2 \in \mathbb{Z}, \qquad \text{as well as} \qquad L_1^2 - \mathcal{D}_f L_2^2 = 4\alpha^4(L^2 + 4K)/9,
$$

which is nonzero by (1-7). By the definition of our height, we also have

$$\text{(6-3)} \qquad \begin{cases} L = O_f(X^{1/2}) \quad \text{and} \quad K = O_f(X) & \text{in all cases,} \\ L_1 = O_f(X^{1/2}) \quad \text{and} \quad L_2 = O_f(X^{1/2}) & \text{if } f \text{ is irreducible.} \end{cases}$$

The latter estimate holds by

$$\begin{cases} \text{(4-5), (4-2)} & \text{if } f \text{ is positive definite,} \\ \text{(4-7), (4-8), (4-2), and } 0 \le t < t_{D_f} & \text{if } f \text{ is indefinite and irreducible,} \end{cases}$$

as well as the fact that L_1 and L_2 are linear in the coefficients of F. Finally, we shall write $d(-)$ for the divisor function.

Proof of Proposition 6.1: first claim. Suppose that $L^2 + 4K = \square$. Then, we have

$$L_1^2 - \mathcal{D}_f L_2^2 = U^2, \quad \text{where } U \in \mathbb{N} \text{ is such that } U = O_f(X^{1/2}).$$

If f is reducible, then $\mathcal{D}_f = \square$ and so clearly there are

$$O_f\Bigg(\sum_{U=1}^{X^{1/2}} d(U^2)\Bigg) = O_{f,\epsilon}\Bigg(\sum_{U=1}^{X^{1/2}} X^{\epsilon}\Bigg) = O_{f,\epsilon}(X^{1/2+\epsilon})$$

choices for the pair (L_1, L_2). If f is irreducible, then note that

$$(L_1/n)^2 - \mathcal{D}_f(L_2/n)^2 = (U/n)^2, \quad \text{where } n = \gcd(L_1, L_2, U),$$

and applying Lemma 6.3 to the ternary quadratic form ξ with matrix

$$M_\xi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\mathcal{D}_f & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \text{with } \begin{cases} \det(M_\xi) = \mathcal{D}_f, \\ \det_0(M_\xi) = 1, \end{cases}$$

we deduce from (6-3) that there are

$$O_f\Bigg(\sum_{n=1}^{X^{1/2}} N_\xi\Big(\frac{X^{1/2}}{n}, \frac{X^{1/2}}{n}, \frac{X^{1/2}}{n}\Big)\Bigg) = O_{f,\epsilon}\Bigg(\sum_{n=1}^{X^{1/2}}\Big(1 + \frac{X^{1/2+\epsilon}}{n^{1+\epsilon}}\Big)\Bigg) = O_{f,\epsilon}(X^{1/2+\epsilon})$$

choices for the pair (L_1, L_2). In both cases, we see that there are

$$O_f(X^{1/2}) \cdot O_{f,\epsilon}(X^{1/2+\epsilon}) = O_{f,\epsilon}(X^{1+\epsilon})$$

choices for (L, L_1, L_2) in total, whence the claim. $\square$

Proof of Proposition 6.1: second claim. Suppose that $(L^2+4K)(2L^2-K)/\mathcal{D}_f = \square$. By Proposition 3.3, we may write

$$\gcd(L^2 + 4K, 4(2L^2 - K)/\mathcal{D}_f) = 9ma^2, \quad \text{where } m, a \in \mathbb{N} \text{ and } m \text{ is square-free.}$$

From the hypothesis, we then easily see that

$$L^2 + 4K = 9mU^2 \quad \text{and} \quad 4(2L^2 - K)/\mathcal{D}_f = 9mV^2, \qquad \text{where } U, V \in \mathbb{N},$$

as well as that m divides L. In particular, a simple calculation yields

$$L^2 = m(U^2 + \mathcal{D}_f V^2), \quad \text{whence } mW^2 = U^2 + \mathcal{D}_f V^2, \text{ where } W \in \mathbb{Z} \text{ with } L = mW.$$

Now, suppose also that $L \neq 0$, in which case $m = O_f(X^{1/2})$ by (6-3). Note also that

$$m(W/n)^2 = (U/n)^2 + \mathcal{D}_f(V/n)^2, \quad \text{where } n = \gcd(W, U, V).$$

Applying Lemma 6.3 to the ternary quadratic form ξ_m with matrix

$$M_{\xi_m} = \begin{pmatrix} m & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\mathcal{D}_f \end{pmatrix}, \quad \text{with } \begin{cases} \det(M_{\xi_m}) = m\mathcal{D}_f, \\ \det_0(M_{\xi_m}) = \gcd(m, \mathcal{D}_f) \leq |\mathcal{D}_f|, \end{cases}$$

we then see from (6-3) that there are

$$\begin{aligned} O_f\biggl(\sum_{n=1}^{X^{1/2}/m} N_{\xi_m}\Bigl(\frac{X^{1/2}}{mn}, \frac{X^{1/2}}{m^{1/2}n}, \frac{X^{1/2}}{m^{1/2}n}\Bigr)\biggr) &= O_{f,\epsilon}\biggl(\sum_{n=1}^{X^{1/2}/m}\Bigl(1 + \frac{X^{1/2+\epsilon}}{(mn)^{1+\epsilon}}\Bigr)m^{\epsilon}\biggr) \\ &= O_{f,\epsilon}\Bigl(\frac{X^{1/2}}{m^{1-\epsilon}} + \frac{X^{1/2+\epsilon}}{m}\Bigr) \end{aligned}$$

choices for (x, u, v) when m is fixed. It follows that we have

$$O_{f,\epsilon}\biggl(\sum_{m=1}^{X^{1/2}}\Bigl(\frac{X^{1/2}}{m^{1-\epsilon}} + \frac{X^{1/2+\epsilon}}{m}\Bigr)\biggr) = O_{f,\epsilon}(X^{1/2+\epsilon})$$

choices for (m, x, u, v) and hence for (L, K).

Next, regard (L, K) as being fixed, and recall that

$$L_1^2 - \mathcal{D}_f L_2^2 = T, \quad \text{where } T = 4\alpha^4(L^2 + 4K)/9.$$

We claim that there are $O_f(d(T))$ choices for (L_1, L_2). If f is positive definite or if f is reducible, then this is clear. If f is indefinite and irreducible, then by Definition 5.2 as well as Propositions 3.1 and 4.3, we have

$$F = (\Psi_f \circ \Phi^{(i)})(L, K, t), \quad \text{where } 0 \leq t < t_{D_f} \text{ and } i \in \{1, 2, 3, 4\}.$$

Since $\mathcal{D}_f > 0$, we must have $L^2 + 4K > 0$ by the hypothesis, and so in fact $i \in \{1, 2\}$. From the proof of Lemma 5.7, we know that

$$L_1 - \sqrt{D_f}L_2 = (-1)^i\sqrt{T}e^{4t} \quad \text{and} \quad L_1 + \sqrt{D_f}L_2 = (-1)^i\sqrt{T}e^{-4t},$$

which implies that

$$L_1 = (-1)^i \sqrt{T} \cosh(4t) \quad \text{and} \quad L_2 = (-1)^i \sqrt{T} \sinh(4t)/\sqrt{D_f}.$$

Since $t = O_f(1)$, we then deduce that indeed there are $O_f(d(T))$ choices for (L_1, L_2). Using the bound $d(T) = O_\epsilon(T^\epsilon) = O_{f,\epsilon}(X^\epsilon)$, we conclude that there are

$$O_{f,\epsilon}(X^{1/2+\epsilon}) \cdot O_{f,\epsilon}(X^\epsilon) = O_{f,\epsilon}(X^{1/2+\epsilon})$$

choices for (L, L_1, L_2) in total, whence the claim. □

Proof of Proposition 6.1: third claim. Suppose that $L = 0$ and that F is in the shape as in (3-1). Using Proposition 3.2, we then deduce that

$$C = (-12\gamma A + 3\beta B)/(2\alpha), \quad \text{and so} \quad K = -9\mathcal{D}_f(\alpha B^2 - 4\beta AB + 16\gamma A^2)/(4\alpha^3).$$

Clearly $-4K2/\Delta(f) = \square$ if and only if $-\Delta(f) = \square$.

We now suppose that $-\Delta(f) = \square$, so in particular f is positive definite. The form F is then determined by $(A, B) \in \mathbb{Z}^2$, and that $|K| \leq X$ implies

$$\left| \left(B - \frac{2\beta}{\alpha} A \right)^2 - \frac{4\mathcal{D}_f}{\alpha^2} A^2 \right| \ll_f X.$$

Hence there are $O_f(X)$ choices for (A, B). It follows that the claim holds. □

6B. ***Proof of Proposition 6.2.*** By Lemma 3.8 and Proposition 6.1, we have

$$\#\{F \in \mathcal{S}_f(X) \cap V^0_{\mathbb{Z},f} : F \text{ is reducible of type } 1\} = O_{f,\epsilon}(X^{1+\epsilon}), \tag{6-4}$$

whence it is enough to consider the reducible forms in $\mathcal{S}_f(X) \cap V^0_{\mathbb{Z},f}$ of type 2; recall Definition 3.7. By definition, such a form has the shape

$$F(x, y) = p_2q_2x^4 + (p_2q_1 + p_1q_2)x^3y + (p_2q_0 + p_1q_1 + p_0q_2)x^2y^2 + (*)xy^3 + (*)y^4,$$

where $p_2, p_1, p_0, q_2, q_1, q_0 \in \mathbb{Z}$, and we have

$$p_0 = (\beta p_1 - 2\gamma p_2)/(2\alpha) \quad \text{and} \quad q_0 = (\beta q_1 - 2\gamma q_2)/(2\alpha)$$

by Lemma 3.10. We have the condition

$$\begin{aligned} &|(\alpha p_1^2 - 2\beta p_1p_2 + 4\gamma p_2^2)/\alpha|, |(\alpha q_1^2 - 2\beta q_1q_2 + 4\gamma q_2^2)/\alpha|, |p_2|, \\ &\qquad\qquad |\alpha p_1 - \beta p_2|, |q_2|, |\alpha q_1 - \beta q_2| \geq 1 \end{aligned} \tag{6-5}$$

since the above numbers are all integers. Using Proposition 3.2(a), we compute that

$$\frac{L_f(F)^2 + 4K_f(F)}{9} = \frac{\alpha p_1^2 - 2\beta p_1p_2 + 4\gamma p_2^2}{\alpha} \cdot \frac{\alpha q_1^2 - 2\beta q_1q_2 + 4\gamma q_2^2}{\alpha}.$$

Now, by the definition of our height, we clearly have

$$|(\alpha p_1^2 - 2\beta p_1p_2 + 4\gamma p_2^2)/\alpha|, \; |(\alpha q_1^2 - 2\beta q_1q_2 + 4\gamma q_2^2)/\alpha| \leq X. \tag{6-6}$$

Observe also that

(6-7) $\quad p_2q_2,\, p_2q_1+p_1q_2,\, p_1q_1 = O_f(X^{1/2}) \quad$ if f is indefinite and irreducible

by (4-7), (4-8), (4-2), and the bound $0 \le t < t_{D_f}$. We then deduce that

(6-8) $\qquad \#\{F \in \mathcal{S}_f(X) \cap V^0_{\mathbb{Z},f} : F \text{ is reducible of type } 2\} \le \#(\mathcal{R}'_f(X) \cap \mathbb{Z}^4),$

where we define

$$\mathcal{R}'_f(X) = \{(p_2, p_1, q_2, q_1) \in \mathbb{R}^4 : \text{(6-5), (6-6), and (6-7)}\}.$$

It is clear that this set is bounded and semialgebraic. Hence, we may apply Proposition 5.1 to estimate the number of integral points it contains.

6B1. *The case when f is irreducible.* Let us define

$$\mathcal{R}''_f(X) = \mathcal{L}_{D_f}(\mathbb{R}'_f(X)), \quad \text{where } \mathcal{L}_{D_f} = \begin{pmatrix} \sqrt{D_f} & 0 & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ 0 & 0 & \sqrt{D_f} & 0 \\ 0 & 0 & -\beta & \alpha \end{pmatrix}.$$

Applying Proposition 5.1, we then obtain

$$\begin{aligned} \#(\mathcal{R}'_f(X) \cap \mathbb{Z}^4) &= \mathrm{Vol}(\mathcal{R}'_f(X)) + O(\max\{\mathrm{Vol}(\overline{\mathcal{R}_f(X)}, 1\}) \\ &= \frac{1}{\det(\mathcal{L}_{D_f})}\,\mathrm{Vol}(\mathcal{R}''_f(X)) + O_f(\max\{\mathrm{Vol}(\overline{\mathcal{R}''_f(X)}), 1\}) \end{aligned}$$

For any $(u_2, u_1, v_2, v_1) \in \mathcal{R}''_f(X)$, from (6-5) and (6-6), we deduce that

$$|u_2|, |u_1|, |v_2|, |v_1| \ge 1$$

as well as that

(6-9) $$\begin{cases} 1 \le |u_1^2+u_2^2|,\, |v_1^2+v_2^2| \le \alpha^4 X & \text{if } f \text{ is positive definite,} \\ 1 \le |u_1^2-u_2^2|,\, |v_1^2-v_2^2| \le \alpha^4 X & \text{if } f \text{ is indefinite.} \end{cases}$$

This, together with (6-7), implies that in fact

$$1 \le |u_2|, |u_1|, |v_2|, |v_1|, |u_2v_2|, |u_1v_1| \ll_f X^{1/2}.$$

We then compute that

$$\begin{aligned} &\mathrm{Vol}(\mathcal{R}''_f(X)) = O_f\left(\prod_{i=1}^{2}\int_1^{X^{1/2}/v_i} du_i\, dv_i\right) = O_f(X(\log X)^2), \\ &\mathrm{Vol}(\overline{\mathcal{R}''_f(X)}) = O_f(X \log X). \end{aligned}$$

The claim now follows from (6-4) and (6-8).

6B2. *The case when f is reducible.* Let us define

$$\mathbb{R}''_f(X) = \mathcal{L}_{0,D_f}(\mathbb{R}'_f(X)), \quad \text{where } \mathcal{L}_{0,D_f} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{D_f} & 0 & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ 0 & 0 & \sqrt{D_f} & 0 \\ 0 & 0 & -\beta & \alpha \end{pmatrix}.$$

Since $D_f = \square$ in this case, we see that

$$\mathcal{L}_{0,D_f}(\mathcal{R}'_f(X) \cap \mathbb{Z}^4) \subset \mathcal{R}''_f(X) \cap \mathbb{Z}^4 \quad \text{and so} \quad \#(\mathcal{R}'_f(X) \cap \mathbb{Z}^4) \leq \#(\mathcal{R}''_f(X) \cap \mathbb{Z}^4).$$

Now, applying Proposition 5.1, we have

$$\#(\mathcal{R}''_f(X) \cap \mathbb{Z}^4) = \mathrm{Vol}(\mathcal{R}''_f(X)) + O(\max\{\mathrm{Vol}(\overline{\mathcal{R}''_f(X)}), 1\}).$$

For any $(z_1, z_2, z_3, z_4) \in \mathcal{R}''_f(X)$, the conditions (6-5) and (6-6) imply that

$$|z_1|, |z_2|, |z_3|, |z_4| \geq 1 \quad \text{and} \quad |z_1 z_2 z_3 z_4| \leq \alpha^4 X,$$

which is analogous to (6-9). We then compute that

$$\begin{aligned} \mathrm{Vol}(\mathcal{R}''_f(X)) &= O_f\left(\int_1^X \int_1^{X/z_4} \int_1^{X/(z_3 z_4)} \int_1^{X/(z_2 z_3 z_4)} dz_1\, dz_2\, dz_3\, dz_4 \right) \\ &= O_f(X (\log X)^3), \\ \mathrm{Vol}(\overline{\mathcal{R}''_f(X)}) &= O_f(X (\log X)^2). \end{aligned}$$

The claim now follows from (6-4) and (6-8).

6C. *Proof of Theorem 1.2.* We have already proven part (d). To prove parts (a) through (c), it remains to compute the volumes in (6-2).

6C1. *The case when f is positive definite.* We have

$$\mathrm{Vol}(\Theta_1(\mathcal{S}_f(X))) = \frac{8\alpha^3}{D_f^{3/2}} \cdot \frac{1}{18} \cdot \mathrm{Vol}(\Omega^+(X) \times [-\pi/4, \pi/4))$$

by Lemma 4.1 and Proposition 4.2(b), as well as

$$\mathrm{Vol}\big(\Omega^+(X) \times [-\pi/4, \pi/4)\big) = \int_{-X^{1/2}}^{X^{1/2}} \int_{-L^2/4}^{X} \frac{\pi}{2}\, dK\, dL = \frac{13\pi}{12} X^{3/2}.$$

Observe also that

$$\mathrm{Vol}(\overline{\Theta_1(\mathcal{S}_f(X))}) = O_f(X)$$

because $\Theta_1(\mathcal{S}_f(X))$ lies in the cube centered at the origin of side length $O_f(X^{1/2})$ by (4-5) and (4-2). We then deduce part (a) from (6-1) and (6-2).

6C2. *The case when f is reducible.* We have

$$\mathrm{Vol}(\Theta_2(\mathcal{S}_f(X))) = \tfrac{1}{18}\cdot 2\cdot \mathrm{Vol}(\Omega^0(X)\times[t_{f,1},t_{f,2}])$$

by Proposition 4.4, as well as

$$\mathrm{Vol}(\Omega^0(X)\times[t_{f,1},t_{f,2}]) = \int_{-X^{1/2}}^{X^{1/2}}\int_{-X}^{X}\frac{1}{4}\log\Big(\frac{20X}{9}\Big)\,dK\,dL = X^{3/2}\log(20X/9).$$

We then deduce part (b) from Lemma 6.4 below as well as (6-1) and (6-2).

Lemma 6.4. *We have* $\mathrm{Vol}(\overline{\Theta_2(\mathcal{S}_f(X))}) = O_f(X^{3/2})$.

Proof. By Definition 5.2, an element in $\Theta_2(\mathcal{S}_f(X))$ takes the form

$$(A,B,C) = (\Theta_2\circ\Phi_f)(L,K,t),\quad \text{where } (L,K,t)\in\Omega^0(X)\times[t_{f,1},t_{f,2}].$$

Let us recall that

$$|L|\le X^{1/2},\quad |K|\le X,\quad 4t_{f,1} = -\log 8,\quad 4t_{f,2} = \log(5X/18). \tag{6-10}$$

Then, from (4-11), we see that 1-dimensional projections of $\Theta_2(\mathcal{S}_f(X))$ have lengths of order $O_f(X)$. As for the 2-dimensional projections, note that (5-1) and (6-10) yield

$$|C| = \beta^2 e^{4t}\quad\text{and}\quad 1\ll_f |C|\ll_f X,$$

as well as the estimates

$$\Big|B-\frac{6\alpha^2C}{\beta^2}\Big|\le\tfrac{1}{2}X^{1/2}\quad\text{and}\quad\Big|A-\frac{\alpha^4C}{\beta^4}\Big|\le\frac{5}{144|C|}X+\frac{\alpha^2}{2\beta^2}X^{1/2}.$$

Hence, the projections of $\Theta_2(\mathcal{S}_f(X))$ onto the BC-plane and AC-plane, respectively, have areas bounded by

$$O_f\Big(\int_1^X X^{1/2}\,dC\Big)\quad\text{and}\quad O_f\Big(\int_1^X\Big(\frac{1}{C}X+X^{1/2}\Big)\,dC\Big).$$

Similarly, from (5-1) and (6-10), we deduce that

$$|2B-L| = 12\alpha^2e^{4t},\quad 1\ll_f|2B-L|\ll_f X,\quad |B|\ll_f X,$$

as well as the estimate

$$\Big|A-\frac{\alpha^2B}{6\beta^2}\Big|\le\frac{5\alpha^2}{12\beta^2}\Big(\frac{1}{|2B-L|}X+X^{1/2}\Big).$$

Note that $|L| \leq X^{1/2}$ also implies that

$$|2B - L| \geq |2|B| - |L|| \geq 2|B| - X^{1/2} \quad \text{when } |B| \geq X^{1/2}/2.$$

Hence, the projection of $\Theta_2(\mathcal{S}_f(X))$ onto the AB-plane has area bounded by

$$O_f\biggl(\int_0^{1+X^{1/2}/2} (X + X^{1/2})\, dB + \int_{1+X^{1/2}/2}^{X} \Bigl(\frac{1}{2B - X^{1/2}} X + X^{1/2}\Bigr)\, dB\biggr).$$

It follows that all of the 2-dimensional projections of $\Theta_2(\mathcal{S}_f(X))$ have areas of order $O_f(X^{3/2})$, and this proves the lemma. □

6C3. *The case when f is indefinite and irreducible.* We have

$$\mathrm{Vol}(\Theta_1(\mathcal{S}_f(X))) = \frac{8\alpha^3}{D_f^{3/2}} \cdot \frac{1}{18} \cdot 2 \cdot \bigl(\mathrm{Vol}(\Omega^+(X) \times [0, t_{D_f})) + \mathrm{Vol}(\Omega^-(X) \times [0, t_{D_f}))\bigr)$$

by Lemma 4.1 and Proposition 4.3, as well as

$$\mathrm{Vol}\bigl(\Omega^+(X) \times [0, t_{D_f})\bigr) = \int_{-X^{1/2}}^{X^{1/2}} \int_{-L^2/4}^{X} t_{D_f}\, dK\, dL = \frac{13 t_{D_f}}{6} X^{3/2},$$
$$\mathrm{Vol}\bigl(\Omega^-(X) \times [0, t_{D_f})\bigr) = \int_{-X^{1/2}}^{X^{1/2}} \int_{-X}^{-L^2/4} t_{D_f}\, dK\, dL = \frac{11 t_{D_f}}{6} X^{3/2},$$

Observe also that

$$\mathrm{Vol}(\overline{\Theta_1(\mathcal{S}_f(X))}) = O_f(X)$$

because $\Theta_1(\mathcal{S}_f(X))$ lies in the cube centered at the origin of side length $O_f(X^{1/2})$ by (4-7), (4-8), (4-2), and the bound on t. We then deduce part (c) from (6-1) and (6-2).

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References

[Altuğ et al. 2017] S. A. Altuğ, A. Shankar, I. Varma, and K. H. Wilson, "The number of quartic D_4-fields ordered by conductor", preprint, 2017. arXiv

[Bhargava 2004] M. Bhargava, "Higher composition laws, III: The parametrization of quartic rings", *Ann. of Math.* (2) **159**:3 (2004), 1329–1360. MR Zbl

[Bhargava 2008] M. Bhargava, "Higher composition laws, IV: The parametrization of quintic rings", *Ann. of Math.* (2) **167**:1 (2008), 53–94. MR Zbl

[Bhargava and Shankar 2015] M. Bhargava and A. Shankar, “Binary quartic forms having bounded invariants, and the boundedness of the average rank of elliptic curves”, *Ann. of Math.* (2) **181**:1 (2015), 191–242. MR Zbl

[Bhargava and Shnidman 2014] M. Bhargava and A. Shnidman, “On the number of cubic orders of bounded discriminant having automorphism group C_3, and related problems”, *Algebra Number Theory* **8**:1 (2014), 53–88. MR Zbl

[Bhargava et al. 2013] M. Bhargava, A. Shankar, and J. Tsimerman, “On the Davenport–Heilbronn theorems and second order terms”, *Invent. Math.* **193**:2 (2013), 439–499. MR Zbl

[Birch and Merriman 1972] B. J. Birch and J. R. Merriman, “Finiteness theorems for binary forms with given discriminant”, *Proc. London Math. Soc.* (3) **24**:3 (1972), 385–394. MR Zbl

[Brumer 1992] A. Brumer, “The average rank of elliptic curves, I”, *Invent. Math.* **109**:1 (1992), 445–472. MR Zbl

[Conrad 2012] K. Conrad, “Galois groups of cubics and quartics (not in characteristic 2)”, expository notes, 2012, http://www.math.uconn.edu/∼kconrad/blurbs/galoistheory/cubicquartic.pdf.

[Cremona 1999] J. E. Cremona, “Reduction of binary cubic and quartic forms”, *LMS J. Comput. Math.* **2** (1999), 64–94. MR Zbl

[Davenport 1951a] H. Davenport, “On a principle of Lipschitz”, *J. London Math. Soc.* **26**:3 (1951), 179–183. MR Zbl

[Davenport 1951b] H. Davenport, “On the class-number of binary cubic forms, I”, *J. London Math. Soc.* **26**:3 (1951), 183–192. MR Zbl

[Davenport 1951c] H. Davenport, “On the class-number of binary cubic forms, II”, *J. London Math. Soc.* **26**:3 (1951), 192–198. MR Zbl

[Davenport and Heilbronn 1971] H. Davenport and H. Heilbronn, “On the density of discriminants of cubic fields, II”, *Proc. Roy. Soc. London Ser. A* **322**:1551 (1971), 405–420. MR Zbl

[Delone and Faddeev 1964] B. N. Delone and D. K. Faddeev, *The theory of irrationalities of the third degree*, Trans. Math. Monographs **10**, Amer. Math. Soc., Providence, RI, 1964. MR Zbl

[Gauss 1801] C. F. Gauss, *Disquisitiones arithmeticae*, Fleischer, Leipzig, 1801.

[Heath-Brown 2002] D. R. Heath-Brown, “The density of rational points on curves and surfaces”, *Ann. of Math.* (2) **155**:2 (2002), 553–598. MR Zbl

[Heath-Brown 2004] D. R. Heath-Brown, “The average analytic rank of elliptic curves”, *Duke Math. J.* **122**:3 (2004), 591–623. MR Zbl

[Klagsbrun and Lemke Oliver 2014] Z. Klagsbrun and R. J. Lemke Oliver, “The distribution of the Tamagawa ratio in the family of elliptic curves with a two-torsion point”, *Res. Math. Sci.* **1**:1 (2014), art. id. 15. MR Zbl

[Nakagawa 1989] J. Nakagawa, “Binary forms and orders of algebraic number fields”, *Invent. Math.* **97**:2 (1989), 219–235. Correction in **105**:2 (1991), 443. MR Zbl

[Newman 1972] M. Newman, *Integral matrices*, Pure Appl. Math. **45**, Academic Press, New York, 1972. MR Zbl

[Siegel 1944] C. L. Siegel, “The average measure of quadratic forms with given determinant and signature”, *Ann. of Math.* (2) **45**:4 (1944), 667–685. MR Zbl

[Taniguchi and Thorne 2013] T. Taniguchi and F. Thorne, “Secondary terms in counting functions for cubic fields”, *Duke Math. J.* **162**:13 (2013), 2451–2508. MR Zbl

[Tsang and Xiao 2017] C. Tsang and S. Y. Xiao, “The number of quartic D_4-fields with monogenic cubic resolvent ordered by conductor”, preprint, 2017. arXiv

[Wood 2012] M. M. Wood, "Quartic rings associated to binary quartic forms", *Int. Math. Res. Not.* **2012**:6 (2012), 1300–1320. MR Zbl

[Xiao 2019] S. Y. Xiao, "On binary cubic and quartic forms", *J. Théor. Nombres Bordeaux* **31**:2 (2019), 323–341.

[Young 2006] M. P. Young, "Low-lying zeros of families of elliptic curves", *J. Amer. Math. Soc.* **19**:1 (2006), 205–250. MR Zbl

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CINDY (SIN YI) TSANG
YAU MATHEMATICAL SCIENCES CENTER
TSINGHUA UNIVERSITY
BEIJING
CHINA
Current address:
SCHOOL OF MATHEMATICS (ZHUHAI)
SUN YAT-SEN UNIVERSITY
TANGJIAWAN, ZHUHAI
GUANGDONG
CHINA
zengshy26@mail.sysu.edu.cn

STANLEY YAO XIAO
MATHEMATICAL INSTITUTE
UNIVERSITY OF OXFORD
UNITED KINGDOM
Current address:
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TORONTO
CANADA
syxiao@math.toronto.edu

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