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CYCLIC η-PARALLEL SHAPE AND RICCI OPERATORS ON REAL HYPERSURFACES IN TWO-DIMENSIONAL NONFLAT COMPLEX SPACE FORMS

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We consider three-dimensional real hypersurfaces in a nonflat complex space form of complex dimension two with cyclic η-parallel shape or Ricci operators and classify such hypersurfaces satisfying some other geometric restrictions. Some results extend those of Ahn et al. (1993), Lim et al. (2013), Kim et al. (2007) and Sohn (2007).

1. Introduction

A complex space form is a Kähler manifold of constant holomorphic sectional curvature c with complex dimension n and is denoted by $M^n(c)$, $n > 1$. If $M^n(c)$ is complete and simply connected, then it is complex analytically isometric to one of the following spaces:

- a complex projective space $\mathbb{C}P^n(c)$ when $c > 0$;
- a complex hyperbolic space $\mathbb{C}H^n(c)$ when $c < 0$;
- a complex Euclidean space $\mathbb{C}^n$ when $c = 0$.

Let M be a real hypersurface in a nonflat complex space form $M^n(c)$ whose Kähler metric and complex structure are denoted by $\bar{g}$ and J respectively. On M there exists an *almost contact metric structure* (ϕ, ξ, η, g) induced from $\bar{g}$ and J (for details see Section 2), where ξ is called a *structure vector field*. Let D be the distribution determined by tangent vectors orthogonal to ξ at each point of M. Let A be the shape operator of M in $M^n(c)$. If the structure vector field ξ is *principal*, that is, $A\xi = \alpha\xi$, where $\alpha = \eta(A\xi)$, then M is called a *Hopf hypersurface* and α is called *Hopf principal curvature*. Hopf real hypersurfaces in nonflat complex space forms with constant principal curvatures were classified in [Cecil and Ryan 1982; Kimura 1986; Takagi 1973; 1975a; 1975b] in the case of $\mathbb{C}P^n(c)$ and in [Berndt 1989] in the case of $\mathbb{C}H^n(c)$, respectively. For simplicity, a real hypersurface M in a nonflat complex space form is said to be of type (A) if it is locally congruent to a type (A_1) or (A_2) hypersurface in $\mathbb{C}P^n(c)$ (see [Takagi 1973; 1975a]) or type (A_0), (A_1)

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or (A_2) hypersurface in $\mathbb{C}H^n(c)$ (see [Berndt 1989]). Similarly, M is said to be of type (B) if it is locally congruent to a type (B) hypersurface in $\mathbb{C}P^n(c)$ or $\mathbb{C}H^n(c)$.

Unlike the case of real space forms, the shape operator of real hypersurfaces in a nonflat complex space form can not be parallel (deduced directly from the Codazzi equation (2-8)). This motivates some authors to consider certain conditions weaker than parallel shape operators. One of the methods is to investigate cyclic parallel shape operators, i.e., $g((\nabla_X A)Y, Z) + g((\nabla_Y A)Z, X) + g((\nabla_Z A)X, Y) = 0$ for any vector fields X, Y and Z. Ki [1988] proved that the shape operator of a real hypersurface in a nonflat complex space form is cyclic parallel if and only if the hypersurface is of type (A); see also [Niebergall and Ryan 1997]. This implies that cyclic parallelism is still too strong and therefore cyclic η-parallelism for the shape operator, i.e., $g((\nabla_X A)Y, Z) + g((\nabla_Y A)Z, X) + g((\nabla_Z A)X, Y) = 0$ for any vector fields X, Y and Z orthogonal to the structure vector field, were studied in [Kim et al. 2007]. Another method is to study η-parallel shape operators, i.e., $g((\nabla_X A)Y, Z) = 0$ for any vector fields X, Y, Z orthogonal to the structure vector field. Because the η-parallelism condition is so weak, in history many authors investigated it together with some other conditions; see [Ahn et al. 1993; Kimura and Maeda 1989; Kon and Loo 2012; Niebergall and Ryan 1997; Suh 1990]. Finally, without any other redundant restrictions, Kon and Loo [2011] proved that a real hypersurface of dimension greater than three in nonflat complex space forms has η-parallel shape operator if and only if it is of type (A) or (B) or a ruled real hypersurface. However, the above result is still open for real hypersurfaces of dimension three. Among others, the third method to extend parallel shape operators was introduced by Cho [2015], who proved that there exist no real hypersurfaces with Killing type shape operator, i.e., $(\nabla_X A)Y + (\nabla_Y A)X = 0$ for any vector fields X, Y. However, there exist real hypersurfaces in a nonflat complex space form $M^n(c)$ with transversal Killing shape operators, i.e., $(\nabla_X A)Y + (\nabla_Y A)X = 0$ for any vector fields X, Y orthogonal to the structure vector field, which are of type (A) (see [Cho 2015] for the case of $n > 2$).

Just like the case of shape operators, the Ricci operator of real hypersurfaces in a nonflat complex space form can not be parallel (see [Niebergall and Ryan 1997] for dimension greater than three and [Kim 2004] for dimension three). However, there exist real hypersurfaces such that the Ricci operator is cyclic parallel (see [Kwon and Nakagawa 1988; Niebergall and Ryan 1997]), i.e.,

$$g((\nabla_X Q)Y, Z) + g((\nabla_Y Q)Z, X) + g((\nabla_Z Q)X, Y) = 0$$

for any vector fields X, Y, Z. Because Ricci cyclic parallelism is too strong, Kwon and Nakagawa [1989] considered cyclic η-parallel Ricci operator, i.e., $g((\nabla_X Q)Y, Z) + g((\nabla_Y Q)Z, X) + g((\nabla_Z Q)X, Y) = 0$ for any vector fields X, Y, Z orthogonal to the structure vector field, and proved that a Hopf real hypersurface of dimension greater than three in a nonflat complex space form has cyclic η-parallel Ricci operator if and only if it is of type (A) or (B). Real hypersurfaces with cyclic η-parallel and η-parallel Ricci operators were also studied in [Kim et al. 2007] and [Kim et al. 2006; Kon 2014; 2017; Maeda 2013; Pérez et al. 2001], respectively.

Many results mentioned above were obtained for hypersurfaces of dimension greater three. This motivates us to generalize those results for three-dimensional real hypersurfaces. Here we aim to classify real hypersurfaces in a nonflat complex space form of complex dimension two with cyclic η-parallel shape or Ricci operators satisfying some other geometric conditions. Some results in this paper are extensions of corresponding earlier results; see [Ahn et al. 1993; Lim et al. 2013a; 2013b; Kim et al. 2007; Sohn 2007].

2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M^n(c)$ and N be a unit normal vector field of M. We denote by $\bar{\nabla}$ the Levi-Civita connection of the metric $\bar{g}$ of $M^n(c)$ and J the complex structure respectively. Let g and ∇ be the induced metric from the ambient space and the Levi-Civita connection of g respectively. Then the Gauss and Weingarten formulas are given respectively as:

$$\bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \bar{\nabla}_X N = -AX \tag{2-1}$$

for any vector fields X and Y tangent to M, where A denotes the shape operator of M in $M^n(c)$. For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi. \tag{2-2}$$

We can define on M an almost contact metric structure (ϕ, ξ, η, g) satisfying

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \tag{2-3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \tag{2-4}$$

for any vector fields X and Y on M. Moreover, applying the parallelism of the complex structure (i.e., $\bar{\nabla} J = 0$) of $M^n(c)$ and using (2-1), (2-2), we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2-5}$$

$$\nabla_X \xi = \phi AX \tag{2-6}$$

for any vector fields X and Y. We denote by R the Riemannian curvature tensor of M. Since $M^n(c)$ is assumed to be of constant holomorphic sectional curvature c, then the Gauss and Codazzi equations of M in $M^n(c)$ are given respectively as:

$$(2\text{-}7)\qquad R(X,Y)Z=\frac{c}{4}\Big\{g(Y,Z)X-g(X,Z)Y+g(\phi Y,Z)\phi X-g(\phi X,Z)\phi Y -2g(\phi X,Y)\phi Z\Big\}+g(AY,Z)AX-g(AX,Z)AY,$$

$$(2\text{-}8)\qquad (\nabla_X A)Y-(\nabla_Y A)X=\frac{c}{4}\{\eta(X)\phi Y-\eta(Y)\phi X-2g(\phi X,Y)\xi\}$$

for any vector fields X, Y on M.

From (2-7) we see that the Ricci operator Q is given by

$$(2\text{-}9)\qquad QX=\frac{c}{4}((2n+1)X-3\eta(X)\xi)+mAX-A^2X$$

for any vector field X tangent to the hypersurface, where $m:=\operatorname{trace}A$ is the mean curvature.

In this paper, all manifolds are assumed to be connected and of class C^∞.

3. Cyclic η-parallel shape operator

Let M be a real hypersurface in a complex space form $M^n(c)$. We put

$$(3\text{-}1)\qquad A\xi=\alpha\xi+\beta U,$$

where $\alpha=\eta(A\xi)$, U is a unit vector field orthogonal to ξ and β is a smooth function. Applying (2-1) and (2-2) we see that $\beta U=-\phi\nabla_\xi\xi$. We put

$$\Omega=\{p\in M \mid \beta(p)\neq 0\}.$$

Then Ω is an open subset of M.

Lemma 3.1 [Panagiotidou and Xenos 2012, Lemma 1]. *Suppose M is a three-dimensional real hypersurface in a nonflat complex plane $M^2(c)$. Then the following relations hold*:

$$(3\text{-}2)\qquad \begin{aligned} AU&=\gamma U+\delta\phi U+\beta\xi,\quad A\phi U=\delta U+\mu\phi U,\\ \nabla_U\xi&=-\delta U+\gamma\phi U,\quad \nabla_{\phi U}\xi=-\mu U+\delta\phi U,\quad \nabla_\xi\xi=\beta\phi U,\\ \nabla_U U&=\kappa_1\phi U+\delta\xi,\quad \nabla_{\phi U}U=\kappa_2\phi U+\mu\xi,\quad \nabla_\xi U=\kappa_3\phi U,\\ \nabla_U\phi U&=-\kappa_1 U-\gamma\xi,\quad \nabla_{\phi U}\phi U=-\kappa_2 U-\delta\xi,\quad \nabla_\xi\phi U=-\kappa_3 U-\beta\xi,\end{aligned}$$

where γ, δ, μ, κ_i, $i=\{1,2,3\}$ are smooth functions on M and $\{\xi,U,\phi U\}$ is an orthonormal basis of the tangent space of M at a point of M.

Applying this lemma, from the Codazzi equation (2-8) for $X = U$ or $X = \phi U$ and $Y = \xi$ we have

(3-3) $$U(\beta) - \xi(\gamma) = \alpha\delta - 2\delta\kappa_3,$$

(3-4) $$\xi(\delta) = \alpha\gamma + \beta\kappa_1 + \delta^2 + \mu\kappa_3 + \tfrac{c}{4} - \gamma\mu - \gamma\kappa_3 - \beta^2,$$

(3-5) $$U(\alpha) - \xi(\beta) = -3\beta\delta,$$

(3-6) $$\xi(\mu) = \alpha\delta + \beta\kappa_2 - 2\delta\kappa_3,$$

(3-7) $$\phi U(\alpha) = \alpha\beta + \beta\kappa_3 - 3\beta\mu,$$

(3-8) $$\phi U(\beta) = \alpha\mu - 2\gamma\mu + 2\delta^2 + \tfrac{c}{2} + \alpha\gamma + \beta\kappa_1.$$

Similarly, from the Codazzi equation for $X = U$ and $Y = \phi U$ we have

(3-9) $$U(\delta) - \phi U(\gamma) = \mu\kappa_1 - \gamma\kappa_1 - \beta\gamma - 2\delta\kappa_2 - 2\beta\mu,$$

(3-10) $$U(\mu) - \phi U(\delta) = \gamma\kappa_2 + \beta\delta - \kappa_2\mu - 2\delta\kappa_1.$$

Moreover, applying Lemma 3.1, from the Gauss Equation (2-7) and the definition of the Riemannian curvature tensor $R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ we have

(3-11) $$U(\kappa_2) - \phi U(\kappa_1) = 2\delta^2 - 2\gamma\mu - \kappa_1^2 - \gamma\kappa_3 - \kappa_2^2 - \mu\kappa_3 - c.$$

(3-12) $$\phi U(\kappa_3) - \xi(\kappa_2) = 2\beta\mu - \mu\kappa_1 + \delta\kappa_2 + \kappa_3\kappa_1 + \beta\kappa_3.$$

The above relations can also be seen in [Panagiotidou and Xenos 2012; Wang 2018].

Proposition 3.2. *The shape operator of a real hypersurface in complex space forms is cyclic η-parallel if and only if it is η-parallel.*

Proof. From the Codazzi equation (2-8), we have $g((\nabla_X A)Y, Z) = g((\nabla_Y A)X, Z)$ for any vector fields X, Y, Z orthogonal to the structure vector field ξ. Since the shape operator is symmetric, we have $g((\nabla_X A)Y, Z) = g((\nabla_X A)Z, Y)$ for any vector fields X, Y, Z. Consequently, if the shape operator is cyclic η-parallel, i.e., $g((\nabla_X A)Y, Z) + g((\nabla_Y A)Z, X) + g((\nabla_Z A)X, Y) = 0$ for any vector fields X, Y, Z orthogonal to ξ, by the previous two properties we obtain $g((\nabla_X A)Y, Z) = 0$ for any vector fields X, Y, Z orthogonal to ξ. The converse is trivial. □

According to Proposition 3.2, we observe that [Kim et al. 2007, Theorem 1.1] and [Ahn et al. 1993, Theorem C] are the same. More precisely, as mentioned before, real hypersurfaces of dimension greater than three with η-parallel shape operators were completely classified by Kon and Loo [2011]. Thus, following their result and Proposition 3.2 we have:

Corollary 3.3. *A real hypersurface in a nonflat complex space form of complex dimension greater than two has cyclic η-parallel shape operator if and only if it is of type (A), (B) or is a ruled real hypersurface.*

However, the classification problem for real hypersurfaces of dimension three with η-parallel shape operators is still an open question; see also [Cho 2015; Niebergall and Ryan 1997]. Considering η-parallel shape operators together with some other conditions, we have:

Lemma 3.4 [Kimura and Maeda 1989; Niebergall and Ryan 1997; Suh 1990]. *A Hopf real hypersurface in a nonflat complex space form $M^n(c)$ with η-parallel shape operator is of type (A) or (B) for all $n \geq 2$.*

The next proposition shows that η-parallelism for shape operators is weak, so people studying this problem require some other geometric conditions (a recent paper on this topic is [Lim et al. 2013a]).

Proposition 3.5. *The shape operator of a real hypersurface in a nonflat complex space form of complex dimension two is η-parallel if and only if*

$$\begin{aligned} U(\gamma)-2\kappa_1\delta-2\beta\delta&=0,\\ U(\delta)+\kappa_1\gamma+\beta\gamma-\kappa_1\mu&=0,\\ \phi U(\delta)-\kappa_2\mu+\kappa_2\gamma+\beta\delta&=0,\\ \phi U(\mu)+2\kappa_2\delta&=0,\\ U(\mu)+2\kappa_1\delta&=0,\\ \phi U(\gamma)-2\beta\mu-2\kappa_2\delta&=0. \end{aligned} \tag{3-13}$$

Proof. According to Lemma 3.1 and relation (3-1), it follows directly that

$$(\nabla_U A)U=(U(\beta)-\alpha\delta)\xi+(U(\gamma)-2\kappa_1\delta-2\beta\delta)U+(U(\delta)+\kappa_1\gamma+\beta\gamma-\kappa_1\mu)\phi U.$$

$$(\nabla_U A)\phi U=(\delta^2-\mu\gamma+\beta\kappa_1+\alpha\gamma)\xi(U(\delta)-\kappa_1\mu+\kappa_1\gamma+\beta\gamma)U+(U(\mu)+2\kappa_1\delta)\phi U.$$

$$\begin{aligned}(\nabla_{\phi U} A)U=&(\phi U(\beta)+\mu\gamma-\alpha\mu-\delta^2)\xi+(\phi U(\gamma)-2\beta\mu-2\kappa_2\delta)U\\ &+(\phi U(\delta)+\kappa_2\gamma+\beta\delta-\kappa_2\mu)\phi U.\end{aligned}$$

$$(\nabla_{\phi U} A)\phi U=(\alpha\delta+\beta\kappa_2)\xi+(\phi U(\delta)-\kappa_2\mu+\kappa_2\gamma+\beta\delta)U+(\phi U(\mu)+2\kappa_2\delta)\phi U.$$

The shape operator is said to be η-parallel if $g((\nabla_X A)Y, Z)=0$ for any X, Y, Z orthogonal to ξ. Thus, the remainder of the proof follows immediately from the previous four equations. □

Next we present some solutions of the system of partial differential equations (3-13). But first, we need the following lemma:

Lemma 3.6 [Ahn et al. 1993; Kimura 1986]. *A real hypersurface in a nonflat space form $M^n(c)$, $n \geq 2$, is a ruled real hypersurface if and only if $g(AX, Y)=0$ for any vector fields X, Y orthogonal to ξ.*

A ruled real hypersurface M in $\mathbb{C}P^n(c)$ or $\mathbb{C}H^n(c)$ can be characterized by using the shape operator A, namely,

$$A\xi = \alpha\xi + \beta U \ (\beta \neq 0), \quad AU = \beta\xi, \quad AZ = 0$$

for any $Z \perp \{\xi, U\}$, where U is a unit vector field orthogonal to ξ, both α and β are functions on M.

Theorem 3.7. *Let M be a real hypersurface in a nonflat complex space form of complex dimension two with η-parallel shape operator. Then, ξ is an eigenvector field of the Ricci operator if and only if M is of type (A), (B) or is a ruled real hypersurface.*

Proof. In view of Lemma 3.4, next we need only to consider the non-Hopf case. Assume that M is non-Hopf and hence

$$\Omega = \{p \in M \mid \beta(p) \neq 0 \text{ in a neighborhood of } p\}$$

is nonempty. On Ω, the applications of (3-1) and (3-2) in (2-9) give

$$Q\xi = \left(\tfrac{1}{2}c + \alpha(\gamma+\mu) - \beta^2\right)\xi + \beta\mu U - \beta\delta\phi U.$$

If ξ is an eigenvector field of the Ricci operator, it follows that $\mu = \delta = 0$ on Ω. In this context, the system of partial differential equations (3-13) becomes

$$U(\gamma) = 0, \quad (\kappa_1 + \beta)\gamma = 0, \quad \kappa_2\gamma = 0, \quad \phi U(\gamma) = 0. \tag{3-14}$$

From Lemma 3.1, we acquire $[U, \phi U] = -\gamma\xi - \kappa_1 U - \kappa_2\phi U$. By virtue of (3-14), taking differentiation of γ along $[U, \phi U]$ implies $\gamma\xi(\gamma) = 0$.

First of all, on Ω we suppose that $\gamma \neq 0$ holds on some open subset $\mathcal{Q}$ of Ω and in view of (3-14) we see that γ is a nonzero constant. Now, (3-14) becomes $\kappa_1 + \beta = 0$ and $\kappa_2 = 0$. With the aid of $\mu = \delta = 0$, the application of this in (3-11) implies $\phi U(\beta) = -\beta^2 - \gamma\kappa_3 - c$, which is compared with (3-8) giving

$$\tfrac{3}{2}c + \alpha\gamma + \gamma\kappa_3 = 0.$$

In view of $\delta = \mu = 0$ and $\kappa_1 = -\beta$, (3-4) becomes $\frac{1}{4}c + \alpha\gamma - 2\beta^2 - \gamma\kappa_3 = 0$, which is compared with the previous equation, giving

$$\tfrac{7}{8}c + \alpha\gamma - \beta^2 = 0. \tag{3-15}$$

Taking differentiation of (3-15) along ϕU yields that $\gamma\phi U(\alpha) = 2\beta\phi(\beta)$ which is analyzed with the aid of (3-7) and (3-8) giving

$$c + \alpha\gamma - \gamma\kappa_3 - 2\beta^2 = 0,$$

where we have applied the assumption $\beta \neq 0$ on Ω. In view of $\kappa_1 = -\beta$ and $\mu = \delta = 0$, comparing the previous equation with (3-4) yields $c = 0$, and we arrive at a contradiction and hence $\mathcal{Q}$ is empty.

Therefore, we conclude that $\gamma = 0$ on Ω. Moreover, in view of $\mu = \delta = 0$, from Lemma 3.1 we see that $g(AX, Y) = 0$ for any vector fields X, Y orthogonal to ξ on Ω. Next, we need only prove that $M - \Omega$ is empty. Actually, when $M - \Omega$ is nonempty, on this subset ξ is principal and hence α is a constant; see [Kon 1979; Maeda 1976; Niebergall and Ryan 1997]. Applying Lemma 3.4, we see that those principal curvatures on $M - \Omega$ have the same property as those of real hypersurfaces of type (A) or (B) in M. Consequently, it follows that all principal curvatures on $M - \Omega$ are constant. In view of continuity of principal curvatures and connectedness of the hypersurface, we conclude that $M - \Omega$ is empty, or equivalently, Ω coincides with the whole of M. Finally, according to Lemma 3.6 we see that the hypersurface is locally congruent to a ruled real hypersurface. The converse is easy to check. This completes the proof. □

We continue to solve the system of partial differential equations (3-13) under some other conditions. First, we consider

$$g((A\phi - \phi A)X, Y) = 0 \text{ for any } X, Y \perp \xi. \tag{3-16}$$

Note that (3-16) was investigated in [Ahn et al. 1993; Lim et al. 2013a; Kim et al. 2007] for real hypersurfaces of dimension greater than three. In what follows, we aim to generalize those results for real hypersurfaces of dimension three.

Theorem 3.8. *Let M be a real hypersurface in a nonflat complex space form of complex dimension two with η-parallel shape operator. Then, M satisfies* (3-16) *if and only if M is of type* (A) *or ruled real hypersurface.*

Proof. According to Lemma 3.1, we see that M satisfies (3-16) if and only if

$$\delta = 0 \quad \text{and} \quad \gamma = \mu. \tag{3-17}$$

In this case, by Proposition 3.5, the shape operator is η-parallel if and only if

$$U(\mu) = \phi U(\mu) = 0 \quad \text{and} \quad \beta\gamma = \beta\mu = 0. \tag{3-18}$$

If the hypersurface M is Hopf, using $\beta = 0$, with the aid of the first term of (3-17) and (3-6), we observe that $\xi(\mu) = 0$ which is combined with the first two terms of (3-18) giving that μ is a constant. This implies $A\phi = \phi A$ and this is a sufficient and necessary condition for a real hypersurface to be of type (A); for more details see [Montiel and Romero 1986; Okumura 1975].

If M is not a Hopf hypersurface, then

$$\Omega = \{p \in M \mid \beta(p) \neq 0 \text{ in a neighborhood of } p\}$$

is nonempty. As seen in proof of Theorem 3.7, we state that Ω coincides with the whole of M. Therefore, on M, using $\beta \neq 0$, from the second term of (3-18) we obtain $\gamma = \mu = 0$. In view of the first term of (3-17), by Lemmas 3.1 and 3.6, we see that M is locally congruent to a ruled real hypersurface. The converse is easy to check. □

Remark 3.9. The above theorem improves [Lim et al. 2013a, Theorem 1.8].

We consider the following condition studied in [Kim et al. 2006; Sohn 2007], i.e.,

$$g((Q\phi - \phi Q)X, Y) = 0 \text{ for any } X, Y \perp \xi. \tag{3-19}$$

By using (3-19), a solution for (3-13) is given and applying this we have:

Theorem 3.10. *Let M be a real hypersurface in a nonflat complex space form of complex dimension two with η-parallel shape operator. Then, M satisfies* (3-19) *if and only if M is of type* (A) *hypersurface.*

Proof. Because the proof is long, we divide the discussion into several steps.

Step 1. We assume that the hypersurface M is non-Hopf and hence

$$\Omega = \{p \in M \mid \beta(p) \neq 0 \text{ in a neighborhood of } p\}$$

is nonempty. On Ω, the application of (2-9) and Lemma 3.1 for the case $n = 2$ gives

$$\begin{aligned} Q\xi &= (\tfrac{1}{2}r - c - \gamma\mu + \delta^2)\xi + \beta\mu U - \beta\delta\phi U, \\ QU &= \beta\mu\xi + (\tfrac{1}{2}r - \tfrac{1}{4}c - \alpha\mu)U + \alpha\delta\phi U, \\ Q\phi U &= -\beta\delta\xi + \alpha\delta U + (\tfrac{1}{2}r - \tfrac{1}{4}c - \alpha\gamma + \beta^2)\phi U, \end{aligned} \tag{3-20}$$

where the scalar curvature is given by

$$r = 3c + 2\alpha\gamma + 2\alpha\mu + 2\gamma\mu - 2\delta^2 - 2\beta^2. \tag{3-21}$$

It is clear that M satisfies (3-19) if and only if

$$\alpha\delta = 0 \quad \text{and} \quad \alpha\mu - \alpha\gamma + \beta^2 = 0. \tag{3-22}$$

We shall prove that from the first term of (3-22) we have $\delta = 0$. In fact, let us assume that $\delta \neq 0$ on a certain open subset of Ω and we shall get a contradiction. On this subset, from (3-22) we have $\alpha = 0$ and hence $\beta = 0$, which is a contradiction. Therefore, (3-19) is true on Ω if and only if $\delta = 0$ and $\alpha\mu - \alpha\gamma + \beta^2 = 0$. Also, in this case (3-13) becomes

$$\begin{aligned} &U(\mu) = \phi U(\mu) = 0, \quad \kappa_1(\mu - \gamma) = \beta\gamma, \\ &\kappa_2(\mu - \gamma) = 0, \quad U(\gamma) = 0, \quad \phi U(\gamma) = 2\beta\mu. \end{aligned} \tag{3-23}$$

As seen before, from Lemma 3.1 we have $[U, \phi U] = -(\gamma + \mu)\xi - \kappa_1 U - \kappa_2\phi U$. With the aid of (3-23) and (3-6), taking differentiation of μ along $[U, \phi U]$ we get

$$(\gamma + \mu)\beta\kappa_2 = 0. \tag{3-24}$$

In view of (3-24), next we consider the following subcases.

If $\kappa_2 \neq 0$ holds on some open subset of Ω, it follows from (3-24) that $\gamma + \mu = 0$ on this subset. Moreover, from (3-23) we have $\gamma = \mu$ which is compared with the previous relation giving $\gamma = \mu = 0$. However, in this context the second term of (3-22) becomes $\beta = 0$, contradicting the assumption. Thus, it follows from (3-24) that $\kappa_2 = 0$ on Ω.

By virtue of $\delta = \kappa_2 = 0$, (3-6) becomes $\xi(\mu) = 0$ and hence by the first two terms of (3-23) we observe that μ is a constant. In this case, from Lemma 3.1 we have $[U, \phi U] = -(\gamma+\mu)\xi - \kappa_1 U$. Taking differentiation of γ along $[U, \phi U]$, with the aid of (3-23), we obtain $2\mu U(\beta) = -(\gamma+\mu)\xi(\gamma)$ which is simplified by (3-3) and $\delta = 0$ giving

$$(\gamma + 3\mu)\xi(\gamma) = 0. \tag{3-25}$$

If $\gamma + 3\mu = 0$, from the second and last term of (3-23) we obtain $\mu = 0$. However, this is impossible because in this case the second term of (3-22) becomes $\beta = 0$, contradicting the assumption. Thus, from (3-25) we obtain $\gamma + 3\mu \neq 0$ and hence $\xi(\gamma) = 0$ on Ω. In this context, from Lemma 3.1 we have $[\xi, U] = (\kappa_3 - \gamma)\phi U$. With the aid of $\xi(\gamma) = 0$, (3-23) and the assumption $\beta \neq 0$ on Ω, the action of $[\xi, U]$ on γ gives

$$(\kappa_3 - \gamma)\mu = 0. \tag{3-26}$$

Because μ is a constant, with the aid of (3-7), (3-8), (3-23) and the assumption $\beta \neq 0$, taking differentiation of the second term of (3-22) along ϕU we get

$$\alpha\mu - \gamma\mu + c + \alpha\gamma + 2\beta\kappa_1 + \mu\kappa_3 - 3\mu^2 - \gamma\kappa_3 = 0.$$

Comparing the above equation with (3-4), with the aid of $\delta = 0$, we obtain

$$\beta\kappa_1 + \tfrac{3}{4}c + \alpha\mu - 3\mu^2 + \beta^2 = 0. \tag{3-27}$$

We shall show that (3-26) implies only one case, i.e., $\mu = 0$. Otherwise, when $\mu \neq 0$ holds on some open subset of Ω, we have $\kappa_3 = \gamma$. Substituting this into (3-4), we have

$$\alpha\gamma + \beta\kappa_1 + \tfrac{1}{4}c - \gamma^2 - \beta^2 = 0. \tag{3-28}$$

Substituting the second term of (3-22) into this equation gives $\frac{1}{4}c + \alpha\mu + \beta\kappa_1 - \gamma^2 = 0$, which is compared with (3-27) giving $\frac{1}{2}c + \gamma^2 - 3\mu^2 + \beta^2 = 0$. In view of $\beta \neq 0$ on Ω and the fact that μ is a constant, taking differentiation of (3-28) along ϕU we get

$$\tfrac{1}{2}c + \alpha\mu + \alpha\gamma + \beta\kappa_1 = 0.$$

Comparing the above equation with (3-28) we obtain $\frac{1}{4}c + \alpha\mu + \gamma^2 + \beta^2 = 0$, which is simplified by the second term of (3-22) giving

$$\tfrac{1}{4}c + \alpha\gamma + \gamma^2 = 0. \tag{3-29}$$

By virtue of $\delta = 0$ and (3-23), (3-9) becomes $\phi U(\gamma) = 2\beta\mu$. By applying this and (3-7) and taking differentiation of (3-29) along ϕU we obtain

$$2\alpha\mu + \alpha\gamma + \gamma^2 + \gamma\mu = 0, \tag{3-30}$$

which is combined with (3-29) giving

$$2\alpha\mu + \gamma\mu - \tfrac{1}{4}c = 0. \tag{3-31}$$

Because μ is a constant, taking differentiation of (3-31) along ϕU, with the aid of (3-7) and $\phi U(\gamma) = 2\beta\mu$, we obtain $\alpha + \gamma = 2\mu$. Consequently, substituting this into (3-30) we obtain

$$\alpha = -\tfrac{3}{2}\gamma \quad \text{and} \quad \mu = -\tfrac{1}{4}\gamma. \tag{3-32}$$

Finally, substituting (3-32) into (3-31) we get $\gamma^2 = \frac{1}{2}c$, a constant. Therefore, from the last term of (3-23) we have $\mu = 0$ because $\beta \neq 0$, contradicting our assumption.

Based on the above analyses, it follows from (3-26) that $\mu = 0$. From the second term of (3-13) we have $\gamma(\kappa_1 + \beta) = 0$ and hence $\gamma = 0$. In fact, if $\gamma \neq 0$ holds on some open subset of Ω, it follows that $\kappa_1 + \beta = 0$ and now the application of this and $\mu = 0$ in (3-27) implies $c = 0$, a contradiction. Taking into account $\gamma = \mu = 0$ and $\beta \neq 0$ on Ω, following Lemmas 3.4, 3.6 and the related statement in the proof of Theorem 3.7, we see that Ω coincides with the whole of M and hence the hypersurface is locally congruent to a ruled real hypersurface. However, with the aid of (3-20), one observes easily that ruled hypersurfaces do not satisfy (3-19). Thus, we conclude that the hypersurface M must be Hopf.

Step 2. Let the hypersurface M be Hopf. Using $\beta = 0$ and (3-20), we see that M satisfies (3-19) if and only if

$$\alpha\delta = 0 \quad \text{and} \quad \alpha(\mu - \gamma) = 0. \tag{3-33}$$

Moreover, using $\beta = 0$, (3-13) becomes

$$\begin{aligned} &U(\mu) = -2\kappa_1\delta, \qquad \phi U(\mu) = -2\kappa_2\delta, \qquad U(\delta) = \kappa_1(\mu - \gamma), \\ &\phi U(\delta) = \kappa_2(\mu - \gamma), \qquad U(\gamma) = 2\kappa_1\delta, \qquad \phi U(\gamma) = 2\kappa_2\delta. \end{aligned} \tag{3-34}$$

First, we show that on M there holds $\gamma = \mu$. If this is not true, then

$$\mathcal{W} = \{q \in M \mid (\gamma - \mu)(q) \neq 0 \text{ in a neighborhood of } q\}$$

is nonempty and an open subset of M. On $\mathcal{W}$, from the second term of (3-33) we have $\alpha = 0$. Using this and $\beta = 0$ on (3-8) we have

$$\delta^2 - \gamma\mu + \frac{c}{4} = 0. \tag{3-35}$$

Applying (3-35) together with $\alpha=\beta=0$ on (3-4) we obtain $\xi(\delta)=(\mu-\gamma)\kappa_3$. From Lemma 3.1 we obtain $[U,\phi U]=-(\gamma+\mu)\xi-\kappa_1 U-\kappa_2\phi U$. With the aid of (3-34), the action of this equation on δ reduces to $(\mu-\gamma)(U(\kappa_2)-\phi U(\kappa_1))=(\gamma-\mu)(\kappa_1^2+\kappa_2^2+(\gamma+\mu)\kappa_3)$. Because on $\mathcal{W}$ we have $\gamma\neq\mu$, it follows that

$$U(\kappa_2)-\phi U(\kappa_1)=-(\kappa_1^2+\kappa_2^2+(\gamma+\mu)\kappa_3),$$

which is simplified by (3-11) giving $\delta^2-\gamma\mu-\frac{c}{2}=0$. Comparing this with (3-35) we get $c=0$, which is a contradiction. This means that $\mathcal{W}$ is empty and we always have $\gamma=\mu$ on M.

Second, we show that on M we have $\delta=0$. If this is not true, then

$$\mathcal{N}=\{q\in M\mid \delta(q)\neq 0 \text{ in a neighborhood of } q\}$$

is nonempty and an open subset of M. It follows from the first term of (3-33) that $\alpha=0$ on $\mathcal{N}$. Notice that on $\mathcal{N}$ equation (3-35) is still true in this situation. The application of this and $\alpha=\beta=0$ on (3-4) gives $\xi(\delta)=0$. Moreover, the application of $\mu=\gamma$ on (3-34) gives $U(\delta)=\phi U(\delta)=0$, that is, δ is a constant.

Using $\alpha=\beta=0$ on $\mathcal{N}$, from (3-3) and (3-6) we have

$$\xi(\gamma)=2\kappa_3\delta \quad \text{and} \quad \xi(\mu)=-2\kappa_3\delta.$$

From the above two relations and (3-34), it is easily seen that $\mu+\gamma$ is a constant. Consequently, in view of constancy of δ, from (3-35) we see that both γ and μ are constant. The application of this on (3-34) and $\xi(\gamma)=2\kappa_3\delta$ yields

$$\kappa_1=\kappa_2=\kappa_3=0$$

on $\mathcal{N}$. However, using the above relations in (3-11) we have $\delta^2-\gamma\mu-\frac{c}{2}=0$, which is compared with (3-35), giving $c=0$, a contradiction. This means that $\mathcal{N}$ is empty and on the whole of M we always have $\delta=0$.

Finally, in view of $\delta=0$ and $\gamma=\mu$, by Lemma 3.1 we obtain $A\phi=\phi A$ on M. Following [Montiel and Romero 1986; Okumura 1975], we observe that the hypersurface M is locally congruent to a type (A) real hypersurface. The converse is easy to check. This completes the proof. □

The classification problem for η-parallel shape operators has existed for a long time, but it is hard to solve. Based on results shown in this section and the introduction, especially those of [Kon and Loo 2011] for dimension greater than three, we propose:

Conjecture 3.11. *A real hypersurface in a nonflat complex space form of complex dimension two has η-parallel shape operator if and only if it is of type (A), (B) or is a ruled real hypersurface.*

4. Cyclic η-parallel Ricci operator

The characterizations of real hypersurfaces in a nonflat complex space form by means of the Ricci operator were studied by many authors; see [Lim et al. 2013a; 2013b; Ki 1988; Kon 2014; 2017; Kwon and Nakagawa 1988; 1989; Niebergall and Ryan 1997]. Among others, Ricci η-parallelism was one of the most often discussed conditions. In this section, as applications of some results in Section 3, we aim to classify three-dimensional real hypersurfaces satisfying cyclic η-parallel Ricci operator. Unlike the case of shape operators, we show that the Ricci cyclic η-parallelism condition is much weaker than Ricci η-parallelism.

Proposition 4.1. *The Ricci operator of a real hypersurface in a nonflat complex space form of complex dimension two is cyclic η-parallel if and only if*

$$U(r-2\alpha\mu)-4\delta\beta\mu-4\alpha\delta\kappa_1=0, \tag{4-1}$$

$$\phi U(r-2\alpha\gamma+2\beta^2)-4\beta\delta^2+4\alpha\delta\kappa_2=0, \tag{4-2}$$

$$\begin{aligned}&4(U(\alpha\delta)+\kappa_1(\alpha\gamma-\alpha\mu-\beta^2)+\beta\mu\gamma+\beta\delta^2)\\&\qquad+\phi U(r-2\alpha\mu)-4\beta\mu^2-4\alpha\delta\kappa_2=0,\end{aligned} \tag{4-3}$$

$$\begin{aligned}&4(\phi U(\alpha\delta)+2\beta\mu\delta+\kappa_2(\alpha\gamma-\beta^2-\alpha\mu))\\&\qquad+U(r-2\alpha\gamma+2\beta^2)-4\beta\delta\gamma+4\alpha\delta\kappa_1=0.\end{aligned} \tag{4-4}$$

Proof. The application of Lemma 3.1 and (3-20) gives

$$\begin{aligned}(\nabla_U Q)U=&\\&\tfrac{1}{4}(4U(\beta\mu)+\delta(3c-4\alpha\mu+4\gamma\mu-4\delta^2)+4\beta\delta\kappa_1-4\alpha\delta\gamma)\xi\\&+\tfrac{1}{2}(U(r-2\alpha\mu)-4\beta\delta\mu-4\alpha\delta\kappa_1)U\\&+(U(\alpha\delta)+\kappa_1(\alpha\gamma-\alpha\mu-\beta^2)+\beta\gamma\mu+\beta\delta^2)\phi U.\end{aligned} \tag{4-5}$$

$$\begin{aligned}(\nabla_U Q)\phi U=&\\&\tfrac{1}{4}(4\alpha\delta^2-4U(\beta\delta)+\gamma(4\alpha\gamma-3c-4\beta^2-4\gamma\mu+4\delta^2)+4\beta\mu\kappa_1)\xi\\&+(U(\alpha\delta)+\beta\delta^2+\kappa_1(\alpha\gamma-\alpha\mu-\beta^2)+\beta\mu\gamma)U\\&+\tfrac{1}{2}(U(r-2\alpha\gamma+2\beta^2)-4\beta\delta\gamma+4\alpha\delta\kappa_1)\phi U.\end{aligned} \tag{4-6}$$

$$\begin{aligned}(\nabla_{\phi U} Q)U=&\\&\tfrac{1}{4}(4\phi U(\beta\mu)+\mu(3c+4\gamma\mu-4\delta^2-4\alpha\mu)-4\alpha\delta^2+4\beta\delta\kappa_2)\xi\\&+\tfrac{1}{2}(\phi U(r-2\alpha\mu)-4\beta\mu^2-4\alpha\delta\kappa_2)U\\&+(\phi U(\alpha\delta)+2\beta\delta\mu+\kappa_2(\alpha\gamma-\alpha\mu-\beta^2))\phi U.\end{aligned} \tag{4-7}$$

(4-8) $(\nabla_{\phi U} Q)\phi U =$

$$\tfrac{1}{4}(4\alpha\mu\delta - 4\phi U(\beta\delta) + 4\beta\mu\kappa_2 + \delta(4\alpha\gamma - 3c - 4\beta^2 - 4\gamma\mu + 4\delta^2))\xi$$
$$+ (\phi U(\alpha\delta) + 2\beta\mu\delta + \kappa_2(\alpha\gamma - \beta^2 - \alpha\mu))U$$
$$+ \tfrac{1}{2}(\phi U(r - 2\alpha\gamma + 2\beta^2) - 4\beta\delta^2 + 4\alpha\delta\kappa_2)\phi U.$$

The Ricci tensor is cyclic η-parallel if and only if

$$g((\nabla_X Q)Y, Z) + g((\nabla_Y Q)Z, X) + g((\nabla_Z Q)X, Y) = 0$$

for any vector fields X, Y, Z orthogonal to ξ. Locally, this is also equivalent to

$$g((\nabla_U Q)U, U) = 0, \quad 2g((\nabla_{\phi U} Q)\phi U, U) + g((\nabla_U Q)\phi U, \phi U) = 0,$$
$$g((\nabla_{\phi U} Q)\phi U, \phi U) = 0, \qquad 2g((\nabla_U Q)U, \phi U) + g((\nabla_{\phi U} Q)U, U) = 0.$$

The proof follows directly from (4-5)–(4-8). □

Some solutions of the system of equations (4-1)–(4-4) are given as follows.

Theorem 4.2. *Let M be a real hypersurface in a nonflat complex space form of complex dimension two with cyclic η-parallel Ricci operator. Then, M satisfies* (3-16) *if and only if it is of type* (A).

Proof. We assume that $\Omega = \{p \in M \mid \beta(p) \neq 0 \text{ in a neighborhood of } p\}$ is nonempty. Suppose that M has a cyclic η-parallel Ricci operator and satisfies (3-16). As seen in the proof of Theorem 3.8, M satisfies (3-16) if and only if $\delta = 0$ and $\gamma = \mu$. With the aid of this, substituting (4-2) into (4-3) gives

$$\beta(\phi U(\beta) + \kappa_1\beta) = 0. \tag{4-9}$$

Next we show that it follows from (4-9) that Ω is empty. Otherwise, on Ω, from (4-9) we have $\phi U(\beta) = -\kappa_1\beta$, which is used in (3-8), giving

$$2\alpha\mu - 2\mu^2 + \frac{c}{2} + 2\beta\kappa_1 = 0, \tag{4-10}$$

where we have used $\gamma = \mu$ and $\delta = 0$. Applying this again and comparing (4-10) with (3-4) implies $\beta = 0$, which is a contradiction. We have proved that the hypersurface M must be Hopf.

In view of $\beta = 0$ and the above statement, we see that M satisfies (3-16) if and only if $\delta = 0$ and $\gamma = \mu$. In this context, from Lemma 3.1 we have $A\phi = \phi A$ and the proof follows directly from [Montiel and Romero 1986; Okumura 1975].

The converse is easy to check. This completes the proof. □

Remark 4.3. Kim, Kim and Sohn [Kim et al. 2007, Theorem 1.3] proved that real hypersurfaces of dimension > 3 satisfying (3-16) and cyclic η-parallel Ricci operators are of type (A). Our Theorem 4.2 extends their results for real hypersurfaces of dimension three.

Before giving another solution of partial differential equations (4-1)–(4-4), we need the following:

Lemma 4.4 [Maeda 2013; Suh 1990]. *Let M be a connected Hopf real hypersurface in $\mathbb{C}P^n(c)$ or $\mathbb{C}H^n(c)$, $c \neq 0$, $n \geq 2$. If M has η-parallel Ricci tensor, then it is locally congruent to a homogeneous type (A) or (B) hypersurface or a nonhomogeneous real hypersurface with vanishing Hopf principal curvature.*

With the aid of the above result, we have:

Theorem 4.5. *Let M be a real hypersurface in a nonflat complex space form of complex dimension two with cyclic η-parallel Ricci operator. Then, M satisfies* (3-19) *if and only if it is of homogeneous type (A) or (B) real hypersurface or a nonhomogeneous real hypersurface with vanishing Hopf principal curvature.*

Proof. We assume that M is non-Hopf, then

$$\Omega = \{p \in M \mid \beta(p) \neq 0 \text{ in a neighborhood of } p\}$$

is nonempty. Suppose that M has cyclic η-parallel Ricci operator and satisfies (3-19). As seen in the proof of Theorem 3.10, M satisfies (3-19) if and only if

$$\alpha\delta = 0 \quad \text{and} \quad \alpha\mu - \alpha\gamma + \beta^2 = 0. \tag{4-11}$$

If $\delta \neq 0$ holds on some open subset of Ω, it follows from (4-11) that $\alpha = \beta = 0$, contradicting the definition of Ω. Thus, on Ω we have $\delta = 0$.

Because on Ω we have $\beta \neq 0$, with the aid of the second term of (4-11) and $\delta = 0$, subtracting (4-3) from (4-2) we obtain

$$\mu(\gamma - \mu) = 0. \tag{4-12}$$

If $\mu = 0$, with the aid of $\delta = 0$, (4-1)–(4-4) and the second term of (4-11), from (4-5)–(4-8) one can check that the Ricci operator is η-parallel. However, from Lemma 4.4 we see that the hypersurface is Hopf, contradicting the assumption. Otherwise, if $\mu \neq 0$ holds on some open subset of Ω, from (4-12) we have $\gamma = \mu$. On such a subset, using $\delta = 0$ and $\gamma = \mu$, by Lemma 3.1 we obtain $A\phi = \phi A$ and hence M is of type (A) Hopf hypersurface (see [Montiel and Romero 1986; Okumura 1975]), a contradiction. Based on the above statement, we see that Ω is empty and the hypersurface M is Hopf.

In this context, M satisfies (3-19) if and only if

$$\alpha\delta = 0 \quad \text{and} \quad \alpha(\mu - \gamma) = 0. \tag{4-13}$$

Moreover, in this case, with the aid of (4-13), (4-1)–(4-4) become

$$\begin{aligned} U(r - 2\alpha\mu) &= U(r - 2\alpha\gamma) = 0, \\ \phi U(r - 2\alpha\mu) &= \phi U(r - 2\alpha\gamma) = 0. \end{aligned} \tag{4-14}$$

According to (4-13) and (4-14), from (4-5)–(4-8) it is easily seen that the Ricci operator is η-parallel. Applying Lemma 4.4, we see that M is locally congruent to a homogeneous type (A) or (B) real hypersurface or a nonhomogeneous real hypersurface with $A\xi = 0$.

The converse is easy to check. This completes the proof. □

Remark 4.6. Sohn [2007, Theorem 2] proved that real hypersurfaces of dimension greater than three satisfying (3-19) and η-parallel Ricci operators are of type (A) or (B). Later, Lim, Sohn and Song [Lim et al. 2013b] proved that three-dimensional real hypersurfaces satisfying (3-19) and η-parallel Ricci operators are of type (A) or satisfy $A\xi = 0$. Obviously, our Theorem 4.5 is an extension of these results.

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