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FINSLER SPHERES WITH CONSTANT FLAG CURVATURE AND FINITE ORBITS OF PRIME CLOSED GEODESICS

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In this paper, we consider a Finsler sphere $(M, F) = (S^n, F)$ with dimension $n > 1$ and flag curvature $K \equiv 1$. The action of the connected isometry group $G = I_o(M, F)$ on M, together with the action of $T = S^1$ shifting the parameter $t \in \mathbb{R}/\mathbb{Z}$ of the closed curve $c(t)$, define an action of $\hat{G} = G \times T$ on the free loop space ΛM of M. In particular, for each closed geodesic, we have a $\hat{G}$-orbit of closed geodesics. We assume the Finsler sphere (M, F) described above has only finite orbits of prime closed geodesics. Our main theorem claims that, if the subgroup H of all isometries preserving each close geodesic is of dimension m, then there exists m geometrically distinct orbits $\mathcal{B}_i$ of prime closed geodesics, such that for each i, the union B_i of geodesics in $\mathcal{B}_i$ is a totally geodesic submanifold in (M, F) with a nontrivial H_o-action. This theorem generalizes and slightly refines the one in a previous work, which only discussed the case of finite prime closed geodesics. At the end, we show that, assuming certain generic conditions, the Katok metrics, i.e., the Randers metrics on spheres with $K \equiv 1$, provide examples with the sharp estimate for our main theorem.

1. Introduction

In the recent work of R. L. Bryant, P. Foulon, S. Ivanov, V. S. Matveev and W. Ziller [Bryant et al. 2017], the authors classified Finsler spheres with constant flag curvature $K \equiv 1$ according to the behavior of geodesics. The Katok metric [1973] provides the most important key model for their classification. The celebrated Anosov conjecture [1975], claiming the minimal number of prime closed geodesics on a Finsler sphere (S^n, F) is $2[(n+1)/2]$, was based on the discovery of Katok metrics with only finite prime closed geodesics. There are many works using Morse theory and index theory to study the closed geodesics and Anosov conjecture in

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Finsler geometry, assuming a pinch condition for the flag curvature, nondegenerating property for all closed geodesics, or using the specialty of low dimensions. See for example [Bangert and Long 2010; Duan 2016; Long and Duan 2009; Duan et al. 2016; Rademacher 1989; Wang 2012; 2015]. From the geometrical point of view, it was much later that people noticed that Katok metrics are Randers metrics on spheres with constant flag curvature [Rademacher 2004]. D. Bao, C. Robles and Z. Shen [Bao et al. 2004] provided a complete classification for all Randers metrics with constant flag curvature. The classification for the non-Randers case is still widely open. Bryant [1996; 1997; 2002]. provided many important examples of Finsler spheres with $K \equiv 1$.

However, one of the most important technique in [Bryant et al. 2017] is from Lie theory. The authors considered the antipodal map ψ for a Finsler sphere with $K \equiv 1$ (see [Bryant et al. 2017; Shen 1996] or Section 2 for its definition). It is a Clifford Wolf translation in the center of the isometry group $I(M, F)$. When ψ has an infinite order, after taking closure, it can be used to generate a closed abelian subgroup of isometries with a positive dimension.

For nonzero Killing vector fields on a Finsler sphere with $K \equiv 1$, we have the following totally geodesic technique. The common zero point set of Killing vector fields, or more generally the fixed point set of isometries, provide closed totally geodesic submanifolds. In particular, when the dimension of such a submanifold is one, it is a reversible geodesic, and when the dimension is even bigger, it is a Finsler sphere inheriting the curvature property and geodesic property from the ambient space. We can use this key observation to set up an inductive argument, when studying the geodesics on (S^n, F) with $n > 2$ and $K \equiv 1$, and generalizing some results in [Bryant et al. 2017] to high dimensions.

For example, we have proved the following lower bound estimate for the number of reversible prime closed geodesics in Finsler spheres with constant flag curvature.

Theorem 1.1 [Xu 2018b]. *Let $(M, F) = (S^n, F)$ with $n > 1$ be a Finsler sphere with $K \equiv 1$ and only finite prime closed geodesics. Then the number of geometrically distinct reversible closed geodesics is at least* $\dim I(M, F)$.

Recall that a geodesic $c(t)$ with constant speed is called *reversible* if $c(-t)$ also provides a geodesic with constant speed after a reparametrization by the new arc length. Two geodesics are *geometrically distinct* if and only if they are different subsets.

The assumption of only finite prime closed geodesics imposes a strong restriction on $I_o(M, F)$, which can only be a torus. A lot of important examples are excluded, for example, the standard unit spheres and the homogeneous non-Riemannian Randers spheres with $K \equiv 1$. So if we want more possibility for $I_o(M, F)$, the geodesic condition could be replaced by the assumption that there exist only finite orbits of prime closed geodesics, or Assumption (F) for simplicity. See Section 3 for its precise definition and detailed discussion.

The main purpose of this paper is to prove the following theorem.

Theorem 1.2. *Let $(M, F) = (S^n, F)$ be a Finsler sphere satisfying $n > 1$, $K \equiv 1$ and Assumption (F). Denote by H the subgroup of $G = I_o(M, F)$ preserving each closed geodesic, H_o its identity component and $m = \dim H$. Then there exist at least m geometrically distinct orbits $\mathcal{B}_i$'s of prime closed geodesics such that each union B_i of geodesics in $\mathcal{B}_i$ is a totally geodesic submanifold in M with a nontrivial H_o-action.*

When (M, F) has only finite prime closed geodesics, then Assumption (F) is satisfied, $H_o = G = I_o(M, F)$, and each orbit of closed geodesics consists of only one closed geodesic. So Theorem 1.2 generalizes Theorem 1.1. It even slightly refines Theorem 1.1 by claiming the totally geodesic B_i's found have nontrivial H_o-actions. So if the common zero point of H_o has a positive dimension, it provides one more totally geodesic B_i, which is either a reversible closed geodesic which length is a rational multiple of π, or isometric to a standard unit sphere.

By Theorem 1.1 and Theorem 1.2 in [Xu 2018a], each submanifold $(B_i, F|_{B_i})$ is in fact a non-Riemannian homogeneous Randers sphere with constant flag curvature. So Theorem 1.2 implies the existence of totally geodesic subspheres in which F has standard restrictions, though F itself may be strange.

This paper is organized as following. In Section 2, we recall some fundamental geometric properties of Finsler spheres with $K \equiv 1$, discussing their antipodal maps and totally geodesic submanifolds. In Section 3, we define Assumption (F), i.e., the assumption of only finite prime closed geodesics. In Section 4, we introduce the subgroup H of isometries which preserves each closed geodesic. In Section 5, we prove Theorem 1.2 by induction. In Section 6, we discuss the Katok metrics, and show that in some cases they provides examples for Theorem 1.2, with a sharp estimate.

2. Preliminaries: from antipodal map to Killing vector field

Let $(M, F) = (S^n, F)$ be a Finsler sphere satisfying the dimension $n > 1$ and the flag curvature $K \equiv 1$. Denote $G = I_o(M, F)$ the connected isometry group, i.e., the identity component of the isometry group $I(M, F)$ of (M, F).

We briefly recall the definition of the exponential map [Bao et al. 2000] and the antipodal map ψ [Bryant et al. 2017; Shen 1996] for (M, F).

For any $x \in M$ and nonzero $y \in T_xM$, the *exponential map* $\mathrm{Exp}_x : T_xM \to M$ is defined by $\mathrm{Exp}_x(y) = c(1)$ where $c(t)$ is the constant speed geodesic with $c(0) = x$ and $\dot{c}(0) = y$. When $y = 0 \in T_xM$, we define $\mathrm{Exp}_x(0) = x$. Notice that Exp_x is C^1 at $y = 0$ and C^∞ elsewhere.

The discussion for the Jacobi fields and conjugation points when $K \equiv 1$ indicates Exp_x maps the sphere

$$S_o^F(\pi) = \{y \in T_xM \mid F(y) = \pi\} \subset T_xM$$

to a single point $x^* \in M$. The map from x to x^* is an isometry of (M, F) in the center of $I(M, F)$ [Bryant et al. 2017]. Further more, it is easy to see that ψ is a Clifford Wolf translation for the (possibly nonreversible) distance $d_F(\cdot, \cdot)$ defined by the Finsler metric F. We will call it the *antipodal map* and always denote it as ψ. It is a generalization for the antipodal map for standard unit spheres but may not be an involution any more.

The above description immediately proves that any connected and simply connected Finsler manifold (M, F) with $\dim M > 1$ and $K \equiv 1$ is homeomorphic to a sphere. A more careful discussion with the local charts shows that the homeomorphism in this statement can be refined to be a diffeomorphism, and the argument is valid not only for M, but also any closed connected totally geodesic submanifold N with $\dim N > 1$, i.e., we have the following lemma.

Lemma 2.1 [Xu 2018b, Lemma 3.2]. *Let (M, F) be a closed connected and simply connected Finsler manifold with $K \equiv 1$ and N a closed connected totally geodesic submanifold with* $\dim N > 1$. *Then both M and N are diffeomorphic to standard spheres*, *and N is an imbedded submanifold in M.*

The fixed point set for a family of isometries in $I(M, F)$ is a closed, possibly disconnected, totally geodesic submanifold. We have the following lemma, indicating the connectedness of N, when its dimension is positive.

Lemma 2.2 [Xu 2018b, Lemma 3.5]. *Let $(M, F) = (S^n, F)$ be a Finsler sphere with $n > 1$ and $K \equiv 1$, and N the fixed point set of a family of isometries of (M, F). Then N must satisfy one of the following*:

(1) *N is a two-points ψ-orbit, i.e., $N = \{x', x''\}$ with $d_F(x', x'') = d_F(x'', x') = \pi$.*

(2) *N is a reversible closed geodesic.*

(3) *$(N, F|_N)$ is a Finsler sphere with* $\dim N > 1$ *and $K \equiv 1$.*

The space of Killing vector fields can be viewed as the Lie algebra of $I(M, F)$. So the common zero set of a family of Killing vector fields on (M, F) is a special case of fixed point sets for isometries.

In later discussions, we will need the following two lemmas for Killing vector fields.

Lemma 2.3. *Assume that X is a Killing vector field of the Finsler space (M, F), $f(\cdot) = F(X(\cdot))$ and $f(x) > 0$ at $x \in M$. Then the integration curve of X passing x is a geodesic if and only if x is a critical point of $f(\cdot)$.*

Lemma 2.4 (corollary of [Deng and Xu 2014, Lemma 3.1]). *Assume that $c = c(t)$ is a geodesic of positive constant speed on the Finsler space (M, F). Then restricted to $c(t)$, any Killing vector field X of (M, F) satisfies*

$$\langle X(c(t)), \dot{c}(t)\rangle^F_{\dot{c}(t)} \equiv \text{const}, \tag{2-1}$$

where $\langle u, v\rangle^F_y = g_{ij}(y)u^i v^j$ for $u, v, y \in T_x M$ and $y \neq 0$ is the inner product defined by the fundamental tensor.

Proof. Whenever the value of X is linearly independent of $\dot{c}(t)$, we can prove (2-1) by choosing a special local chart, such that $c = c(t)$ can be presented as $x^1 = t$ and $x^i = 0$ for $i > 1$, and $X = \partial_{x^2}$. Because X is Killing vector field, $F(x, y)$ is independent of x^2. The condition that $c = c(t)$ is a geodesic implies that for the coefficients $\boldsymbol{G}^i$ of the geodesic spray, we have

$$\begin{aligned}\boldsymbol{G}^i(c(t), \dot{c}(t)) &= \tfrac{1}{4} g^{il}([F^2]_{x^m y^l} y^m - [F^2]_{x^l})\\ &= \tfrac{1}{4} g^{il}([F^2]_{x^1 y^l} - [F^2]_{x^l}) = 0.\end{aligned}$$

In particular, on the geodesic $c = c(t)$, we have

$$\frac{d}{dt}\langle X(c(t)), \dot{c}(t)\rangle^F_{\dot{c}(t)} = \tfrac{1}{2}[F^2]_{x^1 y^2} = \tfrac{1}{2}[F^2]_{x^2} = 0,$$

which proves the lemma in this case.

When X is tangent to $c = c(t)$ for t in an interval I, we can easily get (2-1) for $t \in I$.

Summarizing this two cases and using the continuity, we have proved (2-1) along the whole geodesic $c = c(t)$. □

3. Orbit of closed geodesics and Assumption (F)

Now we define *Assumption (F)*, i.e., the condition that (M, F) has only finite orbits of prime closed geodesics. In later discussion, we will always assume it to be satisfied by (M, F) unless otherwise specified.

The free loop space ΛM of all piecewise smooth path $c = c(t)$ with $t \in \mathbb{R}/\mathbb{Z}$ (sometimes we will simply denote it as c or γ) admits the natural actions of $\hat{G} = G \times T$ such that

$$((g, t')\cdot c)(t) = g\cdot c(t + t'), \quad \text{for all } t.$$

So for each closed geodesic γ of constant speed, we have an $\hat{G}$-orbit $\hat{G}\cdot\gamma$ of closed geodesics with the same speed. The geodesic $c(t)$ (with $t \in \mathbb{R}/\mathbb{Z}$) is *prime*, i.e.,

$$\min\{t \mid t > 0 \text{ and } c(t) = c(0)\} = 1,$$

if and only if all the closed geodesics in $\hat{G}\cdot c$ are prime.

Definition 3.1. We say (M, F) has only finite orbits of prime closed geodesics if it satisfies

Assumption (F) all the prime closed geodesics of positive constant speed can be listed as a finite set of $\hat{G}$-orbits, $\mathcal{B}_i = \hat{G} \cdot \gamma_i$, $1 \leq i \leq k$.

In Definition 3.1, we can equivalently list all the closed geodesics of constant speed $c(t)$ with $t \in \mathbb{R}/\mathbb{Z}$ as $\mathcal{B}_i^j = \hat{G} \cdot \gamma_i^j$, $1 \leq i \leq k$, $j \in \mathbb{N}$. The orbit $\mathcal{B}_i$ in Definition 3.1 coincides with $\mathcal{B}_i^1$, for each i. The closed geodesics γ_i^j is the one which rotates j-times along the prime closed geodesic γ_i in Definition 3.1, i.e., if γ_i is presented as $c_i = c_i(t)$, then γ_i^j is $c_{i,j}(t) = c_i(jt)$.

We denote B_i the union of the geodesics in $\mathcal{B}_i$ or $\mathcal{B}_i^j$ for any $j \in \mathbb{N}$. Then we call $\mathcal{B}_i^j$ and $\mathcal{B}_{i'}^{j'}$ *geometrically distinct* (or *geometrically the same*), if B_i and $B_{i'}$ are different subsets (or the same subsets, respectively) of M.

The Assumption (F) for the ambient space can be inherited by some totally geodesic submanifolds, i.e., we have the following lemma.

Lemma 3.2. *Let (M, F) be any closed compact Finsler manifold satisfying Assumption (F), ϕ_α with $\alpha \in \mathcal{A}$ a family of isometries in the center of $I(M, F)$, and N the fixed point set for all ϕ_α's. Then each orbit of prime closed geodesic for $(N, F|_N)$ is also an orbit of prime closed geodesic for (M, F). In particular, $(N, F|_N)$ also satisfies Assumption (F).*

Proof. The fixed point set N for the isometries ϕ_α with $\alpha \in \mathcal{A}$ is a closed (possibly disconnected) totally geodesic submanifold of (M, F). Because each ϕ_α commutes with all isometries of (M, F), the fixed point set N for all ϕ_α's is preserved by the action of $G = I_o(M, F)$. The restriction of G-action to N defines isometries in $G' = I_o(N, F|_N)$. Denote $\hat{G}' = G' \times T$. Then for each prime closed geodesic γ in N, Assumption (F) implies that $\hat{G}' \cdot \gamma$ is a disjoint finite union of $\hat{G}$-orbits. Both $\hat{G}'$-orbits and $\hat{G}'$-orbits are compact and connected, so we get $\hat{G}' \cdot \gamma = \hat{G} \cdot \gamma$, which proves the first claim. The second claim follows immediately. □

The effect of Assumption (F) can be seen from the behavior of the antipodal map ψ. For example, when ψ has a finite order k, i.e., there exists a positive integer k, such that

$$\psi^k = \mathrm{id}, \quad \text{and} \quad \psi^i \neq \mathrm{id} \quad \text{when } 1 \leq i < k,$$

we have the following lemma.

Lemma 3.3. *Let $(M, F) = (S^n, F)$ be a Finsler sphere satisfying $n > 1$, $K \equiv 1$ and Assumption (F). Assume that the antipodal map ψ has a finite order k. Then F must be the Riemannian metric for a standard unit sphere.*

Proof. Because ψ is a Clifford Wolf translation, and it has a finite order k, each geodesic of (M, F) is closed, and each prime closed geodesic admits a suitable multiple such that the length of the resulting closed geodesic is $k\pi$. By Assumption (F), the subset $B \subset \Lambda M$ of all closed geodesics with the length $k\pi$ can be listed as the disjoint union of $\mathcal{B}_i^{n_i} = \hat{G} \cdot \gamma_i^{n_i}$, $1 \le i \le k$, where each γ_i is a prime closed geodesic. Obviously B is connected and each $\mathcal{B}_i^{n_i}$ is compact, so we must have $k = 1$.

Then we prove (M, F) is G-homogeneous. Assume conversely that it is not, we consider a unit speed geodesic $c(t)$, and the G-orbit N passing $c(0)$, such that

$$\langle \dot{c}(0), T_{c(0)}N \rangle^F_{\dot{c}(0)} = 0. \tag{3-1}$$

Then by Lemma 2.4, for any Killing vector field $X \in \mathfrak{g}$, we have

$$\langle \dot{c}(t), X(c(t)) \rangle^F_{\dot{c}(t)} \equiv 0,$$

i.e., $c(t)$ meets each G-orbit orthogonally in the sense of (3-1). This property is preserved by $\hat{G}$-actions. So its $\hat{G}$-orbit can not exhaust all the geodesics, for example, those which does not satisfy (3-1). This is a contradiction to our previous observation that (M, F) can only have one orbit of prime closed geodesics, and it proves that (M, F) is homogeneous Finsler sphere.

Finally, we prove (M, F) is a standard unit sphere. Because (M, F) is a homogeneous Finsler space, it has at least one homogeneous geodesic $c(t) = \exp(tX) \cdot o$, in which $o \in M$ and $X \in \mathfrak{g} = \mathrm{Lie}(G)$ [Yan and Huang 2018]. Our previous observation that all geodesics belong to a single $\hat{G}$-orbit implies all geodesics are homogeneous. So for any $x \in M$ and any two F-unit tangent vectors y_1 and y_2 in T_xM, we have two unit speed geodesics $c_1(t)$ and $c_2(t)$ such that $c_1(0) = c_2(0) = x$ and $\dot{c}_i(0) = y_i$. Both geodesics belong to the same $\hat{G}$-orbit, so we can find $g_1 \in G$ such that $(g_1 \cdot c_1)(t) \equiv c_2(t + t_0)$ for some fixed t_0. Because the geodesic $c_2(t)$ is homogeneous, we can find another $g_2 \in G$ such that $(g_2 \cdot c_2)(t) = c_2(t - t_0)$. Then we have

$$(g_2 g_1 \cdot c_1)(t) = (g_2 \cdot c_2)(t + t_0) = c_2(t), \quad \text{for all } t.$$

So the isotropy action for (M, F) is transitive at each point. The only homogeneous spheres satisfying this property are Riemannian spheres of constant curvature. □

Using Lemmas 3.2 and 3.3, we can generalize Lemma 3.6 in [Xu 2018b] to the following.

Lemma 3.4. *Let $(M, F) = (S^n, F)$ be a Finsler sphere satisfying $n > 1$, $K \equiv 1$ and Assumption (F). Then the union N of all the finite orbits of ψ in M must be one of the following*:

(1) *A two-points ψ-orbit.*

(2) *A closed reversible geodesic which length is rational multiple of π.*

(3) *A Riemannian sphere of constant curvature isometrically imbedded in* (M, F) *as a totally geodesic submanifold. In this case we have* $k = 2$.

Proof. By the same argument as in the proof of Lemma 3.6 in [Xu 2018b], we can prove N is the fixed point set of ψ^k for some integer k, hence it is totally geodesic in (M, F). When $\dim N = 0$ or 1, we get the cases (1) and (2) respectively. The difference appears when $\dim N > 1$, which may happen with the finite orbit of prime closed geodesics condition. When $\dim N > 1$, by Lemma 2.2, $(N, F|_N)$ is a Finsler sphere satisfying $K \equiv 1$. By Lemma 3.2, $(N, F|_N)$ also satisfies Assumption (F). Then Lemma 3.3 provides the case (2) in the lemma. □

The cases (2) and (3) cover all the possibilities for the $\hat{G}$-orbit of a prime closed geodesic γ such that the length of γ is a rational multiple of π.

Next, we consider the $\hat{G}$-orbit of a prime closed geodesic γ such that the length of γ is an irrational multiple of π.

When the length of γ is an irrational multiple of π, any ψ-orbit in γ is dense. Following this observation, we can easily prove the following lemma.

Lemma 3.5. *Let* $(M, F) = (S^n, F)$ *be a Finsler sphere satisfying* $n > 1$, $K \equiv 1$ *and Assumption (F). Then two geometrically distinct closed geodesics can intersect if and only if they are intersecting geodesics in the totally geodesic submanifold in* (M, F) *which is isometric to a unit sphere, i.e., the case* (3) *in Lemma 3.4.*

Proof. Lemma 3.4 indicates that any two geometrically distinct closed geodesics γ_1 and γ_2 must satisfy one of the following. Either both lengths are 2π or one of them, for example γ_1, has a length which is an irrational multiple of π. In the first case, they are contained in a totally geodesic submanifold of (M, F) which is isometric to a unit sphere. In the second case, the intersection of the two geodesics contains a ψ-orbit, which is dense in γ_1. Both geodesics are closed, so does their intersection. So as subsets of M, we have $\gamma_1 \subset \gamma_2$ and furthermore the equality must happen because γ_2 is a closed connected curve. This is the contradiction ending the proof of the lemma. □

Using above lemmas, we can provide more explicit description for the orbits of prime closed geodesics by the following lemma.

Lemma 3.6. *Assume* $(M, F) = (S^n, F)$ *is a Finsler sphere satisfying* $n > 1$, $K \equiv 1$, *Assumption (F), and that it is not the standard unit sphere. Then we have the following*:

(1) *There exists closed geodesics whose lengths are irrational multiples of* π.

(2) *For the orbit of prime closed geodesics* $\mathcal{B}_i = \hat{G} \cdot \gamma_i$ *such that the length of* γ_i *is an irrational multiple of* π, *the corresponding* B_i *is an orbit for the action of* $G = I_o(M, F)$.

(3) *Two different orbits of prime closed geodesics*, $\mathcal{B}_i$ *and* $\mathcal{B}_j$, *are geometrically distinct if and only if* B_i *and* B_j *do not intersect.*

(4) *Two different orbits of prime closed geodesics* $\mathcal{B}_i$ *and* $\mathcal{B}_j$ *are geometrically the same if and only if we can find* $\gamma_i \in \mathcal{B}_i$ *and* $\gamma_j \in \mathcal{B}_j$ *such that* γ_i *and* γ_j *are the same curve with different directions.*

Proof. By Lemma 3.3 and the assumption that (M, F) is not the standard unit sphere, the antipodal map ψ generates an infinite subgroup in $I(M, F)$, which closure is a subgroup in the center of $I(M, F)$, corresponding to an abelian subalgebra $\mathfrak{c}' \subset \mathfrak{c}(\mathfrak{g})$ with $\dim \mathfrak{c}' > 0$. We can find a nonzero Killing vector field X from $\mathfrak{c}'$ which generates an S^1. Obviously, X is tangent to each closed geodesic. The restriction of X to each closed geodesic which length is a rational multiple of π is zero.

To prove (1), we only need to consider a maximum point x of $f(\cdot) = F(X(\cdot))$. By Lemma 2.3, the integration curve γ of X passing x is a geodesic, restricted to which X is nonzero. Because X generates an S^1, γ is closed. So it is a closed geodesic which length is an irrational multiple of π.

To prove (2), we consider a prime closed geodesic γ_i which length is an irrational multiple of π. Because the restriction of X to γ_i is a nonzero tangent vector field, γ_i is a homogeneous geodesic. In its $\hat{G} = G \times T$-orbit, The T-action on γ_i can be replaced by the actions of $\exp(tX) \in G$. So the union B_i for the geodesics in $\mathcal{B}_i$ is a G-orbit.

The statements (3) and (4) follows immediately from Lemma 3.5. □

Corollary 3.7. *Assume* $(M, F) = (S^n, F)$ *is a homogeneous Finsler sphere satisfying* $n > 1$, $K \equiv 1$ *and Assumption (F). Then all closed geodesics are reversible. Furthermore, one of the following two cases must happen*:

(1) (M, F) *is a standard unit sphere. It has exactly one orbit of prime closed geodesics and all geodesics are closed.*

(2) (M, F) *is a homogeneous non-Riemannian Randers sphere with an odd n and* $K \equiv 1$. *There exists exactly two orbits of prime closed geodesics* $\hat{G} \cdot \gamma_1$ *and* $\hat{G} \cdot \gamma_2$, *in which* γ_1 *and* γ_2 *are the same curve with different directions.*

Proof. If the antipodal map ψ has a finite order, then (M, F) is the standard unit sphere by Lemma 3.3. If ψ has an infinite order, then $G = I_o(M, F)$ has a one-dimensional center $\mathbb{R}X$, and $M = G/H$ must be

$$U(n')/U(n'-1) \quad \text{or} \quad Sp(n'')U(1)/Sp(n''-1)U(1).$$

By Theorems 1.1 and 1.2 in [Xu 2018a], when $K \equiv 1$, (M, F) is a geodesic orbit Finsler sphere and must be Randers. Integration curves of X and $-X$ provide prime closed geodesics whose lengths are different irrational multiples of π, belonging to two different orbits $\mathcal{B}_1$ and $\mathcal{B}_2$ with $B_1 = B_2 = M$. By Lemma 3.6, They are the only orbits of prime closed geodesics. □

4. Isometries preserving each closed geodesic

Assume $(M, F) = (S^n, F)$ is a Finsler sphere satisfying $n > 1$, $K \equiv 1$, and Assumption (F). Let ψ be its antipodal map. By Lemma 3.3, the case that ψ has a finite order is easy, so in the following discussion we assume that ψ has an infinite order.

Let H denote the subgroup of $G = I(M, F)$ which preserves each closed geodesic, H_o its identity component, and $\mathfrak{h}$ its Lie algebra. The group H is intersection of

$$G_\gamma = \{g \in G \mid (g \cdot \gamma)(t) \equiv \gamma(t + t_0) \text{ for some } t_0\}$$

for all closed geodesics γ. Each G_γ is a closed subgroup of G. So is H.

It should be remarked that the claim that G_γ is a closed subgroup of G is an easy fact in this case because γ is closed. In the recent work [Berestovskii and Nikonorov 2019], it has been proved that G_γ is still a Lie group when γ is not closed.

Obviously the antipodal map ψ belongs to H. Because ψ has an infinite order, then after taking closure, it generates an abelian subgroup of positive dimension, i.e., we have $\dim H > 0$. The following lemma claims that H_o commutes with all the G-actions.

Lemma 4.1. *The subgroup H_o is a closed subgroup in the center of $G = I_o(M, F)$.*

Proof. The previous observations have already proved that H_o is a closed subgroup of G. Because G is a compact Lie group, to prove this lemma we only need to prove $\mathfrak{h} = \mathrm{Lie}(G)$ is an abelian ideal of $\mathfrak{g}$.

The Lie algebra $\mathfrak{h} = \mathrm{Lie}(H)$ consists of all the Killing vector fields X which is tangent to each closed geodesic. Because the action of G permutes the closed geodesics in each orbit of prime closed geodesics, any Killing vector field of the form $\mathrm{Ad}(g)X$ for $g \in G$ and $X \in \mathfrak{h}$ is also tangent to each closed geodesic. So conjugations of G preserves $\mathfrak{h}$, i.e., $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.

Then we prove $\mathfrak{h}$ is abelian by contradiction. Assume conversely that $\mathfrak{h}$ is not abelian, then we can find a nonzero vector X from the compact semisimple Lie algebra $[\mathfrak{h}, \mathfrak{h}]$ which generates an S^1-subgroup. The Killing vector field on (M, F) induced by X has trivial restriction on each closed geodesic. By Lemma 2.3, the integration curve of X passing the maximum point of $f(\cdot) = F(X(\cdot))$ is a closed geodesic. This is a contradiction which ends the proof of this lemma. □

A direct consequence of Lemma 4.1 is the following lemma.

Lemma 4.2. *For any Killing vector field $X \in \mathfrak{h}$ and any orbit $\mathcal{B}_i$ of the prime closed geodesic $c = c(t)$, their exists a constant $\rho_{X,i} \in \mathbb{R}$ such that*

$$X|_{c(t)} \equiv \rho_{X,i}\dot{c}(t), \quad \text{for all } c \in \mathcal{B}_i. \tag{4-1}$$

In particular, a Killing vector field $X \in \mathfrak{h}$ vanishes at some point $x \in \mathcal{B}_i$ if and only if $\rho_{X,i} = 0$, and if and only if X vanishes identically on B_i.

The last ingredient for the proof of Theorem 1.2 is the following lemma.

Lemma 4.3. *Let $(M, F) = (S^n, F)$ be a Finsler sphere satisfying $n > 1$, $K \equiv 1$ and Assumption (F). Then we have the following*:

(1) *For any nonzero Killing vector field $X \in \mathfrak{h}$ which generates an S^1, there exists some orbit $\mathcal{B}_i$ of prime closed geodesics such that $\rho_{X,i} > 0$.*

(2) *Any Killing vector field $X \in \mathfrak{h}$ vanishing on all closed geodesics must be a zero vector field.*

(3) *The common zero set of all Killing vector fields in $\mathfrak{h}$ must be the fixed point set of ψ^k for some integer k. To be more precise, it is empty, a two-points ψ-orbit, some B_i which is a reversible closed geodesic which lengths for both directions are rational multiples of π, or some B_i which is a totally geodesic submanifold isometric to a standard unit sphere.*

Proof. (1) We consider the maximum point x for the function $f(\cdot) = F(X(\cdot))$. By Lemma 2.3, the integration curve of X passing x provide a prime closed geodesic γ, for which we have $X(c(t)) \equiv \rho_{X,\gamma}\dot{c}(t)$ with $\rho_{X,\gamma} > 0$.

(2) We assume conversely that there exists a nonzero Killing vector field on (M, F) such that it vanishes on all closed geodesics. Let $\mathfrak{k}$ be the space of all such Killing vector fields. It is a subalgebra of $\mathfrak{h}$ corresponding to a subtorus in H_o. We can find a nonzero Killing vector field X from $\mathfrak{k}$ which generates an S^1. The argument for (1) indicates X is not vanishing on some closed geodesic, which is the contradiction.

(3) Let N be the fixed point set of H_o, and assume N is not empty. By Lemma 2.2, N must be a two-points ψ-orbit, a reversible closed geodesic, or a Finsler sphere with $\dim N > 1$, $K \equiv 1$ isometrically imbedded in (M, F).

Obviously the action of ψ preserves N, i.e., N consists of ψ-orbits. Because H is compact, H/H_o is finite. We also have $\psi \in H$, and thus each ψ-orbit in N is finite. So when $\dim N = 1$, the lengths of N for both directions are rational multiples of π.

When $\dim N > 1$, we see $(N, F|_N)$ satisfies Assumption (F) by Lemma 3.2. Then Lemma 3.3 tells us that $(N, F|_N)$ is a standard unit sphere. □

5. Proof of Theorem 1.2

Now we are ready to prove Theorem 1.2, which applies a similar inductive argument as that for Theorem 1.2 in [Xu 2018b].

When ψ has a finite order, then by Lemma 3.3, (M, F) is the standard unit sphere. Obviously Theorem 1.2 is valid in this case. So in the following discussion, we assume ψ has an infinite order, and thus we have $m = \dim H > 0$.

We will prove Theorem 1.2 by an induction for $n = \dim M$.

When $n = 2$ and the antipodal map ψ has an infinite order, H_o coincides with $G = I_o(M, F) = S^1$. In [Bryant et al. 2017], it has been proved that geometrically there exists exactly one reversible closed geodesic γ with a nontrivial H_o-action. So Theorem 1.2 is valid in this case, and the estimate is sharp.

Now we assume Theorem 1.2 is valid when $n < l$ with $l > 3$ (the inductive assumption) and we will prove the theorem when $n = l$.

Firstly, we prove:

Claim 1. *When* $\dim H = 1$, *there exists at least one totally geodesic B_i with a nontrivial H_o-action.*

Let X be any nonzero Killing vector field from $\mathfrak{h} = \mathrm{Lie}(H)$. We list all the $\hat{G}$-orbits of prime closed geodesics as $\mathcal{B}_i$ with $1 \le i \le k$, such that when $1 \le i \le k'$ the coefficient $\rho_{X,i}$ in (4-1) is positive. Notice that by Lemma 4.3(1), we have $k' > 0$.

If the antipodal map ψ is not contained in H_o, we can find an isometry of (M, F) which is of the form $\phi = \psi \exp(t'X)$ such that its fixed point set contains B_1. By Lemma 2.2 (or see Lemma 3.5 in [Xu 2018b]), the fixed point set N of ϕ is a closed connected totally geodesic submanifold. It must have a positive codimension in M because $\phi \notin H_o$. When $\dim N = 1$, it is a reversible closed geodesic. When $\dim N > 1$, by Lemma 3.2 and the totally geodesic property, $(N, F|_N)$ is a Finsler sphere satisfying $K \equiv 1$ and Assumption (F). Using the inductive assumption, we can find some orbit of prime closed geodesic, $\mathcal{B}_i = \hat{G}' \cdot \gamma_i = \hat{G} \cdot \gamma_i$, where $\hat{G}' = G' \times T$ and $G' = I_o(N, F|_N)$, such that the corresponding $\mathcal{B}'_i$, is totally geodesic in $(N, F|_N)$ as well as in (M, F). The H_o-action on B_i is nontrivial because

$$\exp(t'X)|_{B_i} = \psi^{-1}\phi|_{B_i} = \psi^{-1}|_{B_i},$$

and ψ has no fixed point on any closed geodesic.

To summarize, this proves Claim 1 when $\psi \notin H_o$.

To continue the proof of Claim 1, we may assume $\psi \in H_o$. In this case, we can prove the zero set of X is empty as following. Assume conversely that the zero set of X is not empty, by Lemma 4.3, it is a two-points ψ-orbit, a reversible closed geodesic, or a connected totally geodesic standard unit sphere. For each possibility,

ψ can not be generated by X, which is a contradiction to the assumption $\psi \in H_o$. This fact implies that $f(\cdot) = F(X(\cdot))$ is a smooth function on M. By Lemma 2.3, the critical point set of $f(\cdot)$ consists of exactly all B_i's with $1 \le i \le k'$. Meanwhile, we see the H_o-action on each closed geodesic is nontrivial.

We take a prime closed geodesic $c_i(t)$ with $t \in \mathbb{R}/\mathbb{Z}$ from $\mathcal{B}_i$ for $1 \le i \le k'$, then $X|_{c_i} = \rho_{X,i}\dot{c}_i$ with $\rho_{X,i} > 0$. Because $H_o = S^1$, we can find some $t' > 0$ such that $\exp(t'X) = \mathrm{id}$, then we have

$$n_i = t'\rho_{X,i} \in \mathbb{N}, \quad \text{for all } 1 \le i \le k'.$$

We may reorder these c_i's such that

$$n_1 \le n_2 \le \cdots \le n_{k'}.$$

There are two possibilities, all n_i's are not all the same, or all n_i's are all the same.

Assume all n_i's are not all the same, i.e., $n_1 < n_{k'}$. The fixed point set N of the isometry $\phi = \exp((t'/n_{k'})X) \in H_o$ contains $B_{k'}$ but not B_1. It is either a reversible closed geodesic, or a Finsler sphere satisfying $1 < \dim N < \dim M$, $K \equiv 1$ and Assumption (F). Applying the inductive assumption and Lemma 3.2, we can find a totally geodesic B_i for $(N, F|_N)$, as well as for (M, F).

Assume all n_i's are all the same, then all $\rho_{X,i}$'s are all the same as well. We may choose a suitable t' such that $n_i = 1$ for $1 \le i \le k'$. There exists $t'' \in (0, 1)$ such that $\psi(c_i(0)) = c_i(t'')$, i.e., $d_F(c_i(0), c_i(t'')) = \pi$, for $1 \le i \le k'$. Then we have

$$F(X|_{c_1}) = F(X|_{c_2}) = \cdots = F(X|_{c_{k'}}).$$

The function $f(\cdot) = F(X(\cdot))$ takes the same value on its critical point set, so it is a constant function. By Lemma 2.3, all integration curves of X are closed geodesics, which belongs to one $\hat{G}$-orbit. By Corollary 3.7, (M, F) is a non-Riemannian homogeneous Randers Finsler sphere with $K \equiv 1$ and exactly two $\hat{G}$-orbits of prime closed geodesics, $\mathcal{B}_1 = \hat{G} \cdot \gamma_1$ and $\mathcal{B}_2 = \hat{G} \cdot \gamma_2$ such that γ_1 and γ_2 are the same curve with different directions.

This ends the proof of Claim 1, i.e., Theorem 1.2 is valid when $m = \dim H = 1$.

Next we prove Theorem 1.2 assuming $m = \dim H > 1$.

Claim 2. *There exists at least $m - 1$ geometrically distinct orbits $\mathcal{B}_i$ such that each B_i is a totally geodesic submanifold with a nontrivial H_o-action.*

Let $\mathcal{B}_i$ with $1 \le i \le k'$ be all the geometrically distinct $\hat{G}$-orbits of prime closed geodesics such that the H_o-action on each B_i is not trivial. Let $\mathfrak{h}_i$ be the codimension one subalgebra of $\mathfrak{h}$ which restriction to B_i is zero. By Lemma 4.3, the intersection $\bigcap_{i=1}^{k'} \mathfrak{h}_i = 0$, from which we see that $m \le k'$. We may reorder the orbits $\mathcal{B}_i$'s such that $\bigcap_{i=1}^{m} \mathfrak{h}_i = 0$. Take a nonzero Killing vector field $X \in \bigcap_{i=1}^{m-1} \mathfrak{h}_i$. Then the zero set N of X is a closed connected totally geodesic submanifold in M, containing B_i

for $1 \le i \le m-1$ but not B_m. Let H' be the subgroup of $I_o(N, F|_N)$ preserving all closed geodesics in N, and $\mathfrak{h}'$ its Lie algebra. The restriction from M to N defines a linear map from $\mathfrak{h}$ to $\mathfrak{h}'$ which kernel is spanned by X, so $\dim H' \ge m-1$.

If $\dim N = 1$, then $m = 2$, H_o has no fixed point, and N itself provides the totally geodesic B_i wanted by Claim 2.

If $\dim N > 1$, we can use the inductive assumption to find $m-1$ geometrically distinct orbits $\mathcal{B}_i$ of prime closed geodesics for $(N, F|_N)$, as well as for (M, F) by Lemma 3.2, such that the corresponding B_i's are totally geodesic submanifolds, with nontrivial H'_o-actions. Claim 2 is proved when each of these B_i's also has a nontrivial H_o-action.

But it is possible that there is some B_i in N on which the H'_o-action is nontrivial but the H_o-action is trivial. If it happens, this B_i is unique, and we must have $\dim H' > m-1$. So in this case, we can use the inductive assumption to find m geometrically distinct orbits of prime closed geodesics. At least $m-1$ geometrically distinct totally geodesic B_i's in N have nontrivial H_o-actions.

This proves Claim 2.

To finish the proof of Theorem 1.2 when $n = l$, we only need to find one more totally geodesic B_i with a nontrivial H_o-action.

We may reorder the orbits $\mathcal{B}_i$'s such that the first $m-1$ ones are those provided by Claim 2, and $\bigcap_{i=1}^{m} \mathfrak{h}_i = 0$. The nonzero Killing vector field X from $\bigcap_{i=1}^{m-1} \mathfrak{h}_i$ vanishes on B_i with $1 \le i \le m-1$, but not on B_m. We can find an isometry of the form $\phi = \psi \exp(t'X)$ such that it fixes each point of B_m. On the other hand, the fixed point set N of ϕ does not contain each $\mathcal{B}_i$ for $1 \le i \le m-1$.

The H_o-action on each closed geodesic in N is nontrivial. Assume conversely that there is a closed geodesic in N with a trivial H_o-action. Then the restriction of ψ to this geodesic coincides with that of ϕ, fixing each point of this geodesic. This is not true because ψ has no fixed points.

If $\dim N = 1$ it is a reversible closed geodesic, which is the extra B_i we want. If $\dim N > 1$ it is a Finsler sphere satisfying $K \equiv 1$ and Assumption (F), isometrically imbedded in (M, F) as a totally geodesic submanifold. In this situation we use the inductive assumption one more time, which provides one more totally geodesic B_i.

Summarizing above discussion, we have proved Theorem 1.2 when $n = l$.

This ends the proof of Theorem 1.2 by induction.

6. The example from Katok metrics

We conclude this paper by the examples from Katok metrics for which the estimate in Theorem 1.2 is sharp.

Let $(M, h) = (S^n, h)$ be a standard unit sphere with $n > 1$, W a Killing vector field on (M, h) such that $h(W, W) < 1$ everywhere.

Then the navigation process defines a Randers metric

$$F(y)=\frac{\sqrt{\lambda h(y,y)+h(W,y)^2}}{\lambda}-\frac{h(W,y)}{\lambda}$$

on M, in which $\lambda = 1-h(W,W)$ is positive everywhere.

By the work of Bao, Robles and Shen [Bao et al. 2004], this construction provides all the Randers spheres with $K\equiv 1$. The behavior of the geodesics on (M,F) is determined by the choice of W.

We can find suitable coordinates $x=(\boldsymbol{x}_0, z_1, \ldots, z_k)$ for $x\in\mathbb{R}^{n+1}$, where

$$\boldsymbol{x}_0=(x_{0,1},\ldots,x_{0,n_0})\in\mathbb{R}^{n_0} \quad\text{and}\quad z_i=(z_{i,1},\ldots,z_{i,n_i})\in\mathbb{C}^{n_i}$$

satisfy the following:

(A1) We permit $n_0=0$ and in this case $\boldsymbol{x}_0$ is always 0. All other n_i's are positive.

(A2) (M,h) is naturally identified as the unit sphere $S^n(1)$ defined by

$$|\boldsymbol{x}_0|^2+|z_1|^2+\cdots+|z|^2=1$$

in $\mathbb{R}^{n+1}=\mathbb{R}^{n_0}\oplus\mathbb{C}^{n_1}\oplus\cdots\oplus\mathbb{C}^{n_k}$ with the standard product Euclidean metric.

(A3) W can be presented as

$$W(\boldsymbol{x}_0,z_0,\ldots,z_k)=(0,\sqrt{-1}\lambda_1 z_1,\ldots,\sqrt{-1}\lambda_k z_k), \tag{6-1}$$

such that $0<\lambda_1<\lambda_2<\ldots<\lambda_k<1$.

We further require one of the following is satisfied:

(A4) All λ_i's are irrational numbers. For any $1\le i<j\le k$, 1, λ_i and λ_j are linearly independent over $\mathbb{Q}$.

(A5) All λ_i's are irrational numbers except one, $n_0=0$ and $n_i=1$ if $\lambda_i\in\mathbb{Q}$. If λ_i and λ_j are irrational numbers, 1, λ_i and λ_j are linearly independent.

Then we have

Lemma 6.1. *For the Randers sphere (M,F) described above, satisfying* (A1)–(A3) *and one of* (A4) *and* (A5), *any closed geodesic on (M,F) must be contained in*

$$z_1=\cdots=z_k=0$$

or

$$\boldsymbol{x}_0=0 \quad \textit{and} \quad z_j=0 \quad \textit{when } j\neq i,$$

for some i, $1\le i\le k$.

Proof. Using (6-1), we can present the antipodal map as

$$\psi(\boldsymbol{x}_0,z_1,\ldots,z_k)=(\boldsymbol{x}_0,-e^{\sqrt{-1}\pi\lambda_1}z_1,\ldots,-e^{\sqrt{-1}\pi\lambda_k}z_k).$$

It is easy to check that finite ψ-orbits only appear in the situation that only $\boldsymbol{x}_0$ is nonzero or only z_i with $\lambda_i \in \mathbb{Q}$ is nonzero.

Let $x = (\boldsymbol{x}_0, z_1, \ldots, z_k)$ be a point on the closed geodesic γ. We only need to prove that only one of $\boldsymbol{x}_0$ and z_i's can be nonzero. Assume conversely this is not true. Then the length of γ can not be a rational multiple of π (i.e., consists of finite ψ-orbits), so the ψ-orbit of x is a dense subset in γ. There are three cases we need to consider.

In the first case, λ_i and λ_j are irrational numbers, $z_i \neq 0$, and $z_j \neq 0$. Then the condition that 1, λ_i and λ_j are linearly independent implies that the projection to the z_i- and z_j-factors maps the closed curve γ onto a two dimensional torus, which is a contradiction.

In the second case, λ_i is rational, λ_j is not, $z_i \neq 0$ and $z_j \neq 0$. Then the projection to the z_i-factor maps γ to a finite set with at least two points. This is impossible because γ is connected.

In the third case, $\boldsymbol{x}_0 \neq 0$ and $z_i \neq 0$. Then the projection to the $\boldsymbol{x}_0$-factor maps γ to two points. This is impossible for the same reason as the previous case.

To summarize, we have found contradiction for all the cases, and finished the proof of this lemma. □

Using Lemma 6.1, we can provides examples of Katok metrics such that the estimates in Theorem 1.2 are sharp.

Theorem 6.2. *Let F be the Randers metrics on S^n with $n > 1$ satisfying* (A1)–(A3) *and one of* (A4) *and* (A5). *Then it has only finite orbits of prime closed geodesics. Let H denote the subgroup of isometries preserving each closed geodesic, H_o its identity component, and $m = \dim H$. Then there exist exactly m geometrically distinct $\mathcal{B}_i$, such that the corresponding B_i's are totally geodesic with nontrivial H_o-actions.*

The proof is a case-by-case discussion. For each case, it is not hard to calculate $G = I_o(M, F)$, H_o and all the orbits of prime closed geodesics.

For example, when $n_0 > 2$ and all γ_i's are irrational numbers,

$$G = \mathrm{SO}(n_0) \times U(n_1) \times \cdots \times U(n_k), \quad \text{and}$$
$$H = C(U(n_1) \times \cdots \times U(n_k)) = U(1)^k,$$

so we have $\dim H = k$.

When $1 \leq i \leq k$,

$$B_i = \{x = (\boldsymbol{x}_0, z_1, \ldots, z_k) \in M \text{ with } \boldsymbol{x}_0 = 0 \text{ and } z_j = 0 \text{ when } j \neq i\}$$

is a homogeneous Randers sphere with exactly two orbits of prime closed geodesics. It is isometrically imbedded in (M, F) as a totally geodesic submanifold, because it is the fixed point set of the subgroup of G with the $U(n_i)$-factor removed. They provide all the different totally geodesic B_i's with nontrivial H_o-actions.

There exists one more totally geodesic B_{k+1} with a trivial H_o-action, i.e.,

$$B_{k+1} = \big\{x = (\boldsymbol{x}_0, z_1, \ldots, z_k) \in M \text{ with } z_1 = \cdots = z_k = 0\big\}.$$

It is a standard unit sphere with only one orbit of closed geodesics.

By Lemma 6.1, no other closed geodesics can be found.

Summarizing all these observations, we see that this Randers sphere (M, F) satisfies all the requirements in Theorem 1.2, and the estimate in Theorem 1.2 for the number of totally geodesic B_i's is sharp.

The discussion for other cases is similar, so we skip the details.

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