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**FINSLER SPHERES WITH CONSTANT FLAG CURVATURE  
AND FINITE ORBITS OF PRIME CLOSED GEODESICS**

MING XU

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In this paper, we consider a Finsler sphere  $(M, F) = (S^n, F)$  with dimension  $n > 1$  and flag curvature  $K \equiv 1$ . The action of the connected isometry group  $G = I_o(M, F)$  on  $M$ , together with the action of  $T = S^1$  shifting the parameter  $t \in \mathbb{R}/\mathbb{Z}$  of the closed curve  $c(t)$ , define an action of  $\hat{G} = G \times T$  on the free loop space  $\Lambda M$  of  $M$ . In particular, for each closed geodesic, we have a  $\hat{G}$ -orbit of closed geodesics. We assume the Finsler sphere  $(M, F)$  described above has only finite orbits of prime closed geodesics. Our main theorem claims that, if the subgroup  $H$  of all isometries preserving each close geodesic is of dimension  $m$ , then there exists  $m$  geometrically distinct orbits  $\mathcal{B}_i$  of prime closed geodesics, such that for each  $i$ , the union  $B_i$  of geodesics in  $\mathcal{B}_i$  is a totally geodesic submanifold in  $(M, F)$  with a nontrivial  $H_o$ -action. This theorem generalizes and slightly refines the one in a previous work, which only discussed the case of finite prime closed geodesics. At the end, we show that, assuming certain generic conditions, the Katok metrics, i.e., the Randers metrics on spheres with  $K \equiv 1$ , provide examples with the sharp estimate for our main theorem.

## 1. Introduction

In the recent work of R. L. Bryant, P. Foulon, S. Ivanov, V. S. Matveev and W. Ziller [Bryant et al. 2017], the authors classified Finsler spheres with constant flag curvature  $K \equiv 1$  according to the behavior of geodesics. The Katok metric [1973] provides the most important key model for their classification. The celebrated Anosov conjecture [1975], claiming the minimal number of prime closed geodesics on a Finsler sphere  $(S^n, F)$  is  $2[(n+1)/2]$ , was based on the discovery of Katok metrics with only finite prime closed geodesics. There are many works using Morse theory and index theory to study the closed geodesics and Anosov conjecture in

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Finsler geometry, assuming a pinch condition for the flag curvature, nondegenerating property for all closed geodesics, or using the specialty of low dimensions. See for example [Bangert and Long 2010; Duan 2016; Long and Duan 2009; Duan et al. 2016; Rademacher 1989; Wang 2012; 2015]. From the geometrical point of view, it was much later that people noticed that Katok metrics are Randers metrics on spheres with constant flag curvature [Rademacher 2004]. D. Bao, C. Robles and Z. Shen [Bao et al. 2004] provided a complete classification for all Randers metrics with constant flag curvature. The classification for the non-Randers case is still widely open. Bryant [1996; 1997; 2002]. provided many important examples of Finsler spheres with  $K \equiv 1$ .

However, one of the most important technique in [Bryant et al. 2017] is from Lie theory. The authors considered the antipodal map  $\psi$  for a Finsler sphere with  $K \equiv 1$  (see [Bryant et al. 2017; Shen 1996] or Section 2 for its definition). It is a Clifford Wolf translation in the center of the isometry group  $I(M, F)$ . When  $\psi$  has an infinite order, after taking closure, it can be used to generate a closed abelian subgroup of isometries with a positive dimension.

For nonzero Killing vector fields on a Finsler sphere with  $K \equiv 1$ , we have the following totally geodesic technique. The common zero point set of Killing vector fields, or more generally the fixed point set of isometries, provide closed totally geodesic submanifolds. In particular, when the dimension of such a submanifold is one, it is a reversible geodesic, and when the dimension is even bigger, it is a Finsler sphere inheriting the curvature property and geodesic property from the ambient space. We can use this key observation to set up an inductive argument, when studying the geodesics on  $(S^n, F)$  with  $n > 2$  and  $K \equiv 1$ , and generalizing some results in [Bryant et al. 2017] to high dimensions.

For example, we have proved the following lower bound estimate for the number of reversible prime closed geodesics in Finsler spheres with constant flag curvature.

**Theorem 1.1** [Xu 2018b]. *Let  $(M, F) = (S^n, F)$  with  $n > 1$  be a Finsler sphere with  $K \equiv 1$  and only finite prime closed geodesics. Then the number of geometrically distinct reversible closed geodesics is at least  $\dim I(M, F)$ .*

Recall that a geodesic  $c(t)$  with constant speed is called *reversible* if  $c(-t)$  also provides a geodesic with constant speed after a reparametrization by the new arc length. Two geodesics are *geometrically distinct* if and only if they are different subsets.

The assumption of only finite prime closed geodesics imposes a strong restriction on  $I_o(M, F)$ , which can only be a torus. A lot of important examples are excluded, for example, the standard unit spheres and the homogeneous non-Riemannian Randers spheres with  $K \equiv 1$ . So if we want more possibility for  $I_o(M, F)$ , the geodesic condition could be replaced by the assumption that there exist only finite

orbits of prime closed geodesics, or [Assumption \(F\)](#) for simplicity. See [Section 3](#) for its precise definition and detailed discussion.

The main purpose of this paper is to prove the following theorem.

**Theorem 1.2.** *Let  $(M, F) = (S^n, F)$  be a Finsler sphere satisfying  $n > 1$ ,  $K \equiv 1$  and [Assumption \(F\)](#). Denote by  $H$  the subgroup of  $G = I_o(M, F)$  preserving each closed geodesic,  $H_o$  its identity component and  $m = \dim H$ . Then there exist at least  $m$  geometrically distinct orbits  $B_i$ 's of prime closed geodesics such that each union  $B_i$  of geodesics in  $B_i$  is a totally geodesic submanifold in  $M$  with a nontrivial  $H_o$ -action.*

When  $(M, F)$  has only finite prime closed geodesics, then [Assumption \(F\)](#) is satisfied,  $H_o = G = I_o(M, F)$ , and each orbit of closed geodesics consists of only one closed geodesic. So [Theorem 1.2](#) generalizes [Theorem 1.1](#). It even slightly refines [Theorem 1.1](#) by claiming the totally geodesic  $B_i$ 's found have nontrivial  $H_o$ -actions. So if the common zero point of  $H_o$  has a positive dimension, it provides one more totally geodesic  $B_i$ , which is either a reversible closed geodesic which length is a rational multiple of  $\pi$ , or isometric to a standard unit sphere.

By [Theorem 1.1](#) and [Theorem 1.2](#) in [\[Xu 2018a\]](#), each submanifold  $(B_i, F|_{B_i})$  is in fact a non-Riemannian homogeneous Randers sphere with constant flag curvature. So [Theorem 1.2](#) implies the existence of totally geodesic subspheres in which  $F$  has standard restrictions, though  $F$  itself may be strange.

This paper is organized as following. In [Section 2](#), we recall some fundamental geometric properties of Finsler spheres with  $K \equiv 1$ , discussing their antipodal maps and totally geodesic submanifolds. In [Section 3](#), we define [Assumption \(F\)](#), i.e., the assumption of only finite prime closed geodesics. In [Section 4](#), we introduce the subgroup  $H$  of isometries which preserves each closed geodesic. In [Section 5](#), we prove [Theorem 1.2](#) by induction. In [Section 6](#), we discuss the Katok metrics, and show that in some cases they provides examples for [Theorem 1.2](#), with a sharp estimate.

## 2. Preliminaries: from antipodal map to Killing vector field

Let  $(M, F) = (S^n, F)$  be a Finsler sphere satisfying the dimension  $n > 1$  and the flag curvature  $K \equiv 1$ . Denote  $G = I_o(M, F)$  the connected isometry group, i.e., the identity component of the isometry group  $I(M, F)$  of  $(M, F)$ .

We briefly recall the definition of the exponential map [\[Bao et al. 2000\]](#) and the antipodal map  $\psi$  [\[Bryant et al. 2017; Shen 1996\]](#) for  $(M, F)$ .

For any  $x \in M$  and nonzero  $y \in T_x M$ , the exponential map  $\text{Exp}_x : T_x M \rightarrow M$  is defined by  $\text{Exp}_x(y) = c(1)$  where  $c(t)$  is the constant speed geodesic with  $c(0) = x$  and  $\dot{c}(0) = y$ . When  $y = 0 \in T_x M$ , we define  $\text{Exp}_x(0) = x$ . Notice that  $\text{Exp}_x$  is  $C^1$  at  $y = 0$  and  $C^\infty$  elsewhere.

The discussion for the Jacobi fields and conjugation points when  $K \equiv 1$  indicates  $\text{Exp}_x$  maps the sphere

$$S_o^F(\pi) = \{y \in T_x M \mid F(y) = \pi\} \subset T_x M$$

to a single point  $x^* \in M$ . The map from  $x$  to  $x^*$  is an isometry of  $(M, F)$  in the center of  $I(M, F)$  [Bryant et al. 2017]. Further more, it is easy to see that  $\psi$  is a Clifford Wolf translation for the (possibly nonreversible) distance  $d_F(\cdot, \cdot)$  defined by the Finsler metric  $F$ . We will call it the *antipodal map* and always denote it as  $\psi$ . It is a generalization for the antipodal map for standard unit spheres but may not be an involution any more.

The above description immediately proves that any connected and simply connected Finsler manifold  $(M, F)$  with  $\dim M > 1$  and  $K \equiv 1$  is homeomorphic to a sphere. A more careful discussion with the local charts shows that the homeomorphism in this statement can be refined to be a diffeomorphism, and the argument is valid not only for  $M$ , but also any closed connected totally geodesic submanifold  $N$  with  $\dim N > 1$ , i.e., we have the following lemma.

**Lemma 2.1** [Xu 2018b, Lemma 3.2]. *Let  $(M, F)$  be a closed connected and simply connected Finsler manifold with  $K \equiv 1$  and  $N$  a closed connected totally geodesic submanifold with  $\dim N > 1$ . Then both  $M$  and  $N$  are diffeomorphic to standard spheres, and  $N$  is an imbedded submanifold in  $M$ .*

The fixed point set for a family of isometries in  $I(M, F)$  is a closed, possibly disconnected, totally geodesic submanifold. We have the following lemma, indicating the connectedness of  $N$ , when its dimension is positive.

**Lemma 2.2** [Xu 2018b, Lemma 3.5]. *Let  $(M, F) = (S^n, F)$  be a Finsler sphere with  $n > 1$  and  $K \equiv 1$ , and  $N$  the fixed point set of a family of isometries of  $(M, F)$ . Then  $N$  must satisfy one of the following:*

- (1)  $N$  is a two-points  $\psi$ -orbit, i.e.,  $N = \{x', x''\}$  with  $d_F(x', x'') = d_F(x'', x') = \pi$ .
- (2)  $N$  is a reversible closed geodesic.
- (3)  $(N, F|_N)$  is a Finsler sphere with  $\dim N > 1$  and  $K \equiv 1$ .

The space of Killing vector fields can be viewed as the Lie algebra of  $I(M, F)$ . So the common zero set of a family of Killing vector fields on  $(M, F)$  is a special case of fixed point sets for isometries.

In later discussions, we will need the following two lemmas for Killing vector fields.

**Lemma 2.3.** *Assume that  $X$  is a Killing vector field of the Finsler space  $(M, F)$ ,  $f(\cdot) = F(X(\cdot))$  and  $f(x) > 0$  at  $x \in M$ . Then the integration curve of  $X$  passing  $x$  is a geodesic if and only if  $x$  is a critical point of  $f(\cdot)$ .*

**Lemma 2.4** (corollary of [Deng and Xu 2014, Lemma 3.1]). *Assume that  $c = c(t)$  is a geodesic of positive constant speed on the Finsler space  $(M, F)$ . Then restricted to  $c(t)$ , any Killing vector field  $X$  of  $(M, F)$  satisfies*

$$(2-1) \quad \langle X(c(t)), \dot{c}(t) \rangle_{\dot{c}(t)}^F \equiv \text{const},$$

where  $\langle u, v \rangle_y^F = g_{ij}(y)u^i v^j$  for  $u, v, y \in T_x M$  and  $y \neq 0$  is the inner product defined by the fundamental tensor.

*Proof.* Whenever the value of  $X$  is linearly independent of  $\dot{c}(t)$ , we can prove (2-1) by choosing a special local chart, such that  $c = c(t)$  can be presented as  $x^1 = t$  and  $x^i = 0$  for  $i > 1$ , and  $X = \partial_{x^2}$ . Because  $X$  is Killing vector field,  $F(x, y)$  is independent of  $x^2$ . The condition that  $c = c(t)$  is a geodesic implies that for the coefficients  $G^i$  of the geodesic spray, we have

$$\begin{aligned} G^i(c(t), \dot{c}(t)) &= \frac{1}{4} g^{il} ([F^2]_{x^m y^l} y^m - [F^2]_{x^l}) \\ &= \frac{1}{4} g^{il} ([F^2]_{x^1 y^l} - [F^2]_{x^l}) = 0. \end{aligned}$$

In particular, on the geodesic  $c = c(t)$ , we have

$$\frac{d}{dt} \langle X(c(t)), \dot{c}(t) \rangle_{\dot{c}(t)}^F = \frac{1}{2} [F^2]_{x^1 y^2} = \frac{1}{2} [F^2]_{x^2} = 0,$$

which proves the lemma in this case.

When  $X$  is tangent to  $c = c(t)$  for  $t$  in an interval  $I$ , we can easily get (2-1) for  $t \in I$ .

Summarizing this two cases and using the continuity, we have proved (2-1) along the whole geodesic  $c = c(t)$ .  $\square$

### 3. Orbit of closed geodesics and Assumption (F)

Now we define *Assumption (F)*, i.e., the condition that  $(M, F)$  has only finite orbits of prime closed geodesics. In later discussion, we will always assume it to be satisfied by  $(M, F)$  unless otherwise specified.

The free loop space  $\Lambda M$  of all piecewise smooth path  $c = c(t)$  with  $t \in \mathbb{R}/\mathbb{Z}$  (sometimes we will simply denote it as  $c$  or  $\gamma$ ) admits the natural actions of  $\hat{G} = G \times T$  such that

$$((g, t') \cdot c)(t) = g \cdot c(t + t'), \quad \text{for all } t.$$

So for each closed geodesic  $\gamma$  of constant speed, we have an  $\hat{G}$ -orbit  $\hat{G} \cdot \gamma$  of closed geodesics with the same speed. The geodesic  $c(t)$  (with  $t \in \mathbb{R}/\mathbb{Z}$ ) is *prime*, i.e.,

$$\min\{t \mid t > 0 \text{ and } c(t) = c(0)\} = 1,$$

if and only if all the closed geodesics in  $\hat{G} \cdot c$  are prime.

**Definition 3.1.** We say  $(M, F)$  has only finite orbits of prime closed geodesics if it satisfies

Assumption (F)      all the prime closed geodesics of positive constant speed can be listed as a finite set of  $\hat{G}$ -orbits,  $\mathcal{B}_i = \hat{G} \cdot \gamma_i$ ,  $1 \leq i \leq k$ .

In Definition 3.1, we can equivalently list all the closed geodesics of constant speed  $c(t)$  with  $t \in \mathbb{R}/\mathbb{Z}$  as  $\mathcal{B}_i^j = \hat{G} \cdot \gamma_i^j$ ,  $1 \leq i \leq k$ ,  $j \in \mathbb{N}$ . The orbit  $\mathcal{B}_i$  in Definition 3.1 coincides with  $\mathcal{B}_i^1$ , for each  $i$ . The closed geodesics  $\gamma_i^j$  is the one which rotates  $j$ -times along the prime closed geodesic  $\gamma_i$  in Definition 3.1, i.e., if  $\gamma_i$  is presented as  $c_i = c_i(t)$ , then  $\gamma_i^j$  is  $c_{i,j}(t) = c_i(jt)$ .

We denote  $B_i$  the union of the geodesics in  $\mathcal{B}_i$  or  $\mathcal{B}_i^j$  for any  $j \in \mathbb{N}$ . Then we call  $\mathcal{B}_i^j$  and  $\mathcal{B}_{i'}$  *geometrically distinct* (or *geometrically the same*), if  $B_i$  and  $B_{i'}$  are different subsets (or the same subsets, respectively) of  $M$ .

The Assumption (F) for the ambient space can be inherited by some totally geodesic submanifolds, i.e., we have the following lemma.

**Lemma 3.2.** *Let  $(M, F)$  be any closed compact Finsler manifold satisfying Assumption (F),  $\phi_\alpha$  with  $\alpha \in \mathcal{A}$  a family of isometries in the center of  $I(M, F)$ , and  $N$  the fixed point set for all  $\phi_\alpha$ 's. Then each orbit of prime closed geodesic for  $(N, F|_N)$  is also an orbit of prime closed geodesic for  $(M, F)$ . In particular,  $(N, F|_N)$  also satisfies Assumption (F).*

*Proof.* The fixed point set  $N$  for the isometries  $\phi_\alpha$  with  $\alpha \in \mathcal{A}$  is a closed (possibly disconnected) totally geodesic submanifold of  $(M, F)$ . Because each  $\phi_\alpha$  commutes with all isometries of  $(M, F)$ , the fixed point set  $N$  for all  $\phi_\alpha$ 's is preserved by the action of  $G = I_o(M, F)$ . The restriction of  $G$ -action to  $N$  defines isometries in  $G' = I_o(N, F|_N)$ . Denote  $\hat{G}' = G' \times T$ . Then for each prime closed geodesic  $\gamma$  in  $N$ , Assumption (F) implies that  $\hat{G}' \cdot \gamma$  is a disjoint finite union of  $\hat{G}$ -orbits. Both  $\hat{G}'$ -orbits and  $\hat{G}$ -orbits are compact and connected, so we get  $\hat{G}' \cdot \gamma = \hat{G} \cdot \gamma$ , which proves the first claim. The second claim follows immediately.  $\square$

The effect of Assumption (F) can be seen from the behavior of the antipodal map  $\psi$ . For example, when  $\psi$  has a finite order  $k$ , i.e., there exists a positive integer  $k$ , such that

$$\psi^k = \text{id}, \quad \text{and} \quad \psi^i \neq \text{id} \quad \text{when } 1 \leq i < k,$$

we have the following lemma.

**Lemma 3.3.** *Let  $(M, F) = (S^n, F)$  be a Finsler sphere satisfying  $n > 1$ ,  $K \equiv 1$  and Assumption (F). Assume that the antipodal map  $\psi$  has a finite order  $k$ . Then  $F$  must be the Riemannian metric for a standard unit sphere.*

*Proof.* Because  $\psi$  is a Clifford Wolf translation, and it has a finite order  $k$ , each geodesic of  $(M, F)$  is closed, and each prime closed geodesic admits a suitable

multiple such that the length of the resulting closed geodesic is  $k\pi$ . By [Assumption \(F\)](#), the subset  $B \subset \Lambda M$  of all closed geodesics with the length  $k\pi$  can be listed as the disjoint union of  $\mathcal{B}_i^{n_i} = \hat{G} \cdot \gamma_i^{n_i}$ ,  $1 \leq i \leq k$ , where each  $\gamma_i$  is a prime closed geodesic. Obviously  $B$  is connected and each  $\mathcal{B}_i^{n_i}$  is compact, so we must have  $k = 1$ .

Then we prove  $(M, F)$  is  $G$ -homogeneous. Assume conversely that it is not, we consider a unit speed geodesic  $c(t)$ , and the  $G$ -orbit  $N$  passing  $c(0)$ , such that

$$(3-1) \quad \langle \dot{c}(0), T_{c(0)}N \rangle_{\dot{c}(0)}^F = 0.$$

Then by [Lemma 2.4](#), for any Killing vector field  $X \in \mathfrak{g}$ , we have

$$\langle \dot{c}(t), X(c(t)) \rangle_{\dot{c}(t)}^F \equiv 0,$$

i.e.,  $c(t)$  meets each  $G$ -orbit orthogonally in the sense of (3-1). This property is preserved by  $\hat{G}$ -actions. So its  $\hat{G}$ -orbit can not exhaust all the geodesics, for example, those which does not satisfy (3-1). This is a contradiction to our previous observation that  $(M, F)$  can only have one orbit of prime closed geodesics, and it proves that  $(M, F)$  is homogeneous Finsler sphere.

Finally, we prove  $(M, F)$  is a standard unit sphere. Because  $(M, F)$  is a homogeneous Finsler space, it has at least one homogeneous geodesic  $c(t) = \exp(tX) \cdot o$ , in which  $o \in M$  and  $X \in \mathfrak{g} = \text{Lie}(G)$  [[Yan and Huang 2018](#)]. Our previous observation that all geodesics belong to a single  $\hat{G}$ -orbit implies all geodesics are homogeneous. So for any  $x \in M$  and any two  $F$ -unit tangent vectors  $y_1$  and  $y_2$  in  $T_x M$ , we have two unit speed geodesics  $c_1(t)$  and  $c_2(t)$  such that  $c_1(0) = c_2(0) = x$  and  $\dot{c}_i(0) = y_i$ . Both geodesics belong to the same  $\hat{G}$ -orbit, so we can find  $g_1 \in G$  such that  $(g_1 \cdot c_1)(t) \equiv c_2(t + t_0)$  for some fixed  $t_0$ . Because the geodesic  $c_2(t)$  is homogeneous, we can find another  $g_2 \in G$  such that  $(g_2 \cdot c_2)(t) = c_2(t - t_0)$ . Then we have

$$(g_2 g_1 \cdot c_1)(t) = (g_2 \cdot c_2)(t + t_0) = c_2(t), \quad \text{for all } t.$$

So the isotropy action for  $(M, F)$  is transitive at each point. The only homogeneous spheres satisfying this property are Riemannian spheres of constant curvature.  $\square$

Using [Lemmas 3.2](#) and [3.3](#), we can generalize [Lemma 3.6](#) in [[Xu 2018b](#)] to the following.

**Lemma 3.4.** *Let  $(M, F) = (S^n, F)$  be a Finsler sphere satisfying  $n > 1$ ,  $K \equiv 1$  and [Assumption \(F\)](#). Then the union  $N$  of all the finite orbits of  $\psi$  in  $M$  must be one of the following:*

- (1) A two-points  $\psi$ -orbit.
- (2) A closed reversible geodesic which length is rational multiple of  $\pi$ .



- (3) *A Riemannian sphere of constant curvature isometrically imbedded in  $(M, F)$  as a totally geodesic submanifold. In this case we have  $k = 2$ .*

*Proof.* By the same argument as in the proof of Lemma 3.6 in [Xu 2018b], we can prove  $N$  is the fixed point set of  $\psi^k$  for some integer  $k$ , hence it is totally geodesic in  $(M, F)$ . When  $\dim N = 0$  or  $1$ , we get the cases (1) and (2) respectively. The difference appears when  $\dim N > 1$ , which may happen with the finite orbit of prime closed geodesics condition. When  $\dim N > 1$ , by Lemma 2.2,  $(N, F|_N)$  is a Finsler sphere satisfying  $K \equiv 1$ . By Lemma 3.2,  $(N, F|_N)$  also satisfies Assumption (F). Then Lemma 3.3 provides the case (2) in the lemma.  $\square$

The cases (2) and (3) cover all the possibilities for the  $\hat{G}$ -orbit of a prime closed geodesic  $\gamma$  such that the length of  $\gamma$  is a rational multiple of  $\pi$ .

Next, we consider the  $\hat{G}$ -orbit of a prime closed geodesic  $\gamma$  such that the length of  $\gamma$  is an irrational multiple of  $\pi$ .

When the length of  $\gamma$  is an irrational multiple of  $\pi$ , any  $\psi$ -orbit in  $\gamma$  is dense. Following this observation, we can easily prove the following lemma.

**Lemma 3.5.** *Let  $(M, F) = (S^n, F)$  be a Finsler sphere satisfying  $n > 1$ ,  $K \equiv 1$  and Assumption (F). Then two geometrically distinct closed geodesics can intersect if and only if they are intersecting geodesics in the totally geodesic submanifold in  $(M, F)$  which is isometric to a unit sphere, i.e., the case (3) in Lemma 3.4.*

*Proof.* Lemma 3.4 indicates that any two geometrically distinct closed geodesics  $\gamma_1$  and  $\gamma_2$  must satisfy one of the following. Either both lengths are  $2\pi$  or one of them, for example  $\gamma_1$ , has a length which is an irrational multiple of  $\pi$ . In the first case, they are contained in a totally geodesic submanifold of  $(M, F)$  which is isometric to a unit sphere. In the second case, the intersection of the two geodesics contains a  $\psi$ -orbit, which is dense in  $\gamma_1$ . Both geodesics are closed, so does their intersection. So as subsets of  $M$ , we have  $\gamma_1 \subset \gamma_2$  and furthermore the equality must happen because  $\gamma_2$  is a closed connected curve. This is the contradiction ending the proof of the lemma.  $\square$

Using above lemmas, we can provide more explicit description for the orbits of prime closed geodesics by the following lemma.

**Lemma 3.6.** *Assume  $(M, F) = (S^n, F)$  is a Finsler sphere satisfying  $n > 1$ ,  $K \equiv 1$ , Assumption (F), and that it is not the standard unit sphere. Then we have the following:*

- (1) *There exists closed geodesics whose lengths are irrational multiples of  $\pi$ .*
- (2) *For the orbit of prime closed geodesics  $\mathcal{B}_i = \hat{G} \cdot \gamma_i$  such that the length of  $\gamma_i$  is an irrational multiple of  $\pi$ , the corresponding  $B_i$  is an orbit for the action of  $G = I_o(M, F)$ .*

- (3) *Two different orbits of prime closed geodesics,  $\mathcal{B}_i$  and  $\mathcal{B}_j$ , are geometrically distinct if and only if  $B_i$  and  $B_j$  do not intersect.*
- (4) *Two different orbits of prime closed geodesics  $\mathcal{B}_i$  and  $\mathcal{B}_j$  are geometrically the same if and only if we can find  $\gamma_i \in \mathcal{B}_i$  and  $\gamma_j \in \mathcal{B}_j$  such that  $\gamma_i$  and  $\gamma_j$  are the same curve with different directions.*

*Proof.* By Lemma 3.3 and the assumption that  $(M, F)$  is not the standard unit sphere, the antipodal map  $\psi$  generates an infinite subgroup in  $I(M, F)$ , which closure is a subgroup in the center of  $I(M, F)$ , corresponding to an abelian subalgebra  $\mathfrak{c}' \subset \mathfrak{c}(\mathfrak{g})$  with  $\dim \mathfrak{c}' > 0$ . We can find a nonzero Killing vector field  $X$  from  $\mathfrak{c}'$  which generates an  $S^1$ . Obviously,  $X$  is tangent to each closed geodesic. The restriction of  $X$  to each closed geodesic which length is a rational multiple of  $\pi$  is zero.

To prove (1), we only need to consider a maximum point  $x$  of  $f(\cdot) = F(X(\cdot))$ . By Lemma 2.3, the integration curve  $\gamma$  of  $X$  passing  $x$  is a geodesic, restricted to which  $X$  is nonzero. Because  $X$  generates an  $S^1$ ,  $\gamma$  is closed. So it is a closed geodesic which length is an irrational multiple of  $\pi$ .

To prove (2), we consider a prime closed geodesic  $\gamma_i$  which length is an irrational multiple of  $\pi$ . Because the restriction of  $X$  to  $\gamma_i$  is a nonzero tangent vector field,  $\gamma_i$  is a homogeneous geodesic. In its  $\hat{G} = G \times T$ -orbit, The  $T$ -action on  $\gamma_i$  can be replaced by the actions of  $\exp(tX) \in G$ . So the union  $B_i$  for the geodesics in  $\mathcal{B}_i$  is a  $G$ -orbit.

The statements (3) and (4) follows immediately from Lemma 3.5.  $\square$

**Corollary 3.7.** *Assume  $(M, F) = (S^n, F)$  is a homogeneous Finsler sphere satisfying  $n > 1$ ,  $K \equiv 1$  and Assumption (F). Then all closed geodesics are reversible. Furthermore, one of the following two cases must happen:*

- (1)  *$(M, F)$  is a standard unit sphere. It has exactly one orbit of prime closed geodesics and all geodesics are closed.*
- (2)  *$(M, F)$  is a homogeneous non-Riemannian Randers sphere with an odd  $n$  and  $K \equiv 1$ . There exists exactly two orbits of prime closed geodesics  $\hat{G} \cdot \gamma_1$  and  $\hat{G} \cdot \gamma_2$ , in which  $\gamma_1$  and  $\gamma_2$  are the same curve with different directions.*

*Proof.* If the antipodal map  $\psi$  has a finite order, then  $(M, F)$  is the standard unit sphere by Lemma 3.3. If  $\psi$  has an infinite order, then  $G = I_o(M, F)$  has a one-dimensional center  $\mathbb{R}X$ , and  $M = G/H$  must be

$$U(n')/U(n' - 1) \quad \text{or} \quad Sp(n'')U(1)/Sp(n'' - 1)U(1).$$

By Theorems 1.1 and 1.2 in [Xu 2018a], when  $K \equiv 1$ ,  $(M, F)$  is a geodesic orbit Finsler sphere and must be Randers. Integration curves of  $X$  and  $-X$  provide prime closed geodesics whose lengths are different irrational multiples of  $\pi$ , belonging to

two different orbits  $\mathcal{B}_1$  and  $\mathcal{B}_2$  with  $B_1 = B_2 = M$ . By [Lemma 3.6](#), They are the only orbits of prime closed geodesics.  $\square$

#### 4. Isometries preserving each closed geodesic

Assume  $(M, F) = (S^n, F)$  is a Finsler sphere satisfying  $n > 1$ ,  $K \equiv 1$ , and [Assumption \(F\)](#). Let  $\psi$  be its antipodal map. By [Lemma 3.3](#), the case that  $\psi$  has a finite order is easy, so in the following discussion we assume that  $\psi$  has an infinite order.

Let  $H$  denote the subgroup of  $G = I(M, F)$  which preserves each closed geodesic,  $H_o$  its identity component, and  $\mathfrak{h}$  its Lie algebra. The group  $H$  is intersection of

$$G_\gamma = \{g \in G \mid (g \cdot \gamma)(t) \equiv \gamma(t + t_0) \text{ for some } t_0\}$$

for all closed geodesics  $\gamma$ . Each  $G_\gamma$  is a closed subgroup of  $G$ . So is  $H$ .

It should be remarked that the claim that  $G_\gamma$  is a closed subgroup of  $G$  is an easy fact in this case because  $\gamma$  is closed. In the recent work [\[Berestovskii and Nikonorov 2019\]](#), it has been proved that  $G_\gamma$  is still a Lie group when  $\gamma$  is not closed.

Obviously the antipodal map  $\psi$  belongs to  $H$ . Because  $\psi$  has an infinite order, then after taking closure, it generates an abelian subgroup of positive dimension, i.e., we have  $\dim H > 0$ . The following lemma claims that  $H_o$  commutes with all the  $G$ -actions.

**Lemma 4.1.** *The subgroup  $H_o$  is a closed subgroup in the center of  $G = I_o(M, F)$ .*

*Proof.* The previous observations have already proved that  $H_o$  is a closed subgroup of  $G$ . Because  $G$  is a compact Lie group, to prove this lemma we only need to prove  $\mathfrak{h} = \text{Lie}(G)$  is an abelian ideal of  $\mathfrak{g}$ .

The Lie algebra  $\mathfrak{h} = \text{Lie}(H)$  consists of all the Killing vector fields  $X$  which is tangent to each closed geodesic. Because the action of  $G$  permutes the closed geodesics in each orbit of prime closed geodesics, any Killing vector field of the form  $\text{Ad}(g)X$  for  $g \in G$  and  $X \in \mathfrak{h}$  is also tangent to each closed geodesic. So conjugations of  $G$  preserves  $\mathfrak{h}$ , i.e.,  $\mathfrak{h}$  is an ideal of  $\mathfrak{g}$ .

Then we prove  $\mathfrak{h}$  is abelian by contradiction. Assume conversely that  $\mathfrak{h}$  is not abelian, then we can find a nonzero vector  $X$  from the compact semisimple Lie algebra  $[\mathfrak{h}, \mathfrak{h}]$  which generates an  $S^1$ -subgroup. The Killing vector field on  $(M, F)$  induced by  $X$  has trivial restriction on each closed geodesic. By [Lemma 2.3](#), the integration curve of  $X$  passing the maximum point of  $f(\cdot) = F(X(\cdot))$  is a closed geodesic. This is a contradiction which ends the proof of this lemma.  $\square$

A direct consequence of [Lemma 4.1](#) is the following lemma.

**Lemma 4.2.** *For any Killing vector field  $X \in \mathfrak{h}$  and any orbit  $\mathcal{B}_i$  of the prime closed geodesic  $c = c(t)$ , there exists a constant  $\rho_{X,i} \in \mathbb{R}$  such that*

$$(4-1) \quad X|_{c(t)} \equiv \rho_{X,i} \dot{c}(t), \quad \text{for all } c \in \mathcal{B}_i.$$

In particular, a Killing vector field  $X \in \mathfrak{h}$  vanishes at some point  $x \in \mathcal{B}_i$  if and only if  $\rho_{X,i} = 0$ , and if and only if  $X$  vanishes identically on  $\mathcal{B}_i$ .

The last ingredient for the proof of [Theorem 1.2](#) is the following lemma.

**Lemma 4.3.** *Let  $(M, F) = (S^n, F)$  be a Finsler sphere satisfying  $n > 1$ ,  $K \equiv 1$  and [Assumption \(F\)](#). Then we have the following:*

- (1) *For any nonzero Killing vector field  $X \in \mathfrak{h}$  which generates an  $S^1$ , there exists some orbit  $\mathcal{B}_i$  of prime closed geodesics such that  $\rho_{X,i} > 0$ .*
- (2) *Any Killing vector field  $X \in \mathfrak{h}$  vanishing on all closed geodesics must be a zero vector field.*
- (3) *The common zero set of all Killing vector fields in  $\mathfrak{h}$  must be the fixed point set of  $\psi^k$  for some integer  $k$ . To be more precise, it is empty, a two-points  $\psi$ -orbit, some  $\mathcal{B}_i$  which is a reversible closed geodesic whose lengths for both directions are rational multiples of  $\pi$ , or some  $\mathcal{B}_i$  which is a totally geodesic submanifold isometric to a standard unit sphere.*

*Proof.* (1) We consider the maximum point  $x$  for the function  $f(\cdot) = F(X(\cdot))$ . By [Lemma 2.3](#), the integration curve of  $X$  passing  $x$  provides a prime closed geodesic  $\gamma$ , for which we have  $X(c(t)) \equiv \rho_{X,\gamma} \dot{c}(t)$  with  $\rho_{X,\gamma} > 0$ .

(2) We assume conversely that there exists a nonzero Killing vector field on  $(M, F)$  such that it vanishes on all closed geodesics. Let  $\mathfrak{k}$  be the space of all such Killing vector fields. It is a subalgebra of  $\mathfrak{h}$  corresponding to a subtorus in  $H_o$ . We can find a nonzero Killing vector field  $X$  from  $\mathfrak{k}$  which generates an  $S^1$ . The argument for (1) indicates  $X$  is not vanishing on some closed geodesic, which is the contradiction.

(3) Let  $N$  be the fixed point set of  $H_o$ , and assume  $N$  is not empty. By [Lemma 2.2](#),  $N$  must be a two-points  $\psi$ -orbit, a reversible closed geodesic, or a Finsler sphere with  $\dim N > 1$ ,  $K \equiv 1$  isometrically imbedded in  $(M, F)$ .

Obviously the action of  $\psi$  preserves  $N$ , i.e.,  $N$  consists of  $\psi$ -orbits. Because  $H$  is compact,  $H/H_o$  is finite. We also have  $\psi \in H$ , and thus each  $\psi$ -orbit in  $N$  is finite. So when  $\dim N = 1$ , the lengths of  $N$  for both directions are rational multiples of  $\pi$ .

When  $\dim N > 1$ , we see  $(N, F|_N)$  satisfies [Assumption \(F\)](#) by [Lemma 3.2](#). Then [Lemma 3.3](#) tells us that  $(N, F|_N)$  is a standard unit sphere.  $\square$

## 5. Proof of Theorem 1.2

Now we are ready to prove [Theorem 1.2](#), which applies a similar inductive argument as that for [Theorem 1.2](#) in [\[Xu 2018b\]](#).

When  $\psi$  has a finite order, then by [Lemma 3.3](#),  $(M, F)$  is the standard unit sphere. Obviously [Theorem 1.2](#) is valid in this case. So in the following discussion, we assume  $\psi$  has an infinite order, and thus we have  $m = \dim H > 0$ .

We will prove [Theorem 1.2](#) by an induction for  $n = \dim M$ .

When  $n = 2$  and the antipodal map  $\psi$  has an infinite order,  $H_o$  coincides with  $G = I_o(M, F) = S^1$ . In [\[Bryant et al. 2017\]](#), it has been proved that geometrically there exists exactly one reversible closed geodesic  $\gamma$  with a nontrivial  $H_o$ -action. So [Theorem 1.2](#) is valid in this case, and the estimate is sharp.

Now we assume [Theorem 1.2](#) is valid when  $n < l$  with  $l > 3$  (the inductive assumption) and we will prove the theorem when  $n = l$ .

Firstly, we prove:

**Claim 1.** *When  $\dim H = 1$ , there exists at least one totally geodesic  $B_i$  with a nontrivial  $H_o$ -action.*

Let  $X$  be any nonzero Killing vector field from  $\mathfrak{h} = \text{Lie}(H)$ . We list all the  $\hat{G}$ -orbits of prime closed geodesics as  $\mathcal{B}_i$  with  $1 \leq i \leq k$ , such that when  $1 \leq i \leq k'$  the coefficient  $\rho_{X,i}$  in (4-1) is positive. Notice that by [Lemma 4.3\(1\)](#), we have  $k' > 0$ .

If the antipodal map  $\psi$  is not contained in  $H_o$ , we can find an isometry of  $(M, F)$  which is of the form  $\phi = \psi \exp(t'X)$  such that its fixed point set contains  $B_1$ . By [Lemma 2.2](#) (or see [Lemma 3.5](#) in [\[Xu 2018b\]](#)), the fixed point set  $N$  of  $\phi$  is a closed connected totally geodesic submanifold. It must have a positive codimension in  $M$  because  $\phi \notin H_o$ . When  $\dim N = 1$ , it is a reversible closed geodesic. When  $\dim N > 1$ , by [Lemma 3.2](#) and the totally geodesic property,  $(N, F|_N)$  is a Finsler sphere satisfying  $K \equiv 1$  and [Assumption \(F\)](#). Using the inductive assumption, we can find some orbit of prime closed geodesic,  $\mathcal{B}_i = \hat{G}' \cdot \gamma_i = \hat{G} \cdot \gamma_i$ , where  $\hat{G}' = G' \times T$  and  $G' = I_o(N, F|_N)$ , such that the corresponding  $\mathcal{B}'_i$  is totally geodesic in  $(N, F|_N)$  as well as in  $(M, F)$ . The  $H_o$ -action on  $B_i$  is nontrivial because

$$\exp(t'X)|_{B_i} = \psi^{-1}\phi|_{B_i} = \psi^{-1}|_{B_i},$$

and  $\psi$  has no fixed point on any closed geodesic.

To summarize, this proves [Claim 1](#) when  $\psi \notin H_o$ .

To continue the proof of [Claim 1](#), we may assume  $\psi \in H_o$ . In this case, we can prove the zero set of  $X$  is empty as following. Assume conversely that the zero set of  $X$  is not empty, by [Lemma 4.3](#), it is a two-points  $\psi$ -orbit, a reversible closed geodesic, or a connected totally geodesic standard unit sphere. For each possibility,

$\psi$  can not be generated by  $X$ , which is a contradiction to the assumption  $\psi \in H_o$ . This fact implies that  $f(\cdot) = F(X(\cdot))$  is a smooth function on  $M$ . By [Lemma 2.3](#), the critical point set of  $f(\cdot)$  consists of exactly all  $B_i$ 's with  $1 \leq i \leq k'$ . Meanwhile, we see the  $H_o$ -action on each closed geodesic is nontrivial.

We take a prime closed geodesic  $c_i(t)$  with  $t \in \mathbb{R}/\mathbb{Z}$  from  $\mathcal{B}_i$  for  $1 \leq i \leq k'$ , then  $X|_{c_i} = \rho_{X,i} \dot{c}_i$  with  $\rho_{X,i} > 0$ . Because  $H_o = S^1$ , we can find some  $t' > 0$  such that  $\exp(t'X) = \text{id}$ , then we have

$$n_i = t' \rho_{X,i} \in \mathbb{N}, \quad \text{for all } 1 \leq i \leq k'.$$

We may reorder these  $c_i$ 's such that

$$n_1 \leq n_2 \leq \cdots \leq n_{k'}.$$

There are two possibilities, all  $n_i$ 's are not all the same, or all  $n_i$ 's are all the same.

Assume all  $n_i$ 's are not all the same, i.e.,  $n_1 < n_{k'}$ . The fixed point set  $N$  of the isometry  $\phi = \exp((t'/n_{k'})X) \in H_o$  contains  $B_{k'}$  but not  $B_1$ . It is either a reversible closed geodesic, or a Finsler sphere satisfying  $1 < \dim N < \dim M$ ,  $K \equiv 1$  and [Assumption \(F\)](#). Applying the inductive assumption and [Lemma 3.2](#), we can find a totally geodesic  $B_i$  for  $(N, F|_N)$ , as well as for  $(M, F)$ .

Assume all  $n_i$ 's are all the same, then all  $\rho_{X,i}$ 's are all the same as well. We may choose a suitable  $t'$  such that  $n_i = 1$  for  $1 \leq i \leq k'$ . There exists  $t'' \in (0, 1)$  such that  $\psi(c_i(0)) = c_i(t'')$ , i.e.,  $d_F(c_i(0), c_i(t'')) = \pi$ , for  $1 \leq i \leq k'$ . Then we have

$$F(X|_{c_1}) = F(X|_{c_2}) = \cdots = F(X|_{c_{k'}}).$$

The function  $f(\cdot) = F(X(\cdot))$  takes the same value on its critical point set, so it is a constant function. By [Lemma 2.3](#), all integration curves of  $X$  are closed geodesics, which belongs to one  $\hat{G}$ -orbit. By [Corollary 3.7](#),  $(M, F)$  is a non-Riemannian homogeneous Randers Finsler sphere with  $K \equiv 1$  and exactly two  $\hat{G}$ -orbits of prime closed geodesics,  $\mathcal{B}_1 = \hat{G} \cdot \gamma_1$  and  $\mathcal{B}_2 = \hat{G} \cdot \gamma_2$  such that  $\gamma_1$  and  $\gamma_2$  are the same curve with different directions.

This ends the proof of [Claim 1](#), i.e., [Theorem 1.2](#) is valid when  $m = \dim H = 1$ .

Next we prove [Theorem 1.2](#) assuming  $m = \dim H > 1$ .

**Claim 2.** *There exists at least  $m - 1$  geometrically distinct orbits  $\mathcal{B}_i$  such that each  $B_i$  is a totally geodesic submanifold with a nontrivial  $H_o$ -action.*

Let  $\mathcal{B}_i$  with  $1 \leq i \leq k'$  be all the geometrically distinct  $\hat{G}$ -orbits of prime closed geodesics such that the  $H_o$ -action on each  $B_i$  is not trivial. Let  $\mathfrak{h}_i$  be the codimension one subalgebra of  $\mathfrak{h}$  which restriction to  $B_i$  is zero. By [Lemma 4.3](#), the intersection  $\bigcap_{i=1}^{k'} \mathfrak{h}_i = 0$ , from which we see that  $m \leq k'$ . We may reorder the orbits  $\mathcal{B}_i$ 's such that  $\bigcap_{i=1}^m \mathfrak{h}_i = 0$ . Take a nonzero Killing vector field  $X \in \bigcap_{i=1}^{m-1} \mathfrak{h}_i$ . Then the zero set  $N$  of  $X$  is a closed connected totally geodesic submanifold in  $M$ , containing  $B_i$

for  $1 \leq i \leq m-1$  but not  $B_m$ . Let  $H'$  be the subgroup of  $I_o(N, F|_N)$  preserving all closed geodesics in  $N$ , and  $\mathfrak{h}'$  its Lie algebra. The restriction from  $M$  to  $N$  defines a linear map from  $\mathfrak{h}$  to  $\mathfrak{h}'$  which kernel is spanned by  $X$ , so  $\dim H' \geq m-1$ .

If  $\dim N = 1$ , then  $m = 2$ ,  $H_o$  has no fixed point, and  $N$  itself provides the totally geodesic  $B_i$  wanted by [Claim 2](#).

If  $\dim N > 1$ , we can use the inductive assumption to find  $m-1$  geometrically distinct orbits  $\mathcal{B}_i$  of prime closed geodesics for  $(N, F|_N)$ , as well as for  $(M, F)$  by [Lemma 3.2](#), such that the corresponding  $B_i$ 's are totally geodesic submanifolds, with nontrivial  $H'_o$ -actions. [Claim 2](#) is proved when each of these  $B_i$ 's also has a nontrivial  $H_o$ -action.

But it is possible that there is some  $B_i$  in  $N$  on which the  $H'_o$ -action is nontrivial but the  $H_o$ -action is trivial. If it happens, this  $B_i$  is unique, and we must have  $\dim H' > m-1$ . So in this case, we can use the inductive assumption to find  $m$  geometrically distinct orbits of prime closed geodesics. At least  $m-1$  geometrically distinct totally geodesic  $B_i$ 's in  $N$  have nontrivial  $H_o$ -actions.

This proves [Claim 2](#).

To finish the proof of [Theorem 1.2](#) when  $n = l$ , we only need to find one more totally geodesic  $B_i$  with a nontrivial  $H_o$ -action.

We may reorder the orbits  $\mathcal{B}_i$ 's such that the first  $m-1$  ones are those provided by [Claim 2](#), and  $\bigcap_{i=1}^m \mathfrak{h}_i = 0$ . The nonzero Killing vector field  $X$  from  $\bigcap_{i=1}^{m-1} \mathfrak{h}_i$  vanishes on  $B_i$  with  $1 \leq i \leq m-1$ , but not on  $B_m$ . We can find an isometry of the form  $\phi = \psi \exp(t'X)$  such that it fixes each point of  $B_m$ . On the other hand, the fixed point set  $N$  of  $\phi$  does not contain each  $B_i$  for  $1 \leq i \leq m-1$ .

The  $H_o$ -action on each closed geodesic in  $N$  is nontrivial. Assume conversely that there is a closed geodesic in  $N$  with a trivial  $H_o$ -action. Then the restriction of  $\psi$  to this geodesic coincides with that of  $\phi$ , fixing each point of this geodesic. This is not true because  $\psi$  has no fixed points.

If  $\dim N = 1$  it is a reversible closed geodesic, which is the extra  $B_i$  we want. If  $\dim N > 1$  it is a Finsler sphere satisfying  $K \equiv 1$  and [Assumption \(F\)](#), isometrically imbedded in  $(M, F)$  as a totally geodesic submanifold. In this situation we use the inductive assumption one more time, which provides one more totally geodesic  $B_i$ .

Summarizing above discussion, we have proved [Theorem 1.2](#) when  $n = l$ .

This ends the proof of [Theorem 1.2](#) by induction.

## 6. The example from Katok metrics

We conclude this paper by the examples from Katok metrics for which the estimate in [Theorem 1.2](#) is sharp.

Let  $(M, h) = (S^n, h)$  be a standard unit sphere with  $n > 1$ ,  $W$  a Killing vector field on  $(M, h)$  such that  $h(W, W) < 1$  everywhere.

Then the navigation process defines a Randers metric

$$F(y) = \frac{\sqrt{\lambda h(y, y) + h(W, y)^2}}{\lambda} - \frac{h(W, y)}{\lambda}$$

on  $M$ , in which  $\lambda = 1 - h(W, W)$  is positive everywhere.

By the work of Bao, Robles and Shen [Bao et al. 2004], this construction provides all the Randers spheres with  $K \equiv 1$ . The behavior of the geodesics on  $(M, F)$  is determined by the choice of  $W$ .

We can find suitable coordinates  $x = (x_0, z_1, \dots, z_k)$  for  $x \in \mathbb{R}^{n+1}$ , where

$$x_0 = (x_{0,1}, \dots, x_{0,n_0}) \in \mathbb{R}^{n_0} \quad \text{and} \quad z_i = (z_{i,1}, \dots, z_{i,n_i}) \in \mathbb{C}^{n_i}$$

satisfy the following:

(A1) We permit  $n_0 = 0$  and in this case  $x_0$  is always 0. All other  $n_i$ 's are positive.

(A2)  $(M, h)$  is naturally identified as the unit sphere  $S^n(1)$  defined by

$$|x_0|^2 + |z_1|^2 + \dots + |z_k|^2 = 1$$

in  $\mathbb{R}^{n+1} = \mathbb{R}^{n_0} \oplus \mathbb{C}^{n_1} \oplus \dots \oplus \mathbb{C}^{n_k}$  with the standard product Euclidean metric.

(A3)  $W$  can be presented as

$$(6-1) \quad W(x_0, z_0, \dots, z_k) = (0, \sqrt{-1}\lambda_1 z_1, \dots, \sqrt{-1}\lambda_k z_k),$$

such that  $0 < \lambda_1 < \lambda_2 < \dots < \lambda_k < 1$ .

We further require one of the following is satisfied:

(A4) All  $\lambda_i$ 's are irrational numbers. For any  $1 \leq i < j \leq k$ ,  $1, \lambda_i$  and  $\lambda_j$  are linearly independent over  $\mathbb{Q}$ .

(A5) All  $\lambda_i$ 's are irrational numbers except one,  $n_0 = 0$  and  $n_i = 1$  if  $\lambda_i \in \mathbb{Q}$ . If  $\lambda_i$  and  $\lambda_j$  are irrational numbers,  $1, \lambda_i$  and  $\lambda_j$  are linearly independent.

Then we have

**Lemma 6.1.** *For the Randers sphere  $(M, F)$  described above, satisfying (A1)–(A3) and one of (A4) and (A5), any closed geodesic on  $(M, F)$  must be contained in*

$$z_1 = \dots = z_k = 0$$

or

$$x_0 = 0 \quad \text{and} \quad z_j = 0 \quad \text{when } j \neq i,$$

for some  $i, 1 \leq i \leq k$ .

*Proof.* Using (6-1), we can present the antipodal map as

$$\psi(x_0, z_1, \dots, z_k) = (x_0, -e^{\sqrt{-1}\pi\lambda_1} z_1, \dots, -e^{\sqrt{-1}\pi\lambda_k} z_k).$$



It is easy to check that finite  $\psi$ -orbits only appear in the situation that only  $\mathbf{x}_0$  is nonzero or only  $\mathbf{z}_i$  with  $\lambda_i \in \mathbb{Q}$  is nonzero.

Let  $x = (\mathbf{x}_0, \mathbf{z}_1, \dots, \mathbf{z}_k)$  be a point on the closed geodesic  $\gamma$ . We only need to prove that only one of  $\mathbf{x}_0$  and  $\mathbf{z}_i$ 's can be nonzero. Assume conversely this is not true. Then the length of  $\gamma$  can not be a rational multiple of  $\pi$  (i.e., consists of finite  $\psi$ -orbits), so the  $\psi$ -orbit of  $x$  is a dense subset in  $\gamma$ . There are three cases we need to consider.

In the first case,  $\lambda_i$  and  $\lambda_j$  are irrational numbers,  $\mathbf{z}_i \neq 0$ , and  $\mathbf{z}_j \neq 0$ . Then the condition that 1,  $\lambda_i$  and  $\lambda_j$  are linearly independent implies that the projection to the  $\mathbf{z}_i$ - and  $\mathbf{z}_j$ -factors maps the closed curve  $\gamma$  onto a two dimensional torus, which is a contradiction.

In the second case,  $\lambda_i$  is rational,  $\lambda_j$  is not,  $\mathbf{z}_i \neq 0$  and  $\mathbf{z}_j \neq 0$ . Then the projection to the  $\mathbf{z}_i$ -factor maps  $\gamma$  to a finite set with at least two points. This is impossible because  $\gamma$  is connected.

In the third case,  $\mathbf{x}_0 \neq 0$  and  $\mathbf{z}_i \neq 0$ . Then the projection to the  $\mathbf{x}_0$ -factor maps  $\gamma$  to two points. This is impossible for the same reason as the previous case.

To summarize, we have found contradiction for all the cases, and finished the proof of this lemma.  $\square$

Using [Lemma 6.1](#), we can provides examples of Katok metrics such that the estimates in [Theorem 1.2](#) are sharp.

**Theorem 6.2.** *Let  $F$  be the Randers metrics on  $S^n$  with  $n > 1$  satisfying (A1)–(A3) and one of (A4) and (A5). Then it has only finite orbits of prime closed geodesics. Let  $H$  denote the subgroup of isometries preserving each closed geodesic,  $H_o$  its identity component, and  $m = \dim H$ . Then there exist exactly  $m$  geometrically distinct  $\mathcal{B}_i$ , such that the corresponding  $B_i$ 's are totally geodesic with nontrivial  $H_o$ -actions.*

The proof is a case-by-case discussion. For each case, it is not hard to calculate  $G = I_o(M, F)$ ,  $H_o$  and all the orbits of prime closed geodesics.

For example, when  $n_0 > 2$  and all  $\gamma_i$ 's are irrational numbers,

$$\begin{aligned} G &= \mathrm{SO}(n_0) \times U(n_1) \times \cdots \times U(n_k), \quad \text{and} \\ H &= C(U(n_1) \times \cdots \times U(n_k)) = U(1)^k, \end{aligned}$$

so we have  $\dim H = k$ .

When  $1 \leq i \leq k$ ,

$$B_i = \{x = (\mathbf{x}_0, \mathbf{z}_1, \dots, \mathbf{z}_k) \in M \text{ with } \mathbf{x}_0 = 0 \text{ and } \mathbf{z}_j = 0 \text{ when } j \neq i\}$$

is a homogeneous Randers sphere with exactly two orbits of prime closed geodesics. It is isometrically imbedded in  $(M, F)$  as a totally geodesic submanifold, because

it is the fixed point set of the subgroup of  $G$  with the  $U(n_i)$ -factor removed. They provide all the different totally geodesic  $B_i$ 's with nontrivial  $H_o$ -actions.

There exists one more totally geodesic  $B_{k+1}$  with a trivial  $H_o$ -action, i.e.,

$$B_{k+1} = \{x = (x_0, z_1, \dots, z_k) \in M \text{ with } z_1 = \dots = z_k = 0\}.$$

It is a standard unit sphere with only one orbit of closed geodesics.

By [Lemma 6.1](#), no other closed geodesics can be found.

Summarizing all these observations, we see that this Randers sphere  $(M, F)$  satisfies all the requirements in [Theorem 1.2](#), and the estimate in [Theorem 1.2](#) for the number of totally geodesic  $B_i$ 's is sharp.

The discussion for other cases is similar, so we skip the details.

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## References

- [Anosov 1975] D. V. Anosov, “Geodesics in Finsler geometry”, pp. 293–297 in *Proceedings of the International Congress of Mathematicians, II* (Vancouver, 1974), edited by R. D. James, Canad. Math. Congress, Montreal, 1975. In Russian; translated in *Amer. Math. Soc. Trans.* **109** (1977), 81–85. [MR](#) [Zbl](#)
- [Bangert and Long 2010] V. Bangert and Y. Long, “The existence of two closed geodesics on every Finsler 2-sphere”, *Math. Ann.* **346**:2 (2010), 335–366. [MR](#) [Zbl](#)
- [Bao et al. 2000] D. Bao, S.-S. Chern, and Z. Shen, *An introduction to Riemann–Finsler geometry*, Graduate Texts in Math. **200**, Springer, 2000. [MR](#) [Zbl](#)
- [Bao et al. 2004] D. Bao, C. Robles, and Z. Shen, “Zermelo navigation on Riemannian manifolds”, *J. Differential Geom.* **66**:3 (2004), 377–435. [MR](#) [Zbl](#)
- [Berestovskii and Nikonorov 2019] V. N. Berestovskii and Y. G. Nikonorov, “On homogeneous geodesics and weakly symmetric spaces”, *Ann. Global Anal. Geom.* **55**:3 (2019), 575–589. [MR](#)
- [Bryant 1996] R. L. Bryant, “Finsler structures on the 2-sphere satisfying  $K = 1$ ”, pp. 27–41 in *Finsler geometry* (Seattle, 1995), edited by D. Bao et al., Contemp. Math. **196**, Amer. Math. Soc., Providence, RI, 1996. [MR](#) [Zbl](#)
- [Bryant 1997] R. L. Bryant, “Projectively flat Finsler 2-spheres of constant curvature”, *Selecta Math. (N.S.)* **3**:2 (1997), 161–203. [MR](#) [Zbl](#)
- [Bryant 2002] R. L. Bryant, “Some remarks on Finsler manifolds with constant flag curvature”, *Houston J. Math.* **28**:2 (2002), 221–262. [MR](#) [Zbl](#)
- [Bryant et al. 2017] R. L. Bryant, P. Foulon, S. Ivanov, V. S. Matveev, and W. Ziller, “Geodesic behavior for Finsler metrics of constant positive flag curvature on  $S^2$ ”, 2017. To appear in *J. Differential Geom.* [arXiv](#)

- [Deng and Xu 2014] S. Deng and M. Xu, “Clifford–Wolf translations of Finsler spaces”, *Forum Math.* **26**:5 (2014), 1413–1428. [MR](#) [Zbl](#)
- [Duan 2016] H. Duan, “Two elliptic closed geodesics on positively curved Finsler spheres”, *J. Differential Equations* **260**:12 (2016), 8388–8402. [MR](#) [Zbl](#)
- [Duan et al. 2016] H. Duan, Y. Long, and W. Wang, “The enhanced common index jump theorem for symplectic paths and non-hyperbolic closed geodesics on Finsler manifolds”, *Calc. Var. Partial Differential Equations* **55**:6 (2016), art. id. 145. [MR](#) [Zbl](#)
- [Katok 1973] A. B. Katok, “Ergodic perturbations of degenerate integrable Hamiltonian systems”, *Izv. Akad. Nauk SSSR Ser. Mat.* **37** (1973), 539–576. In Russian; translated in *Math. USSR-Izv.* **7** (1973) 535–572. [MR](#) [Zbl](#)
- [Long and Duan 2009] Y. Long and H. Duan, “Multiple closed geodesics on 3-spheres”, *Adv. Math.* **221**:6 (2009), 1757–1803. [MR](#) [Zbl](#)
- [Rademacher 1989] H.-B. Rademacher, “On the average indices of closed geodesics”, *J. Differential Geom.* **29**:1 (1989), 65–83. [MR](#) [Zbl](#)
- [Rademacher 2004] H.-B. Rademacher, “A sphere theorem for non-reversible Finsler metrics”, *Math. Ann.* **328**:3 (2004), 373–387. [MR](#) [Zbl](#)
- [Shen 1996] Z. Shen, “Finsler manifolds of constant positive curvature”, pp. 83–93 in *Finsler geometry* (Seattle, 1995), edited by D. Bao et al., Contemp. Math. **196**, Amer. Math. Soc., Providence, RI, 1996. [MR](#) [Zbl](#)
- [Wang 2012] W. Wang, “On a conjecture of Anosov”, *Adv. Math.* **230**:4-6 (2012), 1597–1617. [MR](#) [Zbl](#)
- [Wang 2015] W. Wang, “Non-hyperbolic closed geodesics on Finsler spheres”, *J. Differential Geom.* **99**:3 (2015), 473–496. [MR](#) [Zbl](#)
- [Xu 2018a] M. Xu, “Geodesic orbit spheres and constant curvature in Finsler geometry”, *Differential Geom. Appl.* **61** (2018), 197–206. [MR](#) [Zbl](#)
- [Xu 2018b] M. Xu, “The number of geometrically distinct reversible closed geodesics on a Finsler sphere with  $K \equiv 1$ ”, preprint, 2018. [arXiv](#)
- [Yan and Huang 2018] Z. Yan and L. Huang, “On the existence of homogeneous geodesic in homogeneous Finsler spaces”, *J. Geom. Phys.* **124** (2018), 264–267. [MR](#) [Zbl](#)

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MING XU  
SCHOOL OF MATHEMATICAL SCIENCES  
CAPITAL NORMAL UNIVERSITY  
BEIJING  
CHINA  
[mngmngxu@163.com](mailto:mngmngxu@163.com)

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University of California  
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Department of Mathematics  
University of California  
Santa Barbara, CA 93106-3080  
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Department of Mathematics  
The University of Hong Kong  
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Department of Mathematics  
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Los Angeles, CA 90095-1555  
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Mathematics Department  
National University of Singapore  
Singapore 119076  
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Sorin Popa  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[popa@math.ucla.edu](mailto:popa@math.ucla.edu)

Paul Yang  
Department of Mathematics  
Princeton University  
Princeton NJ 08544-1000  
[yang@math.princeton.edu](mailto:yang@math.princeton.edu)

Vyjayanthi Chari  
Department of Mathematics  
University of California  
Riverside, CA 92521-0135  
[chari@math.ucr.edu](mailto:chari@math.ucr.edu)

Kefeng Liu  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[liu@math.ucla.edu](mailto:liu@math.ucla.edu)

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