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DEGENERACY THEOREMS FOR TWO HOLOMORPHIC CURVES IN $\mathbb{P}^n(\mathbb{C})$ SHARING FEW HYPERSURFACES

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In value distribution theory, many uniqueness and degeneracy theorems for holomorphic curves in $\mathbb{P}^n(\mathbb{C})$ sharing hyperplanes or sharing sufficiently many hypersurfaces have been obtained in the last few decades. But there is no result concerning holomorphic curves in $\mathbb{P}^n(\mathbb{C})$ sharing few hypersurfaces. We prove several degeneracy theorems for two algebraically nondegenerate holomorphic curves in $\mathbb{P}^n(\mathbb{C})$ sharing $n+k$ hypersurfaces in general position.

1. Introduction

Since Fujimoto [1975] generalized Nevanlinna's uniqueness theorems of meromorphic functions sharing values to the case of meromorphic maps of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ sharing hyperplanes, plenty of uniqueness and degeneracy results for meromorphic maps sharing hyperplanes have been obtained; see for instance [Smiley 1983; Fujimoto 1998; Fujimoto 1999; Chen and Yan 2009; Si and Le 2015]. Some uniqueness theorems for holomorphic curves in $\mathbb{P}^n(\mathbb{C})$ sharing sufficiently many hypersurfaces have also been proven; see [Dulock and Ru 2008; Phuong 2013; Quang and An 2017].

But as far as we know, there is no result concerning two holomorphic curves in $\mathbb{P}^n(\mathbb{C})$ sharing $n+k$ hypersurfaces. This paper proves some degeneracy theorems for two holomorphic curves in $\mathbb{P}^n(\mathbb{C})$ sharing $n+k$ hypersurfaces.

Now we introduce some notions. A holomorphic map $f:\mathbb{C}\to\mathbb{P}^n(\mathbb{C})$ is said to be linearly (resp. algebraically) nondegenerate if its image is not contained in any proper linear subspace (resp. algebraic subset) of $\mathbb{P}^n(\mathbb{C})$. Hypersurfaces $D_1,\dots,D_q\,(q>n)$ in $\mathbb{P}^n(\mathbb{C})$ are said to be located in general position if $\bigcap_{k=1}^{n+1}\operatorname{Supp} D_{j_k}=\varnothing$ for any $n+1$ distinct indices $j_1,\dots,j_{n+1}\in\{1,\dots,q\}$. For a nonzero meromorphic function h on the complex plane $\mathbb{C}$, let ν_h^0 (resp. ν_h^∞) be the zero (resp. pole) divisor of h, and let $\nu_h=\nu_h^0-\nu_h^\infty$.

We may regard $\mathbb{P}^n(\mathbb{C})\times\mathbb{P}^n(\mathbb{C})$ as a subvariety of $\mathbb{P}^{(n+1)^2-1}(\mathbb{C})$ via the Segre embedding $(a_0:\cdots:a_n)\times(b_0:\cdots:b_n)\mapsto(\dots:a_ib_j:\dots)$. And a holomorphic

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map $F : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ is said to be algebraically degenerate if its image is contained in a proper algebraic subset of $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$.

We state our main theorems now. Let $f, g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be two algebraically nondegenerate holomorphic curves with reduced representations $\tilde{f} = (f_0, \dots, f_n)$ and $\tilde{g} = (g_0, \dots, g_n)$. Let $q > n$ and let D_j, $1 \le j \le q$, be hypersurfaces of degrees d_j in $\mathbb{P}^n(\mathbb{C})$ located in general position. Let $Q_j \in \mathbb{C}[x_0, \dots, x_n]$, $1 \le j \le q$, be the homogeneous polynomials of degrees d_j defining D_j. Let d be the least common multiple of the d_j's and set $\tilde{Q}_j = Q_j^{d/d_j}$ for $1 \le j \le q$.

Theorem 1.1. *Assume that $q = \max\{4, n+2\}$. If*

(a) $f^{-1}(D_i) \cap f^{-1}(D_j) = \varnothing$ *for all* $i \in \{1, \dots, q\}$ *and* $j \in \{1, 2, 3, 4\} \setminus \{i\}$,

(b) $\nu_{Q_j(\tilde{f})} = \nu_{Q_j(\tilde{g})}$ *for* $1 \le j \le 4$ *and* $\min\{\nu_{Q_j(\tilde{f})}, 1\} = \min\{\nu_{Q_j(\tilde{g})}, 1\}$ *for* $4 < j \le q$,

(c) $f = g$ *on* $\bigcup_{j=1}^{q} f^{-1}(D_j)$,

then there are three distinct indices $i, j, k \in \{1, 2, 3, 4\}$ such that

$$\left(\frac{\tilde{Q}_i(\tilde{f}) \cdot \tilde{Q}_k(\tilde{g})}{\tilde{Q}_i(\tilde{g}) \cdot \tilde{Q}_k(\tilde{f})}\right)^s \cdot \left(\frac{\tilde{Q}_j(\tilde{f}) \cdot \tilde{Q}_k(\tilde{g})}{\tilde{Q}_j(\tilde{g}) \cdot \tilde{Q}_k(\tilde{f})}\right)^t \equiv 1$$

for some $(s, t) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$. Consequently $\{f_u g_v\}_{0 \le u, v \le n}$ satisfy a nontrivial homogeneous polynomial equation; thus $f \times g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ is algebraically degenerate.

Since the conditions "$f = g$ on $\bigcup_{j=1}^{q} f^{-1}(D_j)$" and "$f^{-1}(D_i) \cap f^{-1}(D_j) = \varnothing$ for $i \ne j$" are really rigid, it's natural to study the related problem without the two conditions. In this direction, Fujimoto [1999] proved a degeneracy theorem for sharing $2n+2$ hyperplanes with truncated multiplicities. In our case of sharing few hypersurfaces, we can only prove the following.

Theorem 1.2. *Assume that $q = n + 3$. If*

(a) $\nu_{Q_j(\tilde{f})} = \nu_{Q_j(\tilde{g})}$ *for* $1 \le j \le q$,

(b) $f = g$ *on* $\bigcup_{j=1}^{n+2} f^{-1}(D_j)$,

then there are three distinct indices $i, j, k \in \{1, \dots, q\}$ such that

$$\left(\frac{\tilde{Q}_i(\tilde{f}) \cdot \tilde{Q}_k(\tilde{g})}{\tilde{Q}_i(\tilde{g}) \cdot \tilde{Q}_k(\tilde{f})}\right)^s \cdot \left(\frac{\tilde{Q}_j(\tilde{f}) \cdot \tilde{Q}_k(\tilde{g})}{\tilde{Q}_j(\tilde{g}) \cdot \tilde{Q}_k(\tilde{f})}\right)^t \equiv 1$$

for some $(s, t) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$. In particular $f \times g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ is algebraically degenerate.

In fact we prove a stronger theorem (see Theorem 4.1) with a weaker condition than "$f = g$ on $\bigcup_{j=1}^{n+2} f^{-1}(D_j)$" (see Remark 4.2).

If we require further that the order of f (see Definition 2.5) is less than 1, then we can get rid of both the two conditions; namely we have:

Theorem 1.3. *Assume that f is of order < 1. Let $q = n+2$. If $\nu_{Q_j(\tilde{f})} = \nu_{Q_j(\tilde{g})}$ for $j = 1, 2$ and $\min\{\nu_{Q_j(\tilde{f})}, 1\} = \min\{\nu_{Q_j(\tilde{g})}, 1\}$ for $2 < j \le q$, then there exists a nonzero constant C such that*

$$\frac{\tilde{Q}_1(\tilde{f}) \cdot \tilde{Q}_2(\tilde{g})}{\tilde{Q}_1(\tilde{g}) \cdot \tilde{Q}_2(\tilde{f})} \equiv C.$$

In particular, $f \times g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ is algebraically degenerate.

Remark 1.4. If all D_j's are hyperplanes in $\mathbb{P}^n(\mathbb{C})$, then the nondegeneracy assumption on f and g only needs to be linearly nondegenerate.

Our proof is based on the second main theorem for holomorphic curves in $\mathbb{P}^n(\mathbb{C})$ intersecting hypersurfaces, which was first proved by Ru [2004], and a gcd bound for holomorphic units (see [Pasten and Wang 2017, Theorem 3.1]). The technique of using the gcd bound is due to Si [2013].

2. Preliminaries from Nevanlinna theory

For a divisor ν on $\mathbb{C}$, we define the counting function of ν by

$$N(r, \nu) = \int_0^r \frac{n(t, \nu) - n(0, \nu)}{t} dt + n(0, \nu) \log r,$$

where $n(t, \nu) := \sum_{|z| \le t} \nu(z)$.

Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map and let $\tilde{f} = (f_0, \dots, f_n)$ be a reduced representation of f; namely, $f_0, \dots, f_n$ are entire functions on $\mathbb{C}$ without common zeros and $f(z) = [f_0(z) : \cdots : f_n(z)]$ for every $z \in \mathbb{C}$. The characteristic function of f is defined by

$$T_f(r) := \frac{1}{2\pi} \int_0^{2\pi} \log \|\tilde{f}(re^{i\theta})\| d\theta - \log \|\tilde{f}(0)\|,$$

where $\|\tilde{f}(z)\| = \sqrt{|f_0(z)|^2 + \cdots + |f_n(z)|^2}$. This definition is independent of the choice of the reduced representation. Let D be a hypersurface of degree d in $\mathbb{P}^n(\mathbb{C})$ with $f(\mathbb{C}) \not\subseteq D$. Let $Q \in \mathbb{C}[x_0, \dots, x_n]$ be the homogeneous polynomial of degree d defining D. Then the proximity function $m_f(r, D)$ is defined by

$$m_f(r, D) = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\|\tilde{f}(re^{i\theta})\|^d \|Q\|}{|Q(\tilde{f})(re^{i\theta})|} d\theta,$$

where $\|Q\|$ is the maximum of the absolute values of the coefficients of Q. And the counting function of f intersecting D with truncation level M, $M \in \mathbb{Z}^+ \cup \{+\infty\}$, is defined by

$$N_f^{[M]}(r, D) := N(r, \min\{\nu_{Q(\tilde{f})}, M\}).$$

We also write $N_f^{[1]}(r, D) = \overline{N}_f(r, D)$ and $N_f^{[+\infty]}(r, D) = N_f(r, D)$. If H is a hyperplane in $\mathbb{P}^n(\mathbb{C})$ defined by the linear form L, we also write $L(\tilde{f})$ as (f, H).

The Jensen formula (see [Ru 2001, Corollary A1.1.3]) implies the following first main theorem:

Theorem 2.1. *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map and let D be a hypersurface of degree d in $\mathbb{P}^n(\mathbb{C})$. If $f(\mathbb{C}) \not\subseteq D$, then there is a real constant C, such that for all $r > 0$,*

$$m_f(r, D) + N_f(r, D) = dT_f(r) + C.$$

The following is the well known second main theorem for holomorphic curves in $\mathbb{P}^n(\mathbb{C})$ intersecting hyperplanes (see [Ru 2001, Theorem A3.2.2]) which was first proved by H. Cartan.

Theorem 2.2. *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate holomorphic map and $\{H_j\}_{j=1}^q$ be q hyperplanes in $\mathbb{P}^n(\mathbb{C})$ located in general position. Then*

$$\Big\| \; (q - n - 1)T_f(r) \le \sum_{j=1}^{q} N_f^{[n]}(r, H_j) + o(T_f(r)),$$

where the notation "$\|$" means that the assertion holds for all $r > 0$ outside a set of finite Lebesgue measure.

Ru [2004] proved a second main theorem for holomorphic curves in $\mathbb{P}^n(\mathbb{C})$ intersecting hypersurfaces. The following version with truncation was proved in [Yan and Chen 2008; An and Phuong 2009].

Theorem 2.3. *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be an algebraically nondegenerate holomorphic map. Let D_j, $1 \le j \le q$, be hypersurfaces of degrees d_j in $\mathbb{P}^n(\mathbb{C})$ located in general position. Then for any $\epsilon > 0$, there is a positive integer M_ϵ such that*

$$\Big\| \; (q - n - 1 - \epsilon)T_f(r) \le \sum_{j=1}^{q} d_j^{-1} N_f^{[M_\epsilon]}(r, D_j).$$

For a meromorphic function h on the complex plane $\mathbb{C}$, the Nevanlinna's characteristic function of h is defined by

$$T(r, h) := m(r, h) + N(r, h),$$

where $m(r, h) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ |h(re^{i\theta})| d\theta$ with $\log^+ x = \max\{\log x, 0\}$ for $x \ge 0$, and $N(r, h) := N(r, \nu_h^\infty)$. It follows from the definition that for any meromorphic functions h_1, h_2 on $\mathbb{C}$, $T(r, h_1 + h_2) \le T(r, h_1) + T(r, h_2) + \ln 2$ and $T(r, h_1 h_2) \le T(r, h_1) + T(r, h_2)$ for $r \ge 1$. Furthermore we have the following first main theorem for meromorphic functions (see [Ru 2001, Theorem A1.1.5]).

Theorem 2.4. $T(r, h) = T\big(r, \frac{1}{h-a}\big) + O(1)$ *for any meromorphic function h on $\mathbb{C}$ and $a \in \mathbb{C}$ provided that $h \not\equiv a$.*

Definition 2.5. The order of a holomorphic map $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ is defined to be

$$\overline{\lim_{r\to+\infty}} \frac{\log^+ T_f(r)}{\log r}.$$

The order of a meromorphic function h on $\mathbb{C}$ can be similarly defined.

3. Proof of Theorem 1.1

We prove Theorem 1.1 in this section; in fact, we prove the following stronger theorem.

Theorem 3.1. *Let $f, g, \tilde{f}, \tilde{g}, d, D_j, Q_j, \tilde{Q}_j,\ 1 \le j \le q$, be given as in Section 1. Assume there exist $I, J \subseteq \{1, \dots, q\}$ with $\#I \ge n+2$ and for any $i \in I$, $\#(J \setminus \{i\}) \ge 3$, such that the following conditions are satisfied*:

(a) $f^{-1}(D_i) \cap f^{-1}(D_j) = \varnothing$ *for all $i \in I$ and $j \in J \setminus \{i\}$*,

(b) $\nu_{Q_j(\tilde{f})} = \nu_{Q_j(\tilde{g})}$ *for $j \in J$ and* $\min\{\nu_{Q_i(\tilde{f})}, 1\} = \min\{\nu_{Q_i(\tilde{g})}, 1\}$ *for $i \in I$*,

(c) $f = g$ *on* $\bigcup_{i\in I} f^{-1}(D_i)$.

Then there exist three distinct indices $i, j, k \in J$ such that

$$\left(\frac{\tilde{Q}_i(\tilde{f}) \cdot \tilde{Q}_k(\tilde{g})}{\tilde{Q}_i(\tilde{g}) \cdot \tilde{Q}_k(\tilde{f})}\right)^s \cdot \left(\frac{\tilde{Q}_j(\tilde{f}) \cdot \tilde{Q}_k(\tilde{g})}{\tilde{Q}_j(\tilde{g}) \cdot \tilde{Q}_k(\tilde{f})}\right)^t \equiv 1$$

for some $(s, t) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$. In particular, $f \times g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ is algebraically degenerate.

Taking $q = \max\{4, n+2\},\ I = \{1, \dots, q\},\ J = \{1, 2, 3, 4\}$, we get Theorem 1.1. Furthermore, we can deduce the following corollary by taking $q = n+5$, $I = \{4, \dots, q\},\ J = \{1, 2, 3\}$.

Corollary 3.2. *Let $f, g, \tilde{f}, \tilde{g}, d, D_j, Q_j, \tilde{Q}_j,\ 1 \le j \le q$, be given as in Section 1. Assume that $q = n+5$. If*

(a) $f^{-1}(D_i) \cap f^{-1}(D_j) = \varnothing$ *for $i = 1, 2, 3$ and $j = 4, \dots, q$*,

(b) $\nu_{Q_j(\tilde{f})} = \nu_{Q_j(\tilde{g})}$ *for $j = 1, 2, 3$, and* $\min\{\nu_{Q_j(\tilde{f})}, 1\} = \min\{\nu_{Q_j(\tilde{g})}, 1\}$ *for $j = 4, \dots, q$*,

(c) $f = g$ *on* $\bigcup_{j=4}^{q} f^{-1}(D_j)$,

then

$$\left(\frac{\tilde{Q}_1(\tilde{f}) \cdot \tilde{Q}_3(\tilde{g})}{\tilde{Q}_1(\tilde{g}) \cdot \tilde{Q}_3(\tilde{f})}\right)^s \cdot \left(\frac{\tilde{Q}_2(\tilde{f}) \cdot \tilde{Q}_3(\tilde{g})}{\tilde{Q}_2(\tilde{g}) \cdot \tilde{Q}_3(\tilde{f})}\right)^t \equiv 1$$

for some $(s, t) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$. In particular, $f \times g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ is algebraically degenerate.

For our purpose, we need the following lemma on the gcd bound for holomorphic units; for the proof refer to [Pasten and Wang 2017, Theorem 3.1].

Lemma 3.3. *Let F, G be nowhere zero holomorphic functions on $\mathbb{C}$. If $F^s \cdot G^t$ is not constant for all $(s,t) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$, then for any $\epsilon > 0$,*

$$\| \ N(r, F-1, G-1) \le \epsilon \max\{T(r,F), T(r,G)\},$$

where $N(r, F-1, G-1)$ is the counting function of the common 1-points of F and G; namely, $N(r, F-1, G-1) := N(r, \min\{\nu^0_{F-1}, \nu^0_{G-1}\})$.

Remark 3.4. If $F^s \cdot G^t \equiv c \in \mathbb{C} \setminus \{1\}$ for some $(s,t) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$, then F and G have no common 1-points; namely, $N(r, F-1, G-1) \equiv 0$. So the conclusion of the above lemma actually holds when $F^s \cdot G^t \not\equiv 1$ for all $(s,t) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$.

Now we are going to prove Theorem 3.1. We give the following lemma first.

Lemma 3.5. *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map with reduced representation $\tilde{f} = (f_0, \dots, f_n)$. Let $Q_1, Q_2 \in \mathbb{C}[x_0, \dots, x_n]$ be two homogeneous polynomials of same degree $d > 0$ with $Q_2(\tilde{f}) \not\equiv 0$. Then there are constants $C_1, C_2 > 0$ such that for all $r > 0$ large enough,*

$$T\left(r, \frac{Q_1(\tilde{f})}{Q_2(\tilde{f})}\right) \le C_1 T_f(r) + C_2.$$

Proof. Take $k \in \{0, \dots, n\}$ such that $f_k \not\equiv 0$. Write $Q_1(\tilde{f}) = \sum a f_0^{i_0} \cdots f_n^{i_n}$ and $Q_2(\tilde{f}) = \sum b f_0^{j_0} \cdots f_n^{j_n}$, then

$$\frac{Q_1(\tilde{f})}{Q_2(\tilde{f})} = \frac{Q_1(\tilde{f})/f_k^d}{Q_2(\tilde{f})/f_k^d} = \frac{\sum a\left(\frac{f_0}{f_k}\right)^{i_0} \cdots \left(\frac{f_n}{f_k}\right)^{i_n}}{\sum b\left(\frac{f_0}{f_k}\right)^{j_0} \cdots \left(\frac{f_n}{f_k}\right)^{j_n}}.$$

Thus by the first main theorem and the properties of Nevanlinna's characteristic function, we conclude that

$$\begin{aligned} T\left(r, \frac{Q_1(\tilde{f})}{Q_2(\tilde{f})}\right) \le & T\left(r, \sum a\left(\frac{f_0}{f_k}\right)^{i_0} \cdots \left(\frac{f_n}{f_k}\right)^{i_n}\right) \\ & + T\left(r, \sum b\left(\frac{f_0}{f_k}\right)^{j_0} \cdots \left(\frac{f_n}{f_k}\right)^{j_n}\right) + O(1) \\ \le & \tilde{C}_1\left(T\left(r, \frac{f_0}{f_k}\right) + \cdots + T\left(r, \frac{f_n}{f_k}\right)\right) + \tilde{C}_2. \end{aligned}$$

By [Ru 2001, Theorem A3.1.2], we know that $T(r, f_t/f_k) \le T_f(r) + O(1)$ for $t = 0, \dots, n$, this together with the above inequality imply the desired conclusion. □

Proof of Theorem 3.1. Set $h_j = \tilde{Q}_j(\tilde{f})/\tilde{Q}_j(\tilde{g})$ for $j = 1, \dots, q$. Then by condition (b), h_j is a nowhere zero holomorphic function on $\mathbb{C}$ for every $j \in J$.

We argue by the method of contradiction. Assume that the conclusion doesn't hold, then for arbitrary three distinct indices $j_1, j_2, j_3 \in J$,

$$\left(\frac{h_{j_1}}{h_{j_3}}\right)^s \cdot \left(\frac{h_{j_2}}{h_{j_3}}\right)^t \not\equiv 1$$

for all $(s, t) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$. So applying Lemma 3.3 (see Remark 3.4) to the functions h_{j_1}/h_{j_3} and h_{j_2}/h_{j_3}, we conclude that for any $\epsilon > 0$,

$$\| \; N(r, h_{j_1}/h_{j_3} - 1, h_{j_2}/h_{j_3} - 1) \le \epsilon \max\left\{T\left(r, \frac{h_{j_1}}{h_{j_3}}\right), T\left(r, \frac{h_{j_2}}{h_{j_3}}\right)\right\}. \tag{3.6}$$

Note that the $\tilde{Q}_j$'s are all of degree d, so by Lemma 3.5, we see that for any $i, j \in \{1, \dots, q\}$,

$$T\left(r, \frac{h_i}{h_j}\right) \le T\left(r, \frac{\tilde{Q}_i(\tilde{f})}{\tilde{Q}_j(\tilde{f})}\right) + T\left(r, \frac{\tilde{Q}_j(\tilde{g})}{\tilde{Q}_i(\tilde{g})}\right) \le C_1(T_f(r) + T_g(r)) + C_2.$$

Combining this with inequality (3.6), we get that for arbitrary three distinct indices $j_1, j_2, j_3 \in J$, for any $\epsilon > 0$,

$$\| \; N(r, h_{j_1}/h_{j_3} - 1, h_{j_2}/h_{j_3} - 1) \le \epsilon(T_f(r) + T_g(r)). \tag{3.7}$$

Take $i \in I$. By $\#(J \setminus \{i\}) \ge 3$, we can choose three distinct $j_1, j_2, j_3 \in J \setminus \{i\}$. By conditions (a), (b) and (c), if $z \in f^{-1}(D_i)$, then z is not the zero of $\tilde{Q}_{j_k}(\tilde{f})$ and $\tilde{Q}_{j_k}(\tilde{g})$, $k = 1, 2, 3$, and $\tilde{f}(z) = c\tilde{g}(z)$ for some nonzero constant c. So for $k = 1, 2, 3$,

$$h_{j_k}(z) = \frac{\tilde{Q}_{j_k}(\tilde{f})(z)}{\tilde{Q}_{j_k}(\tilde{g})(z)} = \frac{\tilde{Q}_{j_k}(\tilde{f}(z))}{\tilde{Q}_{j_k}(\tilde{g}(z))} = c^d,$$

thus

$$\frac{h_{j_1}}{h_{j_3}}(z) = \frac{h_{j_2}}{h_{j_3}}(z) = \frac{c^d}{c^d} = 1;$$

namely, z is a common 1-point of h_{j_1}/h_{j_3} and h_{j_2}/h_{j_3}. So combining this with inequality (3.7), we have for any $\epsilon > 0$,

$$\| \; \overline{N}_f(r, D_i) \le N(r, \; h_{j_1}/h_{j_3} - 1, \; h_{j_2}/h_{j_3} - 1) \le \epsilon(T_f(r) + T_g(r)).$$

Summing up the above inequality over $i \in I$ and noting that $\overline{N}_f(r, D_i) = \overline{N}_g(r, D_i)$, we get that for any $\epsilon > 0$,

$$\Big\| \; \sum_{i \in I} (\overline{N}_f(r, D_i) + \overline{N}_g(r, D_i)) \le \epsilon(T_f(r) + T_g(r)). \tag{3.8}$$

On the other hand, by the second main theorem for holomorphic curves intersecting hypersurfaces (see Theorem 2.3) and the assumption $\#I \ge n + 2$, and noting that $N_f^{[M]}(r, D) \le M\overline{N}_f(r, D)$, we deduce that there is a positive constant κ such that

$$\Big\| \sum_{i\in I}(\overline{N}_f(r, D_i) + \overline{N}_g(r, D_i)) \ge \kappa(T_f(r) + T_g(r)).$$

This contradicts (3.8).

Therefore we have proved that there exist three distinct indices $i, j, k \in J$ such that

$$\left(\frac{\tilde{Q}_i(\tilde{f}) \cdot \tilde{Q}_k(\tilde{g})}{\tilde{Q}_i(\tilde{g}) \cdot \tilde{Q}_k(\tilde{f})}\right)^s \cdot \left(\frac{\tilde{Q}_j(\tilde{f}) \cdot \tilde{Q}_k(\tilde{g})}{\tilde{Q}_j(\tilde{g}) \cdot \tilde{Q}_k(\tilde{f})}\right)^t \equiv 1$$

for some $(s, t) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$. Now since all $\tilde{Q}_t$'s are of the same degree d, it is easy to see that the $(n+1)^2$ functions $\{f_u g_v\}_{0\le u,v\le n}$ satisfy a nontrivial homogeneous polynomial equation. This shows that the image of $f \times g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ is contained in a proper algebraic subset of $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$; in other words, $f \times g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ is algebraically degenerate.

Furthermore from the above proof, we easily see that if all D_j's are hyperplanes, then the proof still works if f and g are only assumed to be linearly nondegenerate. This completes the proof. □

4. Proof of Theorem 1.2

We prove the following theorem which implies Theorem 1.2.

Theorem 4.1. *Let $f, g, \tilde{f}, \tilde{g}, d, D_j, Q_j, \tilde{Q}_j$, $1 \le j \le q$, be given as in Section 1. Let $q = n+3$ and set $h_j = \tilde{Q}_j(\tilde{f})/\tilde{Q}_j(\tilde{g})$ for $j = 1, \dots, q$. Assume that*

(a) $\nu_{Q_j(\tilde{f})} = \nu_{Q_j(\tilde{g})}$ *for $j = 1, \dots, q$, and*

(b) *for every $i \in \{1, \dots, n+2\}$, the set*

$$A_i := \left\{ \frac{h_j}{h_k}(z) \;\Big|\; z \in f^{-1}(D_i),\ 1 \le j, k \le q \text{ with } z \notin f^{-1}(D_j \cup D_k) \cup g^{-1}(D_j \cup D_k) \right\}$$

is of finite cardinality.

Then there exist distinct indices $i, j, k \in \{1, \dots, q\}$ and constants

$$C_1, C_2 \in A := \{1\} \cup \bigcup_{i=1}^{n+2} A_i$$

such that

$$\left(\frac{\tilde{Q}_i(\tilde{f}) \cdot \tilde{Q}_k(\tilde{g})}{C_1\tilde{Q}_i(\tilde{g}) \cdot \tilde{Q}_k(\tilde{f})}\right)^s \cdot \left(\frac{\tilde{Q}_j(\tilde{f}) \cdot \tilde{Q}_k(\tilde{g})}{C_2\tilde{Q}_j(\tilde{g}) \cdot \tilde{Q}_k(\tilde{f})}\right)^t \equiv 1$$

for some $(s, t) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$. In particular $f \times g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ is algebraically degenerate.

Remark 4.2. By the condition "$f = g$ on $\bigcup_{j=1}^{n+2} f^{-1}(D_j)$" of Theorem 1.2, one deduces as in the proof of Theorem 3.1 that for $i \in \{1, \dots, n+2\}$ and $j, k \in \{1, \dots, q\}$, $(h_j/h_k)(z) = 1$ for every point

$$z \in f^{-1}(D_i) \setminus (f^{-1}(D_j \cup D_k) \cup g^{-1}(D_j \cup D_k)).$$

So $A = \{1\}$. Thus the conclusion of Theorem 1.2 follows from Theorem 4.1.

Proof. By assumption, h_j is a nowhere zero holomorphic function on $\mathbb{C}$ for every $1 \le j \le q$ and A is a nonempty set consisting of finitely many nonzero complex numbers. So we may set $A = \{c_1, \dots, c_p\}$.

Assume that the conclusion doesn't hold, then for any distinct indices $i, j, k \in \{1, \dots, q\}$ and constants $c_u, c_v \in A$,

$$\left(\frac{h_i}{c_u h_k}\right)^s \cdot \left(\frac{h_j}{c_v h_k}\right)^t \not\equiv 1$$

for all $(s, t) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$. Much as in the proof of Theorem 3.1, by making use of Lemmas 3.3 and 3.5, we conclude that for any $\epsilon > 0$,

$$\| \; N\left(r, \frac{h_i}{c_u h_k} - 1, \frac{h_j}{c_v h_k} - 1\right) \le \epsilon(T_f(r) + T_g(r)). \tag{4.3}$$

Let $\nu = \sum_{1 \le i < j < k \le q} \sum_{c_u, c_v \in A} \min\{\nu^0_{h_i/(c_u h_k)-1}, \nu^0_{h_j/(c_v h_k)-1}\}$, then

$$N(r, \nu) = \sum_{1 \le i < j < k \le q} \sum_{c_u, c_v \in A} N\left(r, \frac{h_i}{c_u h_k} - 1, \frac{h_j}{c_v h_k} - 1\right).$$

So (4.3) gives that for any $\epsilon > 0$,

$$\| \; N(r, \nu) \le \epsilon(T_f(r) + T_g(r)). \tag{4.4}$$

Now take $l \in \{1, \dots, n+2\}$. For a point $z \in f^{-1}(D_l)$, by the "in general position" assumption, we know that there are at most $n-1$ distinct $k \in \{1, \dots, q\} \setminus \{l\}$ such that $z \in f^{-1}(D_k)$. Since $q = n+3$, there are three distinct $i, j, k \in \{1, \dots, q\} \setminus \{l\}$ with $i < j < k$ such that $z \notin f^{-1}(D_i \cup D_j \cup D_k) \cup g^{-1}(D_i \cup D_j \cup D_k)$. Then

$$\frac{h_i}{h_k}(z), \frac{h_j}{h_k}(z) \in A_l \subseteq A.$$

Thus there are $c_u, c_v \in A$ such that z is a common 1-point of $h_i/(c_u h_k)$ and $h_j/(c_v h_k)$, so the point z is counted in $N(r, \nu)$. Consequently, for any $\epsilon > 0$,

$$\| \; \overline{N}_f(r, D_l) \le N(r, \nu) \le \epsilon(T_f(r) + T_g(r)).$$

From this we see that for any $\epsilon > 0$,

$$\Big\| \sum_{l=1}^{n+2}(\overline{N}_f(r, D_l) + \overline{N}_g(r, D_l)) \le \epsilon(T_f(r) + T_g(r)).$$

On the other hand, using the second main theorem (see Theorem 2.3), as in the proof of Theorem 3.1, we deduce that there exists a constant $\kappa > 0$ such that

$$\Big\| \kappa(T_f(r) + T_g(r)) \le \sum_{l=1}^{n+2}(\overline{N}_f(r, D_l) + \overline{N}_g(r, D_l)),$$

which contradicts the above inequality. This proves Theorem 4.1. □

Combining the proof of Theorem 4.1 with that of Theorem 3.1, one concludes easily the following theorem which is an improvement of Theorem 3.1.

Theorem 4.5. *Let $f, g, \tilde{f}, \tilde{g}, d, D_j, Q_j, \tilde{Q}_j$, $1 \le j \le q$, be given as in Section 1. Assume that there exist $I, J \subseteq \{1, \dots, q\}$ with $\#I \ge n+2$ and for any $i \in I$, $\#(J \setminus \{i\}) \ge 3$, such that the following conditions are satisfied*:

(a) $\nu_{Q_j(\tilde{f})} = \nu_{Q_j(\tilde{g})}$ *for* $j \in J$ *and* $\min\{\nu_{Q_i(\tilde{f})}, 1\} = \min\{\nu_{Q_i(\tilde{g})}, 1\}$ *for* $i \in I$;

(b) *for every* $i \in I$, *the set*

$$\left\{ \frac{h_j}{h_k}(z) \,\Big|\, z \in f^{-1}(D_i),\ j, k \in J \setminus \{i\} \right\} =: A_i$$

is of finite cardinality.

Then there exist three distinct indices $i, j, k \in J$ and constants

$$C_1, C_2 \in A := \{1\} \cup \bigcup_{u \in I} A_u$$

such that

$$\left(\frac{\tilde{Q}_i(\tilde{f}) \cdot \tilde{Q}_k(\tilde{g})}{C_1 \tilde{Q}_i(\tilde{g}) \cdot \tilde{Q}_k(\tilde{f})} \right)^s \cdot \left(\frac{\tilde{Q}_j(\tilde{f}) \cdot \tilde{Q}_k(\tilde{g})}{C_2 \tilde{Q}_j(\tilde{g}) \cdot \tilde{Q}_k(\tilde{f})} \right)^t \equiv 1$$

for some $(s, t) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$.

5. Proof of Theorem 1.3

We prove Theorem 1.3 and then as a consequence we give a uniqueness theorem.

Proof of Theorem 1.3. Let $f, g, \tilde{f}, \tilde{g}, d, D_j, Q_j, \tilde{Q}_j$, $(1 \le j \le q)$ be given as in Section 1. We set $h_j = \tilde{Q}_j(\tilde{f})/\tilde{Q}_j(\tilde{g})$ for $j = 1, \dots, q$. Then the assumption shows that h_1 and h_2 are nowhere zero holomorphic functions on $\mathbb{C}$. We need to show that h_1/h_2 is constant.

Since $q = n+2$ and $\overline{N}_f(r, D_j) = \overline{N}_g(r, D_j)$ for $j = 1, \dots, q$, it follows from the first and the second main theorem that there are constants $C_1, C_2 > 0$ such that

$$\Big\| \; C_1 T_g(r) \le \sum_{j=1}^{q} \overline{N}_g(r, D_j) = \sum_{j=1}^{q} \overline{N}_f(r, D_j) \le q d T_f(r) + C_2;$$

therefore there is a constant $C > 0$ such that

$$\| \; T_g(r) \le C T_f(r). \tag{5.1}$$

By Lemma 3.5, there is a constant $C_3 > 0$ such that for all large r,

$$T(r, h_1/h_2) \le C_3(T_f(r) + T_g(r)).$$

Combining this with (5.1), we have

$$\| \; T(r, h_1/h_2) \le C_4 T_f(r)$$

for some constant $C_4 > 0$. From this and the assumption that f is of order < 1, it follows that

$$\underline{\lim}_{r\to+\infty} \frac{\log^+ T(r, h_1/h_2)}{\log r} \le \overline{\lim}_{r\to+\infty} \frac{\log^+ T_f(r)}{\log r} < 1. \tag{5.2}$$

Since h_1/h_2 is nowhere zero holomorphic on $\mathbb{C}$, we may write $h_1/h_2 = e^H$ for some entire function H. If h_1/h_2 is nonconstant, then either H is a polynomial of degree ≥ 1 or H is a transcendental entire function; thus by [Yang and Yi 2003, Theorem 1.44] we have

$$\underline{\lim}_{r\to+\infty} \frac{\log^+ T(r, h_1/h_2)}{\log r} \ge 1.$$

This contradicts (5.2). Thus h_1/h_2 is constant, which completes the proof. □

Remark 5.3. From the above proof, we easily see that if all D_j's are hyperplanes, then the conclusion still holds when f and g are only assumed to be linearly nondegenerate. So we have the following uniqueness theorem:

Corollary 5.4. *Let $f, g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be two linearly nondegenerate holomorphic maps. Let $H_1, \dots, H_{n+2}$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ located in general position. Suppose that f is of order < 1. If $\nu_{(f,H_j)} = \nu_{(g,H_j)}$ for every $j = 1, \dots, n+2$, then $f = g$.*

Proof. Take reduced representations $\tilde{f}, \tilde{g}$ for f and g respectively and let L_j, $1 \le j \le n+2$, be the linear forms that define H_j. Set $h_j = L_j(\tilde{f})/L_j(\tilde{g})$, $1 \le j \le n+2$. Then Theorem 1.3 shows that $h_i/h_1 = c_i$ is a constant for any $i \ge 2$, and $L_i(\tilde{f}) = h_1 c_i L_i(\tilde{g})$. Since the H_j's are in general position, we can write $L_1 = \sum_{i=2}^{n+2} b_i L_i$ for some nonzero constants b_i. Thus

$$h_1 \sum_{i=2}^{n+2} b_i L_i(\tilde{g}) = h_1 L_1(\tilde{g}) = L_1(\tilde{f}) = \sum_{i=2}^{n+2} b_i L_i(\tilde{f}) = h_1 \sum_{i=2}^{n+2} b_i c_i L_i(\tilde{g}),$$

which implies that

$$\left(\sum_{i=2}^{n+2} b_i(1-c_i)L_i\right)(\tilde{g}) = 0.$$

Now by the linearly nondegeneracy of g and the fact that $L_2, \ldots, L_{n+2}$ are linearly independent, we conclude that

$$c_{n+2} = c_{n+1} = \cdots = c_2 = 1;$$

namely, $h_1 = h_2 = \cdots = h_{n+2}$. This implies that $f = g$. □

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