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Elena K. Brusyanskaya, Anton A. Klyachko and Andrey V. Vasil'ev

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WHAT DO FROBENIUS'S, SOLOMON'S, AND IWASAKI'S THEOREMS ON DIVISIBILITY IN GROUPS HAVE IN COMMON?

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Our result contains as special cases the Frobenius theorem (1895) on the number of solutions to the equation $x^n = 1$ in a group, the Solomon theorem (1969) on the number of solutions in a group to a system of equations having fewer equations than unknowns, and the Iwasaki theorem (1985) on roots of subgroups. There are other curious corollaries on groups and rings.

0. Introduction

The following result was proved in the nineteenth century.

Frobenius theorem [1895]. *The number of solutions to the equation* $x^n = 1$ *in a finite group is divisible by* GCD(|G|, n) *for any integer n.*

This theorem was generalized in different directions; see, e.g., [Hall 1936; Kulakoff 1938; Sehgal 1962; Brown and Thévenaz 1988; Yoshida 1993; Asai and Takegahara 2001; Asai et al. 2013], and references therein. For example, Frobenius [1903] himself obtained the following generalization:

for any positive integer n and any element g of a finite group G, the number of solutions to the equation $x^n = g$ in G is divisible by the greatest common divisor of n and the order of the centralizer of g.

P. Hall [1936, Theorem II] showed that

in any finite group, the number of solutions to a system of equations in one unknown is divisible by $GCD(|C|, n_1, n_2, ...)$, where C is the centralizer of the set of all coefficients and n_i are exponent sums of the unknown in the *i*-th equation.

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Here as usual, an equation over a group *G* is an expression of form $v(x_1, \ldots, x_m) = 1$, where *v* is a word whose letters are unknowns, their inverses, and elements of *G* (called *coefficients*). In other terms, the left-hand side of an equation is an element of the free product $G * F(x_1, \ldots, x_m)$ of *G* and the free group $F(x_1, \ldots, x_m)$ of rank *m* (where *m* is the number of unknowns).

The following theorem is also about equations in groups and divisibility, but on first view, it is not similar to the Frobenius theorem and its generalizations.

Solomon theorem [1969]. *In any group, the number of solutions to a system of coefficient-free equations is divisible by the order of the group provided the number of equations is less than the number of unknowns.*

This theorem was also generalized in different directions; see [Isaacs 1970; Strunkov 1995; Amit and Vishne 2011; Gordon and Rodriguez-Villegas 2012; Klyachko and Mkrtchyan 2014; 2017], and references therein. For instance, in [Klyachko and Mkrtchyan 2014], it was shown that

in any group, the number of solutions to a system of equations (with coefficients from this group) is divisible by the order of the intersection of centralizers of all coefficients provided the rank of the matrix composed of the exponent sums of the *j*-th unknown in the *i*-th equation is less than the number of unknowns.

Solomon [1969] himself wrote:

"There seems to be no connection between this theorem and the Frobenius theorem on solutions of $x^k = 1$."

Nevertheless, a connection between the Frobenius and Solomon theorems exists.

Theorem 0. In any (not necessarily finite) group, the number of solutions to a (not necessarily finite) system of equations in m unknowns is a multiple of the greatest common divisor of the centralizer of the set of coefficients and the number $\frac{\Delta_m}{\Delta_{m-1}}$, where Δ_i is the greatest common divisor of all minors of order i of the matrix of the system, and the following conventions are assumed: $\Delta_i = 0$ if i is larger than the number of equations, $\Delta_0 = 1$, and $\frac{0}{0} = 0$.

We define the *greatest common divisor* GCD(G, n) of a group G and an integer n as the least common multiple of orders of subgroups of G dividing n. The divisibility is always understood in the sense of cardinal arithmetic: each infinite cardinal is divisible by all smaller nonzero cardinals (and surely zero is divisible by all cardinals and divides only zero). This means that GCD(G, 0) = |G| for any group G and, e.g., $GCD(SL_2(\mathbb{Z}), 2018) = 2$. Although, the reader will not lose much by assuming all group to be finite; in this case, GCD(G, n) = GCD(|G|, n) by the Sylow theorem (and because a finite p-group contains subgroups of all possible orders).

The matrix of a system of equations over a group is the integer matrix $A = (a_{ij})$, where a_{ij} is the exponent sum of the *j*-th unknown in the *i*-th equation. For example, the matrix of the system

$$\begin{cases} xay^{2}[x, y]^{2019}(xby)^{3} = 1, \\ bx^{3}y[x, y]^{100}(xby)^{4} = 1, \\ [x, y^{5}]x^{-2} = 1 \end{cases}$$

(where x and y are unknowns, and a and b are coefficients, i.e., some fixed group elements) has the form

$$\begin{pmatrix} 4 & 5 \\ 7 & 5 \\ -2 & 0 \end{pmatrix}.$$

As usual, the *minors of order i* are determinants of submatrices composed of entries at the intersections of some i rows and i columns. In the example above, there are three minors of order m (up to signs),

$$\det\begin{pmatrix} 4 & 5\\ 7 & 5 \end{pmatrix} = -15, \qquad \det\begin{pmatrix} 4 & 5\\ -2 & 0 \end{pmatrix} = 10, \qquad \det\begin{pmatrix} 7 & 5\\ -2 & 0 \end{pmatrix} = 10,$$

and six minors of order m - 1: 4, 5, 7, 5, -2, 0. Thus, the theorem asserts that (in this example) the number of solutions is divisible by

$$\operatorname{GCD}\left(\frac{\operatorname{GCD}(-15, 10, 10)}{\operatorname{GCD}(4, 5, 7, 5, -2, 0)}, |C(a) \cap C(b)|\right) = \operatorname{GCD}(5, |C(a) \cap C(b)|).$$

Note that the agreements about boundary cases in Theorem 0 are natural. Indeed, we always can add a fictitious equation 1 = 1 to make the number of equations larger than m. We can also add a new variable z and the equation z = 1 (this does not affect the number of solutions and makes m > 1). As for the philosophical question on the interpretation of the fraction $\frac{0}{0}$, it can be understood arbitrarily, e.g., the reader may assume that $\frac{0}{0} = 2019$; in any case, Theorem 0 remains valid (but weaker than under the suggested interpretation).

The meaning of the value $\frac{\Delta_m}{\Delta_{m-1}}$ is as follows. It is well known (see, e.g., [Vinberg 2003]) that invertible integer elementary transformations of rows and columns can transform any integer matrix *A* into a diagonal matrix, where the diagonal entries divide each other (each diagonal entry divides the next one). This diagonal matrix is uniquely determined up to the signs of diagonal elements (and is sometimes called the *Smith form* of *A*); the diagonal elements of the Smith form (sometimes called the *invariant factors* of *A*) equal the ratios $\frac{\Delta_i}{\Delta_{i-1}}$. Thus, in these terms, $\frac{\Delta_m}{\Delta_{m-1}}$ is the *m*-th invariant factor of the matrix of the system of equations. One can also say that

the absolute value of $\frac{\Delta_m}{\Delta_{m-1}}$ is the period (exponent) of the quotient of the free abelian group \mathbb{Z}^m by the subgroup generated by the rows of the matrix of the system of equations

(with the stipulation that this ratio vanishes if and only if the period is infinite).

The Frobenius and Solomon theorems as well as their generalizations stated above are special cases of Theorem 0.

The following theorem is on first view similar to neither the Frobenius theorem nor the Solomon theorem.

Iwasaki theorem [1982]. For any integer *n*, the number of elements of a finite group *G* whose *n*-th powers lie in a subgroup $H \subseteq G$ is divisible by |H|.

This beautiful theorem remains (for some reason) not widely known. In [Sato and Sakurai 2007], it was noticed that the divisibility by |H| still holds for the number of solutions to the "equation" $x^n \in HgH$, where HgH is any double coset of a subgroup H. Clearly, the Iwasaki theorem and its generalizations deal with predicates that are not equations in the usual sense. Let us say that a *generalized equation* over a group G is an expression of the form $w(x_1, \ldots, x_n) \in HgH$, where H is a subgroup of $G \ni g$, and $w(x_1, \ldots, x_m)$ is an element of the free product $G * F(x_1, \ldots, x_m)$ of G and a free group; in other terms, w is a word in the alphabet $G \sqcup \{x_1^{\pm 1}, \ldots, x_m^{\pm 1}\}$. The elements of G occurring in this word are called the *coefficients* of the generalized equation. A system of generalized equations, a solution to this system, and a matrix of this system are defined in a natural way.

In [Klyachko and Mkrtchyan 2017], the following generalization of the Iwasaki theorem was obtained:

the number of solutions to a system of generalized coefficient-free equations whose right-hand sides are double cosets of the same subgroup H(e.g., $\{x^{100}y^{2019}[x, y]^4 \in Hg_1H, [x^5, y^6]^7(xy)^8 \in Hg_2H, ...\}$) is divisible by |H|.

The following theorem includes all results stated above.

Theorem 1. Let S be a (not necessarily finite) system of generalized equations in finitely many unknowns x_1, \ldots, x_m over a group G and let P be its subsystem:

$$S = \{u_i(x_1, \dots, x_m) \in H_i g_i H_i \mid i \in I\} \supseteq P = \{u_i(x_1, \dots, x_m) \in H_j g_j H_j \mid j \in J\},\$$

(where $J \subseteq I$, $u_i \in G * F(x_1, ..., x_m)$, $g_i \in G$, and H_i are subgroups of G). Then the number of solutions to S in G is divisible by the greatest common divisor of the subgroup

$$\widetilde{H} = \left(\bigcap_{j \in J} N(H_j g_j H_j)\right) \cap \left(\bigcap_{i \in I \setminus J} H_i\right) \cap (\text{the centralizer of the set of coefficients of } S)$$

and the number $\frac{\Delta_m}{\Delta_{m-1}}$, where Δ_k is the greatest common divisor of all minors of order k of the matrix of the subsystem P. Henceforth, $N(A) := \{g \in G \mid g^{-1}Ag = A\}$ is the normalizer of a subset A in a group G.

To deduce Theorem 0 from Theorem 1, we rewrite the system of equations in the "generalized" form, i.e., we put $S = P = \{u_1(x_1, \ldots, x_m) \in \{1\}1\{1\}$ and $u_2(x_1, \ldots, x_m) \in \{1\}1\{1\}, \ldots\}$ and note that the normalizer of the trivial subgroup is the whole group.

On the other hand, setting

$$S = \{u_1(x_1, \dots, x_m) \in Hg_1H, u_2(x_1, \dots, x_m) \in Hg_2H, \dots\}$$

(where $u_i \in F(x_1, \dots, x_m)$),

 $P = \emptyset$,

we obtain the above-mentioned generalization (from [Klyachko and Mkrtchyan 2017]) of the Iwasaki theorem.

As a matter of fact, a relation between Solomon's and Iwasaki's theorems was established in [Klyachko and Mkrtchyan 2014; 2017]; our achievement consists only of adding "Frobeniusness". The main theorem of [Klyachko and Mkrtchyan 2017] says that, if we have a group F with a fixed epimorphism onto \mathbb{Z} and some set of homomorphisms from F into another group G, and this set is invariant with respect to some natural transformations (depending on the epimorphisms $F \to \mathbb{Z}$ and a subgroup H of G), then the number of these homomorphisms $F \to G$ is divisible by |H|. Choosing suitable sets of homomorphisms, the Klyachko and Mkrtchyan [2017] obtained Solomon's and Iwasaki's theorems as special cases of their main theorem.

Our main theorem (see Section 1) is a modular analogue of the main theorem of [Klyachko and Mkrtchyan 2017]: we take an epimorphism $F \rightarrow \mathbb{Z}/n\mathbb{Z}$ instead of $F \rightarrow \mathbb{Z}$. One can say that the main theorem of this paper is related to the main theorem of [Klyachko and Mkrtchyan 2017] in the same way as Theorem 0 to the generalization (from [Klyachko and Mkrtchyan 2014]) of the Solomon theorem mentioned in the beginning of this paper. An important role in our argument is played by an elementary (but nontrivial) lemma due to Brauer [1969]. Actually, we need this lemma not to prove the main theorem but rather to explain that its statement per se makes some sense. For readers' convenience, we give a proof of the Brauer lemma. Section 5 contains the proof of the main theorem.

In Section 2, we deduce Theorem 1 from the main theorem. As another corollary, we obtain a theorem on equations in rings (Theorem 4 in Section 3) that implies, e.g., the following fact, which can be considered as a generalization of the Frobenius theorem in another direction:

for any representation $\rho : G \to GL(V)$ of a group G and any words $u_i(x_1, \ldots, x_m) \in F(x_1, \ldots, x_m)$, the number of solutions to the equation

$$\sum_{i=1}^{k} (\rho(u_i(x_1,\ldots,x_m)))^{l_i} = \mathrm{id}$$

is divisible by

$$\begin{cases} \operatorname{GCD}(G, \operatorname{GCD}\{l_i\}) & always, \\ \operatorname{GCD}(G, \operatorname{LCM}\{l_i\}) & if \ k \le m, \\ |G| & if \ k < m. \end{cases}$$

In Section 4, we show that the main theorem implies some fact about the number of crossed homomorphisms, generalizing earlier known results. In the last section, we discuss open questions.

Notations and conventions. We use mainly standard notations and conventions. Note only that, if $k \in \mathbb{Z}$ and x and y are elements of a group, then x^y , x^{ky} , and x^{-y} denote $y^{-1}xy$, $y^{-1}x^ky$, and $y^{-1}x^{-1}y$, respectively. The commutator subgroup of a group G is denoted by G' or [G, G]. If X is a subset of a group, then |X|, $\langle X \rangle$, $\langle \langle X \rangle \rangle$, C(X), and N(X) are the cardinality of X, subgroup generated by X, normal closure of X, centralizer of X, and normalizer of X. The index of a subgroup H of a group G is denoted by |G : H|. The letter \mathbb{Z} denotes the set of integers. If R is an associative ring with unity, then R^* denotes the group of units of this ring. GCD and LCM are the greatest common divisor and least common multiple. The symbol exp G denotes the period (exponent) of a group G if this period is finite; we assume exp G = 0 if the period is infinite. The symbol $\langle g \rangle_n$ denotes the cyclic group of order n generated by an element g. The free group of rank n is denoted by $F(x_1, \ldots, x_n)$ or F_n . The symbol A * B denotes the free product of groups A and B.

Let us recall once again that the finiteness of groups is not assumed by default; the divisibility is always understood in the sense of cardinal arithmetic (an infinite cardinal is divisible by all nonzero cardinals not exceeding it), and GCD(G, n) := $LCM\{|H| \mid H \text{ is a subgroup of } G, \text{ and } |H| \text{ divides } n\}.$

1. Main theorem

A group *F* equipped with an epimorphism $F \to \mathbb{Z}/n\mathbb{Z}$ (where $n \in \mathbb{Z}$) is called an *n*-indexed group. This epimorphism $F \to \mathbb{Z}/n\mathbb{Z}$ is called *degree* and denoted deg. Thus, to any element *f* of an indexed group *F*, an element deg $f \in \mathbb{Z}/n\mathbb{Z}$ is assigned; the group *F* contains elements of all degrees and deg(fg) = deg f + deg *g* for any *f*, $g \in F$. Suppose that $\phi : F \to G$ is a homomorphism from an *n*-indexed group *F* to a group *G* and *H* is a subgroup of *G*. The subgroup

$$H_{\phi} = \bigcap_{f \in F} H^{\phi(f)} \cap C(\phi(\ker \deg))$$

is called the ϕ -core of H [Klyachko and Mkrtchyan 2017]. In other words, the ϕ -core H_{ϕ} of H consists of elements h such that $h^{\phi(f)} \in H$ for all f, and $h^{\phi(f)} = h$ if deg f = 0.

Main theorem. Suppose that an integer n is a multiple of the order of a subgroup H of group G and a set Φ of homomorphisms from an n-indexed group F to G satisfies the following conditions.

(I) Φ is invariant with respect to conjugation by elements of H:

if $h \in H$ and $\phi \in \Phi$, then the homomorphism $\psi : f \mapsto \phi(f)^h$ lies in Φ .

(II) For any $\phi \in \Phi$ and any element h of the ϕ -core H_{ϕ} of H, the homomorphism ψ defined by

$$\psi(f) = \begin{cases} \phi(f) & \text{for all elements } f \in F \text{ of degree zero,} \\ \phi(f)h & \text{for some element } f \in F \text{ of degree one} \\ & (and, hence, for all degree-one elements) \end{cases}$$

belongs to Φ too.

Then $|\Phi|$ *is divisible by* |H|*.*

Note that the mapping ψ from condition (I) is a homomorphism for any $h \in G$, and the formula for ψ from condition (II) defines a homomorphism for any $h \in H_{\phi}$ (as explained below). Thus, conditions (I) and (II) only require these homomorphisms to belong to Φ .

Lemma 2. Suppose that $\phi : F \to G$ is a homomorphism from an n-indexed group F to a group G, $f_1 \in F$ is an element of degree one, and $g \in G$. Then the homomorphism $\psi : F \to G$ such that $\psi(f) = \phi(f)$ for all $f \in F$ of degree zero and $\psi(f_1) = \phi(f_1)g$ exists if and only if $g \in C(\phi(\ker \deg))$ and $(\phi(f_1)g)^n = (\phi(f_1))^n$.

Proof. The group F can be presented in the form

$$F \simeq (F_0 * \langle x \rangle_\infty) / \langle\!\langle \{u^x u^{-f_1} \mid u \in F_0\} \cup \{x^n f_1^{-n}\} \rangle\!\rangle, \quad \text{where } F_0 = \ker \deg.$$

Therefore, the mapping $\psi : F_0 \cup \{x\} \to G$ can be extended to a homomorphism if and only if its restriction to F_0 is a homomorphism and the relations $u^x = u^{f_1}$ (for $u \in F_0$) and $x^n = f_1^n$ are mapped to true equalities in *G*:

(*)
$$\psi(u)^{\psi(x)} = \psi(u^{f_1})$$
 and $\psi(x)^n = \psi(f_1^n)$.

If the restrictions of ψ and ϕ to F_0 coincide and $\psi(x) = \phi(f_1)g$, then the first equality of (*) says that *g* commutes with $\phi(u)$ (for all $u \in F_0$), while the second equality of (*) takes the form $(\phi(f_1)g)^n = (\phi(f_1))^n$.

Recall also the following beautiful (but not widely known) fact.

Brauer lemma [1969]. If U is a finite normal subgroup of a group V, then, for all $v \in V$ and $u \in U$, the elements $v^{|U|}$ and $(vu)^{|U|}$ are conjugate by an element of U.

Proof. The group \mathbb{Z} acts by permutations on the subgroup U:

$$a \circ i = v^{-i} a(vu)^i$$
 (where $i \in \mathbb{Z}$ and $a \in U$).

Let *m* be the minimum length of an orbit. In other words, *m* is the minimum length of a cycle in the decomposition of the permutation $a \mapsto v^{-1}avu$ (of *U*) into the product of independent cycles. The set $X = \{a \in U \mid a \circ m = a\}$ is the union of all orbits of length *m*; therefore, |X| is divisible by *m*. On the other hand, (by definition of the action) $X = \{a \in U \mid v^{-m}a(vu)^m = a\} = \{a \in U \mid a^{-1}v^m a = (vu)^m\}$ and, hence, |X| is the order of the centralizer of v^m in *U* (because, in any group, a nonempty set of the form $\{x \mid x^{-1}yx = z\}$ is a coset of the centralizer of *y*). Thus, |X| divides |U| and, therefore, *m* divides |U| and $a \circ |U| = a$ (if *a* lies in an orbit of length *m*).

These two lemmata imply immediately that the mapping ψ from condition (II) is a homomorphism for any $h \in H_{\phi}$ because $(\phi(f)h)^n = (\phi(f))^n$ by the Brauer lemma applied to $U = H_{\phi} \subset V = H_{\phi} \cdot \langle \phi(f_1) \rangle \ni \phi(f_1) = v$. Indeed, we obtain the equality $(\phi(f_1)h)^{|H_{\phi}|} = (\phi(f_1))^{|H_{\phi}|u}$ for some $u \in H_{\phi}$ and, hence, $(\phi(f_1)h)^n =$ $(\phi(f_1))^{nu} = (\phi(f_1^n))^u$ (because $|H_{\phi}|$ divides n). It remains to note that $u \in H_{\phi}$ commutes with $\phi(f_1^n)$ because deg $f_1^n = n = 0 \in \mathbb{Z}/n\mathbb{Z}$. Thus, we obtain the equality $(\phi(f_1)h)^n = (\phi(f_1))^n$. It remains to refer to Lemma 2.

In the case n = 0 the main theorem was proved by Klyachko and Mkrtchyan [2017]. So, our theorem is a "modular analogue" of their main result. On the other hand, our main theorem is deduced (in Section 5) from this special case n = 0.

Lemma 3. In condition (II) of the main theorem, $\psi(f) \in \phi(f)H_{\phi}$ for all $f \in F$.

Proof. Indeed, if deg f = d, then $f = f_1^d f_0$, where f_1 is the (fixed) element of degree one (from condition (II)) and f_0 is an element of degree zero. Then

$$\psi(f) = \psi(f_1)^d \psi(f_0) = (\phi(f_1)h)^d \phi(f_0) \stackrel{(E)}{=} \phi(f_1)^d \phi(f_0)h' = \phi(f_1^d f_0)h' = \phi(f)h',$$

where the equality (*E*) is valid for some $h' \in H_{\phi}$ because $h \in H_{\phi}$ and $\phi(F)$ normalizes H_{ϕ} .

2. Proof of Theorem 1

Let $L \subseteq G$ be the subgroup generated by all coefficients of the system *S*. Take as *H* any subgroup of the group \widetilde{H} whose order divides $n := \frac{\Delta_m}{\Delta_{m-1}}$, and put

$$F = L * F(x_1, \dots, x_m),$$

$$\Phi = \{\phi : F \to G \mid \phi(f) = f \text{ for } f \in L \text{ and } \phi(u_i) \in H_i g_i H_i \text{ for } i \in I\}.$$

As the indexing deg : $F \to \mathbb{Z}/n\mathbb{Z}$, take an epimorphism whose kernel contains *L* and all u_j , where $j \in J$. Such an epimorphism exists because *n* is the period of the finitely generated abelian group $F/([F, F] \cdot L \cdot \langle \{u_j \mid j \in J\} \rangle)$.

Let us verify that the conditions of the main theorem hold. Condition (I) holds obviously for all $h \in H$ (and even for all $h \in \widetilde{H}$) because (by definition) \widetilde{H} centralizes *L* and normalizes double cosets $H_i g_i H_i$.

Condition (II) holds also for all $h \in H_{\phi}$ because

- on L, the homomorphism ψ coincides with ϕ as L consists of zero-degree elements,
- $\psi(u_j) = \phi(u_j)$ for $j \in J$ because again deg $u_j = 0$, and
- for $i \in I \setminus J$, we have $\psi(u_i) \in \phi(u_i)H_{\phi} \subseteq \phi(u_i)H_i$ (where the inclusion \in follows from Lemma 3).

Thus, the main theorem implies that $|\Phi|$ is divisible by the order of any subgroup $H \subseteq \widetilde{H}$ whose order divides *n*, i.e., $|\Phi|$ is divisible by $\text{GCD}(\widetilde{H}, n)$. It remains to note that $|\Phi|$ is the number of solutions to *S*.

3. Rings and representations

A generalized homogeneous modulo n equation with a set of unknowns X over an associative unital ring R is a finite expression of the form

$$\sum_{i} \prod_{j} c_{ij} x_{ij}^{k_{ij}} = 0,$$

where *coefficients* $c_{ij} \in R$, *unknowns* $x_{ij} \in X$, and *exponents* $k_{ij} \in \mathbb{Z}$,

such that, for some mapping deg : $X \to \mathbb{Z}/n\mathbb{Z}$, the value $\sum_j k_{ij} \deg x_{ij}$ (called the *degree of the equation*) does not depend on *i* (i.e., the "polynomial" in the left-hand side of the equation is homogeneous with respect to some assigning of degrees to variables), and $\langle \{\deg x \mid x \in X\} \rangle = \mathbb{Z}/n\mathbb{Z}$. (This means that the free group F(X) is *n*-indexed with respect to the map deg.)

A system of equations is called generalized homogeneous modulo *n* if all equations of this system are generalized homogeneous modulo *n* (of possibly different degrees) with respect to the same function deg : $X \to \mathbb{Z}/n\mathbb{Z}$.

As we explain below, the set $M = \{n \in \mathbb{Z} \mid a \text{ given system is generalized homogeneous modulo } n\}$ consists of all divisors of a number n_0 , called the *homogeneity modulus* of the system. In other words, the homogeneity modulus is the maximal number from M or zero if M is infinite.

To find the homogeneity modulus, consider a homogenous system of linear equations, where unknowns are degrees of variables and also (the negations of) degrees of equations; these linear equations say that the degree of each monomial equals the degree of the corresponding equation. The matrix of this system (called the *homogeneity matrix* of the initial system of equations) has the following form. Suppose that $X = \{x_1, \ldots, x_m\}$. The *homogeneity matrix of p-th equation* is the integer matrix $A_p = (a_{kl})$ of size

(the total number of monomials in the system) \times (*m*+(the number of equations)),

where, for $l \le m$, the (k, l)-th entry is the exponent sum of the *l*-th unknown in the *k*-th monomial, the (m + p)-th column consists of ones, and the remaining columns are zero for l > m. The homogeneity matrix of the system of equations is composed from the matrices A_p written one under another: $A = (A_1 \ A_2 \ \cdots)^T$. For example, the system of equations $\{ax^3y^2 + y^7bx - 1 = 0, xy^2x + y^7x^5 = 0\}$ (where *x* and *y* are unknowns and $a, b \in R$ are coefficients) has the homogeneity matrix

$$A = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 7 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 1 \\ 5 & 7 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \text{ where } A_1 = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 7 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 2 & 2 & 0 & 1 \\ 5 & 7 & 0 & 1 \end{pmatrix}.$$

Homogeneity-modulus lemma. The homogeneity modulus of a system of s equations in m unknowns over an associative ring with unity is $\frac{\Delta_{m+s}}{\Delta_{m+s-1}}$, where Δ_i is the greatest common divisor of all minors of order *i* of the homogeneity matrix of the system. As always, the following conventions are assumed: $\Delta_i = 0$ if the total number of monomials in all equations is less than *i*, $\Delta_0 = 1$, and $\frac{0}{0} = 0$.

Proof. Let *A* be the homogeneity matrix. We have to find the maximal number *n* such that the system of linear homogeneous equations AX = 0 (in m + s variables) has a solution in $\mathbb{Z}/n\mathbb{Z}$ whose components generate $\mathbb{Z}/n\mathbb{Z}$ as an additive group (or, equivalently, the first *m* components of the solution generate $\mathbb{Z}/n\mathbb{Z}$, because the equations say that the last *s* components are combinations of the first *m* ones). In other words, *n* is the largest order of cyclic quotient of the finitely generated group \mathbb{Z}^{m+s}/N , where *N* is the subgroup generated by rows of *A*. As noted already, the largest cyclic quotient *n* of \mathbb{Z}^{m+s}/N is $\frac{\Delta_{m+s}}{\Delta_{m+s-1}}$, as required.

Theorem 4. Let *R* be an associative ring with unity and let *G* be a subgroup of the multiplicative group of this ring. Then, for each system of equations over *R* in

m unknowns, the number of its solutions lying in G^m is divisible by the greatest common divisor of the homogeneity modulus of the system and the intersection of G with the centralizer of the set of coefficients of the system.

Proof. Let G_0 be the intersection of G and the centralizer of the set of coefficients and let n be the homogeneity modulus. Consider the free group F = F(X) (where X is the set of unknowns) and an epimorphism deg : $F \to \mathbb{Z}/n\mathbb{Z}$.

Let us apply the main theorem taking Φ to be the set of all homomorphisms $\phi: F \to G$ such that the tuple $(\phi(x_1), \ldots, \phi(x_m))$ is a solution to the system of equations (so, the number of solutions is $|\Phi|$). Take *H* to be any subgroup of G_0 of order dividing *n*. Condition (I) of the main theorem obviously holds. To verify condition (II), choose an element $t \in F$ of degree one and write each variable x_i in the form $x_i = t^{\deg x_i} y_i$, where $y_i = t^{-\deg x_i} x_i$ has degree zero. In new notation, each equation $w(x_1, \ldots, x_m) = 0$ takes the form $v(t, y_1, \ldots, y_m) = 0$ and the exponent sum of *t* in each term of this equation is the same (modulo *n*). Now, note that, if $v(\phi(t), \phi(y_1), \ldots, \phi(y_m)) = 0$ and $h \in H_{\phi}$, then $v(\phi(t)h, \phi(y_1), \ldots, \phi(y_m)) = 0$. This result follows from the (right) divisibility of $v(\phi(t)h, \phi(y_1), \ldots, \phi(y_m))$ by $v(\phi(t), \phi(y_1), \ldots, \phi(y_m))$ due to the following fact.

Fact [Klyachko and Mkrtchyan 2017, Lemma 1]. If M is a monoid, b_i , $a, h \in M$, elements a and h are invertible, and the elements $a^{-s}ha^s$, where $s \in \mathbb{Z}$, commute with all b_i , then, for any expression of the form $u(t) = b_0 t^{m_1} b_1 \cdots t^{m_l} b_l$, where $m_i \in \mathbb{Z}$, we have

$$u(ah) = \begin{cases} h^{a^{-1}} h^{a^{-2}} \cdots h^{a^{-k}} u(a) & \text{if } k = \sum m_i > 0, \\ h^{-1} h^{-a} \cdots h^{-a^{-1-k}} u(a) & \text{if } k = \sum m_i < 0, \\ u(a), & \text{if } k = \sum m_i = 0. \end{cases}$$

We apply this fact to each term of v; we also use that t^n has degree zero and $(\phi(t)h)^n = (\phi(t))^n$ according to Lemma 2.

Thus, the main theorem implies that $|\Phi|$ (i.e., the number of solutions to the system of equations) is divisible by |H| as required (because H is an arbitrary subgroup of G_0 whose order divides the homogeneity modulus).

Example. If $\rho : G \to R^*$ is a homomorphism from a finite group *G* to the multiplicative group of an associative ring *R* with unity (e.g., $\rho : G \to GL(V)$ is a linear representation of *G*), then, for any words $u_i(x_1, \ldots, x_m) \in F(x_1, \ldots, x_m)$,

the number of solutions to the equation
$$\sum_{i=1}^{k} (\rho(u_i(x_1, \dots, x_m)))^{l_i} = 1$$

is divisible by
$$\begin{cases} \text{GCD}(G, \text{GCD}\{l_i\}) & \text{always,} \\ \text{GCD}(G, \text{LCM}\{l_i\}) & \text{if } k \le m, \\ |G| & \text{if } k < m. \end{cases}$$

To show this, it suffices to apply Theorem 4 to the subgroup $\rho(G) \subseteq R^*$. The homogeneity matrix of this equation has the form

$$B = \begin{pmatrix} A & 1 \\ \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix},$$

where the last row corresponds to 1 in the right-hand side of the equation, and the *i*-th row of the matrix A corresponds to the *i*-th term in the left-hand side of the equation and, therefore, all elements of this row are divisible by l_i . It remains to note that the *j*-th invariant factor of the matrix B coincides with the (j - 1)-th invariant factor of A and use the following fact, which we leave to readers as an easy exercise:

if the *i*-th row of an integer matrix $k \times m$ is divisible by l_i , then the *m*-th invariant factor of this matrix

$\int is \ divisible \ by \ GCD\{l_i\}$	always,
$\{ is divisible by LCM \{l_i\} \}$	for $k = m$,
vanishes	for $k < m$.

Note that Theorem 0 can be obtained as a corollary of Theorem 4. Indeed, take $R = \mathbb{Z}G$; the group ring contains *G* as a subgroup of the multiplicative group. Any system of equations over *G* can be rewritten in "ring" form: $\{w_i(x_1, ...) - 1 = 0\}$. It remains to note that the value $\frac{\Delta_m}{\Delta_{m-1}}$ from Theorem 0 becomes exactly the homogeneity modulus from the homogeneity-modulus lemma.

4. Crossed homomorphisms

Suppose a group *F* acts (on the right) on a group *B* by automorphisms: $(f, b) \mapsto b^f$. Recall that a *crossed homomorphism* from *F* to *B* with respect to this action is a mapping $\alpha : F \to B$ such that $\alpha(ff') = \alpha(f)^{f'}\alpha(f')$ for all $f, f' \in F$. Saveliy Skresanov noted that the main theorem easily implies the following fact proved in [Asai et al. 2013] (using character theory) for finite groups *F* and *B*.

Theorem 5. If a group F admitting an epimorphism onto $\mathbb{Z}/n\mathbb{Z}$ acts by automorphisms on a group B, then the number of crossed homomorphisms $F \to B$ is divisible by GCD(B, n).

Proof. The set of crossed homomorphisms is in one-to-one correspondence with the set Φ of (usual) homomorphisms from *F* to the semidirect product $G = F \land B$ (with respect to the given action) such that their compositions with the projection $\pi : F \land B \rightarrow F$ is the identity mapping $F \rightarrow F$. We have to show that $|\Phi|$ is a multiple of |H| for any subgroup $H \subseteq B$ whose order divides *n* (by definition of GCD(*B*, *n*)).

The group *F* is *n*-indexed by the hypothesis of Theorem 5. Therefore, the assertion follows immediately from the main theorem. Conditions of the main theorem hold by trivial reasons: condition (I) is fulfilled because $\pi(h^{-1}gh) = \pi(g)$, and condition (II) follows immediately from Lemma 3 because $\pi(gh) = \pi(g)$ (for $g \in G$ and $h \in H$).

5. Proof of the main theorem

Take an element $f_1 \in F$ of degree one, put $F_0 = \ker \deg \subset F$, and consider the semidirect product $\widetilde{F} = \langle a \rangle_{\infty} \land F_0$, where *a* acts on F_0 as f_1 does: $u^a = u^{f_1}$ for $u \in F_0$. The group \widetilde{F} admits a natural indexing (0-indexing) deg : $\widetilde{F} \to \mathbb{Z}$ (denoted by the same symbol deg). The kernel of this map is F_0 and deg a = 1. Moreover, there is a natural epimorphism $\alpha : \widetilde{F} \to F$ mapping *a* to f_1 and identity on F_0 . Let us verify that the conditions of the main theorem hold for the set $\widetilde{\Phi} = \{\phi \circ \alpha \mid \phi \in \Phi\}$ of homomorphisms from \widetilde{F} to *G*.

Condition (I) holds obviously. To verify condition (II), take the degree-one element $a \in \widetilde{F}$ and some homomorphism $\tilde{\phi} = \phi \circ \alpha \in \widetilde{\Phi}$ (where $\phi \in \Phi$). Then the homomorphism $\tilde{\psi}$ from condition (II) has the form

(**)
$$\tilde{\psi}(\tilde{f}) = \begin{cases} \phi(\tilde{f}) & \text{for all elements } \tilde{f} \in F_0, \\ \phi(f_1)h & \text{for } \tilde{f} = a, \end{cases}$$
 where $\phi \in \Phi$ and $h \in H_{\tilde{\varphi}}$.

We have to show that $\tilde{\psi}$ lies in $\tilde{\Phi}$, i.e., has the form $\tilde{\psi} = \phi' \circ \alpha$, where $\phi' \in \Phi$. Note that $H_{\tilde{\varphi}} = H_{\phi}$, because the images of $\tilde{\phi} = \phi \circ \alpha$ and ϕ coincide, and the images of zerodegree elements for these homomorphisms coincide: $\tilde{\phi}(\ker \deg) = \tilde{\phi}(F_0) = \phi(F_0)$. Equation (**) takes the form

$$\tilde{\psi}(\tilde{f}) = \begin{cases} \phi(\tilde{f}) & \text{for } \tilde{f} \in F_0, \\ \phi(f_1)h & \text{for } \tilde{f} = a, \end{cases} \text{ where } \phi \in \Phi \text{ and } h \in H_{\phi}.$$

This means that $\tilde{\psi} = \psi \circ \alpha$, where

$$\psi(f) = \begin{cases} \phi(f) & \text{for } f \in F_0, \\ \phi(f_1)h & \text{for } f = f_1, \end{cases} \text{ where } \phi \in \Phi \text{ and } h \in H_\phi$$

The homomorphism $\psi : F \to G$ lies in Φ by condition (II) of the theorem we are proving. Therefore, $\tilde{\psi} \in \tilde{\Phi}$. Thus, the conditions of the main theorem hold for the set $\tilde{\Phi}$ of homomorphisms from the 0-indexed group \tilde{F} to G. Therefore, $|\tilde{\Phi}|$ is divisible on |H| by virtue of the main theorem of [Klyachko and Mkrtchyan 2017]. It remains to note that $|\Phi| = |\tilde{\Phi}|$ since α is surjective. This completes the proof.

Note that we do not verify here that ψ defines a homomorphism; this is nonobvious but true; see Section 1.

6. Open questions

Theorems 0, 1, 4, and 5 assert that some numbers are multiples of the ratios of two integers. Oddly, we do not know *whether these ratios can be replaced by their numerators*.

Questions 6 and 7. *Is it possible to replace the ratio* Δ_m / Δ_{m-1} *by its numerator* Δ_m *in Theorems 0 and 1*?

For coefficient-free systems of equations, Question 6 is equivalent to the following question posed in [Asai and Yoshida 1993] (for finite groups F and G):

is the number of homomorphisms from a finitely generated group F to a group G divisible by GCD(|F/F'|, G)?

This problem remains unsolved even for finite groups (as far as we know). A survey of some results can be found in [Asai and Takegahara 2001]; e.g., the answer is positive if F is abelian [Yoshida 1993].

Theorem 4 suggests a similar question.

Question 8. *Is it possible, in Theorem 4, to replace the homogeneity modulus by its numerator* Δ_{m+s} (see the homogeneity-modulus lemma)?

As for Theorem 5, it also leads us to a similar question. Indeed, Theorem 5 implies, in particular, the following corollary.

Corollary. If a finitely generated group F acts by automorphisms on a group B, then GCD(exp(F/F'), B) divides the number of crossed homomorphisms $F \rightarrow B$.

Question 9. *Is it possible, in this corollary, to replace the period* $\exp(F/F')$ *by the order of this quotient group?*

This question was posed for the first time in [Asai and Yoshida 1993] (for finite groups *F* and *B*). To show the similarity of Questions 9 and 6, we recall that the absolute value of the ratio Δ_m/Δ_{m-1} in Question 6 is the period of the quotient group of the free abelian group \mathbb{Z}^m by the subgroup generated by the rows of the matrix of the system of equations, while the absolute value of the numerator Δ_m is the order of this quotient group.

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ELENA K. BRUSYANSKAYA FACULTY OF MECHANICS AND MATHEMATICS MOSCOW STATE UNIVERSITY MOSCOW RUSSIA ebrusianskaia@gmail.com ANTON A. KLYACHKO FACULTY OF MECHANICS AND MATHEMATICS MOSCOW STATE UNIVERSITY MOSCOW RUSSIA

klyachko@mech.math.msu.su

ANDREY V. VASIL'EV Sobolev Institute of Mathematics Novosibirsk Russia and

NOVOSIBIRSK STATE UNIVERSITY NOVOSIBIRSK RUSSIA

vasand@math.nsc.ru

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EDITORS

Don Blasius (Managing Editor) Department of Mathematics University of California Los Angeles, CA 90095-1555 blasius@math.ucla.edu

Paul Balmer Department of Mathematics University of California Los Angeles, CA 90095-1555 balmer@math.ucla.edu

Wee Teck Gan Mathematics Department National University of Singapore Singapore 119076 matgwt@nus.edu.sg

Sorin Popa Department of Mathematics University of California Los Angeles, CA 90095-1555 popa@math.ucla.edu

Paul Yang Department of Mathematics Princeton University Princeton NJ 08544-1000 yang@math.princeton.edu

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Matthias Aschenbrenner

Department of Mathematics

University of California

Los Angeles, CA 90095-1555

matthias@math.ucla.edu

Daryl Cooper

Department of Mathematics

University of California

Santa Barbara, CA 93106-3080

cooper@math.ucsb.edu

Jiang-Hua Lu

Department of Mathematics

The University of Hong Kong

Pokfulam Rd., Hong Kong

jhlu@maths.hku.hk

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Kefeng Liu Department of Mathematics University of California Los Angeles, CA 90095-1555 liu@math.ucla.edu

Jie Qing Department of Mathematics University of California Santa Cruz, CA 95064 qing@cats.ucsc.edu

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