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LOCAL ESTIMATES FOR HÖRMANDER'S OPERATORS OF FIRST KIND WITH ANALYTIC GEVREY COEFFICIENTS AND APPLICATION TO THE REGULARITY OF THEIR GEVREY VECTORS

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Following our preceding papers devoted to the case of general Hörmander's operators P with analytic-Gevrey coefficients on an open set Ω in $\mathbb{R}^n$, for which we established local relations of domination by powers of P and derived from it local s'-Gevrey regularity of local s-Gevrey vectors of P (with, furthermore, suitable relations between s, s' and the coefficient of the Sobolev estimate satisfied by P), this article deals with the case of Hörmander's operators of first kind (or of degenerate elliptic kind). We establish, in this case, precise local relations of domination by powers of P which give, when applied to the s'-Gevrey regularity of s-Gevrey vectors of P, in Ω_0, with $\bar{\Omega}_0 \subset \Omega$, an optimal relation between s, s' and the type of $\bar{\Omega}_0$ with respect to the system X of vector fields whose sum of squares is the leading part of P.

1. Introduction

Since the paper of T. Kotake and N. S. Narasimhan [1962] on the analytic regularity of analytic vectors of elliptic operators with analytic coefficients, many articles were published, trying to generalize their result in different directions such as nonelliptic operators, systems, s-Gevrey vectors (which generalize the notion of analytic vectors for $s > 1$, analytic corresponding to $s = 1$). The property proved by Kotake and Narasimhan for elliptic operators with analytic coefficients (also named "iteration property" or even "Kotake–Narasimhan property") was sought to be true for more general operators than elliptic ones and also for systems (see a survey on this subject in [Bolley et al. 1987] or in [Derridj 2017] for a more recent but short one). The "iteration property" is also true for s-Gevrey vectors of elliptic operators with s-Gevrey coefficients, $s \geq 1$, but it was proved by G. Métivier [1978] that it cannot be true for nonelliptic operators, meaning more precisely that if P

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is not elliptic, even with analytic coefficients, then an s-Gevrey vector of P is not necessarily in s-Gevrey class if $s > 1$.

For the value $s = 1$, M. S. Baouendi and Métivier [1982] proved that the iteration property is true for the class of hypoelliptic operators of principal type with analytic coefficients, showing a difference between the cases $s = 1$ and $s > 1$, in this question.

There are also many papers concerning systems of vector fields with analytic coefficients: M. Damlakhi and B. Helffer [1980] showed the "iteration property," in the case $s = 1$, for such real systems satisfying Hörmander's condition, followed by a more precise version of Helffer and C. Mattera [1980].

When the iteration property is not true one can ask for the s'-Gevrey regularity of s-Gevrey vectors of an operator or system, $s' \geq s$. There is a series of papers studying the case of systems of analytic complex vector fields, concerning analytic or Gevrey vectors [Barostichi et al. 2011; Castellanos et al. 2013], where the authors prove such s'-Gevrey regularity of analytic or s-Gevrey vectors with some relation between s, s' and the structure of the system under study.

A more recent paper by N. Braun Rodrigues, G. Chinni, P. Cordaro and M. Jahnke [Braun Rodrigues et al. 2016] was partly devoted to the global Gevrey regularity of global analytic Gevrey vectors of some subclass of Hörmander's operators on a product of tori. In that situation they showed global s'-Gevrey regularity of global analytic or s-Gevrey vectors of such operators, with an optimal result. A little later, we studied in [Derridj 2019b] the case of local Gevrey regularity of local k-Gevrey vectors ($k \in \mathbb{N}^*$) of Hörmander's operators of first kind obtaining also the same relation between s', k and the type of $\overline{\Omega}_0$, with Ω_0 open set in Ω on which we consider the k-Gevrey vectors. More recently, we studied the case of general Hörmander's operators P satisfying an a priori Sobolev estimate done by L. Hörmander [1967] for which we established local relations of domination by powers of P, when the coefficients of P are in $G^s(\Omega)$, $s \geq 1$. From such local relations we deduced the s'-Gevrey regularity of s-Gevrey vectors, with a suitable relation between s, s' and the coefficient σ of the Sobolev estimate, $\sigma = \frac{1}{p}$, $p \in \mathbb{N}^*$ [Derridj 2019a].

Here we study the same question for Hörmander's operators of first kind for which we give precise local relations of domination by powers of P, when the coefficients of P are in $G^s(\Omega)$. From these local relations we deduce an optimal relation between s', s and the type of $\overline{\Omega}_0$, with respect to the system X of the vector fields whose sum of squares is the leading part of P (see detailed theorems in the next sections).

In Section 2 are given some notation and definitions, with some preliminary facts. We recall in Section 3 the basic subelliptic estimate satisfied by Hörmander's operators, established by Hörmander [1967], J. J. Kohn [1978], and L. Rothschild and E. Stein [1976].

We use this basic estimate in order to give in Section 4 a finite family of localized estimates needed in the proof of our local relations of domination by powers of P, when the coefficients of P are in $G^s(\Omega_0)$. Then in Section 6, we prove a theorem (Theorem 6.1) which gives as a corollary the s'-Gevrey regularity in Ω_0, of s-Gevrey vectors of P ($s \geq 1$) on Ω_0 with s' optimal and a relation between s', s and the type of Ω_0 with respect to X (Theorem 6.3).

2. Some notations, definitions and preliminary facts

The differential operators we deal with in this paper are defined in an open set Ω of $\mathbb{R}^n$ and have the form:

$$P = \sum_{j=1}^{m} X_j^2 + Y + b, \tag{2-1}$$

where

$$\begin{aligned} &X = (X_1, \ldots, X_m) \text{ is a system of real smooth vector fields in } \Omega,\\ &Y \text{ is a smooth vector field in } \Omega \text{ such that its imaginary part}\\ &\operatorname{Im} Y \text{ is a linear combination with smooth real coefficients in}\\ &\Omega \text{ of the vector fields } X_j,\ j = 1, \ldots, m, \end{aligned} \tag{2-2}$$

and

$$\operatorname{Im} Y = \sum_{j=1}^{m} b_j X_j, \quad b_j \in C^\infty(\Omega, \mathbb{R}),\ b \in C^\infty(\Omega, \mathbb{C}). \tag{2-3}$$

In the case Y is real and P satisfies the following "Hörmander's condition for hypoellipticity:"

$$\begin{aligned} &\text{The Lie algebra, } \operatorname{Lie}(Y, X_1, \ldots, X_m) \text{, generated by the smooth}\\ &\text{real vector fields } Y, X_1, \ldots, X_m, \text{ is of maximal rank in } \Omega, \end{aligned} \tag{2-4}$$

P is hypoelliptic in Ω [Hörmander 1967].

We studied in [Derridj 2019a] the case where the coefficients of the vector fields $Y, X_1, \ldots, X_m$ and b are in some Gevrey class, and established, under condition (2-4), local relations of domination by powers of P, with application to the Gevrey regularity of analytic-Gevrey vectors of P.

Here we prove precise local relations of domination by powers of P in the case of Hörmander's operators of first kind:

$$\begin{aligned} &\text{The Lie algebra, } \operatorname{Lie}(X_1, \ldots, X_m), \text{ generated by the smooth}\\ &\text{real vector fields } X_1, \ldots, X_m, \text{ is of maximal rank in } \Omega,\\ &P \text{ given by (2-1), satisfying (2-3).} \end{aligned} \tag{2-5}$$

More details are given in the next sections.

Let A be now a linear operator on $\mathcal{D}(\Omega)$, then

$$[P, A] = \sum_{j=1}^{m} (X_j[X_j, A] - [X_j, [X_j, A]]) + [Y + b, A] \quad \text{with } [X, A] = XA - AX. \tag{2-6}$$

In the sequel, our operators will be ordinary derivatives ∂^α, with $\alpha \in \mathbb{N}^n$, or some elementary pseudodifferential operators T_σ or ψT_σ with $\psi \in \mathcal{D}(\Omega)$. Let us recall them and give some related facts.

Given $\sigma \in \mathbb{R}$, one defines T_σ, as operator acting on $\mathcal{S}(\mathbb{R}^n)$ (the Schwartz space on $\mathbb{R}^n$) by:

$$\mathcal{S}(\mathbb{R}^n) \ni u \mapsto T_\sigma u \in \mathcal{S}(\mathbb{R}^n), \quad \text{with} \quad \widehat{T_\sigma u}(\xi) = (1 + |\xi|^2)^{\sigma/2} \hat{u}(\xi). \tag{2-7}$$

As we will work locally, generally in relatively compact open sets in Ω, we consider elementary pseudodifferential operators $\psi T_\sigma u$, with ψ in $\mathcal{D}(\Omega)$. Moreover when working on local regularity of a function u, knowing a property of Pu or of the sequence $P^k u$, $k \in \mathbb{N}$, we can assume by taking $\widetilde{X}_j = \psi X_j$, $\widetilde{Y} = \psi Y$, $\tilde{b} = \psi b$, and $\widetilde{P} = \sum_{j=1}^m \widetilde{X}_j^2 + \widetilde{Y} + \tilde{b}$, that the X_j's, Y and b have compact support, specifying the following:

$$\begin{aligned} &\psi \in \mathcal{D}(\Omega),\ \psi = 1 \text{ on } V(\overline{\Omega}_1), \text{ with } \overline{\Omega}_1 \subset \Omega, \text{ then} \\ &Pu = \widetilde{P}u \quad \text{on } V(\overline{\Omega}_1) = \Omega_2,\ \overline{\Omega}_2 \subset \Omega. \end{aligned} \tag{2-8}$$

The Hörmander's hypothesis will be the same on Ω_1 and so, all the inequalities obtained when using such hypothesis. Coming back to our operator T_σ, we remark that, for $u \in \mathcal{D}(\mathbb{R}^n)$, $T_\sigma u$ is not necessarily in $\mathcal{D}(\mathbb{R}^n)$, but $\psi T_\sigma u$ is, when $\psi \in \mathcal{D}(\Omega) \subset \mathcal{D}(\mathbb{R}^n)$.

The following facts, which we will use along the proof, are, of course, common in the theory of pseudodifferential operators, but here can be proved easily as we work with the above defined simple operators. The Sobolev norms are:

$$\|v\|_\sigma = \|T_\sigma v\|, \qquad \|\cdot\| \text{ being the } L^2 - \text{norm}, \quad v \in \mathcal{S}(\mathbb{R}^n). \tag{2-9}$$

In particular, T_σ and so ψT_σ are linear continuous operators from $H^s(\mathbb{R}^n)$ to $H^{s-\sigma}(\mathbb{R}^n)$. The operator T_σ is of order σ. As we assumed the coefficients of the X_j's, Y and b with compact support (as we used the cut-off function, which has value 1 on Ω_1), we may consider the following:

$$\begin{aligned} &[X_j, T_\sigma] \text{ and } [Y, T_\sigma] \text{ are of order } \sigma, \text{ satisfying} \\ &\qquad \|[X_j, T_\sigma]v\|_\rho \le C_{\rho,\sigma} \|v\|_{\rho+\sigma} \quad \text{for all } v \in \mathcal{D}(\Omega), \\ &\qquad\qquad \text{same for } [Y + b, T_\sigma], \\ &\qquad \|[X_j, [X_j, T_\sigma]v\|_\rho \le C_{\rho,\sigma} \|v\|_{\rho+\sigma} \quad \text{for all } v \in \mathcal{D}(\Omega), \\ &\qquad \|[\psi, T_\sigma]v\|_\rho \le C \|v\|_{\rho+\sigma-1}. \end{aligned} \tag{2-10}$$

The properties in (2-10) are the same replacing T_σ by ψT_σ.

We will use in the next sections the following facts:

(2-11) If Ω_1 is relatively compact in Ω_2 and $\psi \in \mathscr{D}(\Omega_2)$, $\psi|_{\Omega_1} = 1$, then

$$\|v\|_\sigma \le \|\psi T_\sigma v\| + C\|v\|_{\sigma-1}, \quad \text{for all } v \in \mathscr{D}(\Omega_1),\ C = C(\psi).$$

Equation (2-11) follows from: $\|T_\sigma v\| = \|T_\sigma \psi v\| \le \|\psi T_\sigma v\| + \|[T_\sigma, \psi]v\|$.

If $(\cdot,\cdot)$ denotes the scalar product in $L^2(\mathbb{R}^n)$ then

$$\begin{aligned} &(T_\sigma v, w) = (v, T_\sigma w), && v, w \in \mathscr{S}(\mathbb{R}^n) \\ &|(v, w)| \le \text{s.c.}\|v\|_s + \text{l.c.}\|w\|_{-s}, && s \in \mathbb{R}, \end{aligned} \tag{2-12}$$

where s.c. stands for a small constant and l.c. for a corresponding large constant.

As we will need it in the next sections, we recall a relation between the scalar product (Pv, v), $v \in \mathscr{D}(\Omega)$ and the norms $\|X_j v\|$, $j = 1, \dots, m$, for $v \in \mathscr{D}(\Omega)$:

$$(Pv, v) = -\sum_{j=1}^m \|X_j v\|^2 + O\left(\sum_{j=1}^m \|X_j v\|\,\|v\| + \|v\|^2\right) + (Yv, v). \tag{2-13}$$

Now from hypothesis (2-5), made on Y, we see that

$$-\mathrm{Re}\,(Pv, v) = \sum_{j=1}^m \|X_j v\|^2 + O\left(\sum_{j=1}^m \|X_j v\|\,\|v\| + \|v\|^2\right). \tag{2-14}$$

Hence one gets, using that, if X_j is real, $\mathrm{Re}\,(X_j v, v) = O(\|v\|^2)$:

$$\sum_{j=1}^m \|X_j v\|^2 \le C(|(Pv, v)| + \|v\|^2), \quad \text{for all } v \in \mathscr{D}(\Omega). \tag{2-15}$$

We finish this section recalling definitions of "analytic and Gevrey spaces" and "analytic and Gevrey vectors of an operator."

Definition 2.1. Given an open set Ω in $\mathbb{R}^n$, an analytic ($s = 1$) (respectively Gevrey, $s > 1$) function in Ω is a smooth function in Ω such that for every compact K in Ω, there exists $C_k > 0$ such that

$$\|\partial^\alpha u\|_{L^2(K)} \le C_K^{|\alpha|+1}\alpha!^s, \quad \text{for all } \alpha \in \mathbb{N}^n,\ |\alpha| = \sum_{i=1}^n \alpha_i. \tag{2-16}$$

Definition 2.2. Given an operator P_m of order m in Ω, an analytic vector (case $s = 1$) and a Gevrey vector ($s > 1$) of P_m in Ω, is a function $u \in L^2_{\text{loc}}(\Omega)$ such that, for every compact $K \subset \Omega$, there exists a constant $C_K > 0$ such that

$$P_m^k u \in L^2(K) \quad \text{and} \quad \|P_m^k u\|_{L^2(K)} \le C_K^{k+1}(mk)!^s, \qquad \text{for all } k \in \mathbb{N}. \tag{2-17}$$

Remark 2.3. When Ω is compact, just take $K = \Omega$ in (2-16) or in (2-17).

In our case, in the sequel, $m = 2$, with P given by (2-4) and, as P is hypoelliptic [Hörmander 1967], or using directly the basic estimate, one can take $u \in C^\infty(\Omega)$, in Definition 2.2. We write $u \in G^s(\Omega)$ (Definition 2.1), $u \in G^s(P, \Omega)$ in Definition 2.2.

3. The Hörmander–Kohn–Rothschild–Stein basic estimate

In his paper on hypoellipticity, Hörmander [1967] introduced his condition, known as the bracket condition. in the case of operators P of first kind, it reads concretely as follows. Let, for any $i, j \in \{1, \dots, m\}$ and $I = (i_1, \dots, i_\ell)$

$$(3\text{-}1)\quad [X_i, X_j] = X_i \circ X_j - X_j \circ X_i,\qquad X_I = [X_{i_1}, \dots, [X_{i_{\ell-1}}, X_{i_\ell}]]\dots],\quad |I| = \text{length of } I = \ell.$$

For any open subset $\widetilde{\Omega} \subset \Omega$, we set

$$(3\text{-}2)\qquad (H_{\widetilde{\Omega}}):\ \text{For every } x \in \widetilde{\Omega},\ \operatorname{span}\{X_I(x),\ \forall I\} = T_x(\widetilde{\Omega}) \simeq \mathbb{R}^n.$$

Given any subset V contained in $\widetilde{\Omega}$, one can define its type relative to the system X as follows:

$$(3\text{-}3)\quad \operatorname{type}_X(V) = \sup\{\operatorname{type}_X(x) : x \in V\} \in \mathbb{R}_+ \cup \{+\infty\},\qquad \text{where }\ \operatorname{type}_X(x) = \inf\bigl\{k \in \mathbb{N}^* : \operatorname{span}\{X_J(x), |J| \le k\} = \mathbb{R}^n\bigr\}.$$

Then, for the system $X = (X_1, \dots, X_m)$, one has the following basic subelliptic estimate:

Theorem 3.1. *Let* Ω_1 *open*, $\Omega_1 \Subset \Omega$, *such that* $\operatorname{type}_X(\overline{\Omega}_1) = p < +\infty$. *Then*, *if* $\sigma = \frac{1}{p}$, *one has*

$$(3\text{-}4)\qquad \|v\|_\sigma \le C\Bigl(\sum_j \|X_j v\| + \|v\|\Bigr),\qquad C = C(\Omega_1, X),\ \textit{for all } v \in \mathcal{D}(\Omega_1).$$

The estimate (3-4), proved by Hörmander [1967] for $\sigma < \frac{1}{p}$, was improved by Rothschild and Stein [1976]. Kohn [1978] gave a subelliptic estimate with σ smaller, but with a simpler proof $\bigl($in case $p = 2,\ \sigma = \frac{1}{2}$ also$\bigr)$.

In the next sections, once Ω_1 with $\Omega_1 \Subset \Omega$ is fixed, one can assume that the X_j's, Y and b have compact support as we mentioned in (2-8) in the preceding section.

One may deduce from Theorem 3.1 an estimate involving P, which will be more useful to us, as we have information on the $Pu = f$, rather than on the $X_j u$'s. This estimate is

$$(3\text{-}5)\qquad \|v\|_\sigma^2 + \sum_{j=1}^m \|X_j v\|^2 \le C(|(Pv, v)| + \|v\|^2),\quad \text{for all } v \in \mathcal{D}(\Omega_1).$$

Now, given u such that Pu is known, $u \in C^\infty(\Omega_1)$, one way to use (3-5) with u playing some role, is to localize u by a cut-off function $\varphi \in \mathcal{D}(\Omega)$, with $\varphi|_{\Omega_2} = 1$ (so $\varphi u = u$ on Ω_2). So, taking $v = \varphi u$ in (3-5) we will have $(P\varphi u, \varphi u)$ in the second member.

In order to have information on $P\varphi u$, knowing Pu or φPu, if we look at it on Ω_2, is to write

$$P\varphi u = [P, \varphi]u + \varphi Pu, \tag{3-6}$$

which gives, using (2-6),

$$P\varphi u = \textstyle\sum_j \big(2X_j \circ X_j(\varphi) - X_j^2(\varphi)u\big) + \varphi Pu. \tag{3-7}$$

In the sequel, when we study estimates on derivatives of u, knowing locally derivatives of Pu, we will have to deal with the brackets of P with the operators ∂^α. So, from (2-6) we will face the brackets $[X_j, \partial^\alpha]$, $[Y, \partial^\alpha]$, $[X_j, [X_j, \partial^\alpha]]$. These are obviously differential operators of order α. As in our preceding paper [Derridj 2019a], we write them as:

$$\begin{gathered}
[X_j, \partial^\alpha] = \sum_{\substack{\beta<\alpha\\ \ell=1,\dots,n}} a_{j\alpha\beta\ell}\, \partial^{\beta+\ell}, \quad [Y, \partial^\alpha] = \sum_{\substack{\beta<\alpha\\ \ell=1,\dots,n}} b_{\alpha\beta\ell}\, \partial^{\beta+\ell},\\
[X_j, [X_j, \partial^\alpha]] = \sum_{\substack{\beta<\alpha\\ k=1,\dots,n}} d_{jk\alpha\beta}\, \partial^{\beta+k} + \sum_{\substack{\beta<\alpha,\, |\beta|\le|\alpha|-2\\ k,\ell=1,\dots,n}} c_{j\ell k\alpha\beta}\, \partial^{\beta+\ell+k},
\end{gathered} \tag{3-8}$$

where $\beta + \ell$ is the multiindex defined by

$$(\beta+\ell)_i = \beta_i,\ i \neq \ell,\ (\beta+\ell)_\ell = \beta_\ell + 1,$$
$$[b, \partial^\alpha] = \textstyle\sum_{\beta<\alpha} b_{\alpha\beta}\, \partial^\beta.$$

We, often, delete the first subscript j in these coefficients, in the proofs of our estimates, writing for example X instead of X_j's. Let us now recall a proposition giving estimates for the coefficients in (3-8), when the coefficients of P are analytic ($s = 1$), or generally in the Gevrey class $G^s(\Omega_1)$ ($s \geq 1$), Ω_1 open set in Ω; we proved this proposition in [Derridj 2019a], see also [Derridj and Zuily 1973].

Proposition 3.2. *Assume the coefficients are in $G^s(\Omega_1)$, $\overline{\Omega}_1 \subset \Omega$. For every compact $K \subset \Omega_1$, there exists $B = B_K$ such that, if ∇ denotes the gradient operator,*

$$\begin{aligned}
|b_{\alpha\beta}|_K + |b_{\alpha\beta\ell}|_K + |a_{j\alpha\beta\ell}|_K &\le B^{|\alpha-\beta|}\Big(\frac{\alpha!}{\beta!}\Big)^s, && \beta<\alpha,\ 1\le\ell\le n,\ 1\le j\le m,\\
|\nabla b_{\alpha\beta}|_K + |\nabla b_{\alpha\beta\ell}|_K + |\nabla a_{j\alpha\beta\ell}|_K &\le B^{|\alpha-\beta|}\Big(\frac{\alpha!}{\beta!}\Big)^s, && \beta<\alpha,\ 1\le\ell\le n,\ 1\le j\le m,\\
|c_{j\ell\alpha\beta}|_K + |\nabla c_{j\ell\alpha\beta}|_K &\le B^{|\alpha-\beta|}\Big(\frac{(\alpha+k)!}{(\beta+k)!}\Big)^s, && |\beta|\le|\alpha|-2,\ 1\le\ell\le n,\\
|d_{jk\alpha\beta}|_K + |\nabla d_{jk\alpha\beta}|_K &\le B^{|\alpha-\beta|}\Big((|\alpha|+1)\frac{\alpha!}{\beta!}\Big)^s.
\end{aligned} \tag{3-9}$$

Here we recall that $\alpha! = \prod_{i=1}^{n} \alpha_i!$,

$$\beta \le \alpha \Leftrightarrow \beta_i \le \alpha_i \quad \text{for } 1 \le i \le n; \qquad \beta < \alpha \Leftrightarrow \beta \le \alpha, \quad \beta \ne \alpha.$$

As a corollary of Proposition 3.2, we get:

Proposition 3.3. *Assume that the coefficients of the* X_j*'s*, Y *and* b *are in* $G^s(\bar{\Omega}_0)$, *for some* $s \ge 1$. *Then there exists a constant* $B > 0$ *such that for* $j = 1, \dots, m$ *and* $\beta < \alpha$, $1 \le \ell, k \le n$ *and* $0 \le \tau \le 1$:

$$\begin{aligned} \|b_{\alpha\beta} v\|_\tau + \|b_{\alpha\beta\ell} v\|_\tau + \|a_{j\alpha\beta\ell} v\|_\tau &\le B^{|\alpha-\beta|}\Big(\frac{\alpha!}{\beta!}\Big)^s \|v\|_\tau \\ \|c_{j\ell k\alpha\beta} v\|_\tau &\le B^{|\alpha-\beta|}\Big(\frac{(\alpha+k)!}{(\beta+k)!}\Big)^s \|v\|_\tau \qquad (3\text{-}10) \\ \|d_{jk\alpha\beta} v\|_\tau &\le B^{|\alpha-\beta|}\Big(\frac{\alpha!(|\alpha|+1)}{\beta!}\Big)^s \|v\|_\tau, \quad \textit{for all } v \in \mathfrak{D}(\Omega_0). \end{aligned}$$

4. The basic localized estimates

We want to derive from the basic subelliptic estimate (3-5) with $\sigma = \frac{1}{p}$, $p \in \mathbb{N}$, a finite family of localized estimates, which we will use in order to prove our local relations of domination by powers of P. These localized estimates are expressed in the following result.

Proposition 4.1. *Let* Ω_1, $\bar{\Omega}_1 \subset \Omega_0$, *and assume that* (3-5) *is true on* Ω_0. *Then there exists a constant* $C > 0$ *such that for all* $(u, \varphi) \in C^\infty(\Omega_1) \times \mathfrak{D}(\Omega_1)$ *and* $\alpha \in \mathbb{N}^n$:

$$(4\text{-}1)\quad \begin{aligned} &\|\varphi \partial^\alpha u\|_\sigma \\ &\le C\Bigg\{ (\|\varphi\partial^\alpha P u\| \|\varphi \partial^\alpha u\|)^{1/2} + \sum_{|\beta_1|+|\beta_2|\le 2} (\|\varphi^{(\beta_1)}\partial^\alpha u\| \|\varphi^{(\beta_2)}\partial^\alpha u\|)^{1/2} \\ &\quad + \Bigg(\sum_{\substack{|\beta|\le 1\\ j=1,\dots,m}} \|\varphi^{(\beta)}[X_j, \partial^\alpha]u\|^{1/2} + \sum_{j=1}^m \|\varphi[X_j, [X_j, \partial^\alpha]]u\|^{1/2} \\ &\quad + \|\varphi[Y+b, \partial^\alpha]u\|^{1/2}\Bigg) \cdot \|\varphi\partial^\alpha u\|^{1/2} \Bigg\} \end{aligned}$$

and for $1 \le \ell \le p-1$,

$$(4\text{-}2)\quad \begin{aligned} &\|\varphi \partial^\alpha u\|_{(\ell+1)\sigma} \\ &\le C\Bigg\{ \|\varphi\partial^\alpha P u\|_{(\ell-1)\sigma} + \sum_{|\beta|\le 1} \|\varphi^{(\beta)}\partial^\alpha u\|_{\ell\sigma} + \sum_{|\beta|=2} \|\varphi^{(\beta)}\partial^\alpha u\|_{(\ell-1)\sigma} \\ &\quad + \sum_{\substack{|\beta|\le 1\\ j=1,\dots,m}} \|\varphi^{(\beta)}[X_j, \partial^\alpha]u\|_{(\ell-1)\sigma} + \sum_{j=1}^m \|\varphi[X_j, [X_j, \partial^\alpha]u\|_{(\ell-1)\sigma} \\ &\quad + \|\varphi[Y+b, \partial^\alpha]u\|_{(\ell-1)\sigma} \Bigg\}. \end{aligned}$$

It is important to note the difference between the two cases (4-1) and (4-2). The first one has in the second member terms which are square roots of products of two factors and the second one has just norms, but in suitable Sobolev spaces which permit to obtain an optimal result.

Proof of Proposition 4.1. We begin with the proof of (4-1). We first mention that the constant $C > 0$, in the following, may vary from line to line, but as ℓ is in $\{0, \dots, p\}$, at the end we will have a constant $C > 0$, valid for all the estimates in (4-2). Moreover, in all the proof of our proposition, s.c. will denote a small constant and l.c. a large constant, which will be determined along the proof; in order to get a fixed constant $C > 0$, valid for all $\ell \in \{0, \dots, p\}$ and all $\alpha \in \mathbb{N}^n$, we use the basic estimate (3-5) for $v = \varphi\partial^\alpha u$. So

$$\|\varphi\partial^\alpha u\|_\sigma + \sum_{j=1}^{m} \|X_j\varphi\partial^\alpha u\| \le C_0\big(|(P\varphi\partial^\alpha u, \varphi\partial^\alpha u)|^{1/2} + \|\varphi\partial^\alpha u\|\big). \tag{4-3}$$

Then write

$$\begin{aligned}[P, \varphi\partial^\alpha]u &= [P, \varphi]\partial^\alpha u + \varphi[P, \partial^\alpha]u \\ &= \sum_{j=1}^{m}\big(2X_j \circ X_j(\varphi) - X_j^2(\varphi)\big)\,\partial^\alpha u + \varphi\sum_j\big(2X_j[X_j, \partial^\alpha]u - [X_j, [X_j, \partial^\alpha]]u\big) \\ &\qquad + Y(\varphi)\,\partial^\alpha u + \varphi[Y + b, \partial^\alpha]u\end{aligned} \tag{4-4}$$

Now the only terms which are not trivially bounded by the second member of (4-1) are

$$\big|(2X \cdot X(\varphi)\partial^\alpha u, \varphi\partial^\alpha u)\big|^{1/2} \quad \text{and} \quad \big|(2\varphi X[X, \partial^\alpha]u, \varphi\partial^\alpha u)\big|^{1/2}.$$

But

$$\begin{aligned}|(X \circ X(\varphi)\partial^\alpha u, \varphi\partial^\alpha u)|^{1/2} &= |(X(\varphi)\partial^\alpha u, (-X + a)\varphi\partial^\alpha u)|^{1/2} \\ &\le \text{s.c.}\|X\varphi\partial^\alpha u\| + \text{l.c.}(\|\varphi\partial^\alpha u\| + \|X(\varphi)\partial^\alpha u\|)\end{aligned} \tag{4-5}$$

$$\begin{aligned}&|(\varphi X[X, \partial^\alpha]u, \varphi\partial^\alpha u)|^{1/2} \\ &\le |(X(\varphi)[X, \partial^\alpha]u, \varphi\partial^\alpha u)|^{1/2} + |(X \circ \varphi[X, \partial^\alpha]u, \varphi\partial^\alpha u)|^{1/2} \\ &\le \|X(\varphi)[X, \partial^\alpha]u\|^{1/2}\|\varphi\partial^\alpha u\|^{1/2} + |(\varphi[X, \partial^\alpha]u, (-X + a)\varphi\partial^\alpha u)|^{1/2} \\ &\le \text{s.c.}\|X \circ \varphi\partial^\alpha u\| + \text{l.c.}(\|\varphi[X, \partial^\alpha]u)\| + \|\varphi\partial^\alpha u\|) \\ &\qquad + \|X(\varphi)[X, \partial^\alpha]u\|^{1/2}\|\varphi\partial^\alpha u\|^{1/2}\end{aligned} \tag{4-6}$$

Now taking the small constant s.c. less than $\frac{1}{2}C_0$, in view of (4-3), s.c.$\|X(\varphi)\partial^\alpha u\|$ will be absorbed by the left hand, which will be bounded by the right member of

(4-1). Once s.c. is so well chosen, the constant l.c. will be fixed and so get our constant C, needed in (4-1).

In order to prove (4-2), we need our hypothesis that the subelliptic estimate is valid in Ω_0: so in all the sequel we fix a test function $\psi \in \mathcal{D}(\Omega_0)$, $\psi|_{|\Omega_1} = 1$.

We will prove in fact the following estimate:

$$\|\varphi\partial^\alpha u\|_{(\ell+1)\sigma} + \sum_{j=1}^{m} \|X_j\varphi\partial^\alpha u\|_{\ell\sigma} \le C\big\{\text{second member in (4-2)}\big\}. \tag{4-7}$$

Now we set $v = \partial^\alpha u$, and then we get from (2-9)

$$\begin{aligned}\|\varphi v\|_{(\ell+1)\sigma} &+ \sum_{j=1}^{m} \|X_j\varphi v\|_{\ell\sigma} \\ &\le \|\psi\varphi v\|_{(\ell+1)\sigma} + \sum_{j=1}^{m} \|X_j\psi\varphi v\|_{\ell\sigma} \\ &\le \|T_{\ell\sigma}\psi\varphi v\|_{\sigma} + \sum_{j=1}^{m} \|T_{\ell\sigma}X_j\psi\varphi v\| \\ &\le \|[T_{\ell\sigma},\psi]\varphi v\|_{\sigma} + \|\psi T_{\ell\sigma}\varphi v\|_{\sigma} + \sum_{j=1}^{m}\big(\|[T_{\ell\sigma},X_j\psi]\varphi v\| + \|X_j\psi T_{\ell\sigma}\varphi v\|\big).\end{aligned} \tag{4-8}$$

Now, using that $[T_{\ell\sigma},\psi]$ is of order $\ell\sigma - 1 \le 0$, as $\ell \le p$ and $[T_{\ell\sigma}, X_j\psi]$ is of order $\ell\sigma$, we get:

$$\begin{aligned}\|\varphi v\|_{(\ell+1)\sigma} &+ \sum_{j=1}^{m} \|X_j\varphi v\|_{\ell\sigma} \\ &\le \text{s.c.}\|\varphi v\|_{(\ell+1)\sigma} + \text{l.c.}\|\varphi v\| + \|\psi T_{\ell\sigma}\varphi v\|_{\sigma} + \sum_{j=1}^{m} \|X_j\psi T_{\ell\sigma}\varphi v\|.\end{aligned} \tag{4-9}$$

Now we apply the basic estimate to the last two terms:

$$\begin{aligned}\|\psi T_{\ell\sigma}\varphi v\|_{\sigma} &+ \sum_{j} \|X_j\psi T_{\ell\sigma}\varphi v\| \\ &\le C_0\big(|(P\psi T_{\ell\sigma}\varphi v, \psi T_{\ell\sigma}\varphi v)|^{1/2} + \|\varphi v\|_{\ell\sigma}\big) \\ &\le C_0\big(|(P\psi T_{\ell\sigma}\varphi v, \psi T_{\ell\sigma}\varphi v)|^{1/2} + \text{s.c.}\|\varphi v\|_{(\ell+1)\sigma} + \text{l.c.}\|\varphi v\|\big).\end{aligned} \tag{4-10}$$

Gathering (4-8), (4-9) and (4-10), we obtain

$$\begin{aligned}\|\varphi v\|_{(\ell+1)\sigma} &+ \sum_{j=1}^{m} \|X_j\varphi v\| \\ &\le C_0|(P\psi T_{\ell\sigma}\varphi v, \psi T_{\ell\sigma}\varphi v)|^{1/2} + \text{s.c.}\|\varphi v\|_{(\ell+1)\sigma} + \text{l.c.}\|\varphi v\|, \quad \text{with } v = \partial^\alpha u.\end{aligned} \tag{4-11}$$

So we are reduced to study the term $|(P\psi T_{\ell\sigma}\varphi\partial^\alpha u, \psi T_{\ell\sigma}\varphi\partial^\alpha u)|^{1/2}$. We decompose $P\psi T_{\ell\sigma}\varphi\partial^\alpha u$ as follows:

$$\begin{aligned}(4\text{-}12)\quad & P\psi T_{\ell\sigma}\varphi\partial^\alpha u \\ &= \psi T_{\ell\sigma}\varphi\partial^\alpha Pu + [P, \psi T_{\ell\sigma}]\varphi\partial^\alpha u + \psi T_{\ell\sigma}[P,\varphi]\partial^\alpha u + \psi T_{\ell\sigma}\varphi[P,\partial^\alpha]u.\end{aligned}$$

So we are led to bound the following expressions:

(A) A bound to $|(\psi T_{\ell\sigma}\varphi\partial^\alpha Pu, \psi T_{\ell\sigma}\varphi\partial^\alpha u)|^{1/2} = E_1$.
The term E_1 can be bounded as follows, using (2-12):

$$E_1 \le \text{s.c.}\|\varphi\partial^\alpha u\|_{(\ell+1)\sigma} + \text{l.c.}\|\varphi\partial^\alpha Pu\|_{(\ell-1)\sigma}. \tag{4-13}$$

(B) A bound to $|([P, \psi T_{\ell\sigma}]\varphi\partial^\alpha u, \psi T_{\ell\sigma}\varphi\partial^\alpha u)|^{1/2} = E_2$.
Using the expression of the bracket given in (2-6), we have to bound the following three terms $E_{2,1}$, $E_{2,2}$, $E_{3,3}$.

(a) $$E_{2,1} \le \sum_{j=1}^m |(X_j[X_j, \psi T_{\ell\sigma}]\varphi\partial^\alpha u, \psi T_{\ell\sigma}\varphi\partial^\alpha u)|^{1/2}.$$

Using that $X_j^* = -X_j + a_j$ and $[X_j, \psi T_{\ell\sigma}]$ is of order $\ell\sigma$:

$$E_{2,1} \le \text{s.c.}\sum_{j=1}^m \|X_j\psi T_{\ell\sigma}\varphi\partial^\alpha u\| + \text{l.c.}\|\varphi\partial^\alpha u\|_{\ell\sigma}.$$

Similarly:

(b) $E_{2,2} \le \sum_{j=1}^m |([X_j, [X_j, \psi T_{\ell\sigma}]]\varphi\partial^\alpha u, \psi T_{\ell\sigma}\varphi\partial^\alpha u)|^{1/2} \le \text{s.c.}\|\varphi\partial^\alpha u\|_{(\ell+1)\sigma} + \text{l.c.}\|\varphi\partial^\alpha u\|.$

(c) $E_{2,3} \le |([Y+b, \psi T_{\ell\sigma}]\varphi\partial^\alpha, \varphi\partial^\alpha u)|^{1/2} \le \text{s.c.}\|\varphi\partial^\alpha u\|_{(\ell+1)\sigma} + \text{l.c.}\|\varphi\partial^\alpha u\|.$

Hence, as $E_2 = E_{2,1} + E_{2,2} + E_{2,3}$, we obtain

$$E_2 \le \text{s.c.}\|\varphi\partial^\alpha u\|_{(\ell+1)\sigma} + \text{l.c.}\|\varphi\partial^\alpha u\|. \tag{4-14}$$

(C) A bound to $|(\psi T_{\ell\sigma}[P,\varphi]\partial^\alpha u, \psi T_{\ell\sigma}\varphi\partial^\alpha u)|^{1/2} = E_3$.
Using the expression of the bracket $[P,\varphi]$ given in (2-6) we have to bound the following three terms.

(a) $$E_{3,1} \le 2\sum_{j=1}^m |(\psi T_{\ell\sigma}X_j \circ X_j(\varphi)\partial^\alpha u, \psi T_{\ell\sigma}\varphi\partial^\alpha u)|^{1/2}.$$

$$\begin{aligned}E_{3,1} &\le 2\sum_{j=1}^m \big|\big(\psi T_{\ell\sigma}X_j \circ X_j(\varphi)\partial^\alpha u, \psi T_{\ell\sigma}\varphi\partial^\alpha u\big)\big|^{1/2} \\ &\le \sum_{j=1}^m \big(\text{s.c.}\|X_j\psi T_{\ell\sigma}\varphi\partial^\alpha u\| + \text{l.c.}(\|X_j(\varphi)\partial^\alpha u\|_{\ell\sigma} + \|\varphi\partial^\alpha u\|_{\ell\sigma})\big),\end{aligned}$$

where we have used that $X_j^* = -X_j + a_j$.

(b) $$E_{3,2} \le \sum_{j=1}^m |(\psi T_{\ell\sigma} X_j^2(\varphi)\partial^\alpha u, \psi T_{\ell\sigma}\varphi\partial^\alpha u)|^{1/2}.$$

and hence:

$$E_{3,2} \le \text{s.c.}\|\varphi\partial^\alpha u\|_{(\ell+1)\sigma} + \text{l.c.}\sum_{j=1}^m \|X_j^2(\varphi)\partial^\alpha u\|_{(\ell-1)\sigma}.$$

(c) $E_{3,3} \le |(\psi T_{\ell\sigma}[Y+b,\varphi]\partial^\alpha u, \psi T_{\ell\sigma}\varphi\partial^\alpha u)|^{1/2} \le \|Y(\varphi)\partial^\alpha u\|_{\ell\sigma} + \|\varphi\partial^\alpha u\|_{\ell\sigma}$.
Therefore, as $E_3 = E_{3,1} + E_{3,2} + E_{3,3}$, we get

$$\text{(4-15)}\quad \begin{aligned} E_3 &\le \text{s.c.}\Big(\|\varphi\partial^\alpha u\|_{(\ell+1)\sigma} + \sum_{j=1}^m \|X_j\varphi\partial^\alpha u\|_{\ell\sigma}\Big) \\ &\quad + \text{l.c.}\Big(\sum_{j=1}^m \|X_j(\varphi)\partial^\alpha u\|_{\ell\sigma} + \|Y(\varphi)\partial^\alpha u\|_{\ell\sigma} + \|\varphi\partial^\alpha u\|_{\ell\sigma} + \sum_{j=1}^m \|X_j^2(\varphi)\partial^\alpha u\|_{(\ell-1)\sigma}\Big). \end{aligned}$$

(D) A bound to $|(\psi T_{\ell\sigma}\varphi[P,\partial^\alpha]u, \psi T_{\ell\sigma}\varphi\partial^\alpha u)|^{1/2} = E_4$.
Using again the expression of $[P,\partial^\alpha]$, we get

$$\begin{aligned} E_4 &\le \sum_{j=1}^m \big(|(2\psi T_{\ell\sigma}\varphi X_j[X_j,\partial^\alpha]u, \psi T_{\ell\sigma}\varphi\partial^\alpha u)|^{1/2} \\ &\qquad + |(\psi T_{\ell\sigma}\varphi[X_j,[X_j,\partial^\alpha]]u, \psi T_{\ell\sigma}\varphi\partial^\alpha u)|^{1/2}\big), \\ &= E_{4,1} + E_{4,2}, \end{aligned}$$

modulo a term trivially bounded by the second member of (4-2). Now,

$$\begin{aligned} E_{4,1} &\le 2\sum_{j=1}^m \Big\{ |(\psi T_{\ell\sigma} X_j(\varphi)[X_j,\partial^\alpha]u, \psi T_{\ell\sigma}\varphi\partial^\alpha u)|^{1/2} \\ &\qquad + |([\psi T_{\ell\sigma}, X_j]\varphi[X_j,\partial^\alpha]u, \psi T_{\ell\sigma}\varphi\partial^\alpha u)|^{1/2} \\ &\qquad + |(X_j\psi T_{\ell\sigma}\varphi[X_j,\partial^\alpha]u, \psi T_{\ell\sigma}\varphi\partial^\alpha u)|^{1/2} \Big\} \\ &\le \text{s.c.}\Big(\|\varphi\partial^\alpha u\|_{(\ell+1)\sigma} + \sum_{j=1}^m \|X_j\varphi\partial^\alpha u\|_{\ell\sigma}\Big) \\ &\qquad + \text{l.c.}\Big(\sum_{j=1}^m \|X_j(\varphi)[X_j,\partial^\alpha]u\|_{(\ell-1)\sigma} + \sum_{j=1}^m \|\varphi[X_j,\partial^\alpha]u\|_{\ell\sigma}\Big), \end{aligned}$$

$$E_{4,2} \le \text{s.c.}\|\varphi\partial^\alpha u\|_{(\ell+1)\sigma} + \text{l.c.}\|\varphi[X_j[X_j,\partial^\alpha]]u\|_{(\ell-1)\sigma}.$$

Hence

$$(4\text{-}16)\quad E_4 \le \text{s.c.}\Big(\|\varphi\partial^\alpha u\|_{(\ell+1)\sigma} + \sum_{j=1}^m \|X_j\varphi\partial^\alpha u\|_{\ell\sigma}\Big) + \text{l.c.}\Big\{\|\varphi\partial^\alpha u\|_{\ell\sigma} + \sum_{j=1}^m\Big(\|X_j(\varphi)[X_j,\partial^\alpha]u\|_{(\ell-1)\sigma} + \|\varphi[X_j,\partial^\alpha]u\|_{\ell\sigma} + \|\varphi[X_j,[X_j,\partial^\alpha]]u\|_{(\ell-1)\sigma}\Big)\Big\},$$

modulo a term trivially bounded by the second member of (4-2).

Therefore from (4-13)–(4-16), we obtain

$$(4\text{-}17)\quad |(P\psi T_{\ell\sigma}\varphi\partial^\alpha u, \psi T_{\ell\sigma}\varphi\partial^\alpha u)|^{1/2} \le \text{s.c.}\Big(\|\varphi\partial^\alpha u\|_{(\ell+1)\sigma} + \sum_{j=1}^m \|X_j\varphi\partial^\alpha u\|_{\ell\sigma}\Big) + \text{l.c.}\Big\{\|\varphi\partial^\alpha u\|_{\ell\sigma} + \sum_{j=1}^m \|X_j(\varphi)\partial^\alpha u\|_{\ell\sigma} + \|Y(\varphi)\partial^\alpha u\|_{\ell\sigma} + \sum_{j=1}^m \|X_j^2(\varphi)\partial^\alpha u\|_{(\ell-1)\sigma} + \sum_{j=1}^m \|X_j(\varphi)[X_j,\partial^\alpha]u\|_{(\ell-1)\sigma} + \sum_{j=1}^m \|\varphi[X_j[X_j,\partial^\alpha]]u\|_{(\ell-1)\sigma}\Big\},$$

modulo a term trivially bounded by the second member of (4-2).

Now, coming back to (4-11), (4-17) and taking "s.c." small enough, say $\frac{1}{2}$, one gets with the corresponding "l.c." a constant $C > 0$ in the estimates (4-2) (as the coefficients of the X_j and their first derivatives are bounded on Ω_0). The proof of Proposition 4.1 is complete. □

5. Precise local relations of domination by powers of *P*

Before stating our main theorem, in case P has analytic ($s = 1$) or Gevrey coefficients ($s > 1$), giving suitable local bounds of ordinary derivatives of functions under study, let us give some further notations of expressions needed in its proof. As $s \ge 1$ will be fixed, in the statement of the theorem, we do not include it in the notation below, except when it is really needed. First we recall the expression $N^\epsilon_{j,\gamma}(u,\varphi)$ we introduced in [Derridj 2019a]:

$$(5\text{-}1)\quad \text{For } \epsilon > 0,\ \Omega_1 \Subset \Omega,\ j \in \mathbb{N},\ \gamma \in \mathbb{N}^n,\ \text{and } (u,\varphi) \in C^\infty(\Omega_1) \times \mathscr{D}(\Omega_1),\quad N^\epsilon_{j,\gamma}(u,\varphi) = \epsilon^{|\gamma|+2j}|\gamma|!^{-s}(2j)!^{-s}\|\varphi^{(\gamma)}P^j u\|.$$

Now, in this work, we introduce new expressions:

Given $p \in \mathbb{N}$, for every k, we denote

$$(5\text{-}2)\quad \mathcal{I}_k = \Big\{ (j_1, \dots, j_{2^k}, \gamma_1, \dots, \gamma_{2^k}) = (j, \gamma) \in \mathbb{N}^{2^k} \times (\mathbb{N}^n)^{2^k} \quad \text{s.t.} \\ \text{if } |j| = \sum_{\rho=1}^{2^k} j_\rho,\ |\gamma| = \sum_{\rho=1}^{2^k} |\gamma_\rho|,\ |\gamma_\rho| \le (p+1)k,\ \forall \rho,\ |\gamma| + 2|j| \le 2^k pk \Big\}$$

Of course, p in (5-2) will be the type of our considered relatively compact set, as Ω_1, we spoke about.

Then we introduce

(5-3) For $\epsilon > 0$, p as above, $k \in \mathbb{N}$, $(u, \varphi) \in C^\infty(\Omega_1) \times \mathcal{D}(\Omega_1)$,

$$\mathcal{N}_k^{\epsilon, p}(u, \varphi) = \Big(\sum_{(j,\gamma) \in \mathcal{I}_k} \prod_{\rho=1}^{2^k} N_{j_\rho, \gamma_\rho}(u, \varphi) \Big)^{2^{-k}}.$$

(5-4) For $\epsilon > 0$, p as above, $k \in \mathbb{N}$, $\ell = 1, \dots, p$, $(u, \varphi) \in C^\infty(\Omega_1) \times \mathcal{D}(\Omega_1)$,

$$\mathcal{N}_{k,\ell}^{\epsilon, p}(u, \varphi) = \Big(\sum_{(j,\gamma) \in \mathcal{I}_{k,\ell}} \prod_{\rho=1}^{2^{k+1}} N_{j_\rho, \gamma_\rho}(u, \varphi) \Big)^{2^{-(k+1)}},$$

$$\text{where}\quad \mathcal{I}_{k,\ell} = \Big\{ (j_1, \dots, j_{2^{k+1}}, \gamma_1, \dots, \gamma_{2^{k+1}}) : |\gamma_\rho| \le (p+1)k + \ell + 1,\ \forall \rho, \\ |\gamma| + 2|j| \le 2^{k+1}(pk + \ell) \Big\}.$$

Remark 5.1. As p will be fixed with the relatively compact open set considered (say mostly Ω_1 but it may be $\Omega_2 \dots$), it will be deleted in the notation (5-3), (5-4).

Sometimes we denote also, when (u, φ) are specified,

$$(5\text{-}5)\qquad \mathcal{N}_k^\epsilon = \mathcal{N}_k^{\epsilon, p} = \mathcal{N}_{k,0}^\epsilon.$$

When there is no ambiguity we delete ϵ. But looking at the $\mathcal{N}_{k,\ell}^\epsilon$, $\ell = 0, \dots, p$, there is a difference in the expressions, between the case $\ell = 0$ and $1 \le \ell \le p$. Of course, there are some relations between these expressions which we give in the following lemma.

Lemma 5.2. *We have the following inequalities*:

$$(5\text{-}6)\qquad \mathcal{N}_k^\epsilon \le \mathcal{N}_{k,1}^\epsilon \le \dots \le \mathcal{N}_{k,p}^\epsilon = \mathcal{N}_{k+1}^\epsilon$$

Proof. The inequalities $\mathcal{N}_{k,1} \le \dots \le \mathcal{N}_{k,p}$ are trivial as $\mathcal{I}_{k,\ell} \subset \mathcal{I}_{k,\ell+1}$ for $\ell = 1, \dots, p-1$, and $\mathcal{N}_{k,p} = \mathcal{N}_{k+1}$ is clear. So what needs a proof is the first inequality.

In order to prove it, we compare $\mathcal{N}_k^{2^{k+1}}$ and $\mathcal{N}_{k,1}^{2^{k+1}}$, meaning establishing whether or not the following inequality is true:

$$\Big(\sum_{(j,\gamma)\in\mathcal{I}_k}\prod_{\rho=1}^{2^k}N_{j_\rho,\gamma_\rho}\Big)^2\le\sum_{(j,\gamma)\in\mathcal{I}_{k,1}}\prod_{\rho=1}^{2^{k+1}}N_{j_\rho,\gamma_\rho}. \tag{5-7}$$

In the first member of (5-7), we have the following products:

$$\Big(\prod_{\rho=1}^{2^k}N_{j_\rho,\gamma_\rho}\Big)\Big(\prod_{\rho'=1}^{2^k}N_{i_{\rho'},\delta_{\rho'}}\Big),\quad\text{with }(j,\gamma)\in\mathcal{I}_k,\ (i,\delta)\in\mathcal{I}_k.$$

These products are contained in the products in the right member of (5-7), via the map

$$N_{j_\rho,\gamma_\rho}\cdot N_{i_{\rho'},\delta_{\rho'}}\to N_{q_{\rho''},\nu_{\rho''}}, \tag{5-8}$$
$$q=(j_1,\dots,j_{2^k},i_1,\dots,i_{2^k}),\ \nu=(\gamma_1,\dots,\gamma_{2^k},\delta_1,\dots,\delta_{2^k}),$$

after observing that $(q,\nu)\in\mathcal{I}_{k,1}$. □

Now, we will need, in the sequel, to compare the preceding expressions, when one has different couples (u,φ). More precisely, denoting $\mathcal{N}^\epsilon_{\alpha,\ell}=\mathcal{N}^\epsilon_{|\alpha|,\ell}$, we have:

Lemma 5.3. (1) *Let* $(i,\beta)\in\mathbb{N}\times\mathbb{N}^n$ *and* $\mu\in\mathbb{N}$. *Then, for* $(j,\gamma)\in\mathbb{N}\times\mathbb{N}^n$,

$$\mu!^sN^\epsilon_{j,\gamma}(P^iu,\varphi^{(\beta)})\le\epsilon^{-(|\beta|+2i)}(\mu+|\beta|+2i)!^sN_{j+i,\gamma+\beta}(u,\varphi) \tag{5-9}$$
$$\textit{for }|\gamma|+2j\le\mu,\ (u,\varphi)\in C^\infty(\Omega_1)\times\mathcal{D}(\Omega_1).$$

(2) *For* $(u,\varphi)\in C^\infty(\Omega_1)\times\mathcal{D}(\Omega_1)$,

$$\begin{aligned}(p|\alpha|)!^s\mathcal{N}^\epsilon_\alpha(Pu,\varphi)&\le\epsilon^{-2}(p|\alpha|+2)!^s\,\mathcal{N}^\epsilon_{\alpha,1}(u,\varphi),\\(p|\alpha|+\ell)!^s\mathcal{N}^\epsilon_{\alpha,\ell}(Pu,\varphi)&\le\epsilon^{-2}(p|\alpha|+\ell+2)!^s\,\mathcal{N}^\epsilon_{\alpha,\ell+2}(u,\varphi),&&\ell+2\le p,\\(p|\alpha|)!^s\mathcal{N}^\epsilon_\alpha(u,\varphi^{(\beta)})&\le\epsilon^{-|\beta|}(p|\alpha|+|\beta|)!^s\,\mathcal{N}^\epsilon_{\alpha,1}(u,\varphi),&&|\beta|\le2,\\(p|\alpha|+\ell)!^s\mathcal{N}^\epsilon_{\alpha,\ell}(u,\varphi^{(\beta)})&\le\epsilon^{-|\beta|}(p|\alpha|+\ell+|\beta|)!^s\mathcal{N}^\epsilon_{\alpha,\ell+|\beta|}(u,\varphi),&&\ell+|\beta|\le p.\end{aligned} \tag{5-10}$$

For the proof of Lemma 5.3, we need a simple lemma, for which, we give a proof, in order to be complete.

Lemma 5.4. *Let* $q\in\mathbb{N}$ *and* $(a_1,\dots,a_q)\in(\mathbb{R}^n_+)^q$. *Then*

$$\prod_{j=1}^q a_j\le\Big(\frac{\sum_{j=1}^q a_j}{q}\Big)^q. \tag{5-11}$$

Proof of Lemma 5.4. The inequality is trivial for $q=1$. Assume it is true for q. Considering the function $f:\mathbb{R}_+\to\mathbb{R}$ given by

$$f(\lambda)=\Big(\frac{\sum_1^q a_j}{q}\Big)^q\lambda-\Big(\frac{\sum_1^q a_j+\lambda}{q+1}\Big)^{q+1},$$

and computing its derivative, one can see that f takes its maximum value at $\lambda = \frac{q+1}{q}\sum_{j=1}^{q} a_j$, which is zero. Hence $f \le 0$ on $\mathbb{R}_+$. Then taking $\lambda = a_{q+1}$, one deduces (5-11) for $q+1$. □

Proof of Lemma 5.3. The proof of (5-9) is easy to see:

$$N^{\epsilon}_{j,\gamma}(P^i u, \varphi^{|(\beta)}) = \epsilon^{-(|\beta|+2i)}\big\{(|\gamma|+1)\cdots(|\gamma|+|\beta|)(2j+1)\cdots(2j+2i)\big\}^s N^{\epsilon}_{j+i,\gamma+\beta}(u,\varphi). \tag{5-12}$$

Then, if $|\gamma|+2j \le \mu$, (5-12) gives (5-9).

Now, looking at the expression of $\mathcal{N}^{\epsilon}_{\alpha}$, for any $\rho \in \{1,\dots,2^{|\alpha|}\}$, we use (5-12) and consider the product $\prod_{\rho=1}^{2^{|\alpha|}} N^{\epsilon}_{j_\rho,\gamma_\rho}(Pu,\varphi)$:

$$\prod_{\rho=1}^{2^{|\alpha|}} N^{\epsilon}_{j_\rho,\gamma_\rho}(Pu,\varphi) = \prod_{\rho=1}^{2^{|\alpha|}} \epsilon^{-2}\{(2j_\rho+1)(2j_\rho+2)\}^s N_{j_\rho+1,\gamma_\rho}(u,\varphi) \tag{5-13}$$

Then using Lemma 5.4, for $\lambda = 1, 2$,

$$\prod_{\rho=1}^{2^{|\alpha|}} (2j_\rho+\lambda) \le \left(\frac{\sum_1^{2^{|\alpha|}}(2j_\rho+\lambda)}{2^{|\alpha|}}\right)^{2^{|\alpha|}} = \left(\frac{2|j|+\lambda 2^{|\alpha|}}{2^{|\alpha|}}\right)^{2^{|\alpha|}}, \quad \lambda = 1, 2. \tag{5-14}$$

Moreover, in the expression of $\mathcal{N}^{\epsilon}_{\alpha}$, we have $2|j| \le 2^{|\alpha|} p|\alpha|$ and $|\gamma_\rho| \le (p+1)|\alpha|$, $\rho \in \{1,\dots,2^{|\alpha|}\}$. So, from (5-14), we get

$$\prod_{\lambda=1}^{2}\prod_{\rho=1}^{2^{|\alpha|}} (2j_\rho+\lambda) \le \prod_{\lambda=1}^{2} (p|\alpha|+\lambda)^{2^{|\alpha|}}. \tag{5-15}$$

So

$$\prod_{\rho=1}^{2^{|\alpha|}} N_{j_\rho,\gamma_\rho}(Pu,\varphi) \le \prod_{\lambda=1}^{2} (p|\alpha|+\lambda)^{s2^{|\alpha|}} \epsilon^{-2^{|\alpha|+1}} \prod_{\rho=1}^{2^{|\alpha|}} N_{j_\rho+1,\gamma_\rho}(u,\varphi) \tag{5-16}$$

So coming back to the expression of $\mathcal{N}^{\epsilon}_{\alpha}(Pu,\varphi)$:

$$\mathcal{N}^{\epsilon}_{\alpha}(Pu,\varphi) \le \epsilon^{-2}\left(\prod_{\lambda=1}^{2}(p|\alpha|+\lambda)\right)^s \left\{\sum_{(j,\gamma)\in\mathcal{J}_{|\alpha|}} \prod_{\rho=1}^{2^{|\alpha|}} N_{j_\rho+1,\gamma_\rho}(u,\varphi)\right\}^{2^{|\alpha|}}. \tag{5-17}$$

Now we want to prove

$$\left(\sum_{(j,\gamma)\in\mathcal{J}_{|\alpha|}} \prod_{\rho=1}^{2^{|\alpha|}} N_{j_\rho+1,\gamma_\rho}(u,\varphi)\right)^2 \le \sum_{(i,\delta)\in\mathcal{J}_{|\alpha|,1}} \prod_{\rho'=1}^{2^{|\alpha|+1}} N_{i_{\rho'},\delta_{\rho'}}(u,\varphi). \tag{5-18}$$

But this is like what we did in (5-7), after observing that $(j_\rho+1, k_{\rho''+1}, \gamma_\rho, \tilde{\gamma}_{\rho''}) \in \mathcal{J}_{|\alpha|,1}$ when $(j,\gamma) \in \mathcal{J}_{|\alpha|}$, $(k,\tilde{\gamma}) \in \mathcal{J}_{|\alpha|}$.

Taking the two members in (5-18) at a power $2^{-(|\alpha|+1)}$, we obtain the first inequality in (5-10), using

$$(p|\alpha|)!^s\Big(\prod_{\lambda=1}^{2}(p|\alpha|+\lambda)\Big)^s \le (p|\alpha|+2)!^s. \tag{5-19}$$

The proofs of the other inequalities are similar. Let us give that of the last line which seems the "worst." Using (5-9) or (5-12), we get

$$N^{\epsilon}_{j,\gamma}(u,\varphi^{(\beta)}) = \epsilon^{-|\beta|}\big((|\gamma|+1)\cdots(|\gamma|+|\beta|)\big)^s N^{\epsilon}_{j,\gamma+\beta}(u,\varphi). \tag{5-20}$$

So, looking at the expression of $\mathcal{N}^{\epsilon}_{\alpha,\ell}$, for any $\rho\in\{1,\dots,2^{|\alpha|+1}\}$,

$$\prod_{\rho=1}^{2^{|\alpha|+1}} N^{\epsilon}_{j_\rho,\gamma_\rho}(u,\varphi^{(\beta)}) = \prod_{\rho=1}^{2^{|\alpha|+1}} \epsilon^{-|\beta|}\big((|\gamma_\rho|+1)\cdots(|\gamma_\rho|+|\beta|)\big)^s N^{\epsilon}_{j_\rho,\gamma_\rho+\beta}(u,\varphi). \tag{5-21}$$

Using Lemma 5.4, we get, as in (5-14),

$$\prod_{\lambda=1}^{|\beta|}\prod_{\rho=1}^{2^{|\alpha|+1}}(|\gamma_\rho|+\lambda) \le \prod_{\lambda=1}^{|\beta|}\Big(\frac{|\gamma|+\lambda 2^{|\alpha|+1}}{2^{|\alpha|+1}}\Big)^{2^{|\alpha|+1}}. \tag{5-22}$$

Now, in the expression of $\mathcal{N}^{\epsilon}_{\alpha,\ell}$ one has

$$|\gamma|\le 2^{|\alpha|+1}(p|\alpha|+\ell) \quad\text{and}\quad |\gamma_\rho|\le (p+1)|\alpha|+\ell+1.$$

So, from (5-22), we obtain

$$\prod_{\lambda=1}^{|\beta|}\prod_{\rho=1}^{2^{|\alpha|+1}}(|\gamma_\rho|+\lambda) \le \prod_{\lambda=1}^{|\beta|}(p|\alpha|+\ell+\lambda)^{2^{|\alpha|+1}}. \tag{5-23}$$

So,

$$\prod_{\rho=1}^{2^{|\alpha|+1}} N_{j_\rho,\gamma_\rho}(u,\varphi^{(\beta)}) \le \prod_{\lambda=1}^{|\beta|}(p|\alpha|+\ell+\lambda)^{s\,2^{|\alpha|+1}}\epsilon^{-|\beta|2^{|\alpha|+1}}\prod_{\rho=1}^{2^{|\alpha|+1}} N_{j_\rho,\gamma_\rho+\beta}(u,\varphi). \tag{5-24}$$

So, coming back to the expression of $\mathcal{N}^{\epsilon}_{\alpha,\ell}(u,\varphi^{(\beta)})$:

$$\begin{aligned}\mathcal{N}^{\epsilon}_{\alpha,\ell}(u,\varphi^{(\beta)}) &\le \epsilon^{-|\beta|}\prod_{\lambda=1}^{|\beta|}(p|\alpha|+\ell+\lambda)^s\Big\{\sum_{(j,\gamma)\in\mathcal{I}_{|\alpha|,\ell}}\prod_{\rho=1}^{2^{|\alpha|+1}} N_{j_\rho,\gamma_\rho+\beta}(u,\varphi)\Big\}^{2^{-(|\alpha|+1)}}\\ &\le \epsilon^{-|\beta|}\prod_{\lambda=1}^{|\beta|}(p|\alpha|+\ell+\lambda)^s\Big\{\sum_{(i,\delta)\in\mathcal{I}_{|\alpha|,\ell+|\beta|}}\prod_{\rho=1}^{2^{|\alpha|+1}} N_{i_\rho,\delta_\rho}(u,\varphi)\Big\}^{2^{-(|\alpha|+1)}}.\end{aligned} \tag{5-25}$$

Hence the proof is finished using

$$(p|\alpha|)!^s \prod_{\lambda=1}^{|\beta|}(p|\alpha|+\ell+\lambda)^s \le (p|\alpha|+\ell+|\beta|)!^s. \tag{5-26}$$ □

Let us now state our main theorem.

Theorem 5.5. *Let Ω_0 be a relatively compact open subset in Ω such that*

$$\text{type}_X(\overline{\Omega}_0) = p$$

and set $\sigma = 1/p$. Assume that the coefficients of P are in $G^s(\overline{\Omega}_0)$ for some $s \ge 1$. For every $1 \ge \epsilon > 0$, there exists $M_\epsilon = M(\Omega_0, P, \epsilon)$ such that, for every $\alpha \in \mathbb{N}^n$, every couple $(u, \varphi) \in C^\infty(\Omega_0) \times \mathfrak{D}(\Omega_0)$,

$$\|\varphi\partial^\alpha u\| \le M_\epsilon^{2p|\alpha|+1}(p|\alpha|)!^s \mathcal{N}_\alpha^\epsilon(u, \varphi) \tag{5-27}$$

and

$$\|\varphi\partial^\alpha u\|_{\ell\sigma} \le M_\epsilon^{2p|\alpha|+\ell+1}(p|\alpha|+\ell)!^s \mathcal{N}_{\alpha,\ell}^\epsilon(u, \varphi), \quad \ell = 1, \dots, p. \tag{5-28}$$

Proof of Theorem 5.5. Our proof will be inductive on $|\alpha|$. In each step of the induction, we will find a condition or conditions on M_ϵ in order (5-27) and (5-28), $\ell = 1, \dots, p$, are valid. The key point is to show that these conditions do not depend on $|\alpha|$.

(A) Proof of (5-27) and (5-28), for $\alpha = 0$ and $\ell = 1, \dots, p$.

(1) Proof of (5-27) for $\alpha = 0$: We have just to observe that

$$\mathcal{N}_0^\epsilon(u, \varphi) = N_{0,0}^\epsilon(u, \varphi) = \|\varphi u\|.$$

So we just need

$$M_\epsilon \ge 1, \tag{5-29}$$

for the validity of (5-27).

(2) Proof of (5-28), $\alpha = 0$ and $\ell = 1, \dots, p$: In view of the above, $M_\epsilon \ge 1$ gives (5-27); so we have to show (5-28) for $\ell = 1$ and, by induction on ℓ (finite induction here), show that if (5-28) holds for $1 \le \ell \le i$, then (5-28) holds for $\ell = i+1$, under a condition or conditions on M_ϵ. In order to prove (5-28) for $\ell = 1$, we use the localized estimate (4-1), which is

$$\|\varphi u\|_\sigma \le C\Big(\|\varphi Pu\|\|\varphi u\| + \sum_{|\beta_1|+|\beta_2|\le 2} \|\varphi^{(\beta_1)}\|\|\varphi^{(\beta_2)}u\|\Big)^{1/2}. \tag{5-30}$$

But since

$$\mathcal{N}^{\epsilon}_{0,1} = \Bigg(\sum_{(j,\gamma)\in \mathcal{I}_{0,1}} \epsilon^{|\gamma|+2|j|} \prod_{\rho=1}^{2} N_{j_\rho,\gamma_\rho} \Bigg)^{1/2}, \tag{5-31}$$

we see that

$$\|\varphi u\|_{\sigma} \le C\epsilon^{-1}\, \mathcal{N}^{\epsilon}_{0,1}(u,\varphi). \tag{5-32}$$

Hence we get (5-28) for $\ell = 1$ if

$$M_\epsilon \ge (C\epsilon^{-1})^{1/2}. \tag{5-33}$$

To continue the proof of A), we assume that (5-28) is true for $1 \le \ell \le i,\ 1 \le i \le p-1$; we want to prove (5-28) for $\ell = i+1$. For that, we use the localized estimate (4-2) for $\ell = i$:

$$\|\varphi u\|_{(i+1)\sigma} \le C\Bigg(\|\varphi P u\|_{(i-1)\sigma} + \sum_{|\beta|\le 1} \|\varphi^{(\beta)} u\|_{i\sigma} + \sum_{|\beta|=2} \|\varphi^{(\beta)u}\|_{(i-1)\sigma} \Bigg). \tag{5-34}$$

Applying induction to the terms in the second member of (5-34):

$$\|\varphi P u\|_{(i-1)\sigma} \le M^{i}_{\epsilon}(i-1)!^{s}\mathcal{N}^{\epsilon}_{0,i-1}(Pu,\varphi).$$

Then, using properties in (5-10), we get

$$\begin{aligned} \|\varphi P u\|_{(i-1)\sigma} &\le \epsilon^{-2} M^{i}_{\epsilon}(i+1)!^{s}\mathcal{N}^{\epsilon}_{0,i+1}(u,\varphi), \quad \text{and similarly have} \\ \|\varphi^{(\beta)} u\|_{i\sigma} &\le M^{i+1}_{\epsilon}\, \epsilon^{-1}(i+1)!^{s}\mathcal{N}^{\epsilon}_{0,i+1}(u,\varphi), \quad |\beta| \le 1, \\ \|\varphi^{(\beta)} u\|_{(i-1)\sigma} &\le \epsilon^{-2}\, M^{i}_{\epsilon}(i+1)!^{s}\mathcal{N}^{\epsilon}_{0,i+1}(u,\varphi), \qquad |\beta| = 2. \end{aligned} \tag{5-35}$$

So, coming back to (5-34), we obtain

$$\begin{aligned} \|\varphi u\|_{(i+1)\sigma} &\le C((1+n^2)\epsilon^{-2}M^{i}_{\epsilon} + (1+n)\epsilon^{-1}M^{i+1}_{\epsilon})(i+1)!^{s}\mathcal{N}^{\epsilon}_{0,i+1}(u,\varphi) \\ &\le M^{(i+1)+1}_{\epsilon}(i+1)!^{s}\mathcal{N}^{\epsilon}_{0,i+1}(u,\varphi), \end{aligned} \tag{5-36}$$

if the following condition is satisfied, as $1+n \le 1+n^2$:

$$C(1+n^2)(\epsilon M_\epsilon)^{-i}((\epsilon M_\epsilon)^{-1}+1) \le 1. \tag{5-37}$$

So, summarizing the conditions needed in order to prove (A), namely (5-27) and (5-28), $\ell = 1, \dots, p$, these are given in (5-29), (5-33) and (5-37). It is easy to see that (5-37) is deduced from

$$C(\epsilon M_\epsilon)^{-i} \le \frac{1}{2(1+n^2)}, \quad i \in \{1, \dots, p-1\}. \tag{5-38}$$

So, the simple condition (5-38) imply (5-27) and (5-28).

(B) Proof of (5-27) and (5-28) for $|\alpha| \ge 1$.

As we saw above, (5-27) and (5-28) are true for $\alpha = 0$. So we will use induction on $|\alpha| = \mu$; more precisely, assuming that (5-27) and (5-28), $\ell = 1, \dots, p$, are true for $|\alpha| \le \mu$, we will prove them for $|\alpha| = \mu + 1$.

(1) Proof of (5-27) for $\alpha = \beta + k$ for $|\beta| = \mu$ and $k \in \{1, \dots, n\}$: We have

$$\partial^\alpha = \partial^{\beta+k} = \partial_k \partial^\beta, \quad \text{where } \partial_k = \frac{\partial}{\partial x_k}.$$

So

$$\|\varphi \partial^{\beta+k} u\| = \|\varphi \partial_k \partial^\beta u\| \le \|\partial_k(\varphi)\partial^\beta u\| + \|\partial_k \varphi \partial^\beta u\|. \tag{5-39}$$

So,

$$\|\varphi \partial^{\beta+k} u\| \le \|\varphi^{(k)} \partial^\beta u\| + \|\varphi \partial^\beta u\|_{p\sigma}, \quad \text{with } \varphi^{(k)} = \partial_k \varphi. \tag{5-40}$$

We see that is suffices to apply (5-27) (with α replaced by β) for the couple $(u, \varphi^{(k)})$ and (5-28) with $\ell = p$ for the couple (u, φ). Hence,

$$\|\varphi^{(k)} \partial^\beta u\| \le M_\epsilon^{2p|\beta|+1} (p|\beta|)!^s \mathcal{N}_\beta^\epsilon(u, \varphi^{(k)}). \tag{5-41}$$

So from (5-10), we get:

$$\|\varphi^{(k)} \partial^\beta u\| \le M_\epsilon^{2p|\beta|+1} \epsilon^{-1} (p|\beta| + 1)!^s \mathcal{N}_{\beta,1}^\epsilon(u, \varphi). \tag{5-42}$$

Now,

$$\begin{aligned} \|\varphi \partial^\beta u\|_{p\sigma} &\le M_\epsilon^{2p|\beta|+p+1} (p|\beta| + p)!^s N_{\beta,p}^\epsilon(u, \varphi) \\ &\le M_\epsilon^{2p(|\beta|+1)+1} M_\epsilon^{-p} (p(|\beta| + 1))!^s N_{\beta+k}^\epsilon(u, \varphi) \end{aligned} \tag{5-43}$$

The last line in (5-43) is true using $\mathcal{N}_{\beta,p}^\epsilon = \mathcal{N}_{\beta+k}^\epsilon$ (see (5-6)).

Then gathering (5-42) and (5-43), we get, using (5-6),

$$\|\varphi \partial^{\beta+k} u\| \le M_\epsilon^{2p(|\beta|+1)+1} (\epsilon^{-1} M_\epsilon^{-2p} + M_\epsilon^{-p}) (p(|\beta| + 1))!^s \mathcal{N}_{\beta+k}^\epsilon(u, \varphi). \tag{5-44}$$

Hence, finally, we obtain (5-27) for $\alpha = \beta + k$ if

$$(\epsilon^{-1} M_\epsilon^{-p} + 1) M_\epsilon^{-p} \le 1. \tag{5-45}$$

For that, it suffices to take the simple condition

$$(\epsilon M_\epsilon^p)^{-1} \le \tfrac{1}{2}, \tag{5-46}$$

as it is easy to see that $(\epsilon M_\epsilon^p)^{-1} \le \frac{1}{2}$ and $M_\epsilon^{-p} \le \frac{1}{2}$ $(\epsilon \le 1)$. So under the condition (5-46), the inequality (5-27) holds true for $|\alpha| = \mu + 1$.

(2) Proof of (5-28) for $|\alpha| = \mu + 1$: It will be the longest proof of our theorem. Let us stay a moment, saying that we proved (5-27) and (5-28) for $|\alpha| \le \mu$, $\ell = 1 = 1, \dots, p$ and also (5-27) for $|\alpha| = \mu + 1$, properties that we will use in the sequel.

As there is a difference between the localized estimate (4-1) and the localized estimate (4-2) for $\ell = 1, \dots, p-1$, we first prove (5-28) with $\ell = 1$, for all α satisfying $|\alpha| = \mu + 1$. For that, we apply (4-1) in order to bound $\|\varphi\partial^\alpha u\|_\sigma$. Looking at the second member, we need:

(i) *A bound on* $E_1 = (\|\varphi\partial^\alpha Pu\|\|\varphi\partial^\alpha u\|)^{1/2}$. We have to bound two terms, using (5-27) with $|\alpha| = \mu + 1$ respectively to the couple (Pu, φ) and (φ, u). So, using (5-10), we obtain

$$\begin{aligned} \|\varphi\partial^\alpha Pu\| &\le M_\epsilon^{2p|\alpha|+1}(p|\alpha|)!^s \mathcal{N}_\alpha^\epsilon(Pu, \varphi) \\ &\le \epsilon^{-2} M_\epsilon^{2p|\alpha|+1}(p|\alpha|+2)!^s \mathcal{N}_{\alpha,1}^\epsilon(u, \varphi). \end{aligned} \tag{5-47}$$

So we deduce

$$\|\varphi\partial^\alpha Pu\|\|\varphi\partial^\alpha u\| \le (M_\epsilon^{2p|\alpha|+1})^2 \epsilon^{-2}(p|\alpha|+2)!^s (p|\alpha|)!^s \mathcal{N}_\alpha^\epsilon(u, \varphi)\mathcal{N}_{\alpha,1}^\epsilon(u, \varphi). \tag{5-48}$$

Then, using (5-6), we obtain

$$\begin{aligned} (\|\varphi\partial^\alpha Pu\|\|\varphi\partial^\alpha u\|)^{1/2} &\le \epsilon^{-1} M_\epsilon^{2p|\alpha|+1}((p|\alpha|+2)!(p|\alpha|)!)!^{s/2}\mathcal{N}_{\alpha,1}^\epsilon(u, \varphi) \\ &\le 2^{s/2}\epsilon^{-1} M_\epsilon^{2p|\alpha|+1}((p|\alpha|+1)!)!^s \mathcal{N}_{\alpha,1}^\epsilon(u, \varphi) \\ &\le 2^{s/2}\epsilon^{-1} M_\epsilon^{-1}\big(M_\epsilon^{2p|\alpha|+1+1}((p|\alpha|+1)!)!^s \mathcal{N}_{\alpha,1}^\epsilon(u, \varphi)\big) \end{aligned} \tag{5-49}$$

(ii) *A bound on* $E_2 = \sum_{|\beta_1|+|\beta_2|\le 2}(\|\varphi^{(\beta_1)}u\|\|\varphi^{(\beta_2)}u\|)^{1/2}$. Similarly, we get, using (5-10),

$$\begin{aligned} &(\|\varphi^{(\beta_1)}u\|\|\varphi^{(\beta_2)}u\|)^{1/2} \\ &\qquad\le \epsilon^{-1} M_\epsilon^{2p|\alpha|+1}((p|\alpha|+|\beta_1|)!(p|\alpha|+|\beta_2|)!)!^{s/2}\mathcal{N}_{\alpha,1}^\epsilon(u, \varphi) \\ &\qquad\le 2^{s/2}\epsilon^{-1} M_\epsilon^{-1}\big(M_\epsilon^{2p|\alpha|+1+1}((p|\alpha|+1)!)!^s \mathcal{N}_{\alpha,1}^\epsilon(u, \varphi)\big) \end{aligned} \tag{5-50}$$

(iii) *A bound on* $E_3 = \sum_{|\beta|\le 1}(\|\varphi^{(\beta)}[X, \partial^\alpha]u\|\|\varphi\partial^\alpha u\|)^{1/2}$. There are $m(n+1)$ terms in (iii) as X represents the vector fields $X_1, \dots, X_m$. So, let us bound any one of them. Moreover in (3-8), we delete the subscript "j" in the expression of $[X, \partial^\alpha]$: $\sum_{\gamma<\alpha,\, \ell=1,\dots,n} a_{\alpha\gamma\ell}\, \partial^{\gamma+\ell}$ and the estimates (3-9) and (3-10). So we have

$$\|\varphi^{(\beta)}[X, \partial^\alpha]u\| \le \sum_{\substack{\gamma<\alpha \\ \ell=1,\dots,n}} \|a_{\alpha\gamma\ell}\varphi^{(\beta)}\partial^{\gamma+\ell}u\|. \tag{5-51}$$

Hence, using the estimates in Proposition 3.3, we get, with $v = \varphi^{(\beta)}\partial^{\gamma+\ell}u$,

$$\|\varphi^{(\beta)}[X, \partial^\alpha]u\| \le \sum_{\substack{\gamma<\alpha \\ \ell=1,\dots,n}} B^{|\alpha-\gamma|}\Big(\frac{\alpha!}{\beta!}\Big)^s \|\varphi^{(\beta)}\partial^{\gamma+\ell}u\|, \quad |\beta| \le 1.$$

As $\gamma < \alpha$, $|\gamma + \ell| = |\gamma| + 1 \le |\alpha|$. So we have, using (5-27) for the multiindex $\gamma + \ell$,

$$\|\varphi^{(\beta)}[X, \partial^\alpha]u\| \le n \sum_{\gamma<\alpha} B^{|\alpha-\gamma|}\Big(\frac{\alpha!}{\gamma!}\Big)^s M_\epsilon^{2p(|\gamma|+1)+1}(p(|\gamma|+1))!^s \mathcal{N}^\epsilon_{|\gamma|+1}(u, \varphi^{(\beta)}). \tag{5-52}$$

Now, one has, easily,

$$\frac{\alpha!}{\gamma!}(p(|\gamma|+1))! \le (p|\alpha|+1)! \tag{5-53}$$

Hence (5-52) and (5-53) yield

$$\begin{aligned}\|\varphi^{(\beta)}[X, \partial^\alpha]u\| &\le n \sum_{\gamma<\alpha} B^{|\alpha-\gamma|} M_\epsilon^{2p(|\gamma|+1)+1}(p|\alpha|+1)!^s \mathcal{N}^\epsilon_{|\gamma|+1}(u, \varphi^{(\beta)}) \\ &\le n \sum_{\gamma<\alpha} B^{|\alpha\gamma|}\epsilon^{-1} M_\epsilon^{2p(|\gamma|+1)+1}\,(p|\alpha|+2)!^s \mathcal{N}^\epsilon_{\alpha,1}(u, \varphi).\end{aligned} \tag{5-54}$$

The last line is derived from (5-10) and the fact $|\gamma| + 1 \le |\alpha|$. Now we use the following

$$\sum_{\gamma<\alpha} B^{|\alpha-\gamma|} M_\epsilon^{2p(|\gamma|+1)+1} = B \sum_{\gamma<\alpha}\Big(\frac{B}{M_\epsilon^{2p}}\Big)^{|\alpha-\gamma|-1} M_\epsilon^{2p|\alpha|+1} \tag{5-55}$$

Let us recall now the following lemma we proved in [Derridj 2019a, Lemma 4.3].

Lemma 5.6. *There exists $\theta_0 > 0$, independent of α, such that*

$$\sum_{\gamma<\alpha} \lambda^{|\alpha-\gamma|-1} \le n+1, \quad \text{if } 0 \le \lambda < \theta_0. \tag{5-56}$$

So, taking $B < \theta_0 M_\epsilon^{2p}$, we get from (5-54)–(5-56)

$$\|\varphi^{(\beta)}[X, \partial^\alpha]u\| \le n\epsilon^{-1} B(n+1) M_\epsilon^{2p|\alpha|+1}(p|\alpha|+2)!^s \mathcal{N}^\epsilon_{\alpha,1}(u, \varphi). \tag{5-57}$$

We rewrite the above condition we used to get (5-57) as follows:

$$M_\epsilon^{2p} > B\theta_0^{-1}. \tag{5-58}$$

Now we deduce from (5-57),

$$\begin{aligned}\sum_{|\beta|\le 1} (\|\varphi^{(\beta)}[X, \partial^\alpha]u\| \|\varphi\partial^\alpha u\|)^{1/2} &\le (n+1)\big\{n\epsilon^{-1} B(n+1) M_\epsilon^{2p|\alpha|+1}(p|\alpha|+2)!^s \mathcal{N}^\epsilon_{\alpha,1}(u, \varphi)\big\}^{1/2} \\ &\quad \cdot \big\{M_\epsilon^{2p|\alpha|+1}(p|\alpha|)!^s \mathcal{N}^\epsilon_{\alpha}(u, \varphi)\big\}^{1/2}.\end{aligned} \tag{5-59}$$

Using again properties in (5-6) and (5-10) we get

$$\begin{aligned} E_3 &\le (n+1)(n(n+1)\epsilon^{-1}B)^{1/2}M^{2p|\alpha|+1}(p|\alpha|)!^s\,(p|\alpha|+2)!^s\mathcal{N}^{\epsilon}_{\alpha,1}(u,\varphi)\\ &\le (n+1)(n(n+1)\epsilon^{-1}B)^{1/2}2^{s/2}M^{2p|\alpha|+1}(p|\alpha|+1)!^s\mathcal{N}^{\epsilon}_{\alpha,1}(u,\varphi). \end{aligned}$$

Now taking into account that X stands for all X_j's, $j=1,\dots,m$ and Y, we finally get

$$\sum_{j=1}^{m}\sum_{|\beta|\le 1}\big(\|\varphi^{(\beta)}[X_j,\partial^\alpha]u\|\|\varphi\partial^\alpha u\|\big)^{1/2}\le(*), \tag{5-60}$$

where $(*)=m(n+1)(2^s n(n+1)\epsilon^{-1}B)^{1/2}M_\epsilon^{-1}\cdot M^{2p|\alpha|+1+1}(p|\alpha|+1)!^s\mathcal{N}^{\epsilon}_{\alpha,1}(u,\varphi)$,

and $\|\varphi[Y+b,\partial^\alpha]u\|\|\varphi\partial^\alpha u\|^{1/2}\le(*)$.

(iv) *A bound on* $E_4=(\|\varphi[X[X,\partial^\alpha]]u\|\,\|\varphi\partial^\alpha u\|)^{1/2}$. It will be done as for (iii), the only difference being in the expression

$$[X,[X,\partial^\alpha]]=\sum_{\substack{\gamma<\alpha\\ k=1,\dots,n}}d_{k\alpha\gamma}\partial^{\gamma+k}+\sum_{\substack{\gamma<\alpha\\ \ell,k=1,\dots,n}}c_{\ell k\alpha\gamma}\partial^{\gamma+\ell+k}, \tag{5-61}$$

with estimates given in Proposition 3.3.

As in (iii) we have

$$\begin{aligned} &\|\varphi[X,[X,\partial^\alpha]]u\|\\ &\le\sum_{\gamma<\alpha}nB^{|\alpha-\gamma|}\Big((|\alpha|+1)\frac{\alpha!}{\gamma!}\Big)^s M_\epsilon^{2p(|\gamma|+1)+1}(p(|\gamma|+1))!^s\mathcal{N}^{\epsilon}_{|\gamma|+1}(u,\varphi)\\ &\quad+n\sum_{\substack{\gamma<\alpha,\,|\gamma|\le|\alpha|-2\\ k=1,\dots,n}}B^{|\alpha-\gamma|}\Big(\frac{(\alpha+k)!}{(\gamma+k)!}\Big)^s M_\epsilon^{2p(|\gamma|+2)+1}(p(|\gamma|+2))!^s\mathcal{N}^{\epsilon}_{|\gamma|+2}(u,\varphi). \end{aligned} \tag{5-62}$$

All we have now to use are the following ingredients:

$$\begin{aligned} &(|\alpha|+1)\frac{\alpha!}{\gamma!}(p(|\gamma|+1))!\le(p(|\alpha|+2))!, &&\text{when } |\gamma|+1\le|\alpha|.\\ &\frac{(\alpha+k)!}{(\gamma+k)!}(p(|\gamma|+2))!\le(p(|\alpha|+2))!, &&\text{when } |\gamma|+2\le|\alpha|. \end{aligned} \tag{5-63}$$

and under condition (5-58)

$$\begin{aligned} &\sum_{\gamma<\alpha}B^{|\alpha-\gamma|}M_\epsilon^{2p(|\gamma|+1)+1}=B\sum_{\gamma<\alpha}\Big(\frac{B}{M_\epsilon^{2p}}\Big)^{|\alpha-\gamma|-1}M_\epsilon^{2p|\alpha|+1}\le(n+1)BM_\epsilon^{2p|\alpha|+1},\\ &\sum_{\substack{\gamma<\alpha\\ |\gamma|\le|\alpha|-2}}B^{|\alpha-\gamma|}M^{2p(|\gamma|+2)+1}\le B^2\Big(\frac{B}{M^{2p}}\Big)^{|\alpha-\gamma|-2}M_\epsilon^{2p|\alpha|+1}\le(n^2+1)B^2M_\epsilon^{2p|\alpha|+1}. \end{aligned} \tag{5-64}$$

The second line (5-64) follows from Lemma 5.6 with

$$\sum_{\substack{\gamma<\alpha \\ |\gamma|\le|\alpha|-2}} \lambda^{|\alpha-\gamma|} \le n^2+1.$$

So from (5-62)-(5-64), we obtain

$$\begin{aligned}(5\text{-}65)\quad & (\|\varphi[X,[X,\partial^\alpha]]u\|\,\|\varphi\partial^\alpha u\|)^{1/2} \\ &\le \big((n(n+1)B+n^2(n^2+1)B^2)(p|\alpha|+2)!^s\big)^{1/2} \\ &\qquad \cdot (M_\epsilon^{2p|\alpha|+1})^{1/2}(M_\epsilon^{2p|\alpha|+1}(p|\alpha|)!^s)^{1/2}\mathcal{N}_\alpha^\epsilon(u,\varphi).\end{aligned}$$

as $\mathcal{N}^\epsilon_{|\gamma|+1}(u,\varphi)\le\mathcal{N}^\epsilon_\alpha(u,\varphi),\ |\gamma|+1\le|\alpha|,\ \mathcal{N}^\epsilon_{|\gamma|+2}(u,\varphi)\le\mathcal{N}^\epsilon_\alpha(u,\varphi),\ |\gamma|+2\le|\alpha|$.

Then finally we get

$$(\|\varphi[X,[X,\partial^\alpha]]u\|\,\|\varphi\partial^\alpha u\|)^{1/2}\le 2^s n(n+1)BM_\epsilon^{-1}\cdot M_\epsilon^{2p|\alpha|+1+1}(p|\alpha|+1)!^s\mathcal{N}^\epsilon_\alpha(u,\varphi).$$

Now, collecting the bounds obtained in (i)–(iv), we obtain

$$\begin{aligned}(5\text{-}66)\quad & \|\varphi\partial^\alpha u\|_\sigma \\ &\le \big\{2^{s/2+1}\epsilon^{-1}+2^{s/2}mn(n+1)^2(\epsilon^{-1}B)^{1/2}+2^s n(n+1)B\big\} \\ &\qquad \cdot CM_\epsilon^{-1}\cdot M_\epsilon^{2p|\alpha|+1+1}(p|\alpha|+1)!^s\mathcal{N}^\epsilon_{\alpha,1}(u,\varphi),\end{aligned}$$

under the condition (5-58), and C given in (4-1). Now we rewrite in a simpler manner

$$(5\text{-}67)\quad \|\varphi\partial^\alpha u\|_\sigma\le A(s,\epsilon,\Omega_0,P)CM_\epsilon^{-1}\cdot M_\epsilon^{2p|\alpha|+1+1}(p|\alpha|+1)!^s\mathcal{N}^\epsilon_{\alpha,1}(u,\varphi).$$

So we see that (5-28) for $\ell=1$ is true under the condition

$$(5\text{-}68)\qquad M_\epsilon\ge\sup\big\{CA(s,\epsilon,\Omega_0,P),(B\theta_0^{-1})^{1/(2p)}\big\},$$

in view of (5-58), (5-67).

So, our work now will be the proof of (5-28) for $|\alpha|=\mu+1,\ \ell=1,\dots,p$. For that, we assume that (5-28) is true for $|\alpha|=\mu+1,\ 1\le\ell\le i,\ i\in\{1,\dots,p-1\}$; we want to prove (5-28) for $\ell=i+1$. Here the proof will be simpler than the preceding, as when using the localized estimate (4-2) with $\ell\in\{1,\dots,p-1\}$, one has no square root of products of norms. Moreover, some estimates are quite done in the preceding proof of (5-28) for $\ell=1$. We first use (4-2) for $\ell=i$ then looking at the second member of (4-2) with $\ell=i$, we have, as in the preceding, to bound terms, which are, here, simpler as they are norms of some functions in some Sobolev spaces, in place of square roots of products. Let us list them.

(i) *A bound on* $\|\varphi\partial^\alpha Pu\|_{(i-1)\sigma}$. The bound is, simply, given by applying (5-28) with $\ell=i-1$ to the couple (Pu,φ). So

$$(5\text{-}69)\qquad \|\varphi\partial^\alpha Pu\|_{(i-1)\sigma}\le M_\epsilon^{2p|\alpha|+i}(p|\alpha|+i-1)!^s\mathcal{N}^\epsilon_{\alpha,i-1}(Pu,\varphi).$$

Then using (5-10), we get, as $i-1+2=i+1\le p$,

$$\begin{aligned}\|\varphi\partial^\alpha Pu\|_{(i-1)\sigma} &\le \epsilon^{-2}M^{2p|\alpha|+i}(p|\alpha|+i+1)!^s\mathcal{N}^\epsilon_{\alpha,i+1}(u,\varphi)\\ &\le \epsilon^{-2}M_\epsilon^{-2}M_\epsilon^{2p|\alpha|+i+1+1}(p|\alpha|+i+1)!^s\mathcal{N}^\epsilon_{\alpha,i+1}(u,\varphi).\end{aligned} \tag{5-70}$$

(ii) *A bound on* $\sum_{|\beta|\le 1}\|\varphi^{(\beta)}\partial^\alpha u\|_{i\sigma}$. Similarly, we have by applying (5-28) for $\ell=i$ to $(u,\varphi^{(\beta)})$,

$$\|\varphi^{(\beta)}\partial^\alpha u\|_{i\sigma}\le M_\epsilon^{2p|\alpha|+i+1}\,(p|\alpha|+i)!^s\mathcal{N}^\epsilon_{\alpha,i}(u,\varphi^{(\beta)}). \tag{5-71}$$

Then using (5-10), one has, as $i+|\beta|\le i+1\le p$,

$$\sum_{|\beta|\le 1}\|\varphi^{(\beta)}\partial^\alpha u\|_{i\sigma}\le (n+1)\epsilon^{-1}M_\epsilon^{-1}M_\epsilon^{2p|\alpha|+i+1+1}(p|\alpha|+i+1)!^s\mathcal{N}^\epsilon_{\alpha,i+1}(u,\varphi). \tag{5-72}$$

(iii) *A bound on* $\sum_{|\beta|=2}\|\varphi^{(\beta)}\partial^\alpha u\|_{(i-1)\sigma}$. We use exactly the same way, getting successively:

$$\|\varphi^{(\beta)}\partial^\alpha u\|_{(i-1)\sigma}\le M_\epsilon^{2p|\alpha|+i}(p|\alpha|+i-1)!^s\mathcal{N}^\epsilon_{\alpha,i-1}(u,\varphi^{(\beta)}),\qquad \text{as } i-1+|\beta|\le i+1, \tag{5-73}$$

$$\sum_{|\beta|=2}\|\varphi^{(\beta)}\partial^\alpha u\|_{(i-1)\sigma}\le n^2\epsilon^{-2}M_\epsilon^{-2}M_\epsilon^{2p|\alpha|+i+1+1}(p|\alpha|+i+1)!^s\mathcal{N}^\epsilon_{\alpha,i+1}(u,\varphi). \tag{5-74}$$

(iv) *A bound on*

$$\sum_{\substack{|\beta|\le 1\\ j=1,\dots,m}}\|\varphi^{(\beta)}[X_j,\partial^\alpha]u\|_{(i-1)\sigma}\quad\text{and}\quad \|\varphi[Y+b,\partial^\alpha]u\|_{(i-1)\sigma}.$$

We have just to consider $\|\varphi^{(\beta)}[X,\partial^\alpha]u\|_{(i-1)\sigma}$, $|\beta|\le 1$. So:

$$\|\varphi^{(\beta)}[X,\partial^\alpha]u\|_{(i-1)\sigma}\le\sum_{\substack{\gamma<\alpha\\ \ell=1,\dots,m}}B^{|\alpha-\gamma|}\Big(\frac{\alpha!}{\gamma!}\Big)^s\|\varphi^{(\beta)}\partial^{\gamma+\ell}u\|_{(i-1)\sigma}\quad\text{(see (3-8))}. \tag{5-75}$$

So applying (5-28) with $\ell=i-1$ and $\alpha=\gamma+\ell$ ((5-27) with $\alpha=\gamma+\ell$ when $i=1$) to $(u,\varphi^{(\beta)})$:

$$\|\varphi^{(\beta)}[X,\partial^\alpha]u\|_{(i-1)\sigma}\le\sum_{\substack{\gamma<\alpha\\ \ell=1,\dots,n}}B^{|\alpha-\gamma|}\Big(\frac{\alpha!}{\gamma!}\Big)^s M_\epsilon^{2p(|\gamma|+1)+i}\cdot(p(|\gamma|+1)+i-1)!^s\mathcal{N}^\epsilon_{\gamma+\ell,i-1}(u,\varphi^{(\beta)}),\qquad |\beta|\le 1. \tag{5-76}$$

Then using

$$\frac{\alpha!}{\gamma!}(p(|\gamma|+1)+i-1)!\le(p|\alpha|+i)!, \tag{5-77}$$

(5-76) gives, with $|\beta|\le 1$,

$$\begin{aligned}(5\text{-}78)\quad &\|\varphi^{(\beta)}[X,\partial^\alpha]u\|_{(i-1)\sigma}\\ &\le n\sum_{\gamma<\alpha} B^{|\alpha-\gamma|}M_\epsilon^{2p(|\gamma|+1)+i}(p|\alpha|+i)!^s\mathcal{N}^\epsilon_{\gamma+\ell,i-1}(u,\varphi^{(\beta)})\\ &\le n\sum_{\gamma<\alpha}\epsilon^{-1}B^{|\alpha-\gamma|}M_\epsilon^{2p(|\gamma|+1)+i}(p|\alpha|+i+1)!^s\mathcal{N}^\epsilon_{\gamma+\ell,i}(u,\varphi), \qquad (5\text{-}10)\\ &\le n\sum_{\gamma<\alpha}B\Big(\frac{B}{M_\epsilon^{2p}}\Big)^{|\alpha-\gamma|-1}\epsilon^{-1}M_\epsilon^{2p|\alpha|+i}(p|\alpha|+i+1)!^s\mathcal{N}^\epsilon_{\alpha,i}(u,\varphi) \quad (5\text{-}6)\\ &\le n(n+1)\epsilon^{-1}BM_\epsilon^{-2}M_\epsilon^{2p|\alpha|+i+1+1}(p|\alpha|+i+1)!^s\mathcal{N}^\epsilon_{\alpha,i+1}(u,\varphi) \qquad (5\text{-}6)\end{aligned}$$

Hence

$$\begin{aligned}(5\text{-}79)\quad \sum_{|\beta|=1}&\|\varphi^{(\beta)}[X,\partial^\alpha]u\|_{(i-1)\sigma}\\ &\le (n+1)^3\epsilon^{-1}BM_\epsilon^{-2}M_\epsilon^{2p|\alpha|+i+1+1}(p|\alpha|+i+1)!^s\mathcal{N}^\epsilon_{\alpha,i+1}(u,\varphi)\end{aligned}$$

Of course the term $\|\varphi[Y+b,\partial^\alpha]u\|$ is easier to handle.

(v) *A bound on* $\sum_{j=1}^m\|\varphi[X_j,[X_j,\partial^\alpha]]u\|_{(i-1)\sigma}$. As in (5-62) where i corresponds to 1, we get

$$\begin{aligned}(5\text{-}80)\quad &\|\varphi[X,[X,\partial^\alpha]]u\|_{(i-1)\sigma}\\ &\le\sum_{\gamma<\alpha}nB^{|\alpha-\beta|}\Big((|\alpha|+1)\frac{\alpha!}{\beta!}\Big)^s M_\epsilon^{2p(|\gamma|+1)+i}(p(|\gamma|+1)+i-1)!^s\mathcal{N}^\epsilon_{|\gamma|+1,i-1}(u,\varphi)\\ &\quad+n\sum_{\substack{\gamma<\alpha,\,|\gamma|\le|\alpha|-2\\ k=1,\dots,n}}B^{|\alpha-\gamma|}\Big(\frac{(\alpha+k)!}{(\gamma+k)!}\Big)^s M_\epsilon^{2p(|\gamma|+2)+i}(p(|\gamma|+2)+i-1)!^s\\ &\qquad\cdot\mathcal{N}^\epsilon_{|\gamma|+2,i-1}(u,\varphi).\end{aligned}$$

Here, ingredients replacing (5-63) and (5-64) are

$$(5\text{-}81)\quad \begin{aligned}&(|\alpha|+1)\frac{\alpha!}{\gamma!}(p(|\gamma|+1)+i-1)!\le(p|\alpha|+i-1)!, \quad |\gamma|+1\le|\alpha|,\\ &\frac{(\alpha+k)!}{(\gamma+k)!}(p(|\gamma|+2)+i-1)!\le(p|\alpha|+i-1)!, \quad |\gamma|+2\le|\alpha|.\end{aligned}$$

Hence, under condition (5-58), we get

$$\begin{aligned}(5\text{-}82)\quad \sum_{\gamma<\alpha}B^{|\alpha-\gamma|}M_\epsilon^{2p(|\gamma|+1)+i}&=B\sum_{\gamma<\alpha}\Big(\frac{B}{M_\epsilon^{2p}}\Big)^{|\alpha-\gamma|-1}M_\epsilon^{2p|\alpha|+i}\\ &\le(n+1)BM_\epsilon^{2p|\alpha|+i}.\end{aligned}$$

And, again under condition (5-58), we have

$$\sum_{\substack{\gamma<\alpha \\ |\gamma|\le|\alpha|-2}} B^{|\alpha-\gamma|} M_\epsilon^{2p(|\gamma|+2)+i} = B^2 \sum_{\substack{\gamma<\alpha \\ |\gamma|\le|\alpha|-2}} \left(\frac{B}{M_\epsilon^{2p}}\right)^{|\alpha-\beta|-2} M_\epsilon^{2p|\alpha|+i} \le (n^2+1) B^2 M_\epsilon^{2p|\alpha|+i}. \tag{5-83}$$

From (5-80)–(5-83), we deduce

$$\|\varphi[X,[X,\partial^\alpha]]u\|_{(i-1)\sigma} \le \big(n(n+1)BM_\epsilon^{-2} + n(n^2+1)B^2M_\epsilon^{-2}\big) \cdot M_\epsilon^{2p|\alpha|+i+1+1}(p|\alpha|+i+1)!^s \mathcal{N}^\epsilon_{\alpha,i+1}(u,\varphi), \tag{5-84}$$

under condition (5-58). Now we collect all bounds in (i)–(v) to obtain

$$\begin{aligned}\|\varphi\partial^\alpha u\|_{(i+1)\sigma} &\le C\Big\{\epsilon^{-2}M_\epsilon^{-2} + (n+1)\epsilon^{-1}M_\epsilon^{-1} + n^2\epsilon^{-2}M_\epsilon^{-2} + (n+1)^3\epsilon^{-1} \\ &\qquad BM_\epsilon^{-2}(m+1) + (m+1)(n+1)^3B^2M_\epsilon^{-2}\Big\} \\ &\qquad \cdot M_\epsilon^{2p|\alpha|+i+1+1}(p|\alpha|+i+1)!^s \mathcal{N}^\epsilon_{\alpha,i+1}(u,\varphi)\end{aligned} \tag{5-85}$$

So we see that in order to have (5-28) for $\ell = i+1$, we need that the factor of $M^{2p|\alpha|+i+2}\,(p|\alpha|+i+1)!^s\mathcal{N}^\epsilon_{\alpha,i+1}(u,\varphi)$ in (5-85) be less than 1. As $M_\epsilon \ge 1$, $\epsilon \le 1$, so $M_\epsilon^{-2} \le M_\epsilon^{-1}$, it suffices that

$$CM_\epsilon^{-1} \cdot \big\{\epsilon^{-2}(n+2+(m+1)(n+1)^3B) + n^2 + (m+1)(n+1)^3B^2\big\} \le 1. \tag{5-86}$$

So, denoting by $D(\epsilon, B)$ the factor of M_ϵ^{-1} in (5-86):

$$\begin{aligned} &M_\epsilon \ge D(\epsilon,B), \quad D(\epsilon,B) = D(\epsilon,\Omega_0,P), \\ &M_\epsilon \ge (B\theta_0^{-1})^{1/(2p)} \quad \text{(which is condition (5-58)).}\end{aligned} \tag{5-87}$$

Now summarizing all conditions needed along our proof of the theorem, we have:

$$\begin{aligned} &M_\epsilon \ge 1, \quad \text{for validity of (5-27)}, \\ &M_\epsilon \ge (C\epsilon^{-1})^{1/2}, \quad \text{for validity of (5-28)}, \ \ell = 1, \\ &M_\epsilon \ge (2(1+n^2)\epsilon^{-1})^{p-1}, \quad \text{giving condition (5-38)}, \ i = 1,\dots,p-1, \\ &M_\epsilon \ge (2\epsilon^{-1})^{1/p}, \quad \text{which is condition (5-46)}, \\ &M_\epsilon \ge (B\theta_0^{-1})^{1/(2p)}, \quad \text{which is condition (5-58)}, \\ &M_\epsilon \ge \sup\{CA(s,\epsilon,\Omega_0,P), (B\theta_0^{-1})^{1/(2p)}\}, \quad \text{which is (5-68)}, \\ &M_\epsilon \ge D(\epsilon,\Omega_0,P), \quad \text{given in (5-87)}.\end{aligned} \tag{5-88}$$

Then, denoting by $M(\epsilon, s, \Omega_0, P)$ the maximum of all numbers in the list above, any $M_\epsilon \geq M(\epsilon, s, \Omega_0, P)$ satisfies the estimates (5-27) and (5-28) for $\alpha \in \mathbb{N}^n$ and $\ell = 1, \dots, p$ and $(u, \varphi) \in C^\infty(\Omega) \times \mathcal{D}(\Omega)$; so our theorem is completely proved. □

6. Application to Gevrey regularity of analytic (Gevrey) vectors

Before applying Theorem 5.5 to Hörmander's operators of first kind, we want to state a theorem for more general operators of order m satisfying localized estimates that are similar to (5-5) but with a modified definition of $\mathcal{N}^\epsilon_\alpha$, due to the order m of the differential operator still denoted by P, on the open set Ω. The modifications are done in the following, where $(u, \varphi) \in C^\infty(\Omega_0) \times \mathcal{D}(\Omega_0)$:

For $j \in \mathbb{N}$, $\gamma \in \mathbb{N}^n$, $\epsilon \in [0, 1]$, $\quad N^\epsilon_{j,\gamma}(u, \varphi) = \epsilon^{|\gamma|+mj} |\gamma|!^{-s} (mj)!^{-s} \|\varphi^\gamma P^j u\|$,

$$\text{For } \alpha \in \mathbb{N}^n, \quad \mathcal{N}^\epsilon_\alpha(u, \varphi) = \left\{ \sum_{(j,\gamma)\in \mathcal{I}_{|\alpha|}} \prod_{\rho=1}^{2^{|\alpha|}} N^\epsilon_{j_\rho,\gamma_\rho}(u, \varphi) \right\}^{2^{-|\alpha|}}, \tag{6-1}$$

where $\mathcal{I}_{|\alpha|} = (j, \gamma)$, $j = (j_1, \dots, j_{2^{|\alpha|}})$, $\gamma = (\gamma, \dots, \gamma_{2^{|\alpha|}})$ satisfying

$$|\gamma| + mj \leq 2^{|\alpha|} p|\alpha|, \; |\gamma_\rho| \leq q|\alpha|, \text{ for } (p, q) \text{ given in } \mathbb{N}^2.$$

So we define property $(\mathcal{P}_s)$ for the operator P by:

$$\begin{gathered} \text{For every } \epsilon \in [0, 1], \text{ there exists } M_\epsilon, \text{ such that,} \\ \text{for } \alpha \in \mathbb{N}^n, \; (u, \varphi) \in C^\infty(\Omega_0) \times \mathcal{D}(\Omega_0), \\ \|\varphi \partial^\alpha u\| \leq M_\epsilon^{|\alpha|+1} (p|\alpha|)!^s \mathcal{N}^\epsilon_\alpha(u, \varphi). \end{gathered} \tag{$\mathcal{P}_s$}$$

Theorem 6.1. *Assume that P satisfies $(\mathcal{P}_s)$, for some $s \geq 1$. Then every analytic* (case $s = 1$) *or s-Gevrey vector of P in Ω_0, is in $G^{ps}(\Omega_0)$.*

Proof. We distinguish between the cases $s > 1$ and $s = 1$. The case $s > 1$ is simpler as there exist elements in $G^s(\Omega_0)$ with compact support. More precisely, let u be a s-Gevrey vector.

(1) Case $s > 1$. In order to prove that u is in $G^{ps}(\Omega_0)$, we take any relatively compact open set Ω_1 in Ω_0 (i.e., $\Omega_1 \Subset \Omega_0$) and want to find a constant $D = D(\Omega_1)$ (depending on Ω_1) such that, for every $\alpha \in \mathbb{N}^n$,

$$\|\partial^\alpha u\|_{L^2(\Omega_1)} \leq D^{|\alpha|+1} (p|\alpha|)!^s. \tag{6-2}$$

For that, we pick a function $\varphi \in G^s(\Omega_0) \cap \mathcal{D}(\Omega_0)$, $\Omega_0 \Subset \Omega_1$, $\varphi = 1$ on Ω_1. So, there exists $B > 0$ such that

$$|\varphi^{(\beta)}| \leq B^{|\beta|+1} |\beta|!^s, \quad \beta \in \mathbb{N}^n. \tag{6-3}$$

Moreover as u is an s-Gevrey vector of P in Ω_0 ($u \in G^s(P, \Omega_0)$), we have, enlarging B if necessary,

$$\|P^j u\|_{L^2(\Omega_1)} \le B^{mj+1}(mj)!^s, \quad j \in \mathbb{N}. \tag{6-4}$$

So

$$\|\varphi^{(\beta)} P^j u\| \le B^2 B^{|\beta|+mj}|\beta|!^s (mj)!^s, \quad \beta \in \mathbb{N}^n,\ j \in \mathbb{N}. \tag{6-5}$$

Hence

$$N^{\epsilon}_{j,\gamma}(u, \varphi) \le \widetilde{B}(\epsilon\widetilde{B})^{|\gamma|+m|j|}, \quad \widetilde{B} = B^2. \tag{6-6}$$

Therefore

$$\begin{aligned}\mathcal{N}^{\epsilon}_{\alpha}(u, \varphi) &\le \Bigg[\sum_{(j,\gamma)\in\mathcal{I}_{|\alpha|}} \prod_{\rho=1}^{2^{|\alpha|}} \widetilde{B}(\epsilon\widetilde{B})^{|\gamma_\rho|+mj_\rho}\Bigg]^{2^{-|\alpha|}} \\ &\le \widetilde{B}\Bigg[\sum_{|\gamma|+m|j|\le 2^{|\alpha|}p|\alpha|} (\epsilon\widetilde{B})^{|\gamma|+m|j|}\Bigg]^{2^{-|\alpha|}}.\end{aligned} \tag{6-7}$$

Now choose ϵ_0 such that

$$\epsilon_0\widetilde{B} = \tfrac{1}{2} \Leftrightarrow \epsilon_0 = (2\widetilde{B})^{-1}, \tag{6-8}$$

we get

$$\begin{aligned}\mathcal{N}^{\epsilon_0}_{\alpha}(u, \varphi) &\le \widetilde{B}\Bigg\{\sum_{\substack{|\gamma|\le 2^{|\alpha|}p|\alpha| \\ m|j|\le 2^{|\alpha|}p|\alpha|}} \left(\tfrac{1}{2}\right)^{|\gamma|}\left(\tfrac{1}{2}\right)^{m|j|}\Bigg\}^{2^{-|\alpha|}} \\ &\le \widetilde{B}\Bigg\{\sum_{|\gamma|\le 2^{|\alpha|}p|\alpha|} \left(\tfrac{1}{2}\right)^{|\gamma|} \sum_{m|j|\le 2^{|\alpha|}p|\alpha|} \left(\tfrac{1}{2}\right)^{m|j|}\Bigg\}^{2^{-|\alpha|}}.\end{aligned} \tag{6-9}$$

Now in the two sums, $\gamma \in \mathbb{N}^{n2^{|\alpha|}}$ and $j \in \mathbb{N}^{2^{|\alpha|}}$. So

$$\begin{aligned}\sum_{|\gamma|\le 2^{|\alpha|}p|\alpha|} \left(\tfrac{1}{2}\right)^{|\gamma|} = \sum_{k=0}^{2^{|\alpha|}p|\alpha|}\Bigg(\sum_{|\gamma|=k} 1\Bigg)\left(\tfrac{1}{2}\right)^k &\le \sum_{k=0}^{2^{|\alpha|}p|\alpha|} (k+1)^{n2^{|\alpha|}}\left(\tfrac{1}{2}\right)^k \\ &\le (2^{|\alpha}|p|\alpha|+1)^{n2^{|\alpha|}}\sum_{k=0}^{\infty}\left(\tfrac{1}{2}\right)^k \\ &\le 2(2^{|\alpha|}p|\alpha|+1)^{n2^{|\alpha|}},\end{aligned} \tag{6-10}$$

$$\sum_{m|j|\le 2^{|\alpha|}p|\alpha|} \left(\tfrac{1}{2}\right)^{m|j|} \le (2^{|\alpha|}p|\alpha|+1)^{2^{|\alpha|}}\sum_{k=0}^{\infty}\left(\tfrac{1}{2}\right)^k \le 2(2^{|\alpha|}p|\alpha|+1)^{2^{|\alpha|}}.$$

Hence (6-9) and (6-10) imply

$$\text{(6-11)}\quad \mathcal{N}_\alpha^{\epsilon_0}(u,\varphi) \le \widetilde{B}2(2^{|\alpha|}p|\alpha|+1)^n \cdot 2(2^{|\alpha|}p|\alpha|+1) \le 4\widetilde{B}(2^{|\alpha|}p|\alpha|+1)^{n+1} \le A^{|\alpha|+1},$$

for some A. Now using property $(\mathcal{P}_s)$, we get

$$\text{(6-12)}\quad \|\partial^\alpha u\|_{L^2(\Omega_1)} \le \|\varphi\partial^\alpha u\| \le A^{|\alpha|+1}M_{\epsilon_0}^{|\alpha|+1}(p|\alpha|)!^s = (AM_{\epsilon_0})^{|\alpha|+1}(p|\alpha|)!^s.$$

This ends the proof in case $s>1$, as Ω_1 is any relatively compact open set in Ω_0.

(2) Case $s=1$. Now, let u be an analytic vector of P in Ω_0, and let Ω_1, Ω_2 with $\Omega_1 \Subset \Omega_2 \subset \overline{\Omega}_2 \subset \Omega_0$. As, now, we can not consider an analytic function with compact support, we supply by an Ehrenpreis sequence associated to the couple (Ω_1, Ω_2) (we used such a sequence in our preceding work concerning general Hörmander's operators, but with a less precise result [Derridj 2019a, Proposition 5.1]). Let us recall the proposition of L. Ehrenpreis [1960], giving the precise details regarding this sequence:

Proposition 6.2. *Let (Ω_1,Ω_2) be as above. Then there exists a constant $\widetilde{C}>0$ such that*

$$\text{(6-13)}\quad \text{for all } N\in\mathbb{N},\ \text{there exists } \varphi_N \in \mathcal{D}(\Omega_2),\ \varphi_N|_{\Omega_1}=1,\ \text{such that}\quad |\varphi_N^{(\beta)}| \le \widetilde{C}^{|\beta|+1}N^{|\beta|},\quad \text{for } |\beta|\le N.$$

In our proof below, in order to bound $\|\partial^\alpha u\|_{L^2(\Omega_1)}$, we use, in place of φ used in case (1), the function $\varphi_{q|\alpha|}$, where q is given in the definition of $\mathcal{I}_{|\alpha|}$ in (6-1) :

$$\text{(6-14)}\qquad \|\partial^\alpha u\|_{L^2(\Omega_1)} \le \|\varphi_{q|\alpha|}\partial^\alpha u\| \le M_\epsilon^{|\alpha|+1}(p|\alpha|)!^s\ \mathcal{N}_\alpha^\epsilon(u,\varphi_{q|\alpha|}).$$

In view of the definition of $\mathcal{N}_\alpha^\epsilon(u,\varphi_{q|\alpha|})$, we have

$$\text{(6-15)}\quad (\mathcal{N}_\alpha^\epsilon(u,\varphi_{q|\alpha|}))^{2^{|\alpha|}} \le \sum_{m|j|\le 2^{|\alpha|}p|\alpha|} \widetilde{B}^{2^{|\alpha|}}(\epsilon\widetilde{B})^{m|j|} \sum_{\substack{|\gamma_\rho|\le q|\alpha|\\ |\gamma|\le 2^{|\alpha|}p|\alpha|}} \prod_{\rho=1}^{2^{|\alpha|}} \widetilde{C}(\epsilon\widetilde{C})^{|\gamma_\rho|}(q|\alpha|)^{|\gamma_\rho|}\,|\gamma_\rho|!^{-1}$$
$$\le (\widetilde{B}\widetilde{C})^{2^{|\alpha|}} \sum_{m|j|\le 2^{|\alpha|}p|\alpha|} (2\widetilde{B})^{m|j|} \sum_{\substack{|\gamma|\le 2^{|\alpha|}p|\alpha|\\ |\gamma_\rho|\le q|\alpha|}} \prod_{\rho=1}^{2^{|\alpha|}} (\epsilon\widetilde{C}q|\alpha|)^{|\gamma_\rho|}|\gamma_\rho|!^{-1},$$

where $\widetilde{B}$ is given by (6-6). Choosing ϵ_0 such that

$$\text{(6-16)}\qquad \epsilon_0\widetilde{B} \le \tfrac{1}{2},$$

we have

$$(6\text{-}17)\quad (\mathcal{N}_\alpha^{\epsilon_0}(u,\varphi_{q|\alpha|}))^{2^{|\alpha|}} \le (\widetilde{B}\widetilde{C})^{2^{|\alpha|}} 2(2^{|\alpha|}p|\alpha|+1)^{2^{|\alpha|}} \cdot \sum_{|\gamma|\le 2^{|\alpha|}p|\alpha|} \prod_{\rho=1}^{2^{|\alpha|}} (\epsilon_0\widetilde{C}q|\alpha|)^{|\gamma_\rho|}|\gamma_\rho|!^{-1}.$$

So from Lemma 5.4 with $\widetilde{D} = 2\widetilde{B}\widetilde{C}$,

$$(6\text{-}18)\quad (\mathcal{N}_\alpha^{\epsilon_0}(u,\varphi_{q|\alpha|}))^{2^{|\alpha|}} \le \widetilde{D}^{2^{|\alpha|}}(2^{\alpha|}p|\alpha|+1)^{2^{|\alpha|}} \cdot \sum_{|\gamma|\le 2^{|\alpha|}p|\alpha|} \left(\frac{1}{2^{|\alpha|}}\sum_{\rho=1}^{2^{|\alpha|}}(\epsilon_0\widetilde{C}q|\alpha|)^{|\gamma_\rho|}|\gamma_\rho|!^{-1}\right)^{2^{|\alpha|}}$$
$$\le \widetilde{D}^{2^{|\alpha|}}(2^{|\alpha|}p|\alpha|+1)^{2^{|\alpha|}} \cdot \sum_{|\gamma|\le 2^{|\alpha|}p|\alpha|} \left(\frac{1}{2^{|\alpha|}}\exp(\epsilon_0\widetilde{C}q|\alpha|)\right)^{2^{|\alpha|}}.$$

Hence with $\epsilon_0\widetilde{C}q|\alpha| \le (2\widetilde{B})^{-1}\widetilde{C}q|\alpha| = \widetilde{A}|\alpha|$, we get

$$(6\text{-}19)\quad \mathcal{N}_\alpha^{\epsilon_0}(u,\varphi) \le \widetilde{D}(2^{\alpha|}p|\alpha|+1)\cdot\left\{\sum_{|\gamma|\le 2^{|\alpha|}p|\alpha|}\left(\frac{1}{2^{|\alpha|}}\exp(\widetilde{A}|\alpha|)\right)^{2^{|\alpha|}}\right\}^{2^{-|\alpha|}}$$
$$\le \widetilde{D}(2^{\alpha|}p|\alpha|+1)\frac{e^{\widetilde{A}|\alpha|}}{2^{|\alpha|}}\left\{\sum_{|\gamma|\le 2^{|\alpha|}p|\alpha|}1\right\}^{2^{-|\alpha|}}.$$

But

$$(6\text{-}20)\quad \sum_{|\gamma|\le 2^{|\alpha|}p|\alpha|} 1 \le \sum_{k=0}^{2^{|\alpha|}p|\alpha|}\left(\sum_{|\gamma|=k}1\right) \le \sum_{k=0}^{2^{|\alpha|}p|\alpha|}(k+1)^{n2^{|\alpha|}}$$
$$\le (2^{|\alpha|}p|\alpha|+1)(2^{|\alpha|}p|\alpha|+1)^{n2^{|\alpha|}}.$$

Hence

$$(6\text{-}21)\quad \left[\sum_{|\gamma|\le 2^{|\alpha|}p|\alpha|}1\right]^{2^{-|\alpha|}} \le (2^{|\alpha|}p|\alpha|+1)^{n+2^{-|\alpha|}},$$

so

$$\mathcal{N}_\alpha^{\epsilon_0}(u,\varphi_{q|\alpha|}) \le \widetilde{D}(2^{|\alpha|}p|\alpha|+1)^{n+2}\left(\frac{e^{\widetilde{A}}}{2}\right)^{|\alpha|} < A^{|\alpha|+1}, \quad \text{for some } A>0.$$

Then in view of (6-14),

$$(6\text{-}22)\quad \|\partial^\alpha u\|_{L^2(\Omega_1)} \le (AM_{\epsilon_0})^{|\alpha|+1}(p|\alpha|)!^s, \quad \alpha\in\mathbb{N}^n.$$

This finishes the proof of Theorem 6.1 in the remaining case $s = 1$. □

As a corollary of Theorems 5.5 and 6.1, we have:

Theorem 6.3. *Let P be given by* (2-1) *satisfying* (2-2) *and* (2-3) *in an open set* $\Omega \subset \mathbb{R}^n$. *Let Ω_0 be a relatively compact open subset of Ω and $s \geq 1$. Assume that the coefficients of Y, X_j's and b are in $G^s(\bar{\Omega}_0)$ and* $\text{type}_X(\bar{\Omega}_0) = p$. *Then any s-Gevrey vector of P in* Ω_0 (analytic vector when $s = 1$) *belongs to the Gevrey class* $G^{ps}(\Omega_0)$.

Remark 6.4. In [Braun Rodrigues et al. 2016], the authors showed for a particular class of Hörmander's operators in a product of two tori, a global version of Theorem 6.3, also proving its optimality. This implies that Theorem 6.3 is optimal.

Remark 6.5. We proved Theorem 6.3, in case $s \in \mathbb{N}^*$ (in particular for analytic vectors of P in Ω_0) using the method of addition of one variable in [Derridj 2019b]. Let us mention that D. Tartakoff [2018] suggests a different way to attack this question (without a complete proof).

Remark 6.6. Theorem 6.1 shows that estimates (5-27) imply the ps-Gevrey regularity in Ω_0 of any s-Gevrey vector of P in Ω_0. As the ps-Gevrey regularity in Ω_0 is optimal (Remark 6.4), we deduce that the integer p is optimal in the estimates (5-27), so giving optimal (5-27) estimates in that sense.

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