Pacific Journal of Mathematics

BOUNDEDNESS OF SINGULAR INTEGRALS WITH FLAG KERNELS ON WEIGHTED FLAG HARDY SPACES

YONGSHENG HAN, CHIN-CHENG LIN AND XINFENG WU

Volume 302 No. 2 October 2019

PACIFIC JOURNAL OF MATHEMATICS
Vol. 302, No. 2, 2019
dx.doi.org/10.2140/pjm.2019.302.545

BOUNDEDNESS OF SINGULAR INTEGRALS WITH FLAG KERNELS ON WEIGHTED FLAG HARDY SPACES

YONGSHENG HAN, CHIN-CHENG LIN AND XINFENG WU

We establish a weighted Hardy space theory associated with flag structures. This theory differs from those in the classical one-parameter and the product settings, and includes weighted Hardy spaces $H^p_{\mathcal{F},w}$, weighted Carleson measure spaces $CMO^p_{\mathcal{F},w}$ (the dual spaces of $H^p_{\mathcal{F},w}$), and the boundedness of singular integrals with flag kernels on these spaces. We also derive a Calderón–Zygmund decomposition and provide interpolation of operators acting on $H^p_{\mathcal{F},w}$. Examples and counterexamples are constructed to clarify the relations between classes of one-parameter, product and flag A_p weights. The main tool for our approach is the weighted Littlewood–Paley–Stein theory associated with the flag structure.

1. Introduction

The classical singular integral operators are extensions of the Hilbert transform, which have singularity at the origin only. The nature of this singularity leads to the invariance of these singular integral operators under the classical dilations on $\mathbb{R}^n$ given by $\delta x = (\delta x_1, \ldots, \delta x_n)$ for $\delta > 0$. On the other hand, the Calderón–Zygmund

Lin is supported by Ministry of Science and Technology of Taiwan under Grant #MOST 106-2115-M-008-004-MY3 and Wu is partially supported by the National Natural Science Foundation of China (Grant No. 11671397). Wu is the corresponding author.
MSC2010: primary 42B20; secondary 42B30.
Keywords: Calderón–Zygmund decomposition, duality, interpolation, singular integrals with flag kernels, weighted flag Carleson measure spaces, weighted flag Hardy spaces.

product theory of singular integral operators on $\mathbb{R}^n$ is concerned with those singular integral operators which are invariant under the n-fold dilations:

$$\delta x = (\delta_1 x_1, \delta_2 x_2, \ldots, \delta_n x_n),$$

$\delta_j > 0$ for $1 \leq j \leq n$. The product theory of $\mathbb{R}^n$ began with the strong maximal function studied by Zygmund, then continued with the Marcinkiewicz multiplier theorem and has been studied in a variety of directions; for example, product singular integrals and Hardy and BMO spaces studied in [Chang 1979; Chang and Fefferman 1980; 1982; 1985; Fefferman 1986; 1987; 1988; Fefferman and Stein 1982; Gundy and Stein 1979; Journé 1985; 1988; Pipher 1986]. Multiparameter analysis appeared implicitly in [Phong and Stein 1982] and has come to light with the proof by Müller, Ricci and Stein [1995] for the L^p boundedness, $1 < p < \infty$, of Marcinkiewicz multipliers on the Heisenberg group $\mathbb{H}^n$. This is surprising since Marcinkiewicz multipliers, which are invariant under a two-parameter group of dilations on $\mathbb{C}^n \times \mathbb{R}$, are bounded on $L^p(\mathbb{H}^n)$, despite the absence of a two-parameter automorphic group of dilation on $\mathbb{H}^n$. Moreover, Müller, Ricci and Stein showed that Marcinkiewicz multipliers can be characterized by a convolution operator of the form $f * K$, where, however, K is a flag kernel. Sharp L^p estimates for Marcinkiewicz multipliers on H-type groups were established in [Müller et al. 1996]. Nagel, Ricci and Stein [2001] proved that product kernels can be written as finite sums of flag kernels and that flag kernels have good regularity, restriction and composition properties. Applying the theory of singular integrals with flag kernels to the study of the $\Box_b$-complex on certain quadratic CR submanifolds of $\mathbb{C}^n$, they obtained L^p regularity for certain derivatives of the relative fundamental solution of $\Box_b$ and for the corresponding Szegő projections onto the null space of $\Box_b$ by showing that the distribution kernels of these operator are finite sums of flag kernels. Applying a type of singular integral operators whose novel features are related to singular integrals with flag kernels, Nagel and Stein [2006] provided the optimal estimates for solutions of the Kohn–Laplacian for certain classes of model domains in several complex variables. These operators differ essentially from the standard Calderón–Zygmund operators that have been used in these problems hitherto. More recently, Nagel et al. [2012; 2018] further generalized the theory of singular integrals with flag kernels to a more general setting, namely, homogeneous group. They proved that on a homogeneous group, singular integral operators with flag kernels are bounded on L^p, $1 < p < \infty$, and form an algebra. See also [Głowacki 2007] and [Głowacki 2010] and the correction to the latter. Weighted norm inequalities for flag singular integrals on homogeneous groups were established in [Wu 2014a].

As mentioned in [Nagel et al. 2001], on the Euclidean space a singular integral with a flag kernel is a special case of product singular integrals. As a consequence, the L^p, $1 < p < \infty$, boundedness of singular integrals with flag kernels follows automatically from the same result for product singular integrals (see [Fefferman and Stein 1982]). Note that the product theory is not available on the Heisenberg groups; it is interesting to ask: Can one provide the Hardy space boundedness for the Marcinkiewicz multiplier on the Heisenberg groups [Müller et al. 1995]? To answer this, the multiparameter Hardy spaces associated with the flag structure on the Heisenberg groups were developed in [Han et al. 2014]. A kind of atomic decomposition for flag Hardy spaces was provided in [Wu 2014b]. For related function spaces on Euclidean spaces, we refer to [Ding et al. 2010; Han and Lu 2010; Yang 2009].

Another interesting class of operators are flag paraproducts. Such operators were studied in [Muscalu 2007; 2010; Muscalu and Schlag 2013], and are closely related to the lacunary version of bilinear Hilbert transform. Miyachi and Tomita [2016] investigated L^∞ and H^p estimates for trilinear flag paraproducts. Flag paraproducts also appear naturally and play an important role in the study of nonlinear dispersive PDEs; see the work of Germain, Masmoudi and Shatah [2012a; 2012b] and Muscalu and Schlag [2013].

The purpose of this paper is to establish a weighted Hardy space theory associated with flag structures. This theory differs from those in the classical one parameter and the product settings, and includes weighted Hardy spaces $H^p_{\mathcal{F},w}$ and weighted Carleson measure spaces $CMO^p_{\mathcal{F},w}$ (the dual spaces of $H^p_{\mathcal{F},w}$), and the boundedness of singular integrals with flag kernels on these spaces. We will also derive a Calderón–Zygmund decomposition and provide interpolation of operators acting on $H^p_{\mathcal{F},w}$. To achieve this goal, we will employ the following approaches.

(1) Introduce a test function space and distributions: It is well known that in the classical case, the test function space and distributions are important for the development of the Hardy space theory. As in the remarkable work of C. Fefferman and Stein [1972], these are just the Schwartz test functions and tempered distributions. To introduce the Hardy spaces associated with flag kernels, in the current paper, we shall use the *partial* cancellation conditions to define a new test function space, which is different from that used in [Han et al. 2014]. Roughly speaking, any test function satisfies the cancellation conditions in one subvariable only. Such cancellation conditions are fulfilled by flag atoms (see [Wu 2014b]) and were also used by Nagel, Ricci, Stein and Wainger in [Nagel et al. 2012].

(2) Establish discrete Calderón's reproducing formulae: The classical Calderón reproducing formula was first introduced in [Calderón 1964]. Various forms of Calderón reproducing formulae proved to be very powerful tools in both harmonic analysis and wavelet analysis; see, for instance, [David et al. 1985; Frazier and Jawerth 1990; Han 2000; Meyer 1992]. To show the L^p $(1 < p < \infty)$ estimates of flag singular integrals, Nagel, Ricci, Stein and Wainger [Nagel et al. 2012] established a continuous Calderón reproducing formula on homogeneous groups.

In this paper, we shall build two kinds of discrete Calderón's reproducing formulae associated with the flag structure. The first one is expressed in terms of Schwartz functions whose Fourier transforms are compactly supported, and it converges in the above-mentioned test function space and distributions. The second one involves bump functions and converges in L^2 norm. Both formulae will be the main tools for developing the whole theory.

(3) Provide a Plancherel–Pólya type inequality and develop a Littlewood–Paley–Stein theory: The classical Plancherel–Pólya inequality says that the L^p norm of f whose Fourier transform has compact support is equivalent to the ℓ^p norm of the restriction of f at appropriate lattices (see [Plancherel and Pólya 1936]). It is well known that the classical and product Plancherel–Pólya inequalities play a crucial role for developing the Littlewood–Paley–Stein theory (see [Han 1998; Ding et al. 2012]). In this paper, we will provide the Plancherel–Pólya inequality associated with the flag structure and develop the Littlewood–Paley–Stein theory. As a consequence, the weighted flag Hardy spaces are well defined.

(4) Introduce generalized Carleson measure spaces: In the classical one parameter case, it is well known that BMO, the dual of H^1, can be characterized by the Carleson measures. Moreover, applying atomic decompositions of product Hardy spaces, Chang and R. Fefferman [1980] proved that the dual of the product H^1 can be characterized by the product Carleson measure. In this paper, we will characterize the dual of the weighted flag Hardy spaces via generalized Carleson measures. Our approach involves applying techniques of weighted sequence spaces, which enables us to avoid using the atomic decompositions.

(5) Prove a Calderón–Zygmund decomposition for $H^p_{\mathcal{F},w}$: The Calderón–Zygmund decomposition plays a crucial role in developing the Calderón–Zygmund operator theory and has many applications in harmonic analysis and PDEs. Such a decomposition in the product Euclidean spaces was first provided by Chang and R. Fefferman [1982] via atomic decompositions. In this paper, Calderón–Zygmund decomposition is achieved by applying the discrete Calderón's reproducing formula and the weighted flag version of Fefferman–Stein vector-valued maximal inequality. As an application, we derive interpolation results for sublinear operators on $H^p_{\mathcal{F},w}$.

We would like to remark that R. Fefferman [1987] established a criterion for the product H^p to L^p boundedness of product singular integral operators in Journé's class by considering their actions only on rectangle atoms via Journé's lemma. However, R. Fefferman's criterion cannot be extended to three or more parameters without further assumptions on the nature of the operators as shown in Journé [1988]. In fact, Journé provided a counter-example of singular integral operators in the three-parameter setting such that R. Fefferman's criterion breaks down. This means that in the classical product theory, there are substantial differences between the two-parameter case and the setting for three or more parameters. However, our approach works for any parameter case, and hence we shall focus on the case of three parameters and from the proofs given in this paper it is straightforward to extend the theory to the case of k ($k = 2$ and $k > 3$) parameters.

To describe the main results in this paper, we first recall some definitions and notation. A rectangle R in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3} := \mathbb{R}^N$ is called an *acceptable rectangle* (or a *flag rectangle*) if $R = Q_1 \times Q_2 \times Q_3$, where Q_i's are cubes in $\mathbb{R}^{n_i}$ with side-length satisfying

$$\ell(Q_1) \le \ell(Q_2) \le \ell(Q_3).$$

Denote by $\mathcal{R}_{\mathcal{F}}$ the set of all flag rectangles associated with $\mathcal{F}$ and by $\mathcal{R}^d_{\mathcal{F}}$ the set of all *dyadic* flag rectangles associated with $\mathcal{F}$. For $J = (j_1, j_2, j_3)$, the set $\mathcal{R}^J_{\mathcal{F}}$ consists of all dyadic flag rectangles $R = Q_1 \times Q_2 \times Q_3$ of side-length $\ell(Q_1) = 2^{j_1}$, $\ell(Q_2) = 2^{j_1 \vee j_2}$, $\ell(Q_3) = 2^{j_1 \vee j_2 \vee j_3}$, where $a \vee b$ denotes $\max\{a, b\}$.

The following *flag maximal function* was introduced in [Nagel et al. 2012]:

$$\mathcal{M}_{\mathcal{F}}(f)(x) = \sup_{\substack{R \ni x,\\ R \in \mathcal{R}_{\mathcal{F}}}} \frac{1}{|R|} \int_R |f(y)|\, dy.$$

The Muckenhoupt weight class associated with $\mathcal{F}$ can be defined as follows.

Definition 1.1. Let $1 < p < \infty$ and w be a weight function on $\mathbb{R}^N$; that is, a nonnegative locally integrable function on $\mathbb{R}^N$ that take values in $(0, \infty)$ almost everywhere. We say that w is a *flag A_p weight*, denoted by $w \in A^{\mathcal{F}}_p(\mathbb{R}^N)$, if

$$[w]_{A^{\mathcal{F}}_p} := \sup_{R \in \mathcal{R}_{\mathcal{F}}} \left(\frac{1}{|R|} \int_R w(x)\, dx\right)\left(\frac{1}{|R|} \int_R w(x)^{-1/(p-1)}\, dx\right)^{p-1} < \infty.$$

We say that w is in $A^{\mathcal{F}}_1(\mathbb{R}^N)$ if there is a constant C such that

$$\mathcal{M}_{\mathcal{F}}(w)(x) \le C w(x) \qquad \text{a.e. } x \in \mathbb{R}^N.$$

Let $A^{\mathcal{F}}_\infty(\mathbb{R}^N) := \bigcup_{1 \le p < \infty} A^{\mathcal{F}}_p(\mathbb{R}^N)$. If $w \in A^{\mathcal{F}}_\infty$, the *critical index of* w is defined by

$$q_w := \inf\{q : w \in A^{\mathcal{F}}_q(\mathbb{R}^N)\}.$$

We remark that this class of Muckenhoupt weights is different from the classical weight class $A_p(\mathbb{R}^N)$ and the product weight class $A^{\mathrm{pro}}_p(\mathbb{R}^N)$. More precisely, their relations are as follows (see Section 5 for more details):

$$A^{\mathrm{pro}}_p(\mathbb{R}^N) \subsetneq A^{\mathcal{F}}_p(\mathbb{R}^N) \subsetneq A_p(\mathbb{R}^N) \quad \text{for } 1 < p < \infty.$$

To develop the weighted Hardy space theory associated with flag singular integrals, as in the classical case, appropriate test functions and distributions are needed. For this purpose, we define *flag test functions* as follows.

Definition 1.2. A Schwartz function f on $\mathbb{R}^N$ is said to be a *flag test function* in $\mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$ if it satisfies the partial cancellation conditions

$$\int_{\mathbb{R}^{n_3}} f(x_1, x_2, x_3) x_3^\alpha \, dx_3 = 0 \quad \text{for all multi-indices } \alpha \text{ and every } (x_1, x_2) \in \mathbb{R}^{n_1+n_2}.$$

The seminorms on $\mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$ are the same as the ones on $\mathcal{S}(\mathbb{R}^N)$ and these seminorms make $\mathcal{S}_{\mathcal{F}}$ a topological vector space. Let $\mathcal{S}'_{\mathcal{F}}(\mathbb{R}^N)$ denote the topological dual space of $\mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$.

Recall that a Schwartz function f is called a classical test function, denoted by $f \in \mathcal{S}_\infty(\mathbb{R}^N)$, if

$$\int_{\mathbb{R}^{n_1+n_2+n_3}} f(x_1, x_2, x_3)\, x_1^\alpha \, x_2^\beta \, x_3^\gamma \, dx_1 \, dx_2 \, dx_3 = 0 \quad \text{for all multi-indices } \alpha, \beta, \gamma.$$

We say that a Schwartz function f is a product test function, denoted by $f \in \mathcal{S}_\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3})$, if

$$\int_{\mathbb{R}^{n_1}} f(x_1, x_2, x_3) x_1^\alpha \, dx_1 = 0 \quad \text{for all multi-indices } \alpha \text{ and all } x_2 \in \mathbb{R}^{n_2}, x_3 \in \mathbb{R}^{n_3},$$

$$\int_{\mathbb{R}^{n_2}} f(x_1, x_2, x_3) x_2^\beta \, dx_2 = 0 \quad \text{for all multi-indices } \beta \text{ and all } x_1 \in \mathbb{R}^{n_1}, x_3 \in \mathbb{R}^{n_3},$$

and

$$\int_{\mathbb{R}^{n_3}} f(x_1, x_2, x_3) x_3^\gamma \, dx_3 = 0 \quad \text{for all multi-indices } \gamma \text{ and all } x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}.$$

Clearly, $\mathcal{S}_\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}) \subsetneq \mathcal{S}_{\mathcal{F}}(\mathbb{R}^N) \subsetneq \mathcal{S}_\infty(\mathbb{R}^N)$.

Let $N_1 = n_1+n_2+n_3$, $N_2 = n_2+n_3$ and $N_3 = n_3$. For $i = 1, 2, 3$, let $\psi^{(i)} \in \mathcal{S}(\mathbb{R}^{N_i})$ satisfy

$$\text{supp}\, \widehat{\psi^{(i)}} \subset \{\xi^i \in \mathbb{R}^{N_i} : 1/2 < |\xi^i| \le 2\} \tag{1-1}$$

and

$$\sum_{j_i \in \mathbb{Z}} \widehat{\psi^{(i)}}(2^{j_i} \xi^i)^2 = 1 \quad \text{for all } \xi^i \in \mathbb{R}^{N_i} \backslash \{0\}. \tag{1-2}$$

Define $\psi_{j_i}^{(i)}(x^i) = 2^{-j_i N_i} \psi^{(i)}(2^{-j_i} x^i)$, $x^i \in \mathbb{R}^{N_i}$ and $\widetilde{\psi}_{j_i}^{(i)} = \delta_{\mathbb{R}^{N-N_i}} \otimes \psi_{j_i}^{(i)}$, $i = 1, 2, 3$. For $J = (j_1, j_2, j_3) \in \mathbb{Z}^3$, set $\psi_J = \widetilde{\psi}_{j_1}^{(1)} * \widetilde{\psi}_{j_2}^{(2)} * \widetilde{\psi}_{j_3}^{(3)}$. The departure of our approach is the Calderón reproducing formula (see Theorem 2.1 below)

$$f(x) = \sum_{J \in \mathbb{Z}^3} \sum_{R \in \mathcal{R}_{\mathcal{F}}^J} |R| \psi_J(x - x_R) \psi_J * f(x_R),$$

where x_R denotes the "lower left corner" of R (i.e., the corner of R with the least value of each coordinate component), and the series converges in $L^2(\mathbb{R}^N)$, $\mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$ and $\mathcal{S}'_{\mathcal{F}}(\mathbb{R}^N)$ whenever f is in the corresponding space.

Based on the above reproducing formula, the *Littlewood–Paley–Stein square function* of $f \in \mathcal{S}'_{\mathcal{F}}(\mathbb{R}^N)$ is defined by

$$g_{\mathcal{F}}(f)(x) = \left(\sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}^J_{\mathcal{F}}} |\psi_J * f(x_R)|^2 \chi_R(x) \right)^{\frac{1}{2}},$$

where χ_R is the indicator function of R.

Definition 1.3. Let $0 < p < \infty$ and $w \in A^{\mathcal{F}}_\infty(\mathbb{R}^N)$. The *weighted flag Hardy space* $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ is defined by

$$H^p_{\mathcal{F},w}(\mathbb{R}^N) = \{f \in \mathcal{S}'_{\mathcal{F}}(\mathbb{R}^N) : g_{\mathcal{F}}(f) \in L^p_w(\mathbb{R}^N)\}$$

with quasinorm $\|f\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)} := \|g_{\mathcal{F}}(f)\|_{L^p_w(\mathbb{R}^N)}$.

To see that the definition of $H^p_{\mathcal{F},w}$ is independent of the choice of $\{\psi_J\}$, we will prove the following theorem:

Theorem 1.4. *Let $0 < p < \infty$ and $w \in A^{\mathcal{F}}_\infty(\mathbb{R}^N)$. Suppose that both $\{\psi_J\}$ and $\{\varphi_J\}$ satisfy conditions (1-1) and (1-2). Then for $f \in \mathcal{S}'_{\mathcal{F}}(\mathbb{R}^N)$,*

$$\left\| \left\{ \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}^J_{\mathcal{F}}} |\psi_J * f(x_R)|^2 \chi_R \right\}^{\frac{1}{2}} \right\|_{L^p_w(\mathbb{R}^N)} \approx \left\| \left\{ \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}^J_{\mathcal{F}}} |\varphi_J * f(x_R)|^2 \chi_R \right\}^{\frac{1}{2}} \right\|_{L^p_w(\mathbb{R}^N)}.$$

Remark. Note that $\mathcal{S}'_{\mathcal{F}}(\mathbb{R}^N)$ can be identified as the space of tempered distributions on $\mathbb{R}^N$ modulo the tensor product of tempered distributions on $\mathbb{R}^{n_1+n_2}$ and polynomials on $\mathbb{R}^{n_3}$, and thus each element in $H^p_{\mathcal{F},w}$ is given by an equivalent class. As in the classical case, using the equivalent classes rather than distributions ensures that $\|\cdot\|_{H^p_{\mathcal{F},w}}$ is a quasinorm (a norm if $1 \le p < \infty$) and that $H^p_{\mathcal{F},w}$ is a quasi-Banach space (a Banach space if $1 \le p < \infty$). We shall prove in Corollary 2.6 below that $L^2 \cap H^p_{\mathcal{F},w}$ is dense in $H^p_{\mathcal{F},w}$, so that each element $f \in H^p_{\mathcal{F},w}$ can be identified as the limit in $H^p_{\mathcal{F},w}$ of some sequence $\{f_n\} \subset L^2 \cap H^p_{\mathcal{F},w}$. See [Frazier and Jawerth 1990] for similar results in the classical case.

Remark. As mentioned before, it was shown in [Nagel et al. 2001] that flag kernels form a subclass of product kernels. Therefore, singular integrals with flag kernels are bounded automatically on the weighted product Hardy spaces $H^p_w(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3})$, when w is a *product A_∞ weight* (see [Ding et al. 2012]). However, by Proposition 5.1, a flag weight is not necessarily a product weight, so our theory of weighted flag Hardy spaces does not fall under the scope of the product theory. Moreover, all functions $f \in L^2(\mathbb{R}^N) \cap H^p_{\mathcal{F}}(\mathbb{R}^N)$ (the unweighted flag Hardy space) fulfill the partial cancellation condition

$$\int_{\mathbb{R}^{n_3}} f(x_1, x_2, x_3)\, dx_3 = 0 \quad \text{for almost every } (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2},$$

which is different from the cancellation properties for product Hardy spaces; for more details, see the remark after the proof of Theorem 1.10 on page 585. This is an indication that weighted flag Hardy spaces are strictly larger than product ones. Nevertheless, we will prove, in Theorem 1.9 below, that singular integrals with flag kernels are bounded on these larger spaces $H^p_{\mathcal{F},w}(\mathbb{R}^N)$. This, indeed, was the main motivation to develop the weighted Hardy space theory.

Remark. If $1 < p < \infty$ and $w \in A^{\mathcal{F}}_p(\mathbb{R}^N)$, then, by a result in [Wu 2014a] and an argument similar to the proof of Theorem 1.4, the two spaces $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ and $L^p_w(\mathbb{R}^N)$ coincide with comparable norms. However, if $p > 1$ there is a $w \notin A_p$ such that $H^p_{\mathcal{F},w} \neq L^p_w$. To see this, refer to the work of Strömberg and Wheeden [1982]. Indeed, if $u(x) = |q(x)|^p w(x)$, where $q(x)$ is a polynomial and $w(x)$ satisfies the Muckenhoupt A_p condition, they proved that H^p_u and L^p_u can be identified when all the zeros of $q(x)$ are real and that otherwise H^p_u can be identified with a certain proper subspace of L^p_u. Similar results in product spaces were obtained in [Strömberg and Wheeden 1989].

To study the dual of $H^p_{\mathcal{F},w}(\mathbb{R}^N)$, we introduce the following weighted Carleson measure spaces $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$.

Definition 1.5. Let $0 < p \leq 1$, $w \in A^{\mathcal{F}}_\infty(\mathbb{R}^N)$. Suppose that $\{\psi_J\}$ satisfies (1-1) and (1-2). We say that $f \in \mathcal{S}'_{\mathcal{F}}(\mathbb{R}^N)$ belongs to $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$ if

$$\|f\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)} := \sup_{\Omega \subset \mathbb{R}^N} \left\{ \frac{1}{[w(\Omega)]^{2/p-1}} \sum_{J \in \mathbb{Z}^3} \sum_{\substack{R \in \mathcal{R}^J_{\mathcal{F}}, \\ R \subset \Omega}} \frac{|R|^2}{w(R)} |\psi_J * f(x_R)|^2 \right\}^{\frac{1}{2}} < \infty,$$

where the supremum is taken over all open sets Ω with $w(\Omega) < \infty$.

Note that the flag structure is involved in the definition of $CMO^p_{\mathcal{F},w}$. To see that the weighted Carleson measure spaces $CMO^p_{\mathcal{F},w}$ are well defined, we need the following theorem:

Theorem 1.6. *Let $w \in A^{\mathcal{F}}_\infty(\mathbb{R}^N)$. Suppose that both $\{\psi_J\}$ and $\{\varphi_J\}$ satisfy (1-1) and (1-2). Then, for $f \in \mathcal{S}'_{\mathcal{F}}(\mathbb{R}^N)$,*

$$\sup_{\Omega \subset \mathbb{R}^N} \left\{ \frac{1}{[w(\Omega)]^{2/p-1}} \sum_{J \in \mathbb{Z}^3} \sum_{\substack{R \in \mathcal{R}^J_{\mathcal{F}}, \\ R \subset \Omega}} \frac{|R|^2}{w(R)} |\psi_J * f(x_R)|^2 \right\}^{\frac{1}{2}}$$

$$\approx \sup_{\Omega\subset\mathbb{R}^N} \left\{ \frac{1}{[w(\Omega)]^{2/p-1}} \sum_{J\in\mathbb{Z}^3} \sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}},\\ R\subset\Omega}} \frac{|R|^2}{w(R)} |\varphi_J * f(x_R)|^2 \right\}^{\frac{1}{2}},$$

where the suprema run over all open sets Ω with $w(\Omega) < \infty$.

The duality between $H^p_{\mathcal{F},w}$ and $CMO^p_{\mathcal{F},w}$ can be stated as follows.

Theorem 1.7. *Let $0 < p \le 1$. Then $(H^p_{\mathcal{F},w}(\mathbb{R}^N))^* = CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$. More precisely, if $g \in CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$, the mapping ℓ_g given by $\ell_g(f) = \langle f, g\rangle$, defined initially for $f \in \mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$, extends to a unique continuous linear functional on $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ with $\|\ell_g\| \lesssim \|g\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)}$.*

Conversely, for every $\ell \in (H^p_{\mathcal{F},w}(\mathbb{R}^N))^*$, there exists a unique $g \in CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$ such that $\ell = \ell_g$ with $\|g\|_{CMO^p_{\mathcal{F},w}} \lesssim \|\ell\|$.

In order to state the boundedness results for singular integrals with flag kernels on $H^p_{\mathcal{F},w}(\mathbb{R}^N)$, we need to recall some definitions from [Nagel et al. 2001]. A k-normalized bump function on $\mathbb{R}^N$ is a C^k function supported on the unit ball with norm bounded by 1. As pointed out in [Nagel et al. 2001], the definitions given below are independent of the choices of k, and thus we will simply refer to "normalized bump function" without specifying k.

In this paper, we will consider the singular integrals with the following flag kernels. See [Nagel et al. 2012] for this definition on homogeneous groups.

Definition 1.8. A flag kernel is a distribution $\mathcal{K}$ on $\mathbb{R}^N$ which coincides with a C^∞ function away from the coordinate subspace $x_1 = 0$ and satisfies the following:

(i) (differential inequalities) For each $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3$,

$$|\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}\mathcal{K}(x)| \lesssim |x_1|^{-n_1-|\alpha_1|}(|x_1|+|x_2|)^{-n_2-|\alpha_2|}(|x_1|+|x_2|+|x_3|)^{-n_3-|\alpha_3|}$$

for $x_1 \neq 0$.

(ii) (cancellation conditions)

(a) Given normalized bump functions ψ_i, $i = 1, 2, 3$, on $\mathbb{R}^{n_i}$ and any scaling parameter $r > 0$, define a distribution $\mathcal{K}_{\psi_i,r}$ by setting

$$\langle \mathcal{K}_{\psi_i,r}, \varphi\rangle = \langle \mathcal{K}, (\psi_i)_r \otimes \varphi\rangle \tag{1-3}$$

for any test function $\varphi \in \mathcal{S}(\mathbb{R}^{N-n_i})$. Then the distributions $\mathcal{K}_{\psi_i,r}$ satisfy the differential inequalities

$$|\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}\mathcal{K}_{\psi_1,r}(x_2,x_3)| \lesssim |x_2|^{-n_2-|\alpha_2|}(|x_2|+|x_3|)^{-n_3-|\alpha_3|},$$
$$|\partial^{\alpha_1}_{x_1}\partial^{\alpha_3}_{x_3}\mathcal{K}_{\psi_2,r}(x_1,x_3)| \lesssim |x_1|^{-n_1-|\alpha_1|}(|x_1|+|x_3|)^{-n_3-|\alpha_3|},$$
$$|\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\mathcal{K}_{\psi_3,r}(x_1,x_2)| \lesssim |x_1|^{-n_1-|\alpha_1|}(|x_1|+|x_2|)^{-n_2-|\alpha_2|}.$$

(b) For any bump functions $\overline{\psi}_i$ on $\mathbb{R}^{N-n_i}$ and any parameters $r = (r_1, r_2)$, we define the distributions $\mathcal{K}_{\overline{\psi}_i,r}$ by (1-3). Then the distributions $\mathcal{K}_{\overline{\psi}_i,r}$, $i = 1, 2, 3$, are one-parameter kernels and satisfy

$$|\partial_{x_i}^{\alpha_i} \mathcal{K}_{\overline{\psi}_i,r}(x_i)| \lesssim |x_i|^{-n_i-|\alpha_i|}.$$

(c) For any bump function ψ on $\mathbb{R}^N$ and $r_1, r_2, r_3 > 0$, we have

$$|\langle \mathcal{K}, \psi(r_1\cdot, r_2\cdot, r_3\cdot)\rangle| \lesssim 1.$$

Moreover, the corresponding constants that appear in these differential inequalities are independent of r, r_1, r_2.

A *flag singular integral* $T_{\mathcal{F}}$ is of the form $T_{\mathcal{F}}(f) = \mathcal{K} * f$, where $\mathcal{K}$ is a flag kernel on $\mathbb{R}^N$ defined as above.

A typical example of flag kernel adapted to the flag $\mathcal{F}$,

$$\{(0, 0, 0)\} \subset \{(0, 0, z)\} \subset \{(0, y, z)\} \subset \mathbb{R}^3,$$

is

$$\frac{\operatorname{sgn}(y)\operatorname{sgn}(z)}{x\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}$$

(see [Nagel et al. 2001]).

The following result establishes the boundedness of flag singular integrals on the weighted flag Hardy spaces.

Theorem 1.9. *Let $0 < p < \infty$ and $w \in A_\infty^{\mathcal{F}}(\mathbb{R}^N)$. Then the flag singular integral operator $T_{\mathcal{F}}$ is bounded on $H^p_{\mathcal{F},w}(\mathbb{R}^N)$. Moreover, for $f \in H^p_{\mathcal{F},w}(\mathbb{R}^N)$ there exists a constant C_p such that*

$$\|T_{\mathcal{F}}(f)\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)} \le C_p \|f\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}.$$

Remark. As a consequence of Theorem 1.9 and the remark on page 552, we can obtain the boundedness of flag singular integrals on the weighted Lebesgue spaces; that is,

$$\|T_{\mathcal{F}}(f)\|_{L^p_w(\mathbb{R}^N)} \le C_p \|f\|_{L^p_w(\mathbb{R}^N)}, \quad 1 < p < \infty,$$

provided that $w \in A_p^{\mathcal{F}}(\mathbb{R}^N)$ and $f \in L^p_w(\mathbb{R}^N)$.

The following result gives a general principle on the $H^p_{\mathcal{F},w}(\mathbb{R}^N) \to L^p_w(\mathbb{R}^N)$ boundedness of operators.

Theorem 1.10. *Suppose $w \in A_\infty^{\mathcal{F}}(\mathbb{R}^N)$ and $0 < p \le 1$. For any linear operator T which is bounded on both $L^2(\mathbb{R}^N)$ and $H^p_{\mathcal{F},w}(\mathbb{R}^N)$, T is bounded from $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ to $L^p_w(\mathbb{R}^N)$. As a consequence, the flag singular integral operator $T_{\mathcal{F}}$ is bounded from $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ to $L^p_w(\mathbb{R}^N)$.*

Theorem 1.11 gives the $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$ boundedness of flag singular integrals.

Theorem 1.11. *Let $T_{\mathcal{F}}$ be a singular integral with flag kernel. For $0 < p \le 1$ and $w \in A_\infty^{\mathcal{F}}(\mathbb{R}^N)$, $T_{\mathcal{F}}$ extends uniquely to a bounded operator on $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$. Moreover, for any $f \in CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$ there exists a constant C_p such that*

$$\|T_{\mathcal{F}}(f)\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)} \le C_p \|f\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)}.$$

Note that $CMO^1_{\mathcal{F},w}(\mathbb{R}^N) = BMO_{\mathcal{F},w}(\mathbb{R}^N)$, the dual of $H^1_{\mathcal{F},w}(\mathbb{R}^N)$. Therefore, Theorem 1.11 provides the endpoint estimate for singular integrals with flag kernels on $BMO_{\mathcal{F},w}(\mathbb{R}^N)$.

Our last main results are the Calderón–Zygmund decomposition and interpolation for $H^p_{\mathcal{F},w}$.

Theorem 1.12. *Let $w \in A_\infty^{\mathcal{F}}(\mathbb{R}^N)$, $p_1 \in (0,1]$ and $p_1 < p < p_2 < \infty$. Given $f \in H^p_{\mathcal{F},w}(\mathbb{R}^N)$ and $\alpha > 0$, we have the decomposition $f = g + b$, where $g \in H^{p_2}_{\mathcal{F},w}(\mathbb{R}^N)$ and $b \in H^{p_1}_{\mathcal{F},w}(\mathbb{R}^N)$ with*

$$\|g\|^{p_2}_{H^{p_2}_{\mathcal{F},w}(\mathbb{R}^N)} \lesssim \alpha^{p_2-p}\|f\|^p_{H^p_{\mathcal{F},w}(\mathbb{R}^N)} \quad \textit{and} \quad \|b\|^{p_1}_{H^{p_1}_{\mathcal{F},w}(\mathbb{R}^N)} \lesssim \alpha^{p_1-p}\|f\|^p_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}.$$

We would like to point out that the above result was first proved by Chang and Fefferman [1982] for the product Hardy spaces on the product of upper half space.

As an application of Theorem 1.12, we immediately have the following interpolation of operators.

Theorem 1.13. *Let $w \in A_\infty^{\mathcal{F}}(\mathbb{R}^N)$ and $0 < p_1 < p_2 < \infty$. If T is a sublinear operator bounded from $H^{p_1}_{\mathcal{F},w}(\mathbb{R}^N)$ to $L^{p_1}_w(\mathbb{R}^N)$ and bounded from $H^{p_2}_{\mathcal{F},w}(\mathbb{R}^N)$ to $L^{p_2}_w(\mathbb{R}^N)$, then T is bounded from $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ to $L^p_w(\mathbb{R}^N)$ for all $p \in (p_1, p_2)$. Similarly, if T is bounded both on $H^{p_1}_{\mathcal{F},w}(\mathbb{R}^N)$ and $H^{p_2}_{\mathcal{F},w}(\mathbb{R}^N)$, then T is bounded on $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ for all $p \in (p_1, p_2)$.*

Finally, we make the following remark. In contrast with the product case and the unweighted flag case (see [Ding et al. 2012; Han et al. 2014]), the whole theory of weighted flag Hardy spaces is built on the acceptable rectangles. A typical example is the strong maximal operator, which is bounded on the L^p, $1 < p < \infty$, but is not bounded on L^p_w if $w \in A^{\mathcal{F}}_p \backslash A^{\text{pro}}_p(\mathbb{R}^N)$ (see Section 5 for examples of weight functions in $A^{\mathcal{F}}_p \backslash A^{\text{pro}}_p(\mathbb{R}^N)$). In the current paper, we shall use the intrinsic maximal operator $\mathcal{M}_{\mathcal{F}}$, which reflects the geometry of the flag multiparameter structure.

This paper is organized as follows. In Section 2, we establish the weighted theory of flag Hardy and Carleson measure spaces. The boundedness of flag singular integrals on these spaces are proved in Section 3. Section 4 is devoted to the Calderón–Zygmund decomposition and interpolation in these spaces. Finally, in Section 5, we give some examples and counterexamples to clarify the relationships among the classes of flag weights, classical weights and product weights.

2. Weighted Hardy spaces, Carleson measure spaces and the dual theorem

The main purpose of this section is to prove Theorems 1.4, 1.6 and 1.7. To this end, we need the following *discrete Calderón reproducing formula*.

Theorem 2.1. *Suppose that* $\{\psi_J\}$ *satisfy* (1-1) *and* (1-2). *Then*

$$f(x) = \sum_{J\in\mathbb{Z}^3}\sum_{\ell\in\mathbb{Z}^N} 2^{J\cdot n}\psi_J(x-2^J\ell)\psi_J * f(2^J\ell) = \sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^J_{\mathcal{F}}} |R|\psi_J(x-x_R)\psi_J * f(x_R), \tag{2-1}$$

where $2^J\ell = (2^{j_1}\ell_1, 2^{j_1\vee j_2}\ell_2, 2^{j_1\vee j_2\vee j_3}\ell_3) = x_R$ *denotes the lower left corner of* R, $2^{J\cdot n} = 2^{j_1n_1+(j_1\vee j_2)n_2+(j_1\vee j_2\vee j_3)n_3}$ *is the measure of* $R\in\mathcal{R}^J_{\mathcal{F}}$, *and the series converges in* $L^2(\mathbb{R}^N)$, $\mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$ *and* $\mathcal{S}'_{\mathcal{F}}(\mathbb{R}^N)$ (*whenever* f *is in the corresponding space*).

Proof. The proof of the convergence of the series in L^2 is similar to the classical case. Assume $f\in L^2(\mathbb{R}^N)$. By Fourier transform, $f=\sum_{J\in\mathbb{Z}^3}\psi_J*\psi_J*f$ with the series convergent in $L^2(\mathbb{R}^N)$. Similar to the method used in [Frazier et al. 1991], set $g=\psi_J*f$ and $h=\psi_J$. For $\xi\in\mathbb{R}^N$, the Fourier transforms of g and h are, respectively, given by

$$\hat{g}(\xi_1,\xi_2,\xi_3) = \widehat{\psi^{(1)}}(2^{j_1}\xi_1, 2^{j_1}\xi_2, 2^{j_1}\xi_3)\widehat{\psi^{(2)}}(2^{j_2}\xi_2, 2^{j_2}\xi_3)\widehat{\psi^{(3)}}(2^{j_3}\xi_3)\hat{f}(\xi_1,\xi_2,\xi_3),$$
$$\hat{h}(\xi_1,\xi_2,\xi_3) = \widehat{\psi^{(1)}}(2^{j_1}\xi_1, 2^{j_1}\xi_2, 2^{j_1}\xi_3)\widehat{\psi^{(2)}}(2^{j_2}\xi_2, 2^{j_2}\xi_3)\widehat{\psi^{(3)}}(2^{j_3}\xi_3).$$

Note that the Fourier transforms of g and h are both compactly supported in

$$R_j := \{\xi\in\mathbb{R}^N : |\xi_1|\le 2^{-j_1}\pi,\ |\xi_2|\le 2^{-j_1\vee j_2}\pi,\ |\xi_3|\le 2^{-j_1\vee j_2\vee j_3}\pi\}.$$

By first expanding $\hat{g}$ in a Fourier series on the rectangle R_j,

$$\hat{g}(\xi) = \sum_{\ell\in\mathbb{Z}^N} 2^{J\cdot n}(2\pi)^{-N}\left(\int_{R_j}\hat{g}(\xi')e^{i[(2^J\ell)\cdot\xi']}d\xi'\right)e^{-i[(2^J\ell)\cdot\xi]},$$

and then replacing the domain R_J by $\mathbb{R}^N$ as $\hat{g}$ is supported in R_J, we get

$$\hat{g}(\xi) = \sum_{\ell\in\mathbb{Z}^N} 2^{J\cdot n}g(2^J\ell)e^{-i[(2^J\ell)\cdot\xi]}.$$

Multiplying both sides by $\hat{h}(\xi)$ and noticing $\hat{h}(\xi)e^{-i[(2^J\ell)\cdot\xi]} = [h(\cdot-2^J\ell)]\hat{}\,(\xi)$ yields

$$(g*h)(x) = \sum_{\ell\in\mathbb{Z}^N} 2^{J\cdot n}g(2^J\ell)h(x-2^J\ell).$$

Substituting g by ψ_J*f and h by ψ_J into the above identity gives the discrete Calderón reproducing formula (2-1) and the convergence in $L^2(\mathbb{R}^N)$.

To finish the proof, we only need to show that the series in (2-1) converges in $\mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$ if $f \in \mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$; the convergence in $\mathcal{S}'_{\mathcal{F}}(\mathbb{R}^N)$ then follows from a standard duality argument. The key to doing this are the following almost-orthogonality estimates: for any $L, M > 0$ and $f \in \mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$,

$$|f * \psi_J(x)| \lesssim 2^{-(|j_1|+|j_2|+|j_3|)L} \frac{1}{(1+|x|)^M}. \tag{2-2}$$

Assume that (2-2) holds for the moment. Then for any multi-index $\alpha \in \mathbb{N}^N$,

$$\begin{aligned}
&\Big|\sum_{\ell \in \mathbb{Z}^N} 2^{J\cdot n} (\partial^\alpha \psi_J)(x - 2^J \ell) \psi_J * f(2^J \ell)\Big| \\
&\qquad \lesssim 2^{-(|j_1|+|j_2|+|j_3|)L'} \sum_{\ell \in \mathbb{Z}^N} 2^{J\cdot n} \frac{1}{(1+|2^{j_1}\ell_1| + |2^{j_1\vee j_2}\ell_2| + |2^{j_1\vee j_2\vee j_3}\ell_3|)^M} \\
&\qquad\qquad \times \frac{1}{(1+|x_1 - 2^{j_1}\ell_1| + |x_2 - 2^{j_1\vee j_2}\ell_2| + |x_3 - 2^{j_1\vee j_2\vee j_3}\ell_3|)^M} \\
&\qquad \lesssim 2^{-(|j_1|+|j_2|+|j_3|)L'} (1+|x|)^{-M} \quad \text{for some } L' > 0,
\end{aligned}$$

which would further imply that

$$\sum_{|j_1|,|j_2|,|j_3|>k} \sum_{\ell \in \mathbb{Z}^N} 2^{J\cdot n} \psi_J(x - 2^J \ell) \psi_J * f(2^J \ell) \to 0 \quad \text{in } \mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$$

as $k \to +\infty$.

It remains to verify (2-2) under the assumption $f \in \mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$. We note that for any $(x_1, x_2) \in \mathbb{R}^{n_1+n_2}$, $f(x_1, x_2, \cdot) \in \mathcal{S}_\infty(\mathbb{R}^{n_3})$, the space of Schwartz functions with all vanishing moments. It follows from the classical almost-orthogonality estimate on $\mathbb{R}^{n_3}$ (see [Han et al. 2010, p. 2840]) that

$$|\widetilde{\psi}^{(3)}_{j_3} * f(x)| \lesssim 2^{-|j_3|L} (1+|x|)^{-M},$$

which implies

$$|\widetilde{\psi}_J * f(x)| \lesssim 2^{-|j_3|L} 2^{(|j_2|+|j_3|)M} (1+|x|)^{-M}. \tag{2-3}$$

Likewise, using the fact $f(x_1, \cdot, \cdot) \in \mathcal{S}_\infty(\mathbb{R}^{n_2+n_3})$ for any $x_1 \in \mathbb{R}^{n_1}$, we can derive

$$|\widetilde{\psi}_J * f(x)| \lesssim 2^{-|j_2|L} 2^{(|j_1|+|j_3|)M} (1+|x|)^{-M}. \tag{2-4}$$

We finally use $f \in \mathcal{S}_\infty(\mathbb{R}^N)$ to get

$$|\widetilde{\psi}_J * f(x)| \lesssim 2^{-|j_1|L} 2^{(|j_2|+|j_3|)M} (1+|x|)^{-M}. \tag{2-5}$$

Choosing $L > 100M$ in (2-3)–(2-5) and taking the geometric mean, (2-2) follows. □

The following almost orthogonality estimate will be frequently used in the subsequent part of this section. The proof follows directly from the one-parameter orthogonality estimate (see [Han et al. 2010]); see also [Nagel et al. 2012] for similar estimates on homogeneous groups.

Lemma 2.2. *Given positive integers L and M, there exists some constant $C = C(L, M) > 0$ such that*

$$|\psi_J * \varphi_{J'}(x)| \le C 2^{-(|j_1-j_1'|+|j_2-j_2'|+|j_3-j_3'|)L} \prod_{i=1}^{3} \frac{\max_{1\le k\le i} 2^{(j_k \vee j_k')M}}{(\max_{1\le k\le i} 2^{j_k\vee j_k'} + |x_i|)^{n_i+M}},$$

where $\{\psi_J\}$ and $\{\varphi_{J'}\}$ are Schwartz functions satisfying (1-1).

Remark. The conclusion of Lemma 2.2 remains valid if the Schwartz functions $\{\psi^{(i)}\}$ and $\{\varphi^{(i)}\}$, where $i = 1, 2, 3$, have vanishing moments up to order M_0 (see Theorem 3.3 for choosing such an M_0). In such a case, the above inequality holds for any $M > 0$ and $L \le M_0 + 1$.

The following maximal function estimate is also frequently needed.

Lemma 2.3. *Let $J, J' \in \mathbb{Z}^3$, $R = Q_1 \times Q_2 \times Q_3 \in \mathcal{R}_{\mathcal{F}}^J$ and $M > 2N$. Then, for any $x, \bar{x} \in R$ and $\delta \in \left(\frac{N}{M}, 1\right]$, we have*

$$\sum_{R'\in\mathcal{R}_{\mathcal{F}}^{J'}} |R'| \left[\prod_{i=1}^{3} \frac{\max_{1\le k\le i} 2^{(j_k\vee j_k')M}}{(\max_{1\le k\le i} 2^{(j_k\vee j_k')} + |\bar{x}_i - x_i'|)^{n_i+M}}\right] |g(x')| \tag{2-6}$$
$$\le C\left\{\prod_{i=1}^{3} [2^{3n_i(j_i-j_i')} \vee 1]\right\}^{\frac{1}{\delta}-1} \left\{\mathcal{M}_{\mathcal{F}}\left[\left(\sum_{R'\in\mathcal{R}_{\mathcal{F}}^{J'}} |g(x')|^2 \chi_{R'}\right)^{\frac{\delta}{2}}\right](x)\right\}^{\frac{1}{\delta}}, \quad \forall\, x' \in R',$$

where C is a constant depending only on N and M.

Proof. The proof of this lemma is similar to the classical case. We fix an $x' \in R'$ for each $R' \in \mathcal{R}_{\mathcal{F}}^{J'}$. For $i = 1, 2, 3$, set

$$A_0^i = \{Q_i' : |\bar{x}_i - x_i'| \le \max_{1\le k\le i} 2^{(j_k\vee j_k')}\}$$

and

$$A_{r_i}^i = \left\{Q_i' : 2^{r_i-1} < \frac{|\bar{x}_i - x_i'|}{\max_{1\le k\le i} 2^{(j_k\vee j_k')}} \le 2^{r_i}\right\}, \quad \forall\, r_i \in \mathbb{Z}_+.$$

For any fixed $r = (r_1, r_2, r_3) \in \mathbb{N}^3$, for $i = 1, 2, 3$, we set

$$E_r = \left\{(w_1, w_2, w_3) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3} : |w_i - \bar{x}_i| \le 2^{r_1\vee r_2\vee r_3} \max_{1\le k\le i} \{2^{(j_k\vee j_k')+1}\}\right\}.$$

Then for each $R' \in \mathcal{R}_{\mathcal{F}}^{J'}$, $R' \subset E_r$ for some triple $r = (r_1, r_2, r_3)$. Also, if $R' \in \mathscr{A}_r := A_{r_1}^1 \times A_{r_2}^2 \times A_{r_3}^3$, then $R' \subset E_r$. Obviously, $E_r \in \mathcal{R}_{\mathcal{F}}$ and $|E_r| \le C 2^{(r_1\vee r_2\vee r_3)N} \prod_{i=1}^{3} \max_{1\le k\le i} 2^{n_i(j_k\vee j_k')}$. Hence, for any $\delta \in \left(\frac{N}{M}, 1\right]$, the left-hand side of (2-6) is dominated by

$$\text{(2-7)}\quad \sum_{R'\in\mathcal{R}_{\mathcal{F}}^{J'}} |R'|\Bigg[\prod_{i=1}^{3}\frac{\max_{1\le k\le i} 2^{(j_k\vee j_k')M}}{(\max_{1\le k\le i} 2^{(j_k\vee j_k')}+|\bar{x}_i-x_i'|)^{n_i+M}}\Bigg]|g(x')|$$

$$\le C\sum_{r\in\mathbb{N}^3}\Bigg[\prod_{i=1}^{3}2^{-r_i(M+n_i)}(\max_{1\le k\le i}2^{-n_i(j_k\vee j_k')})\Bigg]|R'|\Bigg(\sum_{R'\in\mathscr{A}_r}|g(x')|^{\delta}\Bigg)^{\frac{1}{\delta}}$$

$$= C\sum_{r\in\mathbb{N}^3}\Bigg[\prod_{i=1}^{3}2^{-r_i(M+n_i)}(\max_{1\le k\le i}2^{-n_i(j_k\vee j_k')})\Bigg]|R'|^{1-\frac{1}{\delta}}|E_r|^{\frac{1}{\delta}}$$

$$\times\Bigg(\frac{1}{|E_r|}\int_{E_r}\sum_{R'\in\mathscr{A}_r}|g(x')|^{\delta}\chi_{R'}(y)\,dy\Bigg)^{\frac{1}{\delta}},$$

where $C=2^{3M+N}$. For any $x\in R$, since $\bar{x}$ is also in R, we see that $|x_i-\bar{x}_i|\le 2^{j_i}$ for $i=1,2,3$; hence $x\in E_r$ for any $r\in\mathbb{N}^3$, by the definition of E_r. The expression in the last parentheses above is the average of the function $y\mapsto\sum_{R'\in\mathscr{A}_r}|g(x')|^{\delta}\chi_{R'}(y)$ over a flag rectangle E_r containing x, therefore being bounded by the flag maximal function $\mathcal{M}_{\mathcal{F}}\big(\sum_{R'\in\mathcal{R}_{\mathcal{F}}^{J'}}|g(x')|^{\delta}\chi_{R'}\big)(x)$. Hence we can continue the estimate in (2-7) as

$$\le C\Bigg(\sum_{r\in\mathbb{N}^3}\prod_{i=1}^{3}2^{-r_i\left(M+n_i-\frac{N}{\delta}\right)}\Bigg)\Bigg(\prod_{i=1}^{3}[2^{3n_i(j_i-j_i')}\vee 1]\Bigg)^{\frac{1}{\delta}-1}\Bigg(\mathcal{M}_{\mathcal{F}}\Bigg(\sum_{R'\in\mathcal{R}_{\mathcal{F}}^{J'}}|g(x')|^{\delta}\chi_{R'}\Bigg)(x)\Bigg)^{\frac{1}{\delta}}$$

$$\le C'\Bigg(\prod_{i=1}^{3}[2^{3n_i(j_i-j_i')}\vee 1]\Bigg)^{\frac{1}{\delta}-1}\Bigg(\mathcal{M}_{\mathcal{F}}\Bigg(\sum_{R'\in\mathcal{R}_{\mathcal{F}}^{J'}}|g(x')|^{\delta}\chi_{R'}\Bigg)(x)\Bigg)^{\frac{1}{\delta}},$$

where C' depends only on M and N. This completes the proof of Lemma 2.3. □

For $x=(x_1,x_2,x_3)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{n_3}$, we denote $x^1=x\in\mathbb{R}^{N_1}$, $x^2=(x_2,x_3)\in\mathbb{R}^{N_2}$ and $x^3=x_3\in\mathbb{R}^{N_3}$. For $i=1,2,3$, write $x=(\bar{x}^i,x^i)\in\mathbb{R}^{N-N_i}\times\mathbb{R}^{N_i}$. We say that $w\in A_p^{(i)}(\mathbb{R}^N)$ if $w(\bar{x}^i,\cdot)$ is a classical $A_p(\mathbb{R}^{N_i})$ uniformly in $\bar{x}^i$; that is,

$$\operatorname*{ess\,sup}_{\bar{x}^i\in\mathbb{R}^{N-N_i}}\ \sup_{Q\subset R^{N_i}}\left(\frac{1}{|Q|}\int_Q w(\bar{x}^i,x^i)dx^i\right)\left(\frac{1}{|Q|}\int_Q w(\bar{x}^i,x^i)^{-1/(p-1)}dx^i\right)^{p-1}<\infty.$$

Let $\mathcal{M}_i$ denote the Hardy–Littlewood maximal operator on $\mathbb{R}^{N_i}$. The *lifted maximal operator* $\widetilde{\mathcal{M}}_i$ on $\mathbb{R}^N$ was introduced in [Nagel et al. 2012] by

$$\widetilde{\mathcal{M}}_i:=\delta_{\mathbb{R}^{N-N_i}}\otimes\mathcal{M}_i,$$

where $\delta_{\mathbb{R}^{N-N_i}}$ is the Dirac mass at $0\in\mathbb{R}^{N-N_i}$.

The following result was proved in [Wu 2014a].

Lemma 2.4. *Let $1 < p < \infty$ and w be a weight function. Then the following statements are equivalent*:

(i) $w \in A_p^{\mathcal{F}}(\mathbb{R}^N)$.

(ii) $w \in A_p^{(1)} \cap A_p^{(2)} \cap A_p^{(3)}(\mathbb{R}^N)$.

(iii) $\widetilde{\mathcal{M}}_3 \circ \widetilde{\mathcal{M}}_2 \circ \widetilde{\mathcal{M}}_1$ *is bounded on* $L_w^p(\mathbb{R}^N)$.

(iv) $\mathcal{M}_{\mathcal{F}}$ *is bounded on* $L_w^p(\mathbb{R}^N)$.

Using Lemma 2.4 and applying Rubio de Francia's extrapolation (see [García-Cuerva and Rubio de Francia 1985]), one can easily obtain the following weighted Fefferman–Stein vector-valued inequality.

Corollary 2.5. *Let $w \in A_p^{\mathcal{F}}(\mathbb{R}^N)$ and $\{f_j\}_{j\in\mathbb{Z}} \in L_w^p(\ell^q)$. Then, for all $1 < p, q < \infty$,*

$$\int_{\mathbb{R}^N} |\{\mathcal{M}_{\mathcal{F}}(\{f_j\})(x)\}|_{\ell^q}^p w(x)dx \le C \int_{\mathbb{R}^N} |\{f_j(x)\}|_{\ell^q}^p w(x)dx,$$

where $|\cdot|_{\ell^q}$ means the classical ℓ^q-norm.

We now are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let $f \in \mathcal{S}'_{\mathcal{F}}(\mathbb{R}^N)$ and $w \in A_\infty^{\mathcal{F}}(\mathbb{R}^N)$. In what follows, we use $a \wedge b$ to represent $\min\{a, b\}$. Denote $x_R = 2^J \ell$ and $x_{R'} = 2^{J'}\ell'$. Applying Theorem 2.1, Lemma 2.2 with

$$M > N[((q_w/p) + 1) \vee 2]$$

and $L = 10M$ and Lemma 2.3, we obtain that, for $\frac{N}{M} < \delta < \left(\frac{p}{q_w} \wedge 1\right)$ and for any $x \in R$,

$$\begin{aligned}
&|(\psi_J * f)(x_R)| \\
&\quad\approx \Bigg|\sum_{J'\in\mathbb{Z}^3} \sum_{R'\in\mathcal{R}_{\mathcal{F}}^{J'}} |R'|\psi_J * \varphi_{J'}(x_R - x_{R'})\varphi_J * f(x_{R'})\Bigg| \\
&\quad\lesssim \sum_{J'\in\mathbb{Z}^3} 2^{-(|j_1-j_1'|+|j_2-j_2'|+|j_3-j_3'|)L} \\
&\qquad\times \sum_{R'\in\mathcal{R}_{\mathcal{F}}^{J'}} |R'|\Bigg[\prod_{i=1}^{3} \frac{\max_{1\le k\le i} 2^{(j_k\vee j_k')M}}{(\max_{1\le k\le i} 2^{(j_k\vee j_k')} + |x_{Q_i} - x_{Q_i'}|)^{n_i+M}}\Bigg]|\varphi_{J'} * f(x_{R'})| \\
&\quad\lesssim \sum_{J'\in\mathbb{Z}^3} 2^{-(|j_1-j_1'|+|j_2-j_2'|+|j_3-j_3'|)L'} \Bigg\{\mathcal{M}_{\mathcal{F}}\Bigg[\Bigg(\sum_{R'\in\mathcal{R}_{\mathcal{F}}^{J'}} |\varphi_{J'} * f(x_{R'})|^2 \chi_{R'}\Bigg)^{\frac{\delta}{2}}\Bigg](x)\Bigg\}^{\frac{1}{\delta}},
\end{aligned}$$

where $L' = L - 3N(1/\delta - 1) > 7M > 0$.

Squaring both sides, then multiplying χ_R, summing over all $J \in \mathbb{Z}^3$ and $R \in \mathcal{R}_{\mathcal{F}}^J$, and finally applying Hölder's inequality, for all $x \in \mathbb{R}^N$ and $\frac{N}{M} < \delta < \left(\frac{p}{q_w} \wedge 1\right)$,

$$\begin{aligned}
&\sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_{\mathcal{F}}^J} |\psi_J * f(x_R)|^2 \chi_R \\
&\quad\lesssim \sum_{J\in\mathbb{Z}^3} \Big\{ \sum_{J'\in\mathbb{Z}^3} 2^{-(|j_1-j_1'|+|j_2-j_2'|+|j_3-j_3'|)L'} \Big\} \\
&\qquad \times \Big\{ \sum_{J'\in\mathbb{Z}^3} 2^{-(|j_1-j_1'|+|j_2-j_2'|+|j_3-j_3'|)L'} \Big\{ \mathcal{M}_{\mathcal{F}} \Big[\Big(\sum_{R'\in\mathcal{R}_{\mathcal{F}}^{J'}} |\varphi_{J'} * f(x_{R'})|^2 \chi_{R'} \Big)^{\frac{\delta}{2}} \Big](x) \Big\}^{\frac{2}{\delta}} \Big\} \\
&\quad\lesssim \sum_{J'\in\mathbb{Z}^3} \Big\{ \mathcal{M}_{\mathcal{F}} \Big[\Big(\sum_{R'\in\mathcal{R}_{\mathcal{F}}^{J'}} |\varphi_{J'} * f(x_{R'})|^2 \chi_{R'} \Big)^{\frac{\delta}{2}} \Big](x) \Big\}^{\frac{2}{\delta}},
\end{aligned}$$

where we used the estimates

$$\sum_{J'\in\mathbb{Z}^3} 2^{-(|j_1-j_1'|+|j_2-j_2'|+|j_3-j_3'|)L'} \le C \quad \text{and} \quad \sum_{J\in\mathbb{Z}^3} 2^{-(|j_1-j_1'|+|j_2-j_2'|+|j_3-j_3'|)L'} \le C$$

in the last inequality. Note that $(2 \wedge p)/\delta > q_w$ implies $w \in A_{p/\delta}^{\mathcal{F}}(\mathbb{R}^N)$. Applying Corollary 2.5 with $L_w^{p/\delta}(\ell^{2/\delta})$ yields

$$\Big\| \Big\{ \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_{\mathcal{F}}^J} |\psi_J * f(x_R)|^2 \chi_R \Big\}^{\frac{1}{2}} \Big\|_{L_w^p(\mathbb{R}^N)} \lesssim \Big\| \Big\{ \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_{\mathcal{F}}^J} |\varphi_J * f(x_R)|^2 \chi_R \Big\}^{\frac{1}{2}} \Big\|_{L_w^p(\mathbb{R}^N)}.$$

The converse inequality follows by symmetry. □

As a consequence of Theorem 1.4, we obtain a density result of $H_{\mathcal{F},w}^p$ which will be useful to show the $H_{\mathcal{F},w}^p \to L_w^p$ boundedness of operators, the weak density of $CMO_{\mathcal{F},w}^p$ and the Calderón–Zygmund decomposition for $H_{\mathcal{F},w}^p$.

Corollary 2.6. *Let $0 < p < \infty$ and $w \in A_\infty^{\mathcal{F}}(\mathbb{R}^N)$. Then $\mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$ is dense in $H_{\mathcal{F},w}^p(\mathbb{R}^N)$ and, in consequence, $L^2(\mathbb{R}^N) \cap H_{\mathcal{F},w}^p(\mathbb{R}^N)$ is dense in $H_{\mathcal{F},w}^p(\mathbb{R}^N)$.*

Proof. Let $f \in H_{\mathcal{F},w}^p(\mathbb{R}^N)$. For any fixed $L > 0$, denote

$$\mathscr{E}_L = \{(J, R) : |j_1|, |j_2|, |j_3| \le L,\ R \in \mathcal{R}_{\mathcal{F}}^J,\ R \subset B(0, L)\}$$

and

$$f_L(x) = \sum_{(J,R)\in\mathscr{E}_L} |R| \psi_J(x - x_R) \psi_J * f(x_R).$$

Since f_L is a finite linear combination of $\psi_J(\cdot - x_R) \in \mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$, it is obvious that $f_L \in \mathcal{S}_{\mathcal{F}}(\mathbb{R}^N)$. Repeating the proof of Theorem 1.4, we conclude that $\|f_L\|_{H_{\mathcal{F},w}^p(\mathbb{R}^N)} \lesssim \|f\|_{H_{\mathcal{F},w}^p(\mathbb{R}^N)}$.

To see that f_L tends to f in $H^p_{\mathcal{F},w}(\mathbb{R}^N)$, we use the discrete Calderón reproducing formula to write

$$[g_{\mathcal{F}}(f-f_L)(x)]^2 = \sum_{J'\in\mathbb{Z}^3}\sum_{R'\in\mathcal{R}^{J'}_{\mathcal{F}}}\Bigg|\sum_{(J,R)\in\mathscr{E}^c_L}|R|(\psi_{J'}*\psi_J)(x_{R'}-x_R)(\psi_J*f)(x_R)\Bigg|^2\chi_{R'}(x).$$

Now repeating the same argument as in the proof of Theorem 1.4, we get

$$\|g_{\mathcal{F}}(f-f_L)\|_{L^p_w(\mathbb{R}^N)}\lesssim\Bigg\|\bigg\{\sum_{(J,R)\in\mathscr{E}^c_L}|(\psi_J*f)(x_R)|^2\chi_R\bigg\}^{\frac12}\Bigg\|_{L^p_w(\mathbb{R}^N)},$$

where the last term tends to 0 as L goes to infinity. This implies that f_L tends to f in $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ and hence the proof is finished. □

We follow the classical case (see [Stein 1993; García-Cuerva and Rubio de Francia 1985]) to get the following lemma.

Lemma 2.7. *Suppose $w\in A^{\mathcal{F}}_\infty(\mathbb{R}^n)$ and $q>q_w$. There exist $0<\delta<1<q<\infty$ such that, for all flag rectangles R and all measurable subsets A of R,*

$$\left(\frac{|A|}{|R|}\right)^q\lesssim\frac{w(A)}{w(R)}\lesssim\left(\frac{|A|}{|R|}\right)^\delta.$$

In particular, the measure $w(x)dx$ is doubling with respect to flag rectangles.

Lemma 2.8. *Let $w\in A^{\mathcal{F}}_\infty(\mathbb{R}^N)$. Then, for all flag rectangles R and R' and for $q>q_w$,*

$$\frac{w(R')}{w(R)}\lesssim\prod_{i=1}^3\left(\frac{|Q_i|}{|Q'_i|}\vee\frac{|Q'_i|}{|Q_i|}\right)^q\left(1+\frac{|x_{Q_i}-x_{Q'_i}|}{\ell(Q_i)\vee\ell(Q'_i)}\right)^{n_iq}.$$

Proof. Observe that $Q'_i\subset A_iQ_i,\ i=1,2,3$, where

$$A_i=C[\ell(Q_i)\vee\ell(Q'_i)+|x_{Q_i}-x_{Q'_i}|]/\ell(Q_i).$$

This implies $R'\subset\bar R$, where $\bar R=C[(A_1Q_1)\times(A_2Q_2)\times(A_3Q_3)]$. By Lemma 2.7, for any $q>q_w$,

$$\begin{aligned}\frac{w(R')}{w(R)}\le\frac{w(\bar R)}{w(R)}\lesssim\left[\frac{|\bar R|}{|R|}\right]^q&\lesssim\prod_{i=1}^3\left[\frac{\ell(Q_i)\vee\ell(Q'_i)+|x_{Q_i}-x_{Q'_i}|}{\ell(Q_i)}\right]^{n_iq}\\&\lesssim\prod_{i=1}^3\left[\frac{|Q_i|}{|Q'_i|}\vee\frac{|Q'_i|}{|Q_i|}\right]^q\left[1+\frac{|x_{Q_i}-x_{Q'_i}|}{\ell(Q_i)\vee\ell(Q'_i)}\right]^{n_iq}.\end{aligned}$$

Hence the proof is concluded. □

We now prove Theorem 1.6.

Proof of Theorem 1.6. For $R \in \mathcal{R}_{\mathcal{F}}^{J}$ and $R' \in \mathcal{R}_{\mathcal{F}}^{J'}$, set

$$S_R = |\psi_J * f(x_R)|^2 \quad \text{and} \quad T_{R'} = |\varphi_{J'} * f(x_{R'})|^2.$$

Theorem 2.1 and Lemma 2.2 yield

$$\begin{aligned}
S_R^{\frac{1}{2}} &= \Big| \sum_{J' \in \mathbb{Z}^3} \sum_{R' \in \mathcal{R}_{\mathcal{F}}^{J'}} |R'| \varphi_{J'} * f(x_{R'}) \psi_J * \varphi_{J'}(x_R - x_{R'}) \Big| \\
&\lesssim \sum_{J' \in \mathbb{Z}^3} \sum_{R' \in \mathcal{R}_{\mathcal{F}}^{J'}} 2^{-(|j_1 - j_1'| + |j_2 - j_2'| + |j_3 - j_3'|)NL} |R'| \\
&\qquad \times \Bigg[\prod_{i=1}^{3} \frac{\max_{1 \le k \le i} 2^{(j_k \vee j_k')M}}{(\max_{1 \le k \le i} 2^{(j_k \vee j_k')} + |x_{Q_i} - x_{Q_i'}|)^{n_i + M}} \Bigg] |\varphi_{J'} * f(x_{R'})| \\
&\lesssim \sum_{J' \in \mathbb{Z}^3} \sum_{R' \in \mathcal{R}_{\mathcal{F}}^{J'}} r(R, R') P(R, R') T_{R'}^{\frac{1}{2}},
\end{aligned}$$

where

$$r(R, R') := \prod_{i=1}^{3} \left[\frac{|Q_i|}{|Q_i'|} \wedge \frac{|Q_i'|}{|Q_i|} \right]^L$$

and

$$P(R, R') := \prod_{i=1}^{3} \left(1 + \frac{|x_{Q_i} - x_{Q_i'}|}{\max_{1 \le k \le i} [\ell(Q_k) \vee \ell(Q_k')]} \right)^{-n_i - M}.$$

Squaring both sides, multiplying by $|R|^2 / w(R)$, adding up all the terms over $J \in \mathbb{Z}^3$, $R \in \mathcal{R}_{\mathcal{F}}^{J}$, $R \subset \Omega$ and applying Hölder's inequality, we obtain

$$\begin{aligned}
&\sup_{\Omega} \Bigg\{ \frac{1}{[w(\Omega)]^{2/p-1}} \sum_{\substack{R \in \mathcal{R}_{\mathcal{F}}^{d}, \\ R \subset \Omega}} |R|^2 w(R)^{-1} S_R \Bigg\} \\
&\quad \lesssim \sup_{\Omega} \Bigg\{ \frac{1}{[w(\Omega)]^{2/p-1}} \sum_{\substack{R \in \mathcal{R}_{\mathcal{F}}^{d}, \\ R \subset \Omega}} |R|^2 w(R)^{-1} \Bigg[\sum_{R' \in \mathcal{R}_{\mathcal{F}}^{d}} r(R, R') P(R, R') \Bigg] \\
&\qquad \times \Bigg[\sum_{R' \in \mathcal{R}_{\mathcal{F}}^{d}} r(R, R') P(R, R') T_{R'} \Bigg] \Bigg\} \\
&\quad \lesssim \sup_{\Omega} \Bigg\{ \frac{1}{[w(\Omega)]^{2/p-1}} \sum_{\substack{R \in \mathcal{R}_{\mathcal{F}}^{d}, \\ R \subset \Omega}} \sum_{R' \in \mathcal{R}_{\mathcal{F}}^{d}} |R|^2 w(R)^{-1} r(R, R') P(R, R')) T_{R'} \Bigg\}.
\end{aligned}$$

Here and hereafter, we use $\sum_{R\in\mathcal{R}^d_{\mathcal{F}}}$ to denote $\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^J_{\mathcal{F}}}$ and similarly for $\sum_{R'\in\mathcal{R}^d_{\mathcal{F}}}$. Applying Lemma 2.8, we get

$$\sup_{\Omega}\left\{\frac{1}{[w(\Omega)]^{2/p-1}}\sum_{\substack{R\in\mathcal{R}^d_{\mathcal{F}},\\ R\subset\Omega}}|R|^2w(R)^{-1}S_R\right\} \lesssim \sup_{\Omega}\left\{\frac{1}{[w(\Omega)]^{2/p-1}}\sum_{\substack{R\in\mathcal{R}^d_{\mathcal{F}},\\ R\subset\Omega}}\sum_{R'\in\mathcal{R}^d_{\mathcal{F}}}|R'|^2w(R')^{-1}\tilde{r}(R,R')\widetilde{P}(R,R'))T_{R'}\right\}. \tag{2-8}$$

Here the definition of $\tilde{r}(R,R')$ and $\widetilde{P}(R,R')$ are defined as $r(R,R')$ and $P(R,R')$ with smaller L and M. Since L and M can be chosen arbitrarily large, in what follows, we still use $r(R,R')$ and $P(R,R')$ to denote $\tilde{r}(R,R')$ and $\widetilde{P}(R,R')$, respectively.

To finish the proof, it suffices to show the right-hand side of (2-8) is bounded by

$$C\sup_{\bar{\Omega}}\left\{\frac{1}{[w(\bar{\Omega})]^{2/p-1}}\sum_{\substack{R'\in\mathcal{R}^d_{\mathcal{F}},\\ R'\subset\bar{\Omega}}}|R'|^2w(R')^{-1}T_{R'}\right\}.$$

We point out that $r(R,R')$ and $P(R,R')$ characterize the geometrical properties between two flag rectangles R and R'. Namely, when the difference of the sizes of R and R' grows bigger, $r(R,R')$ becomes smaller; when the distance between R and R' gets larger, $P(R,R')$ becomes smaller. The following argument is quite geometric. More precisely, we shall first decompose the set of dyadic flag rectangles $\{R'\}$ into annuli according to the distance of R and R'. Next, in each annulus, precise estimates are given by considering the difference of the sizes of R and R'. Finally, add up all the estimates in each annulus to finish the proof.

We now turn to details. For $J=(j_1,j_2,j_3)\in\mathbb{N}^3$ and $R\in\mathcal{R}^d_{\mathcal{F}}$, denote

$$R_J=R_{j_1,j_2,j_3}=(2^{j_1}Q_1)\times(2^{j_1\vee j_2}Q_2)\times(2^{j_1\vee j_2\vee j_3}Q_3),\qquad \Omega^J=\Omega^{j_1,j_2,j_3}=\bigcup_{R\subset\Omega}3R_J.$$

For any flag rectangle $R\subset\Omega$ and $J=(j_1,j_2,j_3)\in\mathbb{Z}^3_+$, let

$$\mathscr{A}_{0,0,0}(R)=\{R':3R'_{0,0,0}\cap 3R\neq\varnothing\},$$

$$\mathscr{A}_{j_1,0,0}(R)=\{R':3R'_{j_1,0,0}\cap 3R\neq\varnothing \text{ and } 3R'_{j_1-1,0,0}\cap 3R=\varnothing\},$$

$$\mathscr{A}_{0,j_2,0}(R)=\{R':3R'_{0,j_2,0}\cap 3R\neq\varnothing \text{ and } 3R'_{0,j_2-1,0}\cap 3R=\varnothing\},$$

$$\mathscr{A}_{0,0,j_3}(R)=\{R':3R'_{0,0,j_3}\cap 3R\neq\varnothing \text{ and } R'_{0,0,j_3-1}\cap 3R=\varnothing\},$$

$$\mathscr{A}_{j_1,j_2,0}(R)=\{R':3R'_{j_1,j_2,0}\cap 3R\neq\varnothing,\ 3R'_{j_1-1,j_2,0}\cap 3R=\varnothing \text{ and } 3R'_{j_1,j_2-1,0}\cap 3R=\varnothing\},$$

$$\mathscr{A}_{j_1,0,j_3}(R)=\{R':3R'_{j_1,0,j_3}\cap 3R\neq\varnothing,\ 3R'_{j_1-1,0,j_3}\cap 3R=\varnothing \text{ and } 3R'_{j_1,0,j_3-1}\cap 3R=\varnothing\},$$

$$\begin{aligned}
&\mathscr{A}_{0,j_2,j_3}(R)\\
&\quad=\{R' : 3R'_{0,j_2,j_3}\cap 3R\neq\varnothing,\ 3R'_{0,j_2-1,j_3}\cap 3R=\varnothing \text{ and } 3R'_{0,j_2,j_3-1}\cap 3R=\varnothing\},\\
&\mathscr{A}_{j_1,j_2,j_3}(R)\\
&\quad=\{R' : 3R'_{j_1,j_2,j_3}\cap 3R\neq\varnothing,\ 3R'_{j_1-1,j_2,j_3}\cap 3R=\varnothing, 3R'_{j_1,j_2-1,j_3}\cap 3R=\varnothing\\
&\qquad\qquad \text{and } 3R'_{j_1,j_2,j_3-1}\cap 3R=\varnothing\},
\end{aligned}$$

where all R' are dyadic flag rectangles.

Let $R\subset\Omega$ be any fixed dyadic flag rectangle. For any flag rectangle R', there exists a $J\in\mathbb{N}^3$ such that $R'\in\mathscr{A}_J(R)$, where $\mathscr{A}_J(R)$ represents one of the sets defined above; therefore $\mathcal{R}^d_{\mathcal{F}}=\bigcup_{J\in\mathbb{N}^3}\mathscr{A}_J(R)$. Hence,

$$\begin{aligned}
&\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{\substack{R\in\mathcal{R}^d_{\mathcal{F}},\\ R\subset\Omega}}\sum_{R'\in\mathcal{R}^d_{\mathcal{F}}}|R'|^2w(R')^{-1}r(R,R')P(R,R')T_{R'}\\
&\le\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{\substack{R\in\mathcal{R}^d_{\mathcal{F}},\\ R\subset\Omega}}\sum_{R'\in\mathscr{A}_{0,0,0}(R)}|R'|^2w(R')^{-1}r(R,R')P(R,R')T_{R'}\\
&\quad+\sum_{j_1\in\mathbb{Z}_+}\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{\substack{R\in\mathcal{R}^d_{\mathcal{F}},\\ R\subset\Omega}}\sum_{R'\in\mathscr{A}_{j_1,0,0}(R)}|R'|^2w(R')^{-1}r(R,R')P(R,R')T_{R'}\\
&\quad+\sum_{j_2\in\mathbb{Z}_+}\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{\substack{R\in\mathcal{R}^d_{\mathcal{F}},\\ R\subset\Omega}}\sum_{R'\in\mathscr{A}_{0,j_2,0}(R)}|R'|^2w(R')^{-1}r(R,R')P(R,R')T_{R'}\\
&\quad+\sum_{j_3\in\mathbb{Z}_+}\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{\substack{R\in\mathcal{R}^d_{\mathcal{F}},\\ R\subset\Omega}}\sum_{R'\in\mathscr{A}_{0,0,j_3}(R)}|R'|^2w(R')^{-1}r(R,R')P(R,R')T_{R'}\\
&\quad+\sum_{j_1,j_2\in\mathbb{Z}_+}\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{\substack{R\in\mathcal{R}^d_{\mathcal{F}},\\ R\subset\Omega}}\sum_{R'\in\mathscr{A}_{j_1,j_2,0}(R)}|R'|^2w(R')^{-1}r(R,R')P(R,R')T_{R'}\\
&\quad+\sum_{j_1,j_3\in\mathbb{Z}_+}\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{\substack{R\in\mathcal{R}^d_{\mathcal{F}},\\ R\subset\Omega}}\sum_{R'\in\mathscr{A}_{j_1,0,j_3}(R)}|R'|^2w(R')^{-1}r(R,R')P(R,R')T_{R'}\\
&\quad+\sum_{j_2,j_3\in\mathbb{Z}_+}\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{\substack{R\in\mathcal{R}^d_{\mathcal{F}},\\ R\subset\Omega}}\sum_{R'\in\mathscr{A}_{0,j_2,j_3}(R)}|R'|^2w(R')^{-1}r(R,R')P(R,R')T_{R'}\\
&\quad+\sum_{j_1,j_2,j_3\in\mathbb{Z}_+}\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{\substack{R\in\mathcal{R}^d_{\mathcal{F}},\\ R\subset\Omega}}\sum_{R'\in\mathscr{A}_{j_1,j_2,j_3}(R)}|R'|^2w(R')^{-1}r(R,R')P(R,R')T_{R'}\\
&:=I+II+III+IV+V+VI+VII+VIII.
\end{aligned}$$

In the sequel, we always assume $R, R' \in \mathcal{R}^d_{\mathcal{F}}$. To estimate I, we denote $\mathscr{B}_{0,0,0} = \{R' : 3R' \cap \Omega^{0,0,0} \neq \varnothing\}$. For any $R' \notin \mathscr{B}_{0,0,0}$, we have $3R' \cap \Omega^{0,0,0} = \varnothing$. This implies that $3R' \cap 3R = \varnothing$ for every $R \subset \Omega$ and thus $R' \notin \mathscr{A}_{0,0,0}(R)$. This shows that $\bigcup_{R\subset\Omega} \mathscr{A}_{0,0,0}(R) \subset \mathscr{B}_{0,0,0}$. Hence

$$I \leq \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{R'\in\mathscr{B}_{0,0,0}} \sum_{\substack{R:R\subset\Omega \\ R'\in\mathscr{A}_{0,0,0}(R)}} |R'|^2 w(R')^{-1} r(R, R') P(R, R') T_{R'}.$$

For each integer $h \geq 1$, let $\mathscr{F}_h^{0,0,0} = \{R' \in \mathscr{B}_{0,0,0}, |3R' \cap \Omega^{0,0,0}| \geq 1/2^h |3R'|\}$, $\mathscr{D}_h^{0,0,0} = \mathscr{F}_h^{0,0,0} \backslash \mathscr{F}_{h-1}^{0,0,0}$ and $\Omega_h^{0,0,0} = \bigcup_{R'\in\mathscr{D}_h^{0,0,0}} R'$. Note that $\mathscr{B}_{0,0,0} = \bigcup_{h\geq1} \mathscr{D}_h^{0,0,0}$ and that $P(R, R') \leq 1$ for any R, R'. Thus

$$I \leq \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{h\geq1} \sum_{R'\subset\Omega_h^{0,0,0}} |R'|^2 w(R')^{-1} T_{R'} \sum_{\substack{R:R\subset\Omega, \\ R'\in\mathscr{A}_{0,0,0}(R)}} r(R, R').$$

Note that for $x \in \Omega_h^{0,0,0}$, there exists a dyadic flag rectangle $R \subset \Omega_h^{0,0,0}$ such that $x \in R$. Therefore $\mathcal{M}_{\mathcal{F}}(\chi_{\Omega^{0,0,0}})(x) \geq |3R' \cap \Omega^{0,0,0}|/|3R'| \geq 2^{-h}$. For $q \in (q_w, pL/(2-p))$, we apply the $L^q_w(\mathbb{R}^N)$ boundedness of $\mathcal{M}_{\mathcal{F}}$ and Lemma 2.7 to obtain

$$w(\Omega_h^{0,0,0}) \leq w(\{x : \mathcal{M}_{\mathcal{F}}(\chi_{\Omega^{0,0,0}})(x) \geq 2^{-h}\}) \lesssim 2^{hq} w(\Omega^{0,0,0}) \lesssim 2^{hq} w(\Omega).$$

We claim that if L is sufficiently large, there is a sequence $\{\sigma_1(h)\}_{h\geq1}$ such that

$$\sum_{\substack{R:R\subset\Omega, \\ R'\in\mathscr{A}_{0,0,0}(R)}} r(R, R') \lesssim \sigma_1(h) \quad \text{and} \quad \sum_{h\geq1} \sigma_1(h) 2^{hq\left(\frac{2}{p}-1\right)} \lesssim 1. \tag{2-9}$$

Assume this claim holds for the moment, we can conclude the proof for I as follows.

$$\begin{aligned}
I &\lesssim \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{h\geq1} \sum_{R'\subset\Omega_h^{0,0,0}} \sigma_1(h) \frac{|R'|^2}{w(R')} T_{R'} \\
&\lesssim \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{h\geq1} \sigma_1(h) [w(\Omega_h^{0,0,0})]^{\frac{2}{p}-1} \frac{1}{[w(\Omega_h^{0,0,0})]^{\frac{2}{p}-1}} \sum_{R'\subset\Omega_h^{0,0,0}} \frac{|R'|^2}{w(R')} T_{R'} \\
&\lesssim \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{h\geq1} \sigma_1(h) (2^{hq})^{\frac{2}{p}-1} [w(\Omega)]^{\frac{2}{p}-1} \left(\sup_{\bar{\Omega}} \frac{1}{[w(\bar{\Omega})]^{\frac{2}{p}-1}} \sum_{R'\subset\bar{\Omega}} \frac{|R'|^2}{w(R')} T_{R'} \right) \\
&\lesssim \sup_{\bar{\Omega}} \frac{1}{[w(\bar{\Omega})]^{\frac{2}{p}-1}} \sum_{R'\subset\bar{\Omega}} \frac{|R'|^2}{w(R')} T_{R'}.
\end{aligned}$$

It is sufficient to show claim (2-9). Note that $R' \in \mathscr{A}_{0,0,0}(R)$ implies $3R \cap 3R' \neq \varnothing$. Using an idea of Chang and R. Fefferman [1980], for each R, we consider the following eight cases:

Case 1: $|Q_1'| \geq |Q_1|$, $|Q_2'| \geq |Q_2|$, $|Q_3'| \geq |Q_3|$.

Case 2: $|Q_1'| \geq |Q_1|$, $|Q_2'| \geq |Q_2|$, $|Q_3'| < |Q_3|$.

Case 3: $|Q_1'| \geq |Q_1|$, $|Q_2'| < |Q_2|$, $|Q_3'| \geq |Q_3|$.

Case 4: $|Q_1'| < |Q_1|$, $|Q_2'| \geq |Q_2|$, $|Q_3'| \geq |Q_3|$.

Case 5: $|Q_1'| \geq |Q_1|$, $|Q_2'| < |Q_2|$, $|Q_3'| < |Q_3|$.

Case 6: $|Q_1'| < |Q_1|$, $|Q_2'| \geq |Q_2|$, $|Q_3'| < |Q_3|$.

Case 7: $|Q_1'| < |Q_1|$, $|Q_2'| < |Q_2|$, $|Q_3'| \geq |Q_3|$.

Case 8: $|Q_1'| < |Q_1|$, $|Q_2'| < |Q_2|$, $|Q_3'| < |Q_3|$.

It suffices to verify (2-9) in each case.

We first consider Case 1. In this case,

$$|R| \leq |3R \cap 3R'| \leq |3R' \cap \Omega^{0,0,0}| \leq 2^{1-h}|3R'| \leq 2^{2N+1-h}|R'|,$$

which implies that $|R'| = 2^{h-1-2N+\theta}|R|$ for some integer $\theta \geq 0$. For each fixed θ, the number of such R's must be less than $C(\theta+h)^N 2^{\theta+h}$. Consequently,

$$\sum_{R\in\text{Case }1} r(R,R') \leq C\sum_{\theta\geq 0}\left(\frac{1}{2^{\theta+h}}\right)^L (\theta+h)^N 2^{\theta+h} \leq C2^{-hL'},$$

where $L' = L-(N+1) > 0$. This gives (2-9) with $\sigma_1(h) = 2^{-hL'}$.

We next deal with Case 2. We have

$$|3R'||Q_1 \times Q_2|/(2^{2N}|Q_1' \times Q_2'|) \leq |3R \cap 3R'| \leq 2^{1-h}|3R'|,$$

which implies that $|Q_1' \times Q_2'| = 2^{h+\theta-1-2N}|Q_1 \times Q_2|$ for some integer $\theta \geq 0$. For each fixed θ, the number of such $Q_1 \times Q_2$'s must be less than $C(\theta+h)^N \cdot 2^{\theta+h}$. Similarly, $|Q_3| = 2^{\lambda}|Q_3'|$ for some $\lambda \geq 0$. For each λ, $3Q_3 \cap 3Q_3' \neq \varnothing$ implies that the number of such Q_3's is less than 5^N. It follows that

$$\sum_{R\in\text{Case }2} r(R,R') \lesssim \sum_{\theta\geq 0}\sum_{\lambda\geq 0}\left(\frac{1}{2^{\theta+h+\lambda}}\right)^L (\theta+h)^N 2^{\theta+h} \lesssim 2^{-hL'},$$

which verifies (2-9) with $\sigma_1(h) = 2^{-hL'}$. Cases 3 and 4 can be handled by symmetry.

For Case 5, by the same argument as in Case 2, we get $|Q_1'| = 2^{h+\theta'-1-2N}|Q_1|$ and $|Q_2 \times Q_3| = 2^{\lambda'}|Q_2' \times Q_3'|$ for some θ', $\lambda' \geq 0$. Moreover, for each θ' and λ', the number of Q_1's is less than $C(\theta+h)^N \cdot 2^{\theta+h}$, and the number of $Q_2 \times Q_3$ less than C. Then (2-9) follows in the same way as in Case 2. Cases 6 and 7 can be treated similarly.

For Case 8, we have

$$|R'| \le |3R \cap 3R'| \le |3R' \cap \Omega^{0,0,0}| \le 2^{1-h}|3R'| \le 2^{1-h+2N}|R'|,$$

which implies $h \le 2N+1$. Since in this case $|R'| \le |R|$, we have $|R| = 2^\theta |R'|$ for some integer $\theta \ge 0$. For each fixed θ, the number of such R's must be less than 5^N. Therefore,

$$\sum_{R \in \text{Case 8}} r(R, R') \lesssim \chi_{\mathbb{Z}\cap(0,2N+1]}(h) \sum_{\theta \ge 0} \left(\frac{1}{2^\theta}\right)^L \lesssim \chi_{\mathbb{Z}\cap(0,2N+1]}(h),$$

which verifies (2-9) with $\sigma_1(h) = \chi_{\mathbb{Z}\cap(0,2N+1]}(h)$. This concludes all estimates for I.

We next deal with $VIII$. For $J = (j_1, j_2, j_3)$ with $j_1, j_2, j_3 \ge 1$, set

$$a_J = a_{j_1,j_2,j_3} := \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{R \subset \Omega} \sum_{R' \in \mathscr{A}_J(R)} |R'|^2 w(R')^{-1} r(R, R') P(R, R') T_{R'}$$

and $\mathscr{B}_J := \{R' : R'_J \cap \Omega^{0,0,0} \ne \varnothing\}$. For any $R' \notin \mathscr{B}_J$, we have $R'_J \cap \Omega^{0,0,0} = \varnothing$. Thus for every $R \subset \Omega$, we have $R'_J \cap 3R = \varnothing$, which implies $R' \notin \mathscr{A}_J(R)$ and therefore $\bigcup_{R\subset\Omega} \mathscr{A}_J(R) \subset \mathscr{B}_J$. Hence,

$$a_J \le \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{R' \in \mathscr{B}_J} \sum_{\substack{R: R\subset\Omega, \\ R' \in \mathscr{A}_J(R)}} |R'|^2 w(R')^{-1} r(R, R') P(R, R') T_{R'}.$$

Let $\mathscr{F}_h^J = \{R' \in \mathscr{B}_J : |R'_J \cap \Omega^{0,0,0}| \ge 1/2^h |R'_J|\}$ for $h \ge 0$, $\mathscr{D}_h^J = \mathscr{F}_h^J \setminus \mathscr{F}_{h-1}^J$ for $h \ge 1$, and $\mathscr{D}_0^J = \varnothing$. Note that $\mathscr{B}_J = \bigcup_{h\ge 1} \mathscr{D}_h^J$. Thus,

$$a_J \le \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{h \ge 1} \sum_{R' \in \mathscr{D}_h^J} |R'|^2 w(R')^{-1} T_{R'} \Bigg(\sum_{\substack{R: R\subset\Omega, \\ R' \in \mathscr{A}_J(R)}} r(R, R') P(R, R') \Bigg).$$

Define $\Omega_h^J = \bigcup_{R' \in \mathscr{D}_h^J} R'$ for $h \ge 1$. If $x \in \Omega_h^J$, then $x \in \widetilde{R}'$ for some dyadic flag rectangle $\widetilde{R}' \in \mathscr{D}_h^J$; therefore

$$\mathcal{M}_{\mathcal{F}}(\chi_{\Omega^{0,0,0}})(x) \ge |\widetilde{R}'_J \cap \Omega^{0,0,0}| / |\widetilde{R}'_J| \ge 2^{-h}.$$

Taking $L > n_1 q_w \big(\frac{2}{p} - 1\big) + 2N$, $q \in \big(q_w, \frac{p(L-2N)}{n_1(2-p)}\big)$ and applying the L^q_w boundedness of $\mathcal{M}_{\mathcal{F}}$ and Lemma 2.7, we get

$$w(\Omega_h^J) \le w(\{x : \mathcal{M}_{\mathcal{F}}(\chi_{\Omega^{0,0,0}})(x) \ge 2^{-h}\}) \lesssim 2^{hq} w(\Omega^{0,0,0}) \lesssim 2^{hq} w(\Omega).$$

We claim that for L large enough, $M > NL$, and any $R' \in \mathscr{D}_h^J$, there exists some sequence $\{\sigma_8(h, j_1, j_2, j_3)\}_{h, j_1, j_2, j_3 \geq 1}$ such that

$$\begin{cases} \displaystyle\sum_{h, j_1, j_2, j_3 \geq 1} 2^{qh\left(\frac{2}{p}-1\right)} \sigma_8(h, j_1, j_2, j_3) \lesssim 1, \\ \displaystyle\sum_{\substack{R: R \subset \Omega, \\ R' \in \mathscr{A}_J(R)}} r(R, R') P(R, R') \lesssim \sigma_8(h, j_1, j_2, j_3), \qquad \forall\, j_1, j_2, j_3 \geq 1. \end{cases} \tag{2-10}$$

Assuming (2-10) holds for the moment, we can conclude the proof for *VIII* as follows.

$$\begin{aligned} &\sum_{j_1, j_2, j_3 \geq 1} a_J \\ &\lesssim \sum_{j_1, j_2, j_3, h \geq 1} \sigma(h, j_1, j_2, j_3) \left(\frac{w(\Omega_h^J)}{w(\Omega)} \right)^{\frac{2}{p}-1} \frac{1}{[w(\Omega_h^J)]^{\frac{2}{p}-1}} \sum_{R' \subset \Omega_h^J} |R'|^2 w(R')^{-1} T_{R'} \\ &\lesssim \sum_{j_1, j_2, j_3, h \geq 1} \sigma(h, j_1, j_2, j_3) 2^{qh\left(\frac{2}{p}-1\right)} \left(\sup_{\bar{\Omega}} \frac{1}{[w(\bar{\Omega})]^{\frac{2}{p}-1}} \sum_{R' \subset \bar{\Omega}} |R'|^2 w(R')^{-1} T_{R'} \right) \\ &\lesssim \sup_{\bar{\Omega}} \frac{1}{[w(\bar{\Omega})]^{\frac{2}{p}-1}} \sum_{R' \subset \bar{\Omega}} |R'|^2 w(R')^{-1} T_{R'}. \end{aligned}$$

To finish the estimate of *VIII*, it suffices to show (2-10). For any $j_1, j_2, j_3 \geq 1$, $R' \in \mathscr{A}_J(R)$ implies

$$|x_{Q_i} - x_{Q_i'}| > [2^{\max_{1 \leq k \leq i} j_k} \ell(Q_i')] \vee \ell(Q_i) \quad \text{for } i = 1, 2, 3. \tag{2-11}$$

Similar to the proof for I, we consider the following eight cases:

Case 1: $|2^{j_1} Q_1'| \geq |Q_1|,\ |2^{j_1 \vee j_2} Q_2'| \geq |Q_2|,\ |2^{j_1 \vee j_2 \vee j_3} Q_3'| \geq |Q_3|$.

Case 2: $|2^{j_1} Q_1'| \geq |Q_1|,\ |2^{j_1 \vee j_2} Q_2'| \geq |Q_2|,\ |2^{j_1 \vee j_2 \vee j_3} Q_3'| < |Q_3|$.

Case 3: $|2^{j_1} Q_1'| \geq |Q_1|,\ |2^{j_1 \vee j_2} Q_2'| < |Q_2|,\ |2^{j_1 \vee j_2 \vee j_3} Q_3'| \geq |Q_3|$.

Case 4: $|2^{j_1} Q_1'| < |Q_1|,\ |2^{j_1 \vee j_2} Q_2'| \geq |Q_2|,\ |2^{j_1 \vee j_2 \vee j_3} Q_3'| \geq |Q_3|$.

Case 5: $|2^{j_1} Q_1'| \geq |Q_1|,\ |2^{j_1 \vee j_2} Q_2'| < |Q_2|,\ |2^{j_1 \vee j_2 \vee j_3} Q_3'| < |Q_3|$.

Case 6: $|2^{j_1} Q_1'| < |Q_1|,\ |2^{j_1 \vee j_2} Q_2'| \geq |Q_2|,\ |2^{j_1 \vee j_2 \vee j_3} Q_3'| < |Q_3|$.

Case 7: $|2^{j_1} Q_1'| < |Q_1|,\ |2^{j_1 \vee j_2} Q_2'| < |Q_2|,\ |2^{j_1 \vee j_2 \vee j_3} Q_3'| \geq |Q_3|$.

Case 8: $|2^{j_1} Q_1'| < |Q_1|,\ |2^{j_1 \vee j_2} Q_2'| < |Q_2|,\ |2^{j_1 \vee j_2 \vee j_3} Q_3'| < |Q_3|$.

We shall handle Case 5 first. Since $|(2^{j_1 \vee j_2} Q_2') \times (2^{j_1 \vee j_2 \vee j_3} Q_3')| \leq |Q_2 \times Q_3|$, there exists some $\kappa \geq 0$ such that $2^{(j_1 \vee j_2) n_2 + (j_1 \vee j_2 \vee j_3) n_3 + \kappa} |Q_2' \times Q_3'| = |Q_2 \times Q_3|$. And for each κ, the number of such $Q_2 \times Q_3$'s must be $\lesssim 1$.

For each $h \geq 1$ and $R' \in \mathscr{D}_h^J$, since $|Q_1 \times [2^{j_1 \vee j_2} Q_2'] \times [2^{j_1 \vee j_2 \vee j_3} Q_3']| \leq |3R_J' \cap 3R|$, we get

$$\frac{|Q_1|}{|3 \cdot 2^{j_1} Q_1'|}|3R_J'| \leq |3R_J' \cap 3R| \leq |3R_J' \cap \Omega^{0,0,0}| \leq \frac{1}{2^{h-1}}|3R_J'|,$$

which yields $2^{h-1}|Q_1| \leq 3^{n_1} 2^{j_1 n_1}|Q_1'| \leq 2^{(j_1+2)n_1}|Q_1'|$. We consider two subcases.

Subcase 5.1: $|Q_1'| \geq |Q_1|$. In this subcase, since $2^{h-1-j_1 n_1}|Q_1| \lesssim |Q_1'|$, we have $|Q_1'| \approx 2^{h-1-j_1 n_1+k}|Q_1|$ for some integer $k \geq 0$. And for each fixed k, the number of such Q_1's must be $\lesssim (k+h)^N 2^{k+h}$. By (2-11), $|x_{Q_1} - x_{Q_1'}| \geq 2^{j_1}\ell(Q')$.

Subcase 5.2: $|Q_1'| < |Q_1|$. In this subcase, $|Q_1'| < |Q_1| \leq |2^{j_1} Q_1'|$. So $2^l \ell(Q_1') = \ell(Q_1)$ for some integer l satisfying $1 \leq l \leq j_1$. For each l, the number of Q_1's must be $\lesssim 1$. Moreover, $2^{h-1}2^{l n_1}|Q_1'| = 2^{h-1}|Q_1| \leq 2^{(j_1+2)n_1}|Q_1'|$, which implies $h \leq 3n_1 j_1$. By (2-11),

$$\frac{|x_{Q_1} - x_{Q_1'}|}{\ell(Q_1)} = \frac{|x_{Q_1} - x_{Q_1'}|}{\ell(Q_1')}\frac{\ell(Q_1')}{\ell(Q_1)} \gtrsim 2^{j_1 - l}.$$

These considerations imply that, for $M > n_1 L$,

$$\begin{aligned}
&\sum_{\text{Subcase 5.1}} r(R, R')P(R, R') \\
&\quad\leq \sum_{\text{Subcase 5.1}} \left(\frac{|Q_1|}{|Q_1'|}\right)^L \left(\frac{|Q_2' \times Q_3'|}{|Q_2 \times Q_3|}\right)^L \left(1 + \frac{|x_{Q_1} - x_{Q_1'}|}{\ell(Q_1')}\right)^{-(n_1+M)} \\
&\quad\lesssim \sum_{k,\kappa \geq 0} (k+h)^N 2^{k+h} 2^{-[h+k-j_1 n_1]L} 2^{-[(j_1 \vee j_2)n_2 + (j_1 \vee j_2 \vee j_3)n_3 + \kappa]L} 2^{-(n_1+M)j_1} \\
&\quad\lesssim 2^{-h(L-N-1)} 2^{-j_1(M-n_1 L)} 2^{-[(j_1 \vee j_2)n_2 + (j_1 \vee j_2 \vee j_3)n_3]L}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\text{Subcase 5.2}} r(R, R')P(R, R') \\
&\qquad\leq \sum_{\text{Subcase 5.2}} \left(\frac{|Q_1'|}{|Q_1|}\right)^L \left(\frac{|Q_2' \times Q_3'|}{|Q_2 \times Q_3|}\right)^L \left(1 + \frac{|x_{Q_1} - x_{Q_1'}|}{\ell(Q_1)}\right)^{-(n_1+M)} \\
&\qquad\lesssim \chi_{\mathbb{Z} \cap (0, 3N j_1)}(h) \sum_{l=1}^{j_1} \sum_{\kappa \geq 0} 2^{-n_1 l L} 2^{-[(j_1 \vee j_2)n_2 + (j_1 \vee j_2 \vee j_3)n_3 + \kappa]L} 2^{-M(j_1 - l)} \\
&\qquad\lesssim \chi_{\mathbb{Z} \cap (0, 3N j_1)}(h) 2^{-j_1 n_1 L} 2^{-[(j_1 \vee j_2)n_2 + (j_1 \vee j_2 \vee j_3)n_3]L},
\end{aligned}$$

both of which satisfy the required estimates for σ_8 in (2-10). Cases 6 and 7 can be treated similarly by symmetry.

We next deal with Case 2. Since $|2^{j_1\vee j_2\vee j_3}Q_3'|<|Q_3|$, there exists some $\kappa\geq 0$ such that $|Q_3|=2^{(j_1\vee j_2\vee j_3)n_3+\kappa}|Q_3'|$. From

$$|Q_1\times Q_2\times(2^{j_1\vee j_2\vee j_3}Q_3')|\leq|3R_J'\cap 3R|\leq|3R_J'\cap\Omega^{0,0,0}|\leq\frac{1}{2^{h-1}}|3R_J'|,$$

it follows that

$$2^{h-1}|Q_1\times Q_2|\lesssim 2^{j_1n_1+(j_1\vee j_2)n_2}|Q_1'\times Q_2'|. \tag{2-12}$$

We consider four subcases by comparing $|Q_1|$ with $|Q_1'|$, and $|Q_2|$ with $|Q_2'|$.

Subcase 2.1: $|Q_1'|\geq|Q_1|$ and $|Q_2'|\geq|Q_2|$. By arguments similar to those used in Subcase 5.1, we get $|Q_1'|\approx 2^{h-1-j_1n_1+k_1}|Q_1|$ and $|Q_2'|\approx 2^{h-1-(j_1\vee j_2)n_2+k_2}|Q_2|$ for some $k_1,k_2\geq 0$. For each k_i, $i=1,2$, the number of Q_i's is smaller than $C(k_i+h)2^{k_i+h}$. Hence

$$\begin{aligned}\sum_{\text{Subcase 2.1}} & r(R,R')P(R,R')\\ &\lesssim\sum_{k_1,k_2,\kappa\geq 0}(k_1+h)^N(k_2+h)^N2^{k_1+h}2^{k_2+h}2^{-(h+k_1-j_1n_1)L}2^{-(h+k_2-(j_1\vee j_2)n_2)L}\\ &\qquad\qquad\times 2^{-[(j_1\vee j_2\vee j_3)n_3+\kappa]L}2^{-j_1(n_1+M)-(j_1\vee j_2)(n_2+M)}\\ &\lesssim 2^{-h(L-N-1)}2^{-j_1(M-n_1L)}2^{-(j_1\vee j_2)(M-n_2L)}2^{-(j_1\vee j_2\vee j_3)n_3L}.\end{aligned}$$

Subcase 2.2: $|Q_1'|\geq|Q_1|$ and $|Q_2'|<|Q_2|$. By (2-12),

$$2^{h-1}|Q_1|\lesssim 2^{j_1n_1+(j_1\vee j_2)n_2}|Q_1'|,$$

and therefore $2^{h-1+k}|Q_1|\approx 2^{j_1n_1+(j_1\vee j_2)n_2}|Q_1'|$ for some $k\geq 0$. Moreover, for each $k\geq 0$, the number of Q_1's is less than $C(h+k)^N2^{h+k}$. On the other hand, since $|Q_2'|<|Q_2|\leq|2^{j_1\vee j_2}Q_2'|$, we see that $\ell(Q_2)=2^l\ell(Q_2')$ for some $1\leq l\leq j_1\vee j_2$. For each l, the number of Q_2's must be $\lesssim 1$. Moreover, by (2-11),

$$\frac{|x_{Q_2}-x_{Q_2'}|}{\ell(Q_2)}=\frac{|x_{Q_2}-x_{Q_2'}|}{\ell(Q_2')}\frac{\ell(Q_2')}{\ell(Q_2)}\gtrsim 2^{j_1\vee j_2-l}.$$

By the above considerations, we can deduce that

$$\begin{aligned}\sum_{\text{Subcase 2.2}} & r(R,R')P(R,R')\\ \leq&\sum_{\text{Subcase 2.2}}\left(\frac{|Q_1|}{|Q_1'|}\frac{|Q_2'|}{|Q_2|}\frac{|Q_3'|}{|Q_3|}\right)^L\left(\frac{|x_{Q_1}-x_{Q_1'}|}{\ell(Q_1')}\right)^{-(n_1+M)}\left(\frac{|x_{Q_2}-x_{Q_2'}|}{\ell(Q_2)}\right)^{-(n_2+M)}\\ \lesssim&\sum_{k,\kappa\geq 0}\sum_{l=1}^{j_1\vee j_2}(k+h)^N2^{k+h}2^{-[h+k-j_1n_1-(j_1\vee j_2)n_2]L}2^{-ln_2L}\\ &\qquad\qquad\times 2^{-[(j_1\vee j_2\vee j_3)n_3+\kappa]L}2^{-(n_1+M)j_1}2^{-M(j_1\vee j_2-l)}\\ \lesssim&\, 2^{-h(L-N-1)}2^{-j_1(M-n_1L)}2^{-[(j_1\vee j_2)n_2+(j_1\vee j_2\vee j_3)n_3]L}.\end{aligned}$$

Subcase 2.3: $|Q_1'| < |Q_1|$ and $|Q_2'| \ge |Q_2|$. This is symmetric to Subcase 2.2, and similar arguments yield

$$\sum_{\text{Subcase 2.3}} r(R, R')P(R, R') \lesssim 2^{-h(L-N-1)} 2^{-(j_1\vee j_2)(M-n_2L)} 2^{-[j_1n_1+(j_1\vee j_2\vee j_3)n_3]L}.$$

Subcase 2.4: $|Q_1'| < |Q_1|$ and $|Q_2'| < |Q_2|$. Since $|Q_1'| < |Q_1| \le |2^{j_1} Q_1'|$ and $|Q_2'| < |Q_2| \le |2^{j_1\vee j_2} Q_2'|$, we see that $\ell(Q_1) = 2^{l_1}\ell(Q_1')$ for some $1 \le l_1 \le j_1$ and that $\ell(Q_2) = 2^{l_2}\ell(Q_2')$ for some $1 \le l_2 \le j_1 \vee j_2$. For each l_1 and l_2, the numbers of Q_1's and Q_2's are both $\lesssim 1$. Moreover, by (2-11),

$$\frac{|x_{Q_1} - x_{Q_1'}|}{\ell(Q_1)} \gtrsim 2^{j_1-l_1}, \qquad \frac{|x_{Q_2} - x_{Q_2'}|}{\ell(Q_2)} \gtrsim 2^{j_1\vee j_2-l_2}.$$

By (2-12), $2^{h-1}2^{l_1n_1+l_2n_2}|Q_1' \times Q_2'| = 2^{h-1}|Q_1 \times Q_2| \lesssim 2^{j_1n_1+(j_1\vee j_2)n_2}|Q_1' \times Q_2'|$, and thus $2^h \lesssim 2^{j_1n_1+(j_1\vee j_2)n_2}$. Hence

$$\begin{aligned}
&\sum_{\text{Subcase 2.4}} r(R, R')P(R, R') \\
&\le \sum_{\text{Subcase 2.4}} \left(\frac{|Q_1' \times Q_2'|}{|Q_1 \times Q_2|}\right)^L \left(\frac{|Q_3'|}{|Q_3|}\right)^L \left(1 + \frac{|x_{Q_1} - x_{Q_1'}|}{\ell(Q_1)}\right)^{-(n_1+M)} \\
&\qquad\qquad \times \left(1 + \frac{|x_{Q_2} - x_{Q_2'}|}{\ell(Q_2)}\right)^{-(n_2+M)} \\
&\lesssim \sum_{l_1=1}^{j_1} \sum_{l_2=1}^{j_1\vee j_2} \sum_{\kappa\ge 0} 2^{-(n_1l_1+n_2l_2)L} 2^{-[(j_1\vee j_2\vee j_3)n_3+\kappa]L} 2^{-M(j_1-l_1)} \\
&\qquad\qquad \times 2^{-M(j_1\vee j_2-l_2)} \chi_{\mathbb{Z}\cap(0,\, C[j_1\vee j_2])}(h) \\
&\lesssim 2^{-j_1n_1L} 2^{-(j_1\vee j_2)n_2L} 2^{-(j_1\vee j_2\vee j_3)n_3L} \chi_{\mathbb{Z}\cap(0,\, C[j_1\vee j_2])}(h).
\end{aligned}$$

Cases 3 and 4 can be treated by symmetry. Case 8 is easier, and can be handled similarly. For Case 1, we consider eight subcases by comparing $|Q_i|$ with $|Q_i'|$ for $i = 1, 2, 3$; in each subcase the desired estimate can be proved by using arguments given in Subcases 2.1–2.4. This concludes all estimates for *VIII*.

Finally, the remaining terms *II*–*VII* can be handled similarly. We only consider *VII* and indicate the necessary modifications. As before we consider eight cases by comparing $|Q_1'|$ with $|Q_1|$, $|2^{j_2} Q_2'|$ with $|Q_2|$, and $|2^{j_2\vee j_3} Q_3'|$ with $|Q_3|$, and matters are reduced to showing certain decay estimates for

$$\sum_{\substack{R:R\subset\Omega,\\ R'\in\mathscr{A}_{0,j_1,j_2}(R)}} r(R, R')P(R, R')$$

in each case. Recall that, in dealing with *VIII*, we used (2-11) to derive certain decay factors in j_1, in order to sum over $j_1 \in \mathbb{Z}_+$. But for *VII*, the decay factors in j_1 are no longer needed, and we use the trivial inequality instead

$$P_1(Q_1, Q_1') := \left(1 + \frac{|x_{Q_1} - x_{Q_1'}|}{\ell(Q_1') \vee \ell(Q_1)}\right)^{-(n_1+M)} \leq 1.$$

The rest of the proof is similar to that for *VIII*, and we omit the details.

This completes the proof of Theorem 1.6. □

To show that $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$ is the dual space of $H^p_{\mathcal{F},w}(\mathbb{R}^N)$, we introduce the multiparameter flag weighted sequence spaces.

Definition 2.9. Let $0 < p \leq 1$ and $w \in A_\infty^{\mathcal{F}}(\mathbb{R}^N)$. We use $s^p_w(\mathbb{R}^N)$ to express the collection of all sequences $\{s_R\}$ satisfying

$$\|\{s_R\}\|_{s^p_w(\mathbb{R}^N)} := \left\| \left\{ \sum_{J \in \mathbb{Z}^3} \sum_{R \in \mathcal{R}^J_{\mathcal{F}}} \frac{|s_R|^2}{|R|} \chi_R \right\}^{\frac{1}{2}} \right\|_{L^p_w(\mathbb{R}^N)} < \infty.$$

We also use $c^p_w(\mathbb{R}^N)$ to denote the collection of all sequences $\{t_R\}$ such that

$$\|\{t_R\}\|_{c^p_w(\mathbb{R}^N)} := \sup_{\Omega \subset \mathbb{R}^N} \left\{ \frac{1}{[w(\Omega)]^{2/p-1}} \sum_{J \in \mathbb{Z}^3} \sum_{\substack{R \in \mathcal{R}^J_{\mathcal{F}}, \\ R \subset \Omega}} |t_R|^2 \frac{|R|}{w(R)} \right\}^{\frac{1}{2}} < \infty,$$

where the supremum runs over all open sets Ω with $w(\Omega) < \infty$.

We will show the duality relationship between s^p_w and c^p_w.

Theorem 2.10. *Let* $0 < p \leq 1$. *Then* $(s^p_w(\mathbb{R}^N))^* = c^p_w(\mathbb{R}^N)$. *More precisely, for every* $\{t_R\} \in c^p_w(\mathbb{R}^N)$, *the mapping* $\ell_s : \{s_R\} \mapsto \sum_R s_R \bar{t}_R$ *defines a continuous linear functional on* $s^p_w(\mathbb{R}^N)$ *with operator norm* $\|\ell_s\| \lesssim \|t\|_{c^p_w(\mathbb{R}^N)}$. *Conversely, for every* $\ell \in (s^p_w(\mathbb{R}^N))^*$, *there is a unique* $\{t_R\} \in c^p_w(\mathbb{R}^N)$ *such that* $\ell(s_R) = \sum_R s_R \bar{t}_R$ *and* $\|\{t\}_R\|_{c^p_w} \lesssim \|\ell\|$.

Proof. We first prove $c^p_w(\mathbb{R}^N) \subset (s^p_w(\mathbb{R}^N))^*$. Suppose that $\{t_R\} \in c^p_w(\mathbb{R}^N)$. For $\{s_R\} \in s^p_w(\mathbb{R}^N)$, let

$$\mathcal{G}(\{s_R\})(x) = \left\{ \sum_{J \in \mathbb{Z}^3} \sum_{R \in \mathcal{R}^J_{\mathcal{F}}} |s_R|^2 |R|^{-1} \chi_R(x) \right\}^{\frac{1}{2}}.$$

For $i \in \mathbb{Z}$, set $\Omega_i = \{x \in \mathbb{R}^N : \mathcal{G}(\{s_R\})(x) > 2^i\}$, $\widetilde{\Omega}_i = \{x \in \mathbb{R}^N : \mathcal{M}_{\mathcal{F}}(\chi_{\Omega_i})(x) > 1/2\}$, and $\mathscr{B}_i = \{R \in \mathcal{R}_{\mathcal{F}} : |R \cap \Omega_i| > 1/2|R|,\ |R \cap \Omega_{i+1}| \leq 1/2|R|\}$. If $x \in R \in \mathscr{B}_i$, then $\mathcal{M}_{\mathcal{F}}(\chi_{\Omega_i})(x) \geq \frac{1}{|R|} \int_R \chi_{\Omega_i}(y)dy = |R \cap \Omega_i|/|R| > 1/2$, which implies

$$\bigcup_{R \in \mathscr{B}_i} R \subset \widetilde{\Omega}_i. \tag{2-13}$$

Moreover, for $q > q_w$, by the $L^q_w(\mathbb{R}^N)$ boundedness of $\mathcal{M}_{\mathcal{F}}$,

$$w(\widetilde{\Omega}_i) \lesssim w(\Omega_i), \tag{2-14}$$

and by Lemma 2.7,

$$\frac{w(R\cap(\Omega_i\setminus\Omega_{i+1}))}{w(R)} = \frac{w(R\setminus\Omega_{i+1})}{w(R)} \gtrsim \left(\frac{|R\setminus\Omega_{i+1}|}{|R|}\right)^q \geq \frac{1}{2^q}. \tag{2-15}$$

Suppose $\{t_R\} \in c^p_w(\mathbb{R}^N)$. By (2-13)–(2-15) and Schwarz's inequality,

$$\begin{aligned}
\left|\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^J_{\mathcal{F}}} s_R\bar{t}_R\right| &\lesssim \left|\sum_{i\in\mathbb{Z}}\int_{\widetilde{\Omega}_i\setminus\Omega_{i+1}}\sum_{R\in\mathscr{B}_i}|\bar{t}_R|\frac{|R|^{\frac12}}{w(R)}|s_R|\,|R|^{-\frac12}\chi_R(x)w(x)dx\right| \\
&\leq \sum_{i\in\mathbb{Z}}\left\{\sum_{R\subset\widetilde{\Omega}_i}|t_R|^2\frac{|R|}{w(R)}\right\}^{\frac12}\left\{\int_{\widetilde{\Omega}_i\setminus\Omega_{i+1}}\sum_{R\in\mathscr{B}_i}\frac{|s_R|^2}{|R|}\chi_R(x)w(x)dx\right\}^{\frac12} \\
&\lesssim \|\{t_R\}\|_{c^p_w}\sum_{i\in\mathbb{Z}}[w(\widetilde{\Omega}_i)]^{\left(\frac2p-1\right)\frac12}\left\{\int_{\widetilde{\Omega}_i\setminus\Omega_{i+1}}[\mathcal{G}(\{s_R\})(x)]^2w(x)dx\right\}^{\frac12} \\
&\lesssim \|\{t_R\}\|_{c^p_w}\sum_{i\in\mathbb{Z}}2^i[w(\Omega_i)]^{\frac1p} \\
&\lesssim \|\{t_R\}\|_{c^p_w}\|\mathcal{G}(\{s_R\})\|_{L^p_w} = \|\{t_R\}\|_{c^p_w}\|\{s_R\}\|_{s^p_w},
\end{aligned}$$

which implies the inclusion $c^p_w(\mathbb{R}^N) \subset (s^p_w(\mathbb{R}^N))^*$.

For the converse, we assume that $\ell \in (s^p_w(\mathbb{R}^N))^*$. Then it is clear that $\ell(\{s_R\}) = \sum_R s_R\bar{t}_R$ for some $\{t_R\}$. Now fix an open set $\Omega\subset\mathbb{R}^N$ with $w(\Omega)<\infty$. Let μ be a measure of $\mathcal{R}_{\mathcal{F}}$ such that $\mu(R) = [w(\Omega)]^{1-2/p}|R|[w(R)]^{-1}$ if $R\subset\Omega$ and otherwise $\mu(R)=0$. Set

$$\|\{s_R\}\|_{\ell^2(\Omega,\mu)} = \left\{\sum_{J\in\mathbb{Z}^3}\sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}},\\ R\subset\Omega}}|s_R|^2[w(\Omega)]^{1-\frac2p}\frac{|R|}{w(R)}\right\}^{\frac12}.$$

Then

$$\begin{aligned}
&\left\{\frac{1}{[w(\Omega)]^{2/p-1}}\sum_{J\in\mathbb{Z}^3}\sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}},\\ R\subset\Omega}}|t_R|^2\frac{|R|}{w(R)}\right\}^{\frac12} \\
&\qquad = \|\{t_R\}\|_{\ell^2(\Omega,\mu)} = \sup_{\|\{s_R\}\|_{\ell^2(\Omega,\mu)}\leq 1}\left|\sum_{J\in\mathbb{Z}^3}\sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}},\\ R\subset\Omega}}s_R\bar{t}_R[w(\Omega)]^{1-2/p}\frac{|R|}{w(R)}\right| \\
&\qquad \leq \|\ell\|\sup_{\|\{s_R\}\|_{\ell^2(\Omega,\mu)}\leq 1}\left\|s_R[w(\Omega)]^{1-2/p}\frac{|R|}{w(R)}\right\|_{s^p_w},
\end{aligned}$$

where $\{s_R\}$ satisfies $s_R = 0$ if R is not contained in Ω. However, for such $\{s_R\}$, Hölder's inequality yields

$$\begin{aligned}
&\left\| s_R [w(\Omega)]^{1-2/p} \frac{|R|}{w(R)} \right\|_{s_w^p(\mathbb{R}^N)} \\
&\qquad = \left\{ \int_\Omega \left[\sum_{J\in\mathbb{Z}^3} \sum_{\substack{R\in\mathcal{R}_{\mathcal{F}}^J \\ R\subset\Omega}} |s_R|^2 [w(\Omega)]^{2-4/p} \frac{|R|}{w(R)^2} \chi_R(x) \right]^{\frac{p}{2}} w(x)dx \right\}^{\frac{1}{p}} \\
&\qquad \le [w(\Omega)]^{\frac{1}{p}-\frac{1}{2}} \left\{ \int_\Omega \sum_{J\in\mathbb{Z}^3} \sum_{\substack{R\in\mathcal{R}_{\mathcal{F}}^J, \\ R\subset\Omega}} |s_R|^2 [w(\Omega)]^{2-4/p} \frac{|R|}{w(R)^2} \chi_R(x) w(x)dx \right\}^{\frac{1}{2}} \\
&\qquad = \|\{s_R\}\|_{\ell^2(\Omega,\mu)} \le 1.
\end{aligned}$$

Combining the above estimates yields

$$\|\{t_R\}\|_{c_w^p(\mathbb{R}^N)} \le \|\ell\|,$$

and hence $\{t_R\} \in c_w^p(\mathbb{R}^N)$. □

Now we define a *lifting operator* $\mathcal{L}$ on $\mathcal{S}'_{\mathcal{F}}(\mathbb{R}^N)$ and a *projection operator* $\mathcal{T}$ on sequence spaces by

$$\mathcal{L}(f) := \{|R|^{\frac{1}{2}} \psi_J * f(x_R)\} \quad \text{for } f \in \mathcal{S}'_{\mathcal{F}}(\mathbb{R}^N)$$

and

$$\mathcal{T}(\{t_R\})(x) := \sum_{J\in\mathbb{Z}^3} \sum_{\mathcal{R}\in\mathcal{R}_{\mathcal{F}}^J} |R|^{\frac{1}{2}} \psi_J(x - x_R) t_R,$$

where $\{\psi_J\}$ satisfies (1-1) and (1-2).

To prove Theorem 1.7, we need the following:

Theorem 2.11. *The lifting operator $\mathcal{L}$ is bounded from $H_{\mathcal{F},w}^p(\mathbb{R}^N)$ to $s_w^p(\mathbb{R}^N)$ and bounded from $CMO_{\mathcal{F},w}^p(\mathbb{R}^N)$ to $c_w^p(\mathbb{R}^N)$. The projection operator $\mathcal{T}$ is bounded from $s_w^p(\mathbb{R}^N)$ to $H_{\mathcal{F},w}^p(\mathbb{R}^N)$ and bounded from $c_w^p(\mathbb{R}^N)$ to $CMO_{\mathcal{F},w}^p(\mathbb{R}^N)$. Moreover, $\mathcal{T}\circ\mathcal{L}$ is the identity both on $H_{\mathcal{F},w}^p(\mathbb{R}^N)$ and $CMO_{\mathcal{F},w}^p(\mathbb{R}^N)$.*

Proof. The boundedness of $\mathcal{L}$ from $H_{\mathcal{F},w}^p(\mathbb{R}^N)$ to $s_w^p(\mathbb{R}^N)$ and from $CMO_{\mathcal{F},w}^p(\mathbb{R}^N)$ to $c_w^p(\mathbb{R}^N)$ follows directly from the definition of $\mathcal{L}$.

We now show that $\mathcal{T}$ is bounded from $s_w^p(\mathbb{R}^N)$ to $H_{\mathcal{F},w}^p(\mathbb{R}^N)$. By definition,

$$\|\mathcal{T}(\{t_R\})\|_{H_{\mathcal{F},w}^p(\mathbb{R}^N)} = \left\| \left\{ \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_{\mathcal{F}}^J} |\psi_J * \mathcal{T}(\{t_R\})(x_R)|^2 \chi_R \right\}^{\frac{1}{2}} \right\|_{L_w^p(\mathbb{R}^N)}.$$

A similar argument to the proof of Theorem 1.4 yields

$$\|\mathcal{T}(\{t_R\})\|_{H^p_{\mathcal{F},w}} \lesssim \Bigg\| \Bigg\{ \sum_{J'\in\mathbb{Z}^3} \Bigg\{ \mathcal{M}_{\mathcal{F}}\Bigg[\sum_{R'\in\mathcal{R}^{J'}_{\mathcal{F}}} t^2_{R'}|R'|^{-1}\chi_{R'} \Bigg]^{\frac{\delta}{2}} \Bigg\}^{\frac{2}{\delta}} \Bigg\}^{\frac{1}{2}} \Bigg\|_{L^p_w}$$

$$\lesssim \Bigg\| \Bigg\{ \sum_{J'\in\mathbb{Z}^3} \sum_{R'\in\mathcal{R}^{J'}_{\mathcal{F}}} t^2_{R'}|R'|^{-1}\chi_{R'} \Bigg\}^{\frac{1}{2}} \Bigg\|_{L^p_w} = \|t\|_{s^p_w}.$$

Next, we prove that the operator $\mathcal{T}$ is bounded from $c^p_w(\mathbb{R}^N)$ to $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$. Suppose $\{t_R\} \in c^p_w(\mathbb{R}^N)$. Then, for any open set $\Omega \subset \mathbb{R}^N$ with $w(\Omega) < \infty$,

$$\sum_{J\in\mathbb{Z}^3} \sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}},\\ R\subset\Omega}} |t_R|^2 \frac{|R|}{w(R)} \le C[w(\Omega)]^{\frac{2}{p}-1}.$$

Therefore,

$$\sum_{J\in\mathbb{Z}^3} \sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}},\\ R\subset\Omega}} |\psi_J * \mathcal{T}(\{t_R\})(x_R)|^2 \frac{|R|^2}{w(R)}$$
$$= \sum_{J\in\mathbb{Z}^3} \sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}},\\ R\subset\Omega}} \Bigg(\sum_{J'\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}^{J'}_{\mathcal{F}}} |\psi_J * \psi_{J'}(x_R - x_{R'})| \cdot t_{R'} \cdot |R'|^{\frac{1}{2}} \Bigg)^2 \frac{|R|^2}{w(R)}.$$

Repeating the same argument as in Theorem 1.6, we obtain

$$\|\mathcal{T}(\{t_R\})\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)} \lesssim \sup_{\bar\Omega} \Bigg\{ \frac{1}{[w(\bar\Omega)]^{2/p-1}} \sum_{R'\subset\bar\Omega} |t_{R'}|^2 \frac{|R'|^2}{w(R')} \Bigg\}^{\frac{1}{2}} \approx \|\{t_R\}\|_{c^p_w(\mathbb{R}^N)}.$$

Finally, the fact that $\mathcal{T}\circ\mathcal{L}$ is the identity both on $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ and $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$ follows directly from the discrete Calderón identity in Theorem 2.1. □

Proof of Theorem 1.7. We first prove the inclusion $CMO^p_{\mathcal{F},w}(\mathbb{R}^N) \subset (H^p_{\mathcal{F},w}(\mathbb{R}^N))^*$. Let $g \in CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$. For $f \in \mathcal{S}_\infty(\mathbb{R}^N)$, define the mapping $\ell_g(f) := \langle f, g\rangle$. Applying Theorems 2.1, 2.10 and 2.11, we obtain

$$|\ell_g(f)| = |\langle f, g\rangle| = \Bigg| \Bigg\langle \sum_{J\in\mathbb{Z}^3} \sum_{R\subset\mathcal{R}^J_{\mathcal{F}}} |R|\psi_J(\cdot - x_R)\psi_J * f(x_R), g \Bigg\rangle \Bigg|$$
$$= \Bigg| \sum_{J\in\mathbb{Z}^3} \sum_{R\subset\mathcal{R}^J_{\mathcal{F}}} |R|^{\frac{1}{2}}\psi_J * f(x_R)|R|^{\frac{1}{2}}\psi_J * g(x_R) \Bigg|$$
$$= |\langle \mathcal{L}(f), \mathcal{L}(g)\rangle| \lesssim \|\mathcal{L}(f)\|_{s^p_w(\mathbb{R}^N)} \|\mathcal{L}(g)\|_{c^p_w(\mathbb{R}^N)}$$
$$\lesssim \|f\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)} \|g\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)},$$

where we have chosen $\psi^{(1)}(-x) = \psi^{(1)}(x)$ and $\psi^{(2)}(-x) = \psi^{(2)}(x)$. Since $\mathcal{S}_\infty(\mathbb{R}^N)$

is dense in $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ (by Corollary 2.6), it follows that the mapping $\ell_g(f) = \langle f, g\rangle$ can be extended to a continuous linear functional on $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ and $\|\ell_g\| \lesssim \|g\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)}$.

Conversely, let $\ell \in (H^p_{\mathcal{F},w}(\mathbb{R}^N))^*$ and $\ell_1 = \ell\circ\mathcal{T}$. For $\{s_R\}\in s^p_w(\mathbb{R}^N)$, Theorem 2.11 gives

$$|\ell_1(\{s_R\})| = |\ell(\mathcal{T}(\{s_R\}))| \le \|\ell\|\cdot\|\mathcal{T}(\{s_R\})\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)} \lesssim \|\ell\|\cdot\|\{s_R\}\|_{s^p_w(\mathbb{R}^N)},$$

which implies that $\ell_1 \in (s^p_w(\mathbb{R}^N))^*$. Then by Theorem 2.10, there exists $\{t_R\}\in c^p_w(\mathbb{R}^N)$ such that $\ell_1(\{s_R\}) = \sum_R s_R \bar{t}_R$ for all $\{s_R\}\in s^p_w(\mathbb{R}^N)$ and $\|\{t_R\}\|_{c^p_w(\mathbb{R}^N)} \lesssim \|\ell_1\| \lesssim \|\ell\|$. By Theorem 2.10 again, $\ell = \ell\circ\mathcal{T}\circ\mathcal{L} = \ell_1\circ\mathcal{L}$. Hence,

$$\ell(f) = \ell_1(\mathcal{L}(f)) = \langle\mathcal{L}(f), t\rangle = \langle f, g\rangle,$$

where $g = \sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^J_{\mathcal{F}}} |R|^{\frac12}\, t_R\, \psi_J(x_R - x)$. This implies that $\ell = \ell_g$, and, by Theorem 2.10,

$$\|g\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)} \lesssim \|\{t_R\}\|_{c^p_w(\mathbb{R}^N)} \lesssim \|\ell_g\|.$$

This concludes the proof of Theorem 1.7. □

3. Weighted boundedness of singular integrals with flag kernels

This section is devoted to proving the boundedness results given in Theorems 1.9, 1.10 and 1.11 for flag singular integrals. To prove Theorem 1.9, we need the following orthogonality estimates.

Lemma 3.1. *Let $\varphi \in \mathcal{S}(\mathbb{R}^N)$ satisfy*

$$\begin{cases} \int_{\mathbb{R}^{n_1}} \varphi(x_1,x_2,x_3)\,dx_1 = 0 & \text{for almost every } (x_2,x_3)\in\mathbb{R}^{n_2+n_3},\\ \int_{\mathbb{R}^{n_2}} \varphi(x_1,x_2,x_3)\,dx_2 = 0 & \text{for almost every } (x_1,x_3)\in\mathbb{R}^{n_1+n_3},\\ \int_{\mathbb{R}^{n_3}} \varphi(x_1,x_2,x_3)\,dx_3 = 0 & \text{for almost every } (x_1,x_2)\in\mathbb{R}^{n_1+n_2}, \end{cases} \tag{3-1}$$

and define φ_J by $\varphi_J(x) := 2^{-j_1n_1+j_2n_2+j_3n_3}\varphi(2^{-j_1}x_1, 2^{-j_2}x_2, 2^{-j_3}x_3)$. Also let $\psi_{J'}\in\mathcal{S}(\mathbb{R}^n)$ be defined as in Section 1. Then there exists $\epsilon > 0$ such that, for any $M > 0$,

$$|\varphi_J * \psi_{J'}(x)| \lesssim 2^{-\epsilon(|j_1-j_1'|+|j_2-j_2'|+|j_3-j_3'|)}\left[\prod_{i=1}^{3}\frac{\max_{1\le k\le i} 2^{j_k' M}}{(\max_{1\le k\le i} 2^{j_k'} + |x_i|)^{n_i+M}}\right]. \tag{3-2}$$

Proof. We consider eight cases separately.

Case 1: $j_1 \le j_1',\ j_2 \le j_2',\ j_3 \le j_3'$. By (3-1),

$$|\varphi_J * \psi^{(1)}_{j_1}(x)| = \left| \int_{\mathbb{R}^N} \varphi_J(u)[\psi^{(1)}_{j_1}(x-u) - \psi^{(1)}_{j_1}(x)]\,dy \right|$$
$$\lesssim 2^{-|j_1-j_1'|} \frac{2^{(j_1\vee j_1')M}}{(2^{j_1\vee j_1'}+|x_1|)^{M+n_1}} \frac{2^{(j_2\vee j_1')M}}{(2^{j_2\vee j_1'}+|x_2|)^{M+n_2}} \frac{2^{(j_3\vee j_1')M}}{(2^{j_3\vee j_1'}+|x_3|)^{M+n_3}}.$$

This together with

$$|\psi^{(2)}_{j_2'} *_2 \psi^{(3)}_{j_3'}(x_2,x_3)| \lesssim \frac{2^{j_2'M}}{(2^{j_2'}+|x_2|)^{M+n_2}} \frac{2^{(j_2'\vee j_3')M}}{(2^{j_2'\vee j_3'}+|x_3|)^{M+n_3}}$$

yields

$$\begin{aligned} |\varphi_J * \psi_{J'}(x)| &= |[\varphi_J * \psi^{(1)}_{j_1}] *_2 [\psi^{(2)}_{j_2'} *_2 \psi^{(3)}_{j_3'}](x)| \\ &\lesssim 2^{-|j_1-j_1'|} \left[\prod_{i=1}^{3} \frac{\max_{1\le k\le i} 2^{(j_k\vee j_k')M}}{(\max_{1\le k\le i} 2^{j_k\vee j_k'}+|x_i|)^{n_i+M}} \right] \\ &= 2^{-|j_1-j_1'|} \left[\prod_{i=1}^{3} \frac{\max_{1\le k\le i} 2^{j_k'M}}{(\max_{1\le k\le i} 2^{j_k'}+|x_i|)^{n_i+M}} \right]. \end{aligned} \tag{3-3}$$

The same techniques yield

$$\begin{aligned} |\varphi_J * \psi_{J'}(x)| &= |[\varphi_J *_{2,3} \psi^{(2)}_{j_2}] * [\psi^{(1)}_{j_1'} *_3 \psi^{(3)}_{j_3'}](x)| \\ &\lesssim 2^{-|j_2-j_2'|} \left[\prod_{i=1}^{3} \frac{\max_{1\le k\le i} 2^{j_k'M}}{(\max_{1\le k\le i} 2^{j_k'}+|x_i|)^{n_i+M}} \right] \end{aligned} \tag{3-4}$$

and

$$\begin{aligned} |\varphi_J * \psi_{J'}(x)| &= |[\varphi_J *_3 \psi^{(3)}_{j_3}] * [\psi^{(1)}_{j_1'} *_{2,3} \psi^{(2)}_{j_2'}](x)| \\ &\lesssim 2^{-|j_3-j_3'|} \left[\prod_{i=1}^{3} \frac{\max_{1\le k\le i} 2^{j_k'M}}{(\max_{1\le k\le i} 2^{j_k'}+|x_i|)^{n_i+M}} \right]. \end{aligned} \tag{3-5}$$

Taking the geometric mean of (3-3)–(3-5), we obtain (3-2) with $\epsilon = 1/3$.

Case 2: $j_1 > j_1',\ j_2 \le j_2',\ j_3 \le j_3'$. We use the moment condition of $\psi^{(1)}$ and Taylor's remainder theorem to get

$$\begin{aligned} &|\varphi_J * \psi^{(1)}_{j_1}(x)| \\ &\qquad = \left| \int_{\mathbb{R}^N} [\varphi_J(x-y) - P_{L-1}[\psi_J](x)]\psi^{(1)}_{j_1}(y)\,dy \right| \end{aligned}$$

$$\lesssim \Big(\sum_{L_1+L_2=L} 2^{-|j_1-j_1'|L_1}2^{-|j_2-j_1'|L_2}\Big) \times \frac{2^{(j_1\vee j_1')M}}{(2^{j_1\vee j_1'}+|x_1|)^{M+n_1}}\frac{2^{(j_2\vee j_1')M}}{(2^{j_2\vee j_1'}+|x_2|)^{M+n_2}}\frac{2^{(j_3\vee j_1')M}}{(2^{j_3\vee j_1'}+|x_3|)^{M+n_3}}$$

$$\lesssim 2^{-|j_1-j_1'|L}\frac{2^{(j_1\vee j_1')M}}{(2^{j_1\vee j_1'}+|x_1|)^{M+n_1}}\frac{2^{(j_2\vee j_1')M}}{(2^{j_2\vee j_1'}+|x_2|)^{M+n_2}}\frac{2^{(j_3\vee j_1')M}}{(2^{j_3\vee j_1'}+|x_3|)^{M+n_3}},$$

where in the last inequality we have used the fact that $|j_1-j_1'|\ge|j_2-j_1'|$ and $P_{L-1}[f]$ is the $(L-1)$-th order Taylor's polynomial of f. It follows that

$$\begin{aligned}|\varphi_J*\psi_{J'}(x)| &= |[\varphi_J*\psi^{(1)}_{j_1}]*_2[\psi^{(2)}_{j_2'}*_2\psi^{(3)}_{j_3'}](x)|\\ &\lesssim 2^{-L|j_1-j_1'|}\frac{2^{j_1M}}{(2^{j_1}+|x_1|)^{M+n_1}}\frac{2^{(j_1'\vee j_2')M}}{(2^{j_1'\vee j_2'}+|x_2|)^{M+n_2}}\frac{2^{(j_1'\vee j_2'\vee j_3')M}}{(2^{j_1'\vee j_2'\vee j_3'}+|x_3|)^{M+n_3}}\\ &\lesssim 2^{-(L-M)|j_1-j_1'|}\Bigg[\prod_{i=1}^{3}\frac{\max_{1\le k\le i}2^{j_k'M}}{(\max_{1\le k\le i}2^{j_k'}+|x_i|)^{n_i+M}}\Bigg].\end{aligned}$$

The other cases, $\{j_1\le j_1',\ j_2>j_2',\ j_3\le j_3'\}$, $\{j_1>j_1',\ j_2>j_2',\ j_3\le j_3'\}$, $\{j_1\le j_1',\ j_2\le j_2',\ j_3<j_3'\}$, $\{j_1>j_1',\ j_2\le j_2',\ j_3<j_3'\}$, $\{j_1\le j_1',\ j_2>j_2',\ j_3<j_3'\}$, and $\{j_1>j_1',\ j_2>j_2',\ j_3<j_3'\}$, can be handled in the same manner and details are left to the reader. □

Lemma 3.2. *Let $\mathcal{K}$ be a flag kernel. We have*

$$|\psi_J*\mathcal{K}*\psi_{J'}(x)|\lesssim 2^{-10M(|j_1-j_1'|+|j_2-j_2'|+|j_3-j_3'|)}\prod_{i=1}^{3}\frac{\max_{1\le k\le i}2^{j_k'M}}{(\max_{1\le k\le i}2^{j_k'}+|x_i|)^{1+M}}. \tag{3-6}$$

Proof. It is well known that $\psi^{(i)}_{j_i}*\psi^{(i)}_{j_i'}$ and $2^{-L|j_i-j_i'|}\psi^{(i)}_{j_i\vee j_i'}$ satisfy the same differential inequalities and moment conditions on $\mathbb{R}^{N_i}$. Thus,

$$\psi_J*\psi_{J'}=[\psi^{(1)}_{j_1}*\psi^{(1)}_{j_1'}]*_{2,3}[\psi^{(2)}_{j_2}*\psi^{(2)}_{j_2'}]*_3[\psi^{(3)}_{j_3}*\psi^{(3)}_{j_3'}]$$

satisfies the same properties as $2^{-L(|j_1-j_1'|+|j_2-j_2'|+|j_3\vee j_3'|)}\psi_{J\vee J'}$, where

$$\psi_{J\vee J'}:=\psi^{(1)}_{j_1\vee j_1'}*_{2,3}\psi^{(2)}_{j_2\vee j_2'}*_3\psi^{(3)}_{j_3\vee j_3'}.$$

By [Nagel et al. 2001, Corollary 2.4.4],

$$\mathcal{K}=\sum_{j_1\le j_2\le j_3}\varphi^{(J)}_J,$$

where $\{\varphi^{(J)}\}$ is a bounded collection of C^∞ functions, each of which is supported on $\{|x_i|\le c,\ i=1,2,3\}$ with (3-1), and the series converges in the sense of distributions.

Lemma 3.1 yields

$$\begin{aligned}|\psi_J*\mathcal{K}*\psi_{J'}(x)| &\le \sum_{j_1''\le j_2''\le j_3''}\left|\varphi_{J''}^{(J'')}*[\psi_J*\psi_{J'}](x)\right| \\ &\lesssim \sum_{j_1''\le j_2''\le j_3''} 2^{-L(|j_1''-(j_1\vee j_1')|+|j_2''-(j_2\vee j_2')|+|j_3''-(j_3\vee j_3')|)} \\ &\qquad \times 2^{-L(|j_1-j_1'|+|j_2-j_2'|+|j_3-j_3'|)}\prod_{i=1}^{3}\frac{\max_{1\le k\le i}2^{(j_k\vee j_k')M}}{(\max_{1\le k\le i}2^{j_k\vee j_k'}+|x_i|)^{1+M}} \\ &\lesssim 2^{-(|j_1-j_1'|+|j_2-j_2'|+|j_3-j_3'|)L}\left[\prod_{i=1}^{3}\frac{\max_{1\le k\le i}2^{(j_k\vee j_k')M}}{(\max_{1\le k\le i}2^{(j_k\vee j_k')}+|x_i|)^{n_i+M}}\right].\end{aligned}$$

This finishes the proof of (3-6). □

Proof of Theorem 1.9. By the discrete Calderón reproducing formula,

$$\begin{aligned}&\|T_{\mathcal{F}}(f)\|_{H^p_{\mathcal{F},w}} \\ &= \left\|\left\{\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^J_{\mathcal{F}}}|\psi_J*\mathcal{K}*f(x_R)|^2\chi_R\right\}^{\frac12}\right\|_{L^p_w} \\ &= \left\|\left\{\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^J_{\mathcal{F}}}\left|\sum_{J'\in\mathbb{Z}^3}\sum_{R'\in\mathcal{R}^J_{\mathcal{F}}}|R'|\psi_{J'}*f(x_{R'})\psi_J*\mathcal{K}*\psi_{J'}(x_R-x_{R'})\right|^2\chi_R\right\}^{\frac12}\right\|_{L^p_w}.\end{aligned}$$

Equation (3-6) says that, for each $J, J'\in\mathbb{Z}^3$, $\psi_J*\mathcal{K}*\psi_{J'}$ satisfies the same orthogonality estimate as $\psi_J*\psi_{J'}$. Thus, repeating the same argument as in the proof of Theorem 1.4, we obtain

$$\begin{aligned}&\|Tf\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)} \\ &\lesssim \left\|\left\{\sum_{J'\in\mathbb{Z}^3}\left[\mathcal{M}_{\mathcal{F}}\left(\sum_{R'\in\mathcal{R}^J_{\mathcal{F}}}|\psi_{J'}*f(x_{R'})|^2\chi_{R'}\right)^{\frac{\delta}{2}}\right]^{\frac{2}{\delta}}\right\}^{\frac12}\right\|_{L^p_w(\mathbb{R}^N)} \lesssim \|f\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}.\end{aligned}$$

This concludes the proof of Theorem 1.9. □

To prove Theorem 1.10, we need a new Calderón type identity in terms of bump functions. For $i=1,2,3$, let $\phi^{(i)}, \phi'^{(i)}\in\mathcal{S}(\mathbb{R}^{N_i})$ be such that $\operatorname{supp}\widehat{\phi^{(i)}}$ is compact and bounded away from origin, $\phi'^{(i)}$ is supported on $B(0,2)$, and they satisfy

$$\int_{\mathbb{R}^{N_i}}\phi'^{(i)}(x^i)(x^i)^{\alpha_i}\,dx^i=0 \quad \text{for } 0\le|\alpha|\le M_0,$$

where M_0 is a large positive integer given in Theorem 3.3 below, and

$$\sum_{j_i\in\mathbb{Z}} \widehat{\phi^{(i)}}(2^{j_i}\xi^i)\widehat{\phi'^{(i)}}(2^{j_i}\xi^i)=1 \quad \text{for } \xi^i\in\mathbb{R}^{N_i}\backslash\{0\};$$

see [Frazier and Jawerth 1990, Theorem 4.2]. For $J=(j_1, j_2, j_3)\in\mathbb{Z}^3$, set $\phi_J=(\widetilde{\phi}^{(1)}_{j_1}*\widetilde{\phi}^{(2)}_{j_2}*\widetilde{\phi}^{(3)}_{j_3})$, where $\widetilde{\phi}^{(i)}_{j_i}=\delta_{\mathbb{R}^{N-N_i}}\otimes\phi^{(i)}_{j_i}$, and $\widetilde{\phi}_J$ is defined similarly in terms of $\phi'^{(i)}_{j_i}$.

Theorem 3.3. *Let $0<p\le 1$ and $w\in A^{\mathcal{F}}_\infty(\mathbb{R}^N)$. Let $M_0\ge 10(N\{[q_w/p-1]\vee 2\}+1)$ (here $[\,\cdot\,]$ means the greatest integer function). For a fixed sufficiently large integer K, let $\mathcal{R}^{J,K}_{\mathcal{F}}=\mathcal{R}^{j_1-K,j_2-K,j_3-K}_{\mathcal{F}}$ and let x_R denote the lower left corner of R. Then, for every $f\in L^2(\mathbb{R}^N)\cap H^p_{\mathcal{F},w}(\mathbb{R}^N)$ there exists some function*

$$h=h_f\in L^2(\mathbb{R}^N)\cap H^p_{\mathcal{F},w}(\mathbb{R}^N)$$

depending only on f such that

$$f(x)\overset{L^2}{=}\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^{J,K}_{\mathcal{F}}}|R|\widetilde{\phi}_J(x-x_R)\phi_J*h(x_R).$$

Moreover,

$$\|f\|_{H^p_{\mathcal{F},w}}\approx\|h\|_{H^p_{\mathcal{F},w}}. \tag{3-7}$$

Proof. For $f\in L^2(\mathbb{R}^N)\cap H^p_{\mathcal{F},w}(\mathbb{R}^N)$, applying the Fourier transform gives $f=\sum_{J\in\mathbb{Z}^3}\widetilde{\phi}_J*\phi_J*f$, where the series converges in $L^2(\mathbb{R}^N)$ norm. Using Coifman's idea of the decomposition of the identity operator, we have

$$\begin{aligned} f(x)&=\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^{J,K}_{\mathcal{F}}}|R|\widetilde{\phi}_J(x-x_R)\phi_J*f(x_R)\\ &\quad+\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^{J,K}_{\mathcal{F}}}\int_R[\widetilde{\phi}_J(x-x')(\phi_J*f)(x')-\widetilde{\phi}_J(x-x_R)(\phi_J*f)(x_R)]\,dx'\\ &=: T_K(f)(x)+R_K(f)(x),\end{aligned}$$

where K is a fixed large integer to be determined later.

We can decompose $R_K(f)$ further as

$$\begin{aligned} R_K(f)(x)&=\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^{J,K}_{\mathcal{F}}}\int_R[\widetilde{\phi}_J(x-x')-\widetilde{\phi}_J(x-x_R)](\phi_J*f)(x')\,dx'\\ &\quad+\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^{J,K}_{\mathcal{F}}}\int_R\widetilde{\phi}_J(x-x_R)[(\phi_J*f)(x')-(\phi_J*f)(x_R)]\,dx'\\ &=: R^1_K(f)(x)+R^2_K(f)(x).\end{aligned}$$

We claim that for $k = 1, 2$,

$$\|R_K^k(f)\|_{H^p_{\mathcal{F},w}} \le C2^{-K}\|f\|_{H^p_{\mathcal{F},w}}, \tag{3-8}$$

where C is a constant independent of f, K and x_R.

Assume the claim for the moment. Then choosing sufficiently large K such that $C2^{-K} < 1$ implies that both T_K and $T_K^{-1} = \sum_{n=0}^{\infty}(R_K)^n$ are bounded on $L^2(\mathbb{R}^N)$ and on $H^p_{\mathcal{F},w}(\mathbb{R}^N)$. Setting $h = R_K^{-1}(f)$ gives (3-7). Moreover,

$$f = T_K(T_K^{-1}(f)) = \sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}_{\mathcal{F}}^{J,K}} |R|\widetilde{\phi}_J(\cdot - x_R)(\phi_J * h)(x_R),$$

where the series converges in $L^2(\mathbb{R}^N)$.

To finish the proof of Theorem 1.10, it suffices to verify the claim. Since the proofs for R_K^1 and R_K^2 are similar, we only treat R_K^1. The discrete Calderón reproducing formula in Theorem 2.1 yields

$$\begin{aligned} \psi_{J'} * R_K^1(f)(x) &= \sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}_{\mathcal{F}}^{J,K}} \int_R \psi_{J'} * [\widetilde{\phi}_J(\cdot - x') - \widetilde{\phi}_J(\cdot - x_R)](x)(\phi_J * f)(x')\,dx' \\ &= \sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}_{\mathcal{F}}^{J,K}} \int_R \psi_{J'} * [\widetilde{\phi}_J(\cdot - x') - \widetilde{\phi}_J(\cdot - x_R)](x) \\ &\qquad \times \Bigg(\sum_{J''\in\mathbb{Z}^3}\sum_{R''\in\mathcal{R}_{\mathcal{F}}^{J''}} |R''|\cdot\psi_{J''} * f(x_{R''})\phi_J * \psi_{J''}(x' - x_{R''})\Bigg)\,dx', \end{aligned} \tag{3-9}$$

where $x_{R''} = (x_{Q_1''}, x_{Q_2''}, x_{Q_3''})$ denotes the lower left corner of R''. Set

$$\widetilde{\widetilde{\phi}}_J(u) = \widetilde{\phi}_J(u - x') - \widetilde{\phi}_J(u - x_R).$$

Applying Lemma 2.2 with M sufficiently large (which will be determined later) and $L = 10M$, we obtain that for some constant C (depending only on M, ψ and ϕ, but independent of K),

$$\begin{aligned} &|\psi_{J'} * \widetilde{\widetilde{\phi}}_J(x)| \\ &\qquad \le C2^{-K}2^{-10M(|j_1-j_1'|+|j_2-j_2'|+|j_3-j_3'|)} \prod_{i=1}^{3} \frac{\max_{1\le k\le i} 2^{j_k'M}}{(\max_{1\le k\le i} 2^{j_k'} + |x_i - x_i'|)^{1+M}}, \end{aligned}$$

and, similarly,

$$\begin{aligned} &|\phi_J * \psi_{J''}(x' - x_{R''})| \\ &\qquad \le C2^{-10M(|j_1-j_1''|+|j_2-j_2''|+|j_3-j_3''|)} \prod_{i=1}^{3} \frac{\max_{1\le k\le i} 2^{j_k''M}}{(\max_{1\le k\le i} 2^{j_k''} + |x_i' - x_{Q_i''}|)^{1+M}}. \end{aligned}$$

Substituting both estimates into the last term of (3-9) yields

$$
\begin{aligned}
&|\psi_{J'} * R_K^1(f)(x)| \\
&\qquad \lesssim \sum_{J''\in\mathbb{Z}^3} \sum_{R''\in\mathcal{R}_{\mathcal{F}}^{J''}} |R''|\,|\psi_{J''} * f(x_{R''})| \\
&\qquad\quad \times \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_{\mathcal{F}}^{J,K}} \int_R 2^{-K} \prod_{i=1}^{3} 2^{-|j_i-j_i'|3M} \frac{\max_{1\le k\le i} 2^{j_k' M}}{(\max_{1\le k\le i} 2^{j_k'} + |x_i - x_i'|)^{1+M}} \\
&\qquad\quad \times 2^{-|j_i-j_i''|3M} \frac{\max_{1\le k\le i} 2^{j_k'' M}}{(\max_{1\le k\le i} 2^{j_k''} + |x_i' - x_{Q_i''}|)^{1+M}} dx' \\
&\qquad \lesssim 2^{-K} \sum_{J''\in\mathbb{Z}^3} \sum_{R''\in\mathcal{R}_{\mathcal{F}}^{J''}} 2^{-(|j_1'-j_1''|+|j_2'-j_2''|+|j_3'-j_3''|)M} |R''| \\
&\qquad\quad \times \Bigg(\prod_{i=1}^{3} \frac{\max_{1\le k\le i} 2^{(j_k'\vee j_k'')M}}{(\max_{1\le k\le i} 2^{j_k'\vee j_k''} + |x_i - x_{Q_i''}|)^{1+M}}\Bigg) |\psi_{J''} * f(x_{R''})|.
\end{aligned}
$$

Now we may choose $M = N\{[q_w/p+1]\vee 2\}+1,\ L = 10M$ and $N/M < \delta < 1$. Then $p/\delta > q_w$ so that $w \in A^{\mathcal{F}}_{p/\delta}(\mathbb{R}^N)$. Arguing as in the proof of Theorem 1.4, we obtain

$$
\begin{aligned}
&\|R_K^1(f)\|_{H^p_{\mathcal{F},w}} \\
&\quad \lesssim 2^{-K} \Bigg\| \Bigg\{ \sum_{J''\in\mathbb{Z}^3} \Bigg\{ \mathcal{M}_{\mathcal{F}} \Bigg(\sum_{R''\in\mathcal{R}_{\mathcal{F}}^{J''}} |\psi_{J''} * f(x_{R''})| \chi_{R''} \Bigg)^{\delta} \Bigg\}^{\frac{2}{\delta}} \Bigg\}^{\frac12} \Bigg\|_{L^p_w} \lesssim 2^{-K} \|f\|_{H^p_{\mathcal{F},w}}.
\end{aligned}
$$

This verifies claim (3-8) and hence Theorem 3.3 follows. □

Using a similar argument to the one in the proof of Theorem 1.4, one can prove the following:

Corollary 3.4. *Suppose $w \in A^{\mathcal{F}}_\infty(\mathbb{R}^N)$. Then, for $f \in H^p_{\mathcal{F},w}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ and $0 < p \le 1$, we have*

$$
\|f\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)} \approx \|\tilde{g}_{\mathcal{F}}(f)\|_{L^p_w(\mathbb{R}^N)} := \Bigg\| \Bigg\{ \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_{\mathcal{F}}^{J,K}} |\phi_J * h(x_R)|^2 \chi_R \Bigg\}^{\frac12} \Bigg\|_{L^p_w(\mathbb{R}^N)},
$$

where $h = h_f$ and K are the same as in Theorem 3.3.

The key to the proof of Theorem 1.10 is the following.

Lemma 3.5. *Suppose $0 < p \le 1$ and $w \in A^{\mathcal{F}}_\infty(\mathbb{R}^N)$. If $f \in H^p_{\mathcal{F},w}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, then $f \in L^p_w(\mathbb{R}^N)$ and there is a constant $C_p > 0$ independent of the $L^2(\mathbb{R}^N)$ norm of f such that*

$$
\|f\|_{L^p_w(\mathbb{R}^N)} \le C_p \|f\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}.
$$

Proof. Without loss of generality, we may assume $w \in A_q^{\mathcal{F}}(\mathbb{R}^N)$ for some $q \in [2, \infty)$. Given $f \in L^2(\mathbb{R}^N) \cap H^p_{\mathcal{F},w}(\mathbb{R}^N)$, set $\Omega_i = \{x \in \mathbb{R}^N : \tilde{g}_{\mathcal{F}}(h)(x) > 2^i\}$ where $h = h_f$ is given by Theorem 3.3, and

$$\mathscr{U}_i = \{(J, R) : J \in \mathbb{Z}^3,\ R \in \mathcal{R}^{J,K}_{\mathcal{F}},\ |R \cap \Omega_i| > (1/2)|R|,\ |R \cap \Omega_{i+1}| \le (1/2)|R|\}.$$

By the discrete Calderón reproducing formula in Theorem 3.3,

$$f = \sum_{i\in\mathbb{Z}} \sum_{(J,R)\in\mathscr{U}_i} |R| \widetilde{\phi}_J(\cdot - x_R)\phi_J * h(x_R).$$

We claim that

$$\Big\| \sum_{(J,R)\in\mathscr{U}_i} |R|\widetilde{\phi}_J(\cdot - x_R)\phi_J * h(x_R)\Big\|^p_{L^p_w(\mathbb{R}^N)} \lesssim 2^{pi} w(\Omega_i). \tag{3-10}$$

Since $0 < p \le 1$, the above claim together with Theorem 3.3 yields

$$\begin{aligned}\|f\|^p_{L^p_w(\mathbb{R}^N)} &\le \sum_{i\in\mathbb{Z}} \Big\| \sum_{(J,R)\in\mathscr{U}_i} |R|\widetilde{\phi}_J(\cdot - x_R)\phi_J * h(x_R)\Big\|^p_{L^p_w(\mathbb{R}^N)}\\ &\lesssim \sum_{i\in\mathbb{Z}} 2^{pi} w(\Omega_i) \lesssim \|\tilde{g}(h)\|^p_{L^p_w(\mathbb{R}^N)} \approx \|h\|^p_{H^p_{\mathcal{F},w}(\mathbb{R}^N)} \approx \|f\|^p_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}\end{aligned}$$

and Lemma 3.5 follows.

To show claim (3-10), we note that if $(J, R) \in \mathscr{U}_i$, the function $x \mapsto \widetilde{\phi}_J(x - x_R)$ is supported in $\widetilde{\Omega}_i := \{x : \mathcal{M}_{\mathcal{F}}(\chi_{\Omega_i})(x) > 1/100\}$. By Hölder's inequality,

$$\begin{aligned}\Big\| \sum_{(J,R)\in\mathscr{U}_i} |R|\widetilde{\phi}_J(\cdot - x_R)\phi_J * h(x_R)\Big\|^p_{L^p_w(\mathbb{R}^N)}&\\ \lesssim w(\widetilde{\Omega}_i)^{1-(p/q)} &\Big\| \sum_{(J,R)\in\mathscr{U}_i} |R|\widetilde{\phi}_J(\cdot - x_R)\phi_J * h(x_R)\Big\|^p_{L^q_w(\mathbb{R}^N)}.\end{aligned} \tag{3-11}$$

We now estimate the last L^q_w-norm by a duality argument. For $\zeta \in L^{q'}_{w^{1-q'}}(\mathbb{R}^N)$ with $\|\zeta\|_{L^{q'}_{w^{1-q'}}(\mathbb{R}^N)} \le 1$,

$$\begin{aligned}&\Big|\Big\langle \sum_{(J,R)\in\mathscr{U}_i} |R|\widetilde{\phi}_J(\cdot - x_R)\phi_J * h(x_R), \zeta\Big\rangle\Big| = \Big| \sum_{(J,R)\in\mathscr{U}_i} \int \overline{\phi}_J * \zeta(x_R)\phi_J * h(x_R)\chi_R(x)\,dx\Big|\\ &\quad\le \Big\|\Big\{ \sum_{(J,R)\in\mathscr{U}_i} |\phi_J * h(x_R)|^2\chi_R\Big\}^{\frac12}\Big\|_{L^q_w(\mathbb{R}^N)} \Big\|\Big\{ \sum_{(J,R)\in\mathscr{U}_i} |\overline{\phi}_J * \zeta(x_R)|^2\chi_R\Big\}^{\frac12}\Big\|_{L^{q'}_{w^{1-q'}}(\mathbb{R}^N)}\\ &\quad:= I_1 \times I_2,\end{aligned}$$

where $\overline{\phi}_J(x) = \widetilde{\phi}_J(-x)$.

We first estimate I_2. Since $w \in A_q^{\mathcal{F}}(\mathbb{R}^N)$ implies $w^{1-q'} \in A_{q'}^{\mathcal{F}}(\mathbb{R}^N)$, Corollary 3.4 and the remark on page 552 yield

$$I_2 \lesssim \left\| \left\{ \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_{\mathcal{F}}^{J,K}} |\overline{\phi}_J * \zeta(x_R)|^2 \chi_R \right\}^{\frac{1}{2}} \right\|_{L^{q'}_{w^{1-q'}}(\mathbb{R}^N)} \approx \|\zeta\|_{L^{q'}_{w^{1-q'}}(\mathbb{R}^N)} \le 1. \tag{3-12}$$

As for I_1, note that $\Omega_i \subset \widetilde{\Omega}_i$ and $w(\widetilde{\Omega}_i) \lesssim w(\Omega_i)$ due to the $L^q_w(\mathbb{R}^N)$ boundedness of $\mathcal{M}_{\mathcal{F}}$. For any $(J, R) \in \mathscr{U}_i$ and $x \in R$, $\mathcal{M}_{\mathcal{F}}(\chi_{R\cap\widetilde{\Omega}_i\setminus\Omega_{i+1}})(x) > \frac{1}{2}$. Applying Corollary 2.5 again, we have

$$\begin{aligned} I_1^q &= \int_{\mathbb{R}^N} \left\{ \sum_{(J,R)\in\mathscr{U}_i} |\phi_J * h(x_R)|^2 \chi_R(x) \right\}^{\frac{q}{2}} w(x)\,dx \\ &\lesssim \int_{\mathbb{R}^N} \left\{ \sum_{(J,R)\in\mathscr{U}_i} |\phi_J * h(x_R)\mathcal{M}_{\mathcal{F}}(\chi_{R\cap\widetilde{\Omega}_i\setminus\Omega_{i+1}})(x)|^2 \right\}^{\frac{q}{2}} w(x)\,dx \\ &\lesssim \int_{\widetilde{\Omega}_i\setminus\Omega_{i+1}} \left\{ \sum_{(J,R)\in\mathscr{U}_i} |\phi_J * h(x_R)|^2 \chi_R(x) \right\}^{\frac{q}{2}} w(x)\,dx \\ &\lesssim 2^{iq} w(\widetilde{\Omega}_i) \lesssim 2^{iq} w(\Omega_i). \end{aligned} \tag{3-13}$$

Combining both estimates (3-12) and (3-13), we obtain

$$\left\| \sum_{(J,R)\in\mathscr{U}_i} |R|\widetilde{\phi}_J(\cdot - x_R)\phi_J * h(x_R) \right\|_{L^q_w(\mathbb{R}^N)} \lesssim 2^{iq} w(\Omega_i).$$

Plugging this estimate into (3-11) yields claim (3-10), and so Lemma 3.5 follows. □

Proof of Theorem 1.10. For $f \in L^2(\mathbb{R}^N) \cap H^p_{\mathcal{F},w}(\mathbb{R}^N)$, by Theorem 1.9 and Lemma 3.5,

$$\|T(f)\|_{L^p_w(\mathbb{R}^N)} \le C\|T(f)\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)} \le C\|f\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}.$$

Corollary 2.6 together with a limiting argument yields Theorem 1.10. □

We remark that, as mentioned before, the flag Hardy space $H^p_{\mathcal{F}}(\mathbb{R}^N)$ (where $w \equiv 1$) differs from those of the classical one parameter and the product Hardy space. To see this, by Lemma 3.5, if $f \in L^2(\mathbb{R}^N) \cap H^p_{\mathcal{F}}(\mathbb{R}^N)$, then $f \in L^1(\mathbb{R}^N)$ and

$$\int f(x_1, x_2, x_3)\,dx_3 = 0 \quad \text{for almost every } (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \tag{3-14}$$

Indeed, for $f \in L^2(\mathbb{R}^N) \cap H^p_{\mathcal{F}}(\mathbb{R}^N)$, Lemma 3.5 gives that $f \in L^p(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, and hence $f \in L^1(\mathbb{R}^N)$ by interpolation. To see $\int f(x_1, x_2, x_3)\,dx_3 = 0$, applying the Calderón reproducing formula in Theorem 2.1,

$$f(x)=\sum_{J\in\mathbb{Z}^3}\sum_{R\in\mathcal{R}^J_{\mathcal{F}}}|R|\psi_J(x-x_R)\psi_J*f(x_R)$$

where the series converges in both $L^2(\mathbb{R}^N)$ and $H^p_{\mathcal{F}}(\mathbb{R}^N)$. Let $\mathscr{E}_L$ and f_L be defined as in the proof of Corollary 2.6. Therefore,

$$f-f_L=\sum_{\mathscr{E}^c_L}|R|\psi_J(x-x_R)\psi_J*f(x_R)$$

converges to zero in both $L^2(\mathbb{R}^N)$ and $H^p_{\mathcal{F}}(\mathbb{R}^N)$ as L tends to infinity. Applying Lemma 3.5 and interpolation, we obtain

$$\Big\|\sum_{\mathscr{E}^c_L}|R|\psi_J(x-x_R)\psi_J*f(x_R)\Big\|_{L^1}$$
$$\lesssim\Big\|\sum_{\mathscr{E}^c_L}|R|\psi_J(x-x_R)\psi_J*f(x_R)\Big\|_{L^2}+\Big\|\sum_{\mathscr{E}^c_L}|R|\psi_J(x-x_R)\psi_J*f(x_R)\Big\|_{H^p_{\mathcal{F}}},$$

which implies that $\|f-f_L\|_{L^1}$ tends to zero as L tends to infinity. Therefore, for almost every $(x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$,

$$\int_{\mathbb{R}^{n_3}}f(x_1,x_2,x_3)\,dx_3=\lim_{L\to\infty}\int_{\mathbb{R}^{n_3}}\sum_{\mathscr{E}_L}|R|\psi_J(x-x_R)\psi_J*f(x_R)\,dx_3=0,$$

where the last equality follows from the cancellation property of $\psi_J(x-x_R)$. This indicates that all L^2 elements in $H^p_{\mathcal{F}}$ satisfy only "partial" cancellation property (3-14), which is different from cancellation conditions for classical one parameter and the product Hardy spaces.

Finally, we prove Theorem 1.11. It is known that $L^2(\mathbb{R}^N)\cap H^p_{\mathcal{F},w}(\mathbb{R}^N)$ is dense in $H^p_{\mathcal{F},w}(\mathbb{R}^N)$. This allows us to use the discrete Calderón reproducing formula in Theorem 3.3, which plays a crucial role in the proof of the boundedness of flag singular integral operators on $H^p_{\mathcal{F},w}(\mathbb{R}^N)$. For $CMO^p_{\mathcal{F},w}$, we prove the following so-called *weak density* result which will offer the same service as the above density does in $H^p_{\mathcal{F},w}$.

Lemma 3.6. *Let* $0<p\le 1$ *and* $w\in A^{\mathcal{F}}_\infty(\mathbb{R}^N)$. *Then* $L^2(\mathbb{R}^N)\cap CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$ *is dense in* $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$ *in the weak topology* $\langle H^p_{\mathcal{F},w}(\mathbb{R}^N), CMO^p_{\mathcal{F},w}(\mathbb{R}^N)\rangle$. *More precisely, for any* $f\in CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$, *there exists a sequence*

$$\{f_n\}\subset L^2(\mathbb{R}^N)\cap CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$$

satisfying $\|f_n\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)}\lesssim\|f\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)}$ *and*

$$\lim_{n\to\infty}\langle f_n,g\rangle=\langle f,g\rangle\quad\textit{for any } g\in H^p_{\mathcal{F},w}(\mathbb{R}^N).$$

Proof. Suppose $f \in CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$. Set

$$f_n(x) = \sum_{|j|\le n, |k|\le n} \sum_{R\subset B(0,n)} |R|\psi_J * f(x_R)\psi_J(x-x_R),$$

where $\{\psi_J\}$ satisfy (1-1) and (1-2). It is easy to see that $f_n \in L^2(\mathbb{R}^N)$. Repeating the same proof as the one in Theorem 1.6, we have $\|f_n\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)} \le C\|f\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)}$ and hence $f_n \in L^2(\mathbb{R}^N)\cap CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$. For any $g \in \mathcal{S}_\infty(\mathbb{R}^N)$, the discrete Calderón reproducing formula given in Theorem 2.1 yields

$$\begin{aligned}\langle f - f_n, g\rangle &= \Big\langle \sum_{|j|>n,\text{ or } |k|>n,\text{ or } R\subsetneq B(0,n)} |R|\psi_J * f(x_R)\psi_J(\cdot - x_R), g\Big\rangle \\ &= \Big\langle f, \sum_{|j|>n,\text{ or } |k|>n,\text{ or } R\subsetneq B(0,n)} |R|\psi_J * g(x_R)\psi_J(\cdot - x_R)\Big\rangle.\end{aligned}$$

By Corollary 2.6, the function

$$\sum_{|j|>n,\text{ or } |k|>n,\text{ or } R\subsetneq B(0,n)} |R|\psi_J * g(x_R)\psi_J(x - x_R)$$

belongs to $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ and its $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ norm tends to 0 as $n \to \infty$. Hence, Theorem 1.7 implies that $\langle f - f_n, g\rangle$ tends to zero as $n \to \infty$. Since $\mathcal{S}_\infty(\mathbb{R}^N)$ is dense in $H^p_{\mathcal{F},w}(\mathbb{R}^N)$, a standard limiting argument finishes the proof of Lemma 3.6. □

Now let us show how a flag singular integral operator $T_{\mathcal{F}}$ acts on $CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$. Given $f \in CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$, by Lemma 3.6, there is a sequence

$$\{f_n\} \subset L^2(\mathbb{R}^N) \cap CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$$

such that

$$\begin{cases} \|f_n\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)} \le C\|f\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)}, \\ \lim_{n\to\infty} \langle f_n, g\rangle = \langle f, g\rangle \quad \text{for any } g \in L^2(\mathbb{R}^N) \cap H^p_{\mathcal{F},w}(\mathbb{R}^N). \end{cases} \tag{3-15}$$

We thus define

$$\langle T_{\mathcal{F}}(f), g\rangle = \lim_{n\to\infty} \langle T_{\mathcal{F}}(f_n), g\rangle \quad \text{for any } g \in L^2(\mathbb{R}^N) \cap H^p_{\mathcal{F},w}(\mathbb{R}^N).$$

To see that the limit exists, write $\langle (T_{\mathcal{F}})(f_j - f_k), g\rangle = \langle f_j - f_k, (T_{\mathcal{F}})^*(g)\rangle$ since both $f_j - f_k$ and g belong to $L^2(\mathbb{R}^N)$, and $T_{\mathcal{F}}$ is bounded on $L^2(\mathbb{R}^N)$. By Theorem 1.9, $(T_{\mathcal{F}})^*$ is bounded on $H^p_{\mathcal{F},w}(\mathbb{R}^N)$, and therefore $(T_{\mathcal{F}})^*(g) \in L^2(\mathbb{R}^N) \cap H^p_{\mathcal{F},w}(\mathbb{R}^N)$. Thus, by Lemma 3.6, $\langle f_j - f_k, (T_{\mathcal{F}})^*(g)\rangle$ tends to zero as $j, k \to \infty$. It is also easy to verify that the definition of $T_{\mathcal{F}}(f)$ is independent of the choice of the sequence f_n satisfying the conditions in Lemma 3.6.

We are now ready to prove Theorem 1.11.

Proof of Theorem 1.11. We first show that for $f \in L^2(\mathbb{R}^N) \cap CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$ and any open set Ω with $w(\Omega) < \infty$,

$$\left\{\frac{1}{[w(\Omega)]^{2/p-1}} \sum_{J\in\mathbb{Z}^3} \sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}},\\ R\subset\Omega}} |\psi_J * K_{\mathcal{F}} * f(x_R)|^2 \frac{|R|^2}{w(R)}\right\}^{\frac{1}{2}} \lesssim \|f\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)}. \tag{3-16}$$

Using the discrete Calderón reproducing formula given in Theorem 2.1, we write

$$\begin{aligned}\sum_{J\in\mathbb{Z}^3} \sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}},\\ R\subset\Omega}} &|\psi_J * K_{\mathcal{F}} * f(x_R)|^2 \frac{|R|^2}{w(R)} \\ &= \sum_{J\in\mathbb{Z}^3} \sum_{\substack{R\in\mathcal{R}^J_{\mathcal{F}},\\ R\subset\Omega}} \left| \sum_{J'\in\mathbb{Z}^3} \sum_{R'\in\mathcal{R}^{J'}_{\mathcal{F}}} t_{R'} |R'|^{\frac{1}{2}} \psi_J * \mathcal{K} * \psi_{J'}(x_R - x_{R'})\right|^2 \frac{|R|^2}{w(R)},\end{aligned}$$

where $t_{R'} = \psi_{J'} * f(x_{R'})|R'|^{\frac{1}{2}}$. By (3-6), $\psi_J * \mathcal{K} * \psi_{J'}$ satisfies the same almost-orthogonality estimate as $\psi_J * \psi_{J'}$. Repeating the same argument as in Theorem 1.6 yields (3-16).

For $f \in CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$, there is a sequence

$$\{f_n\} \subset L^2(\mathbb{R}^N) \cap CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$$

satisfying (3-15). By the definition of $T_{\mathcal{F}}(f)$ and the boundedness of $T_{\mathcal{F}}$ on $L^2(\mathbb{R}^N) \cap CMO^p_{\mathcal{F},w}(\mathbb{R}^N)$,

$$\begin{aligned}\|T_{\mathcal{F}}(f)\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)} &\le \liminf_{n\to\infty} \|T_{\mathcal{F}}(f_n)\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)} \\ &\lesssim \liminf_{n\to\infty} \|f_n\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)} \lesssim \|f\|_{CMO^p_{\mathcal{F},w}(\mathbb{R}^N)},\end{aligned}$$

which concludes the proof of Theorem 1.11. □

4. Calderón–Zygmund decomposition and interpolation

We first prove the Calderón–Zygmund decomposition for $H^p_{\mathcal{F},w}$.

Proof of Theorem 1.12. According to Corollary 2.6, $L^2(\mathbb{R}^N) \cap H^p_{\mathcal{F},w}(\mathbb{R}^N)$ is dense in $H^p_{\mathcal{F},w}(\mathbb{R}^N)$. Thus it suffices to prove Theorem 1.12 for $f \in L^2(\mathbb{R}^N) \cap H^p_{\mathcal{F},w}(\mathbb{R}^N)$. Given any fixed $\alpha > 0$, let $\Omega_l = \{x \in \mathbb{R}^N : \tilde{g}_{\mathcal{F}}(f)(x) > \alpha 2^l\}$, $l \in \mathbb{N}$, where $\tilde{g}_{\mathcal{F}}(f)$ and K are the same as in Corollary 3.4. For $J \in \mathbb{Z}^3$, set

$$\begin{aligned}&\mathcal{R}^{J,K}_0 = \left\{R \in \mathcal{R}^{J,K}_{\mathcal{F}} : |R \cap \Omega_0| < \tfrac{1}{2}|R|\right\}, \quad \text{and} \\ &\mathcal{R}^{J,K}_l = \left\{R \in \mathcal{R}^{J,K}_{\mathcal{F}} : |R \cap \Omega_{l-1}| \ge \tfrac{1}{2}|R|,\ |R \cap \Omega_l| < \tfrac{1}{2}|R|\right\} \quad \text{for } l \ge 1.\end{aligned}$$

It follows from Theorem 3.3 that there exists some $h \in L^2 \cap H^p_{\mathcal{F},w}$ such that

$$f(x) = \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_0^{J,K}} |R|\phi_J * h(x_R)\widetilde{\phi}_J(x-x_R) \\ + \sum_{J\in\mathbb{Z}^3} \sum_{l\geq 1} \sum_{R\in\mathcal{R}_l^{J,K}} |R|\phi_J * h(x_R)\widetilde{\phi}_J(x-x_R) =: g(x)+b(x).$$

We first estimate $\|g\|_{H^{p_2}_{\mathcal{F},w}}$. Repeating the same argument as in the proof of Theorem 1.4, we deduce that for $N/M < \delta < \min\{(p_2/q_w), 1\}$,

$$\sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_{\mathcal{F}}^J} |\psi_J * g(x_R)|^2 \chi_R(x) \lesssim \sum_{J'\in\mathbb{Z}^3} \left\{ \mathcal{M}_{\mathcal{F}}\left[\left(\sum_{R'\in\mathcal{R}_0^{J',K}} |\phi_{J'} * g(x_{R'})|^2 \chi_{R'}\right)^{\frac{\delta}{2}}\right](x)\right\}^{\frac{2}{\delta}}.$$

Take the square root on both sides and apply Corollary 2.5 on $L_w^{p_2/\delta}(\ell^{2/\delta})$ (note that $w \in A^{\mathcal{F}}_{p_2/\delta}$) to derive

$$\|g\|_{H^{p_2}_{\mathcal{F},w}(\mathbb{R}^N)} \lesssim \left\| \left\{ \sum_{J'\in\mathbb{Z}^3} \sum_{R'\in\mathcal{R}_0^{J',K}} |\phi_{J'} * h(x_{R'})|^2 \chi_{R'} \right\}^{\frac{1}{2}} \right\|_{L_w^{p_2}(\mathbb{R}^N)}.$$

We claim

$$\int_{\{\tilde{g}_{\mathcal{F}}(f)(x)\leq\alpha\}} [\tilde{g}_{\mathcal{F}}(f)(x)]^{p_2} w(x)\,dx \\ \gtrsim \left\| \left\{ \sum_{J'\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_0^{J',K}} |\phi_{J'} * h(x_{R'})|^2 \chi_{R'} \right\}^{\frac{1}{2}} \right\|^{p_2}_{L_w^{p_2}(\mathbb{R}^N)}, \tag{4-1}$$

which implies

$$\|g\|^{p_2}_{H^{p_2}_{\mathcal{F},w}(\mathbb{R}^N)} \lesssim \int_{\{\tilde{g}_{\mathcal{F}}(f)(x)\leq\alpha\}} [\tilde{g}_{\mathcal{F}}(f)(x)]^{p_2} w(x)\,dx \\ \leq \alpha^{p_2-p} \int_{\{\tilde{g}_{\mathcal{F}}(f)(x)\leq\alpha\}} [\tilde{g}_{\mathcal{F}}(f)(x)]^{p} w(x)\,dx \lesssim \alpha^{p_2-p} \|f\|^p_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}$$

as desired. It suffices to verify claim (4-1). Choose $\delta < \min\{p_2/q_w, 1\}$ and get

$$\int_{\{\tilde{g}_{\mathcal{F}}(f)(x)\leq\alpha\}} [\tilde{g}_{\mathcal{F}}(f)(x)]^{p_2} w(x)\,dx \\ = \int_{\mathbb{R}^N} \left\{ \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_{\mathcal{F}}^{J,K}} |\phi_J * h(x_R)|^2 \chi_{R\cap \complement\Omega_0}(x) \right\}^{\frac{p_2}{2}} w(x)\,dx$$

$$\gtrsim \int_{\mathbb{R}^N} \Big\{ \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_{\mathcal{F}}^{J,K}} (|\phi_J * h(x_R)|^\delta \mathcal{M}_{\mathcal{F}}(\chi_{R\cap\complement\Omega_0})(x))^{\frac{2}{\delta}} \Big\}^{\frac{p_2}{2}} w(x)\,dx$$

$$\gtrsim \bigg\| \Big\{ \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_{\mathcal{F}}^{J,K}} |\phi_J * h(x_R)|^2 \chi_R \Big\}^{\frac{1}{2}} \bigg\|_{L_w^{p_2}(\mathbb{R}^N)}^{p_2},$$

where in the last inequality we used the estimate $\chi_R(x) \le 2^{1/\delta}\mathcal{M}_{\mathcal{F}}(\chi_{R\cap\complement\Omega_0})^{1/\delta}(x)$ for $R\in\mathcal{R}_{\mathcal{F}}^{J,K}$, and the first inequality follows from Corollary 2.5 with $q=2/\delta$ and $p=p_2/\delta$.

Now, we turn to the estimate for the $H^{p_1}_{\mathcal{F},w}(\mathbb{R}^N)$ norm of b. Set $\widetilde{\Omega}_l = \{x\in\mathbb{R}^N : \mathcal{M}_{\mathcal{F}}(\chi_{\Omega_l}) > \frac{1}{2}\}$, $l\in\mathbb{Z}$. Then the desired estimate follows from

$$\bigg\| \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_l^{J,K}} |R|\phi_J * h(x_R)\widetilde{\phi}_J(\cdot - x_R) \bigg\|_{H^{p_1}_{\mathcal{F},w}}^{p_1} \lesssim (2^l\alpha)^{p_1} w(\widetilde{\Omega}_{l-1}), \tag{4-2}$$

for any $0<p_1\le 1$ and $l\ge 1$. Indeed, by the $L^q_w(\mathbb{R}^N)$ $(q>q_w)$ boundedness of $\mathcal{M}_{\mathcal{F}}$,

$$w(\widetilde{\Omega}_{l-1}) \lesssim \int_{\mathbb{R}^N} [\mathcal{M}_{\mathcal{F}}(\chi_{\Omega_{l-1}})(x)]^q w(x)\,dx \lesssim w(\Omega_{l-1}).$$

This fact together with (4-2) yields

$$\begin{aligned} \|b\|^{p_1}_{H^{p_1}_{\mathcal{F},w}(\mathbb{R}^N)} &\lesssim \sum_{l\ge 1} (2^l\alpha)^{p_1} w(\widetilde{\Omega}_{l-1}) \lesssim \sum_{l\ge 1} (2^l\alpha)^{p_1} w(\Omega_{l-1}) \\ &\lesssim \int_{\{\tilde{g}_{\mathcal{F}}(f)(x)>\alpha\}} [\tilde{g}_{\mathcal{F}}(f)(x)]^{p_1} w(x)\,dx \\ &\lesssim \alpha^{p_1-p} \int_{\{\tilde{g}_{\mathcal{F}}(f)(x)>\alpha\}} [\tilde{g}_{\mathcal{F}}(f)(x)]^{p} w(x)\,dx \lesssim \alpha^{p_1-p} \|f\|^p_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}. \end{aligned}$$

Thus to finish the proof, it remains to establish (4-2). Following the same argument as in the estimation of $\|g\|_{H^{p_2}_{\mathcal{F},w}}$, we get that for any $l\ge 1$,

$$\begin{aligned} \bigg\| \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_l^{J,K}} |R|\phi_J * h(x_R)\widetilde{\phi}_J(\cdot - x_R) \bigg\|_{H^{p_1}_{\mathcal{F},w}} \\ \lesssim \bigg\| \Big\{ \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_l^{J,K}} |\phi_J * h(x_R)|^2 \chi_R \Big\}^{\frac{1}{2}} \bigg\|_{L^{p_1}_w}. \end{aligned} \tag{4-3}$$

Note that $R\subset\widetilde{\Omega}_{l-1}$ for $R\in\mathcal{R}_l^{J,K}$. Thus, $|R\cap(\widetilde{\Omega}_{l-1}\setminus\Omega_l)| > \frac{1}{2}|R|$, which implies

$$\chi_R(x) \le 2^{\frac{1}{\delta}} \mathcal{M}_{\mathcal{F}}(\chi_{R\cap(\widetilde{\Omega}_{l-1}\setminus\Omega_l)})^{\frac{1}{\delta}}(x).$$

As in the proof of claim (4-1), picking $\delta < \min\{2,\ p_1/q_w\}$ and applying Corollary 2.5, we have

$$(2^l\alpha)^{p_1} w(\widetilde{\Omega}_{l-1}) \ge \int_{\mathbb{R}^N} \Bigg\{ \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_{\mathcal{F}}^{J,K}} |\phi_J * h(x_R)|^2 \chi_{R\cap(\widetilde{\Omega}_{l-1}\setminus\Omega_l)}(x) \Bigg\}^{\frac{p_1}{2}} w(x)\,dx$$

$$\gtrsim \int_{\mathbb{R}^N} \Bigg\{ \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_l^{J,K}} |\phi_J * h(x_R)|^2 \mathcal{M}_{\mathcal{F}}(\chi_{R\cap(\widetilde{\Omega}_{l-1}\setminus\Omega_l)})^{\frac{2}{\delta}}(x) \Bigg\}^{\frac{p_1}{2}} w(x)\,dx$$

$$\gtrsim \Bigg\| \Bigg\{ \sum_{J\in\mathbb{Z}^3} \sum_{R\in\mathcal{R}_l^{J,K}} |\phi_J * h(x_R)|^2 \chi_R \Bigg\}^{\frac{1}{2}} \Bigg\|_{L_w^{p_1}(\mathbb{R}^N)}^{p_1}.$$

Combining this with (4-3) yields (4-2), and hence Theorem 1.12 follows. □

Proof of Theorem 1.13. Suppose that T is bounded from $H_{\mathcal{F},w}^{p_1}(\mathbb{R}^N)$ to $L_w^{p_1}(\mathbb{R}^N)$ and bounded from $H_{\mathcal{F},w}^{p_2}(\mathbb{R}^N)$ to $L_w^{p_2}(\mathbb{R}^N)$. Given $f \in H_{\mathcal{F},w}^{p}(\mathbb{R}^N)$, $p_1 < p < p_2$, the Calderón–Zygmund decomposition shows that $f = g + b$ with

$$\|g\|_{H_{\mathcal{F},w}^{p_2}(\mathbb{R}^N)}^{p_2} \lesssim \alpha^{p_2-p} \|f\|_{H_{\mathcal{F},w}^{p}(\mathbb{R}^N)}^{p} \quad \text{and} \quad \|b\|_{H_{\mathcal{F},w}^{p_1}(\mathbb{R}^N)}^{p_1} \lesssim \alpha^{p_1-p} \|f\|_{H_{\mathcal{F},w}^{p}(\mathbb{R}^N)}^{p}.$$

Moreover, in the proof of Theorem 1.12, we have shown that

$$\|g\|_{H_{\mathcal{F},w}^{p_2}(\mathbb{R}^N)}^{p_2} \lesssim \int_{\{\tilde{g}_{\mathcal{F}}(f)(x)\le\alpha\}} [\tilde{g}_{\mathcal{F}}(f)(x)]^{p_2} w(x)\,dx$$

and

$$\|b\|_{H_{\mathcal{F},w}^{p_1}(\mathbb{R}^N)}^{p_1} \lesssim \int_{\{\tilde{g}_{\mathcal{F}}(f)(x)>\alpha\}} [\tilde{g}_{\mathcal{F}}(f)(x)]^{p_1} w(x)\,dx.$$

Therefore,

$$\begin{aligned}
\|Tf\|_{L_w^p(\mathbb{R}^N)}^p &\le p\int_0^\infty \alpha^{p-1} w\Big(\Big\{x : |T(g)(x)| > \frac{\alpha}{2}\Big\}\Big)\,d\alpha \\
&\qquad + p\int_0^\infty \alpha^{p-1} w\Big(\Big\{x : |T(b)(x)| > \frac{\alpha}{2}\Big\}\Big)\,d\alpha \\
&\lesssim \int_0^\infty \alpha^{p-1}\bigg(\frac{\|T(g)\|_{L_w^{p_2}}}{\alpha}\bigg)^{p_2} d\alpha + \int_0^\infty \alpha^{p-1}\bigg(\frac{\|T(b)\|_{L_w^{p_1}}}{\alpha}\bigg)^{p_1} d\alpha \\
&\lesssim \int_0^\infty \alpha^{p-p_2-1} \int_{\{\tilde{g}_{\mathcal{F}}(f)(x)\le\alpha\}} [\tilde{g}_{\mathcal{F}}(f)(x)]^{p_2} w(x)\,dx\,d\alpha \\
&\qquad + \int_0^\infty \alpha^{p-p_1-1} \int_{\{\tilde{g}_{\mathcal{F}}(f)(x)>\alpha\}} [\tilde{g}_{\mathcal{F}}(f)(x)]^{p_1} w(x)\,dx\,d\alpha \\
&\lesssim \|f\|_{H_{\mathcal{F},w}^{p}(\mathbb{R}^N)}^{p}.
\end{aligned}$$

Hence T is bounded from $H^p_{\mathcal{F},w}(\mathbb{R}^N)$ to $L^p_w(\mathbb{R}^N)$.

To prove the second assertion that T is bounded on $H^p_{\mathcal{F},w}(\mathbb{R}^N)$, for any given $\alpha > 0$ and $f \in H^p_{\mathcal{F},w}(\mathbb{R}^N)$, we apply the Calderón–Zygmund decomposition again to obtain

$$\begin{aligned} w(\{x : |\tilde{g}_{\mathcal{F}}(Tf)(x)| > \alpha\}) &\le w\Big(\Big\{x : |\tilde{g}_{\mathcal{F}}(Tg)(x)| > \frac{\alpha}{2}\Big\}\Big) \\ &\qquad + w\Big(\Big\{x : |\tilde{g}_{\mathcal{F}}(Tb)(x)| > \frac{\alpha}{2}\Big\}\Big) \\ &\lesssim \alpha^{-p_2} \|T(g)\|^{p_2}_{H^{p_2}_{\mathcal{F},w}(\mathbb{R}^N)} + \alpha^{-p_1} \|T(b)\|^{p_1}_{H^{p_1}_{\mathcal{F},w}(\mathbb{R}^N)} \\ &\lesssim \alpha^{-p_2} \|g\|^{p_2}_{H^{p_2}_{\mathcal{F},w}(\mathbb{R}^N)} + \alpha^{-p_1} \|b\|^{p_1}_{H^{p_1}_{\mathcal{F},w}(\mathbb{R}^N)} \\ &\lesssim \alpha^{-p_2} \int_{\{\tilde{g}_{\mathcal{F}}(f)(x) \le \alpha\}} [\tilde{g}_{\mathcal{F}}(f)(x)]^{p_2} w(x)\, dx \\ &\qquad + \alpha^{-p_1} \int_{\{\tilde{g}_{\mathcal{F}}(f)(x) > \alpha\}} [\tilde{g}_{\mathcal{F}}(f)(x)]^{p_1} w(x)\, dx, \end{aligned}$$

which, as above, implies that $\|\tilde{g}_{\mathcal{F}}(Tf)\|_{L^p_w(\mathbb{R}^N)} \lesssim \|f\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}$, and therefore $\|Tf\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)} \lesssim \|f\|_{H^p_{\mathcal{F},w}(\mathbb{R}^N)}$ by Corollary 3.4. The proof of Theorem 1.13 is complete. $\square$

5. Relations between different classes of weights

In this section, we clarify the relations between different classes of weights by constructing some examples and counterexamples. Our aim is to show the following:

Proposition 5.1. *For* $1 < p < \infty$, $A^{\text{pro}}_p(\mathbb{R}^N) \subsetneq A^{\mathcal{F}}_p(\mathbb{R}^N) \subsetneq A_p(\mathbb{R}^N)$.

A positive measure μ on $\mathbb{R}^n$ is called doubling if there exists a constant $0 < C < \infty$, depending only on n, such that

$$\mu(2B) \le C\mu(B) \tag{5-1}$$

for all balls B.

To prove this proposition, we need the following lemmas.

Lemma 5.2. *Let* $1 < p < \infty$. *If* $a > -n$ *and* $A \ge 0$, *then* $(|x| + A)^a\, dx$ *is a doubling measure on* $\mathbb{R}^n$ *with doubling constant depending on* a *and* n, *but independent of* A.

Proof. If $A = 0$, the conclusion above is well known. If $A > 0$, following the arguments given in [Grafakos 2014, pp. 505–506], we divide all balls $B(x_0, R)$ in $\mathbb{R}^n$ into two categories: balls of type I that satisfy $|x_0| + A \ge 3R$ and type II that satisfy $|x_0| + A < 3R$.

For the balls of type I, we have

$$\int_{B(x_0,2R)} (|x| + A)^a\, dx \le \begin{cases} v_n(2R)^n(|x_0| + A + 2R)^a & \text{if } a \ge 0, \\ v_n(2R)^n(|x_0| + A - 2R)^a & \text{if } a < 0, \end{cases}$$

and

$$\int_{B(x_0,R)} (|x|+A)^a \, dx \geq \begin{cases} v_n R^n (|x_0|+A-R)^a & \text{if } a \geq 0, \\ v_n R^n (|x_0|+A+R)^a & \text{if } a < 0, \end{cases}$$

where v_n denotes the volume of the unit ball in $\mathbb{R}^n$. Since $|x_0|+A \geq 3R$, we have $|x_0|+A+2R \leq 4(|x_0|+A-R)$ and $|x_0|+A-2R \geq \frac{1}{4}(|x_0|+A+R)$. The required doubling property (5-1) then follows with $C = 2^{3n}4^{|a|}$.

For balls of type II, $|x_0|+A < 3R$ implies $B(x_0, 2R) \subset B(0, 5R)$. Therefore

$$\int_{B(x_0,2R)} (|x|+A)^a \, dx \leq \int_{B(0,5R)} (|x|+A)^a \, dx \lesssim R^{n+a}.$$

On the other hand, using the fact that $(|x|+A)^a$ is rapidly increasing if $a \geq 0$, and the inequality $|x|+A \leq |x_0|+A+R < 4R$ if $a < 0$, we have

$$\int_{B(x_0,R)} (|x|+A)^a \, dx \geq \begin{cases} \int_{B(0,R)} |x|^a \, dx & \text{if } a \geq 0, \\ \int_{B(x_0,R)} (4R)^a \, dx & \text{if } a < 0, \end{cases}$$
$$\gtrsim R^{n+a}.$$

This concludes the proof of Lemma 5.2. □

Lemma 5.3. *Let $1 < p < \infty$. If $w_1, w_2 \in A_p(\mathbb{R}^n)$, then $w(x) = \min\{w_1(x), w_2(x)\}$ is also in $A_p(\mathbb{R}^n)$ with $[w]_{A_p} \leq \max(1, 2^{p-2})([w_1]_{A_p} + [w_2]_{A_p})$.*

Proof. This lemma slightly generalizes a result in [Grafakos 2014, p. 513], where the case of w_2 being a constant was considered. For any $Q \subset \mathbb{R}^n$,

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{-\frac{p'}{p}}\right)^{\frac{p}{p'}} \\
&\quad \leq \left(\frac{1}{|Q|}\int_Q [w_1 \wedge w_2]\right)\left(\frac{1}{|Q|}\int_Q w_1^{-\frac{p'}{p}} + \frac{1}{|Q|}\int_Q w_2^{-\frac{p'}{p}}\right)^{\frac{p}{p'}} \\
&\quad \leq \max(1, 2^{p-2})\left\{\left(\frac{1}{|Q|}\int_Q w_1\right) \wedge \left(\frac{1}{|Q|}\int_Q w_2\right)\right\} \\
&\qquad\qquad \cdot \left\{\left(\frac{1}{|Q|}\int_Q w_1^{-\frac{p'}{p}}\right)^{\frac{p}{p'}} + \left(\frac{1}{|Q|}\int_Q w_2^{-\frac{p'}{p}}\right)^{\frac{p}{p'}}\right\} \\
&\quad \leq \max(1, 2^{p-2})\left\{\left(\frac{1}{|Q|}\int_Q w_1\right)\left(\frac{1}{|Q|}\int_Q w_1^{-\frac{p'}{p}}\right)^{\frac{p}{p'}}\right. \\
&\qquad\qquad \left. + \left(\frac{1}{|Q|}\int_Q w_2\right)\left(\frac{1}{|Q|}\int_Q w_2^{-\frac{p'}{p}}\right)^{\frac{p}{p'}}\right\} \\
&\quad \leq \max(1, 2^{p-2})([w_1]_{A_p} + [w_2]_{A_p}),
\end{aligned}$$

where in the second inequality we used the basic inequality

$$(a_1+a_2)^{\frac{p}{p'}} \le \max(1, 2^{p-2})(a_1^{\frac{p}{p'}} + a_2^{\frac{p}{p'}})$$

for $a_1, a_2 > 0$. Hence the required result follows. □

Proof of Proposition 5.1. By definition, it is clear that $A_p^{\mathrm{pro}}(\mathbb{R}^N) \subset A_p^{\mathcal{F}}(\mathbb{R}^N) \subset A_p(\mathbb{R}^N)$.

Now, let us show that these inclusions are proper. For simplicity, we only consider the biparameter case on $\mathbb{R}^N = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, where we denote the flag $\{(0,0)\} \subset \{(0,y)\} \subset \mathbb{R}^{n_1+n_2}$ by $\mathcal{F}_1$.

We first show that $A_p^{\mathrm{pro}}(\mathbb{R}^N) \subsetneq A_p^{\mathcal{F}_1}(\mathbb{R}^N)$. Let $1-n_2 < a < n_2(p-1)$. For $x = (x^{(1)}, \dots, x^{(n_1)}) \in \mathbb{R}^{n_1}$, $y = (y^{(1)}, \dots, y^{(n_2)}) \in \mathbb{R}^{n_2}$, we define

$$w(x,y) = |y|^a (|x^{(1)}| + |y|)^{-1}.$$

Since w is independent of $(x^{(2)}, \dots, x^{(n_1)})$, for notational simplicity, we may assume $n_1 = 1$ and write x instead of $x^{(1)}$. We claim that

$$\operatorname*{ess\,sup}_{y\in\mathbb{R}^{n_2}} [w(\cdot\,, y)]_{A_p(\mathbb{R}^1)} = \infty, \tag{5-2}$$

$$\sup_{x\in\mathbb{R}^1} [w(x, \cdot)]_{A_p(\mathbb{R}^{n_2})} < \infty, \tag{5-3}$$

$$[w]_{A_p(\mathbb{R}^{1+n_2})} < \infty, \tag{5-4}$$

where $[\,\cdot\,]_{A_p}$ denotes the classical A_p characteristic constant. Assume (5-2), (5-3) and (5-4) hold for the moment. Then (5-3) and (5-4) imply $w \in A_p^{(1)}(\mathbb{R}^N) \cap A_p^{(2)}(\mathbb{R}^N)$, and therefore $w \in A_p^{\mathcal{F}_1}$ by Lemma 2.4; while (5-2) implies $w \notin A_p^{\mathrm{pro}}$. Hence $A_p^{\mathrm{pro}}(\mathbb{R}^N) \subsetneq A_p^{\mathcal{F}_1}(\mathbb{R}^N)$.

Now let us prove (5-2), (5-3) and (5-4). We prove first (5-2). Given any $y \in \mathbb{R}^{n_2}$ with $y \neq 0$, we choose an interval $I = (0, r)$ with $r = |y|^2$. Then

$$\begin{aligned}\left(\frac{1}{|I|}\int_I w(x,y)\,dx\right)&\left(\frac{1}{|I|}\int_I [w(x,y)]^{-\frac{p'}{p}}\,dx\right)^{\frac{p}{p'}} \\ &= \left(\frac{1}{r}\int_0^r (x+|y|)^{-1}\,dx\right)\left(\frac{1}{r}\int_0^r (x+|y|)^{\frac{p'}{p}}\,dx\right)^{\frac{p}{p'}} \\ &\gtrsim \frac{1}{r}\ln\left(1+\frac{r}{|y|}\right)\cdot r = \ln(1+|y|) \to +\infty,\end{aligned}$$

as $|y|$ goes to $+\infty$, which establishes (5-2).

Next, let us show (5-3). For $x = 0$, $w(0,y) = |y|^{a-1} \in A_p(\mathbb{R}^{n_2})$. For any fixed $x \neq 0$, $w_1(y) := |y|^{a-1}$ and $w_2(y) := |y|^a |x|^{-1}$ are both in $A_p(\mathbb{R}^{n_2})$, with A_p characteristic constants being independent of x. Since $w(y) \approx \min(w_1(y), w_2(y))$, (5-3) follows immediately from Lemma 5.3.

We now verify (5-4). For any $n \in \mathbb{Z}_+$, we divide all cubes $Q(x_0, l)$ (centered at x_0 of side length l) in $\mathbb{R}^n$ into two categories: cubes of type I that satisfy $|x_0| \geq 3\sqrt{n}\, l$ and type II that satisfy $|x_0| < 3\sqrt{n}\, l$. For any $Q \subset \mathbb{R}^{1+n_2}$, write $Q = Q_1 \times Q_2$, where

$$Q_1 = (x_0 - l, x_0 + l) \subset \mathbb{R} \quad \text{and} \quad Q_2 = Q_2(y_0, l) \subset \mathbb{R}^{n_2}.$$

We consider the following four cases.

Case 1: If both Q_1 and Q_2 are of type I, then for any $(x, y) \in Q$, $w(x, y) \approx |y_0|^a (|x_0| + |y_0|)^{-1}$, and hence the desired result follows easily.

Case 2: If Q_1 is of type I and Q_2 of type II, then for any $(x, y) \in Q$, we have $|x| \approx |x_0| \gtrsim |y|$ and thus $w(x, y) \approx |y|^a |x_0|^{-1}$. Moreover, $|y|^a$ is in $A_p(\mathbb{R}^{n_2})$. Hence,

$$\begin{aligned}
&\left(\frac{1}{|Q|} \iint_Q w(x, y)\, dx\, dy\right)\left(\frac{1}{|Q|} \iint_Q w(x, y)^{-\frac{p'}{p}}\, dx\, dy\right)^{\frac{p}{p'}} \\
&\qquad \approx \left(\frac{1}{|Q_2|} \int_{Q_2} |y|^a\, dy\right)\left(\frac{1}{|Q_2|} \int_{Q_2} |y|^{-\frac{ap'}{p}}\, dy\right)^{\frac{p}{p'}} \leq [|y|^a]_{A_p(\mathbb{R}^{n_1})} \lesssim 1.
\end{aligned}$$

Case 3: If Q_1 is of type II and Q_2 of type I, then for any $(x, y) \in Q$, we have $|y| \approx |x| + |y| \approx |y_0|$ and therefore $w(x, y) \approx |y_0|^{a-1}$. Hence the required estimate follows easily.

Case 4: If both Q_1 and Q_2 are of type II, then by Lemma 5.2,

$$\begin{aligned}
&\left(\frac{1}{|Q|} \iint_Q w(x, y)\, dx\, dy\right)\left(\frac{1}{|Q|} \iint_Q w(x, y)^{-\frac{p'}{p}}\, dx\, dy\right)^{\frac{p}{p'}} \\
&\qquad \lesssim \left(\frac{1}{l^{n_2+1}} \int_{B(0,5\sqrt{n_2}\, l)} |y|^a dy \int_0^{5l} (x + |y|)^{-1}\, dx\right) \\
&\qquad\qquad \times \left(\frac{1}{l^{n_2+1}} \int_{B(0,5\sqrt{n_2}\, l)} |y|^{-\frac{ap'}{p}}\, dy \int_0^{5l} (x + |y|)^{\frac{p'}{p}}\, dx\right)^{\frac{p}{p'}} \\
&\qquad \lesssim \left(\frac{1}{l^{n_2}} \int_{B(0,5\sqrt{n_2}\, l)} |y|^{a-1} dy\right)\left(l^{\frac{p'}{p} - n_2} \int_{B(0,5\sqrt{n_2}\, l)} |y|^{-\frac{ap'}{p}}\, dy\right)^{\frac{p}{p'}} \\
&\qquad \approx l^{a-1} \cdot (l^{(1-a)\frac{p'}{p}})^{\frac{p}{p'}} = 1,
\end{aligned}$$

where $B(0, 5\sqrt{n_2}\, l)$ denotes the ball of radius $5\sqrt{n_2}\, l$ centered at origin of $\mathbb{R}^{n_2}$. This concludes the proof of (5-4), and hence $A_p^{\text{pro}}(\mathbb{R}^N) \subsetneq A_p^{\mathcal{F}_1}(\mathbb{R}^N)$.

Finally, let us show $A_p^{\mathcal{F}_1}(\mathbb{R}^N) \subsetneq A_p(\mathbb{R}^N)$. From (5-4) and (5-2), interchanging the roles of x and y, we see that for $1 - n_1 < b < n_1(p-1)$, $|x|^b (|y^{(1)}| + |x|)^{-1}$ is in $A_p(\mathbb{R}^N)$, but not in $A_p^{\mathcal{F}_1}(\mathbb{R}^N)$. This completes the proof of Proposition 5.1. □

Acknowledgment

The authors would like to express their gratitude to the anonymous referee for the careful reading of the manuscript and for providing very valuable corrections, comments and suggestions, which improved the original manuscript.

References

[Calderón 1964] A.-P. Calderón, "Intermediate spaces and interpolation: the complex method", *Studia Math.* **24** (1964), 113–190. MR Zbl

[Chang 1979] S.-Y. A. Chang, "Carleson measure on the bi-disc", *Ann. of Math.* (2) **109**:3 (1979), 613–620. MR Zbl

[Chang and Fefferman 1980] S.-Y. A. Chang and R. Fefferman, "A continuous version of duality of H^1 with BMO on the bidisc", *Ann. of Math.* (2) **112**:1 (1980), 179–201. MR Zbl

[Chang and Fefferman 1982] S.-Y. A. Chang and R. Fefferman, "The Calderón–Zygmund decomposition on product domains", *Amer. J. Math.* **104**:3 (1982), 455–468. MR Zbl

[Chang and Fefferman 1985] S.-Y. A. Chang and R. Fefferman, "Some recent developments in Fourier analysis and H^p-theory on product domains", *Bull. Amer. Math. Soc.* (*N.S.*) **12**:1 (1985), 1–43. MR Zbl

[David et al. 1985] G. David, J.-L. Journé, and S. Semmes, "Opérateurs de Calderón–Zygmund, fonctions para-accrétives et interpolation", *Rev. Mat. Iberoam.* **1**:4 (1985), 1–56. MR Zbl

[Ding et al. 2010] Y. Ding, G. Z. Lu, and B. L. Ma, "Multi-parameter Triebel–Lizorkin and Besov spaces associated with flag singular integrals", *Acta Math. Sin.* **26**:4 (2010), 603–620 . MR

[Ding et al. 2012] Y. Ding, Y. Han, G. Lu, and X. Wu, "Boundedness of singular integrals on multiparameter weighted Hardy spaces $H^p_w(\mathbb{R}^n \times \mathbb{R}^m)$", *Potent. Anal.* **37**:1 (2012), 31–56. MR Zbl

[Fefferman 1986] R. Fefferman, "Calderón–Zygmund theory for product domains: H^p spaces", *Proc. Nat. Acad. Sci. U.S.A.* **83**:4 (1986), 840–843. MR Zbl

[Fefferman 1987] R. Fefferman, "Harmonic analysis on product spaces", *Ann. of Math.* (2) **126**:1 (1987), 109–130. MR Zbl

[Fefferman 1988] R. Fefferman, "A^p weights and singular integrals", *Amer. J. Math.* **110**:5 (1988), 975–987. MR Zbl

[Fefferman and Stein 1972] C. Fefferman and E. M. Stein, "H^p spaces of several variables", *Acta Math.* **129**:3-4 (1972), 137–193. MR Zbl

[Fefferman and Stein 1982] R. Fefferman and E. M. Stein, "Singular integrals on product spaces", *Adv. Math.* **45**:2 (1982), 117–143. MR Zbl

[Frazier and Jawerth 1990] M. Frazier and B. Jawerth, "A discrete transform and decompositions of distribution spaces", *J. Funct. Anal.* **93**:1 (1990), 34–170. MR Zbl

[Frazier et al. 1991] M. Frazier, B. Jawerth, and G. Weiss, *Littlewood–Paley theory and the study of function spaces*, CBMS Regional Conf. Series in Math. **79**, Amer. Math. Soc., Providence, RI, 1991. MR Zbl

[García-Cuerva and Rubio de Francia 1985] J. García-Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Math. Studies **116**, North-Holland, Amsterdam, 1985. MR Zbl

[Germain et al. 2012a] P. Germain, N. Masmoudi, and J. Shatah, "Global solutions for 2D quadratic Schrödinger equations", *J. Math. Pures Appl.* (9) **97**:5 (2012), 505–543. MR Zbl

[Germain et al. 2012b] P. Germain, N. Masmoudi, and J. Shatah, "Global solutions for the gravity water waves equation in dimension 3", *Ann. of Math.* (2) **175**:2 (2012), 691–754. MR Zbl

[Głowacki 2007] P. Głowacki, "The Melin calculus for general homogeneous groups", *Ark. Mat.* **45**:1 (2007), 31–48. MR Zbl

[Głowacki 2010] P. Głowacki, "Composition and L^2-boundedness of flag kernels", *Colloq. Math.* **118**:2 (2010), 581–585. Correction in **120**:2 (2010), 331. MR Zbl

[Grafakos 2014] L. Grafakos, *Classical Fourier analysis*, 3rd ed., Graduate Texts in Math. **249**, Springer, 2014. MR Zbl

[Gundy and Stein 1979] R. F. Gundy and E. M. Stein, "H^p theory for the poly-disc", *Proc. Nat. Acad. Sci. U.S.A.* **76**:3 (1979), 1026–1029. MR Zbl

[Han 1998] Y.-S. Han, "Plancherel–Pôlya type inequality on spaces of homogeneous type and its applications", *Proc. Amer. Math. Soc.* **126**:11 (1998), 3315–3327. MR Zbl

[Han 2000] Y. Han, "Discrete Calderón-type reproducing formula", *Acta Math. Sin.* **16**:2 (2000), 277–294. MR Zbl

[Han and Lu 2010] Y. Han and G. Lu, "Some recent works on multiparameter Hardy space theory and discrete Littlewood–Paley analysis", pp. 99–191 in *Trends in partial differential equations*, vol. 10, Adv. Lect. Math. (ALM), Int. Press, Somerville, MA, 2010. MR

[Han et al. 2010] Y. Han, M.-Y. Lee, C.-C. Lin, and Y.-C. Lin, "Calderón–Zygmund operators on product Hardy spaces", *J. Funct. Anal.* **258**:8 (2010), 2834–2861. MR Zbl

[Han et al. 2014] Y. Han, G. Lu, and E. Sawyer, "Flag Hardy spaces and Marcinkiewicz multipliers on the Heisenberg group", *Anal. PDE* **7**:7 (2014), 1465–1534. MR Zbl

[Journé 1985] J.-L. Journé, "Calderón–Zygmund operators on product spaces", *Rev. Mat. Iberoam.* **1**:3 (1985), 55–91. MR Zbl

[Journé 1988] J.-L. Journé, "Two problems of Calderón–Zygmund theory on product-spaces", *Ann. Inst. Fourier* (*Grenoble*) **38**:1 (1988), 111–132. MR Zbl

[Meyer 1992] Y. Meyer, *Wavelets and operators*, Cambridge Studies in Adv. Math. **37**, Cambridge Univ. Press, 1992. MR Zbl

[Miyachi and Tomita 2016] A. Miyachi and N. Tomita, "Estimates for trilinear flag paraproducts on L^∞ and Hardy spaces", *Math. Z.* **282**:1-2 (2016), 577–613. MR Zbl

[Müller et al. 1995] D. Müller, F. Ricci, and E. M. Stein, "Marcinkiewicz multipliers and multi-parameter structure on Heisenberg(-type) groups, I", *Invent. Math.* **119**:2 (1995), 199–233. MR Zbl

[Müller et al. 1996] D. Müller, F. Ricci, and E. M. Stein, "Marcinkiewicz multipliers and multi-parameter structure on Heisenberg(-type) groups, II", *Math. Z.* **221**:2 (1996), 267–291. MR Zbl

[Muscalu 2007] C. Muscalu, "Paraproducts with flag singularities, I: A case study", *Rev. Mat. Iberoam.* **23**:2 (2007), 705–742. MR Zbl

[Muscalu 2010] C. Muscalu, "Flag paraproducts", pp. 131–151 in *Harmonic analysis and partial differential equations* (El Escorial, Spain, 2008), edited by P. Cifuentes et al., Contemp. Math. **505**, Amer. Math. Soc., Providence, RI, 2010. MR Zbl

[Muscalu and Schlag 2013] C. Muscalu and W. Schlag, *Classical and multilinear harmonic analysis, II*, Cambridge Studies in Adv. Math. **138**, Cambridge Univ. Press, 2013. MR Zbl

[Nagel and Stein 2006] A. Nagel and E. M. Stein, "The $\overline{\partial}_b$-complex on decoupled boundaries in $\mathbb{C}^n$", *Ann. of Math.* (2) **164**:2 (2006), 649–713. MR Zbl

[Nagel et al. 2001] A. Nagel, F. Ricci, and E. M. Stein, "Singular integrals with flag kernels and analysis on quadratic CR manifolds", *J. Funct. Anal.* **181**:1 (2001), 29–118. MR Zbl

[Nagel et al. 2012] A. Nagel, F. Ricci, E. Stein, and S. Wainger, "Singular integrals with flag kernels on homogeneous groups, I", *Rev. Mat. Iberoam.* **28**:3 (2012), 631–722. MR Zbl

[Nagel et al. 2018] A. Nagel, F. Ricci, E. M. Stein, and S. Wainger, *Algebras of singular integral operators with kernels controlled by multiple norms*, Mem. Amer. Math. Soc. **1230**, Amer. Math. Soc., Providence, RI, 2018. MR

[Phong and Stein 1982] D. H. Phong and E. M. Stein, "Some further classes of pseudodifferential and singular-integral operators arising in boundary value problems, I: Composition of operators", *Amer. J. Math.* **104**:1 (1982), 141–172. MR Zbl

[Pipher 1986] J. Pipher, "Journé's covering lemma and its extension to higher dimensions", *Duke Math. J.* **53**:3 (1986), 683–690. MR Zbl

[Plancherel and Pólya 1936] M. Plancherel and G. Pólya, "Fonctions entières et intégrales de Fourier multiples", *Comment. Math. Helv.* **9**:1 (1936), 224–248. MR Zbl

[Stein 1993] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Math. Series **43**, Princeton Univ. Press, 1993. MR Zbl

[Strömberg and Wheeden 1982] J.-O. Strömberg and R. L. Wheeden, "Relations between H^p_u and L^p_u with polynomial weights", *Trans. Amer. Math. Soc.* **270**:2 (1982), 439–467. MR Zbl

[Strömberg and Wheeden 1989] J.-O. Strömberg and R. L. Wheeden, "Relations between H^p_u and L^p_u in a product space", *Trans. Amer. Math. Soc.* **315**:2 (1989), 769–797. MR Zbl

[Wu 2014a] X. Wu, "Weighted norm inequalities for flag singular integrals on homogeneous groups", *Taiwanese J. Math.* **18**:2 (2014), 357–369. MR Zbl

[Wu 2014b] X. Wu, "An atomic decomposition characterization of flag Hardy spaces $H^p_F(\mathbb{R}^n \times \mathbb{R}^m)$ with applications", *J. Geom. Anal.* **24**:2 (2014), 613–626. MR Zbl

[Yang 2009] D. Yang, "Besov and Triebel–Lizorkin spaces related to singular integrals with flag kernels", *Rev. Mat. Complut.* **22**:1 (2009), 253–302. MR

Received December 13, 2017. Revised January 22, 2019.

YONGSHENG HAN
DEPARTMENT OF MATHEMATICS
AUBURN UNIVERSITY
AUBURN, AL
UNITED STATES
hanyong@auburn.edu

CHIN-CHENG LIN
NATIONAL CENTER FOR THEORETICAL SCIENCES
NATIONAL TAIWAN UNIVERSITY
TAIPEI
TAIWAN
Current address:
DEPARTMENT OF MATHEMATICS
NATIONAL CENTRAL UNIVERSITY
CHUNG-LI
TAIWAN
clin@math.ncu.edu.tw

XINFENG WU
DEPARTMENT OF MATHEMATICS
CHINA UNIVERSITY OF MINING AND TECHNOLOGY
BEIJING
CHINA
wuxf@cumtb.edu.cn

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2019 is US $490/year for the electronic version, and $665/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 302 No. 2 October 2019