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A GENERALIZATION OF MALOO'S THEOREM ON FREENESS OF DERIVATION MODULES

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Let A be a Noetherian local k-domain (k is a Noetherian ring) whose derivation module $\mathrm{Der}_k(A)$ is finitely generated as an A-module, and let $\mathfrak{P}_{A/k} \subset A$ be the corresponding maximally differential ideal. A theorem due to Maloo states that if A is regular and height $\mathfrak{P}_{A/k} \leq 2$, then $\mathrm{Der}_k(A)$ is A-free. In this note we prove the following generalization: if $\mathrm{projdim}_A(\mathrm{Der}_k(A)) < \infty$ and grade $\mathfrak{P}_{A/k}$ = height $\mathfrak{P}_{A/k} \leq 2$, then $\mathrm{Der}_k(A)$ is A-free. We provide several corollaries — to wit, the cases where A contains a field of positive characteristic, A is Cohen–Macaulay, or A is a factorial domain — as well as examples with $\mathrm{Der}_k(A)$ having infinite projective dimension. Moreover, our result connects to the Herzog–Vasconcelos conjecture, raised for algebras essentially of finite type over a field of characteristic zero, which we show to be true if depth $A \leq 2$ in a much more general context.

1. Motivation: Maloo's theorem

The investigation about either necessary or sufficient conditions for the freeness of derivation modules, in the algebrogeometric setting of local rings which are essentially of finite type over a field of characteristic zero, has attracted attention for decades. One of the natural reasons is the fact that derivations can be realized as tangent vector fields on the given algebraic variety. Several conjectures have been proposed on the theme, such as the famous Zariski–Lipman conjecture, which remains open in the nongraded 2-dimensional case, and the so-called Herzog–Vasconcelos conjecture, which is homological in nature and, unlike the former, is known to be true in a few specific situations.

Beyond the traditional context of localizations of affine rings, the problem of understanding freeness of derivation modules of more general Noetherian local algebras has been poorly tackled and is, at present, far from being well understood.

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One of the few results in this direction is given in [Maloo 1999, Theorem 4] in the case of (commutative, unital) Noetherian *regular* local algebras. In order to state Maloo's theorem, let k be a Noetherian ring and let A be a Noetherian regular local k-domain for which the A-module $\mathrm{Der}_k(A)$ formed by the k-derivations of A is finitely generated. Let $\mathfrak{P}_{A/k}$ be the corresponding maximally differential ideal (it exists and, in the local case, is unique). Assume that $\mathfrak{P}_{A/k}$ has height at most 2. Then $\mathrm{Der}_k(A)$ is free as an A-module.

In the present paper our main goal is to generalize Maloo's theorem. After invoking in Section 2 a couple of preparatory facts, due to Maloo himself and to Lequain, we give in Section 3 our central result (Theorem 3.1) which establishes the following: Let k be a Noetherian ring and let A be a Noetherian local k-domain such that $\mathrm{Der}_k(A)$ is a finitely generated A-module and $\operatorname{grade}\mathfrak{P}_{A/k} = \operatorname{height}\mathfrak{P}_{A/k} \le 2$. If $\operatorname{projdim}_A(\mathrm{Der}_k(A)) < \infty$ then $\mathrm{Der}_k(A)$ is free as an A-module.

From this result we derive corollaries where we get rid of the hypothesis $\operatorname{grade}\mathfrak{P}_{A/k} = \operatorname{height}\mathfrak{P}_{A/k}$. Namely, if A is a Noetherian local k-domain such that $\mathrm{Der}_k(A)$ is a finitely generated A-module of finite projective dimension and $\operatorname{height}\mathfrak{P}_{A/k} \le 2$, then $\mathrm{Der}_k(A)$ is a free A-module in the following cases:

- k is a field of positive characteristic contained in A (Corollary 3.3),
- A is Cohen–Macaulay (Corollary 3.5),
- A is a factorial domain (Corollary 3.6).

Note that the last two corollaries independently recover Maloo's theorem (which we state as Corollary 3.8), since a regular ring A is necessarily Cohen–Macaulay, factorial, and satisfies the property that all finitely generated A-modules have finite projective dimension. It is also worth mentioning that factorial non-Cohen–Macaulay domains of characteristic zero do exist (see Remark 3.7).

In Remark 3.9 we point out, for completeness, that the converse of Maloo's result is false; we give an instance in characteristic $p > 0$, and we mention that there exists a difficult example in characteristic zero constructed by Maloo.

In Section 4 we employ our results in order to present explicit examples of hypersurface rings A over a field k, in characteristic zero as well as in prime characteristic, satisfying the property that

$$\operatorname{projdim}_A(\mathrm{Der}_k(A)) = \infty.$$

An auxiliary tool is Lemma 4.1, which furnishes a set of generators together with a test for the nonfreeness of $\mathrm{Der}_k(A)$ in this setting. We are also able to describe the ideal $\mathfrak{P}_{A/k}$ in the examples.

Section 5 deals with the aforementioned Herzog–Vasconcelos conjecture, which precisely predicts that if k is a field with $\operatorname{char} k = 0$ and A is a local ring which is essentially of finite type over k, with the property that $\operatorname{projdim}_A(\mathrm{Der}_k(A)) < \infty$, then $\mathrm{Der}_k(A)$ must be free as an A-module. Herzog provides an excellent survey [Herzog 1994] on homological problems related to certain modules (we point out that in [Vasconcelos 1985, p. 373] there is also a related conjecture which says that under the above conditions the ring A must be a complete intersection). The Herzog–Vasconcelos conjecture has been settled in a few specific cases, and its hypotheses force A to be a normal domain (as well as Gorenstein if A is Cohen–Macaulay; see Remark 5.1). Our contribution is Corollary 5.2, which is stated in greater generality and solves the problem in the case where $\operatorname{depth} A \leq 2$. As a consequence, the conjecture is true for the local ring of any point of an affine algebraic surface over a ground field of arbitrary characteristic. We finish the paper by illustrating Corollary 5.2 in Example 5.3, which gives a three-dimensional non-Cohen–Macaulay normal domain A, essentially of finite type over a field k with $\operatorname{char} k = 0$, such that $\mathrm{Der}_k(A)$ has infinite projective dimension over A.

2. Preliminaries and auxiliary results

All rings in this paper are tacitly assumed to be commutative, unital, and Noetherian. If A is a ring and M is an A-module, a *derivation* of A into M is an additive map $\Delta : A \to M$ such that

$$\Delta(ab) = a\Delta(b) + b\Delta(a) \quad \text{for all } a, b \in A.$$

We denote by $\mathrm{Der}(A, M)$ the set of all derivations of A into M, which is an A-module in a natural way. If A is a k-algebra via a ring homomorphism $\psi : k \to A$, an element of $\mathrm{Der}(A, M)$ is a *k-derivation* if it vanishes on the image of ψ (a typical situation is when ψ is an inclusion). The set formed by all k-derivations of A into M is denoted by $\mathrm{Der}_k(A, M)$, which is seen to be an A-submodule of $\mathrm{Der}(A, M)$. If $\Omega_{A/k}$ is the module of Kähler k-differentials of A, it is well known that

$$\mathrm{Hom}_A(\Omega_{A/k}, M) \simeq \mathrm{Der}_k(A, M).$$

We refer, e.g., to [Matsumura 1986, Chapter 9]. In the case $M = A$ we simplify the notation to $\mathrm{Der}_k(A)$ (which is then the A-dual of $\Omega_{A/k}$). If for instance A is a polynomial ring $k[X_1, \ldots, X_n]$ — or a localization thereof — then $\mathrm{Der}_k(A)$ is a free A-module on the partial derivations $\partial_1, \ldots, \partial_n$.

We invoke a few specific concepts and facts that will play a fundamental role in the sequel.

Definition 2.1. Given a k-algebra A, we say that an ideal $\mathfrak{a} \subseteq A$ is $\mathrm{Der}_k(A)$-*differential* (*differential*, for short) if

$$\Delta(\mathfrak{a}) \subseteq \mathfrak{a} \quad \text{for all } \Delta \in \mathrm{Der}_k(A).$$

Remark 2.2. By Zorn's lemma, the family

$$\mathfrak{F}_{A/k} = \{\mathfrak{b} \mid \mathfrak{b} \text{ is a proper differential ideal of } A\}$$

contains maximal elements. If A is local, then $\mathfrak{F}_{A/k}$ has a unique maximal element [Maloo 1997, p. 82], the so-called *maximally differential ideal* of the k-algebra A, denoted herein by $\mathfrak{P}_{A/k}$. This ideal has interesting properties; besides the results given in the present paper, we mention for example [Brumatti and Lequain 1994; Singh 1983], as well as the connection between $\mathfrak{P}_{A/k}$ and Hironaka's concept of permissibility [Seibt 1980, Theorem 1.2]. On the other hand, certain features of this ideal are quite subtle, for instance its behavior under completion [de Souza Doering and Lequain 1986; Patil and Singh 1983].

Lemma 2.3 [Maloo 1997, Theorem 5]. *If A is a local k-algebra such that* $\mathrm{Der}_k(A)$ *is a finitely generated A-module, then* $\mathrm{Der}_k(A)/\mathfrak{P}_{A/k}\,\mathrm{Der}_k(A)$ *is free as an $A/\mathfrak{P}_{A/k}$-module. In particular, if $\mathfrak{P}_{A/k} = (0)$, then* $\mathrm{Der}_k(A)$ *is free as an A-module.*

Lemma 2.4 [Lequain 1971, Theorem 1.4]. *Let $(A, \mathfrak{m})$ be a local k-algebra.*

(i) *If $A/\mathfrak{P}_{A/k}$ has positive characteristic, then* $\operatorname{rad}\mathfrak{P}_{A/k} = \mathfrak{m}$.

(ii) *If $A/\mathfrak{P}_{A/k}$ has characteristic zero, then $\mathfrak{P}_{A/k}$ is prime.*

3. Main result and corollaries

If A is a ring, by a *finite* A-module we shall mean, as usual, a finitely generated A-module, and the *grade* of an ideal $\mathfrak{a} \subset A$ is the number $\operatorname{grade}\mathfrak{a}$ defined as the maximal length of an A-sequence contained in $\mathfrak{a}$. Recall that $\operatorname{grade}\mathfrak{a}$ is bounded above by the height of $\mathfrak{a}$. Basic facts can be found in [Bruns and Herzog 1998].

In the sequel, k denotes a ring. A *k-domain* is an integral domain with a structure of k-algebra. Our central result is the following:

Theorem 3.1. *Let A be a local k-domain such that* $\mathrm{Der}_k(A)$ *is a finite A-module and* $\operatorname{grade}\mathfrak{P}_{A/k} = \operatorname{height}\mathfrak{P}_{A/k} \le 2$. *If* $\operatorname{projdim}_A(\mathrm{Der}_k(A)) < \infty$, *then* $\mathrm{Der}_k(A)$ *is free as an A-module.*

Proof. For simplicity we write $\mathfrak{P} = \mathfrak{P}_{A/k}$ (which exists and is unique, as mentioned in Remark 2.2) and $D = \mathrm{Der}_k(A)$. We may suppose that $D \neq 0$. As recalled in Section 2, if $\Omega = \Omega_{A/k}$ is the module of Kähler k-differentials of A, we have $\mathrm{Hom}_A(\Omega, M) \simeq \mathrm{Der}_k(A, M)$ for every A-module M.

If p stands for the characteristic of the residue ring $A/\mathfrak{P}$, we distinguish two cases:

(i) $p > 0$. Let $\mathfrak{m}$ be the maximal ideal of A. By Lemma 2.4(i), we have $\operatorname{rad}\mathfrak{P} = \mathfrak{m}$. Hence, $\dim A = \operatorname{height}\mathfrak{P} \le 2$, which yields

$$\operatorname{depth} A \le 2.$$

On the other hand, we may assume that depth $A \geq 1$, as otherwise the domain A must a field K; hence, $D = \mathrm{Der}_k(K)$ is a (finite-dimensional) K-vector space and we are done. Let $a \in \mathfrak{m}$ be a nonzero element. Applying $\mathrm{Hom}_A(\Omega, \cdot)$ to the short exact sequence

$$0 \to A \xrightarrow{\cdot a} A \to A/(a) \to 0$$

we get an exact sequence of derivation modules

$$0 \to D \xrightarrow{\cdot a} D \to \mathrm{Der}_k(A, A/(a)).$$

In particular, a is D-regular and the quotient D/aD can be regarded as a submodule of $\mathrm{Der}_k(A, A/(a))$. Now if $\{a, b\} \subset \mathfrak{m}$ is an A-sequence, then in order to conclude that $\{a, b\}$ is a D-sequence it suffices to show that b is $\mathrm{Der}_k(A, A/(a))$-regular. Also note that $D/(a, b)D \neq 0$ because of Nakayama's lemma. Applying $\mathrm{Hom}_A(\Omega, \cdot)$ to the exact sequence

$$0 \to A/(a) \xrightarrow{\cdot b} A/(a)$$

we obtain an exact sequence

$$0 \to \mathrm{Der}_k(A, A/(a)) \xrightarrow{\cdot b} \mathrm{Der}_k(A, A/(a))$$

as needed. Thus, we have shown that depth $D \geq$ depth A. Since $\mathrm{projdim}_A D < \infty$, the Auslander–Buchsbaum formula forces D to be free.

(ii) $p = 0$. According to Lemma 2.4(ii), the ideal $\mathfrak{P}$ is prime. Consider the local ring $(A_{\mathfrak{P}}, \mathfrak{P}A_{\mathfrak{P}})$. First we claim that the $A_{\mathfrak{P}}$-module $D_{\mathfrak{P}} = D \otimes_A A_{\mathfrak{P}}$ is free. Note that $D_{\mathfrak{P}}$ may not be isomorphic to $\mathrm{Der}_k(A_{\mathfrak{P}})$ since we are not requiring $\Omega_{A/k}$ to be finitely presented.

Let $\boldsymbol{a}$ be an A-sequence of maximal length contained in $\mathfrak{P}$. By hypothesis, this length is at most 2. Of course we may assume that $D_{\mathfrak{P}} \neq 0$. Using the same argument employed in the previous case (i), we get that $\boldsymbol{a}$ is a D-sequence. Hence,

$$\boldsymbol{a}/\mathbf{1} \subset \mathfrak{P}A_{\mathfrak{P}}$$

is a $D_{\mathfrak{P}}$-sequence [Bruns and Herzog 1998, Corollary 1.1.3(a)], and we obtain

$$\mathrm{depth}\, D_{\mathfrak{P}} \geq \mathrm{grade}\, \mathfrak{P} = \mathrm{height}\, \mathfrak{P} = \dim A_{\mathfrak{P}} \geq \mathrm{depth}\, A_{\mathfrak{P}}.$$

The hypothesis $\mathrm{projdim}_A D < \infty$ yields $\mathrm{projdim}_{A_{\mathfrak{P}}} D_{\mathfrak{P}} < \infty$ and then the Auslander–Buchsbaum formula guarantees the freeness of $D_{\mathfrak{P}}$, as claimed.

We proceed to prove that D itself is free as an A-module. As a matter of notation, if $(B, \mathfrak{n})$ is a local domain, then we denote by $\nu_B(N)$ and $\mathrm{rank}_B N$ the minimal number of generators and the generic rank of a finite B-module N, respectively. The former is the dimension of the $B/\mathfrak{n}$-vector space $N/\mathfrak{n}N$, and the latter is the dimension of the L-vector space $N \otimes_B L$, where $L = B_{(0)}$ (the fraction field of B).

It is a standard fact that $\nu_B(N) \geq \mathrm{rank}_B\, N$, with equality if and only if N is a free B-module.

As the domains A and $A_{\mathfrak{P}}$ have clearly the same field of fractions, it is easy to see that $\mathrm{rank}_A\, D = \mathrm{rank}_{A_{\mathfrak{P}}}\, D_{\mathfrak{P}}$, and since the $A_{\mathfrak{P}}$-module $D_{\mathfrak{P}}$ is free, we get

$$\mathrm{rank}_A\, D = \nu_{A_{\mathfrak{P}}}(D_{\mathfrak{P}}).$$

Therefore, in order to prove that D is free, it suffices to verify that $\nu_{A_{\mathfrak{P}}}(D_{\mathfrak{P}}) = \nu_A(D)$. To this end, we apply Lemma 2.3, which gives

$$D/\mathfrak{P}D \simeq (A/\mathfrak{P})^{\oplus r}$$

for some integer $r \geq 1$. Thus,

$$\nu_A(D) = \nu_A(D/\mathfrak{P}D) = \nu_{A/\mathfrak{P}}(D/\mathfrak{P}D) = r.$$

On the other hand, if we denote

$$\kappa(\mathfrak{P}) = A_{\mathfrak{P}}/\mathfrak{P}A_{\mathfrak{P}}$$

(the residue field of $A_{\mathfrak{P}}$), we can write

$$\nu_{A_{\mathfrak{P}}}(D_{\mathfrak{P}}) = \nu_{A_{\mathfrak{P}}}(D_{\mathfrak{P}}/\mathfrak{P}D_{\mathfrak{P}}) = \nu_{A_{\mathfrak{P}}/\mathfrak{P}A_{\mathfrak{P}}}(D_{\mathfrak{P}}/\mathfrak{P}D_{\mathfrak{P}}) = \nu_{\kappa(\mathfrak{P})}(\kappa(\mathfrak{P})^{\oplus r}) = r$$

so that $\nu_{A_{\mathfrak{P}}}(D_{\mathfrak{P}}) = \nu_A(D)$, as needed. $\square$

Remark 3.2. If we assume that $k \subset A$ and

$$\mathfrak{P}_{A/k} \cap k = (0)$$

(e.g., if the subring k is a field), then we have a natural embedding

$$k \hookrightarrow A/\mathfrak{P}_{A/k}$$

and hence $\mathrm{char}(A/\mathfrak{P}_{A/k}) = \mathrm{char}\, k$. Now notice that, in the case (i) of the proof of Theorem 3.1, the hypothesis $\mathrm{grade}\,\mathfrak{P}_{A/k} = \mathrm{height}\,\mathfrak{P}_{A/k}$ is not needed. An immediate byproduct of these observations is Corollary 3.3 below.

Corollary 3.3. *Let A be a local domain containing a field k with* $\mathrm{char}\, k > 0$, *such that* $\mathrm{Der}_k(A)$ *is a finite A-module and* $\mathrm{height}\,\mathfrak{P}_{A/k} \leq 2$. *If* $\mathrm{projdim}_A(\mathrm{Der}_k(A)) < \infty$, *then* $\mathrm{Der}_k(A)$ *is free as an A-module.*

We believe that Corollary 3.3 holds in the case $\mathrm{char}\, k = 0$ as well. It is easy to see that the issue relies on the following question, for which we expect a negative answer.

Question 3.4. Let k be a field with $\mathrm{char}\, k = 0$. Is it possible for a local k-domain A, with $\mathrm{Der}_k(A)$ a finite A-module with finite projective dimension, to be such that $\mathrm{grade}\,\mathfrak{P}_{A/k} = 1$ and $\mathrm{height}\,\mathfrak{P}_{A/k} = 2$?

It is a standard fact that in a Cohen–Macaulay local ring A we have grade $\mathfrak{a} =$ height $\mathfrak{a}$ for every ideal $\mathfrak{a} \subset A$. Thus, we also readily derive from Theorem 3.1 the following consequence.

Corollary 3.5. *Let A be a Cohen–Macaulay local k-domain such that* $\operatorname{Der}_k(A)$ *is a finite A-module and* height $\mathfrak{P}_{A/k} \leq 2$. *If* $\operatorname{projdim}_A(\operatorname{Der}_k(A)) < \infty$, *then* $\operatorname{Der}_k(A)$ *is free as an A-module.*

Furthermore, we consider the class of factorial domains, also called unique factorization domains. We mention en passant that, among several important properties, one of the nice features of a factorial local domain $(B, \mathfrak{n})$ with depth $B \geq 2$ is that its punctured spectrum $\operatorname{Spec}(B) \setminus \{\mathfrak{n}\}$ has trivial Picard group (the same happens to Spec B itself).

Corollary 3.6. *Let A be a factorial local k-domain such that* $\operatorname{Der}_k(A)$ *is a finite A-module and* height $\mathfrak{P}_{A/k} \leq 2$. *If* $\operatorname{projdim}_A(\operatorname{Der}_k(A)) < \infty$, *then* $\operatorname{Der}_k(A)$ *is free as an A-module.*

Proof. Write $\mathfrak{P}_{A/k} = \mathfrak{P}$. If $\mathfrak{P} = (0)$, then $\operatorname{Der}_k(A)$ is free by Lemma 2.3. If height $\mathfrak{P} = 1$, then (since A is a domain) we must have grade $\mathfrak{P} =$ height $\mathfrak{P}$, and the assertion follows from Theorem 3.1. So we may assume that height $\mathfrak{P} = 2$. Moreover, by part (i) in the proof of Theorem 3.1 (since this part does not depend on the condition grade $\mathfrak{P} =$ height $\mathfrak{P}$), we are reduced to the case where $\operatorname{char}(A/\mathfrak{P}) = 0$, so that $\mathfrak{P}$ is prime by Lemma 2.4(ii). Therefore, we can guarantee that $\mathfrak{Q} \subset \mathfrak{P}$ for some prime ideal $\mathfrak{Q} \subset A$ with

$$\text{height}\,\mathfrak{Q} = 1.$$

Since A is factorial, we have $\mathfrak{Q} = (a)$ for some (prime) element a [Matsumura 1986, Theorem 20.1]. It follows that any given

$$b \in \mathfrak{P} \setminus (a)$$

is a non-zero-divisor of $A/(a)$, i.e., $\{a, b\} \subset \mathfrak{P}$ is an A-sequence and hence grade $\mathfrak{P} \geq 2$, which forces grade $\mathfrak{P} = 2 =$ height $\mathfrak{P}$, so that we can once again apply Theorem 3.1. □

Remark 3.7. In virtue of the cases treated in Corollaries 3.3 and 3.5, it is natural to ask about the existence of factorial non-Cohen–Macaulay domains of characteristic zero (and dimension necessarily greater than or equal to 3, since a factorial — hence normal — domain of dimension 2 is Cohen–Macaulay). This is a nontrivial problem but fortunately such rings do exist, as shown by [Freitag and Kiehl 1974, Theorem 5.8], which thus justifies our Corollary 3.6. We also refer the reader to the survey given in [Lipman 1975].

Finally, recall that every regular local ring A is a factorial Cohen–Macaulay domain, and that every finite A-module has finite projective dimension. Thus, both Corollaries 3.5 and 3.6 independently recover [Maloo 1999, Theorem 4], which we state below.

Corollary 3.8. *Let A be a regular local k-algebra such that* $\mathrm{Der}_k(A)$ *is a finite A-module and* height $\mathfrak{P}_{A/k} \leq 2$. *Then* $\mathrm{Der}_k(A)$ *is free as an A-module.*

This result can be illustrated simply by taking A as the local ring of a nonsingular point of an affine algebraic (or algebroid) surface over a perfect field k of any characteristic.

Remark 3.9. As expected, the converse of Corollary 3.8 does not hold. A simple example is the nonregular 2-dimensional local domain

$$A = k[X, Y, Z]_{(X,Y,Z)}/(XY - Z^p)$$

where k is a field with $\mathrm{char}\, k = p > 0$. Letting x, y, z denote the residue classes of the variables, the A-module $\mathrm{Der}_k(A)$ is seen to be free, a basis being $\{\Delta_1, \Delta_2\}$, where

$$\Delta_1 = (p-1)x\partial_x + y\partial_y, \qquad \Delta_2 = \partial_z.$$

The case of characteristic zero is much harder, but [Maloo 1997, p. 84] presents a 1-dimensional nonregular Noetherian local ring containing a field k with $\mathrm{char}\, k = 0$, such that $\mathrm{Der}_k(A)$ is a finite free A-module.

Remark 3.10. One of the ingredients used by Maloo in the proof of his theorem is [Maloo 1999, Lemma 3], which he proves essentially by using that $\mathrm{depth}(\mathrm{Der}_k(A)) \geq \min\{2, \mathrm{depth}\, A\}$ (this is easy to see in the special situation where the differential module $\Omega_{A/k}$ is a finite A-module, and can be derived from [Bruns and Herzog 1998, Exercise 1.4.19]). However, no proof of this property in generality was given therein. The argument is, indeed, basic — relies on the fact that $\mathrm{Der}_k(A)$ is a dual — and is supplied in the proof of our Theorem 3.1. Later on, in Corollary 5.2, this general fact will be used crucially.

4. Examples: derivation modules with infinite projective dimension

Our goal in this section is to apply our results in order to furnish explicit examples of algebras over a field k, in characteristic zero as well as in prime characteristic, whose module of k-derivations has infinite projective dimension. Note that this could not be achieved solely by means of Maloo's theorem (Corollary 3.8).

In order to satisfactorily clarify the examples, which will be built on hypersurface rings, we provide first a (characteristic-free) useful tool. As in the proof of Theorem 3.1, we use $\mathrm{rank}(\,\cdot\,)$ and $\nu(\,\cdot\,)$ to denote rank and minimal number of generators of finite modules over a specified base ring, respectively.

Lemma 4.1. *Let k be an arbitrary field and let S be the localization of a polynomial ring over k at the ideal generated by the indeterminates. Let $\{\partial_1, \dots, \partial_n\}$ be the natural free basis of* $\mathrm{Der}_k(S)$. *For a noninvertible $f \in S$ such that $\partial_j(f) \neq 0$ for some j, let $\mathfrak{J} \subset S$ be the ideal generated by the (ordered, signed) set $\{\partial_1(f), \dots, \partial_n(f), f\}$, giving rise to a free presentation*

$$S^m \xrightarrow{\phi} S^{n+1} \to \mathfrak{J} \to 0$$

where we regard ϕ as a matrix (taken in the canonical bases). Let ϕ' be the submatrix of ϕ resulting from deletion of its last row, and if $A = S/(f)$, let ϕ_A be the matrix formed by the nonzero columns of the matrix $\phi' \otimes \mathrm{Id}_A$ obtained by reducing the entries of ϕ' modulo f. We have:

(i) $\mathrm{Der}_k(A)$ *is generated as an A-module by the derivations corresponding to the column-vectors of ϕ_A.*

(ii) *If f is irreducible and $\nu_A(\mathrm{Der}_k(A)) \geq n$, then* $\mathrm{Der}_k(A)$ *cannot be free as an A-module.*

Proof. (i) Consider the tangential idealizer

$$T_{S/k}(f) = \{\Delta \in \mathrm{Der}_k(S) \mid \Delta(f) \in (f)\},$$

which is a submodule of $\mathrm{Der}_k(S)$, also known as the module of logarithmic derivations of f. By [Miranda-Neto 2017, Proposition 2.3], the derivations corresponding to the column-vectors of ϕ' generate $T_{S/k}(f)$ as an S-module. Now, if as above $A = S/(f)$, there is an isomorphism of A-modules

$$\mathrm{Der}_k(A) \simeq T_{S/k}(f)/f\,\mathrm{Der}_k(S)$$

[Miranda-Neto 2016, Proposition 2.6], where clearly

$$f\,\mathrm{Der}_k(S) = (f)\partial_1 \oplus \cdots \oplus (f)\partial_n \simeq fS^n.$$

Therefore, under the natural identification, the A-module $\mathrm{Der}_k(A)$ can generated by the derivations given by the (nonzero) column-vectors of $\phi' \otimes \mathrm{Id}_A$.

(ii) It suffices to show that the rank of the A-module $\mathrm{Der}_k(A)$ is at most $n-1$. This is certainly known (notably in characteristic zero) but it is instructive to write down an independent, general proof. As the ideal $(f) \subset S$ is prime and f is not killed by some ∂_j, we can easily check that

$$T_{S/k}(f) :_S \mathrm{Der}_k(S) = (f),$$

which means that (f) lies in the support of the S-module $\mathfrak{C} = \mathrm{Der}_k(S)/T_{S/k}(f)$. Because $\mathfrak{C}$ is also an A-module, isomorphic to the cokernel of the injection

$\iota : \mathrm{Der}_k(A) \hookrightarrow A^n$ given by

$$\mathrm{Der}_k(A) \simeq T_{S/k}(f)/f\,\mathrm{Der}_k(S) \subset \mathrm{Der}_k(S)/f\,\mathrm{Der}_k(S) \simeq S^n/fS^n = A^n,$$

we get that $(0) \subset A$ lies in the support of $\mathfrak{C} \simeq A^n/\iota(\mathrm{Der}_k(A))$. This yields $\mathrm{rank}_A\,\mathfrak{C} > 0$ and consequently

$$\mathrm{rank}_A(\mathrm{Der}_k(A)) = n - \mathrm{rank}_A\,\mathfrak{C} < n$$

as needed. □

Example 4.2. Let k be a field with $\mathrm{char}\,k = 3$ and let A be the 2-dimensional local domain $A = S/(f)$, where $S = k[X, Y, Z]_{(X,Y,Z)}$ and

$$f = X^2Y + XYZ + Z^3.$$

Consider the Jacobian ideal

$$\mathfrak{J} = (\partial_X(f), \partial_Y(f), \partial_Z(f), f) = (-XY + YZ, X^2 + XZ, XY, f) \subset S.$$

The given generators yield, explicitly, a (minimal) free presentation of the form

$$S^4 \xrightarrow{\phi} S^4 \to \mathfrak{J} \to 0$$

where

$$\phi = \begin{pmatrix} X & X & Z^2 & 0 \\ Y & 0 & Y^2 & -f \\ Z & X-Z & Z^2 & 0 \\ 0 & 0 & -Y & X^2+XZ \end{pmatrix}.$$

Reducing, modulo f, the matrix ϕ' (with notation as in Lemma 4.1) and denoting by x, y, z the residue classes of the variables, Lemma 4.1(i) guarantees that $\mathrm{Der}_k(A)$ can be generated by the derivations corresponding to the columns of the matrix

$$\phi_A = \begin{pmatrix} x & x & z^2 \\ y & 0 & y^2 \\ z & x-z & z^2 \end{pmatrix}$$

which are seen to be minimal generators. Thus,

$$\nu_A(\mathrm{Der}_k(A)) = 3 = \dim S$$

so that $\mathrm{Der}_k(A)$ is not free as an A-module by Lemma 4.1(ii). The condition height $\mathfrak{P}_{A/k} \le 2$ is automatic since $\dim A = 2$ (here we clearly have $\mathfrak{P}_{A/k} = \mathfrak{m}$). By Corollary 3.3 — or, alternatively, by Corollary 3.5 — we get

$$\mathrm{projdim}_A(\mathrm{Der}_k(A)) = \infty.$$

Remark 4.3. The first-named author claimed in [Miranda-Neto 2017, Example 2.10(ii)] that, if as above $\operatorname{char} k = 3$ and $f = X^2Y + XYZ + Z^3$, then the ideal $\mathfrak{J} = (\mathfrak{G}, f) = (\partial_X(f), \partial_Y(f), \partial_Z(f), f)$ satisfies, in particular, $\nu(\mathfrak{J}) = 3$, by implicitly assuming that it coincides with the gradient ideal $\mathfrak{G}$. However, this is not true, since the conductor of f into $\mathfrak{G}$ is the proper ideal

$$\mathfrak{G} : (f) = (Y, X^2 + XZ)$$

so that $f \notin \mathfrak{G}$ (for a weighted polynomial of weight δ, this pathology may only occur if $\operatorname{char} k$ divides δ). Precisely, by Example 4.2, $\mathfrak{J}$ has 4 minimal generators, and moreover, it can be easily verified that the kernel of the presentation map ϕ is free of rank 1, which yields that $\mathfrak{J}$ has projective dimension 2. As a consequence of this correction, f is *not* a free divisor (i.e., its logarithmic derivation module cannot be free) by [Miranda-Neto 2017, Proposition 2.7].

Example 4.4. Let us investigate an example in higher dimension. Let k be a field with $\operatorname{char} k = 0$. Consider the 3-dimensional local domain

$$A = k[X, Y, Z, W]_{(X,Y,Z,W)}/(X^3 + XY^3 + YZ + Y^2W)$$

and write $A = k[x, y, z, w]_{(x,y,z,w)}$, in terms of the residue classes of the variables modulo the defining equation of A. After computing a free presentation matrix of the Jacobian ideal (built from the natural generators as in Lemma 4.1), deleting its last row and taking images in A, we obtain the matrix

$$\phi_A = \begin{pmatrix} 0 & -y & 0 & -x & 3xy^2 + z + 2yw \\ 0 & 0 & -y & -3y & -3x^2 - y^3 \\ -y & 3x^2 & z & 0 & 0 \\ 1 & y^2 & 3xy + 2w & 7xy + 3w & 0 \end{pmatrix}$$

whose columns give generators for the A-module $\operatorname{Der}_k(A)$, by Lemma 4.1(i). Such generators are seen to be minimal. Thus,

$$\nu_A(\operatorname{Der}_k(A)) = 5 > 4 = \dim S$$

and hence, $\operatorname{Der}_k(A)$ is not free as an A-module, by Lemma 4.1(ii).

Notice, moreover, that the maximal ideal $\mathfrak{m} = (x, y, z, w) \subset A$ is not differential. In fact, if

$$\Delta = -y\partial_z + \partial_w$$

is the derivation corresponding to the first column of ϕ_A, then

$$\Delta(w) = 1 \notin \mathfrak{m}$$

which implies $\mathfrak{P}_{A/k} \neq \mathfrak{m}$. Since $\mathfrak{P}_{A/k}$ is prime (by Lemma 2.4(ii)) and $\dim A = 3$, we get $\operatorname{height} \mathfrak{P}_{A/k} \leq 2$ (here indeed we have $\mathfrak{P}_{A/k} = (x, y, z)$, so that, precisely, height $\mathfrak{P}_{A/k} = 2$). By Corollary 3.5, we conclude that

$$\operatorname{projdim}_A(\operatorname{Der}_k(A)) = \infty.$$

Remark 4.5. In the case of a *complete* local algebra of characteristic zero, there is a serious constraint imposed on its structure if the maximally differential ideal is not equal to the maximal ideal. To be more precise, let $(A, \mathfrak{m})$ be a local k-algebra whose $\mathfrak{m}$-adic completion $\hat{A}$ satisfies $\operatorname{char} \hat{A} = 0$ and

$$\mathfrak{P}_{\hat{A}/k} \neq \widehat{\mathfrak{m}}$$

(a word of caution: the ideal $\mathfrak{P}_{\hat{A}/k}$ may differ from the completion of $\mathfrak{P}_{A/k}$ [de Souza Doering and Lequain 1986; Patil and Singh 1983]), or what amounts to the same, there exists $\vartheta \in \operatorname{Der}_k(\hat{A})$ such that $\vartheta(\widehat{\mathfrak{m}}) \nsubseteq \widehat{\mathfrak{m}}$. Then, by [Zariski 1965, Lemma 4], we have that $\hat{A}$ is a power series ring

$$\hat{A} = \mathfrak{A}[\![T]\!]$$

in 1 indeterminate T over a suitable subring $\mathfrak{A} \subset \{a \in \hat{A} \mid \vartheta(a) = 0\} \subset \hat{A}$. As a consequence, we get that if $\hat{A}$ cannot be expressed as a power series ring over a subring $\mathfrak{A} \subset \hat{A}$, then height $\mathfrak{P}_{\hat{A}/k} = \dim A$.

5. Connection to the Herzog–Vasconcelos conjecture

We recall below a well known — but mostly open — conjecture in the algebrogeometric setting of localizations of affine rings over a field containing the rationals, independently raised by Herzog and Vasconcelos. Notice that, if A is such a k-algebra, then the module of differentials $\Omega_{A/k}$ is automatically a finite A-module and hence so is its A-dual $\operatorname{Der}_k(A)$.

Conjecture (Herzog and Vasconcelos). Let k be a field with $\operatorname{char} k = 0$, and let A be a local ring which is essentially of finite type over k. If $\operatorname{projdim}_A(\operatorname{Der}_k(A)) < \infty$, then $\operatorname{Der}_k(A)$ is free as an A-module.

If as above A is a localization of a finite-type algebra over a field, $\operatorname{Der}_k(A)$ can be regarded as a module of second-order syzygies (being isomorphic to the kernel of the A-linear map of free modules induced by a Jacobian matrix corresponding to A), and therefore, in case A is Cohen–Macaulay with $\dim A \leq 2$, the conjecture is true and follows trivially from the Auslander–Buchsbaum formula. Apart from this easy instance, the problem has been settled in a few situations, to wit, quasihomogeneous complete intersections with isolated singularity [Herzog 1994, Theorem 2.4], Stanley–Reisner rings [Brumatti and Simis 1995], and Buchsbaum affine semigroup rings [Müller and Patil 1999].

Remark 5.1. In the setting above, the celebrated Zariski–Lipman conjecture predicts that A must be regular if $\operatorname{Der}_k(A)$ is free [Lipman 1965]. As in [Herzog

1994, p. 6], we may combine the two conjectures and ask whether A is regular if $\operatorname{projdim}_A(\operatorname{Der}_k(A)) < \infty$. By [Scheja and Storch 1972, Satz 9.1], the finiteness of $\operatorname{projdim}_A(\operatorname{Der}_k(A))$ forces A to be a normal domain, and according to [Herzog 1981, Corollary 3.3] it also implies

$$\operatorname{Hom}_A\left(\bigwedge^d \operatorname{Der}_k(A), A\right) \simeq A$$

where $d = \dim A \geq 1$ — the rank of $\operatorname{Der}_k(A)$ in this case. But, if moreover A is Cohen–Macaulay, the dual module above is isomorphic to a canonical module of A, so that A is in fact Gorenstein (it is conjectured in [Vasconcelos 1985, p. 373] that A must be a complete intersection). This observation gives an analogue, in the case of the Herzog–Vasconcelos conjecture, of the statement [Hochster 1977, Remark 2] to the effect that if the Zariski–Lipman conjecture admits a Cohen–Macaulay counterexample, then there is also a Gorenstein counterexample.

Here, our contribution to the Herzog–Vasconcelos problem is Corollary 5.2 below, which is essentially contained in the proof of Theorem 3.1 and solves the conjecture in case $\operatorname{depth} A \leq 2$. Our statement is indeed more general since the k-algebra A is not necessarily essentially of finite type, and moreover k — which is allowed to have any characteristic — is not required to be a field.

Corollary 5.2. *Let A be a local algebra over a ring k such that* $\operatorname{Der}_k(A)$ *is a finite A-module. Assume that* $\operatorname{depth} A \leq 2$. *If* $\operatorname{projdim}_A(\operatorname{Der}_k(A)) < \infty$, *then* $\operatorname{Der}_k(A)$ *is free as an A-module.*

Proof. If $\operatorname{depth} A = 0$, then by the Auslander–Buchsbaum equality, the hypothesis $\operatorname{projdim}_A(\operatorname{Der}_k(A)) < \infty$ yields that $\operatorname{Der}_k(A)$ is free. Now assume that $\operatorname{depth} A$ is either 1 or 2. Thus, just as in part (i) in the proof of Theorem 3.1, we readily derive that a maximal A-sequence is necessarily a $\operatorname{Der}_k(A)$-sequence, so that $\operatorname{depth}(\operatorname{Der}_k(A)) \geq \operatorname{depth} A$ and once again the Auslander–Buchsbaum formula does the job. □

In particular, we get that the Herzog–Vasconcelos conjecture is true for the local ring A of any point of an affine algebraic surface over a ground field k of arbitrary characteristic. If $\operatorname{char} k = 0$ as in the original statement, this follows alternatively from the fact (recalled in Remark 5.1) that a ring satisfying the conditions of the conjecture must be normal.

We close the paper by working out an instance in higher dimension satisfying the property that the corresponding module of derivations has infinite projective dimension — which, by the preceding discussion, occurs automatically if the ring is nonnormal. It follows that, in order to produce an interesting example by means of Corollary 5.2, we need to start from a (non-Cohen–Macaulay) normal domain. Using Serre's normality criterion it is easy to see, for instance, that if B is a local isolated singularity (i.e., B is a local ring which is locally regular at the nonmaximal prime ideals) with $\operatorname{depth}(B) \geq 2$, then B is a normal domain.

Example 5.3. Let X, Y, Z, T, U, V be indeterminates over a field k with $\operatorname{char} k = 0$. Let $(S, \mathfrak{M})$ denote the localization of the polynomial ring $k[X, Y, Z, T, U, V]$ at the ideal generated by the indeterminates. Set $A = S/\mathfrak{I}$, where $\mathfrak{I} \subset S$ is the ideal generated by the 7 polynomials

$$\begin{gathered} XT - YZ, \qquad XV - YU, \qquad ZV - TU, \qquad X^3 + Z^3 + U^3, \\ X^2Y + Z^2T + U^2V, \qquad XY^2 + ZT^2 + UV^2, \qquad Y^3 + T^3 + V^3. \end{gathered}$$

This is a particular situation of the example given by [Roberts 2010, §6], where it is stated that (the completion of) A is a normal domain. Let us indicate how to verify this feature in the present case. Denote by Θ the Jacobian matrix of the given generators of $\mathfrak{I}$. We have $\operatorname{height} \mathfrak{I} = 3$ (hence A is 3-dimensional). The ideal $I_3(\Theta)$ generated by the minors of order 3 of Θ satisfies

$$\operatorname{rad}(I_3(\Theta) + \mathfrak{I}) = \mathfrak{M},$$

and therefore, A is an isolated singularity. Since $\operatorname{projdim}_S A = 4$, the Auslander–Buchsbaum formula gives

$$\operatorname{depth} A = 2,$$

and thus, A is indeed a (non-Cohen–Macaulay) normal domain.

Since A is a domain which is essentially of finite type over a field containing the rationals, the rank of $\theta = \Theta \otimes \mathrm{Id}_A$ (i.e., the Jacobian matrix with entries taken modulo $\mathfrak{I}$) is known to be equal to the height of $\mathfrak{I}$, and then, as

$$\operatorname{Der}_k(A) \simeq \ker\bigl(A^6 \xrightarrow{\theta} A^7\bigr)$$

we get $\operatorname{rank}(\operatorname{Der}_k(A)) = 6 - \operatorname{height} \mathfrak{I} = 3$.

Now, let $T_{S/k}(\mathfrak{I})$ be the S-module formed by the $\Delta \in \operatorname{Der}_k(S)$ such that $\Delta(\mathfrak{I}) \subset \mathfrak{I}$. We can employ the method presented in [Miranda-Neto 2011, §2] in order to describe generators for $T_{S/k}(\mathfrak{I})$, and recall the fact that $\operatorname{Der}_k(A) \simeq T_{S/k}(\mathfrak{I})/\mathfrak{I} \operatorname{Der}_k(S)$ [Miranda-Neto 2016, Proposition 2.6]. Thus, writing x, y, z, t, u, v for the residue classes of the indeterminates, we obtain that the A-module $\operatorname{Der}_k(A)$ is (minimally) generated by the derivations corresponding to the columns of the matrix

$$\varphi = \begin{pmatrix} 0 & 0 & x & y & -zt & 0 & -uv & -u^2 & 0 & -z^2 \\ x & y & 0 & 0 & -t^2 & 0 & -v^2 & -uv & 0 & -zt \\ 0 & 0 & z & t & xy & -uv & 0 & 0 & -u^2 & x^2 \\ z & t & 0 & 0 & y^2 & -v^2 & 0 & 0 & -uv & xy \\ 0 & 0 & u & v & 0 & zt & xy & x^2 & z^2 & 0 \\ u & v & 0 & 0 & 0 & t^2 & y^2 & xy & zt & 0 \end{pmatrix}$$

so that $\nu(\mathrm{Der}_k(A)) = 10 > \mathrm{rank}(\mathrm{Der}_k(A))$ and hence $\mathrm{Der}_k(A)$ cannot be free as an A-module. By Corollary 5.2, we finally obtain that

$$\mathrm{projdim}_A(\mathrm{Der}_k(A)) = \infty.$$

Notice that this cannot be detected by means of the results given in Section 3; indeed, in this example we have height $\mathfrak{P}_{A/k} = 3 > 2$, since $\mathfrak{P}_{A/k}$ equals the maximal ideal of A, as we can immediately observe from the structure of φ.

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