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**CONTRASTING VARIOUS NOTIONS OF CONVERGENCE
IN GEOMETRIC ANALYSIS**

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We explore the distinctions between L^p convergence of metric tensors on a fixed Riemannian manifold versus Gromov–Hausdorff, uniform, and intrinsic flat convergence of the corresponding sequence of metric spaces. We provide a number of examples which demonstrate these notions of convergence do not agree even for two dimensional warped product manifolds with warping functions converging in the L^p sense. We then prove a theorem which requires L^p bounds from above and C^0 bounds from below on the warping functions to obtain enough control for all these limits to agree.

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1. Introduction

When mathematicians have studied sequences of Riemannian manifolds arising naturally in questions of almost rigidity or when searching for solutions to geometric partial differential equations, they have obtained bounds on the metric tensors of these Riemannian manifolds. When the bounds they obtained on (M^n, g_j) guaranteed a subsequence, $g_j \rightarrow g_\infty$ converging in the C^0 sense or stronger, then the Riemannian manifolds, (M, g_j) , viewed as metric spaces, (M, d_j) , converge uniformly to (M, d_∞) where d_∞ is defined as the infimum of the lengths of curves between points measured using g_∞ . After observing this, Gromov [1981] introduced the Gromov–Hausdorff distance between metric spaces, proving that uniform convergence implies Gromov–Hausdorff convergence of metric spaces. The advantage

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of Gromov–Hausdorff convergence is that one may allow the spaces themselves to change (M_j, d_j) and one may obtain a limit metric space which is not even a manifold. Gromov [1981] proved that if (M_j, g_j) have uniform lower bounds on Ricci curvature and uniform upper bounds on diameter then a subsequence converges in the Gromov–Hausdorff sense to a metric space, and since then many people have analyzed the properties of these limit spaces.

More recently Sormani and Wenger [2011] introduced the intrinsic flat distance between oriented Riemannian manifolds which need not be diffeomorphic. Roughly the intrinsic flat distance is measuring a filling volume between two manifolds. A standard sphere and a sphere with a thin deep well are very close in the intrinsic flat sense based on the filling volume of the well, while they are far apart in the Gromov–Hausdorff distance based on the depth of the well. As soon as this notion was introduced people began asking whether L^p convergence of the metric tensors might in some way be related to intrinsic flat convergence of the metric spaces. After all, a uniform L^n bound on metric tensors implies a uniform upper bound on volume. Wenger [2011] proved that as long as a sequence of oriented Riemannian manifolds has a uniform upper bound on volume and on diameter it has a subsequence converging in the intrinsic flat sense. However it is not known whether the limit space is in anyway related to (M, g_∞) even when g_∞ was smooth. In joint work with Lakzian [Lakzian and Sormani 2013], and work of Lakzian alone [2016] it was shown that even when $g_j \rightarrow g_\infty$ smoothly away from a singular set, the Gromov–Hausdorff and intrinsic flat limits need not be closely related to (M, g_∞) unless one controls volumes, areas, and distances near the singular set.

In this paper we provide a number of examples demonstrating that when metric tensors g_j converge in the L^p sense to a metric tensor g_∞ , then uniform, intrinsic flat and Gromov–Hausdorff limits need not converge to a metric space which is defined by g_∞ using the infimum of lengths over all curves. Our examples include very simple two dimensional warped product Riemannian manifolds whose metric tensors are of the form $dr^2 + f_j(r)^2 d\theta^2$.

In Example 3.4 we find a sequence of warping functions $f_j(r)$ which converge in the L^p sense to a constant function, f_∞ , but the uniform, Gromov–Hausdorff, and intrinsic flat limit of the sequence is not even a Riemannian manifold. In this example the $f_j \leq f_\infty$ but have an increasingly narrow dip downward about $r = 0$ so we say the sequence of manifolds is “cinched” at 0. This is an example with smooth convergence away from a singular set that was not seen in [Lakzian and Sormani 2013]. The limit metric space is described in detail within the example and a proof is given afterwards. In Example 3.5 the $f_j \leq f_\infty$ and L^p converge to f_∞ again, but the cinch moves around so that the f_j do not converge pointwise almost everywhere. This example has no uniform, Gromov–Hausdorff, or intrinsic flat limit unless one takes a subsequence where the cinch’s location converges.

In Examples 3.7–3.9 we also consider warping functions, f_j , that L^p converge to a constant function, f_∞ , but now $f_j \geq f_\infty$. In Example 3.7 we have a single increasingly narrow peak about $r = 0$. We say there is a “ridge” at 0. This is another example with smooth convergence away from a singular set that was not studied in [Lakzian and Sormani 2013]. We observe how the shortest paths between points on the ridge, do not lie on the ridge in Lemma 3.6. In Example 3.8 we have a sequence of manifolds with moving ridges, so there is no pointwise convergence almost everywhere. In Example 3.9 we have increasingly many increasingly dense ridges. In all three of these examples we prove uniform convergence of the distances, d_j , to d_∞ of the isometric product Riemannian manifold with metric tensor $g_\infty = dr^2 + f_j(r)^2 d\theta^2$. We obtain intrinsic flat and Gromov–Hausdorff convergence to this limit as well.

In Example 3.12 we have $f_j \geq f_\infty$ with f_∞ constant and $f_j = f_\infty$ on an increasingly dense set. However, now our f_j do not converge in L^p to f_∞ . For the particular sequence we chose, we obtain uniform, intrinsic flat and Gromov–Hausdorff convergence to a non-Riemannian Finsler manifold we call a minimized R -stretched Euclidean taxi metric space. This metric is defined as an infimum over an interpolation between a Euclidean metric stretched by R in one direction and a taxi metric. Our example demonstrates that the L^p convergence was crucial in the prior examples. As discussed in Remark 3.13, this example shows the necessity of scalar curvature bounds in the statement of the scalar compactness conjecture of Gromov and Sormani (see [Gromov 2018]) to conclude that the limit has Euclidean tangent cones almost everywhere. This conjecture was recently verified in the rotationally symmetric case by Park, Tian, and Wang [Park et al. 2018].

We then prove the following general theorem concerning warped product manifolds $M^n = [r_0, r_1] \times_f \Sigma$ where Σ is an $n-1$ dimensional manifold including also M without boundary that have f periodic with period $r_1 - r_0$ as in (1)):

Theorem 1.1. *Assume the warping factors, $f_j \in C^0(r_0, r_1)$, satisfy the following:*

$$0 < f_\infty(r) - \frac{1}{j} \leq f_j(r) \leq K < \infty \quad \text{and} \quad f_j(r) \rightarrow f_\infty(r) > 0 \quad \text{in } L^2,$$

where $f_\infty \in C^0(r_0, r_1)$.

Then we have GH and \mathcal{F} convergence of the warped product manifolds,

$$\begin{aligned} M_j &= [r_0, r_1] \times_{f_j} \Sigma \rightarrow M_\infty = [r_0, r_1] \times_{f_\infty} \Sigma, \\ N_j &= \mathbb{S}^1 \times_{f_j} \Sigma \rightarrow N_\infty = \mathbb{S}^1 \times_{f_\infty} \Sigma, \end{aligned}$$

and uniform convergence of their distance functions, $d_j \rightarrow d_\infty$.

Remark 1.2. In our theorem we assume L^2 convergence but since we are assuming that the f_j are uniformly bounded this is equivalent to L^p , $p \in [1, \infty)$ convergence.

The proof of this theorem and indeed the proof of all the examples relies on a theorem of the second author with Huang and Lee in the appendix of [Huang et al. 2017] which is reviewed in the background section of this paper. The theorem in [Huang et al. 2017] states that if one has uniform upper and lower bounds on the d_j , a subsequence of the Riemannian manifolds converges in the uniform, Gromov–Hausdorff, and intrinsic flat convergence sense to some common limit space. Thus we need only prove pointwise convergence of the original sequence of d_j to our proposed d_∞ . The method applied to control d_j is different in each proof in this paper. For the theorem, we apply the C^0 lower bound to bound d_j from below and the L^p upper bound is all that is needed to bound d_j from above pointwise. Note that the hypothesis of the theorem immediately implies a uniform upper bound on diameter (Lemma 4.2). We end the paper with Theorem 5.1 concerning warped product manifolds where the warping function depends on two variables.

Applications of these theorems will appear in a paper by the first author with Hernandez-Vazquez, Parise, Payne, and Wang [Allen et al. 2019] on a conjecture of Gromov concerning the almost rigidity of the scalar torus theorem. The first author hopes to apply the techniques developed here in combination with his prior work in [Allen 2018a; 2018b] to prove a special case of Lee and Sormani’s conjecture [2014] on almost rigidity of the positive mass theorem. Additional applications to conjectures involving scalar curvature that were raised by the second author at the Fields Institute and described in [Sormani 2017] will be explored with other teams of students and postdocs in the near future. Anyone interested in joining one of these teams should contact the second author.

2. Review

In this subsection we review what we mean by a warped product space even with a noncontinuous warping function and what one needs to know about Gromov–Hausdorff and intrinsic flat convergence to prove all examples and theorems in this paper. The reader does not need any prior knowledge of these two notions of convergence. Readers who are experts in these notions of convergence are recommended to read just the first and last subsections of this review section of the paper, particularly Theorem 2.4 which combines results of Gromov [1981] and the second author with Huang and Lee [Huang et al. 2017]. All examples and theorems in this paper apply that theorem to prove convergence.

2A. Warped product spaces. Let (Σ^{n-1}, σ) be a compact Riemannian manifold and

$$f : [r_1, r_2] \rightarrow \mathbb{R}^+$$

and define the warped product manifolds

$$(1) \quad M = [r_1, r_2] \times_f \Sigma \quad \text{and} \quad N = \mathbb{S}^1 \times_f \Sigma$$

with warped product metrics defined by

$$(2) \quad g = dr^2 + f^2(r)\sigma,$$

where either $r \in [r_1, r_2]$ or $r \in \mathbb{S}^1$. On such a manifold we define lengths of curves to be

$$(3) \quad L_g(C) = \int_0^1 g(C'(t), C'(t))^{1/2} dt = \int_0^1 \sqrt{|r'(t)|^2 + |f(r(t))|^2 |\theta'(t)|^2} dt$$

which is well defined even when f is only L^1 . We then define distances $d_g^M(p, q)$ and $d_g^N(p, q)$ on M and N respectively as

$$(4) \quad d_g(p, q) = \inf\{L_g(C) : C(0) = p, C(1) = q\}$$

where the value is different on M and N because the selection of curves between points within these two spaces are different.

Remark 2.1. Note that we do not need f to be smooth or even continuous to define a warped product metric space. As long as the function is bounded above, we can define lengths using (3). Following the text of Burago, Burago, and Ivanov [Burago et al. 2001], the distance d defined by (4) is symmetric and satisfies the triangle inequality. It is positive definite as long as f is bounded below by a positive number. Such a metric space is then compact and there are geodesics whose lengths achieve the infimum in (4). Even more general warped products of metric spaces are explored by Alexander and Bishop [2004].

Remark 2.2. Throughout this paper we will assume that our warping function f is continuous. Annegret Burtscher has proven that if a Riemannian manifold has a continuous metric tensor then the distance between points is achieved by an absolutely continuous curve (See Proposition 4.1 and Theorem 4.11 in [Burtscher 2015]). This is achieved by showing that the length of absolutely continuous curves defined by (3) is equivalent to the induced length (see [loc. cit., Definition 2.1]) defined by d_g in [loc. cit., Theorem 4.11]. This will be important for us because we will repeatedly use the fact that the distance between points of M can be achieved by an absolutely continuous curve $C(t)$ and hence we can reparametrize $C(t)$ so that $|C'(t)|_g = 1$ almost everywhere.

For warped products we can show that L^2 convergence of metrics $g_j \rightarrow g_\infty$ is equivalent to L^2 convergence of the warping functions $f_j \rightarrow f_\infty$. For this we fix the background metric $\delta = dr^2 + \sigma$ and an orthonormal basis for this metric

$\{\partial_r, \partial_{\theta_1}, \dots, \partial_{\theta_n}\}$ and compute

$$\begin{aligned} \int_M |g_j - g_\infty|_\delta^2 dm &= \int_M \sum_{i=1}^n |f_j - f_\infty|^2 \sigma(\partial_{\theta_i}, \partial_{\theta_i}) dm \\ &= n \int_{r_1}^{r_2} \int_\Sigma |f_j - f_\infty|^2 d\mu dr = n |\Sigma| \int_{r_1}^{r_2} |f_j - f_\infty|^2 dr, \end{aligned}$$

where dm is the measure on M induced by δ , $d\mu$ is the measure on Σ from σ and $|\Sigma|$ is n -dimensional volume of Σ . This shows that we can just work with L^2 convergence of the warping functions for the sake of this paper.

2B. Gromov–Hausdorff convergence. Gromov–Hausdorff convergence was introduced by Gromov [1981]. See also the text of Burago–Burago–Ivanov [Burago et al. 2001]. It measures a distance between metric spaces. It is an intrinsic version of the Hausdorff distance between sets in a common metric space Z :

$$d_H^Z(A_1, A_2) = \inf\{r : A_1 \subset T_r(A_2) \text{ and } A_2 \subset T_r(A_1)\},$$

where $T_r(A) = \{x \in Z : \exists a \in A \text{ s.t. } d_Z(x, a) < r\}$. Since an arbitrary given pair of compact metric spaces, (X_i, d_i) , might not lie in the same compact metric space, we use distance preserving maps:

$$\varphi_i : X_i \rightarrow Z \quad \text{such that} \quad d_Z(\varphi_i(p), \varphi_i(q)) = d_i(p, q) \quad \text{for all } p, q \in X_i$$

to map them into a common compact metric space, Z .

The Gromov–Hausdorff distance between two compact metric spaces, (X_i, d_i) , is then defined to be

$$d_{\text{GH}}((X_1, d_1), (X_2, d_2)) = \inf\{d_H^Z(\varphi_1(X_1), \varphi_2(X_2)) : \varphi_i : X_i \rightarrow Z\},$$

where the infimum is taken over all compact metric spaces Z and all distance preserving maps, $\varphi_i : X_i \rightarrow Z$.

2C. Warped products as integral current spaces. Intrinsic flat convergence is defined for sequences of integral current spaces in [Sormani and Wenger 2011]. An integral current space is a metric space, (X, d) , endowed with a current structure, T , where T is defined by a collection of bi-Lipschitz charts with weights. If we start with an oriented smooth Riemannian manifold, M , then (X, d) is the standard metric space defined by M using lengths of curves as in (3) and T is defined by the orientation of M ,

$$(5) \quad T(f, \pi_1, \dots, \pi_m) = \int_M f d\pi_1 \wedge \dots \wedge d\pi_m.$$

Here we are considering warped product spaces, M and N , as in (1) allowing our function, $f : [r_1, r_2] \rightarrow \mathbb{R}^+$, to simply have a maximum and a positive minimum and

do not require it to be smooth. In order to confirm that we still may use (5) to define the integral current structure on our space, we need only verify that our standard oriented charts on the isometric product manifold are bi-Lipschitz to the metric d we obtain as in (3)–(4). This is confirmed by showing the identity map between the isometric product manifold, $M_1 = [r_1, r_2] \times_1 \Sigma$, and our warped product space, $M = [r_1, r_2] \times_f \Sigma$, is bi-Lipschitz:

Lemma 2.3. *Suppose the warping function is bounded*

$$f(r) \in [a, b] \quad \text{for all } r \in [r_1, r_2],$$

then the identity map

$$F : M_1 = [r_1, r_2] \times_1 \Sigma \rightarrow M = [r_1, r_2] \times_f \Sigma$$

is bi-Lipschitz:

$$0 < \min\{a, 1\} \leq \frac{d_M(F(p), F(q))}{d_{M_1}(p, q)} \leq (\max\{1, b\}).$$

Proof. This can be seen by observing that

$$\begin{aligned} L_g(C) &= \int_0^1 \sqrt{|r'(t)|^2 + |f(r(t))|^2 |\theta'(t)|^2} dt \\ &\leq (\max\{1, b\}) \int_0^1 \sqrt{|r'(t)|^2 + |\theta'(t)|^2} dt \\ &\leq (\max\{1, b\}) L_{g_1}(C). \end{aligned}$$

Thus

$$d_M(F(p), F(q)) \leq (\max\{1, b\}) d_{M_1}(p, q).$$

For the other direction we have

$$\begin{aligned} L_{g_1}(C) &= \int_0^1 \sqrt{|r'(t)|^2 + |\theta'(t)|^2} dt \\ &\leq (\min\{a, 1\})^{-1} \int_0^1 \sqrt{|r'(t)|^2 + |f(r(t))|^2 |\theta'(t)|^2} dt \\ &\leq (\min\{a, 1\})^{-1} L_g(C). \end{aligned}$$

Thus

$$d_{M_1}(p, q) \leq (\min\{a, 1\})^{-1} d_M(F(p), F(q)). \quad \square$$

2D. Key theorem we apply to prove GH and \mathcal{F} convergence. The following theorem was proven by the second author jointly with Huang and Lee in [Huang et al. 2017] building upon earlier work of Gromov [1981]. This theorem allows us to prove GH and intrinsic flat convergence using only information about the sequence of distance functions. Note that it is a compactness theorem, providing the existence of a converging subsequence once one simply has uniform bi-Lipschitz control

on the metrics. The convergence is not bi-Lipschitz convergence but instead it is uniform convergence of the distance functions and also GH and \mathcal{F} convergence of the spaces.

Theorem 2.4. *Fix a precompact n -dimensional integral current space (X, d_0, T) without boundary (e.g., $\partial T = 0$) and fix $\lambda > 0$. Suppose that d_j are metrics on X such that*

$$(6) \quad \lambda \geq \frac{d_j(p, q)}{d_0(p, q)} \geq \frac{1}{\lambda}.$$

Then there exists a subsequence, also denoted d_j , and a length metric d_∞ satisfying (6) such that d_j converges uniformly to d_∞ :

$$\epsilon_j = \sup\{|d_j(p, q) - d_\infty(p, q)| : p, q \in X\} \rightarrow 0.$$

Furthermore

$$\lim_{j \rightarrow \infty} d_{\text{GH}}((X, d_j), (X, d_\infty)) = 0$$

and

$$\lim_{j \rightarrow \infty} d_{\mathcal{F}}((X, d_j, T), (X, d_\infty, T)) = 0.$$

In particular, (X, d_∞, T) is an integral current space and $\text{set}(T) = X$ so there are no disappearing sequences of points $x_j \in (X, d_j)$.

In fact we have

$$d_{\text{GH}}((X, d_j), (X, d_\infty)) \leq 2\epsilon_j$$

and

$$d_{\mathcal{F}}((X, d_j, T), (X, d_\infty, T)) \leq 2^{(n+1)/2} \lambda^{n+1} 2\epsilon_j \mathbf{M}_{(X, d_0)}(T).$$

Remark 2.5. In order to apply this theorem we will use the following method repeatedly. We will demonstrate that a sequence has pointed convergence of the distance functions and also satisfies the bi-Lipschitz bound in (6). Then by this theorem there is a converging subsequence. However by the pointed convergence we will see that all the subsequences must in fact converge to the same limit space. Thus we obtain \mathcal{F} and GH convergence of the original sequence.

3. Examples

In this section we present our examples. Each example contains a sequence of smooth warped product manifolds which converge in various ways to warped product metric spaces. We first study distances on warped product spaces with deep valleys. We apply this to present our cinched warped product example. We then observe what happens to distances on warped product spaces with peaks.

3A. Distances on warped products with valleys. First let us develop the intuitive picture first. Consider a warped product manifold $[-\pi, \pi] \times_g \mathbb{S}^1$ as in Figure 1 with a warping function

$$f_j(r) = \begin{cases} 1, & r \in [-\pi, -1/j], \\ h(jr), & r \in [-1/j, 1/j], \\ 1, & r \in [1/j, \pi], \end{cases}$$

where h is a smooth even function defining a valley with $h(-1) = 1$ with $h'(-1) = 0$, decreasing to $h(0) = h_0 \in (0, 1]$ and then increasing back up to $h(1) = 1$, $h'(1) = 0$. Keep in mind that the distance between the level sets, $r^{-1}(a)$ and $r^{-1}(b)$ is $|a - b|$ and so we have evenly spaced levels drawn in the figure.

A minimizing geodesic, draw in red in Figure 1, will proceed diagonally towards the valley, climb down into the valley, run along the valley, then climb out and proceed diagonally away from the valley. The climbing parts are very short if the change in r is small (which is true for large j). Since it is more efficient to travel around inside the valley (for the change in θ), it is more efficient to travel almost directly to the valley as in the geodesic in the figure. Observe that the length of this geodesic is bounded above by the length of a curve which goes directly to the valley and straight down, then turns a right angle to stay along the bottom of the valley, and then makes a right angle to climb out and move directly to the end point. Thus

$$d((-\pi, \theta_1), (\pi, \theta_2)) \leq |-\pi - \pi| + f(0) d_{\mathbb{S}^1}(\theta_1, \theta_2) + |\pi - \pi|.$$

In the following lemmas we use this same basic idea to bound distances in warped products with a wide variety of warping functions.

Lemma 3.1. *Given a warped product space M (or respectively N) defined as in (1), suppose $f(r) \geq f(r_0)$ for all $r \in [r_1, r_2]$ (or respectively $r \in \mathbb{S}^1$). If $x_1, x_2 \in r^{-1}(r_0)$ then*

$$d_g(x_1, x_2) = f(r_0) d_\sigma(\theta_2, \theta_1).$$

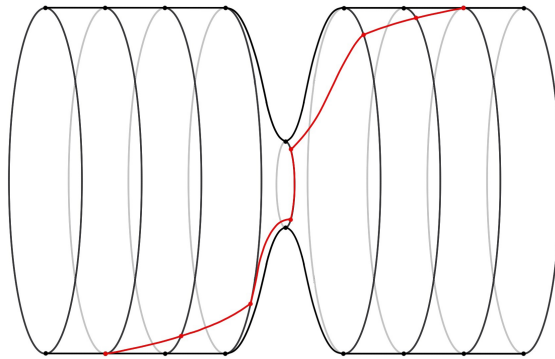


Figure 1. The geodesic will cut across the valley.

Proof. Let $C(t) = (r(t), \theta(t))$ be any curve joining $x_1 = (r_0, \theta_1)$ to $x_2 = (r_0, \theta_2)$. Then

$$\begin{aligned} L(C[0, 1]) &= \int_0^1 \sqrt{|r'(t)|^2 + |f(r(t))|^2 |\theta'(t)|^2} dt \\ &\geq \int_0^1 \sqrt{|0|^2 + |f(r_0)|^2 |\theta'(t)|^2} dt \\ &= f(r_0) \int_0^1 |\theta'(t)| dt \\ &= f(r_0) L_\Sigma(\theta[0, 1]) \\ &\geq f(r_0) d_\sigma(\theta_2, \theta_1). \end{aligned}$$

However if we take the curve $C(t) = (r_0, \theta(t))$, where $\theta(t)$ is a minimizing geodesic in Σ from θ_1 to θ_2 , we have equality everywhere above. So the infimum over all lengths is achieved:

$$d_g(x_1, x_2) = \inf_C L(C[0, 1]) = f(r_0) d_\sigma(\theta_2, \theta_1). \quad \square$$

Lemma 3.2. *Given a warped product space M defined as in (1) and a pair of points $x_1 = (r_1, \theta_1)$ and $x_2 = (r_2, \theta_2)$ with $r_1 < r_2$ then the distance between those points is bounded by*

$$d_{g_j}^M(x_1, x_2) \leq |r_2 - r_1| + D_j(r_1, r_2) d_\sigma(\theta_2, \theta_1),$$

where

$$D_j(r_1, r_2) = \min_{r \in [r_1, r_2]} f_j(r)$$

and d_σ is the distance on (Σ, σ) .

Proof. Let $\hat{r}_j \in (r_1, r_2)$ be chosen so that $f_j(\hat{r}_j) = D_j(r_1, r_2)$. Construct the following curve between the points $x_1, x_2 \in M_j$, where $\alpha \subset \Sigma$ is a geodesic with respect to (Σ, σ) , $\alpha(0) = \theta_1$ and $\alpha(1) = \theta_2$,

$$C_j(t) = \begin{cases} (r_1 + 3(\hat{r}_j - r_1)t, \theta_1), & t \in [0, \frac{1}{3}], \\ (\hat{r}_j, \alpha(3t - 1)), & t \in [\frac{1}{3}, \frac{2}{3}], \\ (\hat{r}_j + 3(r_2 - \hat{r}_j)(t - \frac{2}{3}), \theta_2), & t \in [\frac{2}{3}, 1], \end{cases}$$

and then

$$d_{g_j}^M(x_1, x_2) \leq L_j(C_j) = |r_2 - \hat{r}_j| + f_j(\hat{r}_j) d_\sigma(\theta_2, \theta_1) + |\hat{r}_j - r_1|. \quad \square$$

Almost the same proof can be applied to show the following lemma.

Lemma 3.3. *Given a warped product space N defined as in (1) and a pair of points $x_1 = (r_1, \theta_1)$ and $x_2 = (r_2, \theta_2)$ then the distance between those points is bounded by*

$$d_{g_j}^M(x_1, x_2) \leq d_{\mathbb{S}^1}(r_1, r_2) + D_j(r_1, r_2) d_\sigma(\theta_2, \theta_1),$$

where

$$D_j(r_1, r_2) = \min_{r \in \text{arc}(r_1, r_2)} f_j(r),$$

where $\text{arc}(r_1, r_2)$ is the minor arc between r_1 and r_2 in \mathbb{S}^1 and where d_σ is the distance on (Σ, σ) .

3B. Cinched spaces. Here we see examples of spaces whose warping functions converge in the L^p sense but the GH and SWIF limits do not agree with the L^p limit due to the existence of deep canyons or cinching. See Figure 1 and now imagine that the valley remains equally as deep but becomes very narrow.

Example 3.4. Consider the sequence of smooth functions $f_j(r) : [-\pi, \pi] \rightarrow [1, 2]$

$$f_j(r) = \begin{cases} 1, & r \in [-\pi, -1/j], \\ h(jr), & r \in [-1/j, 1/j], \\ 1, & r \in [1/j, \pi], \end{cases}$$

where h is a smooth even function such that $h(-1) = 1$ with $h'(-1) = 0$, decreasing to $h(0) = h_0 \in (0, 1]$ and then increasing back up to $h(1) = 1$, $h'(1) = 0$. Note that this defines a sequence of smooth Riemannian metrics, g_j , as in (2), with distances, d_j , as in (4) on the manifolds,

$$M_j = [-\pi, \pi] \times_{f_j} \Sigma \quad \text{or} \quad N_j = \mathbb{S}^1 \times_{f_j} \Sigma$$

for any fixed Riemannian manifold Σ . Consider also M_∞ and N_∞ defined as above with $f_\infty(r) = 1$ for all r .

Despite the fact that

$$f_j \rightarrow f_\infty \quad \text{in } L^p,$$

we do not have M_j converging to M_∞ nor N_j to N_∞ in the GH or \mathcal{F} sense. In fact

$$M_j \xrightarrow{\text{GH}} M_0 \quad \text{and} \quad M_j \xrightarrow{\mathcal{F}} M_0$$

and

$$N_j \xrightarrow{\text{GH}} N_0 \quad \text{and} \quad N_j \xrightarrow{\mathcal{F}} N_0,$$

where M_0 and N_0 are warped metric spaces defined as in (1) with warping factor

$$f_0(r) = \begin{cases} 1, & r \in [-\pi, 0), \\ h_0, & r = 0, \\ 1, & r \in (0, \pi]. \end{cases}$$

Proof. First we verify our claim about L^p convergence

$$\left(\int_{-\pi}^{\pi} |f_j - 1|^p dr \right)^{1/p} = \left(\int_{-1/j}^{1/j} |h_j - 1|^p dr \right)^{1/p} \leq \left(\frac{2}{j} \right)^{1/p} \rightarrow 0,$$

where we use the fact that $|h_j - 1|^p \leq 1$ by construction.

Let us consider (M_j, d_j) . Since we have

$$0 < h_0 \leq f_j(r) \leq f_0(r) \leq f_\infty(r) = 1$$

then

$$(h_0)^2 g_\infty \leq g_j \leq g_0 \leq g_\infty$$

and

$$h_0 d_\infty(x_1, x_2) \leq d_j(x_1, x_2) \leq d_0(x_1, x_2) \leq d_\infty(x_1, x_2).$$

Using d_∞ as our background metric we can apply the theorem in the appendix of [Huang et al. 2017] to see that a subsequence of the d_j converges uniformly to some limit, d , such that

$$(7) \quad h_0 d_\infty(x_1, x_2) \leq d(x_1, x_2) \leq d_0(x_1, x_2) \leq d_\infty(x_1, x_2).$$

In addition the subsequences converge in the Gromov–Hausdorff and intrinsic flat sense:

$$(M_j, d_j) \xrightarrow{\text{GH}} (M, d) \quad \text{and} \quad (M_j, d_j, T) \xrightarrow{\mathcal{F}} (M, d, T).$$

We need only prove $d = d_0$ for then no subsequence was necessary and we have proven our example.

Consider $x_1, x_2 \in M$ such that

$$d(x_1, x_2) < \min\{d(x_1, p) + d(p, x_2) : p \in r^{-1}(0)\}.$$

So there exists $\delta > 0$ depending on these two points such that

$$d(x_1, x_2) + \delta \leq \min\{d(x_1, p) + d(p, x_2) : p \in r^{-1}(0)\}.$$

Then for N sufficiently large, and all $j \geq N$ (in our subsequence) we have

$$d_j(x_1, x_2) + \delta/2 \leq \min\{d_j(x_1, p) + d_j(p, x_2) : p \in r^{-1}(0)\}.$$

Thus the L_{g_j} -shortest curve, γ_j , between x_1 and x_2 avoids $r^{-1}(-\delta/4, \delta/4)$. Here we have $g_j = g_0 = g_\infty$ so its length is the same with respect to all three metrics:

$$L_{g_j}(\gamma_j) = L_{g_0}(\gamma_j) = L_{g_\infty}(\gamma_j).$$

So

$$d_j(x_1, x_2) \geq d_0(x_1, x_2)$$

and taking the limit we have

$$d(x_1, x_2) \geq d_0(x_1, x_2)$$

and combining this with (7) we have

$$d(x_1, x_2) = d_0(x_1, x_2).$$

In fact for any L_d -shortest curve γ ,

$$(8) \quad \gamma([t_1, t_2]) \cap r^{-1}(0) = \emptyset \implies d(\gamma(t_1), \gamma(t_2)) = d_0(\gamma(t_1), \gamma(t_2)).$$

We need only confirm that $d(x_1, x_2) = d_0(x_1, x_2)$ for $x_1, x_2 \in M$ such that

$$d(x_1, x_2) = \min\{d(x_1, p) + d(p, x_2) : p \in r^{-1}(0)\}.$$

Taking the L_d -shortest curve γ between x_1 and x_2 , we know that $s_1 \leq s_2$

$$s_1 = \inf\{t : \gamma(t) \in r^{-1}(0)\} \quad \text{and} \quad s_2 = \sup\{t : \gamma(t) \in r^{-1}(0)\}.$$

We have

$$d(x_1, x_2) = L_d(\gamma) = d(\gamma(0), \gamma(s_1)) + d(\gamma(s_1), \gamma(s_2)) + d(\gamma(s_2), \gamma(1))$$

By (8) if $s_1 > 0$ then for all $\delta > 0$ we have

$$d(\gamma(0), \gamma(s_1 - \delta)) = d_0(\gamma(0), \gamma(s_1 - \delta))$$

so

$$d(\gamma(0), \gamma(s_1)) = d_0(\gamma(0), \gamma(s_1)).$$

Similarly

$$d(\gamma(s_2), \gamma(1)) = d_0(\gamma(s_2), \gamma(1)).$$

Thus we need only confirm that $d(x_1, x_2) = d_0(x_1, x_2)$ for $x_1, x_2 \in r^{-1}(0)$. This easily follows by applying Lemma 3.1 to both f_j and f_0 since both functions have minimum $= h_0$ at $r = 0$:

$$d(x_1, x_2) = \lim_{j \rightarrow \infty} d_j(x_1, x_2) = h_0 d_\sigma(\theta_1, \theta_2) = d_0(x_1, x_2).$$

To prove the case where we have a warped product of the form N as in (1) the proof is almost the same. \square

3C. Moving cinches. Here we explore what happens when the warping functions converge in L^p but not pointwise almost everywhere.

Example 3.5. We first construct a classical sequence of smooth functions $f_j : [-\pi, \pi] \rightarrow (0, 1]$ which converge L^p to $f_\infty = 1$ but do not converge pointwise almost everywhere without taking a subsequence. Let

$$f_j(r) = \begin{cases} h((r - t_j)/\delta_j), & r \in [t_j - \delta_j, t_j + \delta_j], \\ 1, & \text{elsewhere,} \end{cases}$$

where h is a smooth even function as in Example 3.4 such that $h(-1) = 1$ with $h'(-1) = 0$, decreasing to $h(0) = h_0 \in (0, 1]$ and then increasing back up to $h(1) = 1$, $h'(1) = 0$, and where

$$\{t_j : j \in \mathbb{N}\} = \left\{ \frac{0}{1}, \frac{1}{1}, \frac{0}{2}, \frac{1}{2}, \frac{2}{2}, \frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \dots \right\}$$

and

$$\{\delta_j : j \in \mathbb{N}\} = \left\{ \frac{1}{1}, \frac{1}{1}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \dots \right\}.$$

Then the cylinders, N_j , defined as in (1) will not converge in the GH or \mathcal{F} sense without taking a subsequence. The tori M_j will converge since each torus in this sequence is isometric to a torus in the sequence of tori in Example 3.4 via an isometry which moves t_j to 0.

Proof. First we check that f_j converges in L^p but not pointwise almost everywhere. To this end we check that

$$\left(\int_{-\pi}^{\pi} |f_j - 1|^p dr \right)^{1/p} = \left(\int_{t_j - \delta_j}^{t_j + \delta_j} |h_0 - 1|^p dr \right)^{1/p} = (2\delta_j)^{1/p} \rightarrow 0$$

since $|h_0 - 1|^p \leq 1$ by construction. Of course we do not find pointwise convergence for any $r \in [0, 1]$ since for every choice of $J > 0$ one can find a $j_1 \geq J$ and a $r \in [-\pi, \pi]$ so that $f_{j_1}(r) = h_0$ and another $j_2 \geq J$ so that $f_{j_2}(r) = 1$.

Now if we take a subsequence where $t_{j_k} = 0$, then exactly as in Example 3.4 we see that N_{j_k} converges in the GH and \mathcal{F} sense to N_0 of that example. On the other hand, if we take a subsequence where $t_{j_k} = 1$, then imitating the proof in Example 3.4 we see that N_{j_k} converges in the GH and \mathcal{F} sense to N'_0 which is a warped product whose warping function is 1 everywhere except at $r = 1$ where it is h_0 . Thus the original sequence of N_j of this example has no GH nor \mathcal{F} limit. \square

3D. Avoiding ridges. The cinched spaces of Example 3.4 did not converge to their L^p limit because their warping functions, f_j , all had a minimum uniformly below the level of their L^p limit, f_∞ . Here we will see there is no corresponding problem when the f_j have a maximum uniformly above the level of their L^p limit.

In the following lemma, we have a ridge as in Figure 2, the minimal geodesic between points, p, q lying on that ridge, will not run along the ridge. In the following we consider f_j with a maximum at r_* and thus there is a ridge along the level set $f_j^{-1}(r_*)$.

Lemma 3.6. *Given $r_*, \hat{r} \in [r_0, r_1]$, the distance between $x_1 = (r_*, \theta_1)$ and $x_2 = (r_*, \theta_2)$ in a warped product space is bounded above by*

$$d(x_1, x_2) \leq 2|\hat{r} - r_*| + f_j(\hat{r})d_\sigma(\theta_1, \theta_2).$$

Thus for a fixed $r_ \in [r_0, r_1]$, if there exists an $\hat{r} \in [r_0, r_1]$ such that*

$$(9) \quad f_j(\hat{r}) < f_j(r_*) - 2 \frac{|\hat{r} - r_*|}{d_\sigma(\theta_1, \theta_2)}$$

then the minimizing geodesic from $x_1 = (r_, \theta_1)$ to $x_2 = (r_*, \theta_2)$, $\theta_1, \theta_2 \in \Sigma$, $\theta_1 \neq \theta_2$, cannot be a curve with constant r -component, $r(t) = r_*$.*

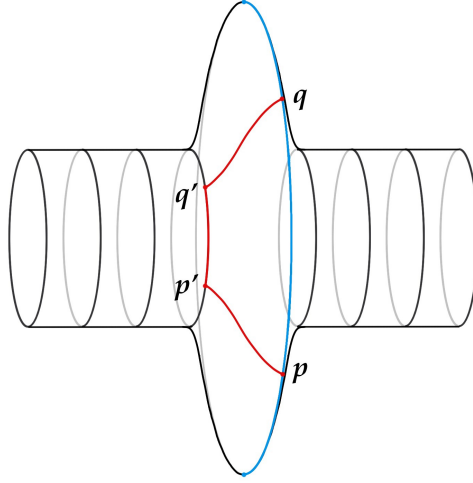


Figure 2. A curve γ from p to q on a ridge, which first cuts down to p' and then runs across to q' before cutting up to q is shorter than curve running along the ridge between p and q if the ridge is narrow enough.

See Figure 2 taking $p = x_1 = (r_*, \theta_1)$ and $q = x_2 = (r_*, \theta_2)$ and $p' = (\hat{r}, \theta_1)$ and $q' = (\hat{r}, \theta_2)$. So $d(p, q) \leq L(\gamma) = d(p, p') + d(p', q') + d(q', q)$, where $d(p, p') = d(q, q') = |r_* - \hat{r}|$.

Proof. Let $x_1, x_2 \in M_j$ with coordinates $x_1 = (r_*, \theta_1)$ to $x_2 = (r_*, \theta_2)$, $\theta_1, \theta_2 \in \Sigma$, $\theta_1 \neq \theta_2$ so that (9) is satisfied for r_* . Let $\alpha \subset \Sigma$ be a curve between θ_1, θ_2 with length $L_\Sigma(\alpha) = d_\sigma(\theta_1, \theta_2)$ and consider the curve

$$\gamma(t) = \begin{cases} (r_* + 3(\hat{r} - r_*)t, \theta_1), & t \in [0, \frac{1}{3}], \\ (\hat{r}, \alpha(3t - 1)), & t \in [\frac{1}{3}, \frac{2}{3}], \\ (\hat{r} + 3(r_* - \hat{r})(t - \frac{2}{3}), \theta_2), & t \in [\frac{2}{3}, 1], \end{cases}$$

as depicted in Figure 2. Then

$$L_j(\gamma) = 2|\hat{r} - r_*| + f_j(\hat{r})d_\sigma(\theta_1, \theta_2).$$

So if we consider $\beta(t) = (r_*, \alpha(t))$ and use the assumption (9) then we find that

$$L_j(\gamma) < L_j(\beta)$$

and hence $\beta(t)$ cannot be the minimizing geodesic. \square

3E. A single ridge disappears. Here we see that a sequence of warped product spaces with a consistently high ridge that is increasingly narrow converges in the L^p , pointwise a.e., GH, and \mathcal{F} sense to an isometric product manifold as if the

ridge simply disappears despite the fact that the warping functions do not converge pointwise to the constant function 1. See Figure 2.

Example 3.7. Consider the sequence of functions $f_j(r) : [-\pi, \pi] \rightarrow [1, 2]$ with

$$f_j(r) = \begin{cases} 1, & r \in [-\pi, -1/j], \\ h(jr), & r \in [-1/j, 1/j], \\ 1, & r \in [1/j, \pi], \end{cases}$$

where $h = h_{ridge}$ is a smooth even function such that $h(-1) = 1$ with $h'(-1) = 0$, increasing to $h(0) = h_0 \in (1, 2]$ and then decreasing back down to $h(1) = 1$, $h'(1) = 0$. Note that this defines a sequence of smooth Riemannian metrics, g_j , as in (2), with distances, d_j , as in (4) on the manifolds,

$$M_j = [-\pi, \pi] \times_{f_j} \Sigma \quad \text{or} \quad N_j = \mathbb{S}^1 \times_{f_j} \Sigma,$$

for any fixed Riemannian manifold Σ . Consider also M_∞ and N_∞ defined as above with $f_\infty(r) = 1$ for all r . Here we have

$$f_j \rightarrow f_\infty = 1 \quad \text{in } L^p \text{ but not pointwise}$$

and yet $M_j \rightarrow M_\infty$ and $N_j \rightarrow N_\infty$ in both the GH and \mathcal{F} sense.

Proof. First we check that f_j converges in L^p to f_∞ . To this end we check that

$$\left(\int_{-\pi}^{\pi} |f_j - f_\infty|^p dr \right)^{1/p} = \left(\int_{-1/j}^{1/j} |h(jr) - 1|^p dr \right)^{1/p} \leq (2/j)^{1/p} \rightarrow 0$$

since $|h_j - 1|^p \leq 1$ by construction. Observe that f_j does not converge pointwise to f_∞ because $f_j(0) = h_0 > 1 = f_\infty(0)$. Let

$$(10) \quad J_\delta = 1/\delta$$

so that $f_j(r) = f_\infty(r)$ on $[0, -1/j] \cup [1/j, 1]$ for all $j \geq J_\delta$.

Next observe that since $2f_\infty(r) \geq f_j(r) \geq f_\infty(r)$ at all $r \in [-\pi, \pi]$, we have

$$(11) \quad d_\infty(p, q) \leq d_j(p, q) \leq 2d_\infty(p, q) \quad \text{for all } p, q.$$

Since our limit space, M_∞ , is an isometric product space, any pair of points $x_1 = (s_1, \theta_1)$ to $x_2 = (s_2, \theta_2)$ with $s_1 < s_2$ is joined by a smooth L_∞ minimizing geodesic, $C : [0, 1] \rightarrow M_\infty$, such that

$$d_\infty(p, q) = L_\infty(C).$$

In fact $C(t) = (r(t), \theta(t))$ where $r : [0, 1] \rightarrow [r_1, r_2]$ is strictly increasing from s_1 to s_2 , and $\theta : [0, 1] \rightarrow \Sigma$ is a geodesic from θ_1 to θ_2 with respect to (Σ, σ) . Let $T_\delta \subset [0, 1]$ be defined as the possibly empty interval

$$T_\delta = \{t : r(t) \in [-\delta, \delta]\}.$$

Observe that the length of C restricted to the interval T_δ satisfies

$$L_\infty(C(T_\delta)) \leq 2\delta L_\infty(C) \leq 2\delta d_\infty(x_1, x_2).$$

For $j \geq J_\delta$ as in (10), we have

$$\begin{aligned} d_j(x_1, x_2) &\leq L_j(C) = \int_0^1 g_j(C'(t), C'(t))^{1/2} dt \\ &\leq \int_{T_\delta} 2g_\infty(C'(t), C'(t))^{1/2} + \int_{[0,1] \setminus T_\delta} g_\infty(C'(t), C'(t))^{1/2} \\ &\leq 2L_\infty(C(T_\delta)) + L_\infty(C[0, 1]) \\ &\leq (1 + 2\delta)d_\infty(x_1, x_2). \end{aligned}$$

Thus for x_1 and x_2 lying on different levels of r we have pointwise convergence $d_j(x_1, x_2) \rightarrow d_\infty(x_1, x_2)$.

Taking points that lie on the same level, $x_1 = (s, \theta_1)$ to $x_2 = (s, \theta_2)$, we know that the minimizing geodesic, C , in our isometric product will have the form $C(t) = (s, \theta(t))$. If the points do not lie on the ridge, $s \neq 0$, and so

$$d_j(x_1, x_2) \leq L_j(C) = L_\infty(C) = d_\infty(x_1, x_2) \quad \text{for all } j \geq J_\delta.$$

So again we have pointwise convergence $d_j(x_1, x_2) \rightarrow d_\infty(x_1, x_2)$.

If the points both lie on the ridge $x_1 = (0, \theta_1)$ to $x_2 = (0, \theta_2)$ then by Lemma 3.6 we have

$$d_j(x_1, x_2) \leq 1d_\Sigma(\theta_1, \theta_2) + 2\delta = d_\infty(x_1, x_2) + 2\delta \quad \text{for all } j \geq J_\delta.$$

And again we have pointwise convergence $d_j(x_1, x_2) \rightarrow d_\infty(x_1, x_2)$.

By Theorem 2.4 combined with (11) we know a subsequence d_{j_k} converges uniformly to some limit distance. Since we have pointwise convergence to d_∞ , we know in fact that the d_j thus converge uniformly to d_∞ without even taking a subsequence. Furthermore we have Gromov–Hausdorff and intrinsic flat convergence.

The proof when we have warped around \mathbb{S}^1 to create N_j is very similar. \square

3F. Moving ridges. Here we see a sequence of spaces which have f_j converging to $f_\infty = 1$ in the L^p sense and $f_j \geq 1$. The sequence does not converge pointwise almost everywhere unless one takes a subsequence. Nevertheless by Theorem 1.1 there is a GH and a SWIF limit without taking a subsequence and indeed the limit is the space warped by f_∞ .

Example 3.8. We first construct a classical sequence of smooth functions $f_j : [-\pi, \pi] \rightarrow [1, 2]$ which converge L^p to $f_\infty = 1$ but do not converge pointwise almost everywhere without taking a subsequence. Let

$$\{s_j : j \in \mathbb{N}\} = \left\{ \frac{0}{1}, \frac{1}{1}, \frac{0}{2}, \frac{1}{2}, \frac{2}{2}, \frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \dots \right\}$$

and

$$\{\delta_j : j \in \mathbb{N}\} = \left\{ \frac{1}{1}, \frac{1}{1}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \dots \right\}.$$

Let

$$f_j(r) = \begin{cases} h((r - s_j)/\delta_j), & r \in [s_j - \delta_j, s_j + \delta_j], \\ 1, & \text{elsewhere,} \end{cases}$$

where h is a smooth even function such that $h(-1) = 1$ with $h'(-1) = 0$, increasing up to $h(0) = h_0 \in (1, 2]$ and then decreasing back down to $h(1) = 1$, $h'(1) = 0$. Note that this defines a sequence of smooth Riemannian metrics, g_j , as in (2), with distances, d_j , as in (4) on the manifolds,

$$(12) \quad M_j = [-\pi, \pi] \times_{f_j} \Sigma \quad \text{or} \quad N_j = \mathbb{S}^1 \times_{f_j} \Sigma,$$

for any fixed Riemannian manifold Σ . Consider also M_∞ and N_∞ defined as above with $f_\infty(r) = 1$ for all r . Here we have

$$(13) \quad f_j \rightarrow f_\infty = 1 \quad \text{in } L^p \text{ but not pointwise}$$

and yet $M_j \rightarrow M_\infty$ and $N_j \rightarrow N_\infty$ in both the GH and \mathcal{F} sense.

Proof. First we check that f_j converges in L^p but not pointwise almost everywhere. To this end we check that

$$\left(\int_{-\pi}^{\pi} |f_j - 1|^p dr \right)^{1/p} = \left(\int_{s_j - \delta_j}^{s_j + \delta_j} |h_j - 1|^p dr \right)^{1/p} = (2\delta_j)^{1/p} \rightarrow 0$$

since $|h_j - 1|^p \leq 1$ by construction. Of course we do not find pointwise convergence for any $r \in [-\pi, \pi]$ since for every choice of $J > 0$ one can find a $j_1 \geq J$ so that $f_{j_1}(r) = 0$ and another $j_2 \geq J$ so that $f_{j_2}(r) > 0$.

The proof of the Gromov–Hausdorff and intrinsic flat convergence follows almost exactly as in Example 3.7 except that we must choose J_δ and T_δ differently. We skip this proof since the convergence follows from Theorem 1.1 anyway. \square

3G. Many ridges. Here we see a sequence of spaces which have f_j converging to $f_\infty = 1$ in the L^p sense and $f_j \geq 1$. The sequence converges pointwise to a nowhere continuous function. Nevertheless by Theorem 1.1 there is a GH and a SWIF limit without taking a subsequence and indeed the limit is the isometric product space.

Example 3.9. We first construct a classical sequence of smooth functions $f_j : [-\pi, \pi] \rightarrow [1, 2]$ as in Figure 3 which converge L^p to $f_\infty = 1$ but do not converge pointwise almost everywhere without taking a subsequence. Let

$$\begin{aligned} S &= \{s_{i,j} = -\pi + 2\pi i/2^j : i = 1, 2, \dots, (2^j - 1), j \in \mathbb{N}\} \\ &= \left\{ -\pi + \frac{2\pi}{2}, -\pi + \frac{2\pi}{4}, -\pi + \frac{2\pi 2}{4}, -\pi + \frac{2\pi 3}{4}, -\pi + \frac{2\pi}{8}, \dots \right\} \end{aligned}$$

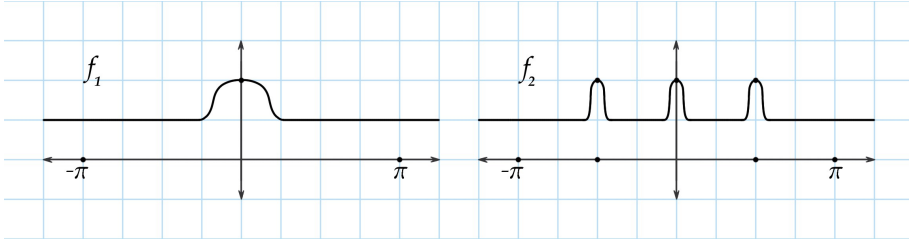


Figure 3. The warping functions of Example 3.9.

which is dense in $[-\pi, \pi]$ and

$$\{\delta_j = \left(\frac{1}{2}\right)^{2j} : j \in \mathbb{N}\} = \left\{\frac{1}{4}, \frac{1}{16}, \frac{1}{32}, \dots\right\}.$$

Let

$$f_j(r) = \begin{cases} h((r - s_{i,j})/\delta_j), & r \in [s_{i,j} - \delta_j, s_{i,j} + \delta_j] \text{ for } i = 1, \dots, 2^j - 1, \\ 1, & \text{elsewhere,} \end{cases}$$

where h is a smooth even function such that $h(-1) = 1$ with $h'(-1) = 0$, increasing up to $h(0) = h_0 \in (1, 2]$ and then decreasing back down to $h(1) = 1$ with $h'(1) = 0$. Note that this defines a sequence of smooth Riemannian metrics, g_j , as in (2), with distances, d_j , as in (4) on the manifolds,

$$M_j = [-\pi, \pi] \times_{f_j} \Sigma \quad \text{or} \quad N_j = \mathbb{S}^1 \times_{f_j} \Sigma$$

for any fixed Riemannian manifold Σ . Consider also M_∞ and N_∞ defined as above with $f_\infty(r) = 1$ for all r . Here we have

$$f_j \rightarrow f_\infty = 1 \quad \text{in } L^p \text{ but not pointwise}$$

and yet $M_j \rightarrow M_\infty$ and $N_j \rightarrow N_\infty$ in both the GH and \mathcal{F} sense.

Proof. First we check that f_j converges in L^p

$$\begin{aligned} \left(\int_{-\pi}^{\pi} |f_j - 1|^p dr \right)^{1/p} &= \left(\sum_{i=1}^{2^j-1} \int_{s_{i,j}-\delta_j}^{s_{i,j}+\delta_j} |f_j - 1|^p dr \right)^{1/p} = ((2^j - 1)(2\delta_j))^{1/p} \\ &= ((2^j - 1)\left(\frac{1}{2}\right)^{2j})^{1/p} \rightarrow 0. \end{aligned}$$

Next observe that f_j converges pointwise on S to h_0 and pointwise to 1 elsewhere. Since S is dense and $h_0 > 1$ the pointwise limit is continuous nowhere.

The proof of the Gromov–Hausdorff and intrinsic flat convergence follows almost exactly as in Example 3.7 except that we must choose J_δ and T_δ differently. We skip this proof since the convergence follows from Theorem 1.1 anyway. \square

3H. Converging to Euclidean-taxi spaces. In Theorem 1.1 we will prove that if $f_j \geq 1$ and $f_j \rightarrow 1$ in the L^p sense then we have Gromov–Hausdorff and intrinsic flat convergence to the isometric product space just as in Examples 3.7, 3.8 and 3.9. We now investigate what might happen if f_j does not converge to 1 in the L^p sense but does have a dense collection of points where f_j converges pointwise to 1. In the example below we see that this does not suffice to prove GH or intrinsic flat convergence to the isometric product space.

Here we will construct a sequence of warped product spaces with increasingly many cinches. The limit metric we obtain in this example is not a Riemannian metric but a metric of the following form:

Definition 3.10. Let M and N be product manifolds as in (1). For any $R > 1$, we define the minimized R -stretched Euclidean taxi metric (R -ET metric) between $x_1 = (s_1, \theta_1)$ and $x_2 = (s_2, \theta_2)$ to be

$$d_{R\text{-ET}}^M(x_1, x_2) = \min_{\Theta \in [0, d_\Sigma(\theta_1, \theta_2)]} \sqrt{|s_1 - s_2|^2 + R^2 \Theta^2} + d_\Sigma(\theta_1, \theta_2) - \Theta,$$

$$d_{R\text{-ET}}^N(x_1, x_2) = \min_{\Theta \in [0, d_\Sigma(\theta_1, \theta_2)]} \sqrt{d_{\mathbb{S}^1}(s_1, s_2)^2 + R^2 \Theta^2} + d_\Sigma(\theta_1, \theta_2) - \Theta.$$

Note that the R -ET metric is smaller than the isometric product metric with the θ direction scaled by R (achieved at $\Theta = d_\Sigma(\theta_1, \theta_2)$), and it is also smaller than the taxi product (achieved at $\Theta = 0$). One may view the R -ET metric as an infimum over lengths of all curves which are partly line segments of the form $\theta = ms + \theta_0$ (whose lengths are measured by stretching the Euclidean metric by R in the θ direction) and partly vertical segments purely in the θ direction (whose lengths are not rescaled). Without stretching, taking $R = 1$, we see the minimum is achieved going purely diagonal with the standard Euclidean metric.

It is not immediately obvious that R -ET metrics are true metrics satisfying positivity, symmetry and the triangle inequality. We prove this in the following lemma.

Lemma 3.11. *When*

$$(14) \quad d_\Sigma(\theta_1, \theta_2) \leq \frac{|s_1 - s_2|}{R\sqrt{R^2 - 1}},$$

the metric is an isometric product

$$(15) \quad d_{R\text{-ET}}^M((s_1, \theta_1), (s_2, \theta_2)) = \sqrt{|s_1 - s_2|^2 + R^2 d_\Sigma(\theta_1, \theta_2)^2},$$

and otherwise the metric is a stretched taxi product:

$$(16) \quad d_{R\text{-ET}}^M((s_1, \theta_1), (s_2, \theta_2)) = |s_1 - s_2| \left(\frac{\sqrt{R^2 - 1}}{R} \right) + d_\Sigma(\theta_1, \theta_2).$$

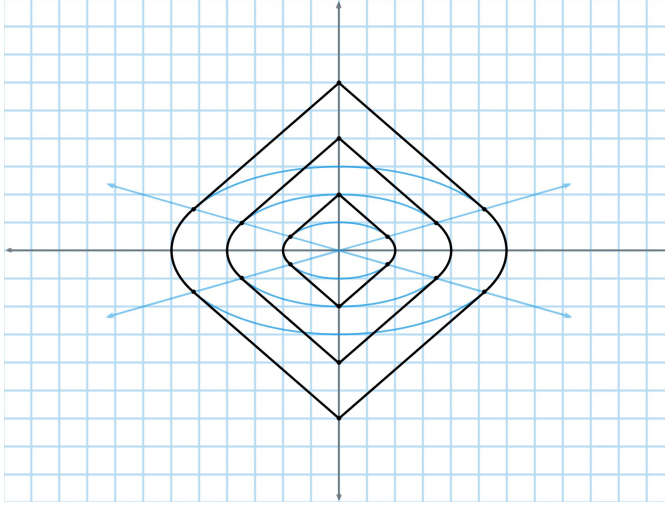


Figure 4. The concentric balls of radius $r = 2, 4,$ and 6 in an R -ET space with $R = 2$ are unions of diamonds, $|s| + \frac{\sqrt{3}}{2}|\theta| < r$, and ellipses, $s^2 + 2\theta^2 < r^2$.

In fact $d_{R\text{-ET}}^M$ is a minimum of these two metrics and is a length metric whose balls are the unions of diamonds and ellipses (as in Figure 4). It is a true metric satisfying positivity, symmetry and the triangle inequality.

Proof. To locate the minimum in the definition of the ET metric, we take the derivative

$$\frac{d}{d\Theta} \sqrt{|s_1 - s_2|^2 + R^2\Theta^2} + d_{\Sigma}(\theta_1, \theta_2) - \Theta = \frac{1}{2}(|s_1 - s_2|^2 + R^2\Theta^2)^{-1/2}(2R^2\Theta) - 1.$$

This derivative is negative at $\Theta = 0$ so the minimum is not achieved by the taxi product metric. The derivative becomes 0 at

$$(17) \quad \Theta_0 = \frac{|s_1 - s_2|}{R\sqrt{R^2 - 1}}$$

and is then positive for $\Theta > \Theta_0$. If (14) holds then Θ_0 does not lie in $(0, d_{\Sigma}(\theta_1, \theta_2))$, so the minimum is achieved at $\Theta = d_{\Sigma}(\theta_1, \theta_2)$ and we have (15).

Otherwise, the minimum is achieved at Θ_0 . Since

$$R^2\Theta_0^2 = |s_1 - s_2|^2/(R^2 - 1) \quad \text{and} \quad 1 + (1/(R^2 - 1)) = R^2/(R^2 - 1)$$

we have

$$\begin{aligned}
 d_{R\text{-ET}}^M((s_1, \theta_1), (s_2, \theta_2)) &\leq \sqrt{|s_1 - s_2|^2 + R^2 \Theta_0^2} + d_\Sigma(\theta_1, \theta_2) - \Theta_0 \\
 &= \frac{|s_1 - s_2| \cdot |R|}{\sqrt{R^2 - 1}} + d_\Sigma(\theta_1, \theta_2) - \frac{|s_1 - s_2|}{R\sqrt{R^2 - 1}} \\
 &= \frac{|s_1 - s_2|(R^2 - 1)}{R\sqrt{R^2 - 1}} + d_\Sigma(\theta_1, \theta_2) \\
 &= |s_1 - s_2| \frac{\sqrt{R^2 - 1}}{R} + d_\Sigma(\theta_1, \theta_2).
 \end{aligned}$$

Thus we have (16).

We also see that $d_{R\text{-ET}}^M((s_1, \theta_1), (s_2, \theta_2))$ is the minimum of the two metrics in (15) and (16). We know that both these metrics are length metrics. Indeed the metric in (15) is the infimum of the lengths of curves, $C(t) = (s(t), \theta(t))$ where

$$L_E(C) = \int_0^1 \sqrt{s'(t)^2 + R^2 g_\Sigma(\theta'(t), \theta'(t))} dt$$

and the metric in (16) is the infimum of the lengths of curves $C(t) = (s(t), \theta(t))$ where

$$L_T(C) = \int_0^1 |s'(t)| \frac{\sqrt{R^2 - 1}}{|R|} + g_\Sigma(\theta'(t), \theta'(t))^{1/2} dt.$$

Thus

$$d_{R\text{-ET}}^M(x_1, x_2) = \min\{\inf_C L_E(C), \inf_C L_T(C)\} = \inf_C L_{R\text{-ET}}(C),$$

where $L_{R\text{-ET}}(C) = \min\{L_E(C), L_T(C)\}$. Thus we have positivity and symmetry (which was easy to see) and now the triangle inequality as well (which was not). \square

We now present our example: a sequence of warped product spaces with increasingly many cinches which converges in the uniform, GH and \mathcal{F} sense to a product space with a minimized R -stretched Euclidean taxi metric. Here we have $R = 5$, but we could easily construct similar sequences converging to any R -ET metric with $R > 1$.

Example 3.12. Let

$$\begin{aligned}
 S &= \{s_{i,j} = -\pi + 2\pi i/2^j : i = 1, 2, \dots, (2^j - 1), j \in \mathbb{N}\} \\
 &= \{-\pi + \frac{2\pi}{2}, -\pi + \frac{2\pi}{4}, -\pi + \frac{2\pi}{4}, -\pi + \frac{2\pi}{4}, -\pi + \frac{2\pi}{4}, -\pi + \frac{2\pi}{8}, \dots\}
 \end{aligned}$$

which is dense in $[-\pi, \pi]$ and

$$\{\delta_j = (\frac{1}{2})^{2^j} : j \in \mathbb{N}\} = \{\frac{1}{4}, \frac{1}{16}, \frac{1}{32}, \dots\}$$

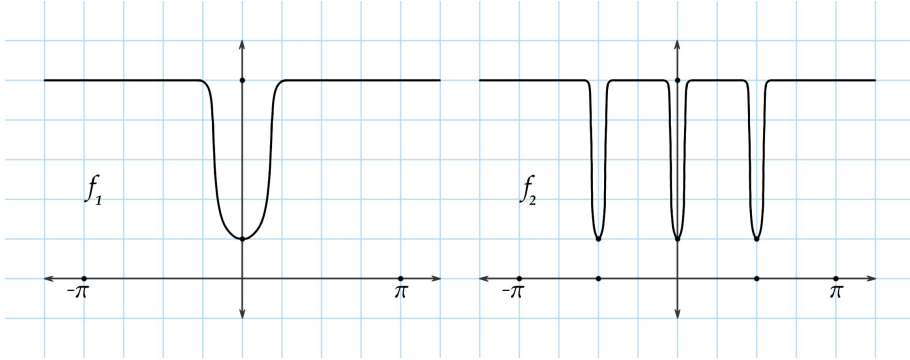


Figure 5. The warping functions of Example 3.12.

Define the functions f_j as in Figure 5 as follows:

$$f_j(r) = \begin{cases} h((r - s_{i,j})/\delta_j), & r \in [s_{i,j} - \delta_j, s_{i,j} + \delta_j] \text{ for } i = 1, \dots, 2^j - 1, \\ 5, & \text{elsewhere,} \end{cases}$$

where h is a smooth even function such that $h(-1) = 5$ with $h'(-1) = 0$, decreasing down to $h(0) = 1$ and then increasing back up to $h(1) = 5$ with $h'(1) = 0$.

Then $f_j(r) \geq 1$ converges pointwise to 1 on the dense set, S .

If we define M_j and N_j as in (1) then they do not converge to isometric products with warping function 1. Instead they converge in the GH and \mathcal{F} sense to a product manifold with an R -ET metric with $R = 5$.

Proof. First we check that $f_j \rightarrow 5$ in L^p by using the fact that $|f_j - 5|^p \leq 4^p$:

$$\begin{aligned} \left(\int_{-\pi}^{\pi} |f_j - 5|^p dr \right)^{1/p} &= \left(\sum_{i=1}^{2^j-1} \int_{s_{i,j}-\delta_j}^{s_{i,j}+\delta_j} |f_j - 5|^p dr \right)^{1/p} \\ &\leq ((2^j - 1)(2\delta_j)4^p)^{1/p} \\ &= 4((2^j - 1)\left(\frac{1}{2}\right)^{2j})^{1/p} \rightarrow 0. \end{aligned}$$

Now observe that since

$$(18) \quad 1 \leq f_j(r) \leq 5 \quad \text{for all } r \in [-\pi, \pi]$$

we have

$$d_1(p, q) \leq d_j(p, q) \leq 5d_1(p, q),$$

where d_1 is the warped product metric with warping function 1. Thus by [Huang et al. 2017], a subsequence of the warped product manifolds converges in the uniform, GH and intrinsic flat sense to some limit metric space with limit metric

d_∞

$$d_1(p, q) \leq d_\infty(p, q) \leq 5d_1(p, q) \quad \text{for all } p, q.$$

We will show that the pointwise limit of the d_j is $d_{5\text{-ET}}$, thus proving that the original sequence of warped product manifolds converges in the uniform, GH and intrinsic flat sense to the Euclidean taxi space.

Let us consider an arbitrary pair of points, $x_i = (s_i, \theta_i)$. If $\theta_1 = \theta_2$ then

$$d_j(x_1, x_2) = |s_1 - s_2| = d_{5\text{-ET}}(s_1, s_2).$$

In general, if $\theta_1 \neq \theta_2$ let $s'_{i,j} \in f_j^{-1}(1)$ with

$$|s'_{i,j} - s_i| < 2\pi/2^j, \quad x'_{i,j} = (s'_{i,j}, \theta_i).$$

By the triangle inequality applied two ways we have

$$(19) \quad |d_j(x_1, x_2) - d_j(x'_{1,j}, x'_{2,j})| \leq d_j(x_1, x'_{1,j}) + d_j(x'_{2,j}, x_2) \\ \leq |s_1 - s'_{1,j}| + |s'_{2,j} - s_2| < 4\pi/2^j$$

and

$$(20) \quad |d_{5\text{-ET}}(x_1, x_2) - d_{5\text{-ET}}(x'_{1,j}, x'_{2,j})| \leq d_{5\text{-ET}}(x_1, x'_{1,j}) + d_{5\text{-ET}}(x'_{2,j}, x_2) \\ \leq |s_1 - s'_{1,j}| + |s'_{2,j} - s_2| < 4\pi/2^j.$$

Recall that to complete the proof we must prove the pointwise limit:

$$\lim_{j \rightarrow \infty} d_j(x_1, x_2) = d_{5\text{-ET}}(x_1, x_2).$$

By (19) we need only show

$$\lim_{j \rightarrow \infty} d_j(x'_{1,j}, x'_{2,j}) = d_{5\text{-ET}}(x_1, x_2).$$

Applying the triangle inequality again, with $x_{1,\theta,j} = (s'_{1,j}, \theta)$, where $\theta \in \Sigma$ so that $d_\Sigma(\theta_2, \theta) \in [0, d_\Sigma(\theta_1, \theta_2)]$, we have

$$d_j(x'_{1,j}, x'_{2,j}) \leq d_j(x'_{1,j}, x_{1,\theta,j}) + d_j(x_{1,\theta,j}, x'_{2,j}) \\ \leq d_\Sigma(\theta_1, \theta) + \sqrt{|s'_{1,j} - s'_{2,j}|^2 + 25d_\Sigma(\theta_2, \theta)^2},$$

where we have used (18) in the last line. Since this is true for any $\theta \in \Sigma$ such that $d_\Sigma(\theta_2, \theta) \in [0, d_\Sigma(\theta_1, \theta_2)]$ we find

$$d_j(x'_{1,j}, x'_{2,j}) \leq d_{5\text{-ET}}(x'_{1,j}, x'_{2,j}).$$

Thus taking the limsup and applying (20) we have

$$(21) \quad \limsup_{j \rightarrow \infty} d_j(x'_{1,j}, x'_{2,j}) \leq \limsup_{j \rightarrow \infty} d_{5\text{-ET}}(x'_{1,j}, x'_{2,j}) = d_{5\text{-ET}}(x_1, x_2).$$

So now we need only show

$$(22) \quad \liminf_{j \rightarrow \infty} d_j(x'_{1,j}, x'_{2,j}) \geq d_{5\text{-ET}}(x_1, x_2).$$

By (20) we need only show

$$(23) \quad \liminf_{j \rightarrow \infty} (d_j(x'_{1,j}, x'_{2,j}) - d_{5\text{-ET}}(x'_{1,j}, x'_{2,j})) \geq 0.$$

If $s'_{1,j} = s'_{2,j}$ then

$$d_j(x'_{1,j}, x'_{2,j}) \geq d_{\Sigma}(\theta_1, \theta_2) = d_{5\text{-ET}}(x'_{1,j}, x'_{2,j}).$$

If $s'_{1,j} \neq s'_{2,j}$, then the L_j shortest path, $C_j(t) = (r(t), \theta(t))$, from $x'_{1,j}$ to $x'_{2,j}$ must pass from one valley over to the other, possibly passing through many valleys in between. Observe that

$$(24) \quad d_j(x'_{1,j}, x'_{2,j}) = L_j(C_j) = L_j(C_j \cap f^{-1}(5)) + L_j(C_j \setminus f^{-1}(5)).$$

The segments of C_j which intersect $f_j^{-1}(5)$ lie in an product space warped by the constant function 5 so

$$(25) \quad L_j(C_j \cap f^{-1}(5)) = \sqrt{R_j^2 + 25\Theta_j^2},$$

where R_j is the sum of changes in r on these segments and where Θ_j is the sum of distances in Σ between the theta values of the endpoints of these segments.

Let $R_0 = |s_1 - s_2|$ which is the total change in r along C_j . By the definition of δ_j ,

$$2^j \delta_j = 2^j \left(\frac{1}{2}\right)^{2j} \rightarrow 0.$$

Since we have at most 2^j intervals where $f_j < 5$, we see that as

$$(26) \quad \lim_{j \rightarrow \infty} R_0 - R_j = 0,$$

the total change in r for the segments in $C_j \setminus f^{-1}(5)$ is converging to 0.

Let $\Theta_0 = d_{\Sigma}(\theta_1, \theta_2)$. Then $\Theta_0 - \Theta_j$ is the sum of distances in Σ between the theta values of the endpoints of the segments in $C_j \setminus f^{-1}(5)$. Since the warping factors $f_j(r) \geq 1$ everywhere, the distance between the endpoints of each segment is \geq distance in Σ between the theta values of the endpoints of the segment. Thus

$$(27) \quad L_j(C_j \setminus f^{-1}(5)) \geq \Theta_0 - \Theta_j.$$

Combining this together with (24) and (25) we have

$$(28) \quad \begin{aligned} d_j(x'_{1,j}, x'_{2,j}) &= L_j(C_j) \geq \sqrt{R_j^2 + 25\Theta_j^2} + \Theta_0 - \Theta_j \\ &\geq \inf_{\Theta \in [0, d_{\Sigma}(\theta_1, \theta_2)]} \sqrt{R_j^2 + 25\Theta^2} + \Theta_0 - \Theta. \end{aligned}$$

Since

$$(29) \quad \lim_{j \rightarrow \infty} \left(\inf_{\Theta \in [0, d_\Sigma(\theta_1, \theta_2)]} \sqrt{R_j^2 + 25\Theta^2} + \Theta_0 - \Theta \right) = \lim_{j \rightarrow \infty} d_{5\text{-ET}}(x'_{1,j}, x'_{2,j})$$

we are done by combining (28) and (29) which shows (23). \square

Remark 3.13. If we take the isometric product of Example 3.12 with a standard circle, $\bar{N}_j^3 = N_j^2 \times \mathbb{S}^1$, $\Sigma = \mathbb{S}^1$, then we have a sequence of 3-manifolds satisfying all the hypotheses of the scalar compactness conjecture of Gromov and Sormani (see [Gromov 2018]), recently proved in the rotationally symmetric case by Park, Tian, and Wang [Park et al. 2018],

$$\text{Vol}(\bar{N}_j) \leq 5 \text{Vol}(\mathbb{T}^3), \quad \text{Diam}(\bar{N}_j) \leq 5 \text{Diam}(\mathbb{T}^3), \quad \text{minA}(\bar{N}_j) \geq \text{minA}(\mathbb{T}^3),$$

except for the scalar curvature bound. Therefore, this example demonstrates that the conclusion of the scalar compactness conjecture, that the SWIF limit have Euclidean tangent cones almost everywhere, requires the scalar curvature bound. We note that the volume and diameter bound follow since $f_j \leq 5$ and the minA bound follows since $f_j \geq 1$.

4. Proof of the main theorem

The goal of this section is to prove our main theorem, Theorem 1.1.

In this theorem, $M_j = [r_0, r_1] \times_{f_j} \Sigma$, where Σ is an $n-1$ dimensional manifold including also M_j without boundary that have f_j periodic with period $r_1 - r_0$ as in (1). We assume that the warping factors, $f_j \in C^0([r_0, r_1])$, satisfy the following:

$$0 < f_\infty - \frac{1}{j} \leq f_j(r) \leq K \quad \text{and} \quad f_j(r) \rightarrow f_\infty(r) \quad \text{in } L^2,$$

where $f_\infty \in C^0([r_0, r_1])$.

The proof of Theorem 1.1 proceeds as follows. In Lemma 4.1 we use the C^0 lower bound to show that

$$\liminf_{j \rightarrow \infty} d_j(p, q) \geq d_\infty(p, q) \quad \text{pointwise.}$$

We use the L^2 convergence of $f_j \rightarrow f_\infty$ in Lemmas 4.3 and 4.6, combined with the estimate of Lemma 4.4, to show that the lengths of fixed curves with respect to M_j and M_∞ converge. We apply this result to a fixed geodesic with respect to g_∞ , to prove that

$$\limsup_{j \rightarrow \infty} d_j(p, q) \leq d_\infty(p, q) \quad \text{pointwise.}$$

Thus in Proposition 4.8 we have the pointwise limit

$$\lim_{j \rightarrow \infty} d_j(p, q) = d_\infty(p, q).$$

To complete the proof of uniform, GH and SWIF convergence using Theorem 2.4, as is done in the examples in Section 3, we need uniform bounds on d_j proven in Lemma 2.3.

4A. Assuming a C^0 lower bound. We have seen in Section 3 that in order to get Gromov–Hausdorff convergence to agree with L^2 convergence we will need a C^0 lower bound on f_j and so now we see the consequence of this assumption for the distance between points.

Lemma 4.1. *Let $p, q \in [r_0, r_1] \times \Sigma$ and assume that*

$$f_j(r) \geq f_\infty - \frac{1}{j} > 0, \quad \text{Diam}(M_j) \leq D.$$

Then

$$\liminf_{j \rightarrow \infty} d_j(p, q) \geq d_\infty(p, q)$$

and furthermore we find the uniform estimate

$$d_{g_j}(p, q) - d_{g_\infty}(p, q) \geq -\frac{\sqrt{2} \max_{[r_0, r_1]} \sqrt{f_\infty} D}{\min_{[r_0, r_1]} f_j(r) \sqrt{j}}.$$

Proof. Let $C_j(t) = (r_j(t), \theta_j(t))$ be the absolutely continuous curve in M_j , parametrized so that $|C_j|_{g_j} = 1$ a.e., realizing the distance between p and q . Then compute

$$\begin{aligned} (30) \quad d_{g_j}(p, q) &= \int_0^{L_j(C_j)} \sqrt{r_j(t)^2 + f_j(r_j(t))^2 |\theta_j'(t)|^2} dt \\ &\geq \int_0^{L_j(C_j)} \sqrt{r_j(t)^2 + (f_\infty(r_j(t)) - \frac{1}{j})^2 |\theta_j'(t)|^2} dt \\ &= \int_0^{L_j(C_j)} \left(r_j(t)^2 + f_\infty(r_j(t))^2 |\theta_j'(t)|^2 \right. \\ &\quad \left. - ((2/j) f_\infty(r_j(t)) |\theta_j'(t)|^2 - (1/j^2) |\theta_j'(t)|^2) \right)^{1/2} dt \end{aligned}$$

Now we use the inequality $\sqrt{|a-b|} \geq |\sqrt{a} - \sqrt{b}| \geq \sqrt{a} - \sqrt{b}$ in succession, employing the fact that the last integrand in (30) is positive and the square roots

that follow are of positive quantities by the assumptions of the lemma.

$$\begin{aligned}
& d_{g_j}(p, q) \\
& \geq \int_0^{L_j(C_j)} \left| \sqrt{r_j(t)^2 + f_\infty(r_j(t))^2 |\theta'_j(t)|^2} - \frac{1}{\sqrt{j}} |\theta'_j(t)| \sqrt{2f_\infty(r_j(t)) - \frac{1}{j}} \right| dt \\
& \geq \int_0^{L_j(C_j)} \sqrt{r_j(t)^2 + f_\infty(r_j(t))^2 |\theta'_j(t)|^2} dt \\
& \quad - \frac{1}{\sqrt{j}} \int_0^{L_j(C_j)} |\theta'_j(t)| \sqrt{2f_\infty(r_j(t)) - \frac{1}{j}} dt \\
& \geq L_{g_\infty}(C_j) - \frac{1}{\sqrt{j}} \int_0^{L_j(C_j)} |\theta'_j(t)| \sqrt{2f_\infty(r_j(t)) - \frac{1}{j}} dt
\end{aligned}$$

Now we notice that

$$\sqrt{f'_j(t)^2 + f_j(r_j(t))^2 |\theta'_j(t)|^2} = 1 \quad \text{a.e.} \quad \Rightarrow \quad |\theta'_j(t)| \leq \frac{1}{\min f_j} \quad \text{a.e.}$$

which allows us to compute

$$d_{g_j}(p, q) \geq d_{g_\infty}(p, q) - \frac{\sqrt{2} \max_{[r_0, r_1]} \sqrt{f_\infty} D}{\min_{[r_0, r_1]} f_j(r) \sqrt{j}},$$

where the diameter bound from the hypotheses is used to conclude that $L_j(C_j) \leq D$. The desired result follows by taking limits. \square

4B. L^2 convergence and convergence of lengths. In this section we would like to observe the consequence of L^2 convergence of $f_j \rightarrow f_\infty$ for convergence of lengths of curves and distances between points in M_j culminating in an estimate on the pointwise limsup of the distance functions (Proposition 4.7).

We start by proving we have uniform bounds on the diameter.

Lemma 4.2. *If $\|f_j - f_\infty\|_{L^2} \leq \delta_j$ and M_j are warped products as in (1) then*

$$(31) \quad \text{Diam}(M_j) \leq 2|r_1 - r_0| + (\|f_\infty\|_{C_0} + \delta_j / \sqrt{r_1 - r_0}) \text{Diam}(\Sigma)$$

Proof. Let $p, q \in M_j$. Recall that the distance between these points is the infimum over lengths of all curves. For any $r \in [r_0, r_1]$ we can take a first path from p radially to the level r , then a second path around that level r , and then a third path from that level to q . The first and third paths each have length $\leq |r_1 - r_0|$, and the middle path has length bounded above by the diameter of the level. Thus we have

$$\begin{aligned}
d_j(p, q) & \leq 2|r_1 - r_0| + f_j(r) \text{Diam}(\Sigma) \\
& \leq 2|r_1 - r_0| + (f_\infty(r) + |f_j(r) - f_\infty(r)|) \text{Diam}(\Sigma).
\end{aligned}$$

Choosing an r such that

$$|f_j(r) - f_\infty(r)|^2 \leq \frac{1}{r_1 - r_0} \int |f_j(s) - f_\infty(s)|^2 ds$$

we have

$$|f_j(r) - f_\infty(r)| \leq \frac{\|f_j - f_\infty\|_{L_2}}{\sqrt{r_1 - r_0}}$$

and $f_\infty(r) \leq \|f_\infty\|_{C_0}$. □

Recall that in warped product manifolds with continuous warping functions we have absolutely continuous curves whose length achieves the distance between two points (Remark 2.2).

We next consider the length of a fixed curve which is monotone in r .

Lemma 4.3. *Fix an absolutely continuous curve $C(t) = (r(t), \theta(t))$, $t \in [0, 1]$, which is monotone in r . If $\|f_j - f_\infty\|_{L^2} \leq \delta = \delta_j$ and M_j are warped products as in (1) then*

$$|L_j(C) - L_\infty(C)| \leq (\delta^2 + 4\|f_\infty\|_{L^2}^2)\delta^{1/2}\Theta(C)$$

where

$$(32) \quad \Theta(C) = \left(\int_{r(0)}^{r(1)} |\theta'(r)|^2 dr \right)^{1/2}.$$

Note also that

$$\|f_j + f_\infty\|_{L^2}^2 \leq (\delta + 2\|f_\infty\|_{L^2})^2.$$

If C is not monotone in r but one knows it has at most N monotone subsegments then we can sum up the segments applying this lemma to each subsegment.

Proof. Since $C(t) = (r(t), \theta(t))$ is such that $r'(t) > 0$ everywhere then we can reparametrize so that $r(t) = r$. Now by comparing two lengths and taking advantage of the inequality $\sqrt{|a-b|} \geq |\sqrt{a} - \sqrt{b}|$ we find

$$\begin{aligned} |L_j(C) - L_\infty(C)| &\leq \int_{r(0)}^{r(1)} \left| \sqrt{1 + f_j^2(r)\theta'(r)^2} - \sqrt{1 + f_\infty^2(r)\theta'(r)^2} \right| dr \\ &\leq \int_{r(0)}^{r(1)} \sqrt{|f_j^2(r) - f_\infty^2(r)|} |\theta'(r)| dr \\ &\leq \left(\int_{r(0)}^{r(1)} |f_j^2(r) - f_\infty^2(r)| dr \right)^{1/2} \left(\int_{r(0)}^{r(1)} |\theta'(r)|^2 dr \right)^{1/2}, \end{aligned}$$

where we used Holder's inequality in the last line.

Now we notice that

$$\begin{aligned} |f_j^2 - f_\infty^2| &= |f_j^2 - f_j f_\infty + f_j f_\infty - f_\infty^2| \\ &= |f_j(f_j - f_\infty) + f_\infty(f_j - f_\infty)| \\ &= |(f_j + f_\infty)(f_j - f_\infty)| = |f_j + f_\infty| |f_j - f_\infty|. \end{aligned}$$

Combining this with Hölder's Inequality we obtain

$$|L_j(C) - L_\infty(C)| \leq \left(\int_{r(0)}^{r(1)} |f_j + f_\infty|^2 dr \right)^{1/4} \left(\int_{r(0)}^{r(1)} |f_j - f_\infty|^2 dr \right)^{1/4} \Theta(C).$$

Lastly, we notice that

$$\begin{aligned} \|f_j + f_\infty\|_{L^2}^2 &= \|f_j - f_\infty + 2f_\infty\|_{L^2}^2 \\ &\leq (\|f_j - f_\infty\|_{L^2} + 2\|f_\infty\|_{L^2})^2 \leq (\delta + 2\|f_\infty\|_{L^2})^2 \end{aligned}$$

which gives us the desired uniform bound. \square

Now that we have obtained a bound on fixed geodesics which are monotone in r we would like to gain some control on the term $\Theta(C)$ from Lemma 4.3 in the case where C is a fixed geodesic with respect to the metric g_j . We note that we will use Lemma 4.4 only in the case where C is a fixed geodesic with respect to g_∞ which is monotone in r but we state it in more generality below since it could be useful for future results.

Lemma 4.4. *Let M_j be a warped product manifold as in (1). Let $C_j(t) = (r(t), \theta(t))$ be a unit speed absolutely continuous geodesic in M_j which is nondecreasing in r and define*

$$m_j = \min_{r \in [r_0, r_1]} f_j(r) > 0.$$

Then Θ of (32) satisfies:

$$\Theta(C_j) \leq \frac{\sqrt{n-1} L_j(C_j)^{1/2}}{m_j}.$$

Proof. We can estimate $\Theta(C_j)$ by rewriting the line integral which defines $\Theta(C_j)$:

$$\Theta(C_j) = \left(\int_{r(0)}^{r(1)} |\vec{\theta}'(r)|^2 dr \right)^{1/2} = \left(\int_0^{L_j(C_j)} |\vec{\theta}'(t)|^2 r'(t) dt \right)^{1/2}.$$

Now by the assumption that $|C_j'|_{g_j} = \sqrt{r'(t)^2 + f_j(r(t))^2 |\vec{\theta}'(t)|^2} = 1$ a.e. and $r'(t) > 0$ we find that $0 < r'(t) \leq 1$ which yields

$$\Theta(C_j) \leq \left(\int_0^{L_j(C_j)} |\vec{\theta}'(t)|^2 dt \right)^{1/2}.$$

Note that $|C'_j|_{g_j} = \sqrt{r'(t)^2 + f_j(r(t))^2 |\vec{\theta}'_j(t)|^2} = 1$ a.e. implies that $|\vec{\theta}'_j(t)| \leq 1/f_j$ a.e. which yields the estimate

$$\Theta(C_j) \leq \left(\int_0^{L_j(C_j)} \frac{1}{f_j(r(t))^2} dt \right)^{1/2} \leq \frac{L_j(C_j)^{1/2}}{m_j}. \quad \square$$

Corollary 4.5. *If the length minimizing absolutely continuous geodesic between $p, q \in M$ with respect to g_∞ is monotone in r and we let $\delta = \|f_j - f_\infty\|_{L^2}$ and $m_\infty = \min_{r \in [r_0, r_1]} f_\infty(r) > 0$ then we find the uniform estimate*

$$d_{g_j}(p, q) - d_{g_\infty}(p, q) \leq (\delta^2 + 4\|f_\infty\|_{L^2}^2) \delta^{1/2} \frac{\sqrt{n} \text{Diam}(M_\infty)}{m_\infty}.$$

Proof. We note that by the fact that C is the length minimizing geodesic between $p, q \in M$ with respect to g_∞ we find

$$d_{g_j}(p, q) - d_{g_\infty}(p, q) \leq L_j(C) - L_\infty(C).$$

Now if we combine Lemmas 4.2, 4.3 and 4.4 then we find

$$d_{g_j}(p, q) - d_{g_\infty}(p, q) \leq (\delta^2 + 4\|f_\infty\|_{L^2}^2) \delta^{1/2} \frac{\sqrt{n} \text{Diam}(M_\infty)}{m_\infty},$$

where $\delta = \|f_j - f_\infty\|_{L^2}$ and $m_\infty = \min_{r \in [r_0, r_1]} f_\infty(r) > 0$. \square

The uniform control of Corollary 4.5 will be used in the proof of Theorem 1.1 below. Now we would like to control the length of geodesics with respect to g_∞ which are constant in r .

Lemma 4.6. *Let $p, q \in [r_0, r_1] \times \Sigma$ and assume that the absolutely continuous geodesic C between p and q with respect to g_∞ is parametrized as $C = (\hat{r}, \theta(t))$, $t \in [0, 1]$, for some fixed $\hat{r} \in [r_0, r_1]$. If $f_j \rightarrow f_\infty$ in L^2 then*

$$\limsup_{j \rightarrow \infty} d_{g_j}(p, q) \leq d_{g_\infty}(p, q).$$

Moreover, we can find an approximating curve C_j^ϵ between p and q so that

$$L_j(C_j^\epsilon) \leq 4\delta_j^\epsilon + L_\infty(C) + \epsilon d_\sigma(\theta(0), \theta(1)),$$

where

$$\delta_j^\epsilon \leq \frac{\|f_j - f_\infty\|_{L^2}^2}{\epsilon^2}.$$

Proof. Since $f_j \rightarrow f_\infty$ in L^2 , if we define

$$S_\epsilon^j = \{x \in [r_0, r_1] : |f_j(x) - f_\infty(x)| \geq \epsilon\}$$

then we know that there exists a $\delta_j > 0$ such that $|S_\epsilon^j| \leq \delta_j$, where $\delta_j \rightarrow 0$ as $j \rightarrow \infty$. This follows since if $|S_\epsilon^j| \geq c > 0$ then

$$\int_{-\pi}^{\pi} |f_j - f_\infty|^2 dr \geq \int_{S_\epsilon^j} |f_j - f_\infty|^2 dr \geq c\epsilon^2$$

which leads to a contradiction. In fact,

$$\begin{aligned} \epsilon |S_j^\epsilon| &\leq \int_{S_j^\epsilon} |f_j - f_\infty| dr \leq |S_j^\epsilon|^{1/2} \left(\int_{S_j^\epsilon} |f_j - f_\infty|^2 dr \right)^{1/2} \\ &\leq |S_j^\epsilon|^{1/2} \left(\int_{-\pi}^{\pi} |f_j - f_\infty|^2 dr \right)^{1/2}, \end{aligned}$$

which implies

$$\delta_j \leq \frac{|f_j - f_\infty|_{L^2}^2}{\epsilon^2}.$$

This implies that we can choose an $r_j \in (\hat{r}, \hat{r} + 2\delta_j)$ or $r_j \in (\hat{r} - 2\delta_j, \hat{r})$ so that $|f_j(r_j) - f_\infty(r_j)| \leq \epsilon$ and so by combining with Lemmas 3.2 and 3.6 we find a curve C_j^ϵ between p and q such that

$$\begin{aligned} d_{g_j}(p, q) &\leq L_j(C_j^\epsilon) \\ &\leq 4\delta_j + f_j(r_j) d_\sigma(\theta(0), \theta(1)) \\ &\leq 4\delta_j + f_\infty(r_j) d_\sigma(\theta(0), \theta(1)) + |f_j(r_j) - f_\infty(r_j)| d_\sigma(\theta(0), \theta(1)). \end{aligned}$$

Now by taking limits as $j \rightarrow \infty$ and using that f_∞ is continuous we find

$$\limsup_{j \rightarrow \infty} d_{g_j}(p, q) \leq f_\infty(\hat{r}) d_\sigma(\theta(0), \theta(1)) + \epsilon d_\sigma(\theta(0), \theta(1)).$$

Since this is true for all $\epsilon > 0$ and $d_{g_\infty}(p, q) = f_\infty(\hat{r}) d_\sigma(\theta(0), \theta(1))$ the desired result follows. \square

We now combine these lemmas into:

Proposition 4.7. *If f_j and f_∞ are positive continuous functions, $f_j \rightarrow f_\infty$ in L^2 , and $M_j = M$ are warped products as in (1) then*

$$\limsup_{j \rightarrow \infty} d_j(p, q) \leq d_\infty(p, q) \quad \text{pointwise.}$$

Proof. Fix p and q in $M_j = M$. Let $C(t)$ be a minimizing curve between p and q with respect to g_∞ :

$$L_\infty(C) = d_\infty(p, q).$$

By Remark 2.2, C is an absolutely continuous curve. It can be broken down into possibly infinitely many segments, each of which is either monotone in r or has constant r component. Let $\mathcal{C} = \{C^\alpha : \alpha \in I\}$, where I is an indexing set, be the

segments which are constant in r with endpoints $(r^\alpha, \theta_1^\alpha), (r^\alpha, \theta_2^\alpha) \in [r_0, r_1] \times \Sigma$ then we can estimate

$$\begin{aligned} L_\infty(C) &\geq \sum_{\alpha \in I} L_\infty(C^\alpha) \\ &= \sum_{\alpha \in I} f_\infty(r^\alpha) d_\sigma(\theta_1^\alpha, \theta_2^\alpha) \geq (\min_{r \in [r_0, r_1]} f_\infty(r)) \sum_{\alpha \in I} d_\sigma(\theta_1^\alpha, \theta_2^\alpha), \end{aligned}$$

and hence

$$(33) \quad \sum_{\alpha \in I} d_\sigma(\theta_1^\alpha, \theta_2^\alpha) \leq \frac{\text{Diam}(M_\infty)}{(\min_{r \in [r_0, r_1]} f_\infty(r))} < \infty.$$

Similarly, if we let $\tilde{C} = \{\tilde{C}^\alpha : \alpha \in I\}$ be the collection of segments of C which are monotone in r , with endpoints $(r_1^\alpha, \theta_1^\alpha), (r_2^\alpha, \theta_2^\alpha) \in [r_0, r_1] \times \Sigma$, then

$$\begin{aligned} L_\infty(C) &\geq \sum_{\alpha \in I} L_\infty(\tilde{C}^\alpha) = \sum_{\alpha \in I} \int_{r_1^\alpha}^{r_2^\alpha} \sqrt{1 + f_\infty(r)^2 \theta'(r)^2} dr \\ &\geq \sum_{\alpha \in I} \int_{r_1^\alpha}^{r_2^\alpha} dr = \sum_{\alpha \in I} |r_1^\alpha - r_2^\alpha|, \end{aligned}$$

which implies

$$(34) \quad \sum_{\alpha \in I} |r_1^\alpha - r_2^\alpha| \leq \text{Diam}(M_\infty).$$

So, by combining (33), (34), and Lemma 3.2 we find for any $\eta > 0$, we can choose $I_\eta \subset I$, $I \setminus I_\eta = K \in \mathbb{N}$, so that

$$(35) \quad \begin{aligned} \sum_{\alpha \in I_\eta} L_\infty(\tilde{C}^\alpha) + \sum_{\alpha \in I_\eta} L_\infty(C^\alpha) \\ \leq \sum_{\alpha \in I_\eta} |r_1^\alpha - r_2^\alpha| + 2 \left(\max_{r \in [r_0, r_1]} f_\infty(r) \right) \sum_{\alpha \in I_\eta} d_\sigma(\theta_1^\alpha, \theta_2^\alpha) \leq \eta \end{aligned}$$

and hence by replacing all but finitely many subsegments of C with finitely many taxi minimizing curves whose g_∞ length is smaller than η we can obtain another curve \bar{C}^η such that

$$L_\infty(\bar{C}^\eta) \leq L_\infty(C) - 2\eta.$$

This can be done so that \bar{C}^η can be broken down into finitely many segments, each of which is either monotone in r or has constant r component. By Lemma 4.6, for each monotone segment \bar{C}^k , $k \in \mathbb{N}$, $k \leq K$ we can find an approximating curve, $\bar{C}_j^{k, \epsilon}$, such that

$$(36) \quad L_j(\bar{C}_j^{k, \epsilon}) \leq 4\delta_j^\epsilon + L_\infty(\bar{C}^k) + \epsilon d_\sigma(\theta_1^k, \theta_2^k),$$

where $\delta_j^\epsilon \leq |f_j - f_\infty|_{L^2}^2 / \epsilon^2$.

Then by Lemmas 4.3, 4.4 and 4.6 we can find a curve $\bar{C}_j^{\eta, \epsilon}$, $\epsilon > 0$ between p and q , by possibly adjusting the monotone segments as in (36), such that

$$(37) \quad \limsup_{j \rightarrow \infty} L_j(\bar{C}_j^{\eta, \epsilon}) \leq L_\infty(C) - 2\eta + \epsilon \frac{\text{Diam}(M_\infty)}{(\min_{r \in [r_0, r_1]} f_\infty(r))}.$$

Since (37) is true for all η , $d_j(p, q) \leq L_j(\bar{C}_j^{\eta, \epsilon})$ and $L_\infty(C) = d_\infty(p, q)$ we have

$$\limsup_{j \rightarrow \infty} d_j(p, q) \leq d_\infty(p, q) + \epsilon \frac{\text{Diam}(M_\infty)}{(\min_{r \in [r_0, r_1]} f_\infty(r))},$$

which is true for all $\epsilon > 0$ and hence the desired result follows. \square

4C. Proof of Theorem 1.1. Recall that in the statement of Theorem 1.1 we have a sequence of warping functions $f_j(r) \geq f_\infty(r) - \frac{1}{j}$ and $f_j(r) \rightarrow f_\infty(r)$ in L^2 . We will prove:

$$\lim_{j \rightarrow \infty} d_j(p, q) = d_\infty(p, q)$$

uniformly by first showing it converges pointwise on a subsequence and then applying Theorem 2.4 which implies uniform convergence, GH and \mathcal{F} convergence to the same space.

Proposition 4.8. *Under the hypothesis of Theorem 1.1 we have pointwise convergence of the distance functions:*

$$\lim_{j \rightarrow \infty} d_j(p, q) = d_\infty(p, q).$$

Proof. Let $p, q \in [r_0, r_1] \times \Sigma$. Applying the C^0 lower bound and Lemma 4.1 we have

$$\liminf_{j \rightarrow \infty} d_j(p, q) \geq d_\infty(p, q).$$

Applying the L^2 upper bound and Proposition 4.7 we also have

$$\limsup_{j \rightarrow \infty} d_j(p, q) \leq d_\infty(p, q).$$

Thus we have pointwise convergence. \square

Proof of Theorem 1.1. By the assumption that $0 < c \leq f_\infty - \frac{1}{j} \leq f_j \leq K$ we can use Lemma 2.3 and choose $\lambda = \max(1/\min(c, 1), \max(1, K)) > 0$ so that for j large enough we find

$$\lambda \geq \frac{d_j(p, q)}{d_1(p, q)} \geq \frac{1}{\lambda},$$

where d_1 is the distance defined with warping factor 1.

Now can apply Theorem 2.4 to conclude that there exists a length metric d'_∞ and a subsequence d_{j_k} such that d_{j_k} converges uniformly to d'_∞ , and hence GH and SWIF converges as well. By the pointwise convergence proven in Proposition 4.8, we know that $d'_\infty = d_\infty$ and hence d_{j_k} must uniformly converge to d_∞ . Since this is true for all the subsequences, we see that d_j uniformly converges to d_∞ . Appealing again to Theorem 2.4 we see it converges in the Gromov–Hausdorff and intrinsic flat sense as well. \square

5. Warping functions with two variables on tori

In this section we give a short exploration of more general warped product manifolds. There are a wealth of new directions one might explore and this section demonstrates how some of our techniques do extend easily. Here we prove the following theorem:

Theorem 5.1. *Let $g_j = dx^2 + dy^2 + f_j(x, y)^2 dz^2$ be a metric on a torus $M_j = \mathbb{S}^1 \times \mathbb{S}^1 \times_{f_j} \mathbb{S}^1$ with coordinates $(x, y, z) \in [-\pi, \pi]^3$, $f_j \in C^0([-\pi, \pi]^2)$. Assume that*

$$f_j \rightarrow f_\infty = c > 0 \quad \text{in } L^2, \quad 0 < f_\infty - \frac{1}{j} \leq f_j \leq K < \infty.$$

Then M_j converges uniformly to M_∞ as well as

$$M_j \xrightarrow{\text{GH}} M_\infty, \quad M_j \xrightarrow{\mathcal{F}} M_\infty.$$

This theorem will be applied in upcoming joint work of a team of doctoral students who are working with the first author: Lisandra Hernandez-Vazquez, Davide Parise, Alec Payne, and Shengwen Wang. Various members of this team which first began working together at the Fields Institute in the Summer of 2017 will explore further theorems in this direction using similar techniques.

The proof of this theorem will be similar to the proof of Theorem 1.1, however we have some additional difficulties arising. The main difficulty is that $f_j \rightarrow f_\infty$ in $L^2([-\pi, \pi]^2)$ does not imply that $f_j \rightarrow f_\infty$ on curves and hence we will not be able to prove the corresponding results to Lemmas 4.3 and 4.4 for this setting. Instead in Lemmas 5.4, 5.5, and 5.6 we will build approximating sequences of curves to a geodesic with respect to g_∞ and show $\limsup_{j \rightarrow \infty} d_j(p, q) \leq d_\infty(p, q)$. The C^0 control on f_j works similarly to Section 4 and hence we are able to show $\liminf_{j \rightarrow \infty} d_j(p, q) \geq d_\infty(p, q)$ in Lemma 5.2. This will imply pointwise convergence of distances which when combined with Theorem 2.4 will show uniform, GH and SWIF convergence, similar to the examples in Section 3.

5A. A lower C^0 bound. We now prove a lemma which shows the consequence of a C^0 lower bound which we have seen is important by the examples in Section 3.

Lemma 5.2. *Let $p, q \in M_j$ and assume that*

$$f_j(x, y) \geq f_\infty(x, y) - \frac{1}{j} > 0 \quad \text{and} \quad \text{Diam}(M_j) \leq D.$$

Then

$$\liminf_{j \rightarrow \infty} d_j(p, q) \geq d_\infty(p, q)$$

Proof. Let $C_j(t) = (x_j(t), y_j(t), z_j(t))$ be the minimizing absolutely continuous geodesic in M_j , parametrized so that $|C'_j(t)|_{g_j} = 1$ a.e., realizing the distance between p and q . Then compute

$$\begin{aligned} g_j(C'_j(t), C'_j(t)) &= x'_j(t)^2 + y'_j(t)^2 + f_j(x_j(t), y_j(t))^2 |z'_j(t)|^2 \\ &\geq x'_j(t)^2 + y'_j(t)^2 + (f_\infty(x_j(t), y_j(t)) - \frac{1}{j})^2 |z'_j(t)|^2 \\ &= x'_j(t)^2 + y'_j(t)^2 + f_\infty(x_j(t), y_j(t))^2 |z'_j(t)|^2 \\ &\quad - ((2/j) f_\infty(x_j(t), y_j(t)) |z'_j(t)|^2 - (1/j^2) |z'_j(t)|^2). \end{aligned}$$

Note that the terms here are positive by the assumptions of the lemma, so that when we take the square root we can apply the inequality

$$\sqrt{|a-b|} \geq |\sqrt{a} - \sqrt{b}| \geq \sqrt{a} - \sqrt{b},$$

before integrating to obtain

$$\begin{aligned} d_{g_j}(p, q) &= \int_0^{L_j(C_j)} \sqrt{g_j(C'_j(t), C'_j(t))} dt \\ &\geq \int_0^{L_j(C_j)} \sqrt{x'_j(t)^2 + y'_j(t)^2 + f_\infty(x_j(t), y_j(t))^2 |z'_j(t)|^2} dt \\ &\quad - \int_0^{L_j(C_j)} \sqrt{(2/j) f_\infty(x_j(t), y_j(t)) |z'_j(t)|^2 - (1/j^2) |z'_j(t)|^2} dt \\ &\geq L_{g_\infty}(C_j) - \frac{1}{\sqrt{j}} \int_0^{L_j(C_j)} |z'_j(t)| \sqrt{(2 f_\infty(x_j(t), y_j(t)) - \frac{1}{j})} dt \end{aligned}$$

Now we notice that

$$\begin{aligned} |C'_j(t)|_{g_j} &= \sqrt{x'_j(t)^2 + y'_j(t)^2 + f_j(x_j(t), y_j(t))^2 |z'_j(t)|^2} = 1 \text{ a.e.} \\ &\Rightarrow |z'_j(t)| \leq \frac{1}{f_j(x_j(t), y_j(t))} \text{ a.e.} \end{aligned}$$

and hence we can then conclude that

$$d_{g_j}(p, q) \geq d_{g_\infty}(p, q) - \frac{\sqrt{2} \max_{[-\pi, \pi]^2} \sqrt{f_\infty} D}{\min_{[-\pi, \pi]^2} f_j \sqrt{j}}.$$

The desired result follows by taking limits. \square

We now prove that we have uniform bounds on the diameter which was used in Lemma 5.2:

Lemma 5.3. *If $\|f_j - f_\infty\|_{L^2} \leq \delta_j$ and M_j are warped products as in Theorem 5.1 then*

$$\text{Diam}(M_j) \leq 4\sqrt{2}\pi + 2\pi(\|f_\infty\|_{C_0} + \delta_j/(2\pi)).$$

Proof. Let $p, q \in M_j$ with $p = (x_1, y_1, z_1)$ and $q = (x_2, y_2, z_2)$. Recall that the distance between these points is the infimum over lengths of all curves. For any $(x_0, y_0) \in [-\pi, \pi]^2$ we can take a first path from p to (x_0, y_0, z_1) which stays in a plane parallel to the xy -plane, then a second path from (x_0, y_0, z_1) to (x_0, y_0, z_2) parallel to the z axis, and then a third path from (x_0, y_0, z_2) to (x_2, y_2, z_2) which stays in a plane parallel to the xy -plane. The first and third paths each have length $\leq 2\sqrt{2}\pi$, and the middle path has length bounded above by 2π with respect to the flat metric. Thus we have

$$\begin{aligned} d_j(p, q) &\leq 4\sqrt{2}\pi + 2\pi f_j(x_0, y_0) \\ &\leq 4\sqrt{2}\pi + 2\pi(f_\infty(x_0, y_0) + |f_j(x_0, y_0) - f_\infty(x_0, y_0)|). \end{aligned}$$

Choosing an (x_0, y_0) such that

$$|f_j(x_0, y_0) - f_\infty(x_0, y_0)|^2 \leq \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f_j(x, y) - f_\infty(x, y)|^2 dx dy$$

we have

$$|f_j(x_0, y_0) - f_\infty(x_0, y_0)| \leq \frac{\|f_j - f_\infty\|_{L^2}}{2\pi}$$

and $f_\infty(x_0, y_0) \leq \|f_\infty\|_{C_0}$. □

5B. L^2 convergence and convergence of distances. In this section we will build sequences of curves whose length approximates the length of a fixed geodesic with respect to g_∞ whose warping function is a constant.

We start by approximating a geodesic which has constant z component which is simple since g_j agrees with g_∞ in the x and y directions.

Lemma 5.4. *Let $p, q \in [-\pi, \pi]^3$ so that $p = (x_1, y_1, z_0)$ and $q = (x_2, y_2, z_0)$. If $f_\infty = c > 0$ then we have that*

$$\limsup_{j \rightarrow \infty} d_j(p, q) \leq d_\infty(p, q).$$

Proof. Let γ be a minimal geodesic with respect to g_∞ from p to q . Since g_∞ is a Euclidean metric it is a straight line segment:

$$\gamma(t) = (x_1(1-t) + x_2t, y_1(1-t) + y_2t, z_0).$$

Note that we can choose coordinate so that this is the minimal geodesic with respect to g_∞ . Then we can compute

$$d_j(p, q) \leq L_j(\gamma) = \int_0^1 \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} dt = d_\infty(p, q),$$

since g_j agrees with g_∞ in the x and y directions, by which the result follows by taking limits. \square

We now construct a sequence of curves which approximates a fixed geodesic with respect to g_∞ which is constant in x and y .

Lemma 5.5. *Assume that $f_j \rightarrow f_\infty = c > 0$ in L^2 and let $p, q \in [-\pi, \pi]^3$ so that $p = (x_0, y_0, z_1)$ and $q = (x_0, y_0, z_2)$ then we have that*

$$\limsup_{j \rightarrow \infty} d_j(p, q) \leq d_\infty(p, q).$$

Proof. We claim that if

$$S_\epsilon^j = \{(x, y) \in [-\pi, \pi]^2 : |f_j(x, y) - f_\infty(x, y)| \geq \epsilon\}$$

then we must have that $|S_\epsilon^j| \leq \delta_j$, where $\delta_j \rightarrow 0$ as $j \rightarrow \infty$ ($|S|$ represents Lebesgue measure of $S \subset [-\pi, \pi]^2$ with respect to the Euclidean metric). If the claim were false then $|S_\epsilon^j| \geq C > 0$ and

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f_j(x, y) - f_\infty(x, y)|^2 dx dy \geq \int_{S_\epsilon^j} |f_j(x, y) - f_\infty(x, y)|^2 dA \geq C\epsilon^2$$

which contradicts $f_j \rightarrow f_\infty$ in L^2 .

Define the set

$$T_\epsilon^j = (B((x_0, y_0), 4\sqrt{\delta_j}) \setminus S_\epsilon^j) \cap [-\pi, \pi]^2.$$

Since eventually

$$\frac{|B((x_0, y_0), 4\sqrt{\delta_j})|}{4} = 4\pi\delta_j > |S_\epsilon^j|,$$

we see that T_ϵ^j is nonempty. Hence we can choose a $(x_\epsilon^j, y_\epsilon^j) \in T_\epsilon^j$.

A minimal geodesic γ from $p = (x_0, y_0, z_1)$ to $q = (x_0, y_0, z_2)$ with respect to g_∞ is purely vertical:

$$\gamma(t) = (x_0, y_0, z_0(1-t) + z_2t),$$

where the addition is mod 2π . Note that $d_\infty(p, q) = c|z_2 - z_1|$. Let

$$p' = (x_\epsilon^j, y_\epsilon^j, z_1) \quad \text{and} \quad q' = (x_\epsilon^j, y_\epsilon^j, z_2).$$

So $d_\infty(p, p') < 4\sqrt{\delta_j}$ and $d_\infty(q, q') < 4\sqrt{\delta_j}$. Also

$$d_\infty(p, q) = c|z_2 - z_1| = d_\infty(p', q').$$

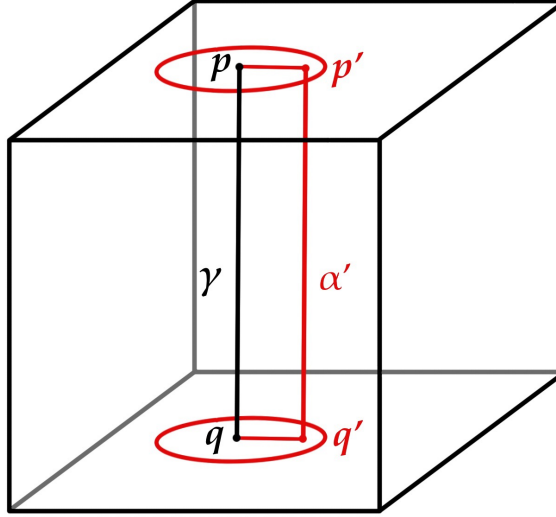


Figure 6. $\alpha' = \alpha_{x_j}^j$ approximates the curve γ between the points p and q .

We can define a curve α_ϵ^j as in Figure 6 which approximates γ . This curve runs minimally with respect to g_∞ from p to p' and then minimally to q' and then minimally to q as follows:

$$\alpha_\epsilon^j(t) = \begin{cases} (x_0(1-3t) + 3x_\epsilon^j t, y_0(1-3t) + 3y_\epsilon^j t, z_1), & 0 \leq t \leq \frac{1}{3}, \\ (x_\epsilon^j, y_\epsilon^j, z_1(2-3t) + z_2(3t-1)), & \frac{1}{3} \leq t \leq \frac{2}{3}, \\ (x_\epsilon^j(3-3t) + x_0(3t-2), y_\epsilon^j(3-3t) + y_0(3t-2), z_2), & \frac{2}{3} \leq t \leq 1, \end{cases}$$

where the addition here is mod 2π .

Now we can compute

$$\begin{aligned} d_j(p, q) &\leq L_j(\alpha_\epsilon^j) \\ &= \int_0^{1/3} \sqrt{|3x_\epsilon^j - 3x_0|^2 + |3y_\epsilon^j - 3y_0|^2} dt + \int_{1/3}^{2/3} |3z_2 - 3z_1| f_j(x_\epsilon^j, y_\epsilon^j) dt \\ &\quad + \int_{2/3}^1 \sqrt{|3x_\epsilon^j - 3x_0|^2 + |3y_\epsilon^j - 3y_0|^2} dt. \end{aligned}$$

Combining this with the definitions of $(x_\epsilon^j, y_\epsilon^j) \in T_\epsilon^j$ and using the continuity of f_∞ we find

$$\begin{aligned} d_j(p, q) &= 2\sqrt{|x_0 - x_\epsilon^j|^2 + |y_0 - y_\epsilon^j|^2} + f_j(x_\epsilon^j, y_\epsilon^j)|z_2 - z_1| \\ &\leq 16\sqrt{\delta_j} + |f_j(x_\epsilon^j, y_\epsilon^j) - f_\infty(x_\epsilon^j, y_\epsilon^j)||z_2 - z_1| + f_\infty(x_\epsilon^j, y_\epsilon^j)|z_2 - z_1| \\ &\leq 16\sqrt{\delta_j} + \epsilon|z_2 - z_1| + c|z_2 - z_1|, \end{aligned}$$

where we are using the hypothesis that $f_\infty = c > 0$.

Now by noticing that $d_\infty(p, q) = c|z_2 - z_1|$ and taking the limit as $j \rightarrow \infty$ we find

$$\limsup_{j \rightarrow \infty} d_j(p, q) \leq \epsilon|z_1 - z_0| + c|z_1 - z_0| = \epsilon|z_1 - z_0| + d_\infty(p, q)$$

and since this is true for all $\epsilon > 0$ the result follows. \square

We now construct a sequence of curves which approximates a fixed geodesic with respect to g_∞ , which does not fall under the hypotheses of Lemmas 5.4 or 5.5.

Lemma 5.6. *Assume that $f_j \rightarrow f_\infty = c > 0$ in L^2 and let $p, q \in [-\pi, \pi]^3$ so that $p = (x_1, y_1, z_1)$, $q = (x_2, y_2, z_2)$ and $(x_1, y_1) \neq (x_2, y_2)$ then*

$$\limsup_{j \rightarrow \infty} d_j(p, q) \leq d_\infty(p, q).$$

Proof. Without loss of generality we may assume that $y_1 \neq y_2$. Let γ be the geodesic with respect to g_∞ which runs from p to q . Since g_∞ is a Euclidean metric, we can choose coordinates on $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ such that

$$\gamma(t) = (\alpha(t), z_1(1-t) + z_2t),$$

where the addition is mod 2π and

$$\alpha(t) = (x_1(1-t) + x_2t, y_1(1-t) + y_2t) \subset [-\pi, \pi]^2.$$

Since $g_\infty = dx^2 + dy^2 + c^2 dz^2$, we have

$$d_\infty(p, q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

We construct a family of geodesics parallel to this geodesic running from $p' = (x'_1, y_1, z_1)$ to $q' = (x_2 + x'_1 - x_1, y_2, z_2)$ where $x'_1 \in B(x_1, 1) \subset [-\pi, \pi]$, as follows:

$$\gamma_{x'_1}(t) = (\alpha'_{x'_1}(t), z_1(1-t) + z_2t),$$

where

$$\alpha'_{x'_1}(t) = (x'_1(1-t) + (x'_1 + x_2 - x_1)t, y_1(1-t) + y_2t),$$

where the addition is mod 2π with values in $[-\pi, \pi]$. Observe that $\alpha: (x', t) \rightarrow (x, y)$ defined by $\alpha(x', t) = \alpha_{x'}(t)$ is

$$\alpha(x', t) = (x' + (x_2 - x_1)t, y_1 + (y_2 - y_1)t)$$

so

$$(38) \quad dx \wedge dy = (dx' + (x_2 - x_1) dt) \wedge (dy' + (y_2 - y_1) dt) = (y_2 - y_1) dx' \wedge dt.$$

Since $f_j \rightarrow f_\infty$ in L^2 we define

$$(39) \quad \bar{f}_j(x') = \int_{\alpha_{x'}} |f_j - f_\infty|^2 dt.$$

We define the set

$$S_\epsilon^j = \{x' \in [-\pi, \pi) : \bar{f}_j(x') \geq \epsilon\} \subset [-\pi, \pi),$$

and the set

$$W = \{\alpha_{x'}(t) : x' \in [-\pi, \pi) \text{ and } t \in [0, 1]\}.$$

By the definition of the line segments, $\alpha_{x'_1}$, we have $W \subset (-\pi, \pi]^2$.

Note that the set

$$T_\epsilon^j = (B(x_1, 4\delta_j) \setminus S_\epsilon^j) \subset [-\pi, \pi]$$

is nonempty where $\delta_j = |S_\epsilon^j|$. We claim $\delta_j \rightarrow 0$ as $j \rightarrow \infty$. Indeed we have

$$\epsilon |S_\epsilon^j| \leq \int_{x' \in S_\epsilon^j} \bar{f}_j(x') dx' \leq \int_{x'=-\pi}^{\pi} \bar{f}_j(x') dx' = \int_{x'=-\pi}^{\pi} \int_{\alpha_{x'}} |f_j - f_\infty|^2 dt dx'.$$

Applying a change of variables as in (38), we have

$$\begin{aligned} \delta_j &= (\epsilon)^{-1} \int \int_W |f_j - f_\infty|^2 |y_2 - y_1|^{-1} dy dx' \\ &\leq (\epsilon)^{-1} |y_2 - y_1|^{-1} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f_j - f_\infty|^2 dy dx, \end{aligned}$$

which converges to 0 by the hypothesis that $f_j \rightarrow f_\infty$ in L^2 .

Since T_ϵ^j is nonempty, we can pick a $x_j \in T_\epsilon^j$. We use this point to choose

$$(40) \quad p' = p'_j = (x_\epsilon^j, y_\epsilon^j, z_1) \quad \text{and} \quad q' = q'_j = (x_\epsilon^j, y_\epsilon^j, z_2).$$

We can define a sequence of curves $\beta_{x_j}^j$ as in Figure 7 which run minimally with respect to g_∞ from p to p' and then minimally to q' and then minimally to q as follows:

$$\beta_{x_j}^j(t) = \begin{cases} (x_1(1-3t) + 3x_j t, y_1, z_1), & 0 \leq t \leq \frac{1}{3}, \\ \gamma_{x_j}(3t-1), & \frac{1}{3} \leq t \leq \frac{2}{3}, \\ ((x_j + x_2 - x_1)(3-3t) + x_2(3t-2), y_2, z_2), & \frac{2}{3} \leq t \leq 1. \end{cases}$$

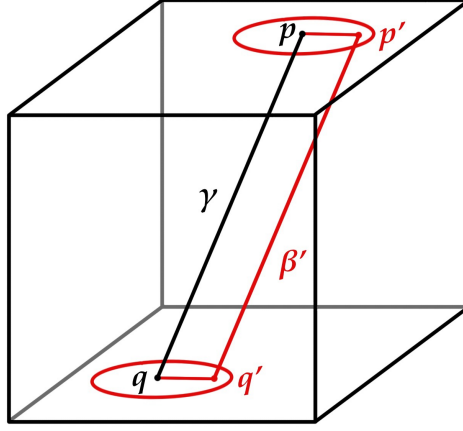


Figure 7. $\beta' = \beta_{x_j}^j$ approximates the curve γ between the points p and q .

The sequence of curves $\beta_{x_j}^j(t)$ is the approximating sequence to γ which can be used to estimate $d_j(p, q)$ as follows

$$\begin{aligned} d_j(p, q) &\leq L_j(\beta_{x_j}) \\ &= \int_0^{1/3} |3x_j - 3x_1| dt' + \int_{1/3}^{2/3} \sqrt{|3\Delta x|^2 + |3\Delta y|^2 + |3\Delta z|^2 f_j^2(\alpha_{x_j}(3t' - 1))} dt' \\ &\quad + \int_{2/3}^1 \sqrt{|3x_2 - 3(x_j + x_2 - x_1)|^2} dt', \end{aligned}$$

where $\Delta x = |x_2 - x_1|$, $\Delta y = |y_2 - y_1|$, and $\Delta z = |z_2 - z_1|$. Integrating the first and last term, and taking $t = 3t' - 1$ we have

$$\begin{aligned} d_j(p, q) &\leq \left(\frac{1}{3} - 0\right)|3x_j - 3x_1| + \left(1 - \frac{2}{3}\right)\sqrt{|3x_2 - 3x_j - 3x_2 + 3x_1|^2} \\ &\quad + \int_0^1 \sqrt{|\Delta x|^2 + |\Delta y|^2 + |\Delta z|^2 f_j^2(\alpha_{x_j}(t))} dt \\ &\leq |x_j - x_1| + |x_j - x_1| + \int_0^1 \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2 f_j^2(\alpha_{x_j}(t))} dt \\ &\leq 2|x_j - x_1| + \int_0^1 \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2 f_\infty^2 + \Delta z^2 (f_j^2(\alpha_{x_j}(t)) - f_\infty^2)} dt \\ &\leq 4\delta_j + \int_0^1 \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2 f_\infty^2} dt + \int_0^1 \Delta z \sqrt{f_j^2(\alpha_{x_j}(t)) - f_\infty^2} dt. \end{aligned}$$

Since g_∞ is Euclidean, the middle term is $d_\infty(p, q)$. Applying Hölder's inequality to the last term of yields

$$(41) \quad d_j(p, q) \leq 4\delta_j + d_\infty(p, q) + \Delta z \left(\int_0^1 |f_j^2(\alpha_{x_j}(t)) - f_\infty^2| dt \right)^{1/2}.$$

Recall that we chose $x_j \in T_\epsilon^j$ near x so that $x_j \notin S_\epsilon^j$. Thus (39) implies that

$$\int_{\alpha_{x_j}} |f_j - f_\infty|^2 dt \int_0^1 |f_j(\alpha_{x_j}(t)) - f_\infty|^2 dt = \bar{f}_j(x_j) < \epsilon.$$

We can apply this to control the final term in (41) by factoring and the applying Hölder's inequality and the triangle inequality

$$\begin{aligned} \left(\int_{\alpha_{x_j}} |f_j^2 - f_\infty^2| dt \right)^{1/2} &\leq \left(\int_{\alpha_{x_j}} |f_j - f_\infty| |f_j + f_\infty| dt \right)^{1/2} \\ &\leq \left(\int_{\alpha_{x_j}} |f_j - f_\infty|^2 dt \right)^{1/4} \left(\int_{\alpha_{x_j}} |f_j + f_\infty|^2 dt \right)^{1/4} \\ &\leq \epsilon^{1/4} \left(\int_{\alpha_{x_j}} |f_j - f_\infty + 2f_\infty|^2 dt \right)^{1/4} \\ &\leq \epsilon^{1/4} \left(\int_{\alpha_{x_j}} (|f_j - f_\infty| + 2|f_\infty|)^2 dt \right)^{1/4} \\ &= \epsilon^{1/4} \left(\int_{\alpha_{x_j}} |f_j - f_\infty|^2 + 4|f_j - f_\infty||f_\infty| + 4|f_\infty|^2 dt \right)^{1/4} \\ &\leq \epsilon^{1/4} \left(\epsilon + 4c \int_{\alpha_{x_j}} |f_j - f_\infty| dt + 4c^2 \right)^{1/4} \\ &\leq \epsilon^{1/4} \left(\epsilon + 4c \left(\int_{\alpha_{x_j}} |f_j - f_\infty|^2 dt \right)^{1/2} + 4c^2 \right)^{1/4} \\ &\leq \epsilon^{1/4} \left(\epsilon + 4c \epsilon^{1/2} + 4c^2 \right)^{1/4}. \end{aligned}$$

Substituting this into (41) we have

$$d_j(p, q) \leq 4\delta_j + d_\infty(p, q) + \Delta z \epsilon^{1/4} (\epsilon + 4c \epsilon^{1/2} + 4c^2)^{1/4}.$$

Now by taking limits as $j \rightarrow \infty$ we find

$$\limsup_{j \rightarrow \infty} d_j(p, q) \leq d_\infty(p, q) + \Delta z \epsilon^{1/4} (\epsilon + 4c \epsilon^{1/2} + 4c^2)^{1/4}.$$

Since this is true for all $\epsilon > 0$ the lemma follows. \square

5C. Proof of Theorem 5.1. In this section we finish the proof of Theorem 5.1, which follows by the results of the last two subsections combined with Theorem 2.4.

Proof. Let $p, q \in [-\pi, \pi]^3$. Then by Lemma 4.1 we have

$$(42) \quad \liminf_{j \rightarrow \infty} d_j(p, q) \geq d_\infty(p, q).$$

By Lemmas 5.4, 5.5 or 5.6 we have

$$\limsup_{j \rightarrow \infty} d_j(p, q) \leq d_\infty(p, q).$$

So by combining with (42) we conclude

$$(43) \quad \lim_{j \rightarrow \infty} d_j(p, q) = d_\infty(p, q),$$

which gives pointwise convergence of distances.

Now by the assumption that $0 < c - \frac{1}{j} \leq f_j \leq K$ we can apply Lemma 2.3 and choose $\lambda = \max(1/\min(c/2, 1), \max(1, K)) > 0$ so that for j chosen large enough we find

$$\lambda \geq \frac{d_j(p, q)}{d_1(p, q)} \geq \frac{1}{\lambda},$$

where d_1 is the distance defined with warping factor 1.

Hence we can apply Theorem 2.4 to conclude that there exists a length metric d'_∞ and a subsequence d_{j_k} such that d_{j_k} converges uniformly to d'_∞ , and GH and SWIF converges as well. By the pointwise convergence (43) we know that $d_\infty = d'_\infty$ and hence d_{j_k} must uniformly converge to d_∞ . Since this is true for all the subsequences, we see that d_j uniformly converges to d_∞ and hence Gromov–Hausdorff and intrinsic flat converges as well. \square

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References

- [Alexander and Bishop 2004] S. B. Alexander and R. L. Bishop, “Curvature bounds for warped products of metric spaces”, *Geom. Funct. Anal.* **14**:6 (2004), 1143–1181. MR Zbl
- [Allen 2018a] B. Allen, “IMCF and the stability of the PMT and RPI under L^2 convergence”, *Ann. Henri Poincaré* **19**:4 (2018), 1283–1306. MR Zbl
- [Allen 2018b] B. Allen, “Stability of the PMT and RPI for asymptotically hyperbolic manifolds foliated by IMCF”, *J. Math. Phys.* **59**:8 (2018), art. id. 082501. MR Zbl
- [Allen et al. 2019] B. Allen, L. Hernandez-Vazquez, D. Parise, A. Payne, and S. Wang, “Warped tori with almost non-negative scalar curvature”, *Geom. Dedicata* **200** (2019), 153–171. MR Zbl
- [Burago et al. 2001] D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*, Graduate Studies in Math. **33**, Amer. Math. Soc., Providence, RI, 2001. MR Zbl
- [Burtscher 2015] A. Y. Burtscher, “Length structures on manifolds with continuous Riemannian metrics”, *New York J. Math.* **21** (2015), 273–296. MR Zbl
- [Gromov 1981] M. Gromov, *Structures métriques pour les variétés riemanniennes*, Textes Math. **1**, CEDIC, Paris, 1981. MR Zbl
- [Gromov 2018] M. Gromov, “Scalar curvature of manifolds with boundaries: natural questions and artificial constructions”, 2018. Notes from the workshop *Emerging topics: scalar curvature and convergence* (IAS Princeton, 2018). arXiv
- [Huang et al. 2017] L.-H. Huang, D. A. Lee, and C. Sormani, “Intrinsic flat stability of the positive mass theorem for graphical hypersurfaces of Euclidean space”, *J. Reine Angew. Math.* **727** (2017), 269–299. MR Zbl
- [Lakzian 2016] S. Lakzian, “On diameter controls and smooth convergence away from singularities”, *Differential Geom. Appl.* **47** (2016), 99–129. MR Zbl
- [Lakzian and Sormani 2013] S. Lakzian and C. Sormani, “Smooth convergence away from singular sets”, *Comm. Anal. Geom.* **21**:1 (2013), 39–104. MR Zbl
- [Lee and Sormani 2014] D. A. Lee and C. Sormani, “Stability of the positive mass theorem for rotationally symmetric Riemannian manifolds”, *J. Reine Angew. Math.* **686** (2014), 187–220. MR Zbl
- [Park et al. 2018] J. Park, W. Tian, and C. Wang, “A compactness theorem for rotationally symmetric Riemannian manifolds with positive scalar curvature”, *Pure Appl. Math. Q.* **14**:3-4 (2018), 529–561.
- [Sormani 2017] C. Sormani, “Scalar curvature and intrinsic flat convergence”, pp. 288–338 in *Measure theory in non-smooth spaces*, edited by N. Gigli, De Gruyter, Warsaw, 2017. MR
- [Sormani and Wenger 2011] C. Sormani and S. Wenger, “The intrinsic flat distance between Riemannian manifolds and other integral current spaces”, *J. Differential Geom.* **87**:1 (2011), 117–199. MR Zbl
- [Wenger 2011] S. Wenger, “Compactness for manifolds and integral currents with bounded diameter and volume”, *Calc. Var. Partial Differential Equations* **40**:3-4 (2011), 423–448. MR Zbl

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
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