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DEGREE-ONE, MONOTONE SELF-MAPS OF THE PONTRYAGIN SURFACE ARE NEAR-HOMEOMORPHISMS

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We prove that a self-map of the closed Pontryagin surface can be approximated by homeomorphisms if and only if it is monotone and has degree ±1. This adds to a body of theorems, each of which characterizes for some space or class of spaces those self-maps which are approximable by homeomorphisms.

1. Introduction

Given a topological space X, one can ask, "Which surjective self-maps of X are near-homeomorphisms (i.e., approximable by homeomorphisms)?" For X either an n-manifold ($n = 2$ [Daverman 1986, §25], $n = 3$ [Armentrout 1971] (in case X is noncompact, this also depends on the solution to the 3-dimensional Poincaré conjecture), $n = 4$ [Freedman and Quinn 1990], and $n > 4$ [Siebenmann 1972]) or a Hilbert cube manifold [Chapman 1973], the answer is the cell-like self-maps. For X an n-dimensional Menger manifold it is the UV^{n-1}self-maps [Bestvina 1988]. This paper establishes a monotone approximation theorem (Theorem 2.2 here) attesting that, for a (connected) Pontryagin surface P of [Mitchell et al. 1992], the near-homeomorphisms are the self-maps which are monotone and have degree plus or minus one.

Not surprisingly, the proof hinges on a shrinking argument, which appears in Section 10 here. The crucial result toward this end, Corollary 10.5, promises that decompositions induced over finite graphs in the target of the usual type of map are shrinkable. That corollary combines with a homeomorphism extension theorem for maps between Pontryagin disks to complete the proof of the monotone approximation theorem. The section also contains a related theorem for maps between Pontryagin disks that restrict to homeomorphisms between their boundaries.

To set up the shrinking argument a great deal of preliminary effort is required. Most of that effort is directed toward the following intermediate result, called the factor theorem: given a self-map f as in the hypothesis and a locally separating, simple arc A in the Pontryagin surface P, the decomposition space X obtained

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from the decomposition of P whose elements are the point-preimages under f of the points in A and singletons in $P - f^{-1}(A)$ is a Pontryagin surface. The factor theorem is stated formally in Section 3; its proof appears at the end of Section 6, based on a related result called the factor reduction theorem. The latter, in turn, is proved in Section 7. Section 8 introduces the notion of a Pontryagin disk, which is a compact subset of a Pontryagin surface that behaves much like a 2-disk in a genuine surface. The main result of the section establishes a controlled equivalence of Pontryagin disks; it has the useful corollary that all homeomorphisms between the boundary curves of Pontryagin disks extend to homeomorphisms between the Pontryagin disks themselves; that result is an essential component of the proof of the monotone approximation theorem. Section 9 introduces the notion of Pontryagin cellularity, a natural analog to the concept of cellularity in 2-manifolds, and a key ingredient in the shrinking arguments.

Pontryagin surfaces and, in particular, Pontryagin disks were introduced by Mitchell, Repovš, and Ščepin [Mitchell et al. 1992], building on a related construction of Pontryagin [1930]. We define Pontryagin surfaces in the next section in a slightly different way than they did, using decompositions into points and figure-eights. Proposition 2.1 attests to the equivalence of this formulation with the original treatment as controlled inverse limits of monotone maps between closed, orientable surfaces. These objects have several interesting features. Connected Pontryagin surfaces are homogeneous [Jakobsche 1991]. A loop L in a locally compact, locally path-connected, locally homologically 1-connected metric space S is null homologous (Borel–Moore homology with $\mathbb{Z}$ coefficients throughout) if and only if it bounds a singular Pontryagin disk in S [Mitchell et al. 1992]. Any map of a Pontryagin disk or Pontryagin surface into a generalized n-manifold, $n > 4$, can be approximated arbitrarily closely by embeddings [Gu 2017].

Any monotone map between closed orientable surfaces must have absolute degree one [Lacher 1977, §7], which might suggest that the degree-one hypothesis in the statement of the monotone approximation theorem is redundant. It is not. However, the construction of the relevant example is quite intricate and the authors will present it in a separate article.

2. Terminology, notation, conventions, and statement of the main result

All maps of spaces will be continuous. A map is *proper* if the preimage of every compact subset of the target is compact. A surjective map is *monotone* if every point preimage is connected. A homotopy f_t of a map $f_0 : X \to Y$ is *supported in* a subset U of Y if, for all $t \in [0, 1]$ and $x \in X - f_0^{-1}(U)$, $f_t(x) = f_0(x)$ and $f_t(f_0^{-1}(U)) \subseteq U$. A map $f : (X, A) \to (Y, B)$ is *split* if $f^{-1}(B) = A$, and a homotopy of such a split map is *admissible* if it is supported in $Y - B$. Given

$f : X \to Y$ and $B \subset Y$, we say f is one-to-one, bijective, onto, etc. over B if $f|f^{-1}(B) : f^{-1}(B) \to B$ is one-to-one, bijective, onto, etc.

A space is *nice* if it is locally compact, locally path-connected, separable, and metrizable (recall that any connected, nice space has an end-point compactification). For a connected, nice space X we denote the one-point compactification, end-point compactification, and space of ends by $\hat{X}$, $\hat{\hat{X}}$, and $e(X)$, respectively (by convention, if X is compact, then $X = \hat{X} = \hat{\hat{X}}$ and $e(X) = \varnothing$). Note that if U is an open, connected subset of a connected, nice space X, then the quotient space $\hat{X}/(\hat{X} - U)$ is homeomorphic to $\hat{U}$. We will often refer to the "quotient-map" $\hat{X} \to \hat{U}$, by which we mean the composition $\hat{X} \to \hat{X}/(\hat{X} - U) \to \hat{U}$ (the maps being those referred to above).

An *exhaustion* of a set X is a sequence $\{X_i\}$ of subsets of X satisfying $X = \bigcup_i X_i$ and, for all i, $X_i \subseteq X_{i+1}$.

We often refer to a collection $\mathcal{E}$ of pairwise-disjoint compacta in a nice space X as a *decomposition* of X. This means the partition of X whose elements are the elements of $\mathcal{E}$ together with all singletons, each of which is contained in no element of $\mathcal{E}$. This partition (or decomposition) is upper semicontinuous (and hence, the associated decomposition space is metrizable) whenever the elements of $\mathcal{E}$ form a null sequence with respect to some metric on X. Such a decomposition space will be denoted as $X/\mathcal{E}$.

Any space homeomorphic to the wedge of two circles is a *figure-eight*.

Definition. A connected, nice space P is a *Pontryagin surface* if there exists a countable family $\mathcal{E}$ of pairwise-disjoint figure-eights in P such that $\mathcal{E}$ is null in $\hat{P}$ and, for any cofinite subfamily $\mathcal{D}$ of $\mathcal{E}$, the image of P under the decomposition map $\hat{P} \to \hat{P}/\mathcal{D}$ is an orientable surface without boundary. Such a family $\mathcal{E}$ is a *sufficient family* for P. (Observe that any closed orientable surface Q is a Pontryagin surface and that any finite family of pairwise-disjoint figure-eights in Q is a sufficient family if and only if it satisfies the following condition: for any element e of the family, the quotient space Q/e is a surface). If, in addition, the image of P in $\hat{P}/\mathcal{E}$ is either planar or a 2-sphere, then $\mathcal{E}$ is a *full* family for P. A nice space is a Pontryagin surface if each of its components is a Pontryagin surface, and a family $\mathcal{E}$ of figure-eights in a Pontryagin surface is a *sufficient family* if the elements of $\mathcal{E}$ in each component Y constitute a sufficient family for Y. A Pontryagin surface is *closed* if it is compact; otherwise, it is *open*.

A subspace C of a Pontryagin surface X is $\mathbb{P}$-*negligible* if X has a sufficient family no element of which meets C. The *manifold set of* X, denoted $M(X)$, is $\{p \in X \mid p \text{ has a neighborhood homeomorphic to } \mathbb{R}^2\}$. X is *rich* if $M(X) = \varnothing$. It should be noted that, unlike [Mitchell et al. 1992], in order to promote greater generality and to accommodate some of our constructions, we do not assume all Pontryagin surfaces have empty manifold set.

A compact space $\mathbb{D}$ is a *Pontryagin disk* if it is homeomorphic to the closure of some complementary component of a separating simple closed curve in a rich, connected Pontryagin surface. (It is important to keep in mind that Pontryagin disks, unlike Pontryagin surfaces, never contain open 2-disks.) Note that, by Corollary 3.2, the frontier of a Pontryagin disk in a Pontryagin surface is $\mathbb{P}$-negligible.

A compact 1-manifold A in a space X is *locally separating* if, given $p \in A - \partial A$ and any neighborhood U of p, there exists a connected neighborhood V of p such that $V \subset U$, $V \cap A$ is connected, and $V - A$ is not connected.

The Čech n-homology with G coefficients of a compact, metrizable space X will be denoted $\check{H}_n(X; G)$ (however, if $G = \mathbb{Z}$, the coefficient group will not be indicated).

A map $f : X \to Y$ of compact, metrizable spaces is an *$\check{H}_2$-isomorphism, monomorphism, etc.* if it induces an isomorphism, monomorphism, etc. on Čech 2-homology.

A surjective map of closed, orientable surfaces is *standard* if it is bijective over the complement of a finite subset F of the target and the preimage of each point in F is a figure-eight.

A map $f : X \to Y$ of one compact, metrizable space onto another is a *near-homeomorphism* if, given a metric ρ on Y and $\varepsilon > 0$, there exists a homeomorphism $h : X \to Y$ such that, for all $x \in X$, $\rho(f(x), h(x)) < \varepsilon$.

The following proposition provides, in effect, an alternate definition of "closed, connected Pontryagin surface".

Proposition 2.1. *A space X is a closed, connected Pontryagin surface if and only if it is the inverse limit of a sequence $\{p_n : R_{n+1} \to R_n\}_{n=1}^{\infty}$ of standard maps between closed, connected, orientable surfaces such that if, for each $n \in \mathbb{N}$, F_n denotes the finite subset of R_n referred to in the definition of standard map, then, for all $n \neq m$, $p_{n,1}(F_n) \cap F_1 = p_{m,1}(F_m) \cap p_{n,1}(F_n) = \varnothing$ (where $p_{n,1} = p_1 \circ p_2 \circ \cdots \circ p_{n-1}$).*

Proof. (Only if) Suppose $\mathcal{E} = \{e_1, e_2, \ldots\}$ is a sufficient family for X. Set $\mathcal{E}_n = \{e_n, e_{n+1}, \ldots\}$ and form the decomposition space $R_n = X/\mathcal{E}_n$. Note that $R_n = R_{n+1}/e_n$. Let $p_n : R_{n+1} \to R_n$ be the obvious map.

(If) Clearly

$$\mathcal{E} = \{e \subset X \mid \text{there exist } n \in \mathbb{N} \text{ and } x \in F_n \text{ such that } p_{\infty,n}^{-1}(x) = e\}$$

(where $p_{\infty,n}$ denotes the projection of X to R_n) is a countable, pairwise-disjoint, null family of figure-eights in X. To verify that it is sufficient, let $\mathcal{E}'$ be a cofinite subfamily of $\mathcal{E}$ and denote $\mathcal{F} = \mathcal{E} - \mathcal{E}'$ and $\mathcal{F}_n = \{e \in \mathcal{E} \mid e \text{ is a component of } p_{\infty,n}^{-1}(F_n)\}$. Choose $N \in \mathbb{N}$ such that $\mathcal{F} \subset \bigcup_{n=1}^{N} \mathcal{F}_n$. The decomposition space $X/\bigcup_{n=N+1}^{\infty} \mathcal{F}_n$ is R_N whose decomposition space obtained from the decomposition $\{p_{\infty,N}(e) \mid e \in \bigcup_{n=1}^{N} \mathcal{F}_n\}$ is R_1. Apply the parenthetical observation in the above definition of Pontryagin surface to complete the proof. □

Definition. We have from Proposition 2.1 that if P is a closed, connected Pontryagin surface, then $\check{H}_2(P) = \mathbb{Z}$ and, for $m \in \mathbb{N}$, $\check{H}_2(P; \mathbb{Z}_m) = \mathbb{Z}_m$. Given a map $f : P \to Q$ of closed, connected Pontryagin surfaces and choices $\mathcal{O}_P$ and $\mathcal{O}_Q$ of generators of $\check{H}_2(P)$ and $\check{H}_2(Q)$ (but $\mathcal{O}_P = \mathcal{O}_Q$ if $P = Q$), the *degree of f* is the integer n such that the induced homomorphism on Čech 2-homology sends $\mathcal{O}_P$ to $n\mathcal{O}_Q$. Note that the absolute value of the degree is independent of the choice of generators. Our interest focuses on maps of degree one, by which we really mean maps of absolute degree one.

Theorem 2.2 (monotone approximation theorem). *A map $f : P \to Q$ of closed, connected, rich Pontryagin surfaces is a near-homeomorphism if and only if it is monotone and has* (*absolute*) *degree one.*

Proof of "only if". That a near homeomorphism must be monotone follows from the well known result [Kuratowski and Lacher 1969] that any uniformly convergent sequence of monotone maps between compact, locally connected metric spaces has a monotone limit. To show that f must have degree one, let $p : Q \to S^2$ be a map arising as the inverse limit of standard maps between closed orientable surfaces and let $f = \lim_{i\to\infty} h_i$ where the $\{h_i : P \to Q\}$ are homeomorphisms. Then $p \circ f = \lim_{i\to\infty}(p \circ h_i)$.

Since S^2 is an ANR there exists an integer $k > 0$ such that $p \circ h_k$ is homotopic to $p \circ f$. Hence, $\deg(p \circ f) = \deg(p) \cdot \deg(f) = \deg(p) \cdot \deg(h_k) = 1$. So $\deg(f) = 1$. □

Applying the Vietoris–Begle mapping theorem, we obtain:

Corollary 2.3. *All cell-like maps between closed, connected, rich Pontryagin surfaces are near-homeomorphisms.*

We adopt the following notational conventions. If A is a subset of a topological space X, then $\operatorname{Fr} A$ and $\operatorname{Int} A$ will denote the frontier and interior of A in X. If A is a manifold, then $\mathring{A}$ denotes $A - \partial A$. $I = [-1, +1] \subset \mathbb{R}$ (but we will also consider I to be the set $[-1, +1] \times \{0\}$ in $\mathbb{R}^2$). $S^1 =$ the unit circle in $\mathbb{R}^2$. $S^2 =$ the one-point compactification of $\mathbb{R}^2$ (so we can regard $\mathbb{R}^2$ as a subset of S^2). $H = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$.

Remarks. Existence, uniqueness up to homeomorphism, and homogeneity of the connected, closed, rich Pontryagin surface (denoted by P in these remarks) are well known. However, it is worth noting that existence follows easily from Proposition 2.1 while uniqueness and homogeneity follow from Corollary 8.2 of this paper (we leave this as an exercise). Note also that any self-map of P constructed as follows is monotone and degree-one but not cell-like. Let $\mathcal{E}$ be a sufficient family for P and $\mathcal{F} \subset \mathcal{E}$ such that the image of $\mathcal{E} - \mathcal{F}$ is dense in $P/\mathcal{E}$ (e.g., $\mathcal{F}$ is finite). It follows that $P/\mathcal{F}$ is a rich Pontryagin surface, so the composition $P \xrightarrow{d} P/\mathcal{F} \xrightarrow{h} P$ where d is the decomposition map and h a homeomorphism is the desired map.

Let S be a compact metric space, G an upper semicontinuous decomposition of S, and $\pi : S \to S/G$ the decomposition map. Then G is *shrinkable* if the following condition, called the Bing shrinkability criterion, is satisfied: for each $\varepsilon > 0$ there exists a homeomorphism $h : S \to S$ such that each $h(g)$ ($g \in G$) has diameter less than ε, and π and πh are ε-close.

The notion of shrinkability was introduced by R. H. Bing [1952]. He exploited it to provide an effective general method for determining the topological type of certain decomposition spaces. R. D. Edwards [1980] gave an elegant proof for the crucial compact case mentioned below; his proof also can be found in [Daverman 1986, Lemma 6.1].

Theorem 2.4. *An upper semicontinuous decomposition G of a compact metric space S is shrinkable if and only if the decomposition map $\pi : S \to S/G$ is a near-homeomorphism.*

Another setting in which upper semicontinuous decompositions arise involves a proper map $f : X \to Y$ defined on a nice space X and a subset C of Y. The decomposition $G(C)$ of X *induced over* C is the partition consisting of the sets $\{f^{-1}(c) \mid c \in C\}$ and the singletons from $X - f^{-1}(C)$. Here $G(C)$ is upper semicontinuous (and $X/G(C)$ is metrizable) whenever C is closed in Y.

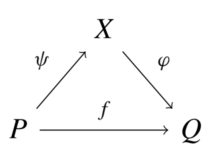
Corollary 2.5. *Let $f : X \to Y$ be a surjective mapping between compact metric spaces and C a closed subset of Y. If the decomposition $G(C)$ induced over C is shrinkable, then f can be approximated, arbitrarily closely, by a surjective map that is injective over C.*

Proof. If $\theta : X \to X/G(C)$ is a homeomorphism very close to the decomposition map $\pi : X \to X/G(C)$, then $F = f\pi^{-1}\theta$ is a map close to f which is 1-1 over C. □

3. The factor theorem

The theorem stated below is a key technical ingredient in the proof of the approximation theorem. Its proof occupies the following four sections.

Theorem 3.1 (the factor theorem). *Suppose the commutative diagram*



of maps and spaces satisfies the following conditions:

(1) *P and Q are closed, connected Pontryagin surfaces.*

(2) *All maps are surjective and f is both monotone and degree one.*

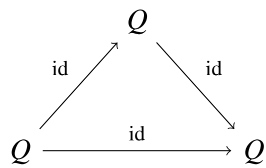
(3) *There is a subspace A of Q which is either a locally separating simple arc or a separating simple closed curve such that φ is injective over A and ψ is injective over $X - \varphi^{-1}(A)$.*

Then X is a Pontryagin surface and $\varphi^{-1}(A)$ is $\mathbb{P}$-negligible in X.

Note. We will prove the factor theorem in detail only for the case in which A is an arc. The proof for A a simple closed curve is essentially the same except for some minor details which we leave to the reader.

Corollary 3.2. *Any locally separating arc or separating simple closed curve in a closed Pontryagin surface is $\mathbb{P}$-negligible.*

Proof. Let Q be the closed Pontryagin surface and apply the factor theorem to the diagram



□

4. Sufficient families

In this brief section we state and prove some results and their consequences concerning sufficient families. First we present some definitions and notation.

A family $\mathcal{D}$ of compacta in a locally compact space X is *locally finite* if, for any compact subset C of X, the set $\{e \in \mathcal{D} \mid e \cap C \neq \varnothing\}$ is finite. If $\mathcal{E}$ is a family of compacta in X and $\mathcal{D} \subset \mathcal{E}$, we say $\mathcal{D}$ is a *locally cofinite* subfamily of $\mathcal{E}$ if $\mathcal{E} - \mathcal{D}$ is locally finite. We denote, for a subset U of X, $\mathcal{E}(U) = \{e \in \mathcal{E} \mid e \subset U\}$.

Observation. *Any locally cofinite subfamily of a sufficient family is sufficient.*

Proposition 4.1. *Any open subset U of a Pontryagin surface P is a Pontryagin surface. Furthermore, if $\mathcal{E}$ is a sufficient family for P, then $\mathcal{E}(U)$ is a sufficient family for U.*

Proof. We can assume that U is connected. Let $\mathcal{D}$ be a locally cofinite subfamily of $\mathcal{E}(U)$. It will suffice to show that if V is an open, connected subset of U such that $\overline{V} \subset U$ and $\overline{V}$ is compact, then the image of V under the decomposition map $\hat{P} \to \hat{P}/\mathcal{D}$ is a surface. Denote

$$\mathcal{F} = \{e \in \mathcal{E} \mid e \text{ meets both } P - U \text{ and } \overline{V} \text{ or } e \subset V \text{ and } e \notin \mathcal{D}\}.$$

Note that $\mathcal{F}$ is finite and $\mathcal{D} \subset \mathcal{E} - \mathcal{F}$. The image of V in $\hat{P}/\mathcal{D}$ is sent homeomorphically by the obvious decomposition map onto its image in $\hat{P}/(\mathcal{E} - \mathcal{F})$, which must be a surface since $\mathcal{E} - \mathcal{F}$ is sufficient for P. □

Lemma 4.2. *Suppose U is a Pontryagin surface with sufficient family $\mathcal{D}$. If A is a closed subset of U with $A \subset M(U)$, then the family $\{e \in \mathcal{D} \mid e \cap A = \varnothing\}$ is sufficient for U.*

Proof. Verify that $\{e \in \mathcal{D} \mid e \cap A \neq \varnothing\}$ is locally finite and apply the above observation. □

Proposition 4.3. *A connected, nice space U is a Pontryagin surface if and only if $\hat{\hat{U}}$ is a Pontryagin surface.*

Proof. (If) This follows from the previous proposition.

(Only if) Let $\mathcal{E}$ be a sufficient family for U and let R denote the image of U under the decomposition map $d : \hat{\hat{U}} \to \hat{\hat{U}}/\mathcal{E}$. So R is an open, connected, orientable surface whose end-point compactification is $\hat{\hat{U}}/\mathcal{E}$. From the classification theorem for open surfaces [Richards 1963] we obtain a locally finite (in R) family $\mathcal{D}$ of figure-eights in R such that $R/\mathcal{D}$ is planar and R is null in $\hat{\hat{U}}/\mathcal{E}$. By a standard general position argument we can choose these figure-eights to avoid $\bigcup_{e \in \mathcal{E}} d(e)$. Then $\mathcal{E} \cup \{d^{-1}(e) \mid e \in \mathcal{D}\}$ is a sufficient family for $\hat{\hat{U}}$. □

5. 2-coherence

This section includes a series of lemmas, propositions, and theorems to be used in the proof of the factor theorem, most of which take as their hypotheses only certain Čech-homological properties of Pontryagin surfaces. Any nice space having these properties will be termed *2-coherent*.

Definition. Suppose X is a connected, nice space and $\check{H}_2(\hat{X}) = \mathbb{Z}$. A family $\mathcal{U}$ of open, connected, nonempty subsets of X is a *coherence family* if, for any $U \in \mathcal{U}$, the following conditions are satisfied:

(1) The quotient map $\hat{X} \to \hat{U}$ is an $\check{H}_2$-isomorphism.

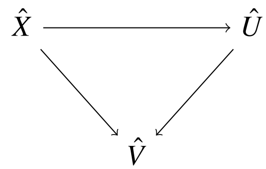
(2) For $n \in \mathbb{N} - \{1\}$, $\check{H}_2(\hat{U}; \mathbb{Z}_n) = \mathbb{Z}_n$ and the homomorphism $\check{H}_2(\hat{U}) \to \check{H}_2(\hat{U}; \mathbb{Z}_n)$ (induced by the coefficient group epimorphism $\mathbb{Z} \to \mathbb{Z}_n$) is an epimorphism.

(3) Any open, connected, nonempty subset of X is exhausted by elements of $\mathcal{U}$.

Definition. A connected, nice space X with $\check{H}_2(\hat{X}) = \mathbb{Z}$ is *2-coherent* if the class of all open, connected, nonempty subsets of X is a coherence family. Among its other benefits, 2-coherence characterizes the 2-manifolds within the class of 2-complexes.

A proper map $f : X \to Y$ of 2-coherent spaces has (*absolute*) *degree one* if $\hat{f} : \hat{X} \to \hat{Y}$ induces an isomorphism on Čech 2-homology with $\mathbb{Z}$ coefficients.

Observation. *If $\mathcal{U}$ is a coherence family for X and $V, U \in \mathcal{U}$ with $V \subset U$, then the quotient map $\hat{U} \to \hat{V}$ is an $\check{H}_2$-isomorphism.* (*To see this, apply $\check{H}_2$ to the diagram*

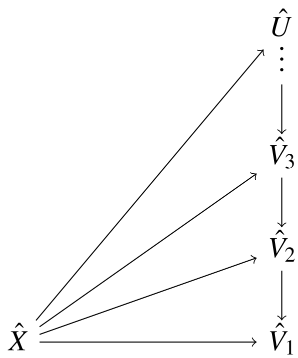


where the maps are quotient maps.) *Moreover, for $n \in \mathbb{N} - \{1\}$, $\check{H}_2(\hat{U}; \mathbb{Z}_n) \to \check{H}_2(\hat{V}; \mathbb{Z}_n)$ is an isomorphism. To see this consider the commutative diagram*

$$\begin{array}{ccc} \check{H}_2(\hat{U}) & \longrightarrow & \check{H}_2(\hat{V}) \\ \downarrow & & \downarrow \\ \check{H}_2(\hat{U}; \mathbb{Z}_n) & \longrightarrow & \check{H}_2(\hat{V}; \mathbb{Z}_n) \end{array}$$

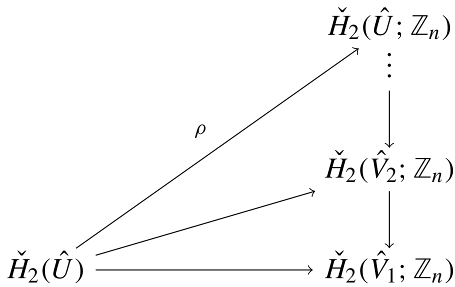
Lemma 5.1. *If a connected nice space X with $\check{H}_2(\hat{X}) = \mathbb{Z}$ has a coherence family, then it is 2-coherent.*

Proof. Let $\mathcal{V}$ denote the coherence family and let U be an open, connected subset of X. Let $\{V_i\}_{i=1}^{\infty}$ be an exhaustion of U with $V_i \in \mathcal{V}$ for all i. To verify that (1) (in the definition of coherence family) holds for U, apply the continuity axiom for Čech homology to the diagram obtained by applying $\check{H}_2$ to the following commutative diagram of spaces and maps:



All maps are quotient maps. Note that $\hat{U} = \varprojlim(\hat{V}_1 \leftarrow \hat{V}_2 \leftarrow \cdots)$.

One obtains from the same diagram that $\check{H}_2(\hat{U}; \mathbb{Z}_n) = \mathbb{Z}_n$. To verify that $\check{H}_2(\hat{U}) \to \check{H}_2(\hat{U}; \mathbb{Z}_n)$ is onto first note that, for all i, the composition $\check{H}_2(\hat{U}) \to \check{H}_2(\hat{V}_i) \to \check{H}_2(\hat{V}_i; \mathbb{Z}_n)$ is onto (the first homomorphism is an isomorphism and the second is onto by hypothesis). Now consider the commutative diagram



The "vertical" maps are isomorphisms and the others (with the possible exception of ρ) are onto. Hence, ρ is onto. □

The proof of the following lemma is left to the reader.

Lemma 5.2. *If C is a closed 0-dimensional subset of a compact, metrizable space X, then the quotient map $X \to X/C$ is an $\check{H}_2$-isomorphism.*

Proposition 5.3. *A connected nice space U is 2-coherent if and only if $\hat{\hat{U}}$ is 2-coherent.*

Proof. (If) This part is left to the reader.

(Only if) We claim that

$$\mathcal{V} = \{V \subset \hat{\hat{U}} \mid V \text{ is connected and open, and } V \cap e(U) \text{ is compact}\}$$

is a coherence family. We verify only condition (1) in the definition of coherence family and leave the rest to the reader. Consider the commutative diagram

$$\begin{array}{ccc} \hat{\hat{U}} & \longrightarrow & \hat{\hat{U}}/(\hat{\hat{U}} - V) = \hat{V} \\ \downarrow & & \downarrow \\ \hat{U} & \longrightarrow & \hat{V} \end{array}$$

where $V \in \mathcal{V}$ and the maps are the obvious quotient maps (the "vertical" map on the right sends $e(U) \cap V$ to $\hat{V} - V$). Applying $\check{H}_2$ to the diagram we have, by hypothesis, that the bottom horizontal homomorphism is an isomorphism and the vertical homomorphisms are isomorphisms by Lemma 5.2, so the top horizontal homomorphism is an isomorphism. □

Proposition 5.4. *Every connected Pontryagin surface is 2-coherent.*

Proof. Observe first that since the end-point compactification of a Pontryagin surface is a Pontryagin surface (Proposition 4.3) and any open connected subset of a 2-coherent space is 2-coherent, we can assume without loss of generality that the Pontryagin surface P of the hypothesis is compact. Use Proposition 2.1 to express P as the inverse limit of standard maps $\{p_n : R_{n+1} \to R_n\}_{n=1}^{\infty}$. We leave it to the reader to verify that the following class of open sets is a coherence family for P:

$$\big\{V \mid \text{there exists a connected compact subsurface } M_n \text{ of } R_n \text{ such that } p_{\infty,n} \text{ is one-to-one over } \partial M_n \text{ and } V \text{ is the interior of } p_{\infty,n}^{-1}(M_n)\big\}. \qquad \square$$

Lemma 5.5. *The proper cell-like image of a* 2*-coherent space is* 2*-coherent.*

Proof. Apply the Vietoris–Begle theorem. □

Lemma 5.6. *Suppose X is a connected* 2*-coherent space. Then*:

(1) *X contains no locally separating point.*

(2) *X contains no separating, closed* 0*-dimensional subset.*

(3) *X contains no separating set which is the union of a simple arc and a closed* 0*-dimensional set.*

(4) *If X is separated by a set which is the union of a simple closed curve α and a closed* 0*-dimensional set, then α separates X.*

Proof. (1) Suppose U is a connected open set in X and $p \in U$ such that $U - \{p\}$ is not connected. Since U is 2-coherent we have that $\hat{\hat{U}}$ is compact, 2-coherent, and separated by p. Denote by C the closure of a component of $\hat{\hat{U}} - \{p\}$ and let D be the closure of the union of all other components of $\hat{\hat{U}} - \{p\}$. Then $\check{H}_2(\hat{\hat{U}}) = \check{H}_2(C) \oplus \check{H}_2(D)$ and so one of the two summands must be trivial, which is impossible by the 2-coherence of $\hat{\hat{U}}$.

(2) The proof is similar to that of (1).

(3) Suppose otherwise and let A denote the arc. By Lemma 5.5, X/A is 2-coherent and is separated by a closed 0-dimensional set, which contradicts (2).

(4) Since $\hat{\hat{X}}$ is 2-coherent by Proposition 5.3 we can assume without loss that X is compact. Denote the 0-dimensional set by C and suppose α does not separate X. Denote $U = X - \alpha$ and note that, by connectivity of U, $\hat{\hat{U}}$ is 2-coherent. Since X/α is the one-point compactification of $X - \alpha$ we have the natural map $\varphi : \hat{\hat{U}} \to X/\alpha$ (from the end-point compactification of any nice space to the one-point compactification of that space). Let x denote the image of α under the quotient map $X \to X/\alpha$ and note that the map $\varphi : (\hat{\hat{U}}, e(U) \cup C) \to (X/\alpha, \{x\} \cup C)$ is split (by abuse of notation C is considered to be a subset of both $\hat{\hat{U}}$ and X/α). However, $e(U) \cup C$ cannot separate $\hat{\hat{U}}$ by (2) and hence $\{x\} \cup C$ cannot separate X/α. But then $\alpha \cup C$ cannot separate X. □

Corollary 5.7. *If X is compact and 2-coherent and A is a cell-like subset of X, then $X - A$ has one end.*

Proof. Otherwise, A would be a locally separating point in X/A (which is 2-coherent by Lemma 5.5). □

Corollary 5.8. *Suppose A is a separating simple closed curve in a compact 2-coherent space X.*

(1) *If U is any component of $X - A$, then $\overline{U} = U \cup A$.*

(2) *A is locally separating.*

Proof. (1) Suppose $A \not\subset \overline{U}$. Then $\overline{U} \cap A$ is contained in some arc α in A which would make α a separating point in the quotient space X/α (which would be 2-coherent).

(2) Use (1). □

Lemma 5.9. *Suppose X is a compact metrizable space, S is a simple closed curve in X, and A and B are the closures in X of two distinct components of $X - S$. Then the inclusion-induced homomorphism $\check{H}_2(A) \oplus \check{H}_2(B) \to \check{H}_2(X)$ is injective.*

Proof. There exists a sequence of nerves $\{p_{n+1,n} : (X_{n+1}, A_{n+1}, B_{n+1}, S_{n+1}) \to (X_n, A_n, B_n, S_n)\}_{n=1}^{\infty}$ such that, for each n, S_n is a simple closed curve, A_n and B_n are closed components of $X_n - S_n$, and for $Z = X, A, B$, or S, $\varprojlim\{p_{n+1,n} : Z_{n+1} \to Z_n\}$ is Z. Now conclude from a Mayer–Vietoris sequence that, for each n, $H_2(A_n) \oplus H_2(B_n) \to H_2(X_n)$ is injective. Since the inverse limit of monomorphisms is a monomorphism, the conclusion follows. □

Observation. *If E is a compact subspace of a 2-coherent space X with $E \neq X$, then the inclusion-induced homomorphism $\check{H}_2(E; G) \to \check{H}_2(\hat{X}; G)$ (where $G = \mathbb{Z}$ or $\mathbb{Z}_n$ for some $n \in \mathbb{N} - \{1\}$) is trivial.*

Proof. Let U be a component of $X - E$ and note that the composition $\check{H}_2(E; G) \to \check{H}_2(\hat{X}; G) \to \check{H}_2(\hat{U}; G)$ (induced by the obvious maps $E \to \hat{X} \to \hat{U}$) is trivial and the second of the two homomorphisms is an isomorphism. □

Lemma 5.10. *Suppose S is a separating simple closed curve in a compact 2-coherent space X and U is a component of $X - S$. Then:*

(1) *$\check{H}_2(\overline{U}) = 0$.*

(2) *$\partial_* : \check{H}_2(\overline{U}, S) \to \check{H}_1(S)$ is an isomorphism.*

(3) *$X - S$ has two components.*

Proof. (1) By Lemma 5.9 (where $A = \overline{U}$) we have $\check{H}_2(\overline{U}) \to \check{H}_2(X)$ is injective. But, by the observation, it is also trivial.

(2) We have homomorphisms $\check{H}_2(\hat{U}) \xrightarrow{\varphi} \check{H}_2(\overline{U}, S) \xrightarrow{\partial_*} \check{H}_1(S)$ where φ is the inverse of the isomorphism induced by the quotient map $\overline{U} \to \overline{U}/S = \hat{U}$. Let $\alpha \in \check{H}_2(\hat{U})$

be a generator and denote $\beta = (\partial_* \circ \varphi)(\alpha)$. We will show that β is a generator of $H_1(S)$ (which we identify with $\mathbb{Z}$). We can assume without loss of generality that $\beta \geq 0$. If $\beta = 0$, then $\varphi(\alpha)$ is in the image of $\check{H}_2(\overline{U}) \to \check{H}_2(\overline{U}, S)$ and hence $\check{H}_2(\overline{U})$ is nontrivial (impossible by (1)). If $\beta > 1$, we have a nontrivial element of $\check{H}_2(\overline{U}; \mathbb{Z}_\beta)$ again violating (1) (note that nontriviality of the element follows from the surjectivity of $\check{H}_2(\hat{U}) \to \check{H}_2(\hat{U}; \mathbb{Z}_\beta)$).

(3) Assume $X - S$ has at least two components U_1 and U_2. If $(X - S) \neq U_1 \cup U_2$, use (2) to argue that $\check{H}_2(\overline{U}_1 \cup \overline{U}_2)$ must be nontrivial, contradicting (1). □

Corollary 5.11. *If X is a noncompact 2-coherent space and R is a separating closed subset of X homeomorphic to $\mathbb{R}$, then $X - R$ has two components.*

Proof. Let α denote the closure in $\hat{\hat{X}}$ of R. By Lemma 5.10 it will suffice to show that α is a simple closed curve, but if α were an arc, then the 2-coherent space $\hat{\hat{X}}/\alpha$ would contain a separating point. □

Definition. A simple arc A in a 2-coherent space X is *2-sided* if, given any subarc α of A, there exists a neighborhood V such that $V \cap A = \mathring{\alpha}$, $V - A$ has two components, and denoting the two components by V_1 and V_2, $\overline{V}_1 \cap \overline{V}_2 = \alpha$ and $\mathrm{Fr}(\overline{V}_i) = \overline{V}_i - V_i$ $(i = 1, 2)$. (Such a neighborhood V of $\mathring{\alpha}$ will be called *dichotomous*.)

Proposition 5.12. *Any locally separating arc A in a 2-coherent space X is 2-sided.*

Proof. By Corollary 5.11 it will suffice to show that $\mathring{A}$ separates some open connected neighborhood of itself. We briefly outline the proof. Construct a family $\mathcal{U} = \{U_n\}_{n \in \mathbb{Z}}$ of open connected sets covering $\mathring{A}$ and satisfying the following properties:

(1) For each n, $U_n - A$ is disconnected.

(2) For each n, $U_n \cap A$ is an open subarc of A whose closure in A is disjoint from ∂A.

(3) $U_i \cap U_j \neq \varnothing$ if and only if $|i - j| \leq 1$.

Now, by a Lebesgue number argument applied infinitely many times, we can choose a second covering $\{V_m\}_{m \in \mathbb{Z}}$ satisfying the same three properties and, in addition, for $|i - j| = 1$, $V_i \cup V_j$ is contained in some element of $\mathcal{U}$. Prove by induction on $N \in \mathbb{N}$ that $\bigcup_{m=-N}^{N} V_m$ is separated by A. Then $\bigcup_{m \in \mathbb{Z}} V_m$ is the desired neighborhood. □

Observation. *If A is a 2-sided simple arc in a 2-coherent space X, then any subarc of A is also 2-sided. Also note that if U is a dichotomous neighborhood of $\mathring{A}$ and V is a connected, open set with $V \subset U$ such that $V \cap A$ is connected, then V is a dichotomous neighborhood of $V \cap A$.*

The following observation is used in the proof of Proposition 5.13. Its proof is left to the reader.

Observation. *The absolute degree of a map of compact* 2*-coherent spaces is "determined locally"; i.e., if $f : X \to Y$ is such a map and V is an open, nonnull, connected subset of Y such that $f^{-1}(V)$ is connected, then the absolute degree of f is the same as the absolute degree of the one-point compactification of the map $f| : f^{-1}(V) \to V$.*

Proposition 5.13 (degree-one proposition). *If X is a compact* 2*-coherent space, then a map $f : X \to S^2$ has degree one if the following conditions are satisfied:*

(1) *$f^{-1}(S^1)$ is the union of a simple closed curve A and a closed* 0*-dimensional set and $f|A : A \to S^1$ is bijective.*

(2) *For C either component of $S^2 - S^1$, $f^{-1}(C) \neq \varnothing$.*

Proof. First note that since $f^{-1}(S^1)$ separates X we have, by Lemma 5.6, that A separates X and hence, by Lemma 5.10, that $X - A$ has two components. Let U be one of them and let D be that component of $S^2 - S^1$ which contains $f(U)$ (and hence by condition (2) we have $f^{-1}(D) = U$). Consider the commutative diagram

$$\begin{array}{ccc} \check{H}_2(\overline{U}, A) & \xrightarrow{\partial_*} & \check{H}_1(A) \\ \Big\downarrow{\scriptstyle (f|\overline{U})_*} & & \Big\downarrow{\scriptstyle (f|A)_*} \\ \check{H}_2(\overline{D}, S^1) & \xrightarrow{\partial_*} & \check{H}_1(S^1) \end{array}$$

By Lemma 5.10, $(f|A)_* \circ \partial_*$ is an isomorphism and ∂_* at the bottom of the diagram is obviously an isomorphism. Hence, $(f|\overline{U})_*$ is an isomorphism. So the map $\check{H}_2(\overline{U}/A) \to \check{H}_2(\overline{D}/S^1)$ (induced by f) is also an isomorphism. To see this, note that $\overline{U}/A = \hat{U}$ and apply the above observation. □

The rest of this section is devoted to the proof of the following theorem.

Theorem 5.14. *If A is either a locally separating simple arc or a separating simple closed curve in a compact* 2*-coherent space X, then there exists a split, degree-one map $f : (X, A) \to (S^2, B)$ which is bijective over B, where B is either I or S^1.*

The principal ingredients in the proof are Proposition 5.13 and the *strong generalized Tietze extension theorem* (SGTE) stated below.

Theorem 5.15 (SGTE). *If A is a closed subset of a compact metrizable space X, then any map $f : A \to S^{n-1}$ ($n \in \mathbb{N}$) has a split extension $g : (X, A) \to (B^n, S^{n-1})$. Furthermore, that extension is unique up to admissible homotopy.*

Proof. The so-called *generalized Tietze extension theorem* guarantees an extension $h : (X, A) \to (B^n, S^{n-1})$ (which however is not, in general, split). Define g as follows. First choose a metric ρ for X and define a second metric ρ' by $\rho'(x, y) = \min\{1, \rho(x, y)\}$. Now let $g(x) = (1 - \rho'(x, A)) \cdot h(x)$ (where B^n is viewed as vectors of norm at most one in $\mathbb{R}^n$, and the dot in the preceding equation indicates scalar multiplication).

Now suppose that g_0 and g_1 are two such split extensions of f. Define $\varphi : (A \times [0,1]) \cup (X \times \{0,1\}) \to (S^{n-1} \times [0,1]) \cup (B^n \times \{0,1\})$ by

$$\varphi(x,t) = \begin{cases} (f(x),t) & \text{if } x \in A, \\ (g_i(x),i) & \text{if } x \in X \text{ and } i \in \{0,1\}. \end{cases}$$

The desired homotopy is a split extension of φ to

$$(X \times I, (A \times I) \cup (X \times \{0,1\})) \to (B^n \times I, \partial(B^n \times I)). \qquad \square$$

Proof of Theorem 5.14. We consider only the case in which A is an arc (the argument for A a closed curve is similar and easier). By Proposition 5.12 we can choose two dichotomous neighborhoods V and U of $\mathring{A}$ in X such that $\overline{U} \subset V \cup \partial A$. Hence, if U_1 and U_2 are the components of $U - A$, we have that $\operatorname{Fr} U_1 - A$ and $\operatorname{Fr} U_2 - A$ are disjoint (since they are in different components of $V - A$). Let C_1 and C_2 denote $\operatorname{Fr} U_1 - \mathring{A}$ and $\operatorname{Fr} U_2 - \mathring{A}$, respectively (note that $\operatorname{Fr} U = C_1 \cup C_2$ and $C_1 \cap C_2 = \partial A$). Let $\alpha : A \to I$ be any homeomorphism and apply the SGTE (Theorem 5.15) to $\alpha|\partial A : \partial A \to \partial I$ to obtain a split map $\beta : (C_1, \partial A) \to (S^1 \cap \{(x_1,x_2) \in \mathbb{R}^2 \subset S^2 \mid x_2 \geq 0\}, \partial I)$. Let B^2_+ and B^2_- be the upper and lower 2-disks in B^2 containing I in their boundaries. Apply the SGTE again to extend $\alpha \cup \beta : \operatorname{Fr} U_1 \to S^2$ to obtain a split map $\varphi_1 : (\overline{U}_1, \operatorname{Fr} U_1) \to (B^2_+, \partial B^2_+)$. Similarly we obtain $\varphi_2 : (\overline{U}_2, \operatorname{Fr} U_2) \to (B^2_-, \partial B^2_-)$. We have $\varphi_1|A = \varphi_2|A = \alpha$. Denote $\varphi = \varphi_1 \cup \varphi_2$. Apply SGTE a final time to extend $\varphi| \operatorname{Fr} U : \operatorname{Fr} U \to S^2$ to a split map $\psi : (\overline{X - U}, \operatorname{Fr} U) \to (S^2 - \mathring{B}^2, \partial B^2)$. Then $\varphi \cup \psi$ is the desired map.

It remains only to verify that the degree of $\varphi \cup \psi$ is one. To see this, consider the end-point compactification of the map $\varphi|U : U \to \mathring{B}^2$ which we denote by $\eta : \hat{\hat{U}} \to Q$. The target is a 2-sphere and the domain a 2-coherent space by Proposition 5.3. It follows easily from the definition of 2-coherence that η and $\varphi \cup \psi$ have the same degree. Denote by L the closure of $\mathring{I}$ in Q (so L is a simple closed curve) and note that $\eta^{-1}(L)$ is the union of a closed set of dimension zero and the closure of $\mathring{A}$ in $\hat{\hat{U}}$ (which must be either a simple arc or a simple closed curve). Since $\eta^{-1}(L)$ must separate $\hat{\hat{U}}$, we conclude from Lemma 5.6 that the closure of $\mathring{A}$ in $\hat{\hat{U}}$ is a simple closed curve. Apply the degree-one proposition to η to complete the proof. $\square$

6. A reduction of the factor theorem

In this section we show that the following result, whose proof is deferred to Section 7, implies the factor theorem. The crucial difference between the factor theorem and this factor reduction theorem is that in the former the complement of the arc (in the intermediate space) is an open Pontryagin surface whereas in the latter the analogous space is a genuine surface.

Theorem 6.1 (factor reduction theorem). *Suppose Y is a compact, connected, metrizable space, A is a closed subset of Y such that $Y - A$ is an open surface, each component of which is orientable, and $f : (Y, A) \to (S^2, C)$ is a split, surjective map which is injective over C and such that one of the following conditions is satisfied:*

(1) *$C = I$, A is 2-sided in Y, $Y - A$ has one end (hence is connected), and $f|Y - A : Y - A \to S^2 - C$ has degree one.*

(2) *$C = S^1$, A is 2-sided in Y, and if R denotes either component of $S^2 - C$, then $f^{-1}(R)$ has one end and the map $f|f^{-1}(R) : f^{-1}(R) \to R$ has degree one.*

Then Y is a Pontryagin surface and A is $\mathbb{P}$-negligible.

(Note that in what follows we will verify the conclusion only for the first of the two conditions in the statement. The proof given the second condition is very similar, though slightly easier at certain points, and is left to the reader.)

We introduce some terminology which will be used only in this section.

Definition. Suppose $\psi : X \to Z$ is a map of compact, metrizable spaces and U is an open subset of Z such that $\psi^{-1}(U)$ is a Pontryagin surface with a sufficient family $\mathcal{E}$ which is null in X. If there exist $\mathcal{D} \subset \mathcal{E}$ with $\mathcal{D}$ sufficient for $\psi^{-1}(U)$ and a homotopy of ψ supported in U to a map ψ' having a factorization

$$\begin{array}{ccc} X & \xrightarrow{\quad\psi'\quad} & Z \\ & {\scriptstyle d}\searrow \quad \nearrow{\scriptstyle \alpha} & \\ & X/\mathcal{D} & \end{array}$$

(where d is the decomposition map), then we say α is a *Euclideanization of ψ over U using $\mathcal{E}$*. (Note that the existence of the factorization for ψ' is equivalent to the condition that for all $e \in \mathcal{D}$, $\psi'(e)$ is a singleton.)

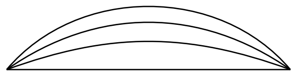
Lemma 6.2. *Suppose $\psi : (Y, C) \to (B, \partial B)$ is a split map where B is a 2-disk, Y is a compact, metrizable space, and $\psi^{-1}(\mathring{B})$ is a Pontryagin surface with sufficient family $\mathcal{E}$ which is null in Y. Then ψ has a Euclideanization over $\mathring{B}$ using $\mathcal{E}$.*

Proof. Let $d : Y \to Y/\mathcal{E}$ be the decomposition map, and note that d is injective over $d(C)$. Apply the SGTE (Theorem 5.15) to $\psi \circ d^{-1}|d(C) : d(C) \to \partial B$ to obtain a split map $\alpha : (Y/\mathcal{E}, d(C)) \to (B, \partial B)$. To obtain the homotopy, apply the uniqueness provision of the SGTE to the maps ψ and $\alpha \circ d$ □

Proposition 6.3. *Suppose that X is a connected, compact, metrizable space, $\varphi : (X, A) \to (S^2, I)$ is a split surjective map which is one-to-one over I, $X - A$ is an open Pontryagin surface, and $\mathcal{E}$ is a sufficient family for $X - A$ which is null in X. Then φ has a Euclideanization over $S^2 - I$ using $\mathcal{E}$.*

Proof. The idea is to choose two 2-disks in S^2 such that the union of their interiors is $S^2 - I$ and then Euclideanize the map over each of the 2-disks in succession. Some care is required.

Choose three disks D, E, and F in S^2 such that $F \subset E \subset D$ and any two of the disks have boundaries intersecting in I as shown below, where I is the horizontal line segment and ∂D is the outermost simple closed curve:



From Proposition 4.1 we have that $\mathcal{E}(\varphi^{-1}(\mathring{D}))$ is sufficient for $\varphi^{-1}(\mathring{D})$. Apply Lemma 6.2 to the map

$$\varphi|(\varphi^{-1}(D), \varphi^{-1}(\partial D)) : (\varphi^{-1}(D), \varphi^{-1}(\partial D)) \to (D, \partial D)$$

to obtain a Euclideanization α of that map using $\mathcal{E}(\varphi^{-1}(\mathring{D}))$. Let $d : X \to X/\mathcal{E}(\varphi^{-1}(\mathring{D}))$ be the decomposition map and denote $\mathcal{D} = \{d(e) \mid e \in \mathcal{E} - \mathcal{E}(\varphi^{-1}(\mathring{D}))$ and $d(e) \in \alpha^{-1}(S^2 - E)\}$. By Lemma 4.2, $\mathcal{D}$ is a sufficient family for $\alpha^{-1}(S^2 - F)$ (where in the application of that lemma we use $U = \alpha^{-1}(S^2 - I)$, $A = \alpha^{-1}(F - I)$, and $B = \alpha^{-1}(E - I)$). Now Euclideanize α over $S^2 - F$ using $\mathcal{D}$. □

Proof that the factor reduction theorem implies the factor theorem. As usual we consider only the case in which the set $A \subset Q$ in the hypothesis of the factor theorem is a simple arc. Note at the outset that monotonicity of f implies monotonicity of φ. As a result, $\varphi^{-1}(A)$ must be 2-sided in X. Set $A' = \varphi^{-1}(A)$.

Let $\mathcal{E}$ be a sufficient family for P. Hence, $\mathcal{E}(P - f^{-1}(A))$ is sufficient for $P - f^{-1}(A)$. Form $\mathcal{E}' = \{\psi(e) \mid e \in \mathcal{E}(P - f^{-1}(A))\}$ and note that $\mathcal{E}'$ is sufficient for $X - A'$ (since the restriction of ψ to $P - f^{-1}(A)$ is 1-1) and is null in X. Here ψ restricts to a homeomorphism

$$P - f^{-1}(A) \to X - \varphi^{-1}(A) = X - A'.$$

Hence, both the end-point and one-point compactifications of $X - A'$ have $\check{H}_2$ isomorphic to $\mathbb{Z}$. As $f = \varphi \circ \psi$ has degree one, $\varphi : (X, A') \to (Q, A)$ must also have degree one.

Use Theorem 5.14 to obtain a split, degree-one map $\alpha : (Q, A) \to (S^2, I)$. Its composition with φ yields a degree-one map $\alpha \circ \varphi : (X, A') \to (S^2, I)$.

Apply Proposition 6.3 to obtain a Euclideanization $\varphi' : X/\mathcal{D} \to S^2$ of $\alpha \circ \varphi$ over $S^2 - I$ using $\mathcal{E}'$. Let $d : X \to X/\mathcal{D} = Y$ be the decomposition map and let $A^* = d(A')$. Note that degree φ' = degree $\alpha \circ \varphi = 1$. We leave it to the reader to verify that A^* is 2-sided in Y (hint: use the fact that the image under d of a dichotomous neighborhood of $\mathring{A}'$ in X which is saturated with respect to $\mathcal{D}$ must be a dichotomous neighborhood of A^* in Y).

Consequently, the hypotheses of the factor reduction theorem are satisfied. From it we conclude that Y is a Pontryagin surface and A^* is $\mathbb{P}$-negligible in Y. By a standard general position argument, we can choose a sufficient family $\mathcal{F}$ for Y, each element of which is disjoint from $A^* \cup \{d(e) \mid e \in \mathcal{D}\}$. It follows easily that $\{d^{-1}(e) \mid e \in \mathcal{F}\} \cup \mathcal{D}$ is a sufficient family for X, no element of which meets A'. □

7. Proof of the factor reduction theorem

We prove the factor reduction theorem (Theorem 6.1) by first producing a convergent sequence of admissible homotopies, starting with the map of the hypothesis, which progressively enlarge the 1-dimensional subspace of S^2 over which the map is bijective. The initial step (the "arc proposition") is the most difficult. It produces an admissible homotopy of the map of the hypothesis to a map which is bijective over the union of C (either I or S^1) and an arc meeting C in a preassigned point. In the proofs of both the arc proposition and the subsequent factor reduction theorem, we consider only the case $C = I$. We leave it to the reader to make the modifications necessary for the case $I = S^1$. In what follows recall that H denotes closed upper half-space in $\mathbb{R}^2$.

Proposition 7.1 (arc proposition). *Let $f : (Y, A) \to (S^2, I)$ be a map as in the hypothesis of the factor reduction theorem. Given $\varepsilon > 0$ and $a \in (-1, +1)$ there exists a homotopy of f to a map g supported in a neighborhood $U \subset H$ for which* $\operatorname{diam} U < \varepsilon$ *and* $\overline{U} \cap (\mathbb{R} \times \{0\}) = \{(a, 0)\}$, *such that g is bijective over* $\{a\} \times [0, r]$ *for some* $r > 0$.

Before proving the arc proposition we introduce some terminology and notation. Suppose $f : X \to Y$ is a map of spaces and $V \subset U$ are subsets of Y. The pair (U, V) is *good for f* (or merely *good* when no ambiguity can result) if $f^{-1}(V)$ is contained in a component of $f^{-1}(U)$. If Y is a metric space, then the *diameter* of the pair (U, V) is the diameter of U. For a point p in a metric space (Y, ρ), $B[p, \varepsilon]$ will denote the closed ball of ρ-radius ε centered at p.

The proof of the arc proposition requires the following five lemmas. The proofs of all but the last are left to the reader.

Lemma 7.2. *Suppose $f : X \to Y$ is a map from a compact nice space to a metric space which is one-to-one over the singleton $\{p\}$ in Y. Then given $\varepsilon > 0$ there exists $\delta > 0$ such that $(B[p, \varepsilon], B[p, \delta])$ is good.*

Lemma 7.3. *Suppose X is a nice space and $A \subset U \subset X$ with A compact and U open and connected. Then there exists a compact, connected subset C of X with $A \subset C \subset U$.*

Lemma 7.4. *Suppose $f : R \to Q$ is a proper, boundary-preserving map of surfaces and (D_1, D_2) is a pair of 2-disks in Q satisfying the following:*

(1) $D_2 \subset \mathring{D}_1 \subset D_1 \subset \mathring{Q}$.

(2) *Both* $f^{-1}(D_1)$ *and* $f^{-1}(D_2)$ *are surfaces.*

Then, if C denotes the union of the components of $f^{-1}(D_1)$ meeting $f^{-1}(D_2)$, there exists a homotopy of f supported in any preassigned neighborhood of D_1 to a map g such that $g^{-1}(D_2) = C$.

Lemma 7.5. *Any degree-one, split map $\varphi : (Q, \partial Q) \to (B^2, \partial B^2)$ of a compact, connected, orientable surface Q is admissibly homotopic to a map which is bijective over a preassigned disk in $\mathring{B}^2$.*

Note. Lemma 7.5 follows easily from the classification theorem for compact, orientable surfaces and is also an immediate consequence of the principal theorem in [Epstein 1966].

Lemma 7.6. *Suppose Q is a noncompact, one-ended, connected, orientable surface such that ∂Q has one noncompact component, $\varphi : (Q, \partial Q) \to (H, \partial H)$ is a proper, degree-one, split map, and Ω is a collar on ∂H in H. Then φ is properly, admissibly homotopic to a map which is bijective over $H - \Omega$.*

Proof. We will treat the case where Q has infinitely many boundary components and infinitely many handles; strategies for dealing with the other possibilities can be inferred from what we do in that slightly more complicated case. For definiteness we assume that $\Omega = \mathbb{R} \times [0, 3] \subset H$. The proof requires some care because ∂Q has two ends while Q has only one.

Let S be the subset of $\mathbb{Z}$ determined as follows (and here we denote by L the noncompact component of ∂Q). If there exists a real number b such that $\varphi(\partial Q - L) \subset (b, +\infty) \subset \partial H$, then $S = \mathbb{N}$; if there exists a number b' such that $\varphi(\partial Q - L) \subset (-\infty, b')$, then $S = \mathbb{Z} - \mathbb{N}$; otherwise, $S = \mathbb{Z}$. Now for each $n \in S$ let D_n be a 2-disk in the interior of $[n, n+1] \times [0, 1]$ and let E_n be a 2-disk in the interior of $[n, n+1] \times [2, 3]$. Also for each $n \in S$, let T_n be a punctured torus in $[n, n+1] \times [2, 3] \times [0, 1]$ with $T_n \cap H = \partial T_n = \partial E_n$ and let A_n be an annulus in $[n, n+1] \times [0, 1] \times [0, 1]$ with one component of ∂A_n equal to ∂D_n, $\partial A_n - \partial D_n \subset (n, n+1) \times \{0\} \times (0, 1)$, and $\mathring{A}_n \subset (n, n+1) \times (0, 1) \times (0, 1)$. Denote $Q' = \big[H - \bigcup_{n \in S} (\mathring{D}_n \cup \mathring{E}_n)\big] \cup \big[\bigcup_{n \in S} (A_n \cup T_n)\big]$. Let $\Phi : Q' \to H$ be the restriction to Q' of the projection map $H \times [0, 1] \to H$. Note that Φ is an admissible map.

By the classification theorem for noncompact surfaces [Prishlyak and Mischenko 2007; Richards 1963], there is a homeomorphism $\theta : Q \to Q'$. Modify θ, if necessary, so that $\Phi\theta|L, \varphi|L : L \to \partial H$ are properly homotopic. Then modify further so that, for each compact component J of ∂Q, $\Phi\theta(J) \subset [0, +\infty) \subset \partial H$ if and only if $\varphi(J) \subset [0, +\infty)$. Now it follows that the straight line homotopy μ_t is a proper homotopy between $\Phi\theta|\partial Q, \varphi|\partial Q : \partial Q \to \partial H$ (as maps to ∂H). By construction, $\Phi\theta$ is injective over $H - \Omega$.

Pass to the one-point compactifications $\hat{Q}$, $\hat{H}$, and $\partial\hat{Q}$ of Q, H, and ∂Q, respectively (the third of these is an admitted abuse of notation), and observe that $\hat{H}$ is a 2-cell. Name compactification points ∞ and ∞' in $\hat{Q}$ and $\hat{H}$, respectively. Let $A \subset \hat{Q} \times [0,1]$ be the subset $(\hat{Q} \times \{0,1\}) \cup (\partial\hat{Q} \times [0,1])$. Define a map $f : A \to \hat{H} \times [0,1]$ as $\Phi\theta$ on $\hat{Q} \times 0$, φ on $\hat{Q} \times 1$, and μ_t on $\partial Q \times [0,1] \subset \partial\hat{Q} \times [0,1]$, and as the map $(\infty, t) \to (\infty', t)$ on $\infty \times [0,1]$. Apply the SGTE (Theorem 5.15) to extend f to a split map $F : (\hat{Q} \times [0,1], A) \to (\hat{H} \times [0,1], \partial(\hat{H} \times [0,1]))$. A restriction of F gives a proper homotopy between $\Phi\theta$ and φ. □

Proof of the arc proposition. Note first that we can assume that ε is small enough so that $f^{-1}(B[(a,0),\varepsilon])$ is contained in a dichotomous neighborhood of $\mathring{A}$. We will also assume without loss of generality that f is transverse to all subsurfaces of $S^2 - I$ constructed below. We denote, for $a \in (-1,+1)$ and $\delta > r > 0$, $M(a,\delta,r) = B[(a,0),\delta] \cap \{(x,y) \in \mathbb{R}^2 \mid y \geq r\}$.

Claim 1. Given $a \in (-1,+1)$ and $\varepsilon > 0$ there exists $\delta \in (0,\varepsilon)$ such that for any $r < \delta$ there exists $s < r$ so that $(M(a,\varepsilon,s), M(a,\delta,r))$ is good.

Proof. From Lemma 7.2 there exists $\delta > 0$ such that $(B[(a,0),\varepsilon], B[(a,0),\delta])$ is good. Since the preimages of these sets lie in a dichotomous neighborhood of $A - \partial A$ we can conclude that $(B[(a,0),\varepsilon] \cap \mathring{H}, B[(a,0),\delta] \cap \mathring{H})$ is also good. Hence, $(B[(a,0),\varepsilon] \cap \mathring{H}, M(a,\delta,r))$ is good. To finish the proof of Claim 1, apply Lemma 7.3 to conclude that, for some $s > 0$, $(M(a,\varepsilon,s), M(a,\delta,r))$ is good. □

Now continuing with the proof of the arc proposition, choose decreasing sequences $\{\delta_i\}$, $\{r_i\}$ in $(0,+\infty)$ converging to 0 and such that, for all i, $\delta_{i+1} > r_i > \delta_{i+2}$.

Denote $N_i = M(a,\delta_i,r_i)$ and observe that, for all i, $N_i \cap N_{i+1}$ is a disk, $N_i \cap N_j = \varnothing$ if $|i-j| > 1$, and $\lim(\operatorname{diam} N_i) = 0$. From Claim 1 we can find (after, in general, deleting the first K entries for some $K \in \mathbb{N}$ and reindexing) a sequence $\{(\varepsilon_i, s_i)\}_{i=1}^{\infty}$ of pairs of positive real numbers such that $\lim \varepsilon_i = 0$ and, for all i, $\varepsilon_i \geq \varepsilon_{i+1}$, $\varepsilon_i > s_i$, $r_i > s_i$, and denoting $M_i = M(a,\varepsilon_i,s_i)$, the pair (M_i, N_i) is good. Denote by Y and Z the components of $[-1,+1] \times [0,+\infty) - \overline{\bigcup_i M_i}$ containing $\{(x,0) \mid -1 < x < a\}$ and $\{(x,0) \mid a < x < 1\}$, respectively. Applying Claim 1 infinitely many times we can choose sequences $\{(A_i, B_i)\}$ and $\{(C_i, D_i)\}$ of good pairs of disks in $\mathring{Y}$ and $\mathring{Z}$, respectively, satisfying the following conditions:

(1) For all $i \neq j$, $A_i \cap A_j = \varnothing$ and $C_i \cap C_j = \varnothing$.

(2) Given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $A_i \cup C_i \subset B[(a,0),\varepsilon]$ whenever $i \geq N$.

Now by Lemmas 7.4 and 7.5, we can assume, without loss of generality, that for all i, f is bijective over $B_i \cup D_i$.

Let R be a closed subset of $H - I$ satisfying the following conditions:

(1) The closure of R in H is $R \cup \{(a,0)\}$ and that closure is a disk.

(2) For every i, ∂R meets each of B_i and D_i transversely in an arc.

(3) $\bigcup_i M_i \subset \mathring{R}$.

Denote $Q = f^{-1}(R)$. From bijectivity of f over $\left(\bigcup_i B_i\right) \cup \left(\bigcup_i D_i\right)$ we conclude that ∂Q has one noncompact component. Denote by Q_0 that component of Q containing the noncompact component of ∂Q. Then $f|Q_0 : Q_0 \to R$ has degree one and is therefore surjective. So for all i, $f(Q_0) \cap N_i \neq \varnothing$ and hence, from goodness of the pairs $\{(M_i, N_i)\}$ we have $f(Q - Q_0) \cap \left(\bigcup_{i=1}^{\infty} N_i\right) = \varnothing$. Now, because $R - \bigcup_{i=1}^{\infty} N_i$ is homeomorphic to $\mathbb{R} \times [0, 1)$, all components of $Q - Q_0$ can be "eliminated" (i.e., in the image, "pushed out of" R) by an admissible homotopy of f fixing f outside any preassigned neighborhood of $Q - Q_0$ in $X - A$ (the details of this argument are left to the reader). So we have established the following claim.

Claim 2. We can assume without loss of generality that Q is connected.

We will show that we can also assume without loss of generality that Q has one end which, by Lemma 7.6, will complete the proof of the arc proposition. To establish this, let $\{W_n\}_{n=1}^{\infty}$ be an exhaustion of R satisfying the following conditions, where Z_n denotes the closure in R of $R - W_n$:

(1) For all n, W_n is a disk such that $W_n \cap \partial R$ is an arc and $W_n \subset \operatorname{Int} W_{n+1}$ (where the interior is with respect to R).

(2) Given n there exists j such that
$$\partial W_n \cap \left(\bigcup_{i=1}^{\infty} N_i\right) = \partial W_n \cap [N_j - (N_{j+1} \cup N_{j-1})]$$
and this set is an arc.

(3) If, for some n and i, $Z_n \cap N_i \neq \varnothing$, then $M_i \subset Z_{n-1}$.

Now, by an argument similar to that which established Claim 2, we can assume the following without loss of generality: $(*)$ for all n, no component of $f^{-1}(Z_n)$ is sent by f into $Z_n - \left(\bigcup_{i=1}^{\infty} N_i\right)$ (we leave this to the reader, but note first that the closure of $Z_n - \left(\bigcup_{i=1}^{\infty} N_i\right)$ in R has two components, each of which is homeomorphic to H).

Now, for a given n, there must exist a component C of $f^{-1}(Z_n)$ such that $f|C : C \to Z_n$ has nonzero degree and is therefore surjective. If C' is another component of $f^{-1}(Z_n)$, we have by $(*)$ that, for some $j \in \mathbb{N}$, $f(C') \cap N_j \neq \varnothing$. By condition (3) for $\{W_n\}$ we then have $M_j \subset Z_{n-1}$ and hence, by goodness of the pair (M_j, N_j), we have that $C' \cup C$ is contained in a component of $f^{-1}(Z_{n-1})$. Hence, Q has one end. □

Notation. To avoid ambiguity in the sequel, the notation (a, b) (where $a, b \in \mathbb{R}$) will be used exclusively for open intervals in $\mathbb{R}$. The map $p_2 : \mathbb{R}^2 \to \mathbb{R}$ is projection to the second coordinate.

In the observations below (used in the proof of Lemma 7.7), R is a compact, orientable surface with boundary.

Observation. (1) *If two split maps from* $(R, \partial R)$ *to* $(B^2, \partial B^2)$ *are equal over* ∂B^2, *then they are admissibly homotopic.*

(2) *If* $0 < a < b < 1$ *and* $\varphi : \partial R \to \partial([0,1]^2)$ *is a map bijective over* $\{0,1\} \times [0,1]$, *then* φ *extends to a split map from* $(R, \partial R)$ *to* $([0,1]^2, \partial([0,1]^2))$ *which is bijective over* $[0,1] \times [a,b]$.

Proof. (1) The straight line homotopy between f and g is admissible.

(2) We leave this to the reader except noting that we can assume without loss of generality that R is planar. To see this, first use the classification of compact surfaces to show that $\mathring{R}$ contains a compact surface S such that R/S is a planar surface. □

Lemma 7.7. *Suppose* R *is a connected, orientable, noncompact surface having one end. Also suppose that* $\varphi : (R, \partial R) \to (Q, \partial Q)$ *is a proper, split map where* $Q = [a,b] \times (0,c]$ *(for some* a, b, c *with* $a < b$ *and* $c > 0$*) which is bijective over* $\{a,b\} \times (0,c]$. *Then there exists* $s < c$ *such that for any* $t < s$ *and* $0 < \varepsilon < \frac{c-t}{2}$ *there is a proper, admissible homotopy of* φ *supported in* $(a,b) \times (t,c)$ *to a map which is bijective over* $[a,b] \times [t+\varepsilon, c-\varepsilon]$.

Addendum. There is a straightforward generalization of Lemma 7.7 which we will need. In the hypothesis of that generalization connectivity of R is replaced by the following: R has finitely many components only one of which is noncompact. Then in the conclusion the number c is replaced by r with $0 < r < c$ such that no compact component of R meets $\varphi^{-1}([a,b] \times (0,r])$.

We leave the full statement and proof of the generalization to the reader. In the sequel, it will be understood that "Lemma 7.7" refers to this generalization.

Proof. By one-endedness of R we can choose s so that if $t \le s$, then the image of only one component of $\varphi^{-1}([a,b] \times [t,c])$ meets both components of $[a,b] \times \{t,c\}$. Let C denote that component. We can assume without loss of generality that the images of all other components of $\varphi^{-1}([a,b] \times [t,c])$ are in $[a,b] \times ([t, t+\varepsilon) \cup (c-\varepsilon, c])$. We leave it to the reader to complete the proof using the above observations. □

Now for the remainder of the proof of the factor reduction theorem we adopt the following notation: for $a > 0$ and $n \in \mathbb{N}$, $E_n = \left\{ \frac{k}{2^n} \;\middle|\; k \in \mathbb{Z} \text{ and } \left|\frac{k}{2^n}\right| < 1 \right\}$, $M_n = \max E_n$, $W\langle n, a\rangle = [-M_n, M_n] \times (0, a]$, and $Z\langle n, a\rangle = (E_n \times (0,a]) \cup ([-M_n, M_n] \times \{a\})$; for $b < 0$, $Z\langle n, b\rangle = \{-\vec{x} \mid \vec{x} \in Z\langle n, -b\rangle\}$ (and similarly for $W\langle n, b\rangle$).

We will show that, for some strictly decreasing sequence $\{a_n\}_{n=1}^{\infty}$ converging to zero, f is admissibly homotopic to a map which is bijective over $\bigcup_{k=1}^{\infty} Z\langle k, a_k\rangle$. We construct such a map as the limit of a sequence $\{f_n\}_{n=1}^{\infty}$ of split maps where, for each n, f_n is bijective over $\bigcup_{k=1}^{n} Z\langle k, a_k\rangle$ (for appropriately chosen $\{a_1, a_2, \dots, a_n\}$) and f_{n+1} is admissibly homotopic to f_n. The following observation will be used to ensure convergence of $\{f_n\}_{n=1}^{\infty}$.

Observation. *A sequence $\{g_n : (Y, A) \to (S^2, I)\}_{n=1}^{\infty}$ of admissible maps converges to a map admissibly homotopic to g_1 if, for each $n > 1$, g_{n-1} is homotopic to g_n by a homotopy supported in an open subset U_n of $S^2 - I$ having finitely many components and compact closure in $S^2 - I$ and satisfying the following: for all $n \neq m$, $U_n \cap U_m = \varnothing$, and*

$$\lim_{n\to\infty} \max\{\operatorname{diam}(\bar{C}) \mid C \text{ is a component of } U_n\} = 0.$$

Now, applying the arc proposition infinitely many times, construct a monotone strictly decreasing sequence $\{b_n\}_{n=1}^{\infty}$ of real numbers converging to zero and a map f_0 admissibly homotopic to f and bijective over $\bigcup_{n=1}^{\infty} E_n \times [0, b_n]$. The map f_0 is itself the limit of a sequence of maps each of which is obtained by an application of the arc proposition to its predecessor. The above observation is used to ensure convergence and to verify that the limit is admissibly homotopic to f. The details are left to the reader.

Now given f_0 we will construct f_1, then briefly indicate the construction of f_2. The induction step in full generality we leave to the reader.

To construct f_1, let F denote the closure in $H - \partial H$ of one of the components of $W\langle 1, b_1\rangle - Z\langle 1, b_1\rangle$ (which is $\left[\left(-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right)\right] \times (0, b_1)$). Apply Lemma 7.7 to $f_0 | f_0^{-1}(F) : f_0^{-1}(F) \to F$ for each choice of F. Choose the r and t (as in Lemma 7.7) to be the same for both applications. We are free to choose r small enough so that each of the two homotopies has support in an open rectangle whose diameter is less than one. Let f_1 be the map which results from the composition of the two homotopies. From the conclusion of Lemma 7.7 we can choose $a_1 < b_1$ such that f_1 is bijective over $\left[-\frac{1}{2}, \frac{1}{2}\right] \times \{a_1\}$.

Now to construct f_2, apply Lemma 7.7 to each map $f_1 | f_1^{-1}(F) : f_1^{-1}(F) \to F$ where $F = \left[\frac{k}{2^2}, \frac{k+1}{2^2}\right] \times (0, c_2]$ where $c_2 = \min\{a_1, b_2\}$ and k is an integer with $-3 \leq k \leq 2$. Choose a common r and t for the six applications of the lemma and furthermore choose r small enough so that each of the homotopies has support in a rectangle of diameter one half and furthermore that support is disjoint from the support of the previously constructed homotopy of f_0 to f_1. The composition of the six homotopies is the homotopy from f_1 to f_2. The conclusion of Lemma 7.7 allows us to choose $a_2 < c_1$ such that f_2 is bijective over $Z\langle 1, a_1\rangle \cup Z\langle 2, a_2\rangle$.

By "symmetry" we can now assume without loss of generality that the map f of the hypothesis is bijective over $\bigcup_{n=1}^{\infty}[Z\langle n, a_n\rangle \cup Z\langle n, b_n\rangle]$ where $\{a_n\}$ and $\{b_n\}$ are monotone strictly decreasing and monotone strictly increasing, respectively, and both sequences converge to zero. Note that the closure in S^2 of $\bigcup_{n=1}^{\infty}[W\langle n, a_n\rangle \cup W\langle n, b_n\rangle]$ is a 2-disk and denote the closed complement of that 2-disk minus the endpoints of I by Q (it is homeomorphic to $[-1,+1]\times\mathbb{R}$). Denote $R=f^{-1}(Q)$. We have that $f|R:R\to Q$ is a proper map of noncompact surfaces and $f|\partial R:\partial R\to\partial Q$ is bijective. Furthermore, it follows from one-endedness of $Y-A$ that $f|R:R\to Q$ is bijective on ends. It follows from the classification theorem for noncompact surfaces [Richards 1963; Prishlyak and Mischenko 2007] (and the special case required here can also be proven by applying Lemma 7.7 infinitely many times) that R can be constructed by first deleting the interiors of a proper family of pairwise-disjoint 2-disks in $\mathbb{R}\times[0,1]$, none of which meets $\mathbb{R}\times\{0,1\}$, and then attaching to the boundary of each 2-disk a once-punctured torus. Denote by S the decomposition space obtained from the decomposition of R whose only nondegenerate elements are the punctured tori. Up to admissible homotopy fixing $f|\partial R$, the map $f|R:R\to Q$ factors through a map $g:S\to Q$ whose end-point compactification is a split boundary-to-boundary map of 2-disks. Hence, from the observation preceding Lemma 7.7, that map is admissibly homotopic (fixing $g(\partial S)$ to a homeomorphism). So we can assume without loss that the map f sends each punctured torus to a point and is injective over the complement of the image of the union of all the punctured tori. So we can easily choose a proper, split embedding $(\mathbb{Z}\times[-1,+1],\mathbb{Z}\times\{-1,+1\})\to(Q,\partial Q)$ such that f is injective over the image of the embedding (which we denote by E). So now we can assume without loss of generality that the map f is bijective over $I\cup E\cup\big(\bigcup_{n=1}^{\infty}[Z\langle n,a_n\rangle\cup Z\langle n,b_n\rangle]\big)$, which we denote by Z.

Note that the closure of any component C of the complement of Z in S^2 is a 2-disk and that $f|f^{-1}(C):f^{-1}(C)\to C$ is a boundary-preserving map of compact, connected orientable surfaces which is bijective over ∂C and hence is homotopic (fixing $f|\partial f^{-1}(C)$) to a standard map. It follows easily that Y is a Pontryagin surface and A is $\mathbb{P}$-negligible.

8. Pontryagin disks

Recall that a *Pontryagin disk* $\mathbb{D}$ is a compact, connected subset of a rich Pontryagin surface P whose frontier relative to P is a simple closed curve. That curve is called the *boundary of* $\mathbb{D}$ and is denoted $\partial\mathbb{D}$. The subset $\mathbb{D}-\partial\mathbb{D}$ is the *interior of* $\mathbb{D}$, written $\operatorname{Int}\mathbb{D}$. By Corollary 3.2 every Pontryagin disk $\mathbb{D}$ has a rich family $\mathcal{E}$ of figure-eights, all of which lie in $\operatorname{Int}\mathbb{D}$; we shall assume that every sufficient family for a Pontryagin disk used here has this property.

Theorem 8.1. *Suppose $\mathbb{D}$ and $\mathbb{D}'$ are Pontryagin disks equipped with sufficient families $\mathcal{E}$ and $\mathcal{E}'$, respectively. Let $S=\mathbb{D}/\mathcal{E}$ and $S'=\mathbb{D}'/\mathcal{E}'$ be the associated decompositions and let $d:\mathbb{D}\to S$ and $d':\mathbb{D}'\to S'$ be the quotient maps. Let Z be a closed subset of S such that $Z\cap d(\mathcal{E})=\varnothing$, and let $h:S\to S'$ be a homeomorphism such that $h(Z)\cap d'(\mathcal{E}')=\varnothing$. Then for any $\varepsilon>0$ there exists a homeomorphism $H:\mathbb{D}\to\mathbb{D}'$ such that hd and $d'H$ are ε-close and equal on $d^{-1}(Z)$.*

The proof of the above theorem, which occupies most of this section, is deferred.

Corollary 8.2. *Every homeomorphism $\psi : \partial\mathbb{D} \to \partial\mathbb{D}'$ between the boundaries of Pontryagin disks $\mathbb{D}$, $\mathbb{D}'$ extends to a homeomorphism $\Psi : \mathbb{D} \to \mathbb{D}'$.*

Proof. Let $\mathcal{E}$ and $\mathcal{E}'$ be full families for $\mathbb{D}$ and $\mathbb{D}'$, respectively. Recall that by convention no elements of $\mathcal{E}$ or $\mathcal{E}'$ meet $\partial\mathbb{D}$ or $\partial\mathbb{D}'$. Let $B = \mathbb{D}/\mathcal{E}$ and $B' = \mathbb{D}'/\mathcal{E}'$ denote the usual decompositions and $d : \mathbb{D} \to B$ and $d' : \mathbb{D}' \to B'$ the quotient maps. Since B and B' are 2-disks the homeomorphism $d' \circ \psi \circ (d|\partial B)^{-1} : \partial B \to \partial B'$ extends to a homeomorphism $h : B \to B'$. Apply Theorem 8.1 with $Z = \partial S$. □

Corollary 8.3. *Let J and J^* denote separating simple closed curves in closed, rich Pontryagin surfaces P and P^*, respectively. Then any homeomorphism $h : J \to J^*$ can be extended to a homeomorphism $H : P \to P^*$.*

Proof. Each of J and J^* bounds two Pontryagin disks in their respective Pontryagin surfaces. Apply Corollary 8.2. □

Theorem 8.1 also supplies an affirmative answer to a question raised by D. Repovš on several occasions back in the 1990s. The argument for Corollary 8.4 below also yields that Cantor sets in connected rich Pontryagin surfaces are homogeneously embedded.

Corollary 8.4. *Suppose that $\mathbb{D}$ and $\mathbb{D}'$ are Pontryagin disks and that $K \subset \operatorname{Int}\mathbb{D}$ and $K' \subset \operatorname{Int}\mathbb{D}'$ are Cantor sets. Then each homeomorphism $h : \partial\mathbb{D} \cup K \to \partial\mathbb{D}' \cup K'$ extends to a homeomorphism $H : \mathbb{D} \to \mathbb{D}'$.*

Proof. Here K is $\mathbb{P}$-negligible in $\operatorname{Int}\mathbb{D}$, so there exists a full collection $\mathcal{E}$ of figure-eights for $\mathbb{D}$, all of which lie in $\operatorname{Int}\mathbb{D} - K$. Similarly, there exists a full collection $\mathcal{E}'$ of figure-eights for $\mathbb{D}'$, all of which lie in $\operatorname{Int}\mathbb{D}' - K'$. Apply Theorem 8.1 using the obvious decompositions. □

Definition. Let $\mathbb{D}$, $\mathcal{E}$, and d be as in Theorem 8.1. A *utilitarian web W for S* is a finite collection $\{B_i\}$ of 2-cells in S that cover S, whose boundaries miss $d(\mathcal{E})$, and for $i \neq j$, $B_i \cap B_j$ is either empty or a connected subset of the boundary of each. (A utilitarian web is a generalized triangulation.) We define utilitarian webs W on appropriate quotients of closed Pontryagin surfaces similarly. We will refer to the union of the boundaries as the 1-*skeleton* of W. We will call two such webs W, W' for S *equivalent* if there exists a homeomorphism $h : S \to S$ that induces a bijection from the cells of W to the cells of W'.

The following is an immediate consequence of Theorem 8.1:

Corollary 8.5. *Under the hypotheses of Theorem 8.1, let W be a utilitarian web for S and let $h : S \to S'$ be a homeomorphism that carries the 1-skeleton T of W into $S' - d'(\mathcal{E}')$. Then there exists a homeomorphism $H : \mathbb{D} \to \mathbb{D}'$ such that $hd(t) = d'H(t)$ for all $t \in d^{-1}(T) \cup \partial\mathbb{D}$.*

We state several definitions before returning to the proof of Theorem 8.1.

Definition. A Pre$\mathbb{P}$ *space* X is a space equipped with two subspaces denoted $E(X)$ and $C(X)$ satisfying the following: X is a compact, connected, orientable surface with connected boundary; $E(X)$ is the union of a finite family of pairwise-disjoint figure-eights in $\mathring{X}$ such that, for each figure-eight $e \in E(X)$, X/e is a surface; $C(X)$ is a countable dense subspace of $\mathring{X}$ disjoint from $E(X)$.

A map $f : X \to Y$ of Pre$\mathbb{P}$ spaces is a Pre$\mathbb{P}$ *map* if it is standard and satisfies $\{y \in Y \mid |f^{-1}(y)| \neq 1\} \subseteq C(Y)$ and $f(E(X) \cup C(X)) = E(Y) \cup C(Y)$. Note that compositions of Pre$\mathbb{P}$ maps are Pre$\mathbb{P}$.

A *diagram* is a set $\mathcal{D}$ of surjective maps of compact metric spaces satisfying the following conditions: the range of no element of $\mathcal{D}$ is the same space as its domain; no two elements of $\mathcal{D}$ have both the same domain and the same range. A *derived* map of a diagram $\mathcal{D}$ is a map which is the composition of elements of $\mathcal{D}$ such that no element of $\mathcal{D}$ appears more than once in the factorization and the domain and range of the composite map are different (subsequently, when we refer to a "factorization" of a derived map, it will be understood that the factorization satisfies this condition).

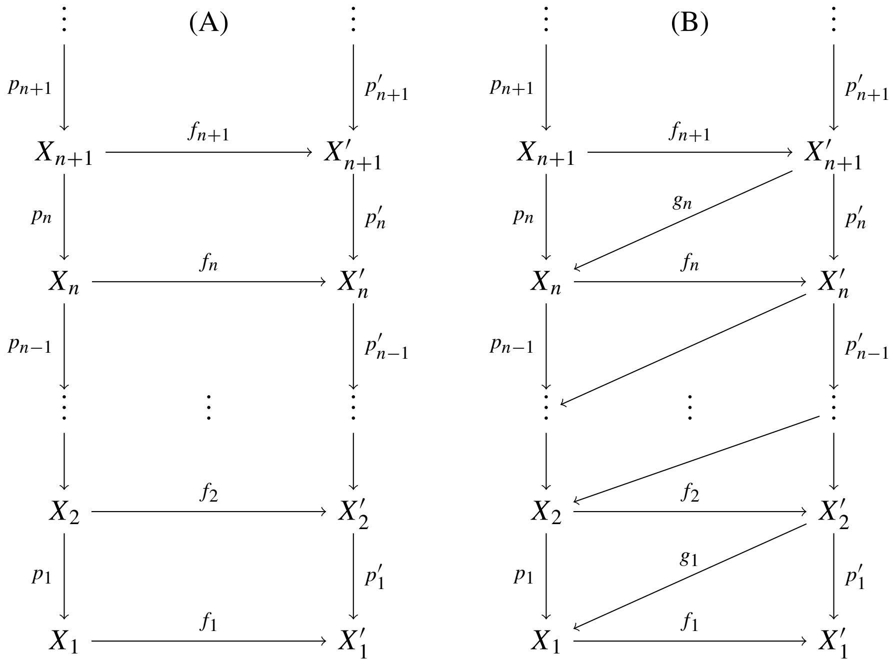
A *modulus of continuity* of a diagram is a function $\delta : (0, +\infty) \to (0, +\infty)$ which is a modulus of continuity for every derived map of the diagram (i.e., given a derived map f of the diagram, $\varepsilon > 0$, and points x and y in the domain of f which are $\delta(\varepsilon)$-close we have that $f(x)$ and $f(y)$ are ε-close). Note that any finite diagram has a modulus of continuity.

A pair $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ of derived maps of a diagram is *allowable* if f and g have factorizations such that if $\mathcal{A}$ and $\mathcal{B}$ denote the sets of spaces appearing in the factorizations of f and g, respectively, then $\mathcal{A} \cap \mathcal{B} = \{X, Y\}$.

A diagram $\mathcal{D}$ is ε-*commutative* if the two maps of any allowable pair are ε-close. An infinite diagram is *asymptotically commutative* if, given $\varepsilon > 0$, there exists a finite subset $\mathcal{D}_\varepsilon$ of $\mathcal{D}$ such that $\mathcal{D} - \mathcal{D}_\varepsilon$ is ε-commutative.

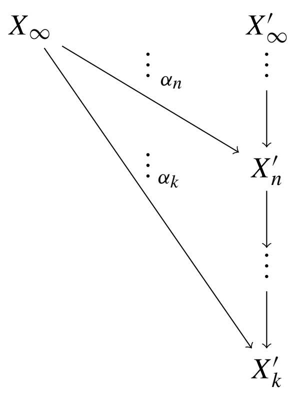
Observation. *Given an inverse sequence $\{p_n : X_{n+1} \to X_n\}_{n=1}^\infty$ of surjective maps of compact metrizable spaces with limit X_∞, there exist metrics $\{\rho_n\}_{n=1}^\infty$ for $\{X_n\}_{n=1}^\infty$ and ρ_∞ for X_∞ so that, for any $x, y \in X_\infty$, the sequence $\{\rho_n(p_{\infty,n}(x), p_{\infty,n}(y))\}_{n=1}^\infty$ (where $p_{\infty,n} : X_\infty \to X_n$ is the projection map) is strictly increasing and has limit $\rho_\infty(x, y)$.* (We leave the proof to the reader.)

Definition. Diagrams of the following two forms will be referred to as *type A* and *type B* diagrams if, in each case, the metrics in the vertical columns (which are inverse sequences) satisfy the condition stated in the above observation:



Lemma 8.6. *Given an asymptotically commutative type A diagram $\mathcal{D}$ (with notation as in the definition) there exists a map $f_\infty : X_\infty \to X'_\infty$ having the following property: for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $p'_{\infty,n} \circ f_\infty$ and $f_n \circ p_{\infty,n}$ are ε-close.*

Proof. From asymptotic commutativity of $\mathcal{D}$ we have that, for any $k \in \mathbb{N}$, the sequence of maps $\{p'_{n,k} \circ f_n \circ p_{\infty,n} : X_\infty \to X'_k\}_{n>k}$ converges uniformly. Denoting the limit by α_k, we have that the diagram

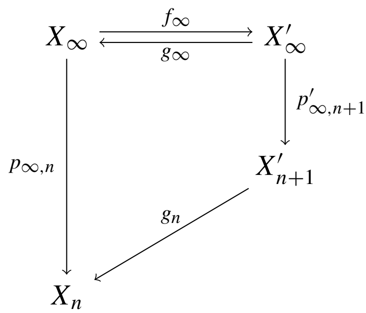


commutes. The inverse limit is f_∞. □

Note. We will refer to the map f_∞ in the conclusion of Lemma 8.6 as the *limit* of $\mathcal{D}$.

Lemma 8.7. *Suppose $\mathcal{D}$ is an asymptotically commutative type B diagram and let f_∞ and g_∞ be the limits, respectively, of the two type A diagrams obtained when the $\{g_i\}$ are deleted from $\mathcal{D}$ and when the $\{f_i\}$ are deleted from $\mathcal{D}$. Then f_∞ and g_∞ are inverses.*

Proof. To show that g_∞ is a left inverse for f_∞ it suffices to show, given $\varepsilon > 0$ and $a \in X_\infty$, that $(g_\infty \circ f_\infty)(a)$ is within ε of a. To see this, extract the subdiagram



We can choose n large enough so that the following conditions are satisfied:

(1) There exists $\delta > 0$ such that if $E \subset X_n$ has diameter less than 2δ, then $p^{-1}_{\infty,n}(E)$ has diameter less than ε.

(2) The subdiagram is δ-commutative.

By condition (2), then $(p_{\infty,n} \circ g_\infty \circ f_\infty)(a)$ is within δ of $(g_n \circ p'_{\infty,n+1} \circ f_\infty)(a)$ and the latter is within δ of $p_{\infty,n}(a)$. The desired inequality follows from condition (1).

The proof that f_∞ is a left inverse for g_∞ is similar. □

Lemma 8.8. *Suppose X is a compact metric space, R is an open surface which is an open subset of X, A and B are countable, dense subsets of R, and $\varepsilon > 0$. Then there exists a split homeomorphism $\varphi : (X, R) \to (X, R)$ supported in R and ε-close to the identity such that $\varphi(A) = B$.*

Proof. Brouwer [1913] and Fréchet [1910], independently, proved that Euclidean space is countable dense homogeneous. This is a mild generalization of their result. We provide some details for completeness.

The idea is to produce φ as a limit of a sequence $\varphi_k : X \to X$ supported in R. For each $k \geq 1$ we will determine a homeomorphism $h_k : X \to X$ supported in a very small 2-disk Δ_k in R and then will set $\varphi_k = h_k \varphi_{k-1}$ (here $\varphi_0 =$ Identity).

List the elements $\{a_1, a_2, \ldots\}$ of A and likewise the elements $\{b_1, b_2, \ldots\}$ of B.

When $k = 2m - 1$, Δ_k will be centered at $\varphi_{k-1}(a_m)$ and will contain no other point of

$$\varphi_{k-1}(\{a_1, a_2, \ldots, a_{m-1}\}) \cup \{b_1, b_2, \ldots, b_{m-1}\} \subset B.$$

If $\varphi_{k-1}(a_m) \in B$. then h_k will be the identity; otherwise, apply density of B to obtain $b_j \in B \cap \operatorname{Int} \Delta_k$ and choose h_k so $h_k \varphi_{k-1}(a_m) = b_j$.

When $k = 2m$, Δ_k will be centered at $\varphi_{k-1}(b_m)$ and contain no other point of

$$\varphi_{k-1}(\{a_1, a_2, \ldots, a_m\}) \cup \{b_1, b_2, \ldots, b_{m-1}\} \subset B.$$

If there exists $a_j \in A$ such that $\varphi_{k-1}(a_j) = b_m$, then h_k will be the identity; otherwise, apply density of A to obtain $a_j \in A \cap \operatorname{Int} \Delta_k$ and choose h_k so $h_k \varphi_{k-1}(a_j) = b_m$.

In short, at odd-numbered stages of the process, a point of A is shifted into B, in orderly fashion, and at even-numbered stages a point of B is caused to be the image of some point of A. Once such arrangements are made, no further adjustment of those special points is allowed at later stages, so those arrangements persist to the limit map φ. Eventually all points of A are moved into B and all from B are covered.

Simply by choosing the Δ_k of diameter less than $\varepsilon/2^k$, we can assure that the sequence $\{\varphi_k\}$ converges uniformly to a continuous function φ ε-close to the identity. Furthermore, φ will restrict to the identity on $X - R$ and will be surjective over X.

At any stage $k > 1$ in this process, we can determine $\eta_{k-1} > 0$ such that points of X at least $1/k$ apart have image under φ_{k-1} at least η_{k-1} apart. Thus, by requiring Δ_k to have diameter less than $\eta_i/2^{k-i}$ for $i = 1, 2, \ldots, k-1$, we assure injectivity of φ. As a result, φ is a split homeomorphism of (X, R) to itself. □

Lemma 8.9. *Suppose $f : X \to X'$ and $p : Y \to X'$ are* Pre$\mathbb{P}$ *maps such that $\{x \in X' \mid |f^{-1}(x)| \neq 1\} \subseteq \{x \in X' \mid |p^{-1}(x)| \neq 1\}$ and Z is a closed subset of X' disjoint from $E(X') \cup C(X')$. Then given $\varepsilon' > 0$ there exists a* Pre$\mathbb{P}$ *map $g : Y \to X$ such that p and $f \circ g$ are ε'-close and equal over Z.*

Proof. Denote $\{x_1, x_2, \ldots, x_n\} = \{x \in X' \mid |p^{-1}(x)| \neq 1\}$. Let $\{D_i\}_{i=1}^n$ be a pairwise-disjoint family of 2-disks in $\mathring{X}'$ such that for each i

$$x_i \in \mathring{D}_i, \qquad \operatorname{diam} D_i < \varepsilon', \qquad \partial D_i \cap [E(X') \cup C(X') \cup Z] = \varnothing.$$

Define g as follows. For $x \notin \bigcup_{i=1}^n p^{-1}(\mathring{D}_i)$ we define $g(x) = f^{-1}(p(x))$ (this can be done since f is injective over the complement of $\bigcup_i \mathring{D}_i$). For each i we define $\alpha_i = g|p^{-1}(D_i) : p^{-1}(D_i) \to f^{-1}(D_i)$ as follows. Note first that, for all i, $p^{-1}(D_i)$ is a disk with a handle and $f^{-1}(D_i)$ is either a disk with a handle or simply a disk. In the first case choose a homeomorphism α_i satisfying the following conditions:

(1) $\alpha_i|p^{-1}(\partial D_i) = f^{-1} \circ p|p^{-1}(\partial D_i)$.

(2) α_i carries $p^{-1}(x_i)$ onto $f^{-1}(x_i)$.

(3) $\alpha_i(C(Y) \cap p^{-1}(D_i)) = C(X) \cap f^{-1}(D_i)$.

Note that (2) can be achieved since the figure-eight in a disk with a handle is unique up to homeomorphism fixing the boundary and (3) can be achieved using Lemma 8.8.

In the second case ($f^{-1}(D_i)$ is a disk), just choose α_i so that $\alpha_i^{-1}(f^{-1}(x_i)) = p^{-1}(x_i)$ and $p^{-1}(x_i)$ is the only nontrivial point preimage. Again Lemma 8.8 allows us to achieve condition (3) above. □

Lemma 8.10. *Suppose $\mathcal{D}$ is a finite ε-commutative diagram, δ is a modulus of continuity for $\mathcal{D}$, and $f : Z \to Y$ and $p : X \to Y$ are maps in $\mathcal{D}$ such that X is neither the domain nor codomain of any map in $\mathcal{D}$ other than p. If $r > 0$, $\varepsilon' \leq \delta(r)$, and $g : X \to Z$ is a map not in $\mathcal{D}$ and such that $f \circ g$ is ε'-close to p, then the diagram $\mathcal{D} \cup \{g\}$ is σ-commutative where $\sigma = \max\{\varepsilon', \varepsilon + r\}$.*

Proof. Suppose $\{\varphi, \psi\}$ is an allowable pair in $\mathcal{D} \cup \{g\}$. If each of p, f, and g is a factor of φ or a factor of ψ, then they must be the only factors in the two factorizations and we are done by hypothesis. If neither of the factorizations of φ and ψ include g, then $\{\varphi, \psi\}$ is an allowable pair in $\mathcal{D}$ and again we are done. The remaining possibility is that g and p are the initial factors of φ and ψ and f is a factor of neither. We will show that, in this case, φ and ψ are $(\varepsilon + r)$-close. We can write $\varphi = \alpha \circ g$ and $\psi = \beta \circ p$ where α and β are derived maps of $\mathcal{D}$. Consider the three maps $\beta \circ p$, $\beta \circ f \circ g$, and $\alpha \circ g$. The first and second are r-close because p and $f \circ g$ are ε'-close (and $\varepsilon' \leq \delta(r)$). The second and third are ε-close since the distance between them is the same as the distance between $\beta \circ f$ and α (an allowable pair in $\mathcal{D}$). The triangle inequality concludes the argument. □

Proof of Theorem 8.1. First note that we can assume (without loss of generality) $\partial S \subset Z$. Using Proposition 2.1 we write

$$\mathbb{D} = X_\infty = \varprojlim\{p_n : X_{n+1} \to X_n\}_{n=1}^{\infty},$$
$$\mathbb{D}' = X'_\infty = \varprojlim\{p'_n : X'_{n+1} \to X'_n\}_{n=1}^{\infty},$$

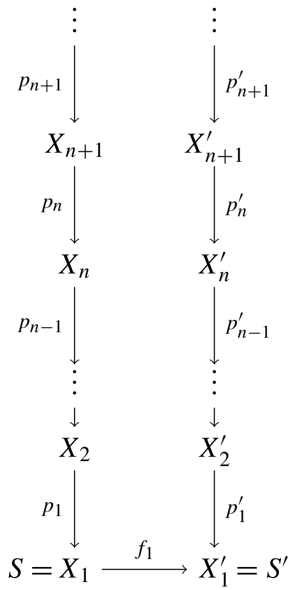
where $X_1 = S$ and $X'_1 = S'$. Furthermore, we can assume that the metrics for these spaces satisfy the conditions stated in the observation following the definition of ε-commutative. For each $n \in \mathbb{N}$ we set

$$C(X_n) = p_{\infty,n}(\{e \in \mathcal{E} \mid |p_{\infty,n}(e)| = 1\}),$$
$$E(X_n) = p_{\infty,n}(\{e \in \mathcal{E} \mid |p_{\infty,n}(e)| \neq 1\})$$

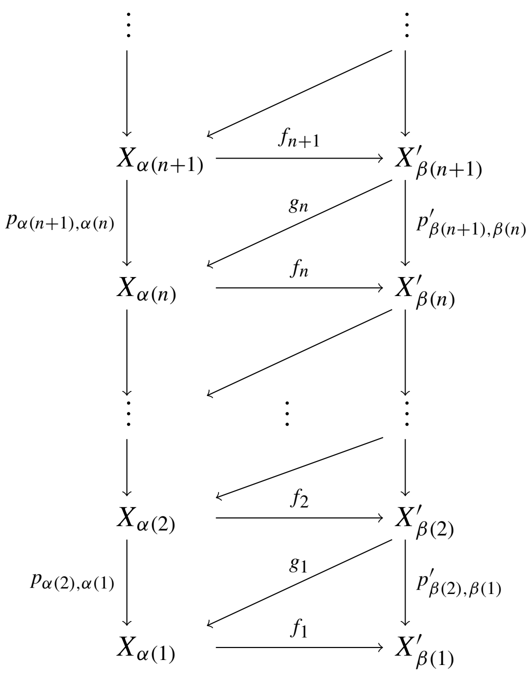
and similarly for $C(X'_n)$ and $E(X'_n)$. This makes the maps $\{p_n\}$ and $\{p'_n\}$ Pre$\mathbb{P}$. Note also that $d = p_{\infty,1}$ and $d' = p'_{\infty,1}$.

The following argument shows that h can be approximated by Pre$\mathbb{P}$ maps and hence we can assume without loss that it is Pre$\mathbb{P}$. Applying Lemma 8.8 to the pair $(X'_1, X'_1 - h(Z))$ and the subsets $h(C(X_1))$ and $C(X'_1)$ we obtain a homeomorphism $\varphi : X'_1 \to X'_1$ fixing $h(Z)$ and throwing $h(C(X_1))$ onto $C(X'_1)$ and ε-close to the identity for any preassigned ε. So $\varphi \circ h$ is Pre$\mathbb{P}$ and arbitrarily close to h. We denote

$h = f_1$ and assume f_1 is Pre$\mathbb{P}$. So we have an infinite diagram of Pre$\mathbb{P}$ maps:



We will construct the infinite diagram of Pre$\mathbb{P}$ maps (denoted $\mathcal{D}$),



where α and β are increasing functions from $\mathbb{N}$ to $\mathbb{N}$ with $\alpha(1) = \beta(1) = 1$.

Before listing the properties which $\mathcal{D}$ will have, we adopt some notation: $\mathcal{D}_k^{(n)}$ (for $1 \le k < n$) will denote the finite subset of $\mathcal{D}$ consisting of all maps, each of which is either f_k, f_n, or a map in $\mathcal{D}$ above f_k and below f_n; define Z_n inductively by $Z_1 = Z$ and $Z_{n+1} = p^{-1}_{\alpha(n+1),\alpha(n)}(Z_n)$.

The diagram $\mathcal{D}$ will satisfy the following two conditions:

(1) For all n and $1 \le k < n$, $\mathcal{D}_k^{(n)}$ is $\varepsilon \cdot \left[\sum_{m=k}^{n} \frac{1}{2^m}\right]$-commutative.

(2) f_n and g_n are injective over $f_n(Z_n)$ and Z_n, respectively.

It follows immediately from the above properties that $\mathcal{D}$ is asymptotically commutative and that the map f_∞ provided by Lemma 8.6 and guaranteed to be a homeomorphism by Lemma 8.7 will serve as the desired H.

The construction of $\mathcal{D}$ is accomplished by producing inductively the sequence $\{\mathcal{D}_1^{(n)}\}_{n=2}^{\infty}$ of subdiagrams. The construction of $\mathcal{D}_1^{(n+1)}$ from $\mathcal{D}_1^{(n)}$ is carried out in two stages. First $\beta(n+1)$ and g_n are chosen, then $\alpha(n+1)$ and f_{n+1}. In each stage, Lemma 8.9 is used to construct the desired map. The choices of ε' (as in Lemma 8.9) which will ensure the necessary approximate commutativity of $\mathcal{D}_k^{(n+1)} (k < n)$ are dictated by Lemma 8.10.

So in determining $\beta(n+1)$ first note that given any finite subset S of $C(X'_{\beta(n)})$ there exists $m \in \mathbb{N}$ with $m > \beta(n)$ such that, for each $x \in S$, $(p'_{m,\beta(n)})^{-1}(x)$ is not a singleton. Let $\beta(n+1)$ be such an m for the set $S = \{x \in C(X'_{\beta(n)}) \mid |f_n^{-1}(x)| \neq 1\}$. This ensures that the maps f_n and $p'_{\beta(n+1),\beta(n)}$ satisfy the hypothesis of Lemma 8.9 (where $\mathcal{D}_1^{(n)} \cup \{p'_{\beta(n+1),\beta(n)}\}$ plays the role of $\mathcal{D}$ and Z_n plays the role of Z in the application of that lemma). For the ε' we choose $\min\left\{\frac{\varepsilon}{2^{n+2}}, \delta_0\left(\frac{\varepsilon}{2^{n+2}}\right)\right\}$ where δ_0 is a modulus of continuity for $\mathcal{D}_1^{(n)} \cup \{p'_{\beta(n+1),\beta(n)}\}$ (and hence also for $\mathcal{D}_k^{(n)}$ for any $k < n$). The application of Lemma 8.9 produces the map g_n and Lemma 8.10 guarantees that, for any $k < n$, $\mathcal{D}_k^{(n)} \cup \{p'_{\beta(n+1),\beta(n)}, g_n\}$ is $\varepsilon \cdot \left(\frac{1}{2} \cdot \frac{1}{2^{n+1}} + \sum_{m=k}^{n} \frac{1}{2^m}\right)$-commutative. Now to construct f_n, first choose $\alpha(n+1)$ larger than $\alpha(n)$ and large enough so that the maps g_n and $p_{\alpha(n+1),\alpha(n)}$ satisfy the hypothesis of Lemma 8.9.

In preparing to apply Lemma 8.10 we choose $r = \varepsilon/2^{n+2}$ and choose $\varepsilon' = \min\{r, \delta_1(r)\}$, where δ_1 is a modulus of continuity for $\mathcal{D}_1^{(n)} \cup \{p'_{\beta(n+1),\beta(n)}, g_n\}$. Upon applying Lemma 8.9 we obtain the map f_{n+1}. We conclude from Lemma 8.10 that, for $k < n+1$, $\mathcal{D}_k^{(n+1)}$ is $\varepsilon \cdot \left(\frac{1}{2} \cdot \frac{1}{2^{n+1}} + \frac{1}{2} \cdot \frac{1}{2^{n+1}} + \sum_{m=k}^{n} \frac{1}{2^m}\right)$-commutative. □

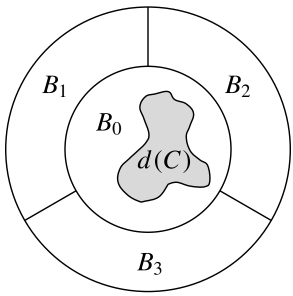
9. Pontryagin cellularity

A compact subset of a Pontryagin surface is *Pontryagin cellular* if it can be expressed as a nested intersection of Pontryagin disks $\mathbb{D}_1, \mathbb{D}_2, \ldots$ where $\mathbb{D}_{i+1} \subset \operatorname{Int} \mathbb{D}_i$ for all i.

Pontryagin cellular subsets of Pontryagin surfaces have some features analogous to those of cellular subsets of genuine surfaces.

Proposition 9.1. *Let C be a compact subset of a rich Pontryagin surface P. Then the decomposition G_C of P whose only nondegenerate element is C is shrinkable if and only if C is Pontryagin cellular in P.*

Proof. The forward implication follows immediately from [Daverman 1986, Proposition 5.12]. For the reverse, given a neighborhood U of C, find a Pontryagin disk $\mathbb{D}$ such that $C \subset \operatorname{Int}\mathbb{D} \subset \mathbb{D} \subset U$. Let $\mathcal{E}$ be a full family of figure-eights for $\mathbb{D}$ and let $d : \mathbb{D} \to B = \mathbb{D}/\mathcal{E}$ denote the quotient map to the resulting disk B. Cover B with a utilitarian web of four disks B_0, B_1, B_2, B_3, as shown below, where $B_0 \subset \mathring{B}$ contains $d(C)$:



Specify a homeomorphism $h : B \to B$ that restricts to the identity on ∂B, that carries B_0 to a disk B_0' whose preimage in P is small, and that sends each of the ∂B_i into $B - d(\mathcal{E})$. Then Corollary 8.5 promises a homeomorphism $H : \mathbb{D} \to \mathbb{D}$ that restricts to the identity on $\partial\mathbb{D}$ and that carries $d^{-1}(B_0)$ to the small set $d^{-1}(B_0')$. Finally, H extends to the rest of P via the identity to give a homeomorphism showing that G_C is shrinkable. □

Proposition 9.2. *A compact subset C of a rich Pontryagin surface P is Pontryagin cellular if and only if C is connected and $P - C$ has an isolated end corresponding to C.*

Proof. The forward implication is routine. For the reverse, note that we can assume P is compact (in view of Proposition 4.3) and connected. Then P/C is both the one-point and end-point compactification of $P - C$. As such, it has a sufficient family $\mathcal{E}_C$ of figure-eights, each of which is contained in $P - C$. Name the quotient map $\psi : P \to P' = P/C$ and the decomposition map $d' : P' \to P'/\mathcal{E}_C$. Given any open subset U of P containing C, one can find a small 2-disk neighborhood B of the point $d'\psi(C)$ in $P'/\mathcal{E}_C$ whose frontier is a simple closed curve missing $d'(\mathcal{E}_C)$, where B satisfies

$$C \subset (d'\psi)^{-1}(\mathring{B}) \subset (d'\psi)^{-1}(B) \subset U.$$

Clearly $(d'\psi)^{-1}(B)$ is a Pontryagin disk. Hence, C is Pontryagin cellular. □

The following observation is used in the proof of the corollary below (other details of which are left to the reader).

Observation. *Suppose X and Y are connected, nice spaces and e is an isolated end of Y. If $f : X \to Y$ is a proper, surjective map which is monotone over some neighborhood of e, then only one end of X is sent to e by f.*

Proof. First note that we can assume without loss that e is the only end of Y (consider $\hat{\hat{f}}| : \hat{\hat{X}} - (\hat{\hat{f}})^{-1}(e) \to \hat{\hat{Y}} - \{e\}$). Supposing that X has more than one end, there exists a neighborhood W of ∞ in X having at least two components which meet ∞. By one-endedness of Y we can find neighborhoods M and N of infinity such that f is monotone over M, $N \subset \mathring{M}$, $\overline{M - N}$ is connected, and $f^{-1}(\overline{M - N}) \subset W$. Hence, $f^{-1}(\overline{M - N})$ is not connected but $f| : f^{-1}(\overline{M - N}) \to \overline{M - N}$ is monotone, thus contradicting the Vietoris–Begle mapping theorem. □

Corollary 9.3. *Let $f : P \to Q$ be a proper, monotone map between Pontryagin surfaces, with P a rich Pontryagin surface. Then each $f^{-1}(q)$, $q \in Q$, is Pontryagin cellular.*

10. Decompositions induced over 1-dimensional subsets and proof of the monotone approximation theorem

The final section of this paper culminates in a proof of the monotone approximation theorem. A key step involves showing how to approximate a given monotone map by one that is injective over certain graphs in the target space.

Proposition 10.1. *Suppose X and Y are compact metrizable spaces and $C_1, C_2, \ldots$ are closed subsets of Y such that, for any surjective monotone map $f : X \to Y$, each of the decompositions $G(C_i)$ induced by f over C_i is shrinkable. Then any such map f can be approximated by a monotone map F that is* 1-1 *over $\bigcup_i C_i$. Moreover, if K is a closed subset of Y such that f is* 1-1 *over K and each of the $G(C_i)$ can be shrunk keeping points over K fixed, then F can be obtained which agrees with f over K.*

Proof. This is a standard Baire category argument. In the complete metric space $\mathscr{S}$ of all surjective, monotone maps $X \to Y$, the collection $O_{j,n}$ of maps f such that diam $f^{-1}(c) < 1/n$ for all $c \in C_j$ is open, for the usual reasons, and is dense by hypothesis. Any map from the dense subset $\bigcap_{j,n} O_{j,n}$ is 1-1 over $\bigcup_j C_j$.

For the additional control over K, take F as above but form the complete metric subspace of $\mathscr{S}$ consisting of monotone maps $X \to Y$ that agree with f over K. □

Proposition 9.1, Corollary 9.3, and Proposition 10.1 combine to yield:

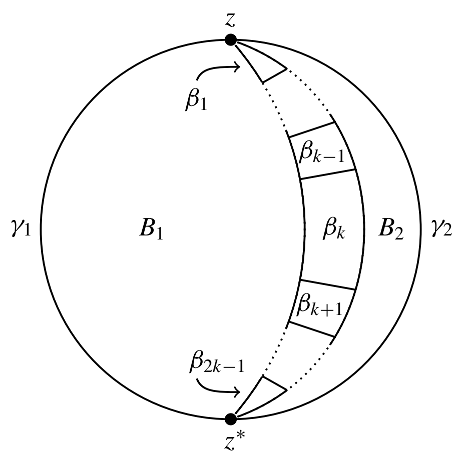
Corollary 10.2. *Let $f : P \to Q$ be a monotone map between rich, closed Pontryagin surfaces, Z a countable subset of Q, and K a closed subset of Q such that f is* 1-1 *over K. Then f can be approximated, arbitrarily closely, by a monotone map F that is* 1-1 *over $Z \cup K$ and agrees with f over K.*

It might be worth mentioning that the next lemma does not show the decomposition under consideration to be shrinkable. The output is merely a homeomorphism that carries decomposition elements to sets of small size — it is not subject to any motion control. A closely related shrinkability result will be established in the subsequent proposition using additional considerations.

Lemma 10.3. *Suppose $\mathbb{D}$ is a Pontryagin disk, $\mathcal{E}$ is a full family of figure-eights for $\mathbb{D}$, $d : \mathbb{D} \to B = \mathbb{D}/\mathcal{E}$ is the decomposition map, and G is a monotone upper semicontinuous decomposition of $\mathbb{D}$ such that the union N_G of all nondegenerate elements of G is a subset of* $\operatorname{Int} \mathbb{D}$ *and the closure of $d(N_G)$ meets ∂B in at most two points. Then for each $\varepsilon > 0$ there exists a homeomorphism $H_\varepsilon : \mathbb{D} \to \mathbb{D}$ such that H_ε restricts to the identity on $\partial\mathbb{D}$ and* $\operatorname{diam} H_\varepsilon(g) < \varepsilon$ *for all $g \in G$.*

Proof. Name the points z, z^* of ∂B containing $\partial B \cap d(ClN_G)$ and let γ_1, γ_2 denote the subarcs of ∂B bounded by these two points.

Identify a disk $B_1 \subset B - \mathring{\gamma}_2$ containing γ_1 such that $\mathring{B}_1$ contains the image under d of every $e \in \mathcal{E}$ with diameter $\varepsilon/6$ or more. Identify another disk $B_2 \subset B$ containing γ_2 that meets B_1 only at the points z and z^*. Cover the rest of B by a chain of 2-cells $\beta_1, \beta_2, \dots, \beta_{2k-1}$ such that β_i and β_j meet if and only if $|i - j| \le 1$, the intersection of successive cells β_i and β_{i+1} is an arc in the boundary of each, $z \in \beta_1$ and $z^* \in \beta_{2k-1}$, these β_i together with B_1, B_2 form a utilitarian web for B, and each $d^{-1}(\beta_i)$ has diameter less than $\varepsilon/3$. Furthermore, we can ensure that the 1-skeleton of this utilitarian web avoids the countable set $d(\mathcal{E})$. See:



Now produce an equivalent utilitarian web (equivalent via a homeomorphism fixing ∂B) in B involving 2-cells $B_1', B_2', \beta_1', \beta_2', \dots, \beta_{2k-1}'$. Here B_1', B_2' should lie very close to γ_1, γ_2, respectively, so as to miss $d(N_G)$. The β_i', except for β_1' and β_{2k-1}', are contained in $\mathring{B}$. In the construction procedure β_k' should be chosen first, and it should meet each of B_1' and B_2' in an arc. Next, β_{k-1}' and β_{k+1}' should be chosen so that any $d(g)$ $(g \in G)$ that meets β_k' lives in $\beta_{k-1}' \cup \beta_k' \cup \beta_{k+1}'$. (This is possible since, by upper semicontinuity and the hypothesis that $d(N_G) \cap \partial B = \varnothing$, elements of G with image near z or z^* are small.) That exposes the general strategy: the 2-cells β'_{k-2} and β'_{k+2} should be chosen, respectively, so that any $d(g)$ that meets β'_{k-1} but not β'_k is contained in $\beta'_{k-2} \cup \beta'_{k-1}$ and so that any $d(g)$ that meets β'_{k+1} but not β'_k is contained in $\beta'_{k+1} \cup \beta'_{k+2}$. The cells $\beta'_{k-3}, \beta'_{k+3}, \ldots$ should be chosen in turn so that, ultimately, any $d(g)$ (g nondegenerate) lies either in $\beta'_{k-1} \cup \beta'_k \cup \beta'_{k+1}$ or in the union $\beta'_{i-1} \cup \beta'_i$ of two successive β'_j. Specify a homeomorphism $h : B \to B$ taking B'_i to B_i and β'_j to β_j and fixing points of ∂B. The homeomorphism H_ε provided by Corollary 8.5 keeps points of $\partial \mathbb{D}$ fixed and shrinks elements of G to size less than ε, since the image of each nondegenerate $g \in G$ lies in some ε diameter set of the form $d^{-1}(\beta_j \cup \beta_{j+1} \cup \beta_{j+2})$. □

Proposition 10.4. *Let $f : P \to Q$ be a degree-one, monotone map between rich, closed Pontryagin surfaces and let A denote any locally separating arc or separating simple closed curve in Q. Then f can be approximated, arbitrarily closely, by monotone maps F that are* 1-1 *over A. Furthermore, the approximations F can be chosen to equal f over any closed subset K of Q such that f is* 1-1 *over K.*

Proof. We will treat only the case in which A is a locally separating arc in Q. The proof for simple closed curves is similar, or can be obtained from the result for arcs plus Proposition 10.1.

By Corollary 10.2 we can approximate f by another monotone map, which we continue to call f, that is 1-1 over a countable, dense subset of A containing ∂A and that agrees with the original f over K. Let $G(A)$ denote the decomposition of P induced by the modified f over A, and let $p : P \to X = P/G(A)$ denote the decomposition map. We show that $G(A)$ is shrinkable fixing points of K.

By the factor theorem (Theorem 3.1) X is a Pontryagin surface and has a full family $\mathcal{E}$ of figure-eights, each of which lives in $X - pf^{-1}(A)$. Let $d : X \to X/\mathcal{E}$ denote the decomposition map associated with the decomposition of X into points and these figure-eights.

Fix $\varepsilon > 0$. Note that d is 1-1 over $A' = dpf^{-1}(A)$. Note also that the closure of each component of $A - K$ has endpoints in $K \cup \partial A$ over which f is one-to-one. It follows easily that only a finite number of components of $A - K$ have preimage under f with diameter at least ε. We let γ denote one of those components. Since we will perform the same operations near each of these components, we assume γ is the only one.

Cover $\gamma' = dpf^{-1}(\gamma) \subset A'$ by a finite collection $B_1, \ldots, B_m$ of 2-cells in the surface $X/\mathcal{E}$. These 2-cells should have pairwise-disjoint interiors and those interiors should miss $dpf^{-1}(K)$, each B_i should meet γ' in an arc whose interior lies in $\operatorname{Int} B_i$, and should be small enough to assure that $d^{-1}(B_i)$ has diameter less than ε. The collection should be arranged so that dp is 1-1 over each $\partial B_i \cap A'$. As a consequence, each $\mathbb{D}_i = (dp)^{-1}(B_i)$ and each $\mathbb{D}'_i = f(dp)^{-1}(B_i)$ is a Pontryagin disk, with $\partial\mathbb{D}_i$ missing all the nondegenerate elements of $G(A)$.

Now apply Lemma 10.3 m times, using the decomposition induced by $f|\mathbb{D}_i : \mathbb{D}_i \to \mathbb{D}'_i$ on each $\mathbb{D}_i$, to obtain a homeomorphism $H_\varepsilon : P \to P$ that sends each $\mathbb{D}_i$ to itself, restricts to the identity on each $\partial\mathbb{D}_i$ as well as outside $\bigcup_i \mathbb{D}_i$, and sends every nondegenerate $g \in G(A)$ to a set of diameter less than ε. Note that, by construction of the $\mathbb{D}'_i$, f and H_ε are ε-close. Hence, H_ε establishes that $G(A)$ satisfies the shrinkability criterion via shrinking homeomorphisms that reduce to the identity over K.

As in the proof of Corollary 2.5, if $\theta : P \to P/G(A)$ is a homeomorphism very close to p, then $F = fp^{-1}\theta$ is a monotone map close to f which is 1-1 over A and which agrees with f over K. □

Corollary 10.5. *Let $f : P \to Q$ be a degree-one, monotone map between rich, closed Pontryagin surfaces, let $\mathcal{E}$ be a sufficient family of figure-eights for Q, with $d : Q \to S = Q/\mathcal{E}$ the quotient map, and let Γ denote the* 1*-skeleton of a utilitarian web for S. Then f can be approximated, arbitrarily closely, by a monotone map F that is* 1-1 *over $d^{-1}(\Gamma)$. Furthermore, if K is a closed subset of Γ such that f is* 1-1 *over K, then F can be chosen to be equal to f over K.*

Proof. Specify locally separating arcs $A_1, \ldots, A_k$ in Γ covering Γ and then employ Propositions 10.4 and 10.1. □

Proof of the monotone approximation theorem (*Theorem 2.2*). Let $f : P \to Q$ be a degree-one, monotone map between closed, connected, rich Pontryagin surfaces. Given $\varepsilon > 0$, specify a full family $\mathcal{E}_Q$ of figure-eights for Q, and let $\mathcal{E}'_Q$ denote the cofinite subcollection consisting of figure-eights of diameter less than $\varepsilon/4$. Let $d_Q : Q \to S = Q/\mathcal{E}'_Q$ be the associated quotient map to a closed surface S. Find a utilitarian web $W = \{B_1, \ldots, B_m\}$ in S with such small mesh that each $(d_Q)^{-1}(B_i)$ has diameter less than $\varepsilon/2$.

Use Corollary 10.5 to obtain another monotone map $F : P \to Q$ such that F is 1-1 over $d_Q^{-1}(\Gamma)$, where Γ is the 1-skeleton of W, and $\rho(F, f) < \varepsilon/2$.

At this juncture Q has been split into m Pontryagin disks $\mathbb{D}'_i = (d_Q)^{-1}(B_i)$ with pairwise-disjoint interiors, each of diameter less than $\varepsilon/2$. The map F lifts them to Pontryagin disks $\mathbb{D}_i = F^{-1}(\mathbb{D}'_i)$ in P, and F determines monotone maps $F_i = F|\mathbb{D}_i : \mathbb{D}_i \to \mathbb{D}'_i$ that restrict to homeomorphisms $\partial\mathbb{D}_i \to \partial\mathbb{D}'_i$. Corollary 8.2 promises the existence of homeomorphisms $\Phi_i : \mathbb{D}_i \to \mathbb{D}'_i$ that agree with F_i on $\partial\mathbb{D}_i$. By construction of $\mathbb{D}'_i$ each Φ_i is $\varepsilon/2$-close to F_i. Hence, $\Phi = \bigcup_i \Phi_i : \mathbb{D} = \bigcup_i \mathbb{D}_i \to \mathbb{D}' = \bigcup_i \mathbb{D}'_i$ is a homeomorphism which is $\varepsilon/2$-close to F and ε-close to f. □

Theorem 10.6. *Let $f : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{D}', \partial\mathbb{D}')$ be a split monotone map between Pontryagin disks and $K \supset \partial\mathbb{D}'$ a closed subset of $\mathbb{D}'$ such that f is* 1-1 *over K. Then*

f can be approximated, arbitrarily closely, by a homeomorphism $\Phi : \mathbb{D} \to \mathbb{D}'$ such that $\Phi | f^{-1}(K) = f | f^{-1}(K)$.

Proof. The only change to the proof of the monotone approximation theorem required in the Pontryagin disks setting is that in applying Corollary 10.5 one should obtain a monotone map $F : \mathbb{D} \to \mathbb{D}'$ that is 1-1 over the 1-skeleton as before and agrees with f over K. □

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