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We introduce a new class of commutative nonnoetherian rings, called n-subperfect rings, generalizing the almost perfect rings that have been studied recently by Fuchs and Salce. For an integer $n \geq 0$, the ring R is said to be n-subperfect if every maximal regular sequence in R has length n and the total ring of quotients of R/I for any ideal I generated by a regular sequence is a perfect ring in the sense of Bass. We define an extended Cohen–Macaulay ring as a commutative ring R that has noetherian prime spectrum and each localization R_M at a maximal ideal M is $\mathrm{ht}(M)$-subperfect. In the noetherian case, these are precisely the classical Cohen–Macaulay rings. Several relevant properties are proved reminiscent of those shared by Cohen–Macaulay rings.

1. Introduction

The Cohen–Macaulay rings play extremely important roles in most branches of commutative algebra. They have a very rich, fast expanding theory and a wide range of applications where the noetherian hypothesis is essential in most aspects. Cohen–Macaulay rings R are usually defined in one of the following ways:

(a) R is a noetherian ring in which ideals generated by elements of regular sequences are unmixed (i.e., have no embedded primes).

(b) R is a noetherian ring such that the grade (the common length of maximal regular sequences in I) of every proper ideal I equals the height of I.

Several branches of the theory of noetherian rings are known to have natural generalizations to the nonnoetherian case, but there is none that still shares more than a few of the many useful properties of Cohen–Macaulay rings. As a matter of fact, there have been several attempts for generalization, a few reached publication, see [Glaz 1994; Hamilton 2004; Hamilton and Marley 2007; Asgharzadeh and Tousi 2009], but a trade-off for generalization of select properties to quite wide classes of nonnoetherian rings has been the sacrifice of Cohen–Macaulay-like behavior in any

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comprehensive fashion. The noetherian condition has never been replaced by any with direct connection to the noetherian property. We believe that a generalization that is closer to the noetherian condition might allow for new applications and capture more features of Cohen–Macaulay rings than the generalizations in the cited references.

In this note, we are looking for a kind of generalization that is very natural and is as close to Cohen–Macaulay rings as possible, but general enough to be amenable to various applications. We break tradition and choose a different approach: one that does not adhere to any of the classical defining properties. Our strategy is to rephrase the definition to one that does not explicitly require the noetherian condition, to replace the condition that implies the noetherian character by a weaker one, and after doing so, to use the modified definition as the base of generalization.

The following simple characterization of Cohen–Macaulay rings is crucial. To underline its relevance and to draw more attention to this characterization, we include the parallel one for Gorenstein rings though this will not be used in this paper.

Theorem 1.1. *For a commutative noetherian ring R, these are equivalent*:

(i) *R is Cohen–Macaulay*;

(ii) *for every ideal I of R generated by a regular sequence, the quotient ring of R/I is artinian* (i.e., 0-dimensional Cohen–Macaulay).

Similarly, R is Gorenstein if and only if, for every ideal I of R generated by a regular sequence, the quotient ring of R/I is quasi-Frobenius (i.e., 0-dimensional Gorenstein).

Proof. (i) ⇒ (ii). Hypothesis (i) implies that the ideal I generated by a regular sequence $x_1, \ldots, x_i$ in R is unmixed. Then the quotient ring $Q(R/I)$ of R/I is semilocal noetherian and zero-dimensional, hence artinian.

(ii) ⇒ (i). It suffices to prove that if (ii) holds, then every ideal I contains a regular sequence of length $\mathrm{ht}(I)$. We show that if $x_1, \ldots, x_t$ is a regular sequence in I and $t < \mathrm{ht}(I)$, then this sequence extends to a regular sequence in I of length $t + 1$. Since the quotient ring $Q(R/I)$ is artinian, there are only finitely many minimal prime ideals $P_1, \ldots, P_m$ of $(x_1, \ldots, x_t)R$, and each element of R not prime to $(x_1, \ldots, x_t)R$ is contained in one of the P_j. As $I/(x_1, \ldots, x_t)R$ has positive height, $I \not\subseteq P_j$ for any j, so $I \not\subseteq P_1 \cup \cdots \cup P_m$ by prime avoidance. Hence there exists $x_{t+1} \in I$ prime to $(x_1, \ldots, x_t)R$, and so $x_1, \ldots, x_t, x_{t+1}$ is a regular sequence in I.

To verify the second claim, recall a characterization of Gorenstein rings by Bass [1963, Theorem, p. 9]; it shows that they are Cohen–Macaulay rings such that the primary components of ideals I generated by regular sequences are irreducible, i.e., not intersections of two larger ideals. This property is equivalent to saying that the ring R/I (that is now a subdirect product of irreducible rings R/L with the primary components L of I; these L have different prime radicals) has no different, isomorphic simple submodules in its socle. This property of the socle is inherited by the (artinian) quotient ring $Q(R/I)$. By Lam [1999, Theorem 15.27], commutative artinian rings with this property are QF rings. □

Using this observation as the point of departure, we follow our strategy, and want to denoetherianize the artinian property. But nothing is simpler than that: we just replace the descending chain condition on *all* ideals by the descending chain condition on *finitely generated* ideals. We do not stop here, but recall that the descending chain condition on finitely generated ideals is equivalent to the same condition on principal ideals [Björk 1969, Theorem 2], and the latter condition characterizes the *perfect* rings, introduced by Bass [1960]. In conclusion, *we will generalize Cohen–Macaulay rings by replacing "artinian" by "perfect."* More precisely, for an integer $n \geq 0$, we will call a ring R (with maximal regular sequences of lengths n) *n-subperfect* ($n \geq 0$) if the ring of quotients of the ring R/I is perfect for every proper ideal I generated by a regular sequence (and add right away that a 0-subperfect ring is the same as a perfect ring in the sense of Bass). Our nonnoetherian Cohen–Macaulay rings are *the extended Cohen–Macaulay rings:* commutative rings R that have noetherian prime spectra and each localization R_M at a maximal ideal M is $\mathrm{ht}(M)$-subperfect. In our discussion we will concentrate on the n-subperfect case for a fixed $n \geq 0$ (which is more general than the local case).

Asgharzadeh and Tousi [2009] review and compare the various nonnoetherian generalizations of Cohen–Macaulay rings in the literature and add their own variants. In a sense, our generalization lies properly between the classical Cohen–Macaulay rings and their generalizations in the literature, at least as far as zero-dimensional rings are concerned. In fact, a zero-dimensional ring is Cohen–Macaulay if and only if it is artinian, while each of the generalizations listed in [Asgharzadeh and Tousi 2009] includes all zero-dimensional rings in their versions of generalized Cohen–Macaulay rings. In our generalization, in the class of zero-dimensional rings only the perfect rings qualify. (A main difference is in the nilradical: T-nilpotency is properly between being just nil and even nilpotent.) Furthermore, every one-dimensional integral domain is included in all of the previously published generalizations. For the Cohen–Macaulayness however, such domains ought to have artinian factor rings modulo any nonzero ideal, while for our 1-subperfectness these factors are required to be perfect rings. Being closer to the classical version, our generalization is expected to share more analogous properties with Cohen–Macaulay rings than the previous generalizations, yet capture fewer classes of rings. To avoid confusion involving these different generalizations of Cohen–Macaulay rings, we assume implicitly in what follows that the term "Cohen–Macaulay ring" always designates a *noetherian* Cohen–Macaulay ring.

Let us point out some relevant features of n-subperfect rings that support our claim that this generalization has a number of properties that are fundamental for Cohen–Macaulay rings in the noetherian setting. (Definitions are recalled later. In the following list, n can be any nonnegative integer.)

- A ring R is n-subperfect if and only if its spectrum is noetherian and the localizations R_M are n-subperfect for all maximal ideals M (Corollary 4.6).
- A ring R is n-subperfect if and only if for each regular sequence $x_1, \dots, x_i$ in R $(0 < i \le n)$, the ring $R/(x_1, \dots, x_i)R$ is $(n-i)$-subperfect (Proposition 3.2).
- An n-subperfect ring is catenary, equidimensional, and of Krull dimension n (Corollary 3.6).
- Direct summand of a direct product of a finite number of n-subperfect rings is n-subperfect (Corollary 4.8).
- A noetherian ring is Cohen–Macaulay if and only if it is an extended Cohen–Macaulay ring as defined above (Corollary 4.4).
- The polynomial ring $R[X_1, \dots, X_n]$, or any of its Veronese subrings, is n-subperfect if and only if R is a perfect ring (Theorems 6.2 and 8.3).
- The grade of a proper ideal I of an n-subperfect ring R (the length t of the longest regular sequence contained in I) is the smallest integer t such that $\operatorname{Ext}^t_R(R/I, R) \neq 0$ (Theorem 3.7).
- If a finite group G operates on an n-subperfect ring R and its order is a unit in R, then the set R^G of ring elements fixed under G is an n-subperfect ring (Corollary 5.2).
- The nilradical N of an n-subperfect ring R is T-nilpotent, and R/N is a Goldie ring (Lemma 2.2, Theorem 5.3).

Our definition leaves ample room for specializations: additional conditions might be added that are not strong enough to enforce the noetherian property, but lead to more pleasant properties of the resulting generalizations (e.g., fixing the injective dimension of the ring as in the Gorenstein case, coherency, or the h-local property might be such a condition). Examples for n-subperfect rings that are not Cohen–Macaulay are abundant; see Section 8.

Our main goal was to get acquainted with the fundamental properties of n-subperfect rings that are analogous to well-known features of Cohen–Macaulay rings. Working in the nonnoetherian situation and in the uncharted territory of subperfect rings meant a challenge in several proofs. We focus our attention to n-subperfectness (i.e., localizations at maximal ideals have the same Krull dimension n — this suffices to explore the general case) in order to avoid dealing with the complicated general situation corresponding to global Cohen–Macaulay rings that would make the main features less transparent. Occasionally, when it does not obscure the main ideas, we work under the global analogue of Cohen-Macaulay rings; these are the regularly subperfect rings defined in Section 2. (See Corollary 4.4.)

While perhaps less familiar in commutative algebra, perfect rings, the cornerstone of our approach, appear throughout the literature on modules and associative algebras. We review these rings briefly in the next section, but see, for example, [Bass 1960; Lam 2001] for more background. Perfect rings were the leading concept in the theories of almost perfect domains by Bazzoni and Salce [2003] and their generalizations, the almost perfect rings, by Fuchs and Salce [2018]: these rings become one-dimensional Cohen–Macaulay once the noetherian condition is imposed. As an application of our approach, we obtain a well-developed Cohen–Macaulay theory of regular sequences in polynomial rings over perfect rings. Thus, while perfect rings help illuminate the workings of Cohen–Macaulay rings, Cohen–Macaulay rings in turn might help shed new light on the class of perfect rings.

We will also establish a close connection with Goldie rings, another important generalization of noetherian rings. It turns out that n-subperfect rings modulo their T-nilpotent radicals are reduced Goldie rings, so Goldie rings appear naturally in the buildup of our new rings. We have not explored this connection to draw conclusions about the structure of n-subperfect rings. Neither have we investigated as yet the possible denoetherianized Gorenstein version of our generalization where for ideals I of R generated by regular sequences, the quotient rings of R/I are self-injective perfect rings.

2. Definitions and notation

All rings considered here are commutative. We mean by a *perfect ring* a ring over which flat modules are projective. Most of the following characterizations of commutative perfect rings can be found in [Bass 1960, Theorem P; Lam 2001, Theorems 23.20, 23.24]. Recall that a module M is *semiartinian* if every nonzero epic image of M contains a simple submodule.

Lemma 2.1. *The following are equivalent for a commutative ring R:*

(a) *R is a perfect ring.*

(b) *R satisfies the descending chain condition on principal ideals.*

(c) *R is a finite direct product of local rings whose maximal ideals N are T-nilpotent* (i.e., for every sequence $y_1, \dots, y_n, \dots$ in N, there is an index m such that $y_1 \cdots y_m = 0$).

(d) *R is semilocal and the localization R_P is perfect for every maximal ideal P.*

(e) *R is semilocal and semiartinian.*

(f) *the finitistic dimension* $\mathrm{Fdim}(R)$ (supremum of finite projective dimensions of R-modules) *is* 0.

(g) *R-modules admit projective covers.* □

We emphasize that "perfect modules and perfect ideals" as they are used, e.g., in [Bruns and Herzog 1998] have nothing to do with perfectness as defined in the preceding lemma.

A ring R is *subperfect* if its total quotient ring $Q(R)$ is perfect, i.e., it is an order in a perfect ring. This is a most essential concept in this paper; it may be viewed as a generalization of the notion of integral domain. All Cohen–Macaulay rings are subperfect. Subperfect rings can be characterized as follows.

Lemma 2.2. *For a commutative ring R, these are equivalent*:

(i) *R is subperfect.*

(ii) *R has only finitely many minimal prime ideals, every zero-divisor in R is contained in a minimal prime ideal, and the nilradical N of R is T-nilpotent.*

(iii) [Gupta 1970] *R satisfies*:

  (a) *the nilradical N of R is T-nilpotent*,

  (b) *R/N is a* (reduced) *Goldie ring* (i.e., it has finite uniform dimension and satisfies the ascending chain condition on annihilators of subsets), *and*

  (c) *R satisfies the regularity condition*: *a regular coset of N can be represented by a regular element of R.* (Moreover, a regular coset of N consists of regular elements of R.)

(iv) [Fuchs and Salce 2018, Lemma 5.5] *The modules over the quotient ring $Q(R)$ are weak-injective as R-modules.*

(v) [Fuchs and Salce 2018, Lemma 5.4] *If M is an R-module of weak dimension ≤ 1, then $Q(R) \otimes_R M$ is a $Q(R)$-projective module.* □

Here an R-module M is said to be *weak-injective* if $\mathrm{Ext}^1_R(A, M) = 0$ for all R-modules A of weak-dimension ≤ 1 [Lee 2006]. The regularity condition with respect to the nilradical was discussed by Small [1966]. His Theorem 2.13 states that a commutative noetherian ring R satisfies this condition if and only if the associated primes of the ideal (0) are the minimal primes of R. (A fourth condition in [Gupta 1970] is automatically satisfied if the ring is commutative.)

It is useful to point out:

Lemma 2.3. *Passing modulo a T-nilpotent ideal preserves subperfectness.*

Proof. If I is a T-nilpotent ideal of a subperfect ring R, then by Lemma 2.2(iii) a regular coset in R/I has a representative that is a regular element of R. Hence it follows that if Q denotes the quotient ring of R, then Q/I is the quotient ring of R/I, which is a perfect ring. □

An ideal I of the commutative ring R is *subperfect* if $Q(R/I)$ is a perfect ring, i.e., R/I is a subperfect ring. A regular sequence is *subperfect* if the ideal it generates is subperfect. We use the conventions that regular sequences are proper and that the empty sequence is considered a regular sequence. Thus the empty sequence in R is subperfect if and only if R is subperfect.

For several results in Section 3, as well as in later arguments, we work with regular sequences that are not necessarily subperfect. We say a ring R is *regularly subperfect* if each regular sequence of R is subperfect. Thus a ring R is regularly subperfect if and only if for each regular sequence $x_1, \ldots, x_i$ in R (including the empty regular sequence), the ring $R/(x_1, \ldots, x_i)R$ is subperfect. In particular, a necessary condition for R to be regularly subperfect is that R itself is subperfect. For an integer $n \geq 0$, the ring R is *n-subperfect* if R is regularly subperfect and every maximal regular sequence has length n. As a consequence, R is 0-subperfect if and only if R is perfect. This is because in a 0-subperfect ring every nonunit is a zero-divisor, so $Q(R) = R$.

The 1-subperfect rings are *"almost perfect rings"* (the only difference is that almost perfect rings might have localizations that are perfect rings). These rings have been studied recently; see [Fuchs and Salce 2018; Fuchs 2019]. They were defined as subperfect rings such that each factor ring modulo a regular ideal (i.e., an ideal containing a nonzero-divisor) is a perfect ring.

Lemma 2.4. *Suppose R is a subperfect ring. The following are equivalent*:

(α) *R is almost perfect.*

(β) *Every nonzero torsion R-module contains a simple submodule.*

(γ) *For every regular proper ideal I of R, R/I contains a simple module.*

(δ) *R is h-local and $Q(R)/R$ is semiartinian.* □

Moreover, almost perfect rings have a number of interesting characteristic properties that are new even for Cohen–Macaulay rings of Krull dimension 1. To wit, we mention the following [Fuchs and Salce 2018; Fuchs 2019]. A subperfect ring R is almost perfect if and only if either of the following conditions is satisfied (in (iii) and (iv), envelopes and covers are understood to be part of a genuine cotorsion pair):

(i) All flat R-modules are strongly flat (strongly flat means that it is a summand of a module that is an extension of a free R-module by a direct sum of copies of the ring of quotients Q of R).

(ii) R-modules of weak dimension ≤ 1 are of projective dimension ≤ 1.

(iii) If R is local: every R-module M has a divisible envelope (i.e., a divisible module containing M and being contained in every divisible module that contains M).

(iv) Each R-module M admits a projective dimension 1 cover (i.e., a module of projective dimension ≤ 1 along with a map α to M such that any map from a module of projective dimension ≤ 1 to M factors through α, and no proper summand has this property).

Next we recall some standard terminology. Let R be a ring (commutative), and $R^\times$ the set of regular (nonzero-divisor) elements of R. An element r of $R \setminus I$ is *prime to an ideal* I of R if whenever $s \in R$ with $rs \in I$, then $s \in I$. The set S of elements prime to I is a saturated multiplicatively closed set. The prime ideals of R that contain I and are maximal with respect to not meeting S are the *maximal prime divisors* of I. The prime ideals of R that are minimal with respect to containing I are the *minimal prime divisors* of I. These ideals do not meet S. It follows that the classical ring of quotients $Q(R/I)$ of R/I is R_S/I_S, and the maximal ideals of $Q(R/I)$ are the extensions to $Q(R/I)$ of the maximal prime divisors of I. Similarly, the minimal prime ideals of $Q(R/I)$ are the extensions of the minimal prime divisors of I.

We say an ideal I of the ring R is *unmixed* if every maximal prime divisor of I is also a minimal prime divisor of I; equivalently, $\dim Q(R/I) = 0$. Thus, I is unmixed if and only if every element of R not in a minimal prime divisor of I is prime to I. In the case where R is noetherian, this agrees with the definition of unmixed ideal given by Bruns and Herzog [1998, p. 59]. If R is noetherian, $Q(R/I)$ is semilocal. However, since nonnoetherian rings are our main focus, in our discussions $Q(R/I)$ need not be semilocal without additional assumptions on I.

We say that an ideal I of R is *finitely unmixed* if $Q(R/I)$ is a semilocal zero-dimensional ring. A regular sequence of R is *finitely unmixed* if the ideal it generates is finitely unmixed. Thus every subperfect regular sequence is finitely unmixed, and every finitely unmixed regular sequence is unmixed.

For unexplained terminology we refer to [Matsumura 1986; Bruns and Herzog 1998].

3. Basic properties

Although the focus for most of the article is on n-subperfect rings, in this section we prove several assertions in greater generality.

For an integer $n \geq 0$, say that a ring R is *n-unmixed* if every regular sequence of R extends to a maximal regular sequence of length n that is unmixed. Let $\mathcal{C}$ be a class of zero-dimensional rings. We call a ring R is *n-unmixed* in $\mathcal{C}$ if every regular sequence extends to a maximal regular sequence of length n and for every regular sequence $x_1, \dots, x_i$ in R, we have $Q(R/(x_1, \dots, x_i)R) \in \mathcal{C}$. Thus a ring R is n-subperfect if and only if R is n-unmixed in the class $\mathcal{C}$ of perfect rings.

The property of being n-unmixed in a class $\mathcal{C}$ of zero-dimensional rings can be inductively described, as in the next lemma.

Lemma 3.1. *Let $\mathcal{C}$ be a class of zero-dimensional rings, and $n \geq 1$. A ring R is n-unmixed in $\mathcal{C}$ if and only if for each $0 < i \leq n$ and for each regular sequence $x_1, \dots, x_i$ in R, the ring $R/(x_1, \dots, x_i)R$ is $(n-i)$-unmixed in $\mathcal{C}$.*

Proof. Suppose R is n-unmixed in $\mathcal{C}$, and let $0 < i \leq n$. Since R is n-unmixed, every regular sequence that begins with $x_1, \dots, x_i$ extends to a maximal regular sequence of length n. It follows that every maximal regular sequence in $R/(x_1, \dots, x_i)R$ has length $n - i$. Also, since every regular sequence in R is unmixed in $\mathcal{C}$, so is every regular sequence in $R/(x_1, \dots, x_i)R$. Thus, $R/(x_1, \dots, x_i)R$ is $(n-i)$-unmixed in $\mathcal{C}$.

Conversely, suppose that for each $0 < i \leq n$ and for each regular sequence $x_1, \dots, x_i$ in R, the ring $R/(x_1, \dots, x_i)R$ is $(n-i)$-unmixed in $\mathcal{C}$. Let $x_1, \dots, x_i$ be a regular sequence in R, and let $j \leq i$. Then the zero ideal in $R/(x_1, \dots, x_j)R$ is by assumption unmixed in $\mathcal{C}$, so $Q(R/(x_1, \dots, x_j)R) \in \mathcal{C}$. Moreover, since $R/(x_1, \dots, x_i)R$ is $(n-i)$-unmixed, every maximal regular sequence in this ring has length $n - i$. Thus every extension of $x_1, \dots, x_i$ $(0 < i \leq n)$ to a maximal regular sequence in R has length n. This proves R is n-unmixed in $\mathcal{C}$. □

Proposition 3.2. *Assume $n \geq 1$. The ring R is n-subperfect if and only if, for each regular sequence $x_1, \dots, x_i$ $(0 < i \leq n)$ in R, the ring $R/(x_1, \dots, x_i)R$ is $(n-i)$-subperfect.*

Proof. Apply Lemma 3.1 to the class $\mathcal{C}$ of perfect rings. □

We record the following corollary that also shows how n-perfectness can be defined by induction on n.

Corollary 3.3. *A ring R is n-subperfect $(n \geq 1)$ if and only if it is subperfect and for each regular element $x \in R$, the ring R/xR is $(n-1)$-subperfect.*

Proof. This is an immediate consequence of Proposition 3.2. □

The next lemma follows at once from Lemma 2.3.

Lemma 3.4. *If I is a T-nilpotent ideal of an n-subperfect ring R, then the ring R/I is also n-subperfect.* □

The property of being n-unmixed also has strong consequences for the dimension theory of the ring.

Proposition 3.5. *Suppose $n \geq 0$. If the ring R is n-unmixed, then* $\dim R = n$ *and all maximal chains of prime ideals of R have the same length n.*

Proof. We first prove by induction on n that $\dim R = n$. If $n = 0$, then the empty regular sequence is unmixed, and so $\dim Q(R) = 0$. In this case regular elements are units, therefore we have $R = Q(R)$. Thus, for $n = 0$, $\dim R = 0$ and the claim is clear.

Suppose that $n > 0$ and for each $0 \le i < n$, every i-unmixed ring has dimension i. We claim that $\dim R = n$. Since R is n-unmixed with $n > 0$, we have $\dim R > 0$. Suppose that $P_0 \subset P_1 \subset \cdots \subset P_m$ is a chain of distinct prime ideals of R with $m > 0$. Since R is n-unmixed with $n > 0$, we have $R \neq Q(R)$ and $\dim Q(R) = 0$. Hence every ideal of R not contained in a minimal prime ideal is regular, so there is a regular $x \in P_1$. By Lemma 3.1, R/xR is $(n-1)$-unmixed. By the induction hypothesis, $\dim R/xR = n - 1$. Since $P_1/xR \subset \cdots \subset P_m/xR$ is a chain of distinct prime ideals of R/xR and $\dim R/xR = n - 1$, we conclude that $m \le n$. Thus no chain of distinct prime ideals of R has length exceeding n, that is, $\dim R \le n$. To see that $n \le \dim R$, use the fact that R has a regular sequence of length n [Kaplansky 1970, Theorem 132]. Therefore, $\dim R = n$.

Next we show that all maximal chains of prime ideals have the same length. The proof is again by induction on n. If $n = 0$, then, as we have established, $\dim R = 0$. In this case the proposition is clear. Let $n > 0$, and suppose the claim holds for all $i < n$. Let $P_0 \subset P_1 \subset \cdots \subset P_k$ and $Q_0 \subset Q_1 \subset \cdots \subset Q_m$ be maximal chains of distinct prime ideals in R. We claim $k = m$. Since the zero ideal of R is unmixed, every nonminimal prime ideal of R is regular. Thus P_1 and Q_1 are regular ideals of R, so there is an $x \in R^\times$ in $P_1 \cap Q_1$. By Lemma 3.1, R/xR is an $(n-1)$-unmixed ring with maximal chains of prime ideals $P_1/xR \subset \cdots \subset P_k/xR$ and $Q_1/xR \subset \cdots \subset Q_m/xR$. By the induction hypothesis on R/xR, we have $k - 1 = m - 1$, thus $k = m$. This means that all chains of maximal length in R have the same length k. It follows that $\dim R = k$, thus $k = \dim R = n$. □

Corollary 3.6. *For every $n \ge 0$, an n-subperfect ring is catenary, equidimensional, and has Krull dimension n.* □

For an ideal I of a ring R, the *I-depth* of R is the smallest positive integer t such that $\mathrm{Ext}^t_R(R/I, R) \neq 0$. If R is noetherian, then the I-depth of R is the length of the longest regular sequence contained in I. Thus a noetherian ring R is Cohen–Macaulay if and only if for each proper ideal I of R, the I-depth of R is equal to the height of I. We show in Theorem 3.7 that this result holds more generally for regularly subperfect rings.

Theorem 3.7. *Let R be a regularly subperfect ring, I a proper ideal of R, and let $n \ge 1$. The following are equivalent:*

(1) *I has height n.*

(2) *Every maximal regular sequence in I has length n.*

(3) *There exists a maximal regular sequence in I of length n.*

(4) $n = \min\{t : \mathrm{Ext}^t_R(R/I, R) \neq 0\}$.

Proof. We first prove the equivalence of (1), (2) and (3). Since the length of a regular sequence in I is at most the height of I, it suffices to show that if $x_1, \dots, x_t$ is a regular sequence in I such that $t < \mathrm{ht}(I)$, then $x_1, \dots, x_t$ extends to a regular sequence in I of length $t+1$. Using the fact that $Q(R/(x_1, \dots, x_t)R)$ is perfect (rather than artinian), we can imitate the proof of Theorem 1.1 to establish the existence of such a regular sequence. Then $x_1, \dots, x_t, x_{t+1}$ is a regular sequence, and the equivalence of (1), (2) and (3) follows.

To see that (4) implies (2), let $x_1, \dots, x_t$ be a regular sequence in I, and $J = (x_1, \dots, x_t)R$. By [Kaplansky 1970, p. 101],

$$\mathrm{Ext}^t_R(R/I, R) \cong \mathrm{Hom}_R(R/I, R/J).$$

Suppose $t < n$. By (4), $\mathrm{Hom}_R(R/I, R/J) = 0$, and hence there does not exist a nonzero element of R/J annihilated by I. Since R is regularly subperfect, R/J is subperfect. By Lemma 2.2 and prime avoidance, I/J is not contained in the set of zero-divisors of R/J, and so $x_1, \dots, x_t$ extends to a regular sequence of length $t+1$. It follows from (4) that $x_1, \dots, x_t$ extends to a regular sequence $x_1, \dots, x_n$ of length n. Since $0 \neq \mathrm{Ext}^n_R(R/I, R) \cong \mathrm{Hom}_R(R/I, R/(x_1, \dots, x_n)R)$, the image of I in R/J consists of zero-divisors. Thus $x_1, \dots, x_n$ is a maximal regular sequence in I.

Finally, to see that (3) implies (4), suppose $x_1, \dots, x_n$ is a maximal regular sequence in I, and let $J = (x_1, \dots, x_n)R$. (Since I has finite height, such a regular sequence must exist.) Then the image of I in R/J consists of zero-divisors. We first show that there is an element $z \in R \setminus J$ such that $zI \subseteq J$.

By Lemma 2.1(c), $Q := Q(R/J)$ contains orthogonal idempotents $e_1, \dots, e_n$ such that $1 = e_1 + \cdots + e_n$ and, for each i, $e_i Q$ is a perfect local ring with identity e_i. For a maximal ideal P of Q containing IQ, there is i such that $e_i P$ is the maximal ideal of $e_i Q$. Since the ring $e_i Q$ is semiartinian by Lemma 2.1(e), there exists $y \in Q$ such that $x = e_i y \neq 0$ and $xP = 0$. Hence $xIQ = 0$. From this it follows that we can find $z \in R \setminus J$ such that $zI \subseteq J$.

Define a homomorphism $f : R/I \to R/J$ by $f(r + I) = rz + J$ for all $r \in R$. Then $f \neq 0$, and so by the above isomorphism $\mathrm{Ext}^n_R(R/I, R) \neq 0$. If $t \leq n$ satisfies $\mathrm{Ext}^t_R(R/I/R) \neq 0$, then since (4) implies (3), $x_1, \dots, x_t$ is a maximal regular sequence in I. By the equivalence of (2) and (3), this yields $t = n$. □

Remark 3.8. From the proof of Theorem 3.7 it is evident that statements (1), (2) and (3) remain equivalent if rather than assuming R is regularly subperfect we assume only that every regular sequence is finitely unmixed.

Corollary 3.9. *Let $n \geq 0$. A ring R is n-subperfect if and only if R is regularly subperfect and each maximal ideal of R has height n.*

Proof. If R is n-subperfect, then each maximal ideal of R has height n by Corollary 3.6. Conversely, if R is regularly subperfect and each maximal ideal has height n, then every maximal regular sequence in R has length n by Theorem 3.7. □

Hamilton and Marley [2007, Definition 4.1] define a ring R to be Cohen–Macaulay if every "strong parameter sequence" on R is a regular sequence. The notion of a strong parameter sequence, which is defined via homology and cohomology of appropriate Koszul complexes, is beyond the scope of our paper. We observe next that regularly subperfect rings are Cohen–Macaulay in this sense.

Corollary 3.10. *Every regularly subperfect ring is Cohen–Macaulay in the sense of Hamilton and Marley.*

Proof. Apply Theorem 3.7 and [Asgharzadeh and Tousi 2009, Theorem 3.4]. □

To verify that a local noetherian ring R of dimension d is Cohen–Macaulay, it is enough to exhibit just one regular sequence of length d. By contrast, the following example shows that in a local domain R of dimension d, the existence of a subperfect regular sequence of length d is not sufficient to guarantee that the domain is d-subperfect.

Example 3.11. Kabele [1971, Example 5] constructs a local domain R having the ring $S = k[[x, y, z]]$ as an integral extension, where k is a field of characteristic 2 with $[k : k^2] = \infty$ and x, y, z are indeterminates for k. The ring R has the property that x, y is not a regular sequence in R, but zR, $(z, x)R$ and $(z, x, y)R$ are distinct prime ideals of R, thus R/zR, $R/(z, x)R$ and $R/(z, x, y)R$ are integral domains, and hence z, x, y is a subperfect regular sequence in R. Moreover, $\dim R = 3$ as S has dimension 3 and is integral over R. Since x, y is not a regular sequence and x is a nonzero-divisor in R, the image of y in R/xR is a zero-divisor. If R is 3-subperfect, then R/xR is subperfect, so y is in a minimal prime ideal P of xR. In this case, Corollary 3.6 implies that $\dim R/P = 2$. Let P' be a prime ideal of S lying over P. S is integral over R, so $\dim S/P' = \dim R/P = 2$ [Kaplansky 1970, Theorem 47, p. 31]. Since S is a catenary domain, this implies $\mathrm{ht}(P') = 1$. However, $(x, y)S$ is a height 2 prime ideal of S contained in P', a contradiction. Therefore, R is not 3-subperfect despite the fact that R has a length 3 maximal regular sequence that is subperfect.

4. Localization and globalization

In this section we consider localization and globalization of the n-subperfect property. In general, issues of localization involving regular sequences are complicated by the fact that a regular sequence in a localization at a prime ideal need not be the image of a regular sequence in R. However, as we observe in the next lemma, this problem can be circumvented for regularly subperfect rings.

Lemma 4.1. *Let R be a regularly subperfect ring, P a prime ideal of R, and let $x_1, \ldots, x_n$ be a regular sequence in R_P. Then there is a regular sequence $y_1, \ldots, y_n \in P$ such that*

$$(x_1, \ldots, x_i)R_P = (y_1, \ldots, y_i)R_P \quad \textit{for each } i = 1, \ldots, n.$$

Proof. Let I and J be the ideals of R defined by

$$I = \{r \in R : (\exists s \in R \setminus P)\, rs \in x_1 R\} \quad \text{and} \quad J = \{r \in R : (\exists s \in R \setminus P)\, rs \in x_1 P\}.$$

Then $IR_P = x_1 R_P$ and $JR_P = IPR_P$. Moreover, $J \subset I$ is a proper inclusion, since the image of x_1 in R_P is a nonzero-divisor. $Q(R)$ is zero-dimensional and semilocal, so R has finitely many minimal prime ideals $P_1, \ldots, P_m$ such that the set of zero-divisors in R is $P_1 \cup \cdots \cup P_m$. Since the image of x_1 in R_P is a nonzero-divisor, $I \not\subseteq P_j$ for any j. By prime avoidance, there is $y_1 \in I$ such that $y_1 \notin J \cup P_1 \cup \cdots \cup P_m$. Since IR_P is a principal ideal and the image of y_1 in R_P is not in JR_P, Nakayama's lemma implies $x_1 R_P = IR_P = y_1 R_P$. By the choice of y_1, we have $y_1 \in R^\times$.

Now suppose $1 < t \leq n$ and there is a regular sequence $y_1, \ldots, y_{t-1}$ with $(x_1, \ldots, x_i)R_P = (y_1, \ldots, y_i)R_P$ for each $1 \leq i \leq t-1$. Then $Q(R/(y_1, \ldots, y_{t-1})R)$ is semilocal and zero-dimensional, so repeating the argument from the first paragraph for the ring $R/(y_1, \ldots, y_{t-1})R$ yields $y_t \in P$ such that $y_1, \ldots, y_{t-1}, y_t$ is a regular sequence in P and $(y_1, \ldots, y_{t-1}, y_t)R_P = (x_1, \ldots, x_{t-1}, x_t)R_P$. □

Theorem 4.2. *Let R be a regularly subperfect ring. For each prime ideal P of R, the ring R_P is regularly subperfect.*

Proof. Let P be a prime ideal of R. Since $Q(R)$ is zero-dimensional, $Q(R_P) = Q(R)_{R \setminus P}$; see [Lipman 1965, Proposition 1 and Corollary 1]. Thus $Q(R_P)$ is perfect since $Q(R)$ is, and so R_P is subperfect. It follows that the localization of a regularly subperfect ring at a prime ideal has the property that the empty regular sequence is subperfect.

We now prove the theorem by induction on the length of regular sequences in R_P. Let $n > 0$, and suppose that for every regularly subperfect ring S and prime ideal L of S, every regular sequence of length $< n$ in S_L is subperfect. Let $x_1, \ldots, x_n$ be a sequence in R whose image in R_P is a regular sequence. By Lemma 4.1 there is $y \in R^\times$ such that $x_1 R_P = y R_P$. Since R/yR is regularly subperfect and the image of the sequence $x_2, \ldots, x_n$ in $R_P/x_1 R_P = R_P/y R_P$ is a regular sequence of length $n-1$, the induction hypothesis implies that $R_P/x_1 R_P$ is regularly subperfect. Therefore, the image of the sequence $x_2, \ldots, x_n$ in $R_P/x_1 R_P$ is a subperfect regular sequence, and hence so is the image of the sequence $x_1, x_2, \ldots, x_n$ in R_P. □

Corollary 4.3. *Let R be a regularly subperfect ring. If P is a prime ideal of finite height n, then R_P is n-subperfect.*

Proof. This follows from Theorems 3.7 and 4.2. □

Corollary 4.4. *The following are equivalent for a noetherian ring R.*

(1) *R is Cohen–Macaulay.*

(2) *R is regularly subperfect.*

(3) *R_M is* $\mathrm{ht}(M)$*-subperfect for each maximal ideal M of R.*

Proof. To see that (1) implies (2), let $x_1, \dots, x_n$ be a regular sequence in R. By the unmixedness theorem [Bruns and Herzog 1998, Theorem 2.1.6, p. 59], $x_1, \dots, x_n$ is unmixed (as is the empty regular sequence). Since R is noetherian, the zero-dimensional ring $Q(R/(x_1, \dots, x_n R)$ is semilocal, hence artinian, hence perfect. Consequently, the sequence $x_1, \dots, x_n$ is subperfect.

That (2) implies (3) follows from Corollary 4.3. That (3) implies (1) is clear. □

A topological space is *noetherian* if its open sets satisfy the ascending chain condition. It follows that every closed subset of a noetherian space is a union of finitely many irreducible components. Thus, if R is a ring for which $\mathrm{Spec}(R)$ is noetherian, then each proper ideal of R has but finitely many minimal prime divisors.

Theorem 4.5. *Let R be a ring of finite Krull dimension. Then R is regularly subperfect if and only if* $\mathrm{Spec}(R)$ *is noetherian and R_M is regularly subperfect for each maximal ideal M of R.*

Proof. Suppose R is regularly subperfect. By Theorem 4.2, R_M is regularly subperfect for each maximal ideal M of R. The proof that $\mathrm{Spec}(R)$ is noetherian is by induction on $\dim R$. If $\dim R = 0$, then R is subperfect, hence perfect, since the ideal (0) of R is generated by the empty regular sequence; thus $\mathrm{Spec}(R)$ is noetherian in this case. Suppose $\dim R > 0$, and for each $0 \le k < \dim R$ every k-dimensional regularly subperfect ring has a noetherian spectrum. Since R is subperfect, R has only finitely many minimal prime ideals $P_1, \dots, P_m$. Thus $\mathrm{Spec}(R)$ is a finite union of the closed sets consisting of the prime ideals containing a given minimal prime ideal P_j. To prove that $\mathrm{Spec}(R)$ is noetherian, we need only verify that each of the spaces $\mathrm{Spec}(R/P_j)$ is noetherian. A space is noetherian if and only if it satisfies the descending chain condition on closed sets, therefore we need only prove that every proper closed subset of $\mathrm{Spec}(R/P_j)$ is noetherian. Every proper closed subset of $\mathrm{Spec}(R/P_j)$ is homeomorphic to a subspace of $\mathrm{Spec}(R/(rR + P_j))$ for some $r \in R \setminus P_j$. Therefore, we treat only spectra of rings of the latter form.

Suppose $r \in R \setminus P_j$ for some $1 \leq j \leq m$, and choose $p_j \in R$ such that p_j is contained in exactly the minimal prime ideals of R that do not contain r. (This is possible by prime avoidance and the fact that there are only finitely many minimal prime ideals of R.) In particular, $p_j \in P_j$. Evidently, $r + p_j \notin P_1 \cup \cdots \cup P_m$, so that $r + p_j \in R^\times$. Thus $R/(r + p_j)R$ inherits from R the property that each regular sequence is subperfect. By the induction hypothesis, $\operatorname{Spec}(R/(r + p_j)R)$ is a noetherian space. As a subspace of a noetherian space, $\operatorname{Spec}(R/(rR + P_j))$ is noetherian. This completes the proof that $\operatorname{Spec}(R)$ is a noetherian space.

Conversely, suppose $\operatorname{Spec}(R)$ is noetherian, and R_M is regularly subperfect for each maximal ideal M of R. Let $x_1, \dots, x_t$ be a (possibly empty) regular sequence in R, and let $I = (x_1, \dots, x_t)R$. For each maximal ideal M containing I, the images of $x_1, \dots, x_t$ in R_M form a regular sequence, so R_M/IR_M is subperfect by assumption. We claim that $Q(R/I)$ is zero-dimensional. Let $r, s \in R$ such that $rs \in I$ and r is not contained in any minimal prime ideal of I. It suffices to show that $s \in I$. If M is any maximal ideal of R containing I, then since R_M/IR_M is subperfect and rR_M is not a subset of any minimal prime ideal of IR_M, we have $sR_M \subseteq IR_M$. Since this is true for each maximal ideal M containing I, we conclude that $s \in I$. This proves that every zero-divisor in R/I is contained in a minimal prime ideal of R/I. Therefore, $Q(R/I)$ is zero-dimensional.

Since $\operatorname{Spec}(R)$ is noetherian, I has only finitely many minimal prime ideals $P_1, \dots, P_m$, so $Q(R/I)$ is also semilocal. For each j, R_{P_j}/IR_{P_j} is T-nilpotent, so it follows that $Q(R/I)$ has T-nilpotent nilradical, and hence $Q(R/I)$ is perfect. This proves that every regular sequence in R (including the empty sequence) is subperfect. □

Corollary 4.6. *Assume $n \geq 0$. A ring R is n-subperfect if and only if* $\operatorname{Spec}(R)$ *is noetherian and R_M is n-subperfect for each maximal ideal M of R.*

Proof. Apply Corollary 3.9 and Theorem 4.5. □

Remark 4.7. The proofs of Lemma 4.1 and Theorems 4.2 and 4.5 show that in the hypotheses of these results the property of being regularly subperfect can be replaced by the more general condition that every regular sequence is finitely unmixed.

We record an immediate consequence of Corollary 4.6 along with the obvious statement on the behavior of n-subperfectness under passing to summands.

Corollary 4.8. *Every summand of a direct product of a finite number of n-subperfect rings is n-subperfect.* □

5. More on n-subperfect rings

We would like to point out several important properties that are shared by n-subperfect rings with Cohen–Macaulay rings. The first of these properties, proved by Hochster and Eagan [1971] for Cohen–Macaulay rings, concerns descent of the n-subperfect property to module direct summands and to rings of invariants of n-subperfect rings.

Theorem 5.1. *Let R be an n-subperfect ring for some $n \geq 0$. If S is a subring of R such that R is integral over S and S is a direct summand of R as an S-module, then S is n-subperfect.*

Proof. First we claim that S is subperfect. Every minimal prime ideal of S is contracted from a minimal prime ideal of R. Since R is subperfect, there are but finitely many minimal prime ideals of R, so there are only finitely many minimal prime ideals of S. Moreover, every zero-divisor in R is an element of a minimal prime ideal of the subperfect ring R, so the same holds for S. Since the nilradical of S is contained in that of R, it is T-nilpotent. Consequently, S is subperfect.

The proof proceeds now by induction on n. Suppose $n = 0$, so that R is perfect. Then $\dim R = 0$, and since R is integral over S, we have $\dim S = 0$. Since S is subperfect, this implies S is perfect, i.e., 0-subperfect.

Now suppose $n > 0$ and that the claim holds for $n - 1$. If every nonzero-divisor of S were a unit, then since S is subperfect, we would have $\dim S = 0$. R is integral over S, whence $\dim R = 0$ would follow. However, R is n-subperfect, so $\dim R = n > 0$ by Corollary 3.6. Therefore, there exist regular sequences in S of length > 0. Let $s \in S^\times$ be a nonunit in S. Since S is a summand of R, it follows that $sR \cap S = sS$; see [Bruns and Herzog 1998, Lemma 6.4.4]. Thus S/sS can be viewed as a direct summand of R/sR. Moreover, R/sR is integral over S/sS.

To see that $s \in R^\times$, suppose to the contrary that s is a zero-divisor in R. Since R is subperfect, s is contained in a minimal prime ideal P_0 of R. By Corollary 3.9, there is a chain of distinct prime ideals $P_0 \subset P_1 \subset \cdots \subset P_n$, with P_n a maximal ideal of R. Since R is integral over S, the chain $P_0 \cap S \subset P_1 \cap S \subset \cdots \subset P_n \cap S$ has length n. Again since R is integral over S, each chain of prime ideals of S has a chain of prime ideals in R lying over it. Therefore, Corollary 3.9 implies that the length of the longest chain of prime ideals in S is n. Consequently, $P_0 \cap S$ is a minimal prime ideal of R. However, $s \in P_0 \cap S$ and $s \in S^\times$, a contradiction that implies $s \in R^\times$.

In view of $s \in R^\times$, we have R/sR is $(n-1)$-subperfect by Proposition 3.2. By the induction hypothesis, S/sS is $(n-1)$-subperfect. Since this is the case for all nonunits $s \in S^\times$, Corollary 3.3 implies S is n-subperfect, completing the induction. □

Corollary 5.2. *Assume G is a finite group acting on an n-subperfect ring R, and the order of G is a unit in R. Then the set of invariants,*

$$R^G = \{r \in R : \ g(r) = r \textit{ for all } g \in G\},$$

is again an n-subperfect ring.

Proof. As in [Bruns and Herzog 1998, pp. 281–283], the hypotheses imply that R^G is a module direct summand of R and R is integral over R^G. Thus we may apply Theorem 5.1 to obtain the corollary. □

Lemma 2.2 makes it possible to get more information on n-subperfect rings once we know more about Goldie rings.

A commutative reduced Goldie ring R is an order in a semisimple ring Q that is the direct product of fields Q_j,

$$Q = Q_1 \times \cdots \times Q_m$$

(see [Lam 1999, Proposition 11.22]). If $X_j = \sum_{i \neq j} Q_i$, then $P_j = X_j \cap R$ $(j = 1, \ldots, m)$ is the set of minimal primes of R. Furthermore, each R/P_j is an integral domain with Q_j as quotient field. Recall that orders R, R' in a ring Q are *equivalent* if $qR \subseteq R'$ and $q'R' \subseteq R$ for some units $q, q' \in Q$.

Theorem 5.3. *A reduced n-subperfect ring R is a Goldie ring. It is a subdirect product of a finite number of integral domains of Krull dimension n. This subdirect product is equivalent to the direct product of the components.*

Proof. Assume R is reduced and n-subperfect; in view of Lemma 2.2, it is a Goldie ring. It has but a finite number of minimal prime ideals $P_1, \ldots, P_m$. From $\bigcap_j P_j = 0$ it follows that R is a subdirect product of the integral domains $D_j = R/P_j$ (with quotient fields Q_j). It is clear that $\dim D_j = n$ for each j.

Choose elements x_j $(j = 0, \ldots, m)$ such that $x_j \in P_i$ for all $i \neq j$, but $x_j \notin P_j$. Then $x = \sum_j x_j \in R$ is a regular element, as it is not contained in any P_j. Therefore,

$$x = (x_1 + P_1, \ldots, x_m + P_m) \in D_1 \oplus \cdots \oplus D_m$$

is a unit in Q. Hence we conclude that R and $R' = D_1 \oplus \cdots \oplus D_m$ are equivalent orders in Q. □

We observe that Theorem 5.3 holds also for the factor ring R/N of an n-subperfect ring R modulo its nilradical N, though R/N need not be n-subperfect. Note that this factor ring is restricted in size inasmuch as R/N must have finite uniform dimension. On the other hand, Example 8.2 will show that the nilradicals of n-subperfect rings can have arbitrarily large cardinalities.

We have failed to establish a stronger result in the preceding theorem (viz. that the domains D_j are also n-subperfect), because passing modulo a minimal prime ideal, regular sequences do not map in general upon regular sequences, though the converse is true for all regularly subperfect rings as is shown by:

Lemma 5.4. *Let R be a regularly subperfect ring, and let P be a minimal prime ideal of R. Then for every regular sequence $y_1, \ldots, y_t$ in $S = R/P$, there is a regular sequence $x_1, \ldots, x_t \in R$ such that $(x_1, \ldots, x_i)S = (y_1, \ldots, y_i)S$ for all $i \leq t$.*

Proof. The proof is by induction on the length of the regular sequence. The claim is clearly true for the empty regular sequence. Suppose that $t \geq 0$ and the claim is true for all regular sequences in S of length t. Let $y_1, \dots, y_t, y_{t+1}$ be a regular sequence in S. Then there is a regular sequence $x_1, \dots, x_t$ in R such that $(x_1, \dots, x_t)S = (y_1, \dots, y_t)S$. Since $R/(x_1, \dots, x_t)R$ is subperfect, $(x_1, \dots, x_t)R$ has but a finite number of minimal prime ideals $L_1, \dots, L_k$. Let $z_{t+1} \in R$ such that $z_{t+1} + P = y_{t+1}$. We observe that $P + z_{t+1}R \not\subseteq L_i$ for any i. Indeed, if $P \subseteq L_i$ for some i, then L_i is a minimal prime ideal of $(x_1, \dots, x_t)R + P$. In this case, since $y_1, \dots, y_{t+1}$ is a regular sequence in S and $(x_1, \dots, x_t)S = (y_1, \dots, y_t)S$, it is impossible to have $y_{t+1} \in L_i/P$. Thus $z_{t+1} \notin L_i$ which shows that $P + z_{t+1}R \not\subseteq L_i$ for every i. By a version of prime avoidance [Kaplansky 1970, Theorem 124], this implies there is $p \in P$ such that $z_{t+1} - p \notin L_i$ for each i. Since $L_1, \dots, L_k$ are the minimal prime ideals of $(x_1, \dots, x_t)R$ and $R/(x_1, \dots, x_t)R$ is subperfect, it follows that $x_1, \dots, x_t, x_{t+1}$ with $x_{t+1} = z_{t+1} - p$ is a regular sequence in R such that $(x_1, \dots, x_t, x_{t+1})S = (y_1, \dots, y_{t+1})S$. This completes the induction and shows that every ideal of S generated by a regular sequence is the image of an ideal of R that is generated by a regular sequence. □

The next theorem shows that for regularly subperfect rings, ideals of the principal class (i.e., ideals I generated by $\mathrm{ht}(I)$ elements) behave like ideals in Cohen–Macaulay rings.

Theorem 5.5. *Let R be a regularly subperfect ring, and let I be an ideal of R generated by t elements. The following are equivalent*:

(1) *I has height t.*

(2) *I has height at least t.*

(3) *I is generated by a regular sequence of length t.*

Proof. That (1) implies (2) is clear, and that (3) implies (1) follows from Theorem 3.7. To see that (2) implies (3), suppose $\mathrm{ht}(I) \geq t$. If $\mathrm{ht}(I) = 0$, then I is generated by the empty regular sequence. The proof now proceeds by induction on $\mathrm{ht}(I)$. Suppose that in a regularly subperfect ring, every ideal $I = (x_1, \dots, x_t)R$ of height at least $\mathrm{ht}(I) - 1$ generated by $\mathrm{ht}(I) - 1$ elements is generated by a regular sequence of length $\mathrm{ht}(I) - 1$. As a subperfect ring, R admits only finitely many minimal prime ideals $P_1, \dots, P_m$. Prime avoidance and the fact that $\mathrm{ht}(I) > 0$ imply that $I \not\subseteq P_1 \cup \cdots \cup P_m$. By [Kaplansky 1970, Theorem 124, p. 90], there exist $r_2, \dots, r_t \in R$ such that $x := x_1 + r_2x_2 + \cdots + r_tx_t \notin P_1 \cup \cdots \cup P_m$. Since R is subperfect, $x \in R^\times$. Moreover, $I = (x, x_2, \dots, x_t)R$. In order to apply the induction hypothesis, we consider next the ring R/xR.

Let P be a minimal prime ideal of I such that $\mathrm{ht}(P) = \mathrm{ht}(I)$. By Theorem 4.2, R_P is $\mathrm{ht}(I)$-subperfect, so Proposition 3.2 implies R_P/xR_P is $(\mathrm{ht}(I)-1)$-subperfect.

By Corollary 3.6, $\dim R_P/xR_P = \mathrm{ht}(I) - 1$, and so P/xR has height $\mathrm{ht}(I) - 1$ in R/xR. Consequently, P/xR is a minimal prime ideal of I/xR of height $\mathrm{ht}(I)-1$ in R/xR. Thus I/xR is an ideal of R/xR that is generated by $t-1$ elements and has height at least $\mathrm{ht}(I) - 1$. By the induction hypothesis, I/xR is generated by a regular sequence in R of length $t-1$. Thus I is generated by a regular sequence of length t. This proves that every ideal of R of height at least t generated by t elements is generated by a regular sequence of length t. Consequently, (2) implies (3). □

6. Polynomial rings

We consider next polynomial rings $S = R[X_1, \ldots, X_n]$ over a perfect ring R. The proof of Theorem 6.2, which shows such rings are n-subperfect, depends on the following lemma.

Lemma 6.1. *Let S be a finitely generated algebra over a perfect ring R. For each proper ideal I of S, the nilradical of S/I is T-nilpotent. If also* $\dim Q(S/I) = 0$, *then S/I is subperfect.*

Proof. Let I be a proper ideal of S. Then the nilradical of S/I is $\sqrt{I}/I$, so to show that this nilradical is T-nilpotent, it suffices to show that for all $a_1, a_2, a_3, \ldots \in \sqrt{I}$, there exists $m > 0$ such that $a_1a_2 \cdots a_m \in I$. We claim first that there is $k > 0$ such that $(\sqrt{I})^k \subseteq I + JS$, where J denotes the Jacobson radical of R. Since R/J is an artinian ring (it is a product of finitely many fields) and S/JS is a finitely generated R/J-algebra, the ring S/JS is noetherian. Thus the image of the ideal $\sqrt{I}$ in S/JS is finitely generated. Letting $f_1, \ldots, f_t \in \sqrt{I}$ such that $\sqrt{I} = (f_1, \ldots, f_t)S + JS$, and choosing $k > 0$ such that $(f_1, \ldots, f_t)^k S \subseteq I$, we obtain $(\sqrt{I})^k \subseteq I + JS$.

For each $i \geq 0$, we have $a_{ik+1}a_{ik+2} \cdots a_{ik+k} \in I + JS$, and so there is a finitely generated ideal $A_i \subseteq J$ such that $a_{ik+1}a_{ik+2} \cdots a_{ik+k} \in I + A_iS$. As a perfect ring, R satisfies the descending chain condition on finitely generated ideals [Björk 1969, Theorem 2], thus there is $t > 0$ such that $A_1A_2 \cdots A_t = A_1A_2 \cdots A_{t+1}$. Since $A_{t+1} \subseteq J$, Nakayama's lemma implies $A_1A_2 \cdots A_t = 0$. It follows that

$$a_1a_2 \cdots a_{tk+k} \in (I + A_0S)(I + A_1S) \cdots (I + A_tS) \subseteq I,$$

which proves the first assertion.

Now suppose $\dim Q(S/I) = 0$. Since R is perfect, $\mathrm{Spec}(R)$ is a finite, hence noetherian, space. As a finitely generated algebra over a ring with noetherian prime spectrum, S also has noetherian prime spectrum [Ohm and Pendleton 1968, Theorem 2.5]. Hence I has but finitely many minimal prime divisors, and so, since $Q(S/I)$ is zero-dimensional, it follows that $Q(S/I)$ is semilocal. The nilradical of $Q(S/I)$ is T-nilpotent as it is extended from the T-nilpotent nilradical of S/I; hence $Q(S/I)$ is perfect. □

We now prove the main theorem of this section. Statement (4) of Theorem 6.2, which is a byproduct of our arguments involving polynomial rings, can be viewed as a characterization of a perfect ring in terms of its multiplicative lattice of ideals.

Theorem 6.2. *Let R denote a commutative ring, and let $X_1, \dots, X_n$ ($n \geq 1$) be indeterminates for R. Then the following are equivalent*:

(1) *R is perfect.*

(2) *$R[X_1, \dots, X_n]$ is n-subperfect.*

(3) *R is semilocal zero-dimensional and $R[X_1, \dots, X_n]$ is subperfect.*

(4) *R is semilocal zero-dimensional and for each sequence $\{I_i\}_{i=1}^{\infty}$ of finitely generated subideals of the Jacobson radical J of R there exists $k > 0$ such that $I_1 I_2 \cdots I_k = 0$.*

Proof. Let $S = R[X_1, \dots, X_n]$, and let J denote the Jacobson radical (= the nilradical) of R.

(1) ⇒ (4). Perfect rings are semilocal and zero-dimensional. Let $\{I_i\}_{i=1}^{\infty}$ be a sequence of finitely generated subideals of J. Since R is perfect, R satisfies the descending chain condition on finitely generated ideals [Björk 1969, Theorem 2], thus there is $k > 0$ such that $I_1 I_2 \cdots I_k = I_1 I_2 \cdots I_{k+1}$. Since $I_{k+1} \subseteq J$, Nakayama's lemma implies $I_1 I_2 \cdots I_k = 0$.

(4) ⇒ (3). Let $f_1/g_1, f_2/g_2, \dots$ be elements of the nilradical of $Q(S)$, where each $f_i \in S$ and each g_i is a nonzero-divisor in S. Then $f_1, f_2, \dots$ are in the nilradical of S, which, since S is a polynomial ring, is the extension JS of the nilradical J of R to S. The ideal I_i generated by the coefficients occurring in f_i is contained in the nilradical of R, so by assumption, there is $k > 0$ such that $I_1 I_2 \cdots I_k = 0$. Since $f_1 f_2 \cdots f_k \in I_1 I_2 \cdots I_k S$, we have $f_1 f_2 \cdots f_k = 0$, thus the nilradical of $Q(S)$ is T-nilpotent. Furthermore, since R is zero-dimensional, so is $Q(S)$ by [Arapović 1983, Proposition 8]. Each prime ideal L in $Q(S)$ contracts to one of the prime ideals P in R. Since $PQ(S) \subseteq L$ is a prime ideal of $Q(S)$ and $Q(S)$ is zero-dimensional, it follows that $PQ(S) = L$. Therefore, since R is semilocal, so is $Q(S)$. This shows that $Q(S)$ is a zero-dimensional semilocal ring with T-nilpotent nilradical; i.e., $Q(S)$ is perfect.

(3) ⇒ (2). Suppose S is subperfect. Let $f_1, \dots, f_t$ be a regular sequence in S, and let $I = (f_1, \dots, f_t)S$. Now R is zero-dimensional and semilocal and I is generated by a regular sequence, therefore — as it is shown in [Olberding 2019] — the ring $Q(S/I)$ is also zero-dimensional and semilocal. By Lemma 6.1, $Q(S/I)$ is a perfect ring, establishing the n-subperfectness of S.

(2) ⇒ (1). Since S is n-subperfect and $X_1, \dots, X_n$ is a maximal regular sequence of S, $S/(X_1, \dots, X_n)R$ is a perfect ring. As a homomorphic image of this ring, R is perfect. □

Let us point out that Coleman and Enochs [1971] prove that if the polynomial rings $R[X]$ and $R'[Y]$ with single indeterminates are isomorphic, and if R is a perfect ring, then $R \cong R'$. It is an open problem if this holds for more indeterminates.

7. The finitistic dimension

The close relation of n-subperfect rings to Goldie rings makes it possible to derive several interesting properties of n-subperfect rings. For details we refer to the literature on Goldie rings, e.g., [Goodearl and Warfield 1989]. As an example we mention that the ring of quotients of a reduced n-subperfect ring is its injective hull.

In view of [Sandomierski 1973], we are able to obtain interesting results on the homological dimensions of n-subperfect rings. We show that in calculating the projective (p.d.), injective (i.d.) and weak (w.d.) dimensions of modules over an n-subperfect ring, only the "Goldie part" of the ring counts (see Lemma 2.2).

Let R be an n-subperfect ring with minimal prime ideals $P_1, \dots, P_m$. Then $N = P_1 \cap \cdots \cap P_m$ is the nilradical of R; it is T-nilpotent. By Theorem 5.3, R/N is a subdirect product of n-dimensional integral domains $D_j = R/P_j$ $(j = 1, \dots, m)$. In the next theorem, D_j-modules are also regarded as R-modules in the natural way.

Theorem 7.1. *Let R denote an n-subperfect ring, and let D_j be as before. Then an R-module M satisfies* $\text{p.d.}_R M \le k$ $(k \ge 0)$ *if and only if* $\text{Ext}_R^{k+1}(M, X) = 0$ *for all D_j-modules X for each $j = 1, \dots, m$.*

Proof. See Theorem 5.3 in [Sandomierski 1973]. □

Theorem 7.2. *Let R be an n-subperfect ring, and $R^* = R/N$. Then for an R-module M we have for any $k \ge 0$:*

(a) $\text{p.d.}_R M \le k$ *if and only if* $\text{Ext}_R^{k+1}(M, X) = 0$ *for all R^*-modules X.*

(b) $\text{i.d.}_R M \le k$ *if and only if* $\text{Ext}_R^{k+1}(R/L, M) = 0$ *for all ideals L containing N.*

(c) $\text{w.d.}_R M \le k$ *if and only if* $\text{Tor}^R_{k+1}(R/L, M) = 0$ *for all ideals L containing N.*

Proof. See Theorems 5.2, 3.2, and 4.2, respectively, in [Sandomierski 1973]. □

Also, [Sandomierski 1973, Proposition 5.4] shows that for a flat R-module F, $\text{p.d.}_R F$ can be calculated as the maximum of the D_j-projective dimensions of the flat D_j-modules $F \otimes_R D_j$, taken for all j.

We would like to have information about the finitistic dimension $\text{Fdim}(R)$ of an n-subperfect ring R. An estimate is given by [Sandomierski 1973, Corollary 1, Section 2] which we cite using the same notation as above.

Theorem 7.3. *For an n-subperfect ring R and for the integral domains D_j we have the inequality*

$$\mathrm{Fdim}(R) \leq \max_j\{\mathrm{p.d.}_R D_j + \mathrm{Fdim}(D_j)\}. \qquad \square$$

We recall (see, e.g., [Jensen 1972, Remarque, p. 44]) that for a Cohen–Macaulay ring R, the finitistic dimension $\mathrm{Fdim}(R)$ is equal either to d or to $d+1$ where $d = \dim R$. For n-subperfect rings we do not have such a precise estimate, but we still have some information, see Theorem 7.5.

In the balance of this section, we will use the notation $\mathcal{P}_n(R)$ for the class of R-modules whose projective dimensions are $\leq n$, and $\mathcal{F}_n(R)$ for the class of modules of weak dimensions $\leq n$. We concentrate on the class $\mathcal{F}_1(R)$ which is more relevant to subperfectness than the class $\mathcal{F}_0(R)$ of flat modules; see, e.g., Lemma 2.2(v).

Next, we verify a lemma (note that $\bar{R}$-modules may be viewed as R-modules).

Lemma 7.4. *Let R be any ring and $\bar{R} = R/rR$ with $r \in R^\times$ a nonunit. Then*

(1) *if $\bar{R}$ is subperfect, then $\mathcal{F}_1(\bar{R}) \subseteq \mathcal{F}_1(R)$;*

(2) *if both R and $\bar{R}$ are subperfect, then $\mathcal{F}_1(R) \subseteq \mathcal{P}_m(R)$ for some $m > 0$ implies $\mathcal{F}_1(\bar{R}) \subseteq \mathcal{P}_{m-1}(\bar{R})$.*

Proof. We start observing that if R is a subperfect ring, then a module H satisfies $\mathrm{Tor}_1^R(H, Y) = 0$ for all torsion-free Y if and only if $H \in \mathcal{F}_1(R)$ (see [Fuchs 2019, Theorem 4.1]); here Y torsion-free means that $\mathrm{Tor}_1^R(R/tR, Y) = 0$ for all $t \in R^\times$. For any commutative ring R, $\mathrm{Tor}_1^R(X, Y) = 0$ for all torsion-free Y implies that $X \in \mathcal{F}_1(R)$ (but not necessarily conversely).

Recall [Cartan and Eilenberg 1956, Chapter VI, Proposition 4.1.1] which states that if an R-module Y satisfies $\mathrm{Tor}_k^R(\bar{R}, Y) = 0$ for all $k > 0$, then

$$\mathrm{Tor}_m^R(\bar{N}, Y) \cong \mathrm{Tor}_m^{\bar{R}}(\bar{N}, Y/rY) \tag{3}$$

holds for all $m > 0$ and for all $\bar{R}$-modules $\bar{N}$. The hypothesis holds if Y is a torsion-free R-module: it holds for $k = 1$ by definition and for $k > 1$ in view of $\mathrm{p.d.}_R \bar{R} = 1$.

First, let $s \in R$ be a divisor of r, and choose $\bar{N} \cong R/sR$. Then the left-hand side Tor vanishes for all torsion-free Y and for $m = 1$, so it follows that Y/rY is a torsion-free $\bar{R}$-module.

(i) Assuming $\bar{R}$ is subperfect, let $\bar{N} \in \mathcal{F}_1(\bar{R})$ and Y a torsion-free R-module. The right hand side of (3) vanishes for $m = 1$, so we can conclude that $\mathrm{Tor}_1^R(\bar{N}, Y) = 0$. This equality holds for all torsion-free R-modules Y, whence we obtain $\bar{N} \in \mathcal{F}_1(R)$.

(ii) Assuming both R and $\bar{R}$ are subperfect, let again $\bar{N} \in \mathcal{F}_1(\bar{R})$. Part (i) implies that $\bar{N} \in \mathcal{F}_1(R)$, so $\bar{N} \in \mathcal{P}_m(R)$ by hypothesis. From a well-known Kaplansky formula for projective dimensions [Kaplansky 1970, Proposition 172] we obtain that $\bar{N} \in \mathcal{P}_{m-1}(\bar{R})$, as claimed. □

Theorem 7.5. *If R is an n-subperfect ring, then* $\mathrm{Fdim}(R) \geq n$.

Proof. According to [Jensen 1972, Proposition 5.6], for any ring R, $\mathcal{F}_0(R) \subseteq \mathcal{P}_{m-1}(R)$ if $m = \mathrm{Fdim}(R) \geq 1$. Hence we have $\mathcal{F}_1(R) \subseteq \mathcal{P}_m(R)$. On the other hand, if R is n-subperfect, then Lemma 7.4 is applicable. Thus if $\mathcal{F}_1(R) \subseteq \mathcal{P}_k(R)$ holds for some k, then we have $\mathcal{F}_1(\bar{R}) \subseteq \mathcal{P}_{k-1}(\bar{R})$, and since R is n-subperfect, we can continue with $\bar{R}$ in the role of R, etc. If $k < n$, then this process would lead us down to $\mathcal{P}_0$, reaching a contradiction that over a subperfect ring of Krull dimension > 0 modules of weak dimension ≤ 1 are projective. Consequently, the inclusion $\mathcal{F}_1(R) \subseteq \mathcal{P}_k(R)$ can hold only if $k \geq n$. □

That we can have strict inequality in the preceding theorem is demonstrated by the following example. Let S denote an almost perfect (i.e., 1-subperfect) domain; it has finitistic dimension 1. If R is defined as in Example 8.1 as $S \oplus D$ with $D \neq 0$ a torsion-free divisible S-module, then $\text{p.d.}_R R/D$ is finite and > 1 (D is flat, but not projective, so $\text{p.d.}_R D = 1$), whence $\mathrm{Fdim}(R) \geq 2$.

The following result shows that in Theorem 7.5 equality may occur for non-Cohen–Macaulay rings as well.

Lemma 7.6. (i) *Let R be any ring, and $S = R[X]$ the polynomial ring over R. Then*

$$\mathcal{F}_1(R) \subseteq \mathcal{P}_n(R) \quad \textit{if and only if} \quad \mathcal{F}_1(S) \subseteq \mathcal{P}_{n+1}(S).$$

(ii) *If R is a perfect ring, then for the polynomial ring $S = R[X_1, \ldots, X_n]$* (which is n-subperfect by Theorem 6.2) *we have*

$$\mathcal{F}_1(S) \subseteq \mathcal{P}_n(S), \quad \textit{but} \quad \mathcal{F}_1(S) \nsubseteq \mathcal{P}_{n-1}(S).$$

Proof. (i) To verify necessity, assume M is a module in $\mathcal{F}_1(S)$. It is easy to see that then $M \in \mathcal{F}_1(R)$ as well, thus $M \in \mathcal{P}_n(R)$ follows by hypothesis. Hence tensoring over R with $R[X]$, we obtain $M[X] \in \mathcal{P}_n(S)$. It remains to refer to the exact sequence $0 \to M[X] \to M[X] \to M \to 0$ of S-modules to conclude that $M \in \mathcal{P}_{n+1}(S)$.

Conversely, working toward contradiction, suppose there are an $F \in \mathcal{F}_1(R)$ and an $H \in$ Mod-R such that $\mathrm{Ext}_R^{n+2}(F, H) \neq 0$. Then also $\mathrm{Ext}_R^{n+2}(F, H[X]) \neq 0$. Since $\mathrm{Tor}_k^R(F, S) = 0$ for all $k > 0$, we have an isomorphism (see [Cartan and Eilenberg 1956, Chapter VI, Proposition 4.1.3])

$$\mathrm{Ext}_S^{n+2}(F \otimes_R S, H[X]) \cong \mathrm{Ext}_R^{n+2}(F, H[X]) \neq 0.$$

Since $F \otimes_R S \in \mathcal{F}_1(S)$, this is in contradiction to $\mathcal{F}_1(S) \subseteq \mathcal{P}_{n+1}(S)$.

(ii) Noticing that $\mathcal{F}_1(R) = \mathcal{P}_0(R)$ if R is perfect, the claim follows by a simple calculation from (i). □

8. Examples

Our final section is devoted to various examples of n-subperfect rings. In the first examples we use n-subperfect domains to construct n-subperfect rings with nontrivial nilradicals. (For examples of nonnoetherian n-subperfect domains, we refer to Example 8.12 and Theorem 8.13 below.)

Example 8.1. Let S denote an n-subperfect domain ($n \geq 1$) with field of quotients H. Let D be a torsion-free divisible S-module. Define the ring R as the *idealization* of D, i.e., $R = S \oplus D$ additively, and multiplication in R is given by the rule

$$(s_1, d_1)(s_2, d_2) = (s_1 s_2, s_1 d_2 + s_2 d_1) \quad (s_i \in S,\ d_i \in D).$$

It is clear that $Q = (H, D)$ is the ring of quotients of R, and $N = (0, D)$ is the nilradical (nilpotent of exponent 2) of both R and Q. We claim that R is an n-subperfect ring.

First we observe that an element $r = (s, d) \in R$ is a zero-divisor if and only if $s = 0$; this is easily seen by direct calculation using the torsion-freeness of D. Hence criterion (iii) in Lemma 2.2 guarantees that R is a subperfect ring. Furthermore, for any $r = (s, d)$, we have $rR = (sS, D)$ (the divisibility of D is relevant). Therefore, we have an isomorphism $R/rR \cong S/sS$ for every regular $r \in R$ (i.e., for every nonzero $s \in S$). Hence we conclude that R/rR is $(n-1)$-subperfect for every regular r (Corollary 3.3). By the same Corollary, we obtain the desired conclusion for R.

Example 8.2. As before choose an n-subperfect ($n \geq 1$) integral domain S. Let A be any commutative S-algebra that is torsion-free and divisible as an S-module, and B a torsion-free divisible S-module containing A. Our ring R is now the ring of upper 3×3-triangular matrices of the form

$$\alpha = \begin{bmatrix} s & a & b \\ 0 & s & a \\ 0 & 0 & s \end{bmatrix} \quad (s \in S,\ a \in A,\ b \in B).$$

It is straightforward to check that $\alpha \in R$ is a zero-divisor if and only if $s = 0$, and that the principal ideal αR equals sR whenever $s \neq 0$. Fix any regular $\alpha_0 \in R$ (i.e., $0 \neq s_0 \in S$ in the diagonal), and consider the homomorphism $\phi : R \to S/s_0 S$ given by $\alpha \mapsto s + s_0 S$ ($\alpha \in R$). Then $\operatorname{Ker} \phi = \alpha_0 R = s_0 R$ leads to the isomorphism $R/\alpha_0 R \cong S/s_0 S$ showing that $R/\alpha_0 R$ is an $(n-1)$-subperfect ring for every regular $\alpha_0 \in R$. To complete the proof that R is n-subperfect, it remains only to show that R is subperfect. By Lemma 2.2(iii) it suffices to observe that the nilradical N of R

is nilpotent of degree 3, and every regular coset mod N consists of regular elements of R.

In order to obtain more general examples of similar kind, in the preceding examples we can choose S as a finite direct sum of n-subperfect domains.

Let R be a perfect ring, and let $S = R[X_1, \dots, X_n]$. By Corollary 5.2 and Theorem 6.2, the ring of invariants S^G of S is n-subperfect for each finite group G acting on S whose order is a unit in R. As in the classical case in which R is a field, more examples of n-subperfect rings can be obtained from S via Veronese subrings: a *Veronese subring* T of S is an R-subalgebra of S generated by all monomials of degree d for some fixed $d > 0$.

Theorem 8.3. *Let R be a ring, and $S = R[X_1, \dots, X_n]$ a polynomial ring over R. A Veronese subring of S is n-subperfect if and only if R is perfect.*

Proof. Let T be a Veronese subring of S generated by the monomials of degree d. Then T is an R-direct summand of S, and S is integral over T. If R is perfect, then T is n-subperfect by Theorems 5.1 and 6.2. Conversely, suppose T is n-subperfect. Then $X_1^d, \dots, X_n^d$ is a maximal regular sequence of T, so $T/(X_1^d, \dots, X_n^d)R$ is a perfect ring. As a homomorphic image of this ring, R is perfect. □

Remark 8.4. Asgharzadeh, Dorreh and Tousi [Asgharzadeh et al. 2017] study Cohen–Macaulay properties for Veronese, determinantal, and Grassmannian rings in the context of polynomial rings in infinitely many variables.

Theorem 6.2 shows that if R is perfect, then the ring $R[X_1, \dots, X_n]$ is n-subperfect. As the next example demonstrates, it need not be the case that for a k-subperfect ring R, $R[X_1, \dots, X_n]$ is $(n+k)$-subperfect.

Example 8.5. Let F be a field, X, Y indeterminates, and $K = F(X)$. Then the ring $R = F + YK[[Y]]$ is an almost perfect domain [Bazzoni and Salce 2003, Example 3.2]. The valuative dimension of R, that is, the maximum of the Krull dimensions of the valuation rings of $Q(R)$ that contain R, is 2. Thus $\dim R[X_1, X_2] = 4$ by [Arnold 1969, Theorem 6]. Although R is 1-subperfect, $R[X_1, X_2]$ is not 3-subperfect, since by Corollary 3.6 the Krull dimension of a 3-subperfect ring is 3.

Example 8.6. *An n-subperfect ($n \geq 1$) Prüfer domain is a Dedekind domain* (hence 1-subperfect). First of all, a Prüfer domain R cannot have a regular sequence of length greater than 1. Indeed, if x, y is a regular sequence in R, then $xR \cap yR = xyR$. If M is a maximal ideal containing x and y, then since R_M is a valuation domain, this implies $xR_M = xyR_M$ or $yR_M = xyR_M$, contradicting that neither x nor y is a unit in R_M. Consequently, an n-subperfect Prüfer domain is an almost perfect domain. But for modules over such domains, w.d. ≤ 1 implies p.d. ≤ 1 (see [Fuchs and Salce 2018, Theorem 6.1]), thus any n-subperfect Prüfer domain — if not a field — must be a Dedekind domain. Dedekind domains are trivially 1-subperfect.

Our next source of examples involves the idealization of a module, as defined in Example 8.1. For an R-module N, we denote by $R \star N$ the idealization of N. It is well known that if R is a Cohen–Macaulay ring and N is a maximal Cohen–Macaulay module, then $R \star N$ is a Cohen–Macaulay ring. In Corollary 8.8, we prove the analogue of this statement for n-subperfect rings. This follows from a more general lifting property of n-subperfectness:

Theorem 8.7. *Let I be an ideal of the ring R such that $I^2 = 0$ and R/I is n-subperfect for some $n \geq 0$. If every (R/I)-regular sequence in R is also I-regular, then R is n-subperfect.*

Proof. First we show that R is subperfect. If N is the nilradical of R, then N/I is the nilradical of the n-subperfect ring R/I, hence T-nilpotent. Therefore, N as an extension of the nilpotent I by the T-nilpotent N/I is T-nilpotent. Suppose $r + N$ ($r \in R$) is a regular element in R/N; then $r + N/I$ is regular in $(R/I)/(N/I)$, so Lemma 2.2(iii) shows that $r + I$ is regular in R/I. Since r is (R/I)-regular, r is I-regular by assumption. If r is both (R/I)-regular and I-regular, then it is regular in R. From Lemma 2.2(iii) we conclude that R is subperfect.

We claim next that each $r \in R^\times$ is (R/I)-regular. Since R/I is subperfect, there are finitely many prime ideals $P_1, \dots, P_m$ of R that are minimal over I and whose images in R/I contain every zero-divisor in R/I. Since I is in the nilradical of R, these primes are also the minimal prime ideals of R. If $r \in R^\times$, then $r \notin P_1 \cup \cdots \cup P_m$, so the image of r in R/I is not a zero-divisor. This shows that the regular elements of R are (R/I)-regular.

We prove now using induction that R is n-subperfect. If $n = 0$, then R/I is perfect and hence zero-dimensional. Since $I^2 = 0$, R is zero-dimensional. We have established that R is subperfect, so from $R = Q(R)$ we conclude that R is perfect.

Now let $n > 0$, and suppose the theorem has been proved for all $k < n$. We have already shown that R is subperfect. We claim that $A := R/rR$ is $(n-1)$-subperfect for every $r \in R^\times$. By the induction hypothesis, it suffices to show

(i) $(IA)^2 = 0$,

(ii) A/IA is $(n-1)$-subperfect, and

(iii) every A/IA-regular sequence in A is IA-regular.

It is clear that $(IA)^2 = 0$. To verify (ii), we use the fact already established that if $r \in R^\times$, then $r + I$ is regular in R/I. Since R/I is n-subperfect, Proposition 3.2 implies $R/(rR + I)$ is $(n-1)$-subperfect. In view of the isomorphism $A/IA \cong R/(rR + I)$, statement (ii) follows.

To verify (iii), suppose $a_1, \dots, a_t$ is an A/IA-regular sequence in A. If we write $a_i = r_i + rR$, then $r_1, \dots, r_t$ is an A/IA-regular sequence in R. Since $r \in R^\times$ and $A/IA \cong R/(rR + I)$, we have that $r, r_1, \dots, r_n$ is an R/I-regular sequence. By assumption, $r, r_1, \dots, r_t$ is also an I-regular sequence, so $r_1, \dots, r_t$ is an I/rI-regular sequence. As established, every regular element of R is a regular element in R/I. Thus $I \cap rR = rI$, and it follows that $IA = (I+rR)/rR \cong I/(I \cap rR) = I/rI$. Since $r_1, \dots, r_t$ is an (I/rI)-regular sequence in R, we conclude that $a_1, \dots, a_t$ is an IA-regular sequence in A. Thus every A/IA-regular sequence in A is IA-regular.

Having verified (i), (ii) and (iii), we conclude from the induction hypothesis that $A = R/rR$ is $(n-1)$-subperfect. Since R is subperfect and R/rR is $(n-1)$-subperfect for each $r \in R^{\times}$, Corollary 3.3 implies R is n-subperfect. □

Corollary 8.8. *Let R be an n-subperfect ring, and let N be an R-module such that every regular sequence in R extends to a regular sequence on N. Then $R \star N$ is an n-subperfect ring.* □

Example 8.9. Corollary 8.8 implies that if R is a local Cohen–Macaulay ring, and if N is a balanced big Cohen–Macaulay R-module, then $R \star N$ is n-subperfect for $n = \dim R$. Choosing N to be an infinite rank free R-module, we obtain a nonnoetherian n-subperfect ring $R \star N$.

More interesting choices are possible for N. For example if R is an excellent local Cohen–Macaulay domain of positive characteristic and positive dimension, and R^+ is the integral closure of R in the algebraic closure of the quotient field of R, then $R \star R^+$ is a nonnoetherian n-subperfect ring, since R^+ is a balanced big Cohen–Macaulay module that is not finitely generated [Hochster and Huneke 1992, Theorem 1.1].

Example 8.10. Let R be an n-subperfect ring and $\{X_i : i \in I\}$ a collection of indeterminates for R. Let

$$S = R[X_i : i \in I]/(X_i : i \in I)^2.$$

The ideal $N = (X_i : i \in I)/(X_i : i \in I)^2$ of S is nilpotent of index 2 and is a free R-module with basis the images of the X_i in N. As $S \cong R \star N$, the ring S is a special case of the construction in Example 8.9; therefore, S is n-subperfect. If the index set I is infinite, then S is not noetherian.

So far, our nonnoetherian examples, at least for $n > 1$, have involved n-subperfect rings with zero-divisors. Our next source of examples produces nonnoetherian n-subperfect domains, albeit in a nontransparent way.

Theorem 8.11. *Let S be a local Cohen–Macaulay domain such that $Q(S)$ is separably generated, and has positive characteristic and uncountable transcendence degree over its prime subfield. If $n := \dim S \geq 1$, then there exists a nonnoetherian n-subperfect subring R of S such that $Q(R) = Q(S)$ and S is integral over R.*

Proof. Let N be a free S-module of infinite rank. Applying [Olberding 2012, Theorem 3.5] to S and N, we obtain a subring R of S such that R is “strongly twisted by N." We omit the definition of this notion here, but we use the fact that by [Olberding 2012, Theorems 4.1 and 4.6] this implies

(i) there is a subring A of R such that S/A is a torsion-free divisible A-module and $I \cap A \neq 0$ for each ideal I of S;

(ii) R has the same quotient field as S and S is an integral extension of R; and

(iii) there is a faithfully flat ring embedding $f : R \to S \star N$ such that for each $0 \neq a \in A$, the induced map $f_a : R/aR \to (S \star N)/a(S \star N)$ is an isomorphism.

We show that the ring $R/(x_1, \dots, x_t)R$ is subperfect for each nonempty regular sequence $x_1, \dots, x_t$ in R. Since f is faithfully flat, $f(x_1), \dots, f(x_t)$ is a regular sequence in $T := S \star N$. By Corollary 8.8, T is an n-subperfect ring. Thus $f(x_1), \dots, f(x_t)$ is a subperfect sequence in T. Since for each $0 \neq a \in A$, the map f_a is an isomorphism, we have $T = f(R) + f(a)T$. By (i) and (ii), the fact that S/R is a torsion R-module implies there is $0 \neq a \in (x_1, \dots, x_t)R \cap A$. Hence

$$T = f(R) + (f(x_1), \dots, f(x_t))T.$$

Moreover, since f is faithfully flat, we have

$$(f(x_1), \dots, f(x_t))T \cap f(R) = (f(x_1), \dots, f(x_t))f(R).$$

Therefore,

$$\begin{aligned} T/(f(x_1), \dots, f(x_t))T &= \big(f(R) + (f(x_1), \dots, f(x_t))T\big)/(f(x_1), \dots, f(x_t))T \\ &\cong f(R)/\big((f(x_1), \dots, f(x_t))T \cap f(R)\big) \\ &= f(R)/(f(x_1), \dots, f(x_t))f(R) \\ &\cong R/(x_1, \dots, x_t)R. \end{aligned}$$

Consequently, since $f(x_1), \dots, f(x_t)$ is a subperfect sequence in T, it follows that $x_1, \dots, x_t$ is a subperfect sequence in R. This proves that every regular sequence in R is subperfect.

Finally, since S is integral over R and S is local, R is also local and has the same Krull dimension as S. By Corollary 3.6, $n = \dim S = \dim R$. Taking into account that every regular sequence in R is subperfect, Corollary 3.9 implies that R is n-subperfect. By [Olberding 2012, Theorem 5.2], the fact that N is a free S-module of infinite rank implies R is not noetherian. □

Example 8.12. Let p be a prime number, and let $\mathbb{F}_p$ denote the field with p elements. Suppose k is a purely transcendental extension of $\mathbb{F}_p$ with uncountable transcendence degree. Then $S = k[X_1, \dots, X_n]_{(X_1, \dots, X_n)}$ is a local n-subperfect domain (in fact, a Cohen–Macaulay ring) meeting the requirements of Theorem 8.11. Thus S contains a nonnoetherian n-subperfect subring R having the same quotient field as S.

Our final source of examples involves local Cohen–Macaulay rings that have a coefficient field. The next theorem shows that restriction to a smaller coefficient field can produce examples of nonnoetherian n-subperfect rings.

Theorem 8.13. *Let S be a local Cohen–Macaulay ring containing a field F such that $S = F + M$, where M is the maximal ideal of S. For each subfield k of F, the local ring $R = k + M$ is n-subperfect for $n = \dim S$. The ring R is noetherian if and only if F/k is a finite extension.*

Proof. Evidently, R is a local ring with maximal ideal M. It is clear that every prime ideal of S is a prime ideal of R. To verify the converse, let P be a nonmaximal prime ideal of R. To show that P is in fact an ideal of S, let $s \in S$. Then $sP \subseteq sM \subseteq R$, and also, $(sP)M = P(sM) \subseteq P$ because $sM \subseteq R$. Since $M \not\subseteq P$, we conclude that $sP \subseteq P$, which proves that P is an ideal of S. To see that P is prime in S, let $x, y \in S$ with $xy \in P$. If one of x or y is a unit in S, then the other is in P. Otherwise, if neither x nor y are units, then necessarily $x, y \in M \subseteq R$, and since P is a prime ideal of R, one of x, y is in P. Thus P is a prime ideal of S, and this shows that the prime ideals of R are precisely those of S.

We show now that R is n-subperfect, where $n = \dim S$. By [Fontana et al. 1997, Lemma 1.1.4, p. 5], $Q(R) = Q(S)$, so R is a subperfect ring, since the total quotient ring $Q(S)$ of the Cohen–Macaulay ring S is artinian. Let $x_1, \dots, x_t$ be a regular sequence in R, and $I = (x_1, \dots, x_t)R$. We claim that R/I is a subperfect ring. The height of I in R is at least t, and since R and S share the same prime ideals, the height of IS is also at least t. Krull's height theorem implies then that the height of the t-generated ideal IS is t. Since S is a Cohen–Macaulay ring, the ideal IS is unmixed. We use this to show next that $Q(R/I)$ is zero-dimensional.

To this end, we prove that every zero-divisor of R/I is contained in a minimal prime ideal of R/I. Let $x, y \in R$ such that $xy \in I$ and $y \notin I$. Suppose by way of contradiction that x is not contained in any minimal prime ideal of I. Since I and IS share the same minimal primes, the image of x in S/IS does not belong to any minimal prime ideal of S/IS. However, IS is unmixed, so necessarily $y \in IS$. Therefore, using the fact that $S = F + M$, we can write

$$y = \alpha_1 x_1 + \cdots + \alpha_t x_t + z \quad \text{for } \alpha_1, \dots, \alpha_t \in F \text{ and } z \in (x_1, \dots, x_t)M.$$

Similarly, since $xy \in I$ and $R = k + M$, we have

$$xy = \beta_1 x_1 + \cdots + \beta_t x_t + w \quad \text{for } \beta_1, \dots, \beta_t \in k \text{ and } w \in (x_1, \dots, x_t)M.$$

Let i be the largest index such that at least one of α_i, β_i is not 0. Using the preceding expressions for y and xy, we obtain

$$\beta_1 x_1 + \cdots + \beta_i x_i + w = \alpha_1 x x_1 + \cdots + \alpha_i x x_i + xz.$$

Therefore,

$$(\beta_i - \alpha_i x)x_i \in (x_1, \ldots, x_{i-1})R.$$

Since $\beta_i - \alpha_i x \in k + M = R$ and $x_1, \ldots, x_i$ is a regular sequence in R, we have $\beta_i - \alpha_i x \in (x_1, \ldots, x_{i-1})R$. The fact that x is a nonunit in R implies $\beta_i \in M$, so $\beta_i = 0$ and hence, by the choice of i, $\alpha_i \neq 0$. Since the prime ideals of S are the same as the prime ideals of R, $\sqrt{(x_1, \ldots, x_{i-1})R}$ is an ideal of S. Also, α_i is a unit in S and $\alpha_i x \in (x_1, \ldots, x_{i-1})R$, so $x \in \sqrt{(x_1, \ldots, x_{i-1})R} \subseteq \sqrt{I}$. However, x was chosen not to be contained in any minimal prime ideal of I. This contradiction implies that x must be in some minimal prime ideal of I, establishing that $Q(R/I)$ is a zero-dimensional ring. Since I and IS share the same minimal prime ideals, I has only finitely many minimal primes, so $Q(R/I)$ is also semilocal.

It remains to show that the nilradical of R/I is T-nilpotent, and to prove this, it suffices to show that some power of $\sqrt{I}$ is contained in I. Since $\sqrt{I}$ is a finitely generated ideal of the noetherian ring S and $\sqrt{I} = \sqrt{IM}$, with IM an ideal of S, there is $t > 0$ such that $(\sqrt{I})^t \subseteq IM \subseteq I$. Therefore, R/I is subperfect, which completes the proof that every regular sequence in R is subperfect. Since R and S share the same maximal ideal, Corollary 4.3 implies R is n-subperfect for $n = \dim S$. Finally, it is straightforward to check that R is noetherian if and only if F/k is a finite field extension; see [Fontana et al. 1997, Proposition 1.1.7, p. 7]. □

Example 8.14. Let $S = F[\![X_1, \ldots, X_n]\!]/I$, where F is a field, $X_1, \ldots, X_n$ are indeterminates for F, and I is an ideal such that S is Cohen–Macaulay. Theorem 8.13 implies that for each subfield k of F,

$$R = \{ f + I \in S : f \in F[\![X_1, \ldots, X_n]\!] \text{ and } f(0, \ldots, 0) \in k \}$$

is an n-subperfect ring.

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References

[Arapović 1983] M. Arapović, "Characterizations of the 0-dimensional rings", *Glas. Mat. Ser. III* **18(38)**:1 (1983), 39–46. MR Zbl

[Arnold 1969] J. T. Arnold, "On the dimension theory of overrings of an integral domain", *Trans. Amer. Math. Soc.* **138** (1969), 313–326. MR Zbl

[Asgharzadeh and Tousi 2009] M. Asgharzadeh and M. Tousi, "On the notion of Cohen–Macaulayness for non-Noetherian rings", *J. Algebra* **322**:7 (2009), 2297–2320. MR Zbl

[Asgharzadeh et al. 2017] M. Asgharzadeh, M. Dorreh, and M. Tousi, "Direct summands of infinite-dimensional polynomial rings", *J. Commut. Algebra* **9**:1 (2017), 1–19. MR Zbl

[Bass 1960] H. Bass, "Finitistic dimension and a homological generalization of semi-primary rings", *Trans. Amer. Math. Soc.* **95** (1960), 466–488. MR Zbl

[Bass 1963] H. Bass, "On the ubiquity of Gorenstein rings", *Math. Z.* **82** (1963), 8–28. MR Zbl

[Bazzoni and Salce 2003] S. Bazzoni and L. Salce, "Almost perfect domains", *Colloq. Math.* **95**:2 (2003), 285–301. MR Zbl

[Björk 1969] J.-E. Björk, "Rings satisfying a minimum condition on principal ideals", *J. Reine Angew. Math.* **236** (1969), 112–119. MR Zbl

[Bruns and Herzog 1998] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, revised ed., Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, 1998. Zbl

[Cartan and Eilenberg 1956] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, 1956. MR Zbl

[Coleman and Enochs 1971] D. B. Coleman and E. E. Enochs, "Isomorphic polynomial rings", *Proc. Amer. Math. Soc.* **27** (1971), 247–252. MR Zbl

[Fontana et al. 1997] M. Fontana, J. A. Huckaba, and I. J. Papick, *Prüfer domains*, Monographs and Textbooks in Pure and Applied Mathematics **203**, Marcel Dekker, New York, 1997. MR Zbl

[Fuchs 2019] L. Fuchs, "Characterizing almost perfect rings by covers and envelopes", *J. Korean Math. Soc.* (online publication September 2019).

[Fuchs and Salce 2018] L. Fuchs and L. Salce, "Almost perfect commutative rings", *J. Pure Appl. Algebra* **222**:12 (2018), 4223–4238. MR Zbl

[Glaz 1994] S. Glaz, "Coherence, regularity and homological dimensions of commutative fixed rings", pp. 89–106 in *Commutative algebra* (Trieste, 1992), World Sci. Publ., River Edge, NJ, 1994. MR Zbl

[Goodearl and Warfield 1989] K. R. Goodearl and R. B. Warfield, Jr., *An introduction to noncommutative Noetherian rings*, London Mathematical Society Student Texts **16**, Cambridge University Press, 1989. MR Zbl

[Gupta 1970] R. N. Gupta, "Characterization of rings whose classical quotient rings are perfect rings", *Publ. Math. Debrecen* **17** (1970), 215–222. MR Zbl

[Hamilton 2004] T. D. Hamilton, "Weak Bourbaki unmixed rings: a step towards non-Noetherian Cohen–Macaulayness", *Rocky Mountain J. Math.* **34**:3 (2004), 963–977. MR Zbl

[Hamilton and Marley 2007] T. D. Hamilton and T. Marley, "Non-Noetherian Cohen–Macaulay rings", *J. Algebra* **307**:1 (2007), 343–360. MR Zbl

[Hochster and Eagon 1971] M. Hochster and J. A. Eagon, "Cohen–Macaulay rings, invariant theory, and the generic perfection of determinantal loci", *Amer. J. Math.* **93** (1971), 1020–1058. MR Zbl

[Hochster and Huneke 1992] M. Hochster and C. Huneke, "Infinite integral extensions and big Cohen–Macaulay algebras", *Ann. of Math.* (2) **135**:1 (1992), 53–89. MR Zbl

[Jensen 1972] C. U. Jensen, *Les foncteurs dérivés de* $\varprojlim$ *et leurs applications en théorie des modules*, Lecture Notes in Mathematics **254**, Springer, 1972. MR Zbl

[Kabele 1971] T. Kabele, "Regularity conditions in nonnoetherian rings", *Trans. Amer. Math. Soc.* **155** (1971), 363–374. MR Zbl

[Kaplansky 1970] I. Kaplansky, *Commutative rings*, Allyn and Bacon, Boston, 1970. MR Zbl

[Lam 1999] T. Y. Lam, *Lectures on modules and rings*, Graduate Texts in Mathematics **189**, Springer, New York, 1999. MR Zbl

[Lam 2001] T. Y. Lam, *A first course in noncommutative rings*, 2nd ed., Graduate Texts in Mathematics **131**, Springer, 2001. MR Zbl

[Lee 2006] S. B. Lee, "Weak-injective modules", *Comm. Algebra* **34**:1 (2006), 361–370. MR Zbl

[Lipman 1965] J. Lipman, "Some properties of localization and normalization", *Proc. Amer. Math. Soc.* **16** (1965), 1120–1122. MR Zbl

[Matsumura 1986] H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics **8**, Cambridge University Press, 1986. MR Zbl

[Ohm and Pendleton 1968] J. Ohm and R. L. Pendleton, "Rings with noetherian spectrum", *Duke Math. J.* **35** (1968), 631–639. MR Zbl

[Olberding 2012] B. Olberding, "A counterpart to Nagata idealization", *J. Algebra* **365** (2012), 199–221. MR Zbl

[Olberding 2019] B. Olberding, "Height theorems and unmixedness for polynomial rings over zero-dimensional rings", preprint, 2019.

[Sandomierski 1973] F. L. Sandomierski, "Homological dimension under change of rings", *Math. Z.* **130** (1973), 55–65. MR Zbl

[Small 1966] L. W. Small, "Orders in Artinian rings", *J. Algebra* **4** (1966), 13–41. MR Zbl

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