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We define an "antiholomorphic involution" of a module M over the Dieudonné ring $\mathcal{E}(k)$ of a finite field k with $q = p^a$ elements to be an involution $\tau : M \to M$ that switches the action of $\mathcal{F}^a$ with that of $\mathcal{V}^a$. The definition extends to include quasi-polarizations of Dieudonné modules. Nontrivial examples exist. The number of isomorphism classes of quasi-polarized Dieudonné modules within a fixed isogeny class is shown to be given by a twisted orbital integral over the general linear group. Earlier (*Pacific J. Math.* 303:1 (2019), 165–215) we considered these notions in the case of ordinary abelian varieties over k, in which case the contribution at p to the number of isomorphism classes within an isogeny class was shown to be given by an ordinary orbital integral over the general linear group. The definitions here are shown to be equivalent to those in our previous paper and, as a consequence, the equality of the orbital integrals of both types is proven.

1. Introduction

Locally symmetric spaces associated to the group $GL_n(\mathbb{R})$ for $n \geq 3$ do not carry a complex structure and do not admit an obvious reduction to characteristic $p > 0$. However, it is known ([Adler 1979; Gross and Harris 1981; Comessatti 1925; 1926; Goresky and Tai 2003a; 2003b; Milne and Shih 1981; Shimura 1975; Silhol 1982; Seppälä and Silhol 1989]) that such locally symmetric spaces parametrize *real* polarized abelian varieties (possibly with level structures). In an effort to find a characteristic p analog for such moduli spaces in [Goresky and Tai 2019] we introduced the notion of a *real structure* on an ordinary abelian variety A (or, rather, on its associated Deligne module T_A) defined over a finite field k: it is an "antiholomorphic" involution, that is, a linear involution that switches the action of the Frobenius and the Verschiebung. If A is the good, ordinary reduction of a CM variety $A/\mathbb{C}$ defined over $\mathbb{R}$ then complex conjugation of $A/\mathbb{C}$ induces such an involution on the Deligne module T_A. Over a finite field there are finitely many isomorphism classes of principally polarized ordinary abelian varieties with real structure and the number of isomorphism classes is given ([Goresky and Tai 2019])

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by a certain sum of orbital integrals over the general linear group $\mathrm{GL}_n \times \mathrm{GL}_1$. It is expected that these (or similar) definitions make sense beyond the "ordinary" case.

In Section 3.2, we extend the notion of a "real structure" to the case of (not necessarily ordinary) Dieudonné modules. We give examples (Section 3.3) to show that real structures often exist, even on supersingular Dieudonné modules. Then we show (Proposition 4.4) that the number of isomorphism classes of principally polarized "real" Dieudonné modules within a single isogeny class is given by a "twisted" orbital integral $TO(\delta)$ over the same general linear group $\mathrm{GL}_n \times \mathrm{GL}_1$.

We show that the constructions in this paper are compatible with those in [Goresky and Tai 2019], which requires an explicit description (Proposition 6.8) of the Dieudonné module (and its polarization) of an ordinary polarized abelian variety. Then we use this description to show (Proposition 6.12) that a real structure in the sense of [Goresky and Tai 2019] on an ordinary abelian variety determines a real structure (in the sense of this paper) on its Dieudonné module. This last step is not automatic: it requires a universal choice of involution on the Witt vectors, as constructed in Appendix A.

The compatibility between these two notions of real structure leads to a simplification of the twisted orbital integral $TO(\delta)$. The number of isomorphism classes of "real" Deligne modules (over $\mathbb{Z}_p$) is given by an (ordinary) orbital integral $O(\gamma)$: it is the component at p in the adèlic orbital integral of [Goresky and Tai 2019]. Using a linear algebra argument, we show (Section 7.5) that the orbital integral $O(\gamma)$ (which counts Deligne modules with real structure) coincides with the twisted orbital integral $TO(\delta)$ (which counts Dieudonné modules with real structure). This equality of orbital integrals is reminiscent of the results in [Kottwitz 1992] (for the symplectic group rather than the general linear group) in which the fundamental lemma for Levi subgroups is used in order to evaluate stable sums of twisted orbital integrals in terms of ordinary orbital integrals (and presumably a similar argument would work in our case as well).

2. Notation and terminology

Throughout this paper we fix a finite field $k = \mathbb{F}_q$ ($q = p^a$) of characteristic p. Let W denote the Witt ring functor, so that $W(k)$, $W(\bar{k})$ are the rings of (infinite) Witt vectors over k, $\bar{k}$, respectively, with fraction fields $K(k) = W(k) \otimes \mathbb{Q}_p$ and $K(\bar{k}) = W(\bar{k}) \otimes \mathbb{Q}_p$, respectively. We may identify $K(k)$ with the unique unramified extension of $\mathbb{Q}_p$ of degree $a = [k : \mathbb{F}_p]$. Let $W_0(\bar{k})$ denote the maximal unramified extension of $W(k)$. We may identify $W(\bar{k})$ with the completion of $W_0(\bar{k})$. Let $\sigma : W(\bar{k}) \to W(\bar{k})$ be the lift of the Frobenius mapping $\sigma : \bar{k} \to \bar{k}$, $\sigma(x) = x^p$ and let $\pi = \sigma^a$ be the topological generator for the Galois group $\mathrm{Gal}(\bar{k}/k) \cong \mathrm{Gal}(K(\bar{k})/K(k))$. Fix an identification, $\mathbb{Q}_p \cong K(\mathbb{F}_p)$ of the p-adic numbers with the fraction field of the Witt vectors of the prime field.

Let R be an integral domain with fraction field K. Let M be a free R-module of rank $2n$ and $V = M \otimes K$. An alternating bilinear form $\omega : M \times M \to R$ is *symplectic* if $\omega \otimes K : V \otimes V \to K$ is nondegenerate. It is *strongly nondegenerate* if the resulting $M \to \mathrm{Hom}_R(M, R)$ is an isomorphism. It is *symplectic up to homothety* if there exists $c \in K^\times$ such that $c\omega$ is strongly nondegenerate. The *standard symplectic form* ω_0 on $R^{2n} \times R^{2n}$ is that whose matrix is $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. Set $G = \mathrm{GSp}_{2n}$ and for convenience denote

$$\Gamma_p = G(\mathbb{Z}_p) \quad \text{and} \quad \Gamma_W = G(W(k)). \tag{2.0.1}$$

The *standard involution* $\tau_0 : R^{2n} \to R^{2n}$ is the linear map with matrix $\left(\begin{smallmatrix} -I & 0 \\ 0 & I \end{smallmatrix}\right)$. Conjugation by τ_0, which we denote by

$$g \mapsto \tilde{g} = \tau_0 g \tau_0^{-1}$$

defines an action of the group $\langle \tau_0 \rangle \cong \mathbb{Z}/2\mathbb{Z}$ on GSp_{2n}. If $2 \in K^\times$ the fixed subgroup is

$$H = \mathrm{GL}_n^* = \left\{ \begin{pmatrix} A & 0 \\ 0 & \lambda\, {}^t A^{-1} \end{pmatrix} \in \mathrm{GSp}_{2n} \;\middle|\; A \in \mathrm{GL}_n; \lambda \in \mathbb{G}_m \right\} \cong \mathrm{GL}_n \times \mathbb{G}_m. \tag{2.0.2}$$

If $\mathcal{C}$ is a $\mathbb{Z}$-linear category then the associated category up to R isogeny is the category $\mathcal{C} \otimes R$ with the same objects but with morphisms $\mathrm{Hom}_{\mathcal{C}\otimes R}(x, y) = \mathrm{Hom}_{\mathcal{C}}(x, y) \otimes R$.

3. Dieudonné modules

3.1. ***Notation.*** Let $\mathcal{E} = \mathcal{E}(k)$ denote the Cartier–Dieudonné ring, that is, the ring of noncommutative $W(k)$-polynomials in two variables $\mathcal{F}$, $\mathcal{V}$, subject to the relations $\mathcal{F}(wx) = \sigma(w)\mathcal{F}(x)$, $\mathcal{V}(wx) = \sigma^{-1}(w)\mathcal{V}(x)$, and $\mathcal{F}\mathcal{V} = \mathcal{V}\mathcal{F} = p$, where $w \in W(k)$ and $x \in \mathcal{E}$. A *Dieudonné module* M is a module over the ring $\mathcal{E}(k)$ that is free and finite rank over $W(k)$.

The *covariant Dieudonné functor* (see, for example, [Chai et al. 2014, §B.3.5.6] or [Goren 2002, p. 245] or [Pink 2005]) assigns to each p-divisible group

$$G = \dots \underset{\twoheadleftarrow}{\hookrightarrow} G_r \underset{\twoheadleftarrow}{\hookrightarrow} G_{r+1} \underset{\twoheadleftarrow}{\hookrightarrow} \dots$$

a corresponding module $M(G) = \varprojlim M(G_r)$ over the Dieudonné ring $\mathcal{E}$.

A *quasi-polarization* (in the sense of [Moonen 2001; Oort 2001] and [Li and Oort 1998, §5.9] following [Oda 1969, p. 101]) of a Dieudonné module M is an alternating $W(k)$-bilinear form $\omega : M \times M \to W(k)$ such that $\omega \otimes K(k)$ is nondegenerate and $\omega(\mathcal{F}x, y) = \sigma\omega(x, \mathcal{V}y)$. (The use of "quasi" reflects the fact that there is no p-adic counterpart to the "positivity" condition found in the definition of a polarization for abelian varieties.) A *$K(k)$-isogeny* of polarized Dieudonné modules $(M, \omega) \to (M', \omega')$ is an element $\phi \in \mathrm{Hom}_{\mathcal{E}}(M, M') \otimes K(k)$ such that $\phi^*(\omega') = c\omega$ for some $c \in K(k)^\times$.

3.2. ***Real structures.*** Let M be a Dieudonné module of finite rank over $W(k)$ (with $k = \mathbb{F}_q$; $q = p^a$). Let ω be a quasi-polarization on M. Define a real structure on (M, ω) to be a $W(k)$-linear mapping $\tau_p : M \to M$ such that for all $x, y \in M$,

$$\tau_p^2 = I, \quad \tau_p \mathcal{F}^a \tau_p^{-1} = \mathcal{V}^a, \quad \omega(\tau_p x, \tau_p y) = -\omega(x, y). \tag{3.2.1}$$

As in [Kottwitz 1990, §12] the action of $\mathcal{F}$ may be expressed as $\delta\sigma$ for some $\delta \in \mathrm{GSp}(M \otimes K(k), \omega)$, so its norm

$$N(\delta) = \delta\sigma(\delta) \cdots \sigma^{a-1}(\delta) \in \mathrm{GSp}(M \otimes K(k), \omega)$$

coincides with the $W(k)$-linear action of $\mathcal{F}^a$. The second condition in (3.2.1) gives

$$\tau_p N(\delta) \tau_p^{-1} = q N(\delta)^{-1}.$$

3.3. ***Manin modules.*** Following [Manin 1963], let us define Dieudonné modules

$$M_{r,s} = \mathcal{E}(k)/\mathcal{E}(k)(\mathcal{F}^r + \mathcal{V}^s)$$

for nonnegative integers r, s. If $\bar{k}$ is an algebraic closure of k and if we extend scalars to

$$\overline{\mathcal{E}}(\bar{k}) = W(\bar{k})\Big[\frac{1}{p}\Big] \otimes \mathcal{E}(k),$$

it is shown in [Manin 1963] that if $\gcd(r, s) = 1$, the resulting modules $\overline{\mathcal{E}}(\bar{k}) \otimes_{\mathcal{E}(k)} M_{r,s}$ are simple and they account for all the simple Dieudonné modules. Elements of $M_{r,s}$ may be represented by (noncommutative) polynomials

$$x = \sum_{i=1}^{s-1} a_{-i}\mathcal{V}^i + a_0 + \sum_{j=1}^{r} a_j \mathcal{F}^j$$

(with $a_t \in W(k)$ and with identifications $\mathcal{F}^r = -\mathcal{V}^s$).

In the following paragraphs we will show that *the Manin modules $M_{r,s} \oplus M_{s,r}$ and the Manin modules $M_{r,r}$ admit quasi-polarizations and real structures.*

First suppose $r \neq s$. The elements $\{1, \mathcal{F}^j, \mathcal{V}^i\}$ $(1 \le j \le r;\ 1 \le i \le s-1)$ form a basis of $M_{r,s}$ over $W(k)$. The module $M_{s,r}$ admits a dual basis by setting

$$(\mathcal{F}^i)^\vee = \mathcal{V}^{r-i}, \quad (\mathcal{V}^j)^\vee = \mathcal{F}^{s-j}.$$

This gives rise to a $W(k)$-linear pairing $T : M_{r,s} \times M_{s,r} \to W(k)$ with

$$T(\mathcal{F}^i, \mathcal{V}^j) = \begin{cases} 1 & \text{if } i + j = r, \\ 0 & \text{otherwise,} \end{cases} \qquad T(\mathcal{V}^i, \mathcal{F}^j) = \begin{cases} 1 & \text{if } i + j = s, \\ 0 & \text{otherwise,} \end{cases}$$

such that $T(\mathcal{F}x, y) = \sigma(T(x, \mathcal{V}y))$. It follows that the alternating bilinear form

$$\omega(x \oplus y, x' \oplus y') = T(x, y') - T(x', y)$$

defines a quasi-polarization on $M_{r,s} \oplus M_{s,r}$. A real structure on this sum is defined by switching the factors and exchanging $\mathcal{F}$ with $\mathcal{V}$. Explicitly, define $\tau_p : M_{r,s} \to M_{s,r}$ by

$$\tau_p\left(\sum_{i=1}^{s-1} a_{-i}\mathcal{V}^i + a_0 + \sum_{j=1}^{r} a_j \mathcal{F}^j\right) = \sum_{i=1}^{s-1} a_{-i}\mathcal{F}^i + a_0 + \sum_{j=1}^{r} a_j \mathcal{V}^j$$

and similarly for $\tau_p : M_{s,r} \to M_{r,s}$. Then $\tau_p^2 = I$ and

$$\tau_p(\mathcal{F}(x \oplus y)) = \sigma^2 \mathcal{V}(\tau_p(x \oplus y))$$

which implies that $\tau_p \mathcal{F}^a = \mathcal{V}^a \tau_p$. Finally, one verifies for $x, y \in M_{r,s}$ and $x', y' \in M_{s,r}$ that

$$\omega(\tau_p(x \oplus y), \tau_p(x' \oplus y')) = -\omega(x \oplus y, x' \oplus y').$$

Now suppose $r = s$. The Manin module

$$M'_{r,r} = \mathcal{E}(k)/\mathcal{E}(k)(\mathcal{F}^r + \mathcal{V}^r)$$

is the Dieudonné module of a supersingular abelian variety. It has a $W(k)$-basis consisting of $\{\mathcal{V}^i, \mathcal{F}^j, \mathcal{V}^0 = \mathcal{F}^0 = 1, \mathcal{V}^r = -\mathcal{F}^r\}$ with $1 \le i, j \le r-1$. It admits a quasi-polarization which for $0 \le i, j \le r$ is well defined as

$$\omega(\mathcal{V}^i, \mathcal{F}^j) = \begin{cases} 1 & \text{if } i + j = r, \\ 0 & \text{otherwise,} \end{cases} \qquad \omega(\mathcal{F}^i, \mathcal{V}^j) = \begin{cases} -1 & \text{if } i + j = r, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\omega(x, y) = -\omega(y, x)$ and $\omega(\mathcal{F}x, y) = \sigma\omega(x, \mathcal{V}y)$ for all $x, y \in M'_{1,1}$. This module admits a real structure by setting $\tau_p(t\mathcal{F}^i) = t\mathcal{V}^i$ for $t \in W(k)$ and $0 \le i \le r$ (and in particular, $\tau_p(t\mathcal{F}^r) = -t\mathcal{F}^r$). It is easy to check that $\tau_p(\mathcal{F}^a x) = \mathcal{V}^a \tau_p(x)$ for all $a \ge 0$ and all $x \in M'_{r,r}$.

3.4. In [Manin 1963] the isogenous module $\mathcal{E}(k)/\mathcal{E}(k)(\mathcal{F}^r - \mathcal{V}^s)$ is used to replace the module $M_{r,s}$. However the "+" sign in the preceding example is crucial.

4. Counting Dieudonné modules

As in (2.0.1) let $\Gamma_W = G(W(k))$ with the standard symplectic form $\omega_0 = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. Let $I_p = \left(\begin{smallmatrix} I & 0 \\ 0 & pI \end{smallmatrix}\right)$. By the theory of Smith normal form for the symplectic group (see [Spence 1972] or [Andrianov 1987, Lemma 3.3.6]), or by the Cartan decomposition for p-adic groups, we have the following:

Lemma 4.1. *Let $L_0 = W(k)^{2n} \subset K(k)^{2n}$ denote the standard lattice. Let $L \subset K(k)^{2n}$ be a $W(k)$-lattice. Then $L = hL_0$ for some $h \in G(K(k))$ and the following statements are equivalent.*

(1) $pL_0 \subset hL_0 \subset L_0$.

(2) $hL_0 \subset L_0, \; ph^{-1}L_0 \subset L_0$.

(3) $h \in \Gamma_W I_p \Gamma_W$. □

4.2. Assume $p \neq 2$. In this section we fix a Dieudonné module $(M, \mathcal{F}, \mathcal{V})$ with a quasi-polarization ω_M and a real structure $\tau_M : M \to M$. Then M is a free module over $W(k)$ of some even rank, say $2n$. Let $M_{\mathbb{Q}} = M \otimes K(k)$. We wish to understand the set X_M of (real) isomorphism classes of principally (quasi-)polarized Dieudonné modules that are $K(k)$-isogenous to M. In Proposition 4.4 below we show that the cardinality $|X_M|$ is given by a twisted orbital integral over the group $H \cong \mathrm{GL}_n \times \mathrm{GL}_1$ of (2.0.2).

Following the method of [Kottwitz 1990] let $\mathcal{X}_M$ denote the set of isomorphism classes in the category $\mathcal{C}_M$ whose objects consist of tuples $(P, \omega_P, \psi, \tau_P)$ where P is a Dieudonné module, ω_P is a principal quasi-polarization of P, where τ_P is a real structure on P and where $\psi \in \mathrm{Hom}_{W(k)}(P, M) \otimes K(k)$ is a $K(k)$ isogeny (meaning that $\psi \otimes K(k) : P_{\mathbb{Q}} \to M_{\mathbb{Q}}$ is an isomorphism) that commutes with $\mathcal{F}$, $\mathcal{V}$, takes τ_P to τ_M and satisfies $\psi^*(\omega_M) = c\omega_P$ for some $c \in K(k)^\times$. A morphism $\phi : P \to P'$ between left $\mathcal{E}(k)$ modules is in $\mathcal{C}_M$ if it is compatible with ω up to scalars, and it commutes with $\mathcal{F}$, $\mathcal{V}$ and the involutions $\tau_P.\tau_{P'}$. So there is a natural identification

$$X_M \cong I(M)\backslash \mathcal{X}_M,$$

where $I(M)$ denotes the group of $K(k)$ self-isogenies of (M, ω_M, τ_M).

4.3. The mapping $(P, \omega_P, \psi, \tau_P) \mapsto L = \psi(P)$ determines an identification between the set $\mathcal{X}_M$ and the set of $W(k)$-lattices $L \subset M_{\mathbb{Q}}$ that are preserved by $\mathcal{F}_M, \mathcal{V}_M, \tau_M$ and such that L is *symplectic up to homothety* meaning that $L^\vee = cL$ for some $c \in K(k)^\times$, where

$$L^\vee = \{x \in M_{\mathbb{Q}} \mid \omega_M(x, y) \in W(k) \text{ for all } y \in L\}.$$

By [Goresky and Tai 2019, Proposition B.4] there exists a $K(k)$-linear isomorphism $M_{\mathbb{Q}} \to K(k)^{2n}$ which takes the quasi-polarization ω_M to the standard symplectic form ω_0 and takes the involution τ_M to the standard involution $\tau_0 = \left(\begin{smallmatrix} -I & 0 \\ 0 & I \end{smallmatrix}\right) \in G(\mathbb{Z})$. From Section 3.2 the action of $\mathcal{F} \circ \sigma^{-1}$ becomes some element $\delta \in G(K(k))$ with multiplier p, that is well defined up to σ-conjugacy. The group $I(M)$ of self-isogenies becomes identified with the twisted centralizer (note that $\delta \notin H(K(k))$):

$$S_\delta(K(k)) = \{z \in H(K(k)) \mid z^{-1}\delta\sigma(z) = \delta\}.$$

Normalize the Haar measure on $H(K(k))$ so that $H(W(k))$ has volume one.

Proposition 4.4. *The choice of isomorphism $M_{\mathbb{Q}} \to K(k)^{2n}$ determines a one-to-one correspondence between the set of lattices $L \subset M_{\mathbb{Q}}$, symplectic up to homothety, that are preserved by $\mathcal{F}$, $\mathcal{V}$, τ_M and the set*

$$\{g \in H(K(k))/H(W(k)) \mid g^{-1}\delta\sigma(g) \in \Gamma_W I_p \Gamma_W\}. \tag{4.4.1}$$

Consequently the number of isomorphism classes

$$|X_M| = |S_\delta(K(k))\backslash \mathcal{X}_M|$$

of principally quasi-polarized real Dieudonné modules within the $K(k)$-isogeny class of M is given by the twisted orbital integral over $H = \mathrm{GL}_n^*$,

$$TO(\delta) = \int_{S_\delta(K)\backslash H(K)} \kappa_W(g^{-1}\delta\sigma(g))\, dg, \tag{4.4.2}$$

where κ_W is the characteristic function of $\Gamma_W I_p \Gamma_W \subset G(K(k))$.

Proof. Let $L_0 = W(k)^{2n} \subset K(k)^{2n}$ be the standard lattice. If $L \subset K(k)^{2n}$ is a $W(k)$-lattice, symplectic up to homothety, then $L = gL_0$ for some $g \in G(K(k))$. If it is preserved by $\mathcal{F}$, $\mathcal{V}$ then

$$pL_0 \subset g^{-1}\delta\sigma(g)L_0 \subset L_0 \tag{4.4.3}$$

which, by Lemma 4.1, is equivalent to $g^{-1}\delta\sigma(g) \in \Gamma_W I_p \Gamma_W$. (In the case of an "ordinary" Dieudonné module, a simpler formula holds; see Proposition 7.3).

If the lattice L is also preserved by the involution τ_0 then $g^{-1}gL_0 = L_0$ so that $\alpha = g^{-1}\tilde{g}$ is a 1-cocycle, defining a class in $H^1(\langle\tau_0\rangle, G(W(k)))$, which is trivial by [Goresky and Tai 2019, Proposition B.4] since $p \neq 2$. Thus, there exists $h \in G(W(k))$ so that $g^{-1}\tilde{g} = h^{-1}\tilde{h}$, hence $g' = gh^{-1} \in H(K(k)) = \mathrm{GL}_n^*(K(k))$ and $L = g'L_0$. Thus we may assume that $g \in H(K(k))$, while elements of $H(W(k))$ act trivially on the homothety class of the lattice L_0. If we normalize Haar measure so that $H(W(k))$ has volume one then the number of such lattices is given by the integral in (4.4.2). □

5. Deligne modules and ordinary abelian varieties

5.1. Recall from [Deligne 1969] that a *Deligne module* of rank $2n$ over the field $k = \mathbb{F}_q$ of q elements is a pair (T, F) where T is a free $\mathbb{Z}$-module of dimension $2n$ and $F : T \to T$ is an endomorphism such that the following conditions are satisfied:

(1) The mapping F is semisimple and all of its eigenvalues in $\mathbb{C}$ have magnitude $\sqrt{q}$.

(2) Exactly half of the eigenvalues of F in $\overline{\mathbb{Q}}_p$ are p-adic units and half of the eigenvalues are divisible by q.

(3) The middle coefficient of the characteristic polynomial of F is coprime to p.

(4) There exists an endomorphism $V : T \to T$ such that $FV = VF = q$.

A morphism $(T_A, F_A) \to (T_B, F_B)$ of Deligne modules is a group homomorphism $\phi : T_A \to T_B$ such that $F_B\phi = \phi F_A$. A polarization ([Howe 1995]) of a Deligne module (T, F) is a symplectic form $\omega : T \times T \to \mathbb{Z}$ (alternating and nondegenerate over $\mathbb{Q}$) such that $\omega(Fx, y) = \omega(x, Vy)$ for all $x, y \in T$, and such that the form

$R(x, y) = \omega(x, \iota y)$ is symmetric and positive definite, where ι is some (and hence, any) totally positive imaginary element of $\mathbb{Q}[F]$ (see [Howe 1995, §4.7]).

5.2. Following [Deligne 1969], for the rest of this paper we fix an embedding

$$\varepsilon : W(\bar{k}) \to \mathbb{C}. \tag{5.2.1}$$

By a theorem of Serre and Tate, [Drinfeld 1976; Katz 1981; Messing 1972; Nori and Srinivas 1987] an ordinary abelian variety A/k has a canonical lift $\bar{A}$ over $W(k)$ which, using (5.2.1) gives rise to a complex variety $A_{\mathbb{C}}$ over $\mathbb{C}$ (which depends only on the restriction $\varepsilon \mid W_0(\bar{k})$ (see [Deligne 1969, p. 239]), which in turn, is determined by $\varepsilon \mid W(k)$). Let $\pi \in \mathrm{Gal}(\bar{k}/k)$ denote the Frobenius. The corresponding morphism $\pi_{A/k}$ (which on the structure sheaf of A is given by the k-linear ring endomorphism $f \mapsto f^q$) lifts to an automorphism F_A on $T = T_A = H_1(A_{\mathbb{C}}, \mathbb{Z})$, and the pair (T_A, F_A) is a Deligne module.

Theorem 5.3 [Deligne 1969; Howe 1995]. *The association $A \to (T_A, F_A)$, determined by the embedding* (5.2.1), *induces an equivalence between the category of n-dimensional ordinary abelian varieties* (*resp. polarized abelian varieties*) *over $k = \mathbb{F}_q$ and the category of Deligne modules* (*resp. polarized Deligne modules*) *over k, of rank $2n$.* □

5.4. In [Goresky and Tai 2019], we define a real structure on a polarized Deligne module (T, F, ω) to be a group homomorphism $\tau : T \to T$ such that

$$\tau^2 = I, \quad \tau F \tau^{-1} = V, \quad \omega(\tau x, \tau y) = -\omega(x, y).$$

The involution τ is a characteristic p analog of complex conjugation. There are finitely many ("real") isomorphism classes of principally polarized Deligne modules (of dimension $2n$ over $k = \mathbb{F}_q$) with real structure and principal level N structure, and a formula for this number is given in [Goresky and Tai 2019]. There, we follow the method of Kottwitz [1990] and show that the number of isomorphism classes of principally polarized Deligne modules with real structure is finite and is given by an adèlic orbital integral.

5.5. In order to conceptualize the contribution at p to this formula it is convenient to define a *Deligne module at p* (over $\mathbb{F}_q$, of rank $2n$) to be a pair (T_p, F_p) where T_p is a free $\mathbb{Z}_p$ module of rank $2n$ and $F_p : T_p \to T_p$ is a semisimple endomorphism whose characteristic polynomial $\sum_{i=0}^{2n} a_i x^i$ is q-palindromic,[1] with middle coefficient a_n a p-adic unit, half of whose roots in $\overline{\mathbb{Q}}_p$ are p-adic units and half of which are divisible by p, such that there exists $V_p : T_p \to T_p$ with $F_p V_p = V_p F_p = q$. (This implies that if λ is a root then so is q/λ.) A polarization of (T_p, F_p) is a

[1] Meaning that $a_i = q^{n-i} a_{2n-i}$ for $0 \le i \le n-1$.

$\mathbb{Z}_p$-valued symplectic form ω_p such that $\omega(F_p x, y) = \omega(x, V_p y)$. (The "positivity condition" does not make sense in this setting.) A real structure τ_p on (T_p, F_p, ω_p) is a symplectic involution of T_p with multiplier -1 that exchanges F_p and V_p. If (T, F, ω, τ) is a (real, polarized) Deligne module then tensoring with $\mathbb{Z}_p$ gives a (real, polarized) Deligne module at p.

5.6. ***The Tate module.*** Let (T, F) be a Deligne module over $k = \mathbb{F}_{p^a}$. From this we define a $\mathrm{Gal}(\bar{k}/k)$ module, for $\ell \neq p$ a (rational) prime,

$$T_\ell(T) = T \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$$

with Galois action determined by the rule that $\pi \in \mathrm{Gal}(\bar{k}/k)$ acts by $F \otimes 1$. A polarization and/or a real structure on (T, F) induces one on $T \otimes \mathbb{Z}_\ell$ in an obvious way.

Let $\ell \neq p$ be prime. If A is an ordinary abelian variety with Tate module $T_\ell(A)$ and Deligne module (T_A, F_A) then there is a natural isomorphism of $\mathrm{Gal}(\bar{k}/k)$ modules $T_\ell(A) \cong T_\ell(T_A) = T_A \otimes \mathbb{Z}_\ell$.

6. The Dieudonné module of an ordinary variety

6.1. Let A be an ordinary abelian variety over $k = \mathbb{F}_{p^a}$. Denote by $M(A) := M(A[p^\infty])$ the covariant Dieudonné module associated to the p-divisible group $A[p^\infty]$. In this section we explicitly construct this Dieudonné module $M(A)$ (and quasi-polarization) directly from the Deligne module (T_A, F_A) (and a polarization). In fact, the Dieudonné module $M(A)$ depends only on the associated Deligne module $(T_p = T_A \otimes \mathbb{Z}_p,\ F_p = F_A \otimes \mathbb{Z}_p)$ at p. Although this material is well known to experts, we require specific equations for these modules that do not appear to be in the literature.

Given a universal choice of involution $\bar{\tau}$ of the Witt vectors (as in Appendix A) we show, in Section 6.11, that a real structure on (T_p, F_p) determines a real structure on $M(A)$.

6.2. Let (T_p, F_p) be a Deligne module at p, over $k = \mathbb{F}_{p^a}$. The same argument as in [Deligne 1969] shows that the endomorphism F_p determines a unique decomposition

$$T_p \cong T' \oplus T'' \tag{6.2.1}$$

preserved by F_p and V_p, such that F_p is invertible on T' and is divisible by q on T''. In fact, the module $T' \otimes \overline{\mathbb{Q}}_p$ is the sum of the eigenspaces of F_p whose eigenvalues in $\overline{\mathbb{Q}}_p$ are p-adic units while $T'' \otimes \overline{\mathbb{Q}}_p$ is the sum of the eigenspaces of F_p whose eigenvalues are divisible by p. For $t = (t', t'') \in T_p$ set

$$A_q(t', t'') = (t', qt'') \quad \text{and} \quad A_p(t', t'') = (t', pt''). \tag{6.2.2}$$

Then $F_p A_q^{-1} = A_q^{-1} F_p : T_p \to T_p$ is an isomorphism. Extend F_p and σ to $T_p \otimes W(\bar{k})$ by $F_p(t \otimes w) = F_p(t) \otimes w$ and $\sigma(t \otimes w) = t \otimes \sigma(w)$.

6.3. ***The Dieudonné module of a Deligne module.*** For a Deligne module (T_p, F_p) at p as above, define the covariant Dieudonné module $M(T_p, F_p)$ (which we denote simply by $M(T_p)$) to be the $\mathrm{Gal}(\bar{k}/k)$-invariant submodule of $T_p \otimes W(\bar{k})$ where $\pi \in \mathrm{Gal}(\bar{k}/k)$ acts as

(6.3.1) $$\pi(t \otimes w) = A_q^{-1} F_p(t) \otimes \sigma^a(w),$$

so to be explicit,

(6.3.2) $$M(T_p) = \{x \in T_p \otimes W(\bar{k}) \mid F_p(x) = A_q \sigma^{-a}(x)\}$$

with actions $\mathcal{F}(t \otimes w) = pA_p^{-1}(t) \otimes \sigma(w)$ and $\mathcal{V}(t \otimes w) = A_p(t) \otimes \sigma^{-1}(w)$.

6.4. The mapping A_q preserves the splitting of T_p which gives a splitting $M(T_p) = M(T') \oplus M(T'')$. The operator $\mathcal{F}$ is σ-linear; it is invertible on $M(T'')$ and it is divisible by p on $M(T')$. If $\alpha \in M(T)$ then

$$F_p(\alpha) = \mathcal{V}^a(\alpha),$$

that is, the mapping F_p has been factored as $F_p = \mathcal{V}^a$. (The preceding paragraphs may be dualized so as to define the *contravariant* Dieudonné modules $N(T) = N(T') \oplus N(T'')$ corresponding to the splitting (6.3.2), in which case the mapping $\mathcal{F}_p$ is invertible on $N(T')$, divisible by p on $N(T'')$ and one has $F_p = \mathcal{F}^a$. Despite this confusion we use the covariant Dieudonné module because the equations are a bit simpler.)

Proposition 6.5. *Let (T_p, F_p) be a Deligne module at p with $\mathbb{Z}_p$-rank equal to $2n$. Then its Dieudonné module $M(T_p)$ is a free module over $W(k)$ whose $W(k)$-rank also equals $2n$ and in fact there exists a $W(k)$-basis of $M(T_p)$ whose elements also form a $W(\bar{k})$ basis of $T_p \otimes W(\bar{k})$.*

The proof will appear in Appendix B. The following lemma will be needed in the proof of Proposition 7.3.

Lemma 6.6. *Let (T_p, F_p) be a Deligne module at p. The operator $\sigma(t \otimes w) = t \otimes \sigma(w)$ on $T_p \otimes W(\bar{k})$ preserves the Dieudonné module $M(T_p) \subset T_p \otimes W(\bar{k})$. Suppose $\Lambda \subset M(T_p) \otimes \mathbb{Q}_p$ is a $W(k)$-lattice. Then the following statements are equivalent.*

(1) *The lattice Λ is preserved by $\mathcal{F}$ and $\mathcal{V}$.*

(2) $p\Lambda \subset \mathcal{F}\Lambda \subset \Lambda$.

(3) $p\Lambda \subset \mathcal{V}\Lambda \subset \Lambda$.

(4) $A_p^{-1}\mathcal{V}\Lambda = \Lambda$.

Such a lattice is also preserved by σ.

Proof. The equivalence of (1), (2), (3) is straightforward. (See also the related (4.4.3) when Λ is symplectic). Such a lattice Λ is also preserved by F_p, V_p so by the argument of [Deligne 1969] it decomposes as $\Lambda = \Lambda' \oplus \Lambda''$ with $\Lambda' = M_{\mathbb{Q}}(T_p)' \cap \Lambda$ and $\Lambda'' = M_{\mathbb{Q}}(T_p)'' \cap \Lambda$. Then $\mathcal{V} \mid \Lambda'$ is invertible: Since $F_p = \mathcal{V}^a$ is invertible on Λ' it follows that $\mathcal{V}$ is surjective on Λ', and it is injective because it is injective on $M_{\mathbb{Q}}(T_p)'$. Similarly $\mathcal{F} \mid \Lambda''$ is invertible which implies (4). Conversely, suppose that $A_p^{-1}\mathcal{V}\Lambda = \Lambda$. Then $\mathcal{V}\Lambda \subset A_p\Lambda \subset \Lambda$ and $\mathcal{F}\Lambda = p\mathcal{V}^{-1}\Lambda = (pA_p^{-1})\Lambda \subset \Lambda$. Finally, the action of $A_p^{-1}\mathcal{V}$ on $M(T_p)$ coincides with that of σ^{-1}, so (4) implies $\sigma\Lambda = \Lambda$. □

6.7. Let A/k be an ordinary abelian variety with Deligne module (T_A, F_A). The associated finite group scheme $A[p^r] = \ker(\cdot p^r)$ decomposes similarly into a sum $A'[p^r] \oplus A''[p^r]$ of an étale-local scheme and a local-étale scheme, with a corresponding decomposition of the associated p-divisible group, $A[p^\infty] = A' \oplus A''$. Over $W(\bar{k})$ the finite étale group scheme $A'[p^r]$ becomes constant so there is a canonical isomorphism

$$A'[p^r] \cong p^{-r}T'_A/T'_A. \tag{6.7.1}$$

Proposition 6.8. *The isomorphism $A'[p^r] \cong p^{-r}T'_A/T'_A$ induces an isomorphism of covariant Dieudonné modules*

$$M(A) \cong M(T_A \otimes \mathbb{Z}_p).$$

6.9. ***Proof of Proposition 6.8.*** The module $M(T_A \otimes \mathbb{Z}_p)$ was defined in Section 6.3, so we need to determine the Dieudonné module $M(A)$ of the abelian variety A. First let us show that

$$M(A') \cong (T'_A \otimes W(\bar{k}))^{\mathrm{Gal}}, \tag{6.9.1}$$

where the action of $\pi = \sigma^a \in \mathrm{Gal}$, of $\mathcal{F}$ and $\mathcal{V}$ is given by

$$\begin{aligned} \pi.(t' \otimes w) &= F_A(t') \otimes \sigma^a(w), \\ \mathcal{F}(t' \otimes w) &= pt' \otimes \sigma(w), \\ \mathcal{V}(t' \otimes w) &= t' \otimes \sigma^{-1}(w). \end{aligned} \tag{6.9.2}$$

From (6.7.1), over $W(\bar{k})$, the covariant Dieudonné module of the finite group scheme $A'[p^r]$ is:

$$\overline{M}(A'[p^r]) = (p^{-r}T'_A/T'_A) \otimes_{\mathbb{Z}} W(\bar{k}) \cong (T'_A/p^rT'_A) \otimes_{\mathbb{Z}} W(\bar{k}) \tag{6.9.3}$$

with $\mathcal{F}(t' \otimes w) = pt' \otimes \sigma(w)$; see [Demazure 1972, p. 68]. Then (see [Demazure 1972, p. 71] or [Chai et al. 2014, §B.3.5.9, p. 350]),

$$\overline{M}(A') = \lim_{\leftarrow} \overline{M}(A'[p^r]). \tag{6.9.4}$$

Therefore

$$\begin{aligned} M(A') &= (\varprojlim (T'_A/p^r T'_A) \otimes W(\bar{k}))^{\mathrm{Gal}} \\ &\cong \left(\varprojlim \left(T'_A \otimes W(\bar{k})/p^r(T'_A \otimes W(\bar{k}))\right)\right)^{\mathrm{Gal}} \\ &\cong (T'_A \otimes W(\bar{k}))^{\mathrm{Gal}} \end{aligned}$$

with (étale) Galois action

$$\pi(t' \otimes w) = \pi(t') \otimes \pi(w) = F_A(t') \otimes \sigma^a(w). \tag{6.9.5}$$

Next, using double duality, we will show that $M(A'') \cong (T''_A \otimes W(\bar{k}))^{\mathrm{Gal}}$ where

$$\begin{aligned} \pi(t'' \otimes w) &= q^{-1} F_A(t'') \otimes \sigma^a(w), \\ \mathcal{F}(t'' \otimes w) &= t'' \otimes \sigma(w), \\ \mathcal{V}(t'' \otimes w) &= pt'' \otimes \sigma^{-1}(w). \end{aligned} \tag{6.9.6}$$

Let B denote the ordinary abelian variety that is dual to A with Deligne module (T_B, F_B) and corresponding p-divisible groups B', B''. Then B' is dual to A'' (and vice versa), hence it follows from (6.9.1) (see also [Chai et al. 2014, §B.3.5.9], [Demazure 1972, p. 72] and [Howe 1995, Proposition 4.5]) that:

$$\begin{aligned} \overline{M}(B') &= T'_B \otimes_{\mathbb{Z}_p} W(\bar{k}),^{2} \\ \overline{M}(A'') &= \mathrm{Hom}_{W(\bar{k})}(\overline{M}(B'), W(\bar{k})),^{3} \\ T'_B &= \mathrm{Hom}_{\mathbb{Z}_p}(T''_A, \mathbb{Z}_p).^{4} \end{aligned}$$

From this, we calculate that the isomorphism

$$\Psi : T''_A \otimes W(\bar{k}) \to \mathrm{Hom}_{W(\bar{k})}(\mathrm{Hom}_{\mathbb{Z}_p}(T''_A, \mathbb{Z}_p) \otimes W(\bar{k}), W(\bar{k})) = \overline{M}(A'')$$

defined by

$$\Psi_{t'' \otimes w}(\phi \otimes u) = \phi(t'').wu$$

(for $t'' \in T''_A$, for $\phi \in \mathrm{Hom}(T''_A, \mathbb{Z}_p)$ and for $w, u \in W(\bar{k})$) satisfies:

$$\begin{aligned} (\pi.\Psi_{t'' \otimes w})(\phi \otimes u) &= \sigma^a \Psi_{t'' \otimes w}(\pi_B^{-1}(\phi \otimes u)) \\ &= \sigma^a \Psi_{t'' \otimes w}(F_B^{-1}\phi \otimes \sigma^{-a}u) \\ &= \sigma^a((F_B^{-1}\phi)(t'').w.\sigma^{-a}u) \\ &= \phi(V_A^{-1}(t'')).\sigma^a(w).u = (\Psi_{V_A^{-1}t'' \otimes \sigma^a(w)})(\phi \otimes u). \end{aligned}$$

[2] $\pi(t' \otimes w) = F_B(t') \otimes \sigma^a(w), \quad \mathcal{F}(t' \otimes w) = pt' \otimes \sigma(w).$

[3] $\pi_A \psi(m) = \sigma^a \psi(\pi_B^{-1}(m)), \quad \mathcal{F}\psi(m) = \sigma\psi(\mathcal{V}(m)).$

[4] $F_B \phi(t') = \phi V_A(t').$

Therefore $\pi(t''\otimes w) = V_A^{-1}(t'')\otimes\sigma^a(w) = q^{-1}F_A(t'')\otimes\sigma^a(w)$. Similarly,

$$(\mathcal{F}.\Psi_{t''\otimes w})(\phi\otimes u) = \Psi_{t''\otimes\sigma(w)}(\phi\otimes u),$$

hence $\mathcal{F}(t''\otimes w) = t''\otimes\sigma(w)$, which proves (6.9.6). Since $M(A'') = (\overline{M}(A''))^{\mathrm{Gal}}$, this together with (6.9.1) verifies that $M(A)$ satisfies the condition in (6.3.2) (with T_p replaced by $T_A\otimes\mathbb{Z}_p$). □

Proposition 6.10. *Let (T_p, F_p) be a Deligne module at p. Let $\omega : T_p\times T_p\to\mathbb{Z}_p$ be a symplectic form such that $\omega(Fx, y) = \omega(x, Vy)$ for all $x, y\in T_p$. Extending scalars to $W(\bar{k})$ then restricting to the Dieudonné module $M(T_p)\subset T_p\otimes W(\bar{k})$ gives a quasi-polarization*

$$\omega_p : M(T_p)\times M(T_p)\to W(k)$$

of $M(T_p)$. If the original form ω is nondegenerate up to homothety then the same is true of the form ω_p, with the same homothety constant.

Proof. The proof is a direct computation using the decomposition $T_p\cong T'\oplus T''$. □

6.11. *Real structures.* Let (T_p, F_p) be a Deligne module at p, with a polarization $\omega : T_p\times T_p\to\mathbb{Z}_p$. Let ω_p denote the resulting quasi-polarization on the covariant Dieudonné module $M(T_p)$. Let $\tau : T_p\to T_p$ be a real structure on (T_p, F_p) that is compatible with the polarization ω. Unfortunately, the mapping τ does not induce an involution on the Dieudonné module $M(T_p)$ without making a further choice.

Following Appendix A, choose and fix, once and for all, a continuous $K(k)$-linear involution $\bar{\tau} : K(\bar{k})\to K(\bar{k})$ that preserves $W(\bar{k})$, so that $\bar{\tau}\sigma^a(w) = \sigma^{-a}\bar{\tau}(w)$. Then the following construction provides a functor from the category of polarized Deligne modules with real structure to the category of quasi-polarized Dieudonné modules with real structure.

Proposition 6.12. *With (T_p, F_p, ω, τ) as above, the mapping*

$$\tau_p : T_p\otimes W(\bar{k})\to T_p\otimes W(\bar{k})$$

defined by $\tau_p(x\otimes w) = \tau(x)\otimes\bar{\tau}(w)$ is continuous and $W(k)$-linear. It preserves the Dieudonné module $M(T_p)$ and it satisfies $\tau_p\mathcal{F}^a = \mathcal{V}^a\tau_p$ and

(6.12.1) $$\omega_p(\tau_p x, \tau_p y) = -\omega_p(x, y)\quad \textit{for all } x, y\in M(T_p).$$

Proof. The mapping τ takes T' to T'' (and vice versa) because it exchanges the eigenvalues of F and V. If $x'\otimes w\in T'\otimes W(\bar{k})$ then

$$\begin{aligned}\tau_p\pi.(x'\otimes w) &= \tau_p(F(x')\otimes\sigma^a(w)) = V\tau(x')\otimes\sigma^{-a}\bar{\tau}(w)\\ &= \pi^{-1}(\tau(x')\otimes\bar{\tau}(w)) = \pi^{-1}\tau_p(x'\otimes w)\end{aligned}$$

which shows that τ_p takes $M(T')$ to $M(T'')$ (and vice versa). Similarly,

$$\tau_p\mathcal{F}^a(x'\otimes w) = \tau_p(x'\otimes q\sigma^a(w)) = \tau(x')\otimes q\sigma^{-a}\bar{\tau}(w)$$
$$= \mathcal{V}^a(\tau(x')\otimes\bar{\tau}(w)) = \mathcal{V}^a\tau_p(x'\otimes w).$$

Similar calculations apply to any element $x''\otimes w \in T''\otimes W(\bar{k})$.

We now wish to verify (6.12.1). Let $Y = T_p\otimes\mathbb{Q}$. It is possible to decompose $Y = Y_1\oplus\cdots\oplus Y_r$ into an orthogonal direct sum of simple $\mathbb{Q}_p[F]$ modules that are preserved by τ (see, for example, [Goresky and Tai 2019, Lemma 4.3]). This induces a similar ω_p-orthogonal decomposition of

$$M(Y) = M(T_p)\otimes_{W(k)} K(k)$$

into submodules $M_i = M(Y_i)$ over the rational Dieudonné ring

$$\mathcal{A}_{\mathbb{Q}} = \mathcal{A}\otimes K(k) = K(k)[F, V]/(\text{relations}),$$

each of which is preserved by τ_p. Since this is an orthogonal direct sum, it suffices to consider a single factor, that is, we may assume that (V_p, F_p) is a simple $\mathbb{Q}_p[F]$-module.

As in (6.2.1) the $\mathbb{Q}_p$ vector space Y decomposes, $Y = Y'\oplus Y''$ where the eigenvalues of $F\,|\,Y'$ are p-adic units and the eigenvalues of $F\,|\,Y''$ are divisible by p. Then the same holds for the eigenvalues of $\mathcal{F}^a$ on each of the factors of

$$M(Y) = M(Y')\oplus M(Y'').$$

Moreover, these factors are cyclic $\mathcal{F}^a$-modules and τ_p switches the two factors. It is possible to find a nonzero vector $y'\in M(Y')$ so that y' is $\mathcal{F}^a$-cyclic in $M(Y')$ and so that $y'' = \tau_p(y)$ is $\mathcal{F}^a$-cyclic in $M(Y'')$. It follows that $y = y'\oplus y''$ is a cyclic vector for $M(Y)$ which is fixed under τ_p, that is, $\tau_p(y) = y$. We obtain a basis of $M(Y)$:

$$y,\ \mathcal{F}^a y, \cdots,\ \mathcal{F}^{a(2n-1)}y.$$

The symplectic form ω_p is determined by its values $\omega_p(y, \mathcal{F}^{aj}y)$ for $1\le j\le 2n-1$. But

$$\omega_p(\tau_p y, \tau_p\mathcal{F}^{aj}y) = \omega_p(y, \tau_p\mathcal{F}^{aj}\tau_p y) = q^j\omega_p(y, \mathcal{F}^{-aj}y)$$
$$= q^j q^{-j}\omega_p(\mathcal{F}^{aj}y, y) = -\omega_p(y, \mathcal{F}^{aj}y). \qquad \square$$

7. Comparing lattices in the ordinary case

7.1. A twisted orbital integral (4.4.2) "counts" (real, symplectic) lattices in a Dieudonné module while an untwisted orbital integral counts (real, symplectic) lattices in a Deligne module. In this section we show that such lattices are in natural one-to-one correspondence. Let (T_p, F_p, ω, τ) be a polarized Deligne module (at p) with a real structure. By [Goresky and Tai 2019, Proposition B.4] there exists an isomorphism $\Phi : T_p \otimes \mathbb{Q}_p \to \mathbb{Q}_p^{2n}$ which takes ω to the standard involution ω_0 and takes τ to the standard involution τ_0. It takes F_p to some element $\gamma \in G(\mathbb{Q}_p)$ and it takes the decomposition (6.2.1) to a decomposition $\mathbb{Q}_p^{2n} = V' \oplus V''$ where γ is invertible on V' and is divisible by q on V''. It also takes the operator A_q of (6.2.2) to an element $\alpha_q \in G(\mathbb{Q})$ in the centralizer $Z_\gamma(\mathbb{Q})$ such that $\alpha_q|V' = I$ and $\alpha_q|V'' = qI$.

The mapping $\overline{\Phi} = \Phi \otimes K(\bar{k})$ is compatible with the action (see Lemma 6.6) of σ, that is, $\overline{\Phi}(t \otimes \sigma(w)) = \sigma\overline{\Phi}(t \otimes w)$, and it takes the rational Dieudonné module $M_\mathbb{Q}(T_p) = M(T_p) \otimes \mathbb{Q}_p$ to the $K(k)$-vector space (see Section 6.3)

$$\mathcal{J}_\mathbb{Q}(\gamma) = \{x \in K(\bar{k})^{2n} \mid \gamma x = \alpha_q \sigma^{-a}(x)\}.$$

In Corollary B.5 we construct a symplectic basis Ψ of $\mathcal{J}_\mathbb{Q}(\gamma)$ giving the diagram

$$\begin{array}{ccccc} T_p \otimes_\mathbb{Z} K(\bar{k}) & \xrightarrow{\overline{\Phi}} & K(\bar{k})^{2n} & \xleftarrow{\Psi \otimes K(\bar{k})} & K(\bar{k})^{2n} \\ \uparrow & & \uparrow & & \uparrow \\ M_\mathbb{Q}(T_p) & \xrightarrow{\cong} & \mathcal{J}_\mathbb{Q}(\gamma) & \xleftarrow{\Psi} & K(k)^{2n} \end{array} \tag{7.1.1}$$

The involution $\tau_p = \tau \otimes \bar{\tau}$ in the first column becomes $\bar{\tau}_0 = \tau_0 \otimes \bar{\tau}$ in the second and third columns. The mapping $\Psi \otimes K(\bar{k}) \in G(K(\bar{k}))$ satisfies $\widetilde{\Psi} = \bar{\tau}_0 \Psi \tau_0^{-1} = \Psi$. As in Sections 3.2 and 4.3, the operator $\mathcal{F}\sigma^{-1}$ (in the first column) on $M_\mathbb{Q}(T_p)$ becomes (in the third column) multiplication by $\delta \in G(K(k))$. Let $u_p = \Psi \alpha_p \Psi^{-1}$. Then $\delta\sigma(w) = \Psi^{-1} p \alpha_p^{-1} \sigma(\Psi w)$ so $\delta = p u_p^{-1} \Psi^{-1} \sigma(\Psi)$ and its norm

$$N(\delta) = \delta\sigma(\delta)\cdots\sigma^{a-1}(\delta) = \Psi^{-1} q \alpha_q^{-1} \sigma^a(\Psi) = \Psi^{-1} q \gamma^{-1} \Psi$$

is $G(K(\bar{k}))$-conjugate to $q\gamma^{-1}$. Similarly, the action of $\mathcal{V}\sigma$ becomes (in the third column) multiplication by $\eta = \Psi^{-1} \alpha_p \sigma^{-1}(\Psi)$ whose norm is stably conjugate to γ. Notations for these operators are summarized in Table 1.

$T \otimes \mathbb{Z}_p$	$T \otimes W(\bar{k}) \to$	$W(\bar{k})^{2n}$	$\leftarrow W(\bar{k})^{2n}$
	$M_\mathbb{Q}(T)$	$\mathcal{J}_\mathbb{Q}(\gamma)$	$K(k)^{2n}$
F_p	F_p	γ	$\Psi^{-1}\gamma\Psi$
A_p	A_p	α_p	u_p
	$\mathcal{F}$	$p\alpha_p^{-1}\sigma$	$\delta\sigma$
	$\mathcal{V}$	$\alpha_p\sigma^{-1}$	$p\sigma^{-1}\delta^{-1}$
ω	ω_p	ω_0	ω_0
τ	$\tau_p = \tau \otimes \bar{\tau}$	$\bar{\tau}_0 = \tau_0 \otimes \bar{\tau}$	$\bar{\tau}_0$

Table 1. Notations for corresponding operators.

7.2. For each $\mathbb{Z}_p$-lattice $L \subset T_p \otimes \mathbb{Q}_p$ that is preserved by F_p and V_p we obtain a $W(k)$-lattice

$$\Lambda = (L \otimes W(\bar{k}))^{\mathrm{Gal}(\bar{k}/k)} \subset M_{\mathbb{Q}}(T_p)$$

where the Galois action is given by $\pi.(t \otimes w) = FA_q^{-1}(t) \otimes \sigma^a(w)$ for $t \in L$ and $w \in W(\bar{k})$ and where $\mathcal{F}$ is given by $\mathcal{F}(t \otimes w) = pA_p^{-1}(t) \otimes \sigma(w)$ from (6.9.2) and (6.9.6).

Proposition 7.3. *Suppose $p \neq 2$. This association $L \mapsto \Lambda$ induces a one-to-one correspondence between*

(A) *the set of $\mathbb{Z}_p$-lattices $L \subset T_p \otimes \mathbb{Q}_p$, symplectic up to homothety, that are preserved by F_p, V_p and τ, and*

(B) *the set of $W(k)$-lattices $\Lambda \subset M_{\mathbb{Q}}(T)$, symplectic up to homothety, that are preserved by $\mathcal{F}$, $\mathcal{V}$, τ_p.*

The choice of basis Φ determines a one-to-one correspondence between (A) *and*

(C) *the set $\{z \in H(\mathbb{Q}_p)/H(\mathbb{Z}_p) \mid z^{-1}\alpha_q^{-1}\gamma z \in G(\mathbb{Z}_p)\}$*

with H as in (2.0.2). The basis Ψ determines a one to one correspondence between (B) *and*

(D) *the set $\{w \in H(K(k))/H(W(k)) \mid w^{-1}p^{-1}u_p\delta\sigma(w) \in \Gamma_W\}$.*

Conjugation by $\Psi \in \mathrm{Sp}_{2n}(K(\bar{k}))$ takes the centralizer $Z_\gamma(\mathbb{Q}_p) \subset H(\mathbb{Q}_p)$ isomorphically to the twisted centralizer

$$S_\delta(K(k)) = \{w \in H(K(k)) \mid z^{-1}\delta\sigma(z) = \delta\} \subset H(K(k)).$$

The correspondence (C) $\leftrightarrow$ (D) *is equivariant with respect to the action of these centralizers.*

Proof. Using the symplectic isomorphism Φ (and $\bar{\Phi}$) the set (A) may be identified with

(A′) the set of $\mathbb{Z}_p$-lattices $L \subset \mathbb{Q}_p^{2n}$, symplectic up to homothety (with respect to the standard symplectic form ω_0), preserved by the standard involution τ_0 and the mappings $\gamma, q\gamma^{-1}$.

Step 1. Let us show that (A′) $\leftrightarrow$ (C). As in [Deligne 1969], the special properties (Section 5.5) of γ determine a decomposition $\mathbb{Q}_p^{2n} = V' \oplus V''$ where γ is invertible on V' and is divisible by q on V''. Then $\alpha_q \mid V' = I$ and $\alpha_q \mid V'' = qI$. The same holds for any lattice $L \subset \mathbb{Q}_p^{2n} = L' \oplus L''$ that is preserved by γ and by $q\gamma^{-1}$. Such a lattice L is also preserved by $q\gamma^{-1}$ if and only if $\alpha_q^{-1}\gamma : L \to L$ is an isomorphism.

Write $L = gL_0$ for some $g \in G(\mathbb{Q}_p)$. If L is also preserved by the involution τ then $g^{-1}\tilde{g}L_0 = L_0$ (where $\tilde{g} = \tau_0 g \tau_0^{-1}$) so $g^{-1}\tilde{g}$ is a 1-cocycle for $H^1(\langle\tau_0\rangle, \mathrm{Sp}_{2n}(\mathbb{Z}_p))$, which is trivial (by [Goresky and Tai 2019, Proposition B.4], and using the fact that $p \neq 2$). So there exists $h \in \mathrm{Sp}_{2n}(\mathbb{Z}_p)$ such that $h^{-1}\tilde{h} = g^{-1}\tilde{g}$, thus $L = zL_0$ where $z = gh^{-1} \in \mathrm{GL}_n^*(\mathbb{Q}_p)$. Therefore we have that $\alpha_q^{-1}\gamma z L_0 = zL_0$ so that $z^{-1}\alpha_q^{-1}\gamma z \in G(\mathbb{Z}_p)$. Replacing z by zt (for any $t \in H(\mathbb{Z}_p)$ gives the same lattice $L = ztL_0$. This proves (C).

The correspondence (B) → (D) is similar (compare Proposition 4.4). By Lemma 6.6, if a lattice $\Lambda \subset M_{\mathbb{Q}}(T)$ is preserved by $\mathcal{F}, \mathcal{V}$ then it splits $\Lambda = \Lambda' \oplus \Lambda''$; both factors are preserved by $\mathcal{F}, \mathcal{V}$; and $p^{-1}A_p\mathcal{F}(\Lambda) = \Lambda$. Translating this into the third column of Table 1, we have a $W(k)$-lattice, $w\Lambda_0 \subset K(k)^{2n}$ (where $\Lambda_0 = W(k)^{2n}$ is the standard lattice) such that $p^{-1}u_p\delta\sigma(w\Lambda_0) = w\Lambda_0$ or $w^{-1}p^{-1}u_p\delta\sigma(w) \in G(W(k))$, which is condition (D).

Step 2. Next, we claim the mapping $L \mapsto \overline{\Lambda} = L \otimes W(\bar{k})$ determines a correspondence between the set (A′) and

(A″) the set of $W(\bar{k})$-lattices $\overline{\Lambda} \subset K(\bar{k})^{2n}$, symplectic up to homothety, that are preserved by $\gamma, q\gamma^{-1}, \tau_0$, and σ.

Given $\overline{\Lambda}$ from (A″) write $\overline{\Lambda} = \beta\overline{\Lambda}_0$ for some $\beta \in G(K(\bar{k}))$, where $\overline{\Lambda}_0 = W(\bar{k})^{2n}$ is the standard lattice. Then $\beta^{-1}\sigma(\beta) \in \mathrm{Sp}_{2n}(W(\bar{k})^{2n})$ is a 1-cocycle for the Galois cohomology $H^1(\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p), \mathrm{Sp}_{2n}(W(\cdot)))$, that is, the cohomology which forms an index set for the collection of all $\mathbb{F}_p$-isomorphism classes of $\mathbb{F}_p$-forms of nondegenerate skew symmetric bilinear forms on $W(\bar{k})^{2n}$, of which there is only one, by [Milnor and Husemoller 1973, §3.5]. So it is trivial, which implies that $\overline{\Lambda} = z\overline{\Lambda}_0$ for some $z \in G(\mathbb{Q}_p)$. (That is, $\beta^{-1}\sigma(\beta) = s^{-1}\sigma(s)$ for some $s \in G(W(\bar{k}))$; take $z = \beta s^{-1}$.)

The element $z^{-1}\alpha_q^{-1}\gamma z$ is in $G(W(\bar{k}))$ and it is fixed under σ so it lies in $G(\mathbb{Z}_p)$. This implies $\alpha_q^{-1}\gamma z L_0 = zL_0$, hence L is preserved by γ and by $q\gamma^{-1}$. Moreover, $\overline{\Lambda}^{\perp} = c\overline{\Lambda}$ where $c^{-1} \in \mathbb{Q}_p^{\times}$ is the multiplier of z, so the lattice $\overline{\Lambda}$ comes from the lattice $L = z\mathbb{Z}_p^{2n}$ and the homothety constant may be taken to lie in $\mathbb{Q}_p^{\times}$. Finally, since $\tau_0(\overline{\Lambda}) = \overline{\Lambda}$, the same argument as in Step 1 implies that z may be chosen to lie in $H(\mathbb{Q}_p)$, hence the lattice L is also preserved by τ_0.

Step 3. According to Section 6.3, the mapping $\overline{\Phi} : T_p \otimes K(\bar{k}) \to K(\bar{k})^{2n}$ takes the Dieudonné module $M(T_p) \otimes \mathbb{Q}_p$ to the module

$$\mathcal{J}_{\mathbb{Q}}(\gamma) = \{x \in K(\bar{k})^{2n} \mid \gamma x = \alpha_q\sigma^{-a}(x)\}$$

on which the mappings $\mathcal{F}, \mathcal{V}$ become the following (for which we use the same symbols): $\mathcal{F}(x) = p\alpha_p^{-1}\sigma(x)$ and $\mathcal{V}(x) = \alpha_p\sigma^{-1}(x)$. Consider

(B′) the set of $W(k)$-lattices $\Lambda \subset \mathcal{J}_{\mathbb{Q}}(\gamma)$, symplectic up to homothety, that are preserved by $\mathcal{F}, \mathcal{V}, \tau_0$.

We claim functors (A$''$) $\leftrightarrow$ (B$'$) defined by

$$\overline{\Lambda} \mapsto \Lambda = \overline{\Lambda} \cap \mathcal{J}_{\mathbb{Q}}(\gamma)$$
$$\overline{\Lambda} = \Lambda \otimes W(\bar{k}) \leftarrow \Lambda$$

define a one-to-one correspondence between lattices $\overline{\Lambda}$ of (A$''$) and lattices Λ of (B$'$).

Given $\overline{\Lambda}$ from the set (A$''$), the set $\Lambda = \overline{\Lambda} \cap \mathcal{J}_{\mathbb{Q}}(\gamma)$ is clearly preserved by $\mathcal{F}$, $\mathcal{V}$, τ_0, but we need to prove that it is a lattice. In fact, it is a free $W(k)$-module of maximal rank, which follows from the same proof (Appendix B) as that of Proposition 6.5.

On the other hand, given a lattice Λ from the set B$'$ we obtain a lattice

$$\overline{\Lambda} = \Lambda \otimes W(\bar{k}) \subset K(\bar{k})^{2n}.$$

It is clearly preserved by F, V, τ_0. It follows from Lemma 6.6 that it is also preserved by σ, so it is in the set A$''$. We claim that $\overline{\Lambda} \cap (\mathcal{J}_{\mathbb{Q}}(\gamma))) = \Lambda$. Choose a $W(k)$-basis $b_1, b_2, \cdots, b_{2n} \in T_p \otimes K(\bar{k})$ of Λ. If $v = \sum_i s_i b_i \in \overline{\Lambda} \cap (\mathcal{J}_{\mathbb{Q}}(\gamma))$ with $s_i \in W(\bar{k})$ then

$$v = \sum_i s_i b_i = \gamma^{-1}\sigma^{-a}\alpha_q \sum_i s_i b_i = \sum_i \sigma^{-a}(s_i)\gamma^{-1}\alpha_q\sigma^{-a}(b_i) = \sum_i \sigma^{-a}(s_i) b_i$$

which implies that $s_i \in W(k)$. Therefore $v \in \Lambda$.

In fact every lattice in the set (A$''$) arises in this way: given $\overline{\Lambda}$ let $\Lambda = \overline{\Lambda} \cap \mathcal{J}_{\mathbb{Q}}(\gamma)$. Then Proposition 6.5 implies that Λ admits a $W(k)$ basis whose elements form a $W(\bar{k})$ basis of $\overline{\Lambda}$. So the inclusion $\Lambda \to \overline{\Lambda}$ induces an isomorphism $\Lambda \otimes W(\bar{k}) \cong \overline{\Lambda}$. This completes the verification of (A$''$) $\leftrightarrow$ (B$'$).

Step 4. The correspondence between (B) and (B$'$) is straightforward.

Step 5. Suppose $z \in Z_\gamma(\mathbb{Q}_p)$. Then z preserves the eigenspace decomposition $\mathbb{Q}_p^{2n} = V' \oplus V''$ so it commutes with α_p. Then $w = \Psi^{-1} z \Psi \in S_\delta$ because

$$w\delta\sigma(w)^{-1} = \Psi^{-1} p\alpha_p^{-1}\sigma(\Psi) = \delta.$$

Conversely if $w \in S_\delta(K(\bar{k}))$, applying the norm gives $wN(\delta)w^{-1} = N(\delta)$ so $z = \Psi w \Psi^{-1} \in Z_\gamma(K(\bar{k}))$. Moreover z commutes with α_p (as above). Substituting $\delta = \Psi^{-1} p\alpha_p^{-1}\sigma(\Psi)$ into the equation $w\delta\sigma(w)^{-1} = w$ gives $z\sigma(z)^{-1} = 1$, so $z \in Z_\gamma(\mathbb{Q}_p)$.

The equivariance statement in Proposition 7.3 is easily verified. □

7.4. As in Lemma 4.1, the theory of Smith normal form (or rational canonical form) gives a one-to-one correspondence between the set (A$'$) and

(C$'$) the set $\{g \in H(\mathbb{Q}_p)/H(\mathbb{Z}_p) \mid g^{-1}\gamma g \in \Gamma_p I_q \Gamma_p\}$,

where $\Gamma_p = G(\mathbb{Z}_p)$, and as in (4.4.1), an identification between (B$'$) and

(D$'$) the set $\{g \in H(K(k))/H(W(k)) \mid g^{-1}\delta\sigma(g) \in \Gamma_W A_p \Gamma_W\}$.

7.5. Using the same procedure (due to [Kottwitz 1990]) as in Sections 4.2 and 4.3, we may identify the set of isomorphism classes of principally polarized Deligne modules at p with real structure that are $\mathbb{Q}_p$-isogenous to (T_p, F_p, ω, τ) with the quotient

$$Y(T_p) = I(T_p)\backslash\mathcal{Y}(T_p),$$

where $\mathcal{Y}(T_p)$ denotes the set of $\mathbb{Z}_p$-lattices $L \subset T_p \otimes \mathbb{Q}_p$ that are symplectic up to homothety (that is, $L^\vee = cL$ for some $c \in \mathbb{Q}_p^\times$) and preserved by F_p, V_p, and τ (that is, the set (A) of Proposition 7.3), and where $I(T_p)$ denotes the group of self isogenies of (T_p, F_p, ω, τ).

So the correspondence (A) $\leftrightarrow$ (C) $\leftrightarrow$ (C)$'$ $\leftrightarrow$ (B) $\leftrightarrow$ (B)$'$ of Proposition 7.3 and 7.4 means that the number of such isomorphism classes

$$|Y(T_p)| = |Z_\gamma(\mathbb{Q}_p)\backslash\mathcal{Y}(T_p)|$$

is given by any of the integrals

$$\text{(C)} \qquad \int_{Z_\gamma(\mathbb{Q}_p)\backslash H(\mathbb{Q}_p)} \chi(z^{-1}\alpha_q^{-1}\gamma z)\,dz$$

$$\text{(C}'\text{)} \qquad = \int_{Z_\gamma(\mathbb{Q}_p)\backslash H(\mathbb{Q}_p)} \kappa(g^{-1}\gamma g)\,dg$$

$$\text{(B)} \qquad = \int_{S_\delta(K(k))\backslash H(K(k))} \chi_W(w^{-1}p^{-1}u_p\delta\sigma(w))\,dw$$

$$\text{(B}'\text{)} \qquad = \int_{S_\delta(K(k))\backslash H(K(k))} \kappa_W(g^{-1}\delta\sigma(g))\,dg$$

where χ is the characteristic function on $G(\mathbb{Q}_p)$ of $\Gamma_p = G(\mathbb{Z}_p)$, χ_W is the characteristic function of $G(W(k))$, κ is the characteristic function on $G(\mathbb{Q}_p)$ of $\Gamma_p I_q \Gamma_p$, κ_W is the characteristic function on $G(K(k))$ of $\Gamma_W I_p \Gamma_W$ and where $H = \mathrm{GL}_n^* \subset G$ (note that $\gamma, \delta \notin H$).

Appendix A: Involutions on the Witt vectors

A.1. Fix a finite field $\Bbbk$ of characteristic $p > 0$ having $q = p^a = |\Bbbk|$ elements. Fix an algebraic closure $\overline{\Bbbk}$ and let $W(\Bbbk)$, $W(\overline{\Bbbk})$ denote the ring of (infinite) Witt vectors. These are lattices within the corresponding fraction fields, $K(\Bbbk)$ and $K(\overline{\Bbbk})$. Let $W_0(\overline{\Bbbk})$ be the valuation ring in the maximal unramified extension $K_0(\overline{\Bbbk})$ of $\mathbb{Q}_p \subset K(\Bbbk)$. We may canonically identify $W(\overline{\Bbbk})$ with the completion of $W_0(\overline{\Bbbk})$. Denote by $\pi : \overline{\Bbbk} \to \overline{\Bbbk}$ the Frobenius $\pi(x) = x^q$. It has a unique lift, which we also denote by $\pi : W(\overline{\Bbbk}) \to W(\overline{\Bbbk})$, and the cyclic group $\langle\pi\rangle \cong \mathbb{Z}$ is dense in the Galois group $G = \mathrm{Gal}(K_0(\overline{\Bbbk})/K(\Bbbk)) \cong \mathrm{Gal}(\overline{\Bbbk}/\Bbbk)$. If $L \supset \Bbbk$ is a finite extension, for simplicity we write $\mathrm{Gal}(L/\Bbbk)$ in place of $\mathrm{Gal}(K(L)/K(\Bbbk))$ and we write $\mathrm{Trace}_{L/\Bbbk}$ for the trace $W(L) \to W(\Bbbk)$.

Proposition A.2. *There exists a continuous $W(\Bbbk)$-linear mapping $\bar\tau : W(\bar{\Bbbk}) \to W(\bar{\Bbbk})$ such that*:

(1) $\bar\tau^2 = I$.

(2) $\bar\tau\pi = \pi^{-1}\bar\tau$.

(3) *For any finite extension $E/\Bbbk$, the mapping $\bar\tau$ preserves $W(E) \subset W(\bar{\Bbbk})$.*

(4) *For any finite extension $L \supset E \supset \Bbbk$, the following diagrams commute*:

$$\begin{array}{ccc} W(L) & \xrightarrow[\bar\tau]{} & W(L) \\ {\scriptstyle \mathrm{Trace}_{L/E}}\downarrow & & \downarrow{\scriptstyle \mathrm{Trace}_{L/E}} \\ W(E) & \xrightarrow[\bar\tau]{} & W(E) \end{array} \qquad \begin{array}{ccc} W(L) & \xrightarrow[\bar\tau]{} & W(L) \\ \uparrow & & \uparrow \\ W(E) & \xrightarrow[\bar\tau]{} & W(E) \end{array}$$

Such an involution will be referred to as an *antialgebraic involution of the Witt vectors*.

Proof. Let $E \supset \Bbbk$ be a finite extension of degree r. Recall that an element $\theta_E \in W(E)$ is a *normal basis generator* if the collection $\theta_E, \pi\theta_E, \pi^2\theta_E, \cdots, \pi^{r-1}\theta_E$ forms a basis of the lattice $W(E)$ over $W(\Bbbk)$. By simplifying and extending the argument in [Lenstra 1985], P. Lundström [1999] showed that there exists a compatible collection $\{\theta_E\}$ of normal basis generators of $W(E)$ over $W(\Bbbk)$, where E varies over all finite extensions of $\Bbbk$, and where "compatible" means that $\mathrm{Trace}_{L/E}(\theta_L) = \theta_E$ for any finite extension $L \supset E \supset \Bbbk$. Let us fix, once and for all, such a collection of generators. This is equivalent to fixing a "normal basis generator" θ of the free rank one module

$$\varprojlim_E W(E)$$

over the group ring

$$W[\![G]\!] = \varprojlim_E W(\Bbbk)[\mathrm{Gal}(E/\Bbbk)].$$

For each finite extension $E/\Bbbk$, define $\tau_E : W(E) \to W(E)$ by

$$\tau_E\left(\sum_{i=0}^{r-1} a_i\pi^i\theta_E\right) := \sum_{i=0}^{r-1} a_i\pi^{-i}\theta_E = \sum_{i=0}^{r-1} a_i\pi^{r-i}\theta_E,$$

where $a_0, a_1, \cdots, a_{r-1} \in W(\Bbbk)$. Then $\tau_E^2 = I$ and $\tau_E\pi = \pi^{-1}\tau_E$. We refer to τ_E as an antialgebraic involution of $W(E)$. The mapping τ_E is an isometry (hence, continuous) because it takes units to units. To see this, suppose $v \in W(E)$ is a unit and set $\tau_E(v) = p^r u$ where $u \in W(E)$ is a unit. Then

$$v = \tau_E^2(v) = p^r\tau_E(u) \in p^r W(E)$$

is a unit, hence $r = 0$.

Next, we wish to show, for every finite extension $L \supset E \supset \Bbbk$, that $\tau_L | W(E) = \tau_E$ (so that τ_E is well defined) and that $\tau_E \circ \mathrm{Trace}_{L/E} = \mathrm{Trace}_{L/E} \circ \tau_L$. We have an exact sequence

$$1 \to \mathrm{Gal}(L/E) \to \mathrm{Gal}(L/\Bbbk) \to \mathrm{Gal}(E/\Bbbk) \to 1.$$

For each $h \in \mathrm{Gal}(E/\Bbbk)$ choose a lift $\hat{h} \in \mathrm{Gal}(L/\Bbbk)$ so that

$$\mathrm{Gal}(L/\Bbbk) = \{\hat{h}g : \ h \in \mathrm{Gal}(E/\Bbbk), \, g \in \mathrm{Gal}(L/E)\}.$$

Let $x = \sum_{h \in \mathrm{Gal}(E/\Bbbk)} a_h h\theta_E \in W(E)$ where $a_h \in W(\Bbbk)$. Then

$$x = \sum_{h \in \mathrm{Gal}(E/\Bbbk)} a_h h \sum_{g \in \mathrm{Gal}(L/E)} g\theta_L = \sum_{h \in \mathrm{Gal}(E/\Bbbk)} a_h \sum_{g \in \mathrm{Gal}(L/E)} \hat{h}g\theta_L$$

so that

$$\begin{aligned}\tau_L(x) &= \sum_{h \in \mathrm{Gal}(E/\Bbbk)} a_h \sum_{g \in \mathrm{Gal}(L/E)} \hat{h}^{-1} g^{-1} \theta_L \\ &= \sum_{h \in \mathrm{Gal}(E/\Bbbk)} a_h \hat{h}^{-1} \sum_{g \in \mathrm{Gal}(L/E)} g^{-1}\theta_L = \sum_{h \in \mathrm{Gal}(E/\Bbbk)} a_h h^{-1} \theta_E = \tau_E(x).\end{aligned}$$

To verify that $\tau_E \circ \mathrm{Trace}_{L/E}(x) = \mathrm{Trace}_{L/E} \circ \tau_L(x)$, it suffices to consider basis vectors $x = \hat{h}g\theta_L$ where $g \in \mathrm{Gal}(L/E)$ and $h \in \mathrm{Gal}(E/\Bbbk)$. Then $\mathrm{Trace}_{L/E}(x) = h\theta_E$ and

$$\begin{aligned}\mathrm{Trace}_{L/E}(\tau_L(x)) &= \sum_{y \in \mathrm{Gal}(L/E)} y\hat{h}^{-1} g^{-1} \theta_L = \hat{h}^{-1} \sum_{z \in \mathrm{Gal}(L/E)} z\theta_L \\ &= h^{-1}\,\mathrm{Trace}(\theta_L) = \tau_E \,\mathrm{Trace}_{L/E}\,(x).\end{aligned}$$

It follows that the collection of involutions $\{\tau_E\}$ determines an involution

$$\bar{\tau} : W_0(\bar{\Bbbk}) \to W_0(\bar{\Bbbk})$$

of the maximal unramified extension of $W(\Bbbk)$. It is a continuous isometry (so it takes units to units) and it satisfies the conditions (1)–(4). Therefore it extends uniquely and continuously to the completion $W(\bar{\Bbbk})$. □

Appendix B: Applications of Galois cohomology

B.1. Throughout this section let k be a finite field with an algebraic closure $\bar{k}$ with Galois group $\mathrm{Gal} = \mathrm{Gal}(\bar{k}/k)$. Let $W(k)$ be the ring of Witt vectors over k. A bilinear form on a free finite-dimensional $W(k)$ module V is (strongly) nondegenerate if it induces an isomorphism $V \to \mathrm{Hom}_{W(k)}(V, W(k))$. Let ω_0 be the standard symplectic form whose matrix is $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. In this section we recall some standard facts and applications from Galois cohomology.

Proposition B.2. *The Galois cohomology set* $H^1(\mathrm{Gal}(\bar{k}/k), \mathrm{GL}_n(W(\bar{k})))$ *is trivial.*

Proof. The proof follows from [SGA 3_{III} 1970] Exp. XXIV, Prop. 8.1(ii) and [Grothendieck 1968] Thm. 11.7 and Remark 11.8.3 although it takes some work to translate these very general results of Grothendieck into this setting. □

Proposition B.3. *The Galois cohomology set* $H^1(\mathrm{Gal}(\bar{k}/k), \mathrm{Sp}_{2n}(W(\bar{k}))$ *is trivial.*

Proof. The proof also follows from [SGA 3_{III} 1970] and [Grothendieck 1968] but it also follows directly from Proposition B.2 as follows. There is a natural one-to-one correspondence between the set of $W(k)$-isomorphism classes of (strongly) nondegenerate alternating bilinear forms on $W(\bar{k})^{2n}$ and elements of

$$\ker\big(H^1\big(\mathrm{Gal}(\bar{k}/k), \mathrm{Sp}_{2n}(W(\bar{k}))\big) \to H^1\big(\mathrm{Gal}(\bar{k}/k), \mathrm{GL}_{2n}(W(\bar{k}))\big)\big).$$

In fact, if $\{\xi_\theta\}$ is a 1-cocycle (with $\theta \in \mathrm{Gal}$) which lies in this kernel then there exists $g \in \mathrm{GL}_{2n}(W(\bar{k}))$ so that $\xi_\theta = \theta(g)g^{-1}$ (for all $\theta \in \mathrm{Gal}$). It may be used to twist the standard symplectic form ω_0 to give a new symplectic form with matrix $B = {}^t g J g^{-1}$. Then $\theta(B) = B$ so it defines a symplectic form on $W(k)^{2n}$ which is nondegenerate over $K(k)$ and also over $W(\bar{k})$, which implies that it is nondegenerate over $W(k)$, i.e., strongly nondegenerate.

If R is a principal ideal domain, it is well known (see, for example, [Goresky and Tai 2019, Lemma B.2]) that all strongly nondegenerate symplectic forms on R^{2n} are isomorphic over R. It follows that the above kernel contains a single element. By Proposition B.2 above, this implies that $H^1(\mathrm{Sp}_{2n}(W(\bar{k}))$ is trivial. □

Proposition B.4. *Define an action of the group* $\langle\tau_0\rangle \cong \mathbb{Z}/(2)$ *on* $\mathrm{Sp}_{2n}(W(k))$ *where the nontrivial element acts as conjugation by* $\tau_0 = \left(\begin{smallmatrix} -I & 0 \\ 0 & I \end{smallmatrix}\right)$. *If* $\mathrm{char}(k) \neq 2$ *then the nonabelian cohomology set* $H^1(\langle\tau_0\rangle, \mathrm{Sp}_{2n}(W(k))$ *is trivial.*

Proof. This follows from the same method as [Goresky and Tai 2019, Propositions B.4 and D.2]: since $W(k)$ is a principal ideal domain containing $1/2$, every involution of $\mathrm{Sp}_{2n}(W(k))$ with multiplier equal to -1 is conjugate to the standard involution $\tilde{g} = \tau_0 g \tau_0^{-1}$. The above nonabelian cohomology set counts the number of conjugacy classes of such involutions. □

Corollary B.5. *Let* V *be a finite-dimensional free* $W(\bar{k})$ *module together with a semilinear action of* $\mathrm{Gal}(\bar{k}/k)$. *Let* V^{Gal} *be the* $W(k)$*-module of Galois invariant elements.*

(1) *The module* V^{Gal} *is free over* $W(k)$ *and there exists a* $W(k)$*-basis of* V^{Gal} *which is also a* $W(\bar{k})$*-basis of* V.

(2) *If* ω *is a (strongly nondegenerate)* $W(\bar{k})$*-valued symplectic form on* V *such that* $\omega(\theta x, \theta y) = \theta\omega(x, y)$ *for all* $\theta \in \mathrm{Gal}(\bar{k}/k)$ *then* ω *restricts to a strongly nondegenerate* $W(k)$*-valued symplectic form on* V^{Gal} *and there exists a symplectic* $W(k)$*-basis of* V^{Gal} *that is also a symplectic* $W(\bar{k})$*-basis of* V.

(3) *In addition to* (2), *if* $\mathrm{char}(k) \neq 2$, *if* $\tau_p : V \to V$ *is an involution such that* $\tau_p \theta = \theta^{-1} \tau_p$ *for all* $\theta \in \mathrm{Gal}(\bar{k}/k)$ *and* $\omega(\tau_p x, \tau_p y) = -\omega(x, y)$ *then* τ_p *restricts to an involution on* V^{Gal} *and the symplectic basis* $\{e_1, \cdots, e_n, e_1^*, \cdots, e_n^*\}$ *of* V^{Gal} *may be chosen so that* $\tau_p(e_i) = -e_i$ *and* $\tau_p(e_i^*) = e_i^*$.

Proof. For part (1), let $m = \mathrm{rank}(V)$. Since the conclusion holds in the case that $V = W(\bar{k})^m$ it suffices to show that there exists a $\mathrm{Gal}(\bar{k}/k)$-equivariant isomorphism $V \to W(\bar{k})^m$. Choose any $W(\bar{k})$ isomorphism $\phi : V \to W(\bar{k})^m$ where $m = \dim(V)$. Then $\theta \mapsto \theta(\phi)\phi^{-1} \in \mathrm{GL}_m(W(\bar{k}))$ is a 1-cocycle so it equals $\theta(B)B^{-1}$ for some $B \in \mathrm{GL}_m(W(\bar{k}))$ by Proposition B.2. It follows that the isomorphism

$$\phi' = B^{-1}\phi : V \to W(\bar{k})^m$$

is Galois equivariant.

For part (2), let $m = 2n$ in the preceding argument. The conclusions of the argument hold for the standard symplectic form ω_0 on $W(\bar{k})^{2n}$ so it suffices to construct a $\mathrm{Gal}(\bar{k}/k)$-equivariant symplectic isomorphism $V \to W(\bar{k})^{2n}$. The same argument works: choose the original isomorphism $\phi : V \to W(\bar{k})^{2n}$ so as to take the symplectic form ω to the standard symplectic form ω_0. The same argument (using Proposition B.3 this time) gives $B \in \mathrm{Sp}_{2n}(W(\bar{k}))$ so the resulting isomorphism $\phi' = B^{-1}\phi : V \to W(\bar{k})^m$ is equivariant and symplectic.

For part (3), first use (2) to obtain a symplectic isomorphism $\phi : V^{\mathrm{Gal}} \to W(k)^{2n}$. The conclusions of the argument hold for the standard involution τ_0 so it suffices to modify this isomorphism so as to be equivariant with respect to the involutions τ_p and τ_0. The same argument (using Proposition B.4 this time) also works: set $\tilde{\phi} = \tau_0 \phi \tau_p^{-1}$. Then $\tilde{\phi}\phi^{-1} \in \mathrm{Sp}_{2n}(W(k))$ is a 1-cocycle for the action of $\langle \tau_0 \rangle$ and since the cohomology vanishes, the mapping ϕ may be modified so as to become equivariant with respect to the involutions. □

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MARK GORESKY
SCHOOL OF MATHEMATICS
INSTITUTE FOR ADVANCED STUDY
PRINCETON, NJ
UNITED STATES
goresky@ias.edu

YUNG SHENG TAI
DEPARTMENT OF MATHEMATICS
HAVERFORD COLLEGE
HAVERFORD, PA
UNITED STATES
ystai@comcast.net

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