

*Pacific  
Journal of  
Mathematics*

**SPECTRAHEDRAL REPRESENTATIONS  
OF PLANE HYPERBOLIC CURVES**

MARIO KUMMER, SIMONE NALDI AND DANIEL PLAUMANN

Volume 303 No. 1

November 2019



## SPECTRAHEDRAL REPRESENTATIONS OF PLANE HYPERBOLIC CURVES

MARIO KUMMER, SIMONE NALDI AND DANIEL PLAUMANN

**We describe a new method for constructing a spectrahedral representation of the hyperbolicity region of a hyperbolic curve in the real projective plane. As a consequence, we show that if the curve is smooth and defined over the rational numbers, then there is a spectrahedral representation with rational matrices. This generalizes a classical construction for determinantal representations of plane curves due to Dixon and relies on the special properties of real hyperbolic curves that interlace the given curve.**

### Introduction

Determinantal representations of plane curves are a classical topic in algebraic geometry. Given a form  $f$  (i.e., a homogeneous polynomial) of degree  $d$  in three variables with complex coefficients and a general form  $g$  of degree  $d - 1$ , there exists a  $d \times d$  linear matrix  $M = xA + yB + zC$  such that  $f$  is the determinant of  $M$  and  $g$  a principal minor of size  $d - 1$  (see for example [Dolgachev 2012, Chapter 4]). The matrix  $M$  can be chosen to be symmetric if  $g$  is a *contact curve*, which means that all intersection points between the curves defined by  $f$  and  $g$  have even multiplicity. The construction of  $M$  from  $f$  and  $g$  is due to Dixon [1902] (following Hesse's much earlier study of the case  $d = 4$ ). We refer to this construction as the *Dixon process*.

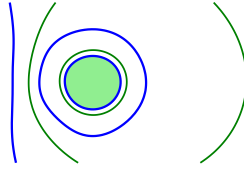
For real curves, the most interesting case for us is that of *hyperbolic curves*. The smooth hyperbolic curves are precisely the curves whose real points contain a set of  $\lfloor d/2 \rfloor$  nested ovals in the real projective plane (plus a pseudoline if  $d$  is odd). A form  $f \in \mathbb{R}[x, y, z]$  is hyperbolic if and only if it possesses a real symmetric determinantal representation  $f = \det(M)$  such that  $M(e) = e_1A + e_2B + e_3C$  is (positive or negative) definite for some point  $e \in \mathbb{P}^2(\mathbb{R})$ . This is the *Helton–Vinnikov theorem*, which confirmed a conjecture by Peter Lax [Helton and Vinnikov 2007].

The Helton–Vinnikov theorem received a lot of attention in the context of semi-definite programming, which was also part of the original motivation: the set of

---

*MSC2010:* 14H50, 14P99, 52A10.

*Keywords:* real algebraic curves, determinantal representations, spectrahedra, linear matrix inequalities.



**Figure 1.** A quintic hyperbolic curve (blue), a quartic interlacer (green), and the hyperbolicity region (green region).

points  $a \in \mathbb{R}^3$  for which the matrix  $M(a)$  is positive semidefinite is a *spectrahedron*

$$\mathcal{S}(M) = \{a \in \mathbb{R}^3 : M(a) \succeq 0\}.$$

It coincides with the *hyperbolicity cone*  $C(f, e)$  of  $f = \det(M)$  in direction  $e$ , that is, the closure of the connected component of  $\{a \in \mathbb{R}^3 : f(a) \neq 0\}$  containing  $e$ . This is a convex cone in  $\mathbb{R}^3$ , whose image in  $\mathbb{P}^2$  is the region enclosed by the convex innermost oval of the curve (see Figure 1). A triple of real symmetric matrices  $A, B, C$  is a *spectrahedral representation* of  $C(f, e)$  if  $M = xA + yB + zC$  satisfies

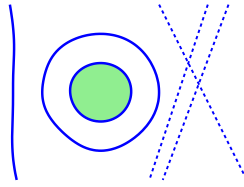
$$C(f, e) = \mathcal{S}(M).$$

It has been pointed out by several authors [Vinnikov 2012; Plaumann and Vinzant 2013] that the proof of the Helton–Vinnikov theorem becomes much simpler if one requires the matrix  $M$  to be only hermitian, rather than real symmetric. In that case,  $M$  can be constructed via the Dixon process starting from any *interlacer* of  $f$ , that is, any hyperbolic form  $g$  of degree  $d - 1$  whose ovals are nested between those of the curve defined by  $f$  (see Figure 1). One downside of this apparent simplification is that the corresponding determinantal representation  $f = \det(M)$  with principal minor  $g$  is harder to construct explicitly, since one has to find the intersection points of  $f$  and  $g$ , while this can be avoided if  $g$  is a contact curve. We refer to [Vinnikov 2012] for a survey of these results.

In this paper, we study a modification of the Dixon process, which can be described as follows: given a form  $f$  of degree  $d$ , hyperbolic with respect to  $e$ , and an interlacer  $g$  of degree  $d - 1$ , we construct a real symmetric matrix pencil  $M$  with the properties that

- the determinant  $\det(M)$  is divisible by  $f$ ,
- the principal minor  $\det(M_{11})$  is divisible by  $g$ ,
- the extra factors  $\det(M)/f$  and  $\det(M_{11})/g$  are products of linear forms, and
- the spectrahedron defined by  $M$  coincides with  $C(f, e)$ .

The extra factor in our spectrahedral representation of  $C(f, e)$  is an arrangement of real lines, as in Figure 2. Informally speaking, these additional lines correct the



**Figure 2.** The extra factor (dashed blue lines) giving the spectral representation of the hyperbolicity region.

failure of  $g$  to be a contact curve by passing through the intersection points of  $g$  with  $f$  that are not of even multiplicity.

The precise statement is Theorem 2.2. The size of  $M$  is at most quadratic in  $d$ . Thus, while  $M$  may not be the smallest or simplest determinantal representation of (some multiple of)  $f$ , it is easier to construct and may better reflect properties of the hyperbolicity region  $C(f, e)$ : as a corollary, we show that if  $f$  has coefficients in  $\mathbb{Q}$ , then  $C(f, e)$  can be represented by a linear matrix inequality with coefficients in  $\mathbb{Q}$  (Theorem 2.10). We may also view Theorem 2.2 in the context of the *generalized Lax conjecture*, which states that every hyperbolicity region (in any dimension) is spectrahedral. While various stronger forms of this conjecture have been disproved, it remains open as stated. One obstacle for constructing symmetric determinantal representations in higher dimensions is the nonexistence of contact interlacers for general hyperbolic hypersurfaces. Since our generalized Dixon process does not require the interlacer to be contact, it is possible that a spectrahedral description of the hyperbolicity cone could be constructed in a similar way, but this is currently purely speculative. In Section 3 we point out how our construction is related to sum-of-squares decompositions of Bézout matrices and the construction in [Kummer 2017].

Even in the original Dixon process for plane curves, details are somewhat subtle: for the construction to succeed as stated, the curve defined by  $f$  must be smooth, and the existence of a contact curve satisfying the required genericity assumption (equivalent to the existence of a nonvanishing even theta characteristic) was not rigorously established until somewhat later. Additionally, the case of singular curves was, to our knowledge, only fully settled and explicitly stated by Beauville [2000]. Likewise, in our generalized Dixon process, we need to treat degenerate cases with care and need some genericity assumptions.

Our generalized Dixon process has the additional feature that the size of the matrix  $M$  decreases if the interlacer  $g$  has real contact points with  $f$ . In particular, if  $g$  is an interlacer with *only real intersection points*, our statement reduces to that of the Helton–Vinnikov theorem. This leads us to the study of interlacers with real intersection (i.e., contact) points. Such interlacers are necessarily on the boundary of the cone  $\text{Int}(f, e)$  of all interlacers of  $f$ . An extreme ray of that cone

will necessarily have a certain number of real contact points (Lemma 1.3). However, we do not know whether there always exists an interlacer with the maximal number  $d(d-1)/2$  of real contact points. Even in the case  $d=4$ , we only obtain a partial answer to this question (see the subsection beginning on page 248). There remain interesting (and easily stated) open questions concerning interlacing curves and the geometry of the interlacer cone.

### 1. Extremal interlacers

Let  $f \in \mathbb{R}[x, y, z]$  be homogeneous of degree  $d$  and hyperbolic with respect to  $e = (0 : 0 : 1)$ , with  $f(e) > 0$ . Let  $C = \mathcal{V}_{\mathbb{C}}(f)$  be the plane projective curve defined by  $f$ . We denote by  $C(f, e)$  the closed hyperbolicity region of  $f$  with respect to  $e$  in the real projective plane.

**Definition 1.1.** Let  $f, g \in \mathbb{R}[t]$  be univariate polynomials with only real zeros and with  $\deg(g) = \deg(f) - 1$ . Let  $\alpha_1 \leq \dots \leq \alpha_d$  be the roots of  $f$ , and let  $\beta_1 \leq \dots \leq \beta_{d-1}$  be the roots of  $g$ . We say that  $g$  *interlaces*  $f$  if  $\alpha_i \leq \beta_i \leq \alpha_{i+1}$  holds for all  $i = 1, \dots, d-1$ . If all these inequalities are strict, we say that  $g$  *strictly interlaces*  $f$ .

If  $f \in \mathbb{R}[x, y, z]$  is hyperbolic with respect to  $e$  and  $g$  is homogeneous of degree  $\deg(f) - 1$ , we say that  $g$  *interlaces  $f$  with respect to  $e$*  if  $g(te + v)$  interlaces  $f(te + v)$  for every  $v \in \mathbb{R}^3$ . This implies that  $g$  is also hyperbolic with respect to  $e$ . We say that  $g$  *strictly interlaces  $f$*  if  $g(te + v)$  strictly interlaces  $f(te + v)$  for every  $v \in \mathbb{R}^3$  not in  $\mathbb{R}e$ .

With  $f$  as above, let  $g$  be any form in  $\mathbb{R}[x, y, z]$  coprime to  $f$ . We say that an intersection point  $p \in \mathcal{V}_{\mathbb{C}}(f, g)$  is a *contact point* of  $g$  with  $f$  if the intersection multiplicity  $\text{mult}_p(f, g)$  is even. If all intersection points are contact points, then  $g$  is called a *contact curve* of  $f$ . A *curve of real contact* is a curve  $g$  for which all real intersection points are contact points, without any assumption on nonreal intersection points. Any interlacer is a curve of real contact.

Interlacers of  $f$  appear naturally in the context of determinantal representations of  $f$  [Plaumann and Vinzant 2013; Kummer et al. 2015]. For example, if  $f = \det(xA + yB + zC)$  is a real symmetric and definite determinantal representation of  $f$ , then every principal  $(d-1) \times (d-1)$  minor of  $xA + yB + zC$  is an interlacer of  $f$  [Plaumann and Vinzant 2013, Theorem 3.3]. Furthermore, such a minor defines a contact curve (see, e.g., [Plaumann and Vinzant 2013, Proposition 3.2]). Conversely, given any interlacer of  $f$  that is also a contact curve, one can construct a definite determinantal representation of  $f$  and therefore a spectrahedral representation of its hyperbolicity region of size  $d \times d$ . However, for computational purposes, it is very difficult to actually find such an interlacer, even though its existence is guaranteed by the Helton–Vinnikov theorem [2007]. In Section 2, we will introduce

a method for constructing from an arbitrary interlacer a spectrahedral representation of possibly larger size. We denote by

$$\text{Int}(f, e) = \{g \in \mathbb{R}[x, y, z]_{d-1} : g \text{ interlaces } f \text{ and } g(e) > 0\}$$

the set of interlacers of  $f$ . It is shown in [Kummer et al. 2015, Corollary 2.7] that this is a closed convex cone. Every boundary point of this cone has at least one contact point. In order to find interlacers with many contact points, it is therefore natural to consider extreme rays of this cone.

**Definition 1.2.** Let  $f$  be hyperbolic with respect to  $e$ . By an *extremal interlacer* of  $f$  we mean an extreme ray of the cone  $\text{Int}(f, e)$ .

The next lemma gives a lower bound on the number of real contact points of an extremal interlacer.

**Lemma 1.3.** Assume that  $f$  defines a smooth curve of degree  $d$ . Any extremal interlacer of  $f$  has at least

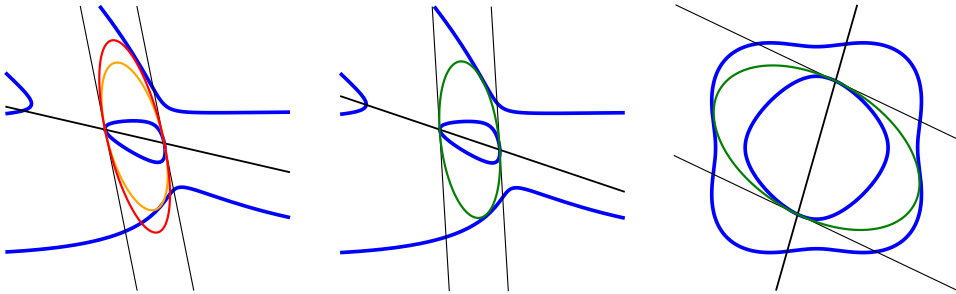
$$\left\lceil \frac{(d+1)d-2}{4} \right\rceil$$

real contact points with  $f$ , counted with multiplicity.

*Proof.* Let  $g$  be an extremal interlacer, and let  $k$  be the number of real contact points of  $g$ . By definition, the real part of the divisor  $\text{div}_C(g)$  is even, say  $2D$ , with  $D$  real and effective of degree  $k$ . The space  $V$  of forms  $h$  of degree  $d-1$  with  $\text{div}_C(h) \geq 2D$  has dimension at least  $n = (d+1)d/2 - 2k$  and contains  $g$ . If  $n > 1$ , then  $V$  contains another form  $h$  linearly independent of  $g$ . We conclude that  $g \pm \varepsilon h \in \text{Int}(f, e)$  for sufficiently small  $\varepsilon$ . Thus,  $g$  is not extremal. Therefore, we must have  $n \leq 1$ , which gives  $k \geq ((d+1)d-2)/4$ .  $\square$

**Remark 1.4.** For smooth  $f$ , given any  $d-1$  real points on the curve, there is an extremal interlacer touching the curve in (at least) the given points. Indeed, it is clear from the above proof that it suffices to show that there is an interlacer passing through these  $d-1$  points. The quadratic system of interlacers considered in [Plaumann and Vinzant 2013, Definition 3.1] has dimension  $d$ , so we can prescribe  $d-1$  points.

**Remark 1.5.** We do not know whether every hyperbolic curve possesses an *irreducible* extremal interlacer. This is true if  $C$  is a smooth cubic: for any two distinct points  $p$  and  $q$  on  $C$ , there is an extremal interlacing conic  $Q$  passing through  $p$  and  $q$ , by the preceding remark. If  $Q$  is reducible, it must factor into the two tangent lines to  $C$  at  $p$  and  $q$ . But  $Q$  is a contact curve by Lemma 1.3; hence, the intersection point of the two tangents must lie on  $C$ . Clearly, this will not be the case for a generic choice of  $p$  and  $q$ . This observation will be used at one point later on. It does not seem clear how to generalize this argument to higher degrees.



**Figure 3.** Quadrics touching hyperbolic quartics in real points.

The following table shows the expected number of real contact points of an extremal interlacer compared with the number of points for a full contact curve:

$d$	2	3	4	5	6	...
$\lceil ((d + 1)d - 2)/4 \rceil$	1	3	5	7	10	...
$d(d - 1)/2$	1	3	6	10	15	...

An interlacer can have many more real contact points than the estimate given by Lemma 1.3, and we do not know whether there is always one with only real intersection points.

**Question 1.6.** Does every hyperbolic plane curve have an interlacer that intersects the curve only in real points?

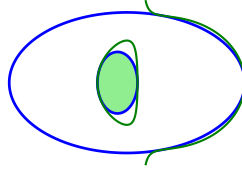
Even without the interlacing condition, it seems to be unknown whether a real curve always possesses a real contact curve with only real contact points. In the case of plane quartic curves we have some partial answers to that question.

**The case of quartics.** Let  $C \subseteq \mathbb{P}^2$  be a smooth hyperbolic quartic that has a real bitangent touching  $C$  in only real points. We will show that in this case there is a contact interlacer touching  $C$  only in real points. It suffices to show that there is a conic touching both ovals in two real points. This, together with the above bitangent, will be the desired totally real interlacer.

Assume that  $C(\mathbb{R})$  is contained in the affine chart  $z \neq 0$  (for smooth quartic curves this is not a restriction). Let  $l \in \mathbb{R}[x, y]_1$  be a nonzero linear form. Maximizing and minimizing  $l$  on the hyperbolicity region gives us two different linear polynomials  $l_1$  and  $l_2$  that are parallel and whose zero sets are tangent to the inner oval at some points  $p_1$  and  $p_2$  (see Figure 3).

Choose the signs such that both  $l_1$  and  $l_2$  are nonnegative on the inner oval. We consider the pencil of conics whose zero sets pass through  $p_1$  and  $p_2$  such that the tangent lines of the conics at  $p_1$  and  $p_2$  are defined by  $l_1$  and  $l_2$ , respectively. This pencil is given by  $q_\lambda = g^2 - \lambda l_1 l_2$ ,  $\lambda \in \mathbb{R}$ , where  $g$  is the line spanned by  $p_1$  and  $p_2$ .





**Figure 4.** A hyperbolic quartic curve (in blue) and a cubic interlacer (in green) with only real intersection points.

The zero set of  $q_\lambda$  is completely contained in the interior of the outer oval for small  $\lambda > 0$ . Label the two half spaces defined by  $g$  by 1 and 2, and let  $\lambda_i > 0$  be the smallest positive number such that the zero set of  $q_{\lambda_i}$  intersects the outer oval in the half space labeled by  $i$ . We observe that both  $q_{\lambda_i}$  have three real contact points with  $C$ . If  $\lambda_1 = \lambda_2$ , then we are done.

Now we let the linear form  $l$ , which we started with, vary continuously and we also keep track of the labels of the half spaces in a continuous manner. The resulting conic  $q_{\lambda_1}(l)$  depends continuously on  $l$ , and we note that  $q_{\lambda_1}(-l) = q_{\lambda_2}(l)$ . Note that one of the zero sets of  $q_{\lambda_1}(l)$  and of  $q_{\lambda_1}(-l)$  on  $C$  contains a pair of complex conjugate points (the orange oval in Figure 3, left) whereas the other one contains only real points of  $C$  (the red oval in Figure 3, left). Therefore, there must be a linear form  $l_0$  such that  $q_{\lambda_1}(l_0)$  has the desired properties (Figure 3, center and right).

If there is no bitangent touching the quartic in two real points, we do not know whether there always exists an interlacer intersecting the curve in only real points. The next example shows that this is at least sometimes the case.

**Example 1.7.** We consider the smooth plane quartic defined by

$$f = 1250000x^4 - 1749500x^3y - 2250800x^2y^2 - 4312500x^2z^2 \\ + 69260xy^3 + 786875xyz^2 + 88176y^4 + 1141000y^2z^2 + 1687500z^4.$$

Its real locus consists of two nested ovals both of which are convex (Figure 4), meaning that there is no bitangent touching the curve in two real points. Nevertheless, the interlacer given by

$$g = 500x^3 - 800x^2y - 740xy^2 - 625xz^2 + 176y^3 + 1000yz^2$$

intersects the quartic curve only in real points. Indeed, its divisor is given by

$$4 \cdot (4 : -5 : 0) + 2 \cdot (11 : 5 : 0) + 2 \cdot (1 : 5 : 0) + 2 \cdot (7 : 10 : -10) + 2 \cdot (7 : 10 : 10).$$

## 2. A generalized Dixon process

Given a real hyperbolic form  $f$  of degree  $d$  and an interlacer  $g$  of degree  $d - 1$ , we wish to produce a real symmetric determinantal representation of  $f$  with a principal

minor divisible by  $g$ . If  $g$  is a contact curve, this is achieved through the classical Dixon process. We will extend the procedure in such a way that the resulting representation will reflect any real contact points between  $f$  and  $g$ , relating to our discussion of extremal curves of real contact in the previous section.

Let  $f$  be irreducible and hyperbolic with respect to  $e \in \mathbb{P}^2(\mathbb{R})$ , and assume that the plane curve  $\mathcal{V}_{\mathbb{C}}(f)$  is smooth. Let  $g$  be an interlacer of  $f$  with  $r$  real contact points  $p_1, \dots, p_r$ , counted with multiplicities. Consider the  $d(d-1) - 2r$  further intersection points, which are nonreal and therefore come in complex conjugate pairs, say  $q_1, \dots, q_s, \bar{q}_1, \dots, \bar{q}_s$ , so that  $d(d-1) = 2r + 2s$ . For each  $i = 1, \dots, s$  let  $\ell_i$  be a linear form defining the unique (real) line joining  $q_i$  and  $\bar{q}_i$ . We will make the assumptions that

- (G1) no three of the intersection points of  $f$  with  $g$  lie on a line,
- (G2) no three of the  $\ell_i$  pass through the same point, and
- (G3)  $f$  does not vanish on any point where two of the  $\ell_i$  intersect.

We begin by showing that such an interlacer always exists.

**Lemma 2.1.** *There exists a strict interlacer for which the genericity assumptions (G1), (G2), and (G3) are satisfied.*

*Proof.* Every choice of  $k = \frac{1}{2}d(d+1) - 1$  points on the zero set of  $f$  that pose linearly independent conditions on forms of degree  $d-1$  determines a unique such form. The other zeros of this  $(d-1)$ -form on the zero set of  $f$  depend continuously on the choice of the  $k$  points. By the general position theorem [Arbarello et al. 1985, Chapter III, §1], any neighborhood of the given interlacer contains a strict interlacer  $g$  with the property that its zero set intersects the one of  $f$  in  $d(d-1)$  distinct points, any  $k$  of which pose linearly independent conditions on forms of degree  $d-1$ . Then we can slightly perturb any subset of  $k$  points in this intersection, and thus  $g$ , so that the number of triples of points in the intersection that lie on a line decreases. Thus, we can find a strict interlacer of  $f$  with the property that no three intersection points with the zero set of  $g$  lie on a line, so that genericity condition (G1) is satisfied. By the same argument, we can satisfy condition (G3).

For condition (G2), we need to move six points spanning three of the lines. Thus, the same argument applies, provided that  $k \geq 6$ , which means  $d > 3$ . The case  $d \leq 2$  being trivial, we are left with condition (G2) for cubics ( $d = 3$ ). In this case, we argue as follows. Suppose there is no interlacing conic satisfying condition (G2). Since the condition is Zariski-open, this would imply that condition (G2) is violated for any conic, strictly interlacing or not. But Lemma 1.3 and the subsequent Remark 1.5 imply that there exists an irreducible conic  $g$  touching  $f$  in three real points. Considering  $g$  as the limit of forms all of whose intersection points with  $f$  are simple, the assumption will imply that the three tangents to  $\mathcal{V}(g)$  at the contact

points meet in one point. But since  $g$  is irreducible of degree 2, this is impossible. This contradiction shows the claim.  $\square$

Under these genericity assumptions, we will construct a symmetric linear determinantal representation  $M$  of  $\ell_1 \cdots \ell_s \cdot f$  such that  $\mathcal{S}(M)$  is the hyperbolicity region of  $f$ . Furthermore, the interlacer  $g$  divides a principal minor of  $M$ . The main result of this section is as follows.

**Theorem 2.2.** *Let  $f$  be an irreducible form of degree  $d$  that is hyperbolic with respect to  $e \in \mathbb{P}^2(\mathbb{R})$ , and assume that the plane curve  $\mathcal{V}(f)$  is smooth. Let  $g$  be an interlacer of  $f$  with  $r$  real contact points, counted with multiplicities, that satisfies the genericity assumptions (G1), (G2), and (G3). Then there exists a symmetric linear matrix pencil  $M$  of size*

$$m = \frac{d^2 + d - 2r}{2}$$

which is positive definite at  $e$  and such that  $C(f, e) = \mathcal{S}(M)$ . We can choose  $M$  in such a way that  $g$  divides the principal minor  $M_{1,1}$  of  $M$  and  $\det(M)/f$  is a product of  $m - d$  linear forms. Furthermore, each  $(m - 1) \times (m - 1)$  minor  $M_{1l}$ ,  $1 \leq l \leq m$ , of  $M$  is also divisible by the product of these  $m - d$  linear forms.

The proof will consist of an algorithm that produces the desired representation given  $f$  and  $g$ .

We begin with some preliminaries. Given any two real ternary forms  $f, g$  of degrees  $d$  and  $d'$ , respectively, without common components, we denote by  $(f.g)$  the intersection cycle of  $f$  and  $g$ , consisting of the intersection points of the curves  $\mathcal{V}(f)$  and  $\mathcal{V}(g)$  in  $\mathbb{P}^2(\mathbb{C})$ . It is a 0-cycle, i.e., an element of the free abelian group over the points of  $\mathbb{P}^2(\mathbb{C})$ . Explicitly,  $(f.g) = \sum_{i=1}^k m_i p_i$ , with  $\mathcal{V}(f) \cap \mathcal{V}(g) = \{p_1, \dots, p_k\}$  and  $m_i$  positive integers, the intersection multiplicities. By Bézout’s theorem, we have  $\sum_{i=1}^k m_i = dd'$ . Intersection cycles are additive, i.e.,  $((f_1 \cdot f_2).g) = (f_1.g) + (f_2.g)$ . Furthermore, there is a natural partial order on 0-cycles, by comparing coefficients. We need the following classical result from the theory of plane curves, which we restate in the form we require.

**Theorem 2.3** (Max Noether). *Let  $f, g, h$  be real ternary forms. Assume that  $f$  is irreducible and does not divide  $gh$ , and that the curve  $\mathcal{V}(f) \subset \mathbb{P}^2(\mathbb{C})$  is smooth. If  $(h.f) \geq (g.f)$ , then there exist real forms  $a$  and  $b$  such that*

$$h = af + bg.$$

*Proof.* See [Fulton 1989, §5.5, Proposition 1].  $\square$

Now let  $f$  and  $g$  be given as in the statement of Theorem 2.2, with intersection points  $p_1, \dots, p_r, q_1, \dots, q_s, \bar{q}_1, \dots, \bar{q}_s$  as before, and let  $\ell_i$  be the linear form

defining the line between  $q_i$  and  $\bar{q}_i$ , for  $i = 1, \dots, s$ , under the genericity assumptions (G1)–(G3).

Put  $h = \ell_1 \cdots \ell_s$ , and consider the polynomial  $fh$  of degree  $(d^2 + d - 2r)/2 = m$ , which is hyperbolic with respect to  $e$ . Furthermore, since each line  $\ell_i$  meets  $C$  in the nonreal point  $q_i$ , none of the lines pass through  $C(f, e)$ , so that  $C(fh, e) = C(f, e)$ .

It therefore suffices to construct a symmetric linear determinantal representation of  $fh$  which is definite at  $e$ . This can be carried out with a modification of Dixon's method, which we now describe in several steps.

(1) Let  $V$  be the linear space of real forms of degree  $d - 1$  vanishing at  $p_1, \dots, p_r$ . We have  $\dim(V) \geq (d + 1)d/2 - r = d + s$ , and we pick linearly independent forms  $a_1, \dots, a_{d+s} \in V$ , with  $a_1 = g$ . We introduce names for all the occurring intersection points:

$$\begin{aligned} (a_1 \cdot f) &= (g \cdot f) = 2 \sum_{j=1}^r p_j + \sum_{j=1}^s (q_j + \bar{q}_j), \\ (a_i \cdot f) &= \sum_{j=1}^r p_j + \sum_{j=1}^{r+2s} p_{ij} \quad \text{for } i \geq 2, \\ (\ell_i \cdot f) &= q_i + \bar{q}_i + \sum_{j=1}^{d-2} r_{ij}, \\ (\ell_i \cdot \ell_j) &= s_{ij} \quad \text{for } i \neq j. \end{aligned}$$

(2) Fix  $k, l \in \{2, \dots, d + s\}$  with  $k \leq l$ . We wish to find a real form  $b_{kl}$  of degree  $d + s - 1$  such that

$$(2.4) \quad b_{kl}g - ha_k a_l \in (f)$$

by applying Max Noether's theorem: we compute the intersection cycles

$$\begin{aligned} (ha_k a_l \cdot f) &= 2 \sum_{j=1}^r p_j + \sum_{j=1}^s (q_j + \bar{q}_j) + \sum_{j=1}^s \sum_{j'=1}^{d-2} r_{jj'} + \sum_{j=1}^{r+2s} p_{kj} + \sum_{j=1}^{r+2s} p_{lj}, \\ (g \cdot f) &= 2 \sum_{j=1}^r p_j + \sum_{j=1}^s (q_j + \bar{q}_j) \end{aligned}$$

and thus find  $b_{kl}$  with

$$(b_{kl} \cdot f) = \sum_{j=1}^s \sum_{j'=1}^{d-2} r_{jj'} + \sum_{j=1}^{r+2s} p_{kj} + \sum_{j=1}^{r+2s} p_{lj}.$$

(3) Assume that  $k = l$ . Then we will produce a real form  $q$  of degree  $s - 1$  such that  $c_{kk} := b_{kk} + qf$  satisfies

$$(c_{kk} \cdot \ell_i) = (b + qf \cdot \ell_i) = \sum_{j=1}^{d-2} r_{ij} + \sum_{j \neq i} s_{ij} + 2t_{ki}$$

for some real point  $t_{ki} \in \ell_{ki}$ , for all  $i = 1, \dots, s$ . To this end, we let  $\ell_0$  be a linear form which does not vanish on any of the  $s_{ij}$ . Let  $h_{ij} = (\ell_0 \cdots \ell_s) / (\ell_i \ell_j)$  and  $\alpha_{ij} = -b_{kk}(s_{ij}) / (h_{ij}(s_{ij}) f(s_{ij}))$  for  $1 \leq i < j \leq s$ . Note that  $h_{ij}$  vanishes on all  $s_{mn}$  except for  $s_{ij}$ . After replacing  $b_{kk}$  by  $b_{kk} + \sum_{i,j} \alpha_{ij} h_{ij} f$ , we can thus assume that  $b_{kk}$  vanishes on all the  $s_{ij}$ .

Next, we consider

$$q_\alpha = \sum_{j=1}^s \alpha_j \frac{\ell_1 \cdots \ell_s}{\ell_j}$$

with  $\alpha_1, \dots, \alpha_s \in \mathbb{R}$ . The form  $q_\alpha$  satisfies  $q_\alpha(s_{ij}) = 0$  for all  $j \neq i$  for any choice of the  $\alpha_j$ . If we now take  $q = \tilde{q} + q_\alpha$ , we find

$$(b_{kk} + qf \cdot \ell_i) = \sum_{j=1}^{d-2} r_{ij} + \sum_{j \neq i} s_{ij} + u_i + v_i$$

with  $u_i$  and  $v_i$  depending on  $\alpha$ . Restricting to  $\ell_i$  we therefore get  $b_{kk} + qf = P \cdot (\tilde{b} + \alpha_i \tilde{f})$  where  $P$  is a nonzero polynomial whose roots are the  $r_{ij}$  and  $s_{ij}$ , and where  $\tilde{b}$  and  $\tilde{f}$  are polynomials of degree two. After possibly replacing  $\alpha_i$  by its negative, we can assume that  $\tilde{f}$  is strictly positive on  $\ell_i$  since it has no real zeros on  $\ell_i$ . Therefore, we can choose  $\alpha_i$  in such a way that  $\tilde{b} + \alpha_i \tilde{f}$  has a double zero  $t_{ki}$  and that makes the product of  $b_{kk} + qf$  and  $f \cdot ((\ell_1 \cdots \ell_s) / \ell_i) \ell_i$  nonnegative on  $\ell_i$ . The reasons for the latter requirement will become clear in a later step.

(4) Similarly, if  $k < l$ , we can find a real form  $q$  of degree  $s - 1$  such that  $c_{kl} := b_{kl} + qf$  satisfies

$$(c_{kl} \cdot \ell_i) = (b_{kl} + qf \cdot \ell_i) = \sum_{j=1}^{d-2} r_{ij} + \sum_{j \neq i} s_{ij} + t_{ki} + t'_{ki}$$

for some real point  $t'_{ki} \in \ell_i$ . In fact, we even have that  $t'_{ki} = t_{li}$ . Indeed, this follows from (2.4) and the following lemma applied to each  $\ell_i$ .

**Lemma 2.5.** *Let  $f \in \mathbb{R}[t]$  be a polynomial of degree two without real zeros. Let  $a, b, c \in \mathbb{R}[t]$  be polynomials of degree at most two such that  $a$  and  $c$  both have a double zero,  $ac$  is nonnegative, and  $b$  vanishes at the zero of  $a$ . If  $ac = b^2 \pmod{f}$ , then  $b$  vanishes at the zero of  $c$  as well.*

*Proof.* Let  $a = \alpha(t - \beta)^2$ ,  $c = \alpha'(t - \beta')^2$ , and  $b = \gamma(t - \beta)(t - \beta'')$  for some  $\alpha, \alpha', \beta, \beta', \beta'', \gamma \in \mathbb{R}$  with  $\alpha\alpha' \geq 0$ . We have by assumption

$$\alpha\alpha'(t - \beta)^2(t - \beta')^2 = \gamma^2(t - \beta)^2(t - \beta'')^2 \pmod{f}.$$

Since  $\mathbb{R}[t]/(f)$  is isomorphic to the field of complex numbers, it follows that

$$\alpha\alpha'(t - \beta')^2 = \gamma^2(t - \beta'')^2 \pmod{f}.$$

If  $\gamma \neq 0$ , then  $\alpha\alpha' > 0$  and  $t - \beta' = \pm\sqrt{\gamma^2/(\alpha\alpha')} \cdot (t - \beta'') \pmod{f}$ . Finally, it follows that  $\alpha\alpha' = \gamma$  and that  $\beta' = \beta''$  because  $1, t \in \mathbb{R}[t]/(f)$  are  $\mathbb{R}$ -linearly independent.  $\square$

If  $k > l$ , we let  $c_{kl} = c_{lk}$ .

(5) We now put  $c_{1k} = c_{k1} = ha_k$  and consider the matrix  $N$  with entries  $c_{kl}$ , for  $k, l = 1, \dots, d + s$ . By construction, the  $(2 \times 2)$ -minors

$$c_{11}c_{kl} - c_{1k}c_{1l} = hgc_{kl} - h^2a_k a_l = h(gc_{kl} - ha_k a_l)$$

are divisible by  $fh$ . Since the first row of  $N$  is not divisible by  $f$ , it follows that all  $(2 \times 2)$ -minors of  $N$  are divisible by  $f$ . We need to show that all  $(2 \times 2)$ -minors  $c_{kl}c_{k'l'} - c_{k'l}c_{k'l'}$  are also divisible by  $h$ . Let  $u$  be such a minor, and fix  $i \in \{1, \dots, s\}$ . Note that  $u$  has degree  $2d + 2s - 2$  and vanishes (with multiplicities) on the  $2d + 2s - 2$  points  $2 \sum_{j=1}^{d-2} r_{ij}$ ,  $2 \sum_{j \neq i} s_{ij}$ , and  $(t_{ki} + t_{k'i} + t_{li} + t_{l'i})$  on  $\ell_i$ , since both products  $c_{kl}c_{k'l'}$  and  $c_{k'l}c_{k'l'}$  vanish at those points. Since  $u$  is divisible by  $f$ , it also vanishes at  $q_i + \bar{q}_i$ . Thus,  $u$  vanishes identically on  $\ell_i$  for each  $i$ , which implies  $h \mid u$ .

(6) In this step we show that  $c_{22}$  interlaces  $fh$ . This can be done by proving that  $c_{22} \cdot D_e(fh)$  is nonnegative on the zero set of  $fh$  [Kummer et al. 2015, Theorem 2.1]. Here  $D_e(fh)$  denotes the derivative of  $fh$  in direction  $e$ . We have

$$D_e(fh) = h \cdot D_e f + f \cdot \sum_{i=1}^s \ell_i(e) \frac{\ell_1 \cdots \ell_s}{\ell_i}.$$

We can rewrite this modulo  $f$  and find

$$c_{22} \cdot D_e(fh) = c_{22} \cdot h \cdot D_e f = \frac{ha_2^2}{g} \cdot h \cdot D_e f = \frac{D_e f}{g} h^2 a_2^2 \pmod{f}$$

by (2.4). This is nonnegative on the zero set of  $f$  because both  $D_e f$  and  $g$  are interlacers. On the other hand, modulo  $\ell_i$  we obtain

$$c_{22} \cdot D_e(fh) = c_{22} \cdot \ell_i(e) \cdot \frac{\ell_1 \cdots \ell_s}{\ell_i} \pmod{\ell_i},$$

which is nonnegative on the line defined by  $\ell_i$  by the choices made in step (3).

(7) Now we proceed as in the usual Dixon process, referring to [Plaumann and Vinzant 2013] for details. Since all  $(2 \times 2)$ -minors of the  $(d + s) \times (d + s)$ -matrix  $N$

are divisible by  $fh$ , its maximal minors are divisible by  $(fh)^{d+s-2}$  (see for example [Plaumann and Vinzant 2013, Lemma 4.7]). The signed maximal minors of  $N$  have degree  $(d + s - 1)^2$  and are the entries of the adjugate matrix  $N^{\text{adj}}$ . It follows that

$$M = (fh)^{2-d-s} \cdot N^{\text{adj}}$$

has linear entries. Using the familiar identity  $NN^{\text{adj}} = \det(N) \cdot I_{d+s}$ , we conclude

$$\det(M) = \gamma \cdot fh$$

for some constant  $\gamma \in \mathbb{R}$ . It remains to show that  $\gamma \neq 0$ . Suppose  $\gamma = 0$ ; then  $\det(M)$  is identically zero and hence so is  $\det(N)$ . In particular, the matrix  $N(e)$  is singular. Let  $\lambda \in \mathbb{R}^{d+s}$  be a nontrivial vector in the kernel of  $N(e)$ , and consider the polynomial  $\tilde{g} = \lambda^t N \lambda$ . It follows from the linear independence of the entries of the first row of  $N$  that  $\tilde{g}$  is not the zero polynomial [Plaumann and Vinzant 2013, Lemma 4.8]. Since  $c_{22}$  interlaces  $fh$  by (6), so does  $\tilde{g}$  [Plaumann and Vinzant 2013, Theorem 3.3, (1)  $\implies$  (2)], contradicting  $\tilde{g}(e) = 0$ . That  $M(e)$  is definite also follows from the fact that  $c_{22}$  interlaces  $fh$ , by [Plaumann and Vinzant 2013, Theorem 3.3, (2)  $\implies$  (3)]. Note that the result in [Plaumann and Vinzant 2013] is stated only for irreducible curves. However, the same argument will apply here, since  $c_{22}$  is coprime to  $fh$  (unlike  $c_{11}$ , which is divisible by  $h$ ). Indeed, we have chosen  $c_{22}$  in step (3) in such a way that it does not vanish entirely on any of the lines  $l_i$ . Thus,  $c_{22}$  is coprime to  $h$ . Moreover,  $c_{22}$  is congruent to  $b_{22}$  modulo  $(f)$ . Thus, if  $f$  divided  $c_{22}$ , it would also divide  $a_2$  by (2.4), which is not the case.

This finishes the construction of the determinantal representation  $M$  of  $fh$ . Finally, we note that the spectrahedron  $\mathcal{S}(M)$  coincides with the hyperbolicity region  $C(f, e)$  of  $f$ . Since  $\det(M) = f \cdot \ell_1 \cdots \ell_s$ , this simply amounts to the fact that the lines  $\ell_1, \dots, \ell_s$  do not pass through  $C(f, e)$ . Indeed, each  $\ell_j$  has two nonreal intersection points with  $C$ , while lines passing through the hyperbolicity region will meet  $C$  in only real points. This completes the proof of Theorem 2.2.

**Remark 2.6.** Clearly, the corank of the constructed matrix pencil  $M$  is at least one at each point where  $fh$  vanishes. It can have corank more than one only at singularities of  $fh$ , i.e., in our case the points where two components intersect. Since the adjugate  $N = M^{\text{adj}}$  vanishes identically at the points  $r_{ij}$  and  $s_{ij}$  and because these are ordinary nodes, the corank of  $M$  at these points is exactly two. On the other hand, we have constructed  $N$  in such a way that it is not entirely zero at the points  $q_j$  and  $\bar{q}_j$ . Thus,  $M$  has corank one at these points. This shows in particular that  $M$  is not equivalent to a block diagonal matrix with more than one block.

**Remark 2.7.** The vector space  $V$  in step (1) of our construction can be found without computing all the real contact points  $p_1, \dots, p_r$ . Indeed, by genericity assumption (G1) the  $q_i, \bar{q}_i$  are all simple intersection points. Therefore, the  $p_i$  can

be computed as the singular locus of the zero-dimensional scheme cut out by  $f$  and  $g$  via the Jacobian criterion.

Next we observe that the genericity assumption in the theorem, as well as the smoothness assumption on  $f$ , can be dropped for strict interlacers by applying a limit argument.

**Corollary 2.8.** *Let  $f$  be a real form of degree  $d$  that is hyperbolic with respect to  $e \in \mathbb{P}^2(\mathbb{R})$ , and let  $g$  be a strict interlacer of  $f$ . Then there exists a symmetric linear matrix pencil  $M$  of size  $(d^2 + d)/2$  which is definite at  $e$  and such that  $C(f, e) = \mathcal{S}(M)$ . We can choose  $M$  in such a way that  $g$  divides a principal minor of  $M$  and  $\det(M)/f$  is a product of  $(d^2 - d)/2$  linear forms.*

*Proof.* Let  $m = (d^2 + d)/2$ . We may assume that  $f(e) = 1$  and consider only monic representations  $f = \det(M)$ , i.e., with  $M(e) = I_m$ . The determinant map taking a monic symmetric real linear matrix pencil of size  $m \times m$  to its determinant is proper; hence, its image is closed (see for example [Plaumann and Vinzant 2013, Lemma 3.4]). If  $g$  is a strict interlacer of  $f$ , the pair  $(f, g)$  is in the closure of the set of pairs  $(\tilde{f}, \tilde{g})$ , where  $\tilde{f}$  is hyperbolic with respect to  $e$ ,  $\mathcal{V}(\tilde{f})$  is smooth, and  $\tilde{g}$  is a strict interlacer of  $\tilde{f}$  satisfying the genericity assumptions (G1)–(G3). Therefore, there exists a sequence  $(\tilde{f}_n, \tilde{g}_n)$  converging to  $(f, g)$  together with representations  $\tilde{f}_n = \det(\tilde{M}_n)$  with  $\tilde{g}_n$  dividing the first principal minor of  $\tilde{M}$  and  $\det(\tilde{M})/\tilde{f}$  a product of  $m - d$  linear forms, by Theorem 2.2. The sequence  $\tilde{M}_n$  then has a subsequence converging to a matrix pencil  $M$ , which is the desired determinantal representation of  $f$ . □

**Remark 2.9.** The procedure of approximating a given hyperbolic form together with an interlacer as in the proof above may be difficult to carry out in practice. However, the generalized Dixon process can often be applied (with small modifications if needed) even when the genericity assumptions fail.

As a further consequence, we can prove the following rationality result.

**Theorem 2.10.** *Let  $f \in \mathbb{Q}[x, y, z]_d$  be a polynomial hyperbolic with respect to  $e \in \mathbb{R}^3$  whose real projective zero set is smooth. Then its hyperbolicity cone is of the form*

$$\{(x, y, z) \in \mathbb{R}^3 : xA + yB + zC \succeq 0\}$$

where  $A, B, C$  are symmetric matrices with rational entries of size at most  $\binom{d+1}{2}$ .

*Proof.* Let  $m \in \mathbb{Q}[x, y, z]_{d-1}^e$  be the vector of all monomials of degree  $d - 1$ , and let  $e = \binom{d+1}{2}$ . The equation

$$(2.11) \quad (xA + yB + zC) \cdot m = f \cdot v$$



poses linear conditions on the entries of the symmetric  $e \times e$  matrices  $A, B, C$  and on the entries of  $v \in \mathbb{R}^e$ . These linear conditions are defined over the rational numbers.

We now apply the above construction to  $f$  with a strict interlacer  $g$  satisfying the genericity assumptions (G1), (G2), and (G3) whose existence is guaranteed by Lemma 2.1. The vector space  $V$  from step (1) is just the vector space of all ternary forms of degree  $d - 1$ , and the  $a_i$  form a basis of  $V$ . Thus, we can find an invertible matrix  $S \in \text{GL}_e(\mathbb{R})$  that maps the vector  $a = (a_1, \dots, a_N)^t$  to the vector  $m$  of all monomials of degree  $d - 1$ . In step (7) of our construction we have seen that there is a matrix  $N$  such that our symmetric determinantal representation  $M$  of  $f \cdot h$  satisfies  $M \cdot N = \gamma \cdot fh \cdot I_e$ . Moreover, the first column of  $N$  is  $h \cdot a$ . Thus, we have that  $M \cdot a = \gamma \cdot f \cdot \delta_1$ . Now we get the identity

$$S^{-t}MS^{-1} \cdot m = \gamma \cdot f \cdot S^{-t}\delta_1.$$

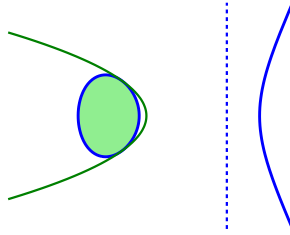
This is a solution over  $\mathbb{R}$  to (2.11) with  $e_0A + e_1B + e_2C$  positive definite and  $\det(xA + yB + zC) = h \cdot f$  where  $h$  is a product of linear forms whose zero set does not intersect the hyperbolicity cone of  $f$ .

Since the rational solutions to (2.11) are dense in the solution set over the real numbers, we can find rational matrices  $A, B, C$  satisfying (2.11) with  $e_0A + e_1B + e_2C$  being positive definite, as well. Then  $\det(xA + yB + zC)$  is not the zero polynomial and is divisible by  $f$ , since the pencil has a nonzero kernel vector whenever  $f$  vanishes at  $(x, y, z)$  by (2.11). If  $A, B, C$  are chosen close enough to our original solution, the other factor of  $\det(xA + yB + zC)$  will not intersect the hyperbolicity cone of  $f$  either. □

**Remark 2.12.** One might be tempted to generalize Theorem 2.10 to singular  $f$  using the determinantal representation  $\det(M) = h \cdot f$ , where  $h$  is a product of linear forms, obtained in Corollary 2.8 by a limit argument. In order to make the arguments from the preceding proof work, the first column of  $M^{\text{adj}}$  would have to be of the form  $h \cdot a$  where  $a$  is a vector whose entries span a subspace of  $\mathbb{R}[x, y, z]_{d-1}$  that is defined over the rationals. It is not clear whether this is always the case.

The next example shows that the smallest size of a rational spectrahedral representation is in general larger than the degree of the curve.

**Example 2.13.** Consider the univariate polynomial  $p = x^3 - 6x - 3 \in \mathbb{Q}[x]$ . It has three distinct real zeros but is irreducible over the rational numbers by Eisenstein's criterion. The plane elliptic curve defined by  $y^2 = p(x)$  is hyperbolic. Its hyperbolicity cone has the following spectrahedral representation with rational  $4 \times 4$



**Figure 5.** Hyperbolic cubic (in blue), an interlacer touching in 2 real points (in green) and the linear factor (dashed in blue).

matrices:

$$\left\{ (x, y, z) \in \mathbb{R}^3 : \begin{pmatrix} 3z & y & -x - z & -3x + z \\ y & -x + 2z & 0 & -y \\ -x - z & 0 & z & x + 4z \\ -3x + z & -y & x + 4z & -x + 18z \end{pmatrix} \geq 0 \right\}.$$

This was obtained by applying our construction to the interlacer  $y^2 + 3xz + z^2$  with two real contact points (Figure 5).

It also has a  $3 \times 3$  spectrahedral representation with real matrices by the Helton–Vinnikov theorem. It does, however, not have such a representation with rational  $3 \times 3$  matrices. Indeed, any such representation would yield a contact interlacer defined over the rational numbers by taking some principal  $2 \times 2$  minor. This interlacer would give rise to a divisor  $D$  defined over the rational numbers with  $2D = 6P_\infty$  where  $P_\infty$  is the point of the curve at infinity. Thus,  $D - 3P_\infty$  would be an even theta characteristic defined over the rationals. On the other hand, the three even theta characteristics of the curve are given by  $P_i - P_\infty$  for  $P_1, P_2, P_3$  the three intersection points of the curve with the  $x$ -axis. These are clearly not defined over the rationals.

### 3. Bézout matrices

Let  $f, g \in \mathbb{R}[t]$  be two univariate polynomials having degrees  $\deg(f) = d$  and  $\deg(g) = d - 1$ . The *Bézout matrix* of  $f$  and  $g$  is defined as follows. We write

$$\frac{f(s)g(t) - f(t)g(s)}{s - t} = \sum_{i,j=1}^d b_{ij} s^{i-1} t^{j-1}$$

for some real numbers  $b_{ij}$ . Then the Bézout matrix is defined as  $B(f, g) = (b_{ij})_{ij}$ . Note that  $B(f, g)$  is always a real symmetric matrix. The Bézout matrix can be used to detect the properties of being real-rooted and interlacing.

**Theorem 3.1** (see, e.g., [Kreĭn and Naĭmark 1981, §2.2]). *Let  $f, g \in \mathbb{R}[t]$  be univariate polynomials with  $d = \deg(f) = \deg(g) + 1$ . Then the following are equivalent:*

- (i) *the Bézout matrix  $B(f, g)$  is positive semidefinite and*
- (ii) *the polynomial  $g$  interlaces  $f$ .*

*Furthermore, the Bézout matrix has full rank if and only if  $f$  and  $g$  have no common zero.*

In the multivariate case we can proceed analogously. Let  $f, g \in \mathbb{R}[x_0, \dots, x_n]$  be homogeneous polynomials of degrees  $d$  and  $d - 1$ , respectively. We assume that  $f$  and  $g$  do not vanish at  $e = (1, 0, \dots, 0)$ . Then, writing  $x = (x_1, \dots, x_n)$ , we have

$$\frac{f(s, x)g(t, x) - f(t, x)g(s, x)}{s - t} = \sum_{i,j=1}^d b_{ij}s^{i-1}t^{j-1}$$

for some homogeneous polynomials  $b_{ij} \in \mathbb{R}[x_1, \dots, x_n]$  of degree  $2d - (i + j)$ . Again, we define the Bézout matrix as  $B(f, g) = (b_{ij})_{ij}$ . It follows from the above theorem that  $B(f, g)$  is positive definite for every  $0 \neq x \in \mathbb{R}^n$  if and only if  $f$  is hyperbolic with respect to  $e$  and  $g$  is a strict interlacer of  $f$ .

**Remark 3.2.** The Bézout matrix  $B(f, g)$  is closely related to the *Wronskian polynomial*  $W(f, g) = D_e f \cdot g - f \cdot D_e g$ . Namely, if we let  $w = (1, x_0, \dots, x_0^{d-1})^t$ , then  $W(f, g) = w^t \cdot B(f, g) \cdot w$ . Indeed, by the definition of the Bézout matrix the right-hand side equals

$$\lim_{s \rightarrow t} \left( \frac{f(s, x)g(t, x) - f(t, x)g(s, x)}{s - t} \right) = W(f, g).$$

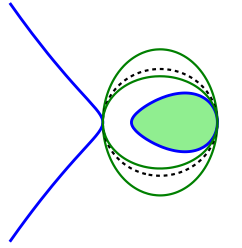
We also note that, for square-free polynomials  $f$ , the polynomial  $g$  of degree  $\deg(f) - 1$  is uniquely determined by  $W(f, g)$ .

We can use the Wronskian polynomial  $W(f, g)$  to describe the set  $\text{Int}(f, e)$  of interlacers of  $f$  in direction  $e$ , which is a convex cone. By [Kummer et al. 2015, Corollary 2.7],  $\text{Int}(f, e)$  can be represented as a linear image of a section of the cone of positive polynomials of degree  $2d - 2$ , where  $d = \deg f$ :

$$\text{Int}(f, e) = \{g \in \mathbb{R}[x, y, z]_{d-1} : W(f, g) \geq 0\}.$$

Whenever  $W(f, g)$  is a sum of squares, the cone  $\text{Int}(f, e)$  can be sampled by solving a linear matrix inequality as shown in the following example.

**Example 3.3.** The cubic  $f = x^3 + 2x^2y - xy^2 - 2y^3 - xz^2$  is hyperbolic with respect to  $e = (1, 0, 0)$ , and  $C(f, e)$  is the green region in Figure 6.



**Figure 6.** A cubic hyperbolic curve (in blue) with three interlacers, one defined over  $\mathbb{Q}$  (in dashed black) and two over an extension of degree 4 (in green). The dashed blue line is the extra factor in the determinantal representation.

Let  $g = x^2 + g_{110}xy + g_{101}xz + g_{020}y^2 + g_{011}yz + g_{002}z^2$  be a generic quadratic form such that  $g(e) = 1$ . The Wronskian of  $f, g$  in direction  $e$  is the ternary quartic

$$\begin{aligned} W(f, g) = & 2g_{110}x^3y + 2g_{110}x^2y^2 + 2g_{110}y^4 + 2g_{101}x^3z + 2g_{101}x^2yz + 2g_{101}y^3z \\ & + 3g_{020}x^2y^2 + 4g_{020}xy^3 - g_{020}y^4 - g_{020}y^2z^2 + 3g_{011}x^2yz + 4g_{011}xy^2z \\ & - g_{011}y^3z - g_{011}yz^3 + 3g_{002}x^2z^2 + 4g_{002}xyz^2 - g_{002}y^2z^2 - g_{002}z^4 \\ & + x^4 + x^2y^2 + x^2z^2 + 4xy^3. \end{aligned}$$

Let  $G = (G_{ij})$  be a symmetric  $6 \times 6$  matrix of unknowns, and consider the linear system  $W(f, g) = m^t \cdot G \cdot m$ , where  $m$  is the vector of monomials of degree 2 in  $x, y, z$ . We obtain that  $G$  (the *Gram matrix* of  $W(f, g)$  [Powers and Wörmann 1998]) has the form

$$G = \begin{bmatrix} 1 & g_{110} & g_{101} & G_{14} & g_{101} - G_{23} + \frac{3}{2}g_{011} & G_{16} \\ g_{110} & H_{22} & G_{23} & 2g_{020} + 2 & -G_{34} + 2g_{011} & -G_{35} + 2g_{002} \\ g_{101} & G_{23} & 1 + 3g_{002} - 2G_{16} & G_{34} & G_{35} & 0 \\ G_{14} & 2g_{020} + 2 & G_{34} & 2g_{110} - g_{020} & g_{101} - \frac{1}{2}g_{011} & G_{46} \\ H_{15} & 2g_{011} - G_{34} & G_{35} & g_{101} - \frac{1}{2}g_{011} & -2G_{46} - g_{020} - g_{002} & -\frac{1}{2}g_{011} \\ G_{16} & 2g_{002} - G_{35} & 0 & G_{46} & -\frac{1}{2}g_{011} & -g_{002} \end{bmatrix},$$

where

$$H_{15} = g_{101} - G_{23} + \frac{3}{2}g_{011} \quad \text{and} \quad H_{22} = 3g_{020} + 2g_{110} + 1 - 2G_{14}.$$

Let  $p_1 = (1, 1, 0)$  and  $p_2 = (1, -1, 0)$ . Interlacers in  $\text{Int}(f, e)$  vanishing in  $p_1$  and  $p_2$  can be computed through the quantified linear matrix inequality

$$(3.4) \quad \text{there exists } G_{ij} : g(p_1) = g(p_2) = 0, G \succeq 0.$$

Solving (3.4) symbolically using [Henrion et al. 2019] yields the parametrization of an interlacer  $g = x^2 - y^2 + t \cdot z^2$ , where  $t$  is any of the two real roots  $t_1, t_2$  of  $q(t) = 49t^4 - 20t^3 + 22t^2 + 12t + 1$  (the green curves in Figure 6).

Since the matrices  $G$  corresponding to the two interlacers have rank two, the corresponding Wronskian polynomials are sums of two squares. Choosing a rational  $t_1 < r < t_2$  gives a rational interlacer, for instance  $g = x^2 - y^2 - \frac{1}{3}z^2$ .

As in Example 2.13, our construction yields rational  $4 \times 4$  determinantal representations of  $f$  times a rational linear polynomial that can be built from the interlacer  $g = x^2 - y^2 - \frac{1}{3}z^2$ :

$$\frac{24}{125} f \cdot (2x - y) = \det \begin{pmatrix} 5x + 10y & -x - 2y & -4z & 2z \\ -x - 2y & x & 0 & 0 \\ -4z & 0 & 4x + 2y & -2x - 4y \\ 2z & 0 & -2x - 4y & 4x + 2y \end{pmatrix}.$$

The matrix on the right-hand side of the previous equality gives a spectrahedral representation of  $C(f, e)$  (the green region in Figure 6).

In the following, we show how our construction gives a *sum-of-squares decomposition*, i.e., a representation  $B(f, g) = S^t S$  for some (not necessarily square) matrix  $S$  with polynomial entries, for any curve  $f$  hyperbolic with respect to  $(1, 0, 0)$  and any strict interlacer  $g$ .

We have seen that there is a basis  $g_1, \dots, g_N$  of  $\mathbb{R}[x, y, z]_{d-1}$  with  $g_1 = g$  and real symmetric matrices  $A, B, C$  of size  $N$  such that  $A$  is positive definite and

$$(3.5) \quad (xA + yB + zC) \cdot v = \delta_1 \cdot f$$

where  $v = (g_1, \dots, g_N)^t$  and  $\delta_1 \in \mathbb{R}^N$  is the first unit vector. Let us write  $v = h_0 x^{d-1} + \dots + h_{d-1}$  for some  $h_i \in \mathbb{R}[y, z]_i^N$ , and let  $S$  be the matrix with columns  $h_d, \dots, h_0$ . We claim that  $B(f, g) = S^t AS$ . Indeed, by [Kummer 2017, §3], we have that  $B(f, \tilde{g}) = S^t AS$  for some  $\tilde{g} \in \mathbb{R}[x, y, z]_{d-1}$ . Furthermore, taking the derivative of (3.5) yields

$$A \cdot v + (xA + yB + zC) \cdot D_e v = \delta_1 \cdot D_e f.$$

Now it follows by multiplying with  $v^t$  from the left and another application of (3.5) that

$$v^t \cdot A \cdot v + f \cdot \delta_1^t \cdot D_e v = v^t \cdot A \cdot v + v^t \cdot (xA + yB + zC) \cdot D_e v = v^t \cdot \delta_1 \cdot D_e f.$$

Thus, by Remark 3.2, applied with  $v = Sw$ , we find

$$W(f, \tilde{g}) = w^t B(f, \tilde{g}) w = w^t S^t AS w = v^t \cdot \delta_1 D_e f - f \cdot \delta_1^t D_e v = W(f, g),$$

which implies  $g = \tilde{g}$ , again by Remark 3.2, since  $f$  is square-free.

**Remark 3.6.** It has been shown in [Kummer 2017] that a sum-of-squares representation  $B(f, g) = S^t A S$  of a Bézout matrix of a hyperbolic polynomial  $f$  with a positive definite matrix  $A$  as above gives rise to a definite determinantal representation of some multiple of  $f$ . Now we have seen that for every strict interlacer of a hyperbolic curve there is a sum-of-squares decomposition of the corresponding Bézout matrix which even gives rise to a spectrahedral representation of the hyperbolicity cone.

### Acknowledgements

We thank the Max Planck Institute for Mathematics in the Sciences in Leipzig for its hospitality, where this work was started. We would also like to thank the referee for useful remarks that helped us improve the exposition. Simone Naldi acknowledges the support of the Fondation Mathématique Jacques Hadamard through the PGMO project 2018-0061H.

### References

- [Arbarello et al. 1985] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris, *Geometry of algebraic curves*, vol. I, Grundlehren der Mathematischen Wissenschaften **267**, Springer, 1985. MR Zbl
- [Beauville 2000] A. Beauville, “Determinantal hypersurfaces”, *Michigan Math. J.* **48** (2000), 39–64. MR Zbl
- [Dixon 1902] A. C. Dixon, “Note on the reduction of a ternary quantic to a symmetrical determinant”, *P. Cambridge Philos. Soc.* **11** (1902), 350–351. Zbl
- [Dolgachev 2012] I. V. Dolgachev, *Classical algebraic geometry: a modern view*, Cambridge University, 2012. MR Zbl
- [Fulton 1989] W. Fulton, *Algebraic curves: an introduction to algebraic geometry*, Addison-Wesley, Redwood City, CA, 1989. MR Zbl
- [Helton and Vinnikov 2007] J. W. Helton and V. Vinnikov, “Linear matrix inequality representation of sets”, *Comm. Pure Appl. Math.* **60**:5 (2007), 654–674. MR Zbl
- [Henrion et al. 2019] D. Henrion, S. Naldi, and M. Safey El Din, “SPECTRA — a Maple library for solving linear matrix inequalities in exact arithmetic”, *Optim. Methods Softw.* **34**:1 (2019), 62–78. MR Zbl
- [Kreĭn and Naĭmark 1981] M. G. Kreĭn and M. A. Naĭmark, “The method of symmetric and Hermitian forms in the theory of the separation of the roots of algebraic equations”, *Linear and Multilinear Algebra* **10**:4 (1981), 265–308. MR Zbl
- [Kummer 2017] M. Kummer, “Determinantal representations and Bézoutians”, *Math. Z.* **285**:1–2 (2017), 445–459. MR Zbl
- [Kummer et al. 2015] M. Kummer, D. Plaumann, and C. Vinzant, “Hyperbolic polynomials, interlacers, and sums of squares”, *Math. Program. B* **153**:1 (2015), 223–245. MR Zbl
- [Plaumann and Vinzant 2013] D. Plaumann and C. Vinzant, “Determinantal representations of hyperbolic plane curves: an elementary approach”, *J. Symbolic Comput.* **57** (2013), 48–60. MR Zbl
- [Powers and Wörmann 1998] V. Powers and T. Wörmann, “An algorithm for sums of squares of real polynomials”, *J. Pure Appl. Algebra* **127**:1 (1998), 99–104. MR Zbl

[Vinnikov 2012] V. Vinnikov, “LMI representations of convex semialgebraic sets and determinantal representations of algebraic hypersurfaces: past, present, and future”, pp. 325–349 in *Mathematical methods in systems, optimization, and control*, edited by H. Dym et al., Operator Theory: Advances and Applications **222**, Springer, 2012. MR Zbl

Received August 28, 2018. Revised May 30, 2019.

MARIO KUMMER  
INSTITUT FÜR MATHEMATIK  
TECHNISCHE UNIVERSITÄT BERLIN  
BERLIN  
GERMANY  
kummer@tu-berlin.de

SIMONE NALDI  
XLIM  
UNIVERSITÉ DE LIMOGES  
LIMOGES  
FRANCE  
simone.naldi@unilim.fr

DANIEL PLAUMANN  
FAKULTÄT FÜR MATHEMATIK  
TECHNISCHE UNIVERSITÄT DORTMUND  
DORTMUND  
GERMANY  
daniel.plaumann@math.tu-dortmund.de





# PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

[msp.org/pjm](http://msp.org/pjm)

## EDITORS

Don Blasius (Managing Editor)  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[blasius@math.ucla.edu](mailto:blasius@math.ucla.edu)

Matthias Aschenbrenner  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[matthias@math.ucla.edu](mailto:matthias@math.ucla.edu)

Daryl Cooper  
Department of Mathematics  
University of California  
Santa Barbara, CA 93106-3080  
[cooper@math.ucsb.edu](mailto:cooper@math.ucsb.edu)

Jiang-Hua Lu  
Department of Mathematics  
The University of Hong Kong  
Pokfulam Rd., Hong Kong  
[jhlu@maths.hku.hk](mailto:jhlu@maths.hku.hk)

Paul Balmer  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[balmer@math.ucla.edu](mailto:balmer@math.ucla.edu)

Wee Teck Gan  
Mathematics Department  
National University of Singapore  
Singapore 119076  
[matgwt@nus.edu.sg](mailto:matgwt@nus.edu.sg)

Sorin Popa  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[popa@math.ucla.edu](mailto:popa@math.ucla.edu)

Paul Yang  
Department of Mathematics  
Princeton University  
Princeton NJ 08544-1000  
[yang@math.princeton.edu](mailto:yang@math.princeton.edu)

Vyjayanthi Chari  
Department of Mathematics  
University of California  
Riverside, CA 92521-0135  
[chari@math.ucr.edu](mailto:chari@math.ucr.edu)

Kefeng Liu  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[liu@math.ucla.edu](mailto:liu@math.ucla.edu)

Jie Qing  
Department of Mathematics  
University of California  
Santa Cruz, CA 95064  
[qing@cats.ucsc.edu](mailto:qing@cats.ucsc.edu)

## PRODUCTION

Silvio Levy, Scientific Editor, [production@msp.org](mailto:production@msp.org)

## SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI  
CALIFORNIA INST. OF TECHNOLOGY  
INST. DE MATEMÁTICA PURA E APLICADA  
KEIO UNIVERSITY  
MATH. SCIENCES RESEARCH INSTITUTE  
NEW MEXICO STATE UNIV.  
OREGON STATE UNIV.

STANFORD UNIVERSITY  
UNIV. OF BRITISH COLUMBIA  
UNIV. OF CALIFORNIA, BERKELEY  
UNIV. OF CALIFORNIA, DAVIS  
UNIV. OF CALIFORNIA, LOS ANGELES  
UNIV. OF CALIFORNIA, RIVERSIDE  
UNIV. OF CALIFORNIA, SAN DIEGO  
UNIV. OF CALIF., SANTA BARBARA

UNIV. OF CALIF., SANTA CRUZ  
UNIV. OF MONTANA  
UNIV. OF OREGON  
UNIV. OF SOUTHERN CALIFORNIA  
UNIV. OF UTAH  
UNIV. OF WASHINGTON  
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

---

See inside back cover or [msp.org/pjm](http://msp.org/pjm) for submission instructions.

---

The subscription price for 2019 is US \$490/year for the electronic version, and \$665/year for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).


---

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

---

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2019 Mathematical Sciences Publishers

# PACIFIC JOURNAL OF MATHEMATICS

Volume 303    No. 1    November 2019

---

Contrasting various notions of convergence in geometric analysis	1
BRIAN ALLEN and CHRISTINA SORMANI	
Explicit formulae and discrepancy estimates for $a$ -points of the Riemann zeta-function	47
SIEGFRED BALUYOT and STEVEN M. GONEK	
Diffeological vector spaces	73
J. DANIEL CHRISTENSEN and ENXIN WU	
Degree-one, monotone self-maps of the Pontryagin surface are near-homeomorphisms	93
ROBERT J. DAVERMAN and THOMAS L. THICKSTUN	
Denoetherianizing Cohen–Macaulay rings	133
LÁSZLÓ FUCHS and BRUCE OLBERDING	
Ordinary points mod $p$ of $GL_n(\mathbb{R})$ -locally symmetric spaces	165
MARK GORESKY and YUNG SHENG TAI	
Real structures on polarized Dieudonné modules	217
MARK GORESKY and YUNG SHENG TAI	
Spectrahedral representations of plane hyperbolic curves	243
MARIO KUMMER, SIMONE NALDI and DANIEL PLAUMANN	
Deformations of linear Lie brackets	265
PIER PAOLO LA PASTINA and LUCA VITAGLIANO	
A mod- $p$ Artin–Tate conjecture, and generalizing the Herbrand–Ribet theorem	299
DIPENDRA PRASAD	
Transitive topological Markov chains of given entropy and period with or without measure of maximal entropy	317
SYLVIE RUETTE	
Restricted sum formula for finite and symmetric multiple zeta values	325
HIDEKI MURAHARA and SHINGO SAITO	
Frobenius–Schur indicators for near-group and Haagerup–Izumi fusion categories	337
HENRY TUCKER	
Compactness theorems for 4-dimensional gradient Ricci solitons	361
YONGJIA ZHANG	



0030-8730(201911)303:1;1-J