Pacific Journal of Mathematics

FROBENIUS–SCHUR INDICATORS FOR NEAR-GROUP AND HAAGERUP–IZUMI FUSION CATEGORIES

HENRY TUCKER

Volume 303 No. 1 November 2019

PACIFIC JOURNAL OF MATHEMATICS
Vol. 303, No. 1, 2019
dx.doi.org/10.2140/pjm.2019.303.337

FROBENIUS–SCHUR INDICATORS FOR NEAR-GROUP AND HAAGERUP–IZUMI FUSION CATEGORIES

HENRY TUCKER

Dedicated to Susan Montgomery

Ng and Schauenburg generalized higher Frobenius–Schur indicators to pivotal fusion categories and showed that these indicators may be computed utilizing the modular data of the Drinfel'd center of the given category. We consider two classes of fusion categories generated by a single noninvertible simple object: near groups, those fusion categories with one noninvertible object, and Haagerup–Izumi categories, those with one noninvertible object for every invertible object. Examples of both types arise as representations of finite or quantum groups or as Jones standard invariants of finite-depth Murray–von Neumann subfactors. We utilize the computations of the tube algebras due to Izumi and to Evans and Gannon to obtain formulae for the Frobenius–Schur indicators of objects in both of these families.

1. Introduction

Fusion categories appear in a wide variety of mathematics and physics. Their objects have the properties of complex representations of finite groups; in particular, they are semisimple and have duals and tensor products. Important examples of fusion categories come from the representations of Drinfel'd–Jimbo quantum groups and Jones standard invariants of Murray–von Neumann subfactors. From the point of view of these examples fusion categories encode symmetry data in the quantum setting in the same way that finite groups do in the classical setting. Classification problems for these categories do not come without considerable difficulty; therefore, it is of great interest to find and understand categorical invariants.

The classical Frobenius–Schur indicator for finite groups was introduced in 1906. It determines if and how a given group representation is self-dual. This was generalized to the setting of semisimple Hopf algebras by Linchenko and Montgomery [2000] and further to the setting of quasi-Hopf algebras [Mason and

MSC2010: primary 18D10, 16T05; secondary 46L37.
Keywords: tensor category, fusion rules, Frobenius–Schur indicator, Drinfel'd center, modular data, Haagerup subfactor, Hopf algebras.

Ng 2005; Ng and Schauenburg 2008] and to pivotal tensor categories [Ng and Schauenburg 2007b].

The FS indicators are a *complete invariant* for the Tambara–Yamagami categories [Basak and Johnson 2015]. These are the fusion categories having exactly one noninvertible simple object ρ where $\mathrm{Hom}_{\mathcal{C}}(\rho \otimes \rho, \rho) = 0$. In the present paper we consider the *near-group categories*: those with exactly one noninvertible simple object ρ where $\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathcal{C}}(\rho \otimes \rho, \rho)) = m$. (The Tambara–Yamagami categories are near groups with $m = 0$.) We provide the required background on this in Section 2.

Letting G be the group of invertible objects in our near-group category, we find in Section 3 that for the near-group categories with $m = |G| - 1$ the indicators are a complete invariant:

Corollary 3.3. *The near-group categories with $m = |G| - 1$ are completely distinguished by their Frobenius–Schur indicators.*

To make the computations here we utilize [Ng and Schauenburg 2007a, Theorem 4.1]: the Frobenius–Schur indicators of a spherical fusion category can be computed using the *ribbon structure* of the Drinfel'd center of the category. A complete list of near-group fusion categories in the case where $m = |G| \leq 13$ was found in [Evans and Gannon 2014]. In each of these examples the modular data for the Drinfel'd centers are given by quadratic forms. From this we get in Section 4:

Theorem 4.8. *In all known near-group categories with $m = |G|$ the noninvertible object has Frobenius–Schur indicators given by quadratic Gauss sums.*

This theorem provides new evidence for [Evans and Gannon 2014, Conjecture 2]: the *modular data* (matrix invariants from the braiding) of the centers of these near groups are always given by quadratic forms. The form of the indicators strongly suggests that these centers are formed from some "crossed product" construction for modular categories. See also the "pasting" of modular data developed in [Evans and Gannon 2011].

Finally, in Section 5, we observe a similar result which supports a similar conjecture for the *Haagerup–Izumi categories*, which are a related family of singly generated fusion categories having one noninvertible object for *each* invertible object:

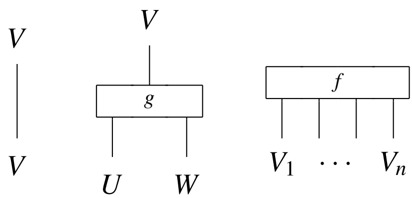
Theorem 5.4. *All known Haagerup–Izumi categories have Frobenius–Schur indicators given by quadratic Gauss sums.*

2. Categorical invariants

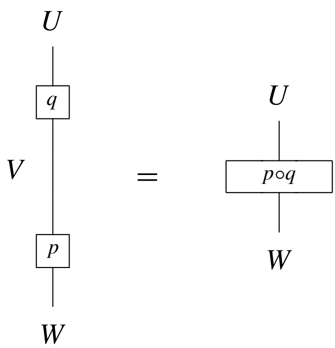
Tensor categories are abelian monoidal categories $(\mathcal{C}, \otimes, \mathbb{1})$ enriched over complex vector spaces; see [Etingof et al. 2015] or [Bakalov and Kirillov 2001] for the specifics of these definitions. The Mac Lane strictness theorem allows us the working assumption of *strictness*: the associativity natural isomorphism is the identity morphism for every triple of objects. Thus, we may use diagrammatic notation for the morphisms in these categories. Our notation is read from top to bottom, tensor products are given by side-by-side concatenation, and $\mathbb{1}$ is not written at all. For examples, the morphisms

$$\mathrm{id}_V : V \to V, \qquad g : V \to U \otimes W, \qquad f : \mathbb{1} \to V_1 \otimes \cdots \otimes V_n$$

are rendered in diagrammatic notation, respectively, as



Composition of morphisms is given by stacking; for example, given morphisms $p : V \to W$ and $q : U \to V$, their composition is given by



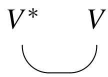
Categorifications of semisimple rings. Tensor categories should be thought of as a *categorification* of the notion of a unital algebra. The abelian and monoidal categorical structures are analogues of addition and multiplication, respectively. This point of view asks an obvious question:

What are the tensor categories that categorify a given ring?

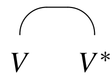
This question has produced several different interesting classification results for *semisimple* tensor categories: those where every object is a direct sum of some irreducible objects [Tambara and Yamagami 1998; Izumi 2001; Evans and Gannon 2014; 2017]. The set of (isomorphism classes) of the irreducible objects is denoted $\mathcal{I}rr(\mathcal{C})$.

Here we consider *fusion categories*: these are semisimple tensor categories $(\mathcal{C}, \otimes, \mathbb{1})$ that are additionally:

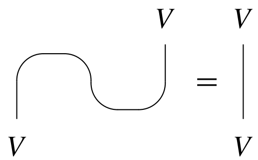
- *finitely* semisimple: $|\mathcal{I}rr(\mathcal{C})| < \infty$ and $\mathbb{1} \in \mathcal{I}rr(\mathcal{C})$.
- *rigid*: objects $V \in \mathcal{C}$ have duals $V^* \in \mathcal{C}$ with corresponding maps $\mathrm{ev}_V : V^* \otimes V \to \mathbb{1}$ and $\mathrm{db}_V : \mathbb{1} \to V \otimes V^*$. These are given, respectively, by the diagrams



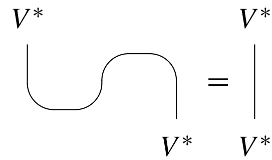
and



satisfying the relations



and



These two requirements are meant to make the objects of $\mathcal{C}$ behave like group representations. Indeed, the tensor category $\mathrm{Rep}(G)$ of complex representations of a finite group G is the prototypical example of a fusion category: Maschke's theorem gives finite semisimplicity and the contragredient representation gives rigidity.

Now we make precise the notion of categorification. The *Grothendieck ring* $K_0(\mathcal{C})$ of a fusion category $\mathcal{C}$ is the $\mathbb{Z}$-based ring with basis $\mathcal{I}rr(\mathcal{C})$, multiplication given by the tensor product in $\mathcal{C}$, and addition given by the direct sum in $\mathcal{C}$; that is, $K_0(\mathcal{C})$ captures the ring structure of the category and forgets the morphisms. In the example of $\mathrm{Rep}(G)$ it is the character ring $R(G)$. We say that $\mathcal{C}$ *categorifies* a ring K if $K_0(\mathcal{C}) = K$.

The simplest class of based rings to consider are the group rings $\mathbb{Z}G$, which are categorified precisely by the *pointed* fusion categories. These are the categories Vec_G^{ω} of G-graded vector spaces where the associativity morphism for the tensor product of three irreducible objects is given by a 3-cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$. These categories are classified to equivalence by the cohomology class of $[\omega] \in H^3(G, \mathbb{C}^\times)$. These facts are due to Mac Lane [Etingof et al. 2015, Proposition 4.10.3].

Here we will consider another level of complication, based rings with (only) one noninvertible object:

Definition 2.1. Let G be a finite group. A fusion category $\mathcal{C}$ is a *near group* if its Grothendieck ring is given by

$$K_0(\mathcal{C}) = \mathrm{NG}(G, m) := \mathbb{Z}[G \cup \{\rho\}]$$

where multiplication is given by the group law and, where $g \in G$,

$$\rho g = \rho = g\rho \quad \text{and} \quad \rho^2 = m\rho + \sum_{h \in G} h.$$

Remark 2.2 [Evans and Gannon 2014, Theorem 2(a)]. When G is abelian the multiplicity m is restricted to the values

- $m = |G| - 1$ or
- $m = k|G|$ for some $k \in \mathbb{N}$.

Consider the following important examples.

(1) The representation categories for the dihedral group of order 8 and the quaternion group of order 8 both categorify $\mathrm{NG}(\mathbb{Z}/(2) \times \mathbb{Z}/(2), 0)$. These are examples of *Tambara–Yamagami* categories, which are the near groups with $m = 0$ [Tambara and Yamagami 1998].

(2) $\mathrm{Rep}(S_3)$ and $\mathrm{Rep}(A_4)$ categorify $\mathrm{NG}(\mathbb{Z}/(2), 1)$ and $\mathrm{NG}(\mathbb{Z}/(3), 2)$, respectively. These are examples where $m = |G| - 1$.

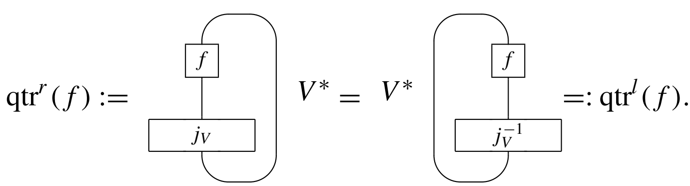
(3) The principal even sectors of the D_5 Murray–von Neumann subfactor also categorify $\mathrm{NG}(\mathbb{Z}/(2), 1)$.

(4) $\mathrm{Rep}(\mathrm{AGL}_1(\mathbb{F}_q))$ categorifies $\mathrm{NG}(\mathbb{Z}/(q-1), q-2)$.

(5) The principal even sectors of the A_4, E_6, and Izumi–Xu subfactors categorify $\mathrm{NG}(\mathbb{Z}/(1), 1)$, $\mathrm{NG}(\mathbb{Z}/(2), 2)$, and $\mathrm{NG}(\mathbb{Z}/(3), 3)$, respectively.

Frobenius–Schur indicators. It is known that the Grothendieck ring and the associativity natural isomorphism completely determine a fusion category up to monoidal equivalence [Etingof et al. 2015, §§4.9–4.10]. The associativity data is encoded by the *6j symbols*, which are the matrix components of the linear maps induced by the associativity natural isomorphism on triples of simple objects. Directly classifying all 6j symbols having a given Grothendieck ring is difficult in general as it requires finding solutions to large systems of nonlinear equations.

The near groups are *spherical* fusion categories. These are the fusion categories equipped with a natural isomorphism of monoidal functors $j : \mathrm{Id}_{\mathcal{C}} \xrightarrow{\sim} (\,\cdot\,)^{**}$ (that is, a *pivotal structure*) whose associated left and right *quantum* (*or categorical*) *trace* functions agree for all objects $V \in \mathcal{Irr}(\mathcal{C})$ and morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(V, V)$:



Note that these are complex numbers since we take V to be a simple object. Since the traces agree we are able to define a *quantum* (*or categorical*) *dimension* of objects $V \in \mathcal{C}$ by

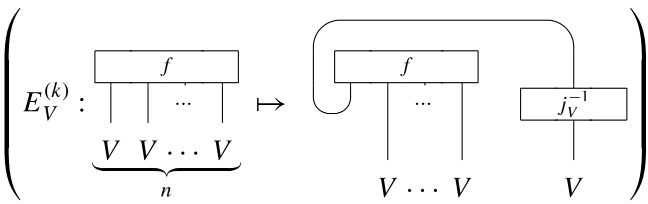
$$\mathrm{qdim}(V) := \mathrm{qtr}(\mathrm{id}_V) = \mathrm{qtr}^r(\mathrm{id}_V) = \mathrm{qtr}^l(\mathrm{id}_V).$$

The name *spherical* is motivated by imagining the strings in the morphism diagrams to be inhabiting a *sphere* rather than a plane — this allows for the strands on either side of the quantum traces to rotate around, giving the equality pictured above in our planar diagrams.

For $\mathcal{C}$ a pivotal fusion category we can define a finer categorical invariant than the Grothendieck ring:

Definition 2.3 [Ng and Schauenburg 2007b]. For $V \in \mathcal{C}$ we define the *k-th Frobenius–Schur indicator* by the linear trace

$\nu_k(V) = \mathrm{Tr}$



with $E_V^{(k)}$ a linear endomorphism of finite-dimensional vector space $\mathrm{Hom}(\mathbb{1}, V^{\otimes n})$ taking $V^{\otimes n}$ to be n-fold tensor product of V with all parentheses to the right.

The Tambara–Yamagami categories are an example of a fusion category family where the Frobenius–Schur indicators are a finer invariant than the Grothendieck ring [Ng and Schauenburg 2008]. In [Basak and Johnson 2015] it was shown that the indicators are a *complete* invariant for the Tambara–Yamagami categories. That is, the monoidal equivalence classes of fusion categories associated to the ring $\mathrm{NG}(G, 0)$ are completely distinguished by their Frobenius–Schur indicators. We give this property a name:

Definition 2.4. A ring K exhibits *FS indicator rigidity* if its categorifications can all be distinguished by their Frobenius–Schur indicators.

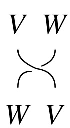
The central question that motivates the present article is immediate:

What rings have FS indicator rigidity?

We will see in Corollary 3.3 that the near-group rings $\mathrm{NG}(G, |G| - 1)$ exhibit this property.

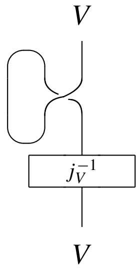
Drinfel'd centers and modular data. The Drinfel'd center $\mathcal{Z}(\mathcal{C})$ of a spherical fusion category $\mathcal{C}$ is *modular* [Müger 2003, Proposition 5.10]; that is, it is again spherical with a *nondegenerate braiding* $c_{V,W} : V \otimes W \to V \otimes W$ which is given in diagrams by

$c_{V,W} =$



Combined with the spherical structure the braiding also endows the Drinfel'd center with a *ribbon structure*, that is, a natural isomorphism θ of the identity functor (satisfying some coherence axioms) given by

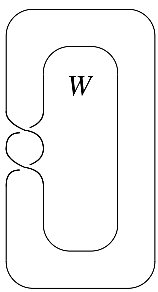
$\theta_V =$



Note that this is a scalar when V is a simple object.

Modular categories come with a projective representation of the modular group called *modular data*. The representation is defined by sending the generators $\mathfrak{s}, \mathfrak{t} \in SL_2(\mathbb{Z})$ to the *S- and T-matrices*

$S = \Big($



$\Big)_{V,W \in \mathcal{I}rr(\mathcal{C})}$ and $T = \mathrm{Diag}(\theta_V)_{V \in \mathcal{I}rr(\mathcal{C})}$.

Most crucially, we can obtain the Frobenius–Schur indicators from the modular data of the Drinfel'd center:

Theorem 2.5 [Ng and Schauenburg 2007a, Theorem 4.1]. *Let $\mathcal{C}$ be a spherical fusion category, and let $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ be the forgetful functor. Then*

$$\nu_k(X) = \frac{1}{\mathrm{qdim}(\mathcal{C})} \sum_{V \in \mathcal{I}rr(\mathcal{Z}(\mathcal{C}))} \theta_V^k \,\mathrm{qdim}(V) \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathcal{C}}(F(V), X))$$

where θ_V are the entries of the T-matrix for $\mathcal{Z}(\mathcal{C})$.

Izumi's classification program. The classification parameters for the Tambara–Yamagami categories were obtained by direct solution of the equations resulting from the pentagon axiom for the associativity. This method is not feasible for more complicated categories.

Masaki Izumi was able to extend this classification by using a fundamental result due to Popa: every *unitary* (or C^*) fusion category tensor-generated by one object can be embedded in the category of sectors of the hyperfinite type-III

Murray–von Neumann subfactor R. Sectors are unitary equivalence classes of endomorphisms of R; the tensor product of sectors is composition. Note that the 6j symbols can be obtained from Izumi's classification data; see [Suzuki and Wakui 2002] for the near-group category $\mathcal{C}$ with $K_0(\mathcal{C}) = \mathrm{NG}(\mathbb{Z}/(3), 3)$ coming from the E_6 subfactor.

Izumi [2001] and Evans and Gannon [2014; 2017] have obtained the classification parameters for the near-group families used in the sequel via this program; hence, our fusion categories will be unitary. In particular, this means that a *canonical spherical structure* can be chosen such that the *quantum dimension and the Frobenius–Perron dimension agree*.

3. Frobenius–Schur indicators for near groups with $m = |G| - 1$

Let $\mathcal{C}$ be a fusion category such that $K_0(\mathcal{C}) = \mathrm{NG}(G, |G|-1)$. It is shown in [Evans and Gannon 2014, Proposition 2] that such a fusion category can only exist if $G \cong \mathbb{F}^{\times}_{|G|+1}$ is the multiplication group of a finite field. (So G is cyclic, and thus, $H^2(G, \mathbb{T}) = 1$.) Let $p = \mathrm{char}(\mathbb{F}_{|G|+1})$.

Consider again the category $\mathrm{Rep}(\mathrm{AGL}_1(\mathbb{F}_{|G|+1}))$. These provide the main examples of $m = |G| - 1$ near groups. In fact, by [Etingof et al. 2004, Corollary 7.4; Evans and Gannon 2014, Proposition 5], these are the *only* fusion categories with this Grothendieck ring unless $|G| = 1, 2, 3, 7$.

Indicators for $\mathcal{C} \simeq$ Rep(AGL$_1$($\mathbb{F}_q$)). We may use classical methods to determine the indicators for $\mathcal{C}$ that is tensor equivalent to the category of representations of an affine general linear group of degree 1 over the finite field $\mathbb{F}_q$. Recall that $\theta_k^G(h) = |\{g \in G \mid g^k = h\}|$.

Proposition 3.1. *Suppose $\mathcal{C}$ is such that $K_0(\mathcal{C}) = \mathrm{NG}(G, |G| - 1)$ and $|G| \neq 1, 2, 3, 7$. Then $\mathcal{C} \simeq_{\otimes} \mathrm{Rep}(\mathrm{AGL}_1(\mathbb{F}_{|G|+1}))$ and*

$$\nu_k(\rho) = \theta_k^G(e) - 1 + \delta_{\lfloor \frac{k}{p} \rfloor, \frac{k}{p}}.$$

Proof. Let $|G| + 1 = q$. Since $\mathrm{AGL}_1(\mathbb{F}_q) \cong \mathbb{F}_q^+ \rtimes GL_1(\mathbb{F}_q) \cong \mathbb{F}_q^+ \rtimes \mathbb{F}_q^{\times}$ we may use Serre's method of little groups [1977, §8.2, Proposition 25] to see that the character ρ for the irreducible representation with degree > 1 is given by

$$\rho(a, b) = \frac{\delta_{1,b}}{q} \sum_{(x,y) \in \mathrm{AGL}_1(\mathbb{F}_q)} \eta(y^{-1}a)$$

for any *nontrivial* linear character $\eta \in \widehat{\mathbb{F}_q^+}$.

Now we may apply the classical formula for $\nu_k(\rho)$ [Isaacs 1976, Lemma 4.4]:

$$\begin{aligned}\nu_k(\rho) &= \frac{1}{q(q-1)} \sum_{(a,b)\in\mathbb{F}_q\rtimes\mathbb{F}_q^\times} \rho((a,b)^k) \\ &= \frac{1}{q(q-1)} \sum_{(a,b),\, b^k=1} \rho((1+b+b^2+\cdots+b^{k-1})a,1) \\ &= \frac{1}{q(q-1)}\Bigg(\sum_{(a,b),\, b^k=1,\, b\neq 1} \rho(0,1) + \sum_{n\in\mathbb{F}_q} \rho(kn,1)\Bigg).\end{aligned}$$

Since ρ is a degree-$(q-1)$ character, the left-hand sum in the last expression above is equal to $q(q-1)(\theta_k^{\mathbb{F}_q^\times}(1)-1)$. The right-hand sum in the same expression is equal to

$$\sum_{n\in\mathbb{F}_q}\sum_{b\in\mathbb{F}_q^\times} \eta(b^{-1}kn) = \begin{cases} q(q-1) & \text{if } p \mid k, \\ 0 & \text{if } p\nmid k.\end{cases}$$

The $p \mid k$ case is clear since then $\eta(b^{-1}kn)$ is identically 1. On the other hand, $\eta(b^{-1}kn) = b\cdot\eta(kn)$ under the transpose of the left regular action of $\mathbb{F}_q^\times \cong GL_1(\mathbb{F}_q)$ on $\widehat{\mathbb{F}_q} \cong \mathbb{F}_q$. Since $(p,k)=1$ we have that

$$\sum_{n\in\mathbb{F}_q} b\cdot\eta(kn) = \sum_{n\in\mathbb{F}_q} b\cdot\eta(n),$$

and since the action is faithful by definition, we know that $b\cdot\eta$ is not the trivial representation for any $b\in\mathbb{F}_q^\times$. Hence, by orthogonality of characters the sum is 0. The formula is now clear since the given Kronecker delta is 1 if $p \mid k$ and is 0 otherwise. □

Indicators in general from modular data of $\mathcal{Z}(\mathcal{C})$. For $|G| = 1, 3, 7$ there is 1 additional monoidal equivalence class, and for $|G| = 2$ there are 2 additional monoidal equivalence classes. The modular data for Drinfel'd centers of unitary $m = |G|-1$ near groups was computed in [Evans and Gannon 2014, Theorem 5]. We will appeal to Theorem 2.5 to compute the indicators for a general unitary $m = |G|-1$ near group.

Let $\epsilon\in\widehat{G}$ be the trivial character, and let $\mathbb{F}_{|G|+1}^+$ be the additive group of the finite field. Excluding the case where $|G|=7$ and $s=-1$ we have the following data for the center $\mathcal{Z}(\mathcal{C})$:

$X \in \mathcal{I}rr(\mathcal{Z}(\mathcal{C}))$		$F(X)$	$c_{X,\cdot}$ given by	θ_X
A_g	$(g \in G)$	g	1	1
Σ		$\bigoplus_{x\in G} x$	1	1
B_g^ω	$(g \in G)$	$\rho + g$	$\omega \in \widehat{G}\setminus\{\epsilon\}$	$\overline{\omega(g)}$
C^ψ	$(\psi \in \widehat{\mathbb{F}^+_{\lvert G\rvert+1}})$	ρ	$\psi \in \widehat{\mathbb{F}^+_{\lvert G\rvert+1}}$	$\overline{\zeta_1\psi(1)}$

where the half-braiding for C^ψ on occurrences of ρ in objects of $\mathcal{Z}(\mathcal{C})$ is a morphism

$$e_{C^\psi}(\rho) \in \mathrm{Hom}_{\mathcal{C}}(\rho\otimes\rho, \rho\otimes\rho) \cong \mathbb{C}^{|G|} \oplus M_m(\mathbb{C})$$

given by

$$e_{C^\psi}(\rho) = \zeta_1\psi(1)\left(\bigoplus_{k\in G}(-1)^{mk}\,\mathrm{Id}_k\right) \oplus [\zeta_\gamma(\psi\circ\sigma)(\gamma)\delta_{\sigma^2(\gamma)^*,\mu}\,\mathrm{Id}_\rho]_{\gamma,\mu}.$$

For the case where $|G| = 7$ and $s = -1$ we have:

$X \in \mathcal{I}rr(\mathcal{Z}(\mathcal{C}))$		$F(X)$	$c_{X,\cdot}$ given by	θ_X
A_g	$(g \in G)$	g	1	1
Σ		$\bigoplus_{x\in G} x$	1	1
B_g^ω	$(g \in G,\ \omega \in \widehat{G}\setminus\{\epsilon\})$	$\rho + g$	$\omega \in \widehat{G}\setminus\{\epsilon\}$	$\overline{\omega(g)}$
E_1		$\rho+\rho$	1	i
E_2		$\rho+\rho$	1	$-i$

With the preceding data in hand we may now apply Theorem 2.5 to see:

Theorem 3.2. *Suppose that $\mathcal{C}$ is a unitary fusion category such that $K_0(\mathcal{C}) =$ $\mathrm{NG}(G, |G|-1)$. Then the indicators for the noninvertible object ρ are given by*:

(1) *If $|G| \neq 7$ or $s = 1$, then*

$$\nu_k(\rho) = (\theta_k^G(e) - 1) + \overline{\zeta_1}^k \delta_{\lfloor \frac{k}{p}\rfloor, \frac{k}{p}}.$$

(2) *If $|G| = 7$ and $s = -1$, then*

$$\nu_k(\rho) = (\theta_k^G(e) - 1) + (-1)^{k/2} \delta_{\lfloor \frac{k}{2}\rfloor, \frac{k}{2}}.$$

Proof. (1) Suppose $|G| \neq 7$ or $s = 1$. Then we have

$$\begin{aligned}\nu_k(\rho) &= \frac{1}{\mathrm{qdim}(\mathcal{C})}\left(\sum_{\substack{g\in G\\ \omega\in\widehat{G}\setminus\{\epsilon\}}} \theta^k_{B_g^\omega}\,\mathrm{qdim}(B_g^\omega) + \sum_{\psi\in\widehat{\mathbb{F}^+_{|G|+1}}} \theta^k_{C^\theta}\,\mathrm{qdim}(C^\psi)\right)\\ &= \frac{1}{\mathrm{qdim}(\mathcal{C})}\left((|G|+1)\sum_{\substack{g\in G\\ \omega\in\widehat{G}\setminus\{\epsilon\}}} \overline{\omega(g)}^k + |G|\overline{\zeta_1}^k \sum_{\psi\in\widehat{\mathbb{F}^+_{|G|+1}}} \overline{\psi(1)}^k\right).\end{aligned}$$

Consider the first summand. Since G is abelian we may choose an isomorphism $h \mapsto \chi_h$ from $G \to \widehat{G}$. Then we have

$$\begin{aligned}\sum_{\substack{g\in G\\ \omega\in\widehat{G}\setminus\{\epsilon\}}} \overline{\omega(g)}^k &= \Bigl(\sum_{g\in G}\sum_{\omega\in\widehat{G}} \overline{\omega(g)}^k\Bigr) - \Bigl(\sum_{g\in G} \overline{\epsilon(g)}^k\Bigr)\\ &= \Bigl(\sum_{g\in G}\sum_{h\in G} \overline{\chi_h(g)}^k\Bigr) - |G|\\ &= \Bigl(|G|\sum_{h\in G} \overline{\nu_k(\chi_h)}\Bigr) - |G|\\ &= |G|(\theta_k^G(e)-1).\end{aligned}$$

Consider the second summand. Since $\mathbb{F}_{n+1}^+$ is the additive group of a finite field we have that $n+1 = p^l$ for some prime p and positive integer l and that $\mathbb{F}_{n+1}^+ \cong (\mathbb{Z}_p)^l$ as groups. Under this identification the multiplicative unit $1 \in \mathbb{F}_{n+1}^+$ is a direct sum of generators of the copies of $\mathbb{Z}_p$:

$$\begin{aligned}\sum_{\psi\in\widehat{\mathbb{F}_{|G|+1}^+}} \overline{\psi(1)}^k = \sum_{\psi\in\widehat{\mathbb{F}_{|G|+1}^+}} \overline{\psi(k1)} &= \begin{cases} 0 & \text{if } k1 \neq 0,\\ p^l & \text{if } k1 = 0\end{cases}\\ &= \begin{cases} 0 & \text{if } p \nmid k,\\ |G|+1 & \text{if } p \mid k\end{cases}\\ &= (|G|+1)\delta_{\lfloor \frac{k}{p}\rfloor, \frac{k}{p}}.\end{aligned}$$

(2) Now suppose that $|G| = 7$ and $s = -1$. Then

$$\begin{aligned}\nu_k(\rho) &= \frac{1}{\mathrm{qdim}(\mathcal{C})}\Bigl(\sum_{\substack{g\in G\\ \omega\in\widehat{G}\setminus\{\epsilon\}}} \theta_{B_g^\omega}^k \,\mathrm{qdim}(B_g^\omega) + 2\sum_{i=1}^{2} \theta_{E_t}^k \,\mathrm{qdim}(E_t)\Bigr)\\ &= \theta_k^G(e) - 1 + \frac{4|G|i^k(1+(-1)^k)}{|G|+|G|^2}\\ &= \theta_k^G(e) - 1 + \frac{i^k(1+(-1)^k)}{2}\\ &= \theta_k^G(e) - 1 + (-1)^{k/2}\delta_{\lfloor \frac{k}{2}\rfloor, \frac{k}{2}}.\end{aligned}$$

□

Corollary 3.3. *The near-group fusion ring* NG$(G, |G|-1)$ *exhibits Frobenius–Schur indicator rigidity.*

Proof. The statement is vacuous in all but the cases where $|G| = 1, 2, 3, 7$. We shall consider them now.

If $|G|=1,3,7$, then there is one additional tensor equivalence class corresponding to $s=-1$. By [Evans and Gannon 2014, p. 41] if $|G|+1$ is even (i.e., a power of 2), then $\zeta_1^2=s$; hence, $\nu_2(\rho)=s$ in each of these three cases.

If $|G|=2$, then $s=1$ but instead $b=\mu$ where μ is some third root of unity. The two nontrivial possibilities for μ correspond to the two additional tensor equivalence classes for this type. By [Evans and Gannon 2014, p. 42] if $\mu=\exp\left(\pm\frac{2\pi i}{3}\right)$, then $\zeta_1=\exp\left(\mp\frac{2\pi i}{3}\right)$; hence, $\nu_3(\rho)=\mu$. □

4. Frobenius–Schur indicators for near groups with $m=|G|$

For the rest of this article the group operation in G will be written *additively*. This will be a more convenient notation for working with bilinear and quadratic forms.

Metric groups and the Fourier transform. Shimizu observed that the Fourier transform for finite groups appears when computing Frobenius–Shur indicators for fusion categories [Shimizu 2011]. A finite abelian group G is isomorphic to its linear dual $\widehat{G}$ via a nondegenerate symmetric *bicharacter* $\langle\cdot,\cdot\rangle$ with the identification

$$G\to\widehat{G},$$
$$g\mapsto\langle g,\cdot\rangle.$$

Symmetric bicharacters $\langle\cdot,\cdot\rangle : G\times G\to\mathbb{T}$ are in one-to-one correspondence with bilinear forms $\beta : G\times G\to\mathbb{Q}/\mathbb{Z}$ via the exponential

$$\langle g,h\rangle=e^{2\pi i\beta(g,h)}.$$

A *quadratic form* is a function $q : G\to\mathbb{Q}/\mathbb{Z}$ with $q(-g)=q(g)$ such that

$$\partial q(g,h):=q(g)+q(h)-q(gh)$$

is a symmetric bilinear form. A pair (G,q) is called a *premetric group*. If the bilinear form ∂q is nondegenerate, then it is called a *metric group*.

Remark 4.1. If $|G|$ is odd, then the correspondence between quadratic forms and bilinear forms given by ∂ is one-to-one. If $|G|$ is even, then the correspondence is $|G/2G|$-to-one.

Now, using the bicharacter $\langle\cdot,\cdot\rangle$ we can define the *Fourier transform* for complex function $f : G\to\mathbb{C}$ on finite abelian groups:

$$\hat{f}(g)=\frac{1}{\sqrt{|G|}}\sum_{h\in G}\overline{\langle g,h\rangle}f(h).$$

The Fourier transform of the exponent of a quadratic form q at the unit element of the group defines a very important invariant of premetric groups:

Definition 4.2. Let (G, q) be a premetric group. Then the Fourier transform of $e^{2\pi i q}$ at $0 \in G$ defines the *Gauss sum*:

$$\Theta(G, q) = \frac{1}{\sqrt{|G|}} \sum_{g \in G} e^{2\pi i q(g)}.$$

The Gauss sum is multiplicative over the (obviously defined) orthogonal direct product of metric groups:

$$\Theta(G \perp G', q + q') = \Theta(G, q)\Theta(G', q').$$

(This identity is for *metric* groups; hence, the quadratic forms must all be nondegenerate.)

Izumi's classification of $m = |G|$ near groups. Izumi completely classified unitary near-group fusion categories with $m = |G|$ and where $H^2(G, \mathbb{C}^\times) = 1$ in [Izumi 2000; 2001; 2017].

Theorem 4.3 [Izumi 2001, Theorem 5.3]. *Unitary fusion categories $\mathcal{C}$ such that $K_0(\mathcal{C}) = \mathrm{NG}(G, |G|)$ and $H^2(G, \mathbb{C}^\times) = 1$ are classified up to monoidal equivalence by the group G, a metric group structure $\langle \cdot, \cdot \rangle$ on G, and the following complex parameters*:

(1) $a : G \to \mathbb{T}$ *such that* $a(g) = e^{2\pi i q(g)}$ *for a quadratic form* q *with* $\langle g, h \rangle = e^{2\pi i \partial q(g,h)}$, *i.e.*,

$$a(0) = 1, \qquad a(g) = a(-g), \qquad \frac{a(g+h)}{a(g)a(h)} = \langle g, h \rangle.$$

(2) $b : G \to \mathbb{C}$ *and* $c \in \mathbb{T}$ *such that*

$$\Theta(G, q) = \hat{a}(0) = \frac{1}{c^3},$$
$$b(g) = \overline{a(g)b(-g)},$$
$$\hat{b}(0) = \frac{-c}{\mathrm{qdim}(\rho)}, \qquad \hat{b}(g) = c\overline{b(g)}, \qquad |\hat{b}(g)|^2 = \frac{1}{|G|} - \frac{\delta_{g,0}}{\mathrm{qdim}(X)},$$
$$\sum_{x \in G} b(x+g)b(x+h)\overline{b(x)} = \langle g, h \rangle b(g)b(h) - \frac{c}{\mathrm{qdim}(\rho)\sqrt{|G|}}.$$

Two such fusion categories $\mathcal{NG}(G_1, \langle \cdot, \cdot \rangle_1, a_1, b_1, c_1)$, $\mathcal{NG}(G_2, \langle \cdot, \cdot \rangle_2, a_2, b_2, c_2)$ *are monoidally equivalent if and only if*

$$c_1 = c_2$$

and there is an isomorphism of metric groups $\phi : (G_1, \langle \cdot, \cdot \rangle_1) \to (G_2, \langle \cdot, \cdot \rangle_2)$ *such that*

$$a_2 = a_1 \circ \phi \quad \textit{and} \quad b_2 = b_1 \circ \phi.$$

Remark 4.4. If the requirement that $H^2(G, \mathbb{C}^\times) = 1$ is relaxed, then a solution of Izumi's equations in Theorem 4.3 is sufficient to produce a near-group category with the required K_0 ring, but *not* necessary.

Indicators from modular data of $\mathcal{Z}(\mathcal{C})$. Izumi found the simple objects of $\mathcal{Z}(\mathcal{C})$ along with their twists and half-braidings in [Izumi 2001, Theorem 6.8], which is given as follows, where $<$ is a chosen order on G:

$X \in \mathcal{I}rr(\mathcal{Z}(\mathcal{C}))$		$F(X)$	$c_{X,\cdot}$ given by	θ_X
A_g	$(g \in G)$	g	1	$\langle g, g\rangle$
B_g	$(g \in G)$	$\rho + g$	1	$\langle g, g\rangle$
$C_{g,h}$	$(g < h \in G)$	$\rho + g + h$	1	$\langle g, h\rangle$
E_j	for $j = 1, \dots, \frac{1}{2}\lvert G\rvert(\lvert G\rvert + 3)$	ρ		ω_j

The $\omega_j \in \mu_\infty \subseteq \mathbb{T}$ are solutions to the system of equations (6.18)-(6.20) in [Izumi 2001, §6] parametrized by $g \in G$ with coefficients given by the complex values $a(g), b(g), c \in \mathbb{C}$.

Proposition 4.5. *Suppose $\mathcal{C}$ is unitary fusion category with Grothendieck ring $K_0(\mathcal{C}) = \mathrm{NG}(G, \lvert G\rvert)$ and noninvertible object ρ. Let q be a quadratic form such that $\langle g, h\rangle = e^{2\pi i \partial q(g,h)}$. Then the indicators for ρ are given by*

$$\nu_k(\rho) = \tfrac{1}{2}\theta_k^G(e) + \frac{\mathrm{qdim}(\rho)}{\mathrm{qdim}(\mathcal{C})}\left(\frac{\sqrt{\lvert G\rvert}}{2}\Theta(G, 2kq) + \sum_{j=1}^{\lvert G\rvert(\lvert G\rvert+3)/2} \omega_j^k\right).$$

Proof. Let $d_\rho := \mathrm{qdim}(\rho)$, and let $<$ be an arbitrary ordering on the finite group G. Again applying Theorem 2.5 we have

$$\begin{aligned}\nu_k(\rho) &= \frac{1}{\mathrm{qdim}(\mathcal{C})}\left((1+d_\rho)\sum_{g\in G}\theta_{B_g}^k + (2+d_\rho)\sum_{\substack{g,h\in G\\ g<h}}\theta_{C_{g,h}}^k + d_\rho \sum_{j=1}^{\lvert G\rvert(\lvert G\rvert+3)/2}\theta_{E_j}^k\right)\\ &= \frac{1}{\mathrm{qdim}(\mathcal{C})}\left(\frac{d_\rho}{2}\sum_{g\in G}\langle g, g\rangle^k + \frac{2+d_\rho}{2}\sum_{g,h\in G}\langle g, h\rangle^k + d_\rho\sum_j \omega_j^k\right)\end{aligned}$$

where the second equality is due to the symmetry of $\langle\cdot,\cdot\rangle$.

Now we consider the middle sum:

$$\sum_{g,h\in G}\langle g, h\rangle^k = \sum_{g\in G}\sum_{h\in G}\langle g, h^k\rangle = \lvert G\rvert\sum_{g\in G}\nu_k^{\mathrm{groups}}(\langle g, \cdot\rangle) = \lvert G\rvert\theta_k^G(e).$$

The second equality is by definition of the Frobenius–Schur indicator for finite groups (denoted ν_k^{groups}) [Isaacs 1976, (4.4)] and the third equality is by [Isaacs 1976, p. 49].

Now let q be a quadratic form such that $\langle g, h\rangle = e^{2\pi i \partial q(g,h)}$, and consider the first sum:

$$\sum_{g\in G}\langle g, g\rangle^k = \sum_{g\in G} e^{2\pi i(2kq(g))} = \sqrt{|G|}\Theta(G, 2kq).$$

Hence, the formula is now clear. ☐

Modular data for pointed modular categories. Recall that any *pointed* fusion category is equivalent to Vec_G^ω for some $[\omega] \in H^3(G, \mathbb{C}^\times)$. Now we consider pointed *modular* categories. Since modular categories are also fusion categories they will be equivalent as fusion categories to Vec_G^ω with G abelian. The braiding induces a quadratic form $c_{g,g} = e^{2\pi i q(g)}$, which gives G the structure of a metric group. Then these categories are classified under *braided* equivalence up to isomorphism of *premetric* groups. Note that in the case of *odd-order* groups if ω admits a braiding it will be unique; the notational convention $\mathrm{Vec}_G^{(\omega,c)}$ includes the braiding c. (This is because the Eilenburg–Mac Lane abelian cohomology $H^3_{\mathrm{ab}}(G, \mathbb{C}^\times)$ is isomorphic to the group of quadratic forms on G [Eilenberg and Mac Lane 1953; 1954]. See [Etingof et al. 2015, §8.4] for an outline of the proof in a more modern context.)

We now give the modular data for a pointed modular category. Define the bicharacter $\langle g, h\rangle_q := e^{2\pi i \partial q(g,h)}$. The modular data are given by the *Weil representation* associated to the premetric group (G, q):

$$S = S^q := \frac{1}{\sqrt{|G|}}(\langle g, h\rangle_q)_{g,h\in G}, \qquad T = T^q := (\delta_{g,h}e^{2\pi i q(g)})_{g,h\in G}.$$

Indicators when $|G|$ is odd. When $|G|$ is odd we have a one-to-one between quadratic forms and bilinear forms given by the map $q \mapsto \partial q$. Let q be the quadratic form on G such that $\langle g, h\rangle = e^{2\pi i \partial q(g,h)}$. Then we define

$$\mathcal{NG}(G, q, b, c) := \mathcal{NG}(G, \langle\cdot,\cdot\rangle_q, e^{2\pi i q}, b, c),$$

the corresponding near-group fusion category via the notation from Theorem 4.3.

Conjecture 4.6 [Evans and Gannon 2014, Conjecture 2]. *Suppose $|G|$ is odd. Then there exists a metric group (G', q') of order $|G|+4$ such that*:

(1) *Simple objects E_j in the subsection starting on page 350 are indexed by $g \in G$ and $x \in G' \setminus \{e\}$ where $E_{g,x} = E_{g,x^{-1}}$ and*

$$\theta_{E_{g,x}} = \langle g, g\rangle e^{2\pi i \partial q'(x)}.$$

(2) *The modular data are given by the Kronecker product of the Weil representation for (G, q) with another pair of modular data (S', T') for a rank $|G|+3$ modular category*:

$$S^{q,q'} := S^q \otimes S', \qquad T^{q,q'} := T^q \otimes T'$$

where we have

$$T' = \mathrm{Diag}(1, 1, \langle g, g\rangle_q, \langle x, x\rangle_{q'})_{g\in G, x\in G'}.$$

(See [Evans and Gannon 2014, Proposition 7] for the definition of S'.)

Remark 4.7. Evans and Gannon [2014] show that the conjecture is true for near groups with $|G| \le 13$ odd.

Theorem 4.8. *Suppose a unitary fusion category $\mathcal{C}$ with $K_0(\mathcal{C}) = \mathrm{NG}(G, |G|)$ and $|G|$ odd satisfies Conjecture 4.6. Then*

$$\nu_k(\rho) = \tfrac{1}{2}\theta_k^G(e) + \tfrac{1}{2}\Theta(G, 2kq)\Theta(G', 2kq').$$

Proof. Let $N = (|G'|-1)/2$, and enumerate G' as

$$G' = \{e, x_1, \dots, x_N, x_1^{-1}, \dots, x_N^{-1}\}.$$

Let $d_\rho := \mathrm{qdim}(\rho)$ and $\langle x, y\rangle_{q'} := e^{2\pi i \partial q(x,y)}$. Starting with Proposition 4.5 we have

$$\begin{aligned}
\nu_k(\rho) &= \tfrac{1}{2}\theta_k^G(e) + \frac{d_\rho}{\mathrm{qdim}(\mathcal{C})}\left(\frac{\sqrt{|G|}}{2}\Theta(G, 2kq) + \sum_{\substack{g\in G\\ 1\le i\le N}} \langle g, g\rangle_q^k \langle x_i, x_i\rangle_{q'}^k\right)\\
&= \tfrac{1}{2}\theta_k^G(e) + \frac{d_\rho\sqrt{|G|}\Theta(G, 2kq)}{|G|(2+d_\rho)}\left(\tfrac{1}{2} + \sum_{1\le i\le N} \langle x_i, x_i\rangle_{q'}^k\right)\\
&= \tfrac{1}{2}\theta_k^G(e) + \frac{d_\rho\sqrt{|G|}\Theta(G, 2kq)}{|G|(2+d_\rho)}\bigl(\tfrac{1}{2} + \tfrac{1}{2}(\Theta(G', 2kq')\sqrt{|G|+4} - 1)\bigr)\\
&= \tfrac{1}{2}\theta_k^G(e) + \frac{d_\rho\sqrt{|G|}\sqrt{|G|+4}}{2|G|(2+d_\rho)}\Theta(G, 2kq)\Theta(G, 2kq'),
\end{aligned}$$

and using the fact that $d_\rho^2 = |G| + |G|d_\rho$ we have

$$\frac{d_\rho\sqrt{|G|}\sqrt{|G|+4}}{2|G|(2+d_\rho)} = \frac{d_\rho(2d_\rho - |G|)}{2|G|(2+d+\rho)} = \frac{1}{2}$$

and then the formula for $\nu_k(\rho)$ is clear. □

As a corollary to the preceding theorem we obtain an easy and more natural proof of [Evans and Gannon 2014, Proposition 7(b)]:

Corollary 4.9. *The matrices $(S^{q,q'}, T^{q,q'})$ are modular data for a near-group center only if $\Theta(G, 2q)\Theta(G', 2q') = -1$.*

Proof. Suppose a near-group category $\mathcal{C} = \mathcal{NG}(G, \langle\cdot,\cdot\rangle_q, e^{2\pi i q}, b, c)$ has a center $\mathcal{Z}(\mathcal{C})$ with modular data given by $(S^{q,q'}, T^{q,q'})$. Since the simple object ρ cannot contain a copy of $\mathbb{1}$ we know that $\nu_1(\rho) = 0$. Therefore, by Theorem 4.8, we must have $\Theta(G, 2q)\Theta(G', 2q') = -1$. □

Let $\left(\frac{p}{q}\right)$ be the Jacobi symbol.

Corollary 4.10. *For* $|G|$ *odd and* k *such that* $\gcd(k, |G|\cdot|G'|) = 1$ *we have*

$$\nu_k(\rho) = \frac{1}{2}\left(1 - \left(\frac{k}{|G|\cdot|G'|}\right)\right).$$

Proof. Using the decomposition into irreducible metric groups given in [Wall 1963] it is easy to see that

$$\Theta(G, kq)\Theta(G', kq') = \left(\frac{k}{|G|\cdot|G'|}\right)\Theta(G, q)\Theta(G', q').$$

See also [Basak and Johnson 2015, §3 and Lemma 3.2]. □

Corollary 4.11. $\Theta(G', q') = -c^3$ *where* (G', q') *is the metric group associated to the center of* $\mathcal{NG}(G, q, b, c)$.

Proof. We've seen in Theorem 4.3 that $\Theta(G, q) = \frac{1}{c^3}$; hence, the above follows by the Corollary 4.9. □

The complete list of near-group categories with $K_0(\mathcal{C}) = \mathrm{NG}(G, |G|)$ for odd $|G| \leq 13$ was obtained in [Evans and Gannon 2014, Proposition 6] by finding solutions to Izumi's equations in Theorem 4.3. They also used Izumi's methods from [Izumi 2001, §6; 2017] to produce the modular data of their Drinfel'd centers; see [Evans and Gannon 2014, §§4.3–4.4 and Table 2]. Collected below are the modular data they found along with the Frobenius–Schur indicators of ρ for each of these categories. Since $|G|$ is odd, let q be the *unique* quadratic form associated to the bicharacter $\langle\,\cdot\,,\cdot\,\rangle$ from the classification parameters.

The data uses the following notation:

- *Column 1*. $\mathcal{C} = \mathcal{NG}(G, q, b, c))$ with $|G|$ odd as in the above notation. (For clearer presentation, the parameters b and c will be given only if they are necessary to establish in-equivalence.)
- *Column 2*. (G', q') is the metric group from the modular data of $\mathcal{Z}(\mathcal{C})$ from Conjecture 4.6. Recall $|G'| = |G| + 4$.
- $\zeta_k = \exp\left(\frac{2\pi i}{k}\right) \in \mathbb{T}$ primitive k-th root of unity.

$\|G\| = 3$	(G', q')	$\nu_3(\rho)$	$\nu_7(\rho)$
$\mathcal{NG}(\mathbb{Z}/(3), \frac{1}{3}g^2, \cdot, \cdot)$	$(\mathbb{Z}/(7), \frac{1}{7}g^2)$	$\frac{3+i\sqrt{3}}{2}$	$\frac{1+i\sqrt{7}}{2}$
$\mathcal{NG}(\mathbb{Z}/(3), -\frac{1}{3}g^2, \cdot, \cdot)$	$(\mathbb{Z}/(7), -\frac{1}{7}g^2)$	$\frac{3-i\sqrt{3}}{2}$	$\frac{1-i\sqrt{7}}{2}$

$\lvert G\rvert=5$	(G',q')	$\nu_3(\rho)$	$\nu_5(\rho)$	$\nu_9(\rho)$
$\mathcal{NG}(\mathbb{Z}/(5),\frac{2}{5}g^2,\cdot,\zeta_3)$	$(\mathbb{Z}/(9),\ \frac{2}{9}g^2)$	$1+\overline{\zeta_3}$	$\frac{5+\sqrt{5}}{2}$	-1
$\mathcal{NG}(\mathbb{Z}/(5),\frac{2}{5}g^2,\cdot,\overline{\zeta_3})$	$(\mathbb{Z}/(9),-\frac{2}{9}g^2)$	$1+\zeta_3$	$\frac{5+\sqrt{5}}{2}$	-1
$\mathcal{NG}(\mathbb{Z}/(5),\frac{1}{5}g^2,\cdot,1)$	$((\mathbb{Z}/(3))^2,\frac{1}{3}(g^2+h^2))$	-1	$\frac{5-\sqrt{5}}{2}$	2

$\lvert G\rvert=7$	(G',q')	$\nu_7(\rho)$	$\nu_{11}(\rho)$
$\mathcal{NG}(\mathbb{Z}/(7),\ \frac{1}{7}g^2,\cdot,\cdot)$	$(\mathbb{Z}/(11),-\frac{2}{11}g^2)$	$\frac{7-i\sqrt{7}}{2}$	$\frac{1+i\sqrt{11}}{2}$
$\mathcal{NG}(\mathbb{Z}/(7),-\frac{1}{7}g^2,\cdot,\cdot)$	$(\mathbb{Z}/(11),\ \frac{2}{11}g^2)$	$\frac{7+i\sqrt{7}}{2}$	$\frac{1-i\sqrt{11}}{2}$

$\lvert G\rvert=9$	(G',q')	$\nu_3(\rho)$	$\nu_9(\rho)$	$\nu_{13}(\rho)$
$\mathcal{NG}(\mathbb{Z}/(9),\ \frac{1}{9}g^2,\cdot,\cdot)$	$(\mathbb{Z}/(13),-\frac{2}{13}g^2)$	$1-\zeta_3$	3	$\frac{1+\sqrt{13}}{2}$
$\mathcal{NG}(\mathbb{Z}/(9),-\frac{1}{9}g^2,\cdot,\cdot)$	$(\mathbb{Z}/(13),\ \frac{2}{13}g^2)$	$1-\overline{\zeta_3}$	3	$\frac{1+\sqrt{13}}{2}$
$\mathcal{NG}((\mathbb{Z}/(3))^2,\frac{1}{3}(g^2-h^2),\cdot,\cdot)$	$(\mathbb{Z}/(13),\ \frac{2}{13}g^2)$	3	3	$\frac{1+\sqrt{13}}{2}$

$\lvert G\rvert=11$	(G',q')	$\nu_3(\rho)$	$\nu_5(\rho)$	$\nu_{11}(\rho)$	$\nu_{15}(\rho)$
$\mathcal{NG}(\mathbb{Z}/(11),\ \frac{1}{11}g^2,\cdot,\zeta_{12}^7)$	$(\mathbb{Z}/(15),\ \frac{2}{15}g^2)$	$\frac{1-i\sqrt{3}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{11-i\sqrt{11}}{2}$	$\frac{1+i\sqrt{15}}{2}$
$\mathcal{NG}(\mathbb{Z}/(11),\ \frac{1}{11}g^2,\cdot,\overline{\zeta_{12}})$	$(\mathbb{Z}/(15),\ \frac{1}{15}g^2)$	$\frac{1+i\sqrt{3}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{11-i\sqrt{11}}{2}$	$\frac{1+i\sqrt{15}}{2}$
$\mathcal{NG}(\mathbb{Z}/(11),-\frac{1}{11}g^2,\cdot,\zeta_{12})$	$(\mathbb{Z}/(15),-\frac{1}{15}g^2)$	$\frac{1-i\sqrt{3}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{11+i\sqrt{11}}{2}$	$\frac{1-i\sqrt{15}}{2}$
$\mathcal{NG}(\mathbb{Z}/(11),-\frac{1}{11}g^2,\cdot,\zeta_{12}^5)$	$(\mathbb{Z}/(15),-\frac{2}{15}g^2)$	$\frac{1+i\sqrt{3}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{11+i\sqrt{11}}{2}$	$\frac{1-i\sqrt{15}}{2}$

$\lvert G\rvert=13$	(G',q')	$\nu_{13}(\rho)$	$\nu_{17}(\rho)$
$\mathcal{NG}(\mathbb{Z}/(13),\frac{1}{13}g^2,b_1,-1)$	$(\mathbb{Z}/(17),\frac{3}{17}g^2)$	$\frac{13-\sqrt{13}}{2}$	$\frac{1+\sqrt{17}}{2}$
$\mathcal{NG}(\mathbb{Z}/(13),\frac{1}{13}g^2,b_2,-1)$	$(\mathbb{Z}/(17),\frac{3}{17}g^2)$	$\frac{13-\sqrt{13}}{2}$	$\frac{1+\sqrt{17}}{2}$
$\mathcal{NG}(\mathbb{Z}/(13),\frac{2}{13}g^2,b_3,\ 1)$	$(\mathbb{Z}/(17),\frac{1}{17}g^2)$	$\frac{13+\sqrt{13}}{2}$	$\frac{1-\sqrt{17}}{2}$
$\mathcal{NG}(\mathbb{Z}/(13),\frac{2}{13}g^2,b_4,\ 1)$	$(\mathbb{Z}/(17),\frac{1}{17}g^2)$	$\frac{13+\sqrt{13}}{2}$	$\frac{1-\sqrt{17}}{2}$

Remark 4.12. See that for $G=\mathbb{Z}/(13)$ we have two pairs of inequivalent fusion categories with the same indicators; hence, the near-group fusion ring $\mathrm{NG}(\mathbb{Z}/(13),13)$ does not have FS indicator rigidity. Note that the lesser odd order groups *do* exhibit FS indicator rigidity.

5. Frobenius–Schur indicators for Haagerup–Izumi fusion categories

Near groups are examples of *quadratic* fusion categories: those tensor-generated by a single noninvertible simple object ρ where the set of simple objects is given by

$$G\cup\{\bar{g}\otimes\rho\mid\bar{g}\text{ a coset representative in }G/H\}$$

where H is some subgroup of G. Near groups correspond to $H = G$. On the other end of the spectrum, the Haagerup–Izumi fusion categories correspond to $H = \{e\}$.

Definition 5.1. $\mathcal{C}$ is a *Haagerup–Izumi* fusion category if

$$K_0(\mathcal{C}) = \mathrm{HI}(G) := \mathbb{Z}[G \cup \{g\rho \mid g \in G\}]$$

where multiplication is given by the group law and

$$g(h\rho) = (gh)\rho = (h\rho)g^{-1},$$
$$(g\rho)(h\rho) = gh^{-1} + \sum_{x \in G} x\rho.$$

Classification and examples. The complete lists of Haagerup–Izumi categories for $G = \mathbb{Z}/(3)$ and $G = \mathbb{Z}/(5)$ were found by Evans and Gannon [2017] without assuming unitarity by generalizing Izumi's methods to endomorphisms of Leavitt algebras. These categories are classified up to isomorphism of the group G and the parameters

- a sign $\pm$,
- ω a third root of unity, and
- $A \in M_{|G|}(\mathbb{C})$ a complex matrix

all satisfying some relations given in [Evans and Gannon 2017, Theorem 1]. An Haagerup–Izumi category with the above parameters will be denoted

$$\mathcal{HI}(G, \pm, \omega, A).$$

The notion of equivalence for the parameters is given in [Evans and Gannon 2017, Theorem 2(b)]. In particular, the category is unitary if and only if both the sign is $+$ and A is hermitian [Evans and Gannon 2017, Theorem 2(c)].

The most important examples of HI categories are the Yang–Lee system of sectors, which is the unique nonunitary such category with G the trivial group, and the system of sectors for the Haagerup subfactor, which is a unitary HI category with $G = \mathbb{Z}/(3)$.

Indicators when $|G|$ is odd. When $\mathcal{C}$ is a Haagerup–Izumi fusion category the modular data for the center $\mathcal{Z}(\mathcal{C})$ was computed by Evans and Gannon [2017, §6.3] and is given as follows:

$X \in \mathcal{I}rr(\mathcal{Z}(\mathcal{C}))$		$F(X)$	braiding	θ_X
$\mathbb{1}$		$\mathbb{1}$	1	1
B		$\mathbb{1}+\sum_{g\in G} g\otimes\rho$	1	1
$A_\psi = A_{\overline{\psi}}$		$2\mathbb{1}+\sum_{g\in G} g\otimes\rho$	$\psi\in\widehat{G}\setminus\{1\}$	1
$C_\phi^{(h)}$	$(h\in G_+)$	$h+h^{-1}+\sum_{g\in G} g\otimes\rho$	$\phi\in\widehat{G}$	$\phi(h)$
D_j	$(1\le j\le \frac{1}{2}(\lvert G\rvert^2+3))$	$\sum_{g\in G} g\otimes\rho$		ζ_j

In the preceding table G_+ is defined by a partition $G = G_+ \sqcup \{e\} \sqcup G_-$ where $(G_+)^{-1} = G_-$, which is always possible since $|G|$ is odd.

The ζ_j are a solutions to a system of equations with coefficients given by $\pm$, ω, A. These equations are (6.14) and (6.16)–(6.19) in [Evans and Gannon 2017, §6.2]. See [Evans and Gannon 2017, Proposition 2]. For G odd order they make another conjecture:

Conjecture 5.2 [Evans and Gannon 2017, Conjecture 1]. *Suppose $|G|$ is odd. Then there exists a metric group (H, q'') of order $|G|^2+4 = 2m+1$ such that the simple objects D_j are indexed by $h \in H\setminus\{e\}$ where $D_h = D_{h^{-1}}$ and*

$$\theta_{D_h} = e^{2\pi i m q''(h)}.$$

Remark 5.3. Evans and Gannon [2017, Theorem 3] show that the conjecture is true for Haagerup–Izumi fusion categories with $|G| = 1, 3, 5$.

Theorem 5.4. *Suppose $\mathcal{C}$ is a Haagerup–Izumi fusion category with $|G|$ odd satisfying Conjecture 5.2. Then*

$$\nu_k(\rho) = \tfrac{1}{2}\theta_k^G(e) + \tfrac{1}{2}\Theta(H, kmq'').$$

Proof. Let $d = \operatorname{qdim}(\rho)$ in the category $\mathcal{C}$. Again by using Theorem 2.5

$$\nu_k(\rho) = \frac{1}{\operatorname{qdim}(\mathcal{C})}\Biggl(\operatorname{qdim}(B) + \sum_{\overline{\psi}\neq\psi\in\widehat{G}} \operatorname{qdim}(A_\psi) + \sum_{e\neq h^{-1}\neq h\in G}\sum_{\phi\in\widehat{G}} \theta^k_{C^h_\phi}\operatorname{qdim}(C^h_\phi) + \sum_{\gamma^{-1}\neq\gamma\in H} \theta_{D_\gamma}\operatorname{qdim}(D_\gamma)\Biggr).$$

Letting $|G| = 2n+1$ and $|H| = 2m+1$ we may enumerate these odd order groups as

$$G = \{e, g_i, g_i^{-1} \mid 1\le i\le n\} \quad\text{and}\quad H = \{e, h_j, h_j^{-1} \mid 1\le j\le m\},$$

which gives us

$$\nu_k(\rho) = \frac{1}{\operatorname{qdim}(\mathcal{C})}\Biggl(|G| + |G|d + |G|dn + (2+|G|d)\sum_{g_i,\phi}\phi(g_i)^k + |G|d\sum_{h_j\in H}\zeta_j^k\Biggr).$$

By the same argument as in the proofs of Theorems 3.2 and 4.8 we can see

$$\sum_{g_i,\phi} \phi(g_i)^k = \frac{|G|}{2}(\theta_k^G(e) - 1).$$

Hence, using the expression for the center's ribbon structure from Conjecture 5.2 and the fact that $|H| = |G|^2 + 4$ and $\operatorname{qdim}(\mathcal{C}) = 2|G| + d|G|^2$ we see

$$\begin{aligned}
\nu_k(\rho) &= \frac{|G|}{\operatorname{qdim}(\mathcal{C})}\left(\frac{2+|G|d}{2}\theta_k^G(e) + d + dn - \frac{|G|d}{2} + d\sum_{h_j \in H} e^{2\pi i k m q''(h_j)}\right) \\
&= \tfrac{1}{2}\theta_k^G(e) + \frac{|G|}{2\operatorname{qdim}(\mathcal{C})}\big(2d + 2dn - |G|d + d(\sqrt{|H|}\Theta(H, kmq'') - 1)\big) \\
&= \tfrac{1}{2}\theta_k^G(e) + \frac{|G|d\sqrt{|G|^2+4}}{2\operatorname{qdim}(\mathcal{C})}\Theta(H, kmq'') \\
&= \tfrac{1}{2}\theta_k^G(e) + \tfrac{1}{2}\Theta(H, kmq''). \qquad \square
\end{aligned}$$

Now we collect in the tables below the values of the Frobenius–Schur indicators for the Haagerup–Izumi categories constructed in [Evans and Gannon 2011]:

$G = \mathbb{Z}/(3)$	(H, q'')	$\nu_k(\rho)$	$\nu_3(\rho)$	$\nu_{13}(\rho)$
$\mathcal{HI}(\mathbb{Z}/(3), +, 1, A_1)$	$(\mathbb{Z}/(13), \frac{1}{13}g^2)$	$\frac{1}{2}(1 - (\frac{1}{13}k))$	1	$\frac{1+\sqrt{13}}{2}$
$\mathcal{HI}(\mathbb{Z}/(3), +, 1, A_2)$	$(\mathbb{Z}/(13), \frac{1}{13}g^2)$	$\frac{1}{2}(1 - (\frac{1}{13}k))$	1	$\frac{1+\sqrt{13}}{2}$
$\mathcal{HI}(\mathbb{Z}/(3), -, 1, A_3)$	$(\mathbb{Z}/(13), \frac{2}{13}g^2)$	$\frac{1}{2}(1 + (\frac{1}{13}k))$	2	$\frac{1+\sqrt{13}}{2}$
$\mathcal{HI}(\mathbb{Z}/(3), -, 1, A_4)$	$(\mathbb{Z}/(13), \frac{2}{13}g^2)$	$\frac{1}{2}(1 + (\frac{1}{13}k))$	2	$\frac{1+\sqrt{13}}{2}$

In the preceding table the integer k must be relatively prime to $3 \cdot 13 = 39$.

$G = \mathbb{Z}/(5)$	(H, q'')	$\nu_k(\rho)$	$\nu_5(\rho)$	$\nu_{29}(\rho)$
$\mathcal{HI}(\mathbb{Z}/(5), +, 1, A_6)$	$(\mathbb{Z}/(29), \frac{1}{29}g^2)$	$\frac{1}{2}(1 - (\frac{1}{29}k))$	2	$\frac{1+\sqrt{29}}{2}$
$\mathcal{HI}(\mathbb{Z}/(5), +, 1, A_7)$	$(\mathbb{Z}/(29), \frac{1}{29}g^2)$	$\frac{1}{2}(1 - (\frac{1}{29}k))$	2	$\frac{1+\sqrt{29}}{2}$
$\mathcal{HI}(\mathbb{Z}/(5), -, 1, A_8)$	$(\mathbb{Z}/(29), \frac{2}{29}g^2)$	$\frac{1}{2}(1 + (\frac{1}{29}k))$	3	$\frac{1+\sqrt{29}}{2}$
$\mathcal{HI}(\mathbb{Z}/(5), -, 1, A_9)$	$(\mathbb{Z}/(29), \frac{2}{29}g^2)$	$\frac{1}{2}(1 + (\frac{1}{29}k))$	3	$\frac{1+\sqrt{29}}{2}$

In the preceding table the integer k must be relatively prime to $5 \cdot 13 = 65$.

Remark 5.5. See that for each of $\mathbb{Z}/(3)$ and $\mathbb{Z}/(5)$ we have two pairs of inequivalent fusion categories with the same indicators; hence, the Haagerup–Izumi fusion rings do not have FS rigidity.

Note that in this case as well as the $m = |G| = 13$ near-group case the pairs have centers with the same modular data (although it is not established whether the centers are equivalent). In view of this we formulate the following conjecture.

Conjecture 5.6. *Two fusion categories with a given Grothendieck ring that are also Morita equivalent cannot be distinguished by their Frobenius–Schur indicators.*

Acknowledgments

The author wishes to thank his PhD advisor Susan Montgomery; this article comprises the results of the author's dissertation. The author expresses much gratitude to Masaki Izumi, Richard Ng, and Peter Schauenburg for many useful conversations during the development of this work. Thanks are also due to David Penneys for pointing out the paper [Evans and Gannon 2014] to the author and to Vaughan Jones for introducing the author to the work of Izumi.

References

[Bakalov and Kirillov 2001] B. Bakalov and A. Kirillov, Jr., *Lectures on tensor categories and modular functors*, University Lecture Series **21**, Amer. Math. Soc., Providence, RI, 2001. MR Zbl

[Basak and Johnson 2015] T. Basak and R. Johnson, "Indicators of Tambara–Yamagami categories and Gauss sums", *Algebra Number Theory* **9**:8 (2015), 1793–1823. MR Zbl

[Eilenberg and Mac Lane 1953] S. Eilenberg and S. Mac Lane, "On the groups $H(\Pi, n)$, I", *Ann. of Math.* (2) **58** (1953), 55–106. MR Zbl

[Eilenberg and Mac Lane 1954] S. Eilenberg and S. Mac Lane, "On the groups $H(\Pi, n)$, II: Methods of computation", *Ann. of Math.* (2) **60** (1954), 49–139. MR Zbl

[Etingof et al. 2004] P. Etingof, S. Gelaki, and V. Ostrik, "Classification of fusion categories of dimension pq", *Int. Math. Res. Not.* **2004**:57 (2004), 3041–3056. MR Zbl

[Etingof et al. 2015] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, *Tensor categories*, Mathematical Surveys and Monographs **205**, Amer. Math. Soc., Providence, RI, 2015. MR Zbl

[Evans and Gannon 2011] D. E. Evans and T. Gannon, "The exoticness and realisability of twisted Haagerup–Izumi modular data", *Comm. Math. Phys.* **307**:2 (2011), 463–512. MR Zbl

[Evans and Gannon 2014] D. E. Evans and T. Gannon, "Near-group fusion categories and their doubles", *Adv. Math.* **255** (2014), 586–640. MR Zbl

[Evans and Gannon 2017] D. E. Evans and T. Gannon, "Non-unitary fusion categories and their doubles via endomorphisms", *Adv. Math.* **310** (2017), 1–43. MR Zbl

[Isaacs 1976] I. M. Isaacs, *Character theory of finite groups*, Pure and Applied Mathematics **69**, Academic, New York, 1976. MR Zbl

[Izumi 2000] M. Izumi, "The structure of sectors associated with Longo–Rehren inclusions, I: General theory", *Comm. Math. Phys.* **213**:1 (2000), 127–179. MR Zbl

[Izumi 2001] M. Izumi, "The structure of sectors associated with Longo–Rehren inclusions, II: Examples", *Rev. Math. Phys.* **13**:5 (2001), 603–674. MR Zbl

[Izumi 2017] M. Izumi, "A Cuntz algebra approach to the classification of near-group categories", pp. 222–343 in *Proceedings of the 2014 Maui and 2015 Qinhuangdao conferences in honour of Vaughan F. R. Jones' 60th birthday*, edited by S. Morrison and D. Penneys, Proc. Centre Math. Appl. Austral. Nat. Univ. **46**, Austral. Nat. Univ., Canberra, 2017. MR Zbl

[Linchenko and Montgomery 2000] V. Linchenko and S. Montgomery, "A Frobenius–Schur theorem for Hopf algebras", *Algebr. Represent. Theory* **3**:4 (2000), 347–355. MR Zbl

[Mason and Ng 2005] G. Mason and S.-H. Ng, "Central invariants and Frobenius–Schur indicators for semisimple quasi-Hopf algebras", *Adv. Math.* **190**:1 (2005), 161–195. MR Zbl

[Müger 2003] M. Müger, "From subfactors to categories and topology, II: The quantum double of tensor categories and subfactors", *J. Pure Appl. Algebra* **180**:1–2 (2003), 159–219. MR Zbl

[Ng and Schauenburg 2007a] S.-H. Ng and P. Schauenburg, "Frobenius–Schur indicators and exponents of spherical categories", *Adv. Math.* **211**:1 (2007), 34–71. MR Zbl

[Ng and Schauenburg 2007b] S.-H. Ng and P. Schauenburg, "Higher Frobenius–Schur indicators for pivotal categories", pp. 63–90 in *Hopf algebras and generalizations*, edited by L. H. Kauffman et al., Contemp. Math. **441**, Amer. Math. Soc., Providence, RI, 2007. MR Zbl

[Ng and Schauenburg 2008] S.-H. Ng and P. Schauenburg, "Central invariants and higher indicators for semisimple quasi-Hopf algebras", *Trans. Amer. Math. Soc.* **360**:4 (2008), 1839–1860. MR Zbl

[Serre 1977] J.-P. Serre, *Linear representations of finite groups*, Graduate Texts Math. **42**, Springer, 1977. MR Zbl

[Shimizu 2011] K. Shimizu, "Frobenius–Schur indicators in Tambara–Yamagami categories", *J. Algebra* **332** (2011), 543–564. MR Zbl

[Suzuki and Wakui 2002] K. Suzuki and M. Wakui, "On the Turaev–Viro–Ocneanu invariant of 3-manifolds derived from the E_6-subfactor", *Kyushu J. Math.* **56**:1 (2002), 59–81. MR Zbl

[Tambara and Yamagami 1998] D. Tambara and S. Yamagami, "Tensor categories with fusion rules of self-duality for finite abelian groups", *J. Algebra* **209**:2 (1998), 692–707. MR Zbl

[Wall 1963] C. T. C. Wall, "Quadratic forms on finite groups, and related topics", *Topology* **2** (1963), 281–298. MR Zbl

Received June 9, 2017. Revised January 20, 2019.

HENRY TUCKER
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, SAN DIEGO
LA JOLLA, CA
UNITED STATES

hjtucker@ucsd.edu

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2019 is US $490/year for the electronic version, and $665/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 303 No. 1 November 2019